Update of Section 7.4 **10.7.14**

This is an update of Section 7.4, written and installed on 10.7.14.

7.4 Second Sign of Trouble: Infinite B fields as ω→0

Recall from (D.11.7) the E fields for small ω,

$$E_z(r,m) = (1/2)\,\eta_m\,\mathcal{B}\,(\omega/k)\,(r/a)^m\,(m+1) \qquad (D.11.7) \qquad (7.4.1)$$
$$E_r(r,m) = (j/4)\,\eta_m\,\mathcal{B}\,(\omega a)\,[(r/a)^{m+1} + (r/a)^{m-1}]$$
$$E_\theta(r,m) = (1/4)\,\eta_m\,\mathcal{B}\,(\omega a)\,[(r/a)^{m+1} - (r/a)^{m-1}] \qquad m > 0$$

$$E_z(r,0) = \mathcal{B}\,(\omega/k) \qquad \mathcal{B} \equiv (\xi_d/\varepsilon_d)\,CV\,R_{dc}$$
$$E_r(r,0) = (j/2)\,\mathcal{B}\,(\omega r)$$
$$E_\theta(r,0) = 0 \qquad m = 0 \qquad G \geq 0$$

Using (D.4.7), which expresses the Maxwell equation **B** = (j/ω) curl **E** in cylindrical coordinates,

$$B_r(r,m) = (j/\omega)\,[\text{curl } E]_r = (j/\omega)\,[r^{-1}jmE_z + jkE_\theta]$$
$$B_\theta(r,m) = (j/\omega)\,[\text{curl } E]_\theta = (j/\omega)[-jkE_r - \partial_r E_z]$$
$$B_z(r,m) = (j/\omega)\,[\text{curl } E]_z = (j/\omega)\,[r^{-1}\partial_r(rE_\theta) - r^{-1}jmE_r]\,, \qquad (D.4.7)$$

we shall have Maple compute the corresponding B fields.

Enter the Bi expressions

```
Br := (j/omega)*((j*m/r)*Ez + j*k*Eth);
```

$$Br := \frac{j\left(\frac{j\,m\,Ez}{r} + j\,k\,Eth\right)}{\omega}$$

```
Bth := (j/omega)*(-j*k*Er - Diff(Ez,r));
```

$$Bth := \frac{j\left(-j\,k\,Er - \left(\frac{\partial}{\partial r}Ez\right)\right)}{\omega}$$

```
Bz := (j/omega)*(1/r)*( Diff(r*Eth,r) - j*m*Er);
```

$$Bz := \frac{j\left(\left(\frac{\partial}{\partial r}r\,Eth\right) - j\,m\,Er\right)}{\omega\,r}$$

Before continuing, look at the first term above for B_r which has the form $\frac{j}{\omega}\,\frac{jmE_z}{r}$. If we install E_z from (7.4.1), the (ω/k) factor in E_z results in a 1/k factor in B_r. This is a preview of our upcoming problem.

Momentarily omitting the common factor $\eta_m\,\mathcal{B}$, we enter the E fields for m > 0 from (7.4.1) above,

Enter the Ei expressions for m > 0

```
Ez := (1/2)*(omega/k)*(r/a)^m*(m+1);
```

$$Ez := \frac{1}{2}\frac{\omega \left(\frac{r}{a}\right)^{m} (m+1)}{k}$$

```
Er := (j/4)*(omega*a)*( (r/a)^(m+1) + (r/a)^(m-1));
```

$$Er := \frac{1}{4} j\, \omega\, a \left(\left(\frac{r}{a}\right)^{(m+1)} + \left(\frac{r}{a}\right)^{(m-1)}\right)$$

```
Eth := (1/4)*(omega*a)*( (r/a)^(m+1) - (r/a)^(m-1));
```

$$Eth := \frac{1}{4} \omega\, a \left(\left(\frac{r}{a}\right)^{(m+1)} - \left(\frac{r}{a}\right)^{(m-1)}\right)$$

Maple then computes the resulting B fields:

```
Br: simplify(%);
```

$$\frac{1}{4}\frac{\left(\frac{r}{a}\right)^{m} (-2m^2 - 2m - k^2 r^2 + k^2 a^2)}{r\,k}$$

```
value(Bth): simplify(%);
```

$$\frac{1}{4}\frac{j\left(\frac{r}{a}\right)^{m} (k^2 r^2 + k^2 a^2 - 2m^2 - 2m)}{r\,k}$$

```
value(Bz): simplify(%);
```

$$\frac{1}{2} j \left(\frac{r}{a}\right)^{m} (m+1)$$

Using $(r/a)^m/r = (r/a)^m (a/r)(1/a) = (r/a)^{m-1}(1/a)$, the Maple results for m > 0 are,

$B_r = (1/4)\,(r/a)^{m-1}\,(1/ak)[\,-2m(m+1) + k^2(a^2-r^2)]$
$B_\theta = (j/4)\,(r/a)^{m-1}\,(1/ak)\,[\,-2m(m+1) + k^2(r^2+a^2)]$
$B_z = (j/2)\,(r/a)^m\,(m+1)$

We repeat the effort for m = 0,

Enter the Ei expressions for m = 0

```
Ez := (omega/k);
```

$$Ez := \frac{\omega}{k}$$

```
Er := (j/2)*omega*r;
```

$$Er := \frac{1}{2} j\, \omega\, r$$

```
Eth := 0;
```

$$Eth := 0$$

to get

```
m := 0;
```

$$m := 0$$

```
Br: simplify(%);
```

$$0$$

```
value(Bth): simplify(%);
```

$$\frac{1}{2} j k r$$

```
value(Bz): simplify(%);
```

$$0$$

Thus, restoring the $\eta_m \mathcal{B}$ factor, for small ω the B fields inside the round wire are given by

B fields in round wire for small ω (7.4.2)

$B_z(r,m) = (j/2)\, \eta_m \mathcal{B}\, (r/a)^m\, (m+1)$ $\qquad m > 0$

$B_r(r,m) = (1/4)\, \eta_m \mathcal{B}\, (r/a)^{m-1}\, (1/ak)[\, -2m(m+1) + k^2(a^2-r^2)]$

$B_\theta(r,m) = (j/4)\, \eta_m \mathcal{B}\, (r/a)^{m-1}\, (1/ak)\, [\, -2m(m+1) + k^2(r^2+a^2)]$

$B_z(r,0) = 0$ $\qquad m = 0$

$B_r(r,0) = 0$

$B_\theta(r,0) = (j/2)\, \mathcal{B}\, kr$ $\qquad \mathcal{B} \equiv (\xi_d/\varepsilon_d)\, CV\, R_{dc}$ $\qquad G \geq 0$

These same B fields can be obtained by taking the small-x limit of the full B fields shown in (D.9.39).

So far we have no infinite B problem. Recall that in Appendix D, the parameter k is an arbitrary complex number. But in order to take the "small ω limit" as we did in Section D.11, we have to assume at least that k is "small" for small ω, so that then $\beta'^2 = \beta^2 - k^2 = -j\omega\mu\sigma - k^2$ will be small, allowing a power series expansion for the various Bessel functions (recall $x = \beta' r$). So as long as k is some small number (in magnitude), we know that the E and B fields given in (7.4.1) and (7.4.2) represent a solution to the Helmholtz equation for E, the div E = 0 equation, all four Maxwell equations, and in fact also the Helmholtz equation for B, though we did not demonstrate this fact. The fields also satisfy the two boundary conditions (D.2.26) and (D.2.27). By assuming small ω and small k and hence small β', we were able to replace the Bessel functions with their leading expansion terms resulting in the various simple polynomial terms in (7.4.1) and (7.4.2).

The trouble now arises if we *further* assume that our small k value of the last paragraph is identified with the ω→0 value of $k = -j\sqrt{zy} = -j\sqrt{(R+j\omega L)(G+j\omega C)}$. This is the k value for the wave proceeding down the dielectric of the transmission line, *based on* the transmission line equations (4.12.17).

G = 0 Case

We first assume that G = 0 which in turn implies $(\xi_d/\varepsilon_d) = 1$. According to (Q.4.9), and assuming that $k = -j\sqrt{(R+j\omega L)(G+j\omega C)}$, we see that as $\omega \to 0$,

$$k \to \sqrt{R_{dc2}C/2}\ \sqrt{\omega}\ (1-j) = \sqrt{R_{dc2}C}\ \sqrt{\omega}\ e^{-j\pi/4} \qquad \text{as } \omega \to 0 \qquad (7.4.3)$$

We are now using R_{dc} as the resistance per length of our round wire conductor, while R_{dc2} is the resistance of both transmission line conductors per length. Note that $k \to 0$ as $\sqrt{\omega}$. The above B fields (7.4.2) then become

$$B_z(r,m) = (j/2)\,\eta_m\,CV\,R_{dc}\,(r/a)^m\,(m+1) \qquad m>0 \qquad (7.4.4)$$
$$B_r(r,m) = (1/4)\,\eta_m\,CV\,R_{dc}\,(r/a)^{m-1}\,(1/a)\,[-2m(m+1)] / [\sqrt{R_{dc2}C}\ \sqrt{\omega}\ e^{-j\pi/4}]$$
$$B_\theta(r,m) = (j/4)\,\eta_m\,CV\,R_{dc}\,(r/a)^{m-1}\,(1/a)\,[-2m(m+1)] / [\sqrt{R_{dc2}C}\ \sqrt{\omega}\ e^{-j\pi/4}]$$

$$B_z(r,0) = 0 \qquad m = 0$$
$$B_r(r,0) = 0$$
$$B_\theta(r,0) = (j/2)\,CV\,R_{dc}\,r\,[\sqrt{R_{dc2}C}\ \sqrt{\omega}\ e^{-j\pi/4}] \qquad G = 0$$

As $\omega \to 0$, the three m = 0 B fields go to zero. This is as expected since the current vanishes:

$$I = CV\,(\omega/k) = CV\,\omega / [\sqrt{R_{dc2}C}\ \sqrt{\omega}\ e^{-j\pi/4}] = CV\sqrt{\omega}\ / [\sqrt{R_{dc2}C}\ e^{-j\pi/4}] \to 0 \quad . \qquad (7.4.5)$$

However, as $\omega \to 0$, the m> 0 B fields all diverge as $1/\sqrt{\omega}$!! In the DC limit, there will still be some asymmetric $n(\theta)$ on the round wire surface just because this round wire is part of a long capacitor connected to a battery of voltage V, so the η_m coefficients do not vanish for m > 0. Looking at (7.4.1) we see that as $\omega \to 0$, all the E field components vanish in all partial waves m, as we would expect since the transmission line impedance goes to ∞. It seems a bit unphysical for our round wire at DC to have no E fields, no current, but some infinite internal B fields. One can show that curl B is finite in the $\omega \to 0$ limit (and div B = 0), but this is not much respite.

<u>G > 0 Case</u>

We start again with (7.4.2), but now

$$(\xi_d/\varepsilon_d) = 1 + (G/j\omega C) \ \to\ (G/j\omega C) \qquad \text{as } \omega \to 0 \qquad (7.4.6)$$

Moreover, according to (Q.4.6), and assuming that $k = -j\sqrt{(R+j\omega L)(G+j\omega C)}$, we see that

$$k \to -j\sqrt{R_{dc2}G} \ \equiv\ k_1 \qquad \text{as } \omega \to 0 \qquad \text{// a very small constant value} \qquad (7.4.7)$$

Then (7.4.2) reads,

$$B_z(r,m) = (j/2)\,\eta_m\ (G/j\omega)\,V\,R_{dc}\,(r/a)^m\,(m+1) \qquad m > 0 \qquad (7.4.8)$$
$$B_r(r,m) = (1/4)\,\eta_m\,(G/j\omega)\,V\,R_{dc}\,(r/a)^{m-1}\,(1/ak_1)\,[-2m(m+1) + k_1^2(a^2-r^2)]$$

$$B_\theta(r,m) = (j/4)\,\eta_m\,(G/j\omega)\,V\,R_{dc}\,(r/a)^{m-1}\,(1/ak_1)\,[\,-2m(m+1) + k_1^2(r^2+a^2)]$$

$B_z(r,0) = 0$ $\qquad m = 0$

$B_r(r,0) = 0$

$B_\theta(r,0) = (j/2)\,(G/j\omega C)\;CV\,R_{dc}\,k_1 r$ $\qquad G > 0$

We find that for G > 0, all the non-zero B field components diverge as $1/\omega$ as $\omega \to 0$!! This is even worse than that G = 0 case where divergence was $1/\sqrt{\omega}$.

This then is our second sign of trouble: B fields are going infinite as $\omega \to 0$. It seems clear that the physical B field of a transmission line operating at DC should be finite since there are no infinite currents anywhere. Thus, the theory of Appendix D combined with the idea that $k = -j\sqrt{(R+j\omega L)(G+j\omega C)}$ is *invalid* as $\omega \to 0$.