Archive this rejected section of Chapter 7 retired on 9/21/14

7.4 A Second Sign of Trouble: Infinite B fields as ω→0

In Appendix D we compute the E fields inside a round wire as outlined above, then the corresponding B fields are computed from the Maxwell curl E equation,

$$\text{curl } \mathbf{E} = -j\omega\mathbf{B} \qquad \Rightarrow \qquad \mathbf{B} = (j/\omega)\ \text{curl } \mathbf{E}$$

In the partial wave m space, and in cylindrical coordinates, this Maxwell equation reads,

$$\mathbf{B} = (j/\omega)\ \{\ \hat{\mathbf{r}}\ [\ r^{-1}jmE_z + jkE_\theta] + \hat{\boldsymbol{\theta}}\ [-jkE_r - \partial_r E_z] + \hat{\mathbf{z}}\ [\ r^{-1}\partial_r(rE_\theta) - r^{-1}jmE_r\ ]\ \}$$

or

$$B_r(r,m) = (j/\omega)\ [\ r^{-1}jmE_z(r,m) + jkE_\theta(r,m)]$$
$$B_\theta(r,m) = (j/\omega)\ [-jkE_r(r,m) - \partial_r E_z(r,m)]$$
$$B_z(r,m) = (j/\omega)\ [r^{-1}\partial_r(rE_\theta(r,m)) - r^{-1}jmE_r(r,m)]$$

where as usual we have replaced $\partial_\theta \rightarrow jm$ and $\partial_z \rightarrow -jk$.

We are interested in examining these B fields as ω→0. Near this limit, we know the behavior of k from Appendix Q.

We first assume that $G_{dc} = 0$ (vacuum dielectric) which means $\omega_d = 0$ in Appendix Q. We then find -- assuming that the equation $k = k(\omega) \equiv -j\sqrt{(R+j\omega L)(G+j\omega C)}$ is valid in this limit -- that

Fact 4: The small ω limit for k(ω), assuming $\omega_d = 0$, is given by (Q.4.9)

$$\text{Re}(k) \approx +\sqrt{R_{dc}C/2}\ \sqrt{\omega}\ (\ 1 - \tan_L/2)\ + O(\omega^{3/2})$$
$$\text{Im}(k) \approx -\sqrt{R_{dc}C/2}\ \sqrt{\omega}\ (\ 1 + \tan_L/2)\ + O(\omega^{3/2})$$

where R_{dc} is the total resistance per length of both transmission line conductors. Ignoring the small loss tangent, this says

$$k(\omega) = \sqrt{R_{dc}C/2}\ \sqrt{\omega}\ (1 - j) = \sqrt{R_{dc}C}\ \sqrt{\omega}\ e^{-j\pi/4}$$

$$k^2 = -j\ R_{dc}C\ \omega\ .$$

Meanwhile, we know from ** that $\beta^2 = -j\mu\sigma\omega$. It then follows that

$$\beta'^2 \equiv \beta^2 - k^2 \rightarrow -j\mu\sigma\omega + j\ R_{dc}C\ \omega = j\ [-\mu\sigma + R_{dc}C]\ \omega$$

As ω→ 0, β' becomes very small, and $x_a \equiv \beta' a$ and $x \equiv \beta' r$ (for $r \le a$) become << 1. We can therefore use the E field expressions given in (D.11.7) ,

$$E_z(r,m) = (1/2)\,\eta_m\, I\, R\, (r/a)^{|m|}\,(|m|+1)$$
$$E_r(r,m) = (j/4)\,\eta_m\, I\, R\, (ak)\,[(r/a)^{|m|+1} + (r/a)^{|m|-1}] \qquad (D.11.7)$$
$$E_\theta(r,m) = (1/4)\,\eta_m\, I\, R\, (ak)\,[(r/a)^{|m|+1} - (r/a)^{|m|-1}]$$

$$E_z(r,0) = I\,R \qquad R = 1/(\sigma\pi a^2)$$
$$E_r(r,0) = (j/2)\, I\, R\, (ak)\,(r/a)$$
$$E_\theta(r,0) = 0 \qquad \text{// low } \omega \text{ E fields}$$

Whereas R_{dc} is the DC resistance of both conductors, R is the DC resistance just of the round wire.

In these expressions, we may write, using ** and **,

$$I = 2\pi a(\omega/k)N_0 = 2\pi a(\omega/k)\{q/2\pi a\} = q\,(\omega/k) = CV\,(\omega/k)$$

But for small ω,

$$(\omega/k) = \omega / [\sqrt{R_{dc}C}\,\sqrt{\omega}\; e^{-j\pi/4}\,] = e^{j\pi/4}\,\sqrt{1/(R_{dc}C)}\,\sqrt{\omega}$$

so

$$I = CV\,(\omega/k) = \{\, e^{j\pi/4}\sqrt{C/R_{dc}}\,\sqrt{\omega}\,\}\, V$$

This implies that $Z_0^{-1} = I/V = \{\, e^{j\pi/4}\sqrt{C\omega/R_{dc}}\,\}$, something we can quickly verify (as $\omega \to 0$) :

$$Z_0 = \sqrt{(R+j\omega L)/(G+j\omega C)} = \sqrt{(R_{dc})/(\,j\omega C)}$$

$$\Rightarrow\; Z_0^{-1} = \sqrt{j\omega C/R_{dc}} = e^{j\pi/4}\,\sqrt{\omega C/R_{dc}}\;.$$

Notice that $I \to$ constant $* \sqrt{\omega}$ and $Z_0 \to$ constant $/ \sqrt{\omega}$. In the final $\omega \to 0$ limit, $I \to 0$ and $Z_0 \to \infty$, but we are interested in what happens very close to that limit.

We can rewrite our E fields for small ω using I = CV (ω/k). On 3 lines we replace (ω/k) (ak) = a, while on two lines we use $C\,(\omega/k) = e^{j\pi/4}\,\sqrt{C/R_{dc}}\,\sqrt{\omega}$. The resulting small ω fields are

$$E_z(r,m) = (1/2)\,\eta_m\, V\, e^{j\pi/4}\,\sqrt{C/R_{dc}}\,\sqrt{\omega}\; R\; (r/a)^{|m|}\,(|m|+1)$$
$$E_r(r,m) = (j/4)\,\eta_m\, CV\,(\omega)\, R\; (a)\,[(r/a)^{|m|+1} + (r/a)^{|m|-1}] \qquad (D.11.7)$$
$$E_\theta(r,m) = (1/4)\,\eta_m\, CV\,(\omega)\, R\; (a)\,[(r/a)^{|m|+1} - (r/a)^{|m|-1}]$$

$$E_z(r,0) = V\, e^{j\pi/4}\,\sqrt{C/R_{dc}}\,\sqrt{\omega}\; R$$
$$E_r(r,0) = (j/2)\, CV\,(\omega)\, R\; (a)\,(r/a)$$
$$E_\theta(r,0) = 0 \qquad \text{// low } \omega \text{ E fields}$$

In preparation for evaluating the B field, we compute these quantities

$\partial_r E_z(r,0) = 0$ m=0

$\partial_r E_z(r,m) = (1/2)\,\eta_m\, V\, e^{j\pi/4}\,\sqrt{C/R_{dc}}\,\sqrt{\omega}\;R\;|m|\,(r/a)^{|m|-1}\,(1/a)\,(|m|+1)$ m>0

$r\,\partial_r(rE_\theta(r,0)) = 0$

$r\,\partial_r(rE_\theta(r,m)) = r\,\partial_r\,[(1/4)\,\eta_m\,CV\,(\omega)\,R\;(a)\;[(r/a)^{|m|+2} - (r/a)^{|m|}]]$ m=0

$= r\,(1/4)\,\eta_m\,CV\,(\omega)\,R\;(a)\;\{\;(|m|+2)\,(r/a)^{|m|+1} - (|m|)\,(r/a)^{|m|-1}\;\}\,(1/a)$

$= (1/4)\,\eta_m\,CV\,(\omega)\,R\;(a)\;\{\;(|m|+2)\,(r/a)^{|m|+1} - (|m|)\,(r/a)^{|m|-1}\;\}\,(r/a)$

$= (1/4)\,\eta_m\,CV\,(\omega)\,R\;(a)\;\{\;(|m|+2)\,(r/a)^{|m|+2} - (|m|)\,(r/a)^{|m|}\;\}$ m > 0

As a space-saving measure, we redisplay the above fields showing only the ω dependence, making up dummy names for the combination of factors.

$E_z(r,m) = A_m\sqrt{\omega}$
$E_r(r,m) = B_m\,\omega$ *(D.11.7)*
$E_\theta(r,m) = C_m\,\omega$

$E_z(r,0) = A_0\sqrt{\omega}$
$E_r(r,0) = B_0\,\omega$
$E_\theta(r,0) = 0$ // low ω E fields

which we simplify to read

$E_z(r,m) = A_m\sqrt{\omega}$
$E_r(r,m) = B_m\,\omega$ // for all m, and $C_0 = 0$
$E_\theta(r,m) = C_m\,\omega$

We may now insert these expressions for the small-ω fields into our **B** = (j/ω) curl **E** equations above,

$B_r(r,m) = (j/\omega)\,[\,r^{-1}jmE_z(r,m) + jkE_\theta(r,m)]$
$B_\theta(r,m) = (j/\omega)\,[-jkE_r(r,m) - \partial_r E_z(r,m)]$
$B_z(r,m) = (j/\omega)\,[r^{-1}\partial_r(rE_\theta(r,m)) - r^{-1}jmE_r(r,m)]$

For B_r

$B_r(r,m) = (j/\omega)\,[r^{-1}jm\,(1/2)\,\eta_m\,V\,e^{j\pi/4}\,\sqrt{C/R_{dc}}\,\sqrt{\omega}\;R\;(r/a)^{|m|}\,(|m|+1)$

The E fields computed in Appendix D may be written as

Second summary of the E field solutions : $R_{dc} = \frac{1}{\sigma \pi a^2}$ $\beta'^2 = \beta^2 - k^2$ *(D.2.33)*

$E_z(r,m) = (1/4)\,\eta_m\, I\, R_{dc}\, (a\beta')\, f_m$ $f_m = \left[\frac{J_m(x)}{J_{m+1}(x_a)} - \frac{J_m(x)}{J_{m-1}(x_a)}\right]$ $x = \beta' r$

$E_r(r,m) = (j/4)\,\eta_m\, I\, R_{dc}\, (ak)\, g_m$ $g_m = \left[\frac{J_{m+1}(x)}{J_{m+1}(x_a)} + \frac{J_{m-1}(x)}{J_{m-1}(x_a)}\right]$ $x_a = \beta' a$

$E_\theta(r,m) = (1/4)\,\eta_m\, I\, R_{dc}\, (ak)\, h_m$ $h_m = \left[\frac{J_{m+1}(x)}{J_{m+1}(x_a)} - \frac{J_{m-1}(x)}{J_{m-1}(x_a)}\right]$

This is too messy, I want another way to show this problem.