**A Case Study** PhL 11.13.19

Fully reviewed.

Here I am going to look at the A solution for the round wire with uniform current in two ways, which I will call the Problem I Way and the Problem II Way. This is all done at $\omega = 0$ and in the 2D way

**Problem I Way.** [ this method is supportable ] The problem is described separately for regions 1 and 2, there is no common PDE, there is no image current on the boundary.

$(\nabla^2 + \beta_1^2)\, \mathbf{A}^{\mathbf{I}} = 0$ x in region 1

$(\nabla^2 + \beta_1^2)\, \mathbf{A}^{\mathbf{I}} = -\mu_2\, \mathbf{J_{c2}}$ x in region 2

$(A_z^{\mathbf{I}})_2 = (A_z^{\mathbf{I}})_1$

$(1/\mu_2)\,(\partial_n A_z^{\mathbf{I}})_2 = (1/\mu_1)\,(\partial_n A_z^{\mathbf{I}})_1$

To solve this problem I make these *assumptions*:

1. The solution in region 2 is the particular solution plus a constant d. Thus

$$A^{\mathbf{I,region2}} = (1/2\pi) \int dV' \,[\mu_2 \mathbf{J_c(x')}] \ln(1/R) \quad + \quad d \;=\; W + d$$

where constant d is a homo solution to $(\nabla^2 + \beta_1^2)\, \mathbf{u} = 0$ .

2. The solution in region 1 is a homo solution of this form:

$$A^{\mathbf{I,region1}} = A \ln r$$

3. I then evaluate the integral W. Just for fun, I do it for both region 1 and region 2, but I only need it right now for region 2 where r < a.

$$W = (1/2\pi) \int dV' \,[\mu_2 \mathbf{J_c(x')}] \ln(1/R) \;= -(1/4\pi)\, \mu_2 \int dV' [J_{\mathbf{cz}}(\mathbf{x'})] \ln(R^2) \;\} \;= -(1/4\pi)\, \mu_2 J_{\mathbf{cz}} \int dV' \ln(R^2) \;\}$$

$$= -(1/4\pi)\, \mu_2 [I/\pi a^2] \begin{cases} 2\pi a^2 \ln(r) & r > a \\ 2\pi a^2 \ln a + \pi(r^2 - a^2) & r < a \end{cases} \qquad \text{// using } \int dV' \ln(R^2) \text{ from App B}$$

$$= -(1/4\pi)\, \mu_2 [I/\pi a^2]\, 2\pi a^2 \begin{cases} \ln(r) & r > a \\ \ln a + (r^2/a^2 - 1)/2 & r < a \end{cases}$$

$$= -(I/2\pi)\, \mu_2 \begin{cases} \ln(r) & r > a \\ \ln a + (r^2/a^2 - 1)/2 & r < a \end{cases}$$

The relevant part is in region 2 where I use the integral above, where then

$$= \; -(I/2\pi)\, \mu_2 \,[\, \ln a + (1/2)(r^2/a^2 - 1) \,] \qquad \text{// integral evaluated in region 2}$$

4. Here then is what the solutions look like in the two regions at this point:

$A^{I,region1}(r) = A \ln r$
$A^{I,region2}(r) = d - (I/2\pi)\,\mu_2\,[\ln a + (1/2)(r^2/a^2-1)]$

Notice these facts which will be used in the next little section

$A^{I,region1}(a) = A \ln a$
$A^{I,region2}(a) = d - (I/2\pi)\,\mu_2 \ln a$

$\partial_r A^{I,region1}(r) = A/r$
$\partial_r A^{I,region2}(r) = - (I/2\pi)\,\mu_2 * (1/2a^2)2r = - (I/2\pi a^2)\,\mu_2\, r$

$\partial_r A^{I,region1}(a) = A/a$
$\partial_r A^{I,region2}(a) = - (I/2\pi a^2)\,\mu_2\, a$

4. I then match the boundary conditions

$A^{I,region2}(a) = A^{I,region1}(a)$
$(1/\mu_2)\,\partial_r A^{I,region2}(a) = (1/\mu_1)\,\partial_r A^{I,region1}(a)$

which conditions then say

$d - (I/2\pi)\,\mu_2 \ln a = A \ln a$
$(1/\mu_2)\,[- (I/2\pi a^2)\,\mu_2\, a] = (1/\mu_1)[A/a]$

or

$d - (I/2\pi)\,\mu_2 \ln a = A \ln a$
$- (I/2\pi) = (1/\mu_1)A$

The solution to these 2 equations in 2 unknowns is

$A = -\mu_1\,(I/2\pi)$
$d = (I/2\pi)\,\mu_2 \ln a - \mu_1\,(I/2\pi) \ln a = (I/2\pi)(\mu_2-\mu_1)\ln a$

Installing these constants, we then obtain our final solution

$A^{I,region2}(r) = W+d = -(I/2\pi)\,\mu_2\,[\ln a + (1/2)(r^2/a^2-1)] + (I/2\pi)(\mu_2-\mu_1)\ln a$
$A^{I,region1}(r) = A\ln r = -\mu_1\,(I/2\pi)\ln r$

or

$A^{I,region2}(r) = -(I/2\pi)\,\mu_2 \ln a - (I/4\pi)\,\mu_2(r^2/a^2-1) + (I/2\pi)(\mu_2-\mu_1)\ln a$
$A^{I,region1}(r) = -\mu_1\,(I/2\pi)\ln r$

or

$A^{I,region2}(r) = -(I/2\pi)\,\mu_1 \ln a - (I/4\pi)\,\mu_2(r^2/a^2-1)$
$A^{I,region1}(r) = -\mu_1\,(I/2\pi)\ln r$

or

$A^{I,region2}(r) = -(I/2\pi)\,[\,\mu_1 \ln a + (1/2)\mu_2(r^2/a^2-1)$

$$A^{I,region1}(r) = -\mu_1 (I/2\pi)\ln r$$

Comment: This is the problem solution obtained by treating each region separately and making assumptions about the assumed forms then finding the constants. [ By the way, this solution happens to agree with the Appendix B Version 2 solution. ]

**Problem II Way.** [ this method works, but is not supportable] The problem is described *uniformly* for region 1 + 2 = region R, but an image surface current is added at the boundary whose value matches the known mag current there.

$$(\nabla^2 + \beta_1^2)\, \mathbf{A}^{II} = -\mu_0 \mathbf{J}_m - \mu_2 \mathbf{J}_{c2} \qquad x \text{ in region R} \qquad \text{// image current present!}$$

$$A^{II,region2}(a) = A^{II,region1}(a)$$
$$(1/\mu_2)\, \partial_r A^{II,region2}(a) - (1/\mu_1)\, \partial_r A^{II,region1}(a) = K_z^{image} = [(\mu_1-\mu_2)/\mu_0]\, I/(2\pi a) \quad (*) \text{ wrong!}$$

Question: Below do I ever make use of equation (*) ?
Answer: No, I never use this, and *moreover*, it is not true! [ correct] See comments in Plan C of Disaster Log 2.

To solve this problem I make these *assumptions*:

1. The *entire solution* is the particular solution of the uniform PDE

$$A^{II,regionR} = (1/2\pi) \int dV' [\mu_2 \mathbf{J}_c(\mathbf{x}') + \mu_0 \mathbf{J}_m(\mathbf{x}')] \ln(1/R) = W + Z$$

where W is the same integral we saw in the Problem I analysis and which we happened to evaluate in both regions 1 and 2, to wit,

$$W = -(I/2\pi)\, \mu_2 \begin{cases} \ln(r) & r > a \\ \ln a + (r^2/a^2 - 1)/2 & r < a \end{cases} .$$

The new integral here is then

$$Z = (1/2\pi) \int dV' J_{zm}(\mathbf{x}')] \ln(1/R) = (1/2\pi)\, \mu_0 \int dS'\, K_z^m(x') \ln(1/R)$$

$$= (1/2\pi)\, \mu_0\, [(\mu_1-\mu_2)/\mu_0]\, I/(2\pi a) \int dS' \ln(1/R)$$

$$= (1/2\pi)\, (\mu_1-\mu_2)\, I/(2\pi a) \int dS' \ln(1/R)$$

$$= -(1/2\pi)\, (\mu_1-\mu_2)\, I/(2\pi a)(1/2) \{ \int dS' \ln(R^2) \}$$

$$= -(1/2\pi)\, (\mu_1-\mu_2)\, I/(2\pi a)(1/2)\, 4\pi a \begin{cases} \ln(a) & a > r \\ \ln(r) & a < r \end{cases} \qquad \text{// } \{..\} \text{ from Appendix B}$$

$$= -(I/2\pi)\, (\mu_1-\mu_2) \begin{cases} \ln(a) & a > r \\ \ln(r) & a < r \end{cases}$$

$$= -(I/2\pi)(\mu_1-\mu_2)\begin{cases} \ln(r) & r > a \\ \ln(a) & r < a \end{cases}$$

Then the unified solution over R is this

$$A^{II,regionR} = W + Z$$

$$= -(I/2\pi)\,\mu_2 \begin{cases} \ln(r) & r > a \\ \ln a + (r^2/a^2-1)/2 & r < a \end{cases} -(I/2\pi)(\mu_1-\mu_2)\begin{cases} \ln(r) & r > a \\ \ln(a) & r < a \end{cases}$$

$$= -(I/2\pi)\left[\ \mu_2 \begin{cases} \ln(r) & r > a \\ \ln a + (r^2/a^2-1)/2 & r < a \end{cases} + (\mu_1-\mu_2)\begin{cases} \ln(r) & r > a \\ \ln(a) & r < a \end{cases}\ \right]$$

We then write out the two region solutions separately

$$A^{II,regionR}(r>a) = -(I/2\pi)\ [\ \mu_2 \ln r + (\mu_1-\mu_2)\ln r\ ] = -(I/2\pi)\,\mu_1 \ln r$$

$$A^{II,regionR}(r<a) = -(I/2\pi)[\ \mu_2(\ln a + (1/2)(r^2/a^2-1)) + (\mu_1-\mu_2)\ln a\ ]$$
$$= -(I/2\pi)[\ \mu_2 \ln a + \mu_2\,(1/2)(r^2/a^2-1) + (\mu_1-\mu_2)\ln a\ ]$$
$$= -(I/2\pi)[\ \mu_1 \ln a + \mu_2\,(1/2)(r^2/a^2-1)\ ]$$

and once again

$$A^{II,regionR}(r>a) = -(I/2\pi)\,\mu_1 \ln r \qquad \text{region 1}$$
$$A^{II,regionR}(r<a) = -(I/2\pi)[\ \mu_1 \ln a + \mu_2\,(1/2)(r^2/a^2-1)\ ] \qquad \text{region 2}$$

[ Note: these agree with the "correct" solutions obtained in the Disaster Log 2 "Resume 11.21.13. ]
Thus, I at least have A METHOD of solving this problem and getting the correct answer!]

**Comparison of the solutions done the two ways**

$$A^{I,region1}(r) = -\mu_1\,(I/2\pi)\ln r$$
$$A^{I,region2}(r) = -(I/2\pi)\,[\ \mu_1 \ln a + (1/2)\mu_2(r^2/a^2-1)$$

$$A^{II,region1}(r) = -\mu_1\,(I/2\pi)\ln r$$
$$A^{II,region2}(r) = -(I/2\pi)[\ \mu_1 \ln a + \mu_2\,(1/2)(r^2/a^2-1)\ ]$$

We observe that the solution to this problem is EXACTLY THE SAME using either method of computation:

I. Smythian form segregated method in each region. Simple homo form is selected for region 1, and a simple homo form constant adder is selected for region 2, and these are sufficient to solve the problem. Each region has a homo contribution.

II. Unified region R solution but an image current has been added at the boundary. There is no homo contribution at all.

This is my Prototype Example of the two Ways.

The question is: WHY do these two Ways give the exact same answer? Is it just a coincidence for this very simple geometry, or is it a more general fact? [ I cannot prove it as general fact, so "cloy" ]

Comment: At r = 0 we get

$$A^{II,region2}(0) = -(I/2\pi)[\ \mu_1 \ln a + \mu_2 (1/2)(-1)\ ] = -(I/2\pi)[\ \mu_1 \ln a - \mu_2/2\ ]$$
$$A^{II,region2}(a) = -(I/2\pi)[\ \mu_1 \ln a + \mu_2 (1/2)(1^2-1)\ ] = -(I/2\pi)[\ \mu_1 \ln a\ ]$$
]

**Problem III Way ?**

Blue shows the original method

$$\text{curl } \mathbf{H} = \partial_t \mathbf{D} + \mathbf{J} \qquad \text{// Maxwell (1.1.1)}$$

$$(1/\mu)\ \text{curl curl } \mathbf{A} = \mu\varepsilon\, \partial_t \mathbf{E} + \mathbf{J} \qquad \text{// } \mathbf{H} = \mathbf{B}/\mu \text{ , } \mathbf{B} = \text{curl } \mathbf{A} \text{ from (1.3.1), and } \mathbf{D} = \varepsilon\mathbf{E}$$

$$\text{grad div}\mathbf{A} - \nabla^2\mathbf{A} = \mu\varepsilon\, \partial_t[-\text{ grad } \varphi - \partial_t \mathbf{A}] + \mu\mathbf{J} \qquad \text{// vector identity and } \mathbf{E} = -\text{ grad } \varphi - \partial_t \mathbf{A}$$

$$(\nabla^2 - \mu\varepsilon\, \partial_t^2)\ \mathbf{A} = \text{grad}\ [\mu\varepsilon\, \partial_t \varphi + \text{div}\mathbf{A}\ ] - \mu\mathbf{J} \qquad [ = \text{Jackson (6.11)} ] \qquad (1.3.3)$$

$$(\nabla^2 - \mu\varepsilon\, \partial_t^2)\mathbf{A} = -\mu\mathbf{J}\ . \qquad \text{// apply same gauge choice div}\mathbf{A} = -\mu\varepsilon\, \partial_t \varphi$$

Black shows a possible different path:

$$\text{curl } \mathbf{B} = \mu_0(\partial_t \mathbf{D} + \mathbf{J}_c + \mathbf{J}_m) \qquad \text{// Maxwell (1.1.24)}$$

$$\text{grad div}\mathbf{A} - \nabla^2\mathbf{A} = \mu_0(\partial_t \varepsilon\mathbf{E} + \mathbf{J}_c + \mathbf{J}_m)$$

$$\text{grad div}\mathbf{A} - \nabla^2\mathbf{A} = \mu_0(\varepsilon\partial_t[-\text{ grad } \varphi - \partial_t \mathbf{A}] + \mathbf{J}_c + \mathbf{J}_m)$$

$$\text{grad div}\mathbf{A} - \nabla^2\mathbf{A} = -\mu_0\varepsilon\, \partial_t \text{grad } \varphi - \mu_0\varepsilon\partial_t^2\mathbf{A} + \mu_0\mathbf{J}_c + \mu_0\mathbf{J}_m$$

$$-\text{ grad div}\mathbf{A} + \nabla^2\mathbf{A} = \mu_0\varepsilon\, \partial_t \text{grad } \varphi + \mu_0\varepsilon\partial_t^2\mathbf{A} - \mu_0\mathbf{J}_c - \mu_0\mathbf{J}_m$$

$$\nabla^2\mathbf{A} - \mu_0\varepsilon\partial_t^2\mathbf{A} = \mu_0\varepsilon\, \partial_t \text{grad } \varphi + \text{grad div}\mathbf{A} - \mu_0\mathbf{J}_c - \mu_0\mathbf{J}_m$$

$$(\nabla^2\mathbf{A} - \mu_0\varepsilon\partial_t^2)\ \mathbf{A} = \text{grad}\ [\text{divA} + \mu_0\varepsilon\partial_t \varphi] - \mu_0\mathbf{J}_c - \mu_0\mathbf{J}_m$$

Still have not made gauge choice, but I can see that this is NOT the wave operator I want to see. This is totally at odds with the King wave operator, so this Way III is just not useful. Here I am trying to

"expose" the mag current so it gets picked up. [ I pursue this more in Disaster 2, Plan E1. It led I think to me finding my big bug, but it is a lousy gauge to work in. ]