Disaster Repair Log 1 PhL 11.17.13

The main thrust of this doc was to try to get the Stak Dirichlet "boundary surface" integral term into the picture in hopes that it might shed some light on something related to my mystery that wrongly adding the J_m term to the wave equation causes the particular integral to give the exact right answer for the round wire. The idea was maybe to associate the added J_m term with this Stak Dirichlet second integral. I know now that in fact both give purely homogenous solutions, so there is some sense in the comparison. In this analogy there is a surface integral in both cases, but Stak wants $\partial_n g$ to be in there (the Poisson kernel) whereas what I have is E there, the 2D propagator thing. Also Stak has the exact Green's g, whereas I am dealing with the free space E. So I give it a try here, but I really can't connect the two concepts.

Comment: Very early on, I had this wonderment as to how a current deep in a conductor would make itself felt in the dielectric. Would it diffract through the boundary as in the Jackson problem? Did you need two propagators to handle the two parts of the "voyage", one for each medium? At least in my latest round wire situation (as of 11.21.13 after I found my Big Bug with J_m having bulk values, this concept still persists, but it is resolved by saying you solve things inside the conductor and outside, and you match the A_z boundary conditions. Nothing really propagates from one medium to the other. Or you consider both regions together as region R, despite their different μ_i values. There is a uniform propagator for both regions in the King gauge. The different μ_i values are then handled in those boundary conditions. I will no doubt continue to ponder these views of what is going on. One view is just dumb matching BV's, while the other view is propagation and the generation of mag currents which then have an effect, similar to the Jackson dielectric problem. A deep point current induces a charge perhaps on a boundary which in term causes re-propagation into the new medium. In Jackson it was induced polarization charge, while in the magnetic world it is current creates magnetization which in turn creates magnetization currents on the boundaries. The difference is that these don't enter into the wave equation, but they certain affect B, just the way E is effected in the Jackson problem. As I say, this is an ongoing background thread.

Review of this doc is complete.

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Note: Today 11.17 I revamped lines Ch 1 to include surface current K_z stuff, and added better pictures, changed

The problem arose when I noticed that the current J in my A wave equation did not include surface currents J_m since it came from curl H = J. The implication then is that "A does not see J_m", and the implication of that is that my Appendix B computation of B from A must be wrong. I was happy about this computation because it correctly predicted the B field both inside and outside the wire, each with its

respective μ_i, since A was computed for both inside and outside. I think this is the correct place to start in attempting repairs, and let other implications rest for a while.

[In other words, I was adding the surface part of J_m to the source J_c in the wave equation, knowing full well that this is "wrong" because that wave equation does not see any J_m pieces. But by doing so, the particular integral gave the exact right answer! So this was the start of the Disaster! The implication was that maybe if I don't add that "wrong" term, I will get the wrong answer.]

1. Review derivation of the A wave equation. It is done leading up to (1.3.3), here it is:

[I have done this "review" about 10 times now!]

$$\text{curl } \mathbf{H} = \partial_t \mathbf{D} + \mathbf{J} \qquad \text{// Maxwell (1.1.1)}$$

$$(1/\mu)\ \text{curl curl } \mathbf{A} = \mu\varepsilon\, \partial_t \mathbf{E} + \mathbf{J} \qquad \text{// } \mathbf{H} = \mathbf{B}/\mu \text{ , } \mathbf{B} = \text{curl } \mathbf{A} \text{ from (1.3.1), and } \mathbf{D} = \varepsilon\mathbf{E}$$

$$\text{grad div}\mathbf{A} - \nabla^2\mathbf{A} = \mu\varepsilon\, \partial_t[-\text{ grad } \varphi - \partial_t \mathbf{A}] + \mu\mathbf{J} \qquad \text{// vector identity and } \mathbf{E} = -\text{ grad } \varphi - \partial_t\mathbf{A}$$

$$(\nabla^2 - \mu\varepsilon\, \partial_t^2)\ \mathbf{A} = \text{grad } [\mu\varepsilon\, \partial_t \varphi + \text{div}\mathbf{A}\ ] - \mu\mathbf{J} \qquad [\text{ = Jackson (6.11) }] \qquad (1.3.3)$$

$$(\nabla^2 - \mu\varepsilon\, \partial_t^2)\mathbf{A} = -\mu\mathbf{J}\ . \qquad \text{// apply same gauge choice div}\mathbf{A} = -\mu\varepsilon\, \partial_t\varphi$$

It is nice to have the track laid out so clearly for purposes of debug. Here in the last step I use the Lorenz gauge. The "region" is unspecified, but let's think of it as dielectric region 1 for the moment. Then later below (1.3.18) I use instead the King gauge instead in region 1. I say that J includes all currents, but that is wrong, and so e the following conclusion is wrong which is (1.3.19).

Here are the corrected facts:

- below 1.3.3 Lorenz gauge: $(\nabla^2 - \mu\varepsilon\, \partial_t^2)\mathbf{A} = -\mu\mathbf{J}_c$
- corrected (1.3.19): $(\nabla^2 - \mu_1\varepsilon_1\, \partial_t^2 - \mu_1\sigma_1\partial_t)\ \mathbf{A} = 0$ region 1
  with $\text{div}\ \mathbf{A} = -\mu_1\varepsilon_1\, \partial_t\varphi - \mu_1\sigma_1\varphi$

So then I am back to King Problem #2 that the A wave equation has no source. [groan!] I guess I now had better face up to this fact [yup, time to face up] When I went on to do regions 2 and 3 using J_2 and J_3, those were conduction currents, and there was then no error made. So here then is the corrected (1.3.21):

$$(\nabla^2 - \mu_1\varepsilon_1\, \partial_t^2 - \mu_1\sigma_1\partial_t\ )\ \mathbf{A} = \mathbf{0} \quad (\text{not } -\mu_1\mathbf{J}_m) \qquad \text{region 1}$$
$$(\nabla^2 - \mu_1\varepsilon_1\, \partial_t^2 - \mu_1\sigma_1\partial_t\ )\ \mathbf{A} = -\mu_2\mathbf{J}_2 \qquad \text{region 2}$$
$$(\nabla^2 - \mu_1\varepsilon_1\, \partial_t^2 - \mu_1\sigma_1\partial_t\ )\ \mathbf{A} = -\mu_3\mathbf{J}_3 \qquad \text{region 3} \qquad (1.3.21)$$

I can still combine these into a single unified equation for all three regions of region R

$$(\nabla^2 - \mu_1\varepsilon_1 \partial_t^2 - \mu_1\sigma_1)\,\mathbf{A} = -\mu_2\mathbf{J}_2 - \mu_3\mathbf{J}_3 \qquad \text{all of region R} \qquad (1.3.22)$$

where as noted the two J_i here are conduction currents only. So at least this equation does have a source.

[This is correct. You could then do the particular integral for 1.3.22 and add homo terms as needed to meet both those boundaries' BC pair.]

2. Maybe the answer lies with the boundary conditions which I have ignored in my entire doc, hoping I could avoid them. But after my long time with Stak, I am in fact an expert on such BC's and should think of this as an "opportunity" to apply what I learned over so many months of effort. Recall from "King meets Stak" this following equation which applies to the Poisson world (and which we trivially generalize to get into Helmholtz world)

[this was not a bad path to try out]

Note: Am making a second pass here, adding ε where appropriate:

$$u_I(x) = (1/\varepsilon) \int_R d\xi\, g(x|\xi)\, q(\xi) \;-\; (1/\varepsilon) \int_\sigma dS_\xi\, f(\xi)\, \partial_{\xi n} g(x|\xi)\,. \qquad \text{// note second BC term}$$

which is a solution to this problem:

$-\nabla^2 u_I(x) = q(x)$	with potential f(ξ) specified on a boundary σ	Stak (6.80)
$-\nabla^2 g(x\|\xi) = \delta(x-\xi)/\varepsilon$	with g(x|ξ) = 0 on that same boundary.	Stak (6.62) $g = u_{II}$

Here is my vague idea: In the above solution $u_I(x)$, it might be true that source q(ξ) does not in include "surface currents" (which for Poisson would be polarization surface charges I think) , but those "surface currents" will appear in the second boundary condition term, and THAT is how $u_I(x)$ "sees" those surface currents. [Well for Problem I, $\partial_{\xi n} g(x|\xi)$ in the second term is a Poisson kernel thing and not a charge distribution (read surface current) thing. And we are doing a Dirichlet problem. Still, one notes that such an extra term is a homo solution that you add to the main or particular term.]

But is this idea even viable? I really have no *interpretation* for the "second term" above. It is just the term that has to be there to solve a Dirichlet problem. It has no propagator. As noted below, $\partial_{\xi n} g(x|\xi)$ has the interpretation of charge induced on boundary *in the Green problem II*, but in *problem I* it does not have that interpretation at all. It is "just" the Poisson kernel that you use to get a Dirichlet solution. That being the case, it really is difficult to identify this second term with some kind of induced charge of any kind, much less polarization charge. To get that charge interpretation, you really need to use the E(x|ξ) free space propagator formalism, whatever it is. Maybe in that formalism, the boundary is the Great Sphere and that second term is just not present since g decays going out there. So then the situation is really this

$$u_{III}(x) = (1/\varepsilon) \int_R d\xi\, E(x|\xi)\, q(\xi)$$

where you include in q all charge that is included in $-\nabla^2 u_I(x) = q(x)$. If q(x) includes polarization charge, then the integral includes that as well.

Notice how I have clearly labeled u_I as the Dirichlet solution - f(ξ) specified - and u_{II} as the Green solution g. Two different problems associated with the same geometrical situation.

Now go off and do a little side problem. At a metal boundary, you know that $E_\perp = n/\varepsilon$, the surface charge. This follows from my lines result, where s is a point on the boundary [here s+ means just inside region]

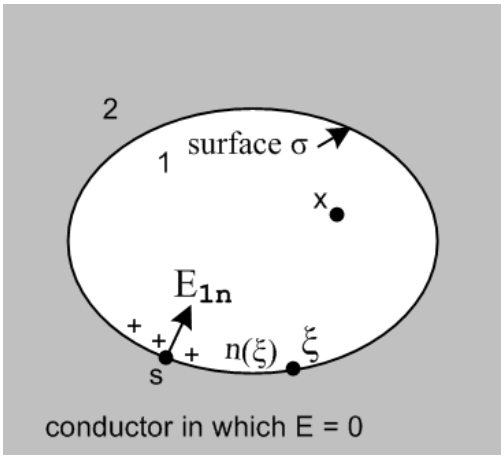
$$[\varepsilon_1 E_\perp^{(1)}(s+) - \varepsilon_2 E_\perp(s-)^{(2)}] = n(s) \qquad (1.1.22)$$

$$\varepsilon_1 E_{1n} - \varepsilon_2 E_{2n} = n(s) \qquad (1.1.27) \text{ in newer notation}$$

where we set $E_\perp^{(2)}(s-) = 0$ inside the *conductor*. Now $E_\perp(s+) = -\partial_n u(s+)$ where u is the potential. Thus, if you were to study Problem II with just a point charge at ξ_1 so g is the solution, then in that problem you could say that $E_\perp(s) = -\partial_n g(s|\xi_1) = n(s)/\varepsilon$ and then $I(s|\xi_1) \equiv -\partial_{\xi n} g(s|\xi_1) = n(s)/\varepsilon$ = the induced surface charge over ε! Again, ξ_1 = location of point charge, s = point on σ.

So, when it is evaluated with the first argument a point s on σ, we can interpret the quantity $-\partial_n g$ appearing in the surface integral as the induced charge (over ε) for Problem II. The induced charge in the actual problem I would be $-\partial_{\xi n} u(s|\xi)$ and you have to solve the problem to know that quantity.

I feel (as always) the need for some kind of picture to go with the above text for the u_{II} = g solution:



So we have $E_{1n} = n(s)/\varepsilon_1$ and if n(s) is positive as shown, then E_{1n} has the direction shown. We know that

$$E_{1n}(s) = -(\partial_n \varphi)^{s+} = (-\partial_n g(x|\xi))^{s+} = n(s)/\varepsilon_1 \qquad \text{where s+ means s + } \varepsilon \text{ = inside region 1}$$

Therefore, for the u_{II} = g problem, we can identify $n(s) = \varepsilon_1 (-\partial_n g)^{s+}$ as the induced charge density. Now this little ε_1 factor arises here if we assume $-\nabla^2 g(x|\xi) = \delta(x-\xi)$ in the canonical manner. If we instead

were to write $-\nabla^2 G(x|\xi) = \delta(x-\xi)/\varepsilon_1$, then we would get $n(s) = \varepsilon_1 (-\partial_n g)^{s+} = \varepsilon_1 (-\partial_n[\varepsilon_1 G])^{s+} = (-\partial_n G)^{s+}$ and then there is no ε_1 sitting in the $n(s) = (-\partial_n G)^{s+}$ equation.

Now let's return to our u_I solution:

$$u_I(x) = \int_R d\xi\, g(x|\xi)\, q(\xi) - \int_\sigma dS_\xi\, f(\xi)\, \partial_{\xi n} g(x|\xi) .$$

Fact: We can say that $(-\partial_{\xi n} g(x|\xi)) = n_{II}(s)$, meaning that in problem I, $(-\partial_{\xi n} g(x|\xi))$ equals the induced surface charge of problem II where the point source is located at **x**.

[OK, this is the 6th time perhaps I have considered problems I and II and the meaning of $\partial_{\xi n} g(x|\xi)$. I think I more or less do understand it.]

3. Question: What happens to my "verification" method when the surface term is present?

$$u_I(x) = \int_R d\xi\, g(x|\xi)\, q(\xi) - \int_\sigma dS_\xi\, f(\xi)\, \partial_{\xi n} g(x|\xi)$$

$$-\nabla^2 u_I(x) = q(x)$$

$$-\nabla^2 g(x|\xi) = \delta(x-\xi)$$

I get

$$-\nabla^2 u_I(x) = \int_R d\xi\, [-\nabla^2 g(x|\xi)]\, q(\xi) - [-\nabla^2 \{ \int_\sigma dS_\xi\, f(\xi)\, \partial_{\xi n} g(x|\xi) \}]$$

$$= q(x) + \nabla^2 \{ \int_\sigma dS_\xi\, f(\xi)\, \partial_{\xi n} g(x|\xi) \}$$

How now do I argue that the second term vanishes? That is to say, how do I *show* it is a solution to the homogeneous equation $-\nabla^2 f = 0$? I know from Green's identity or whatever that the u_I equation is true, and therefore if $q = 0$, then the surface integral MUST be a homo solution. But I am tempted to do this

$$\nabla^2 \{ \int_\sigma dS_\xi\, f(\xi)\, \partial_{\xi n} g(x|\xi) \} = \int_\sigma dS_\xi\, f(\xi) \{ \nabla^2 \partial_{\xi n} g(x|\xi) \}$$

since the surface is fixed in place. Now recall in the II problem we have

$$\{ \nabla^2 \partial_{\xi n} g(x|\xi) \} = -\nabla^2 \partial_{\xi n} \varphi(x|\xi) = -\partial_{\xi n} [\nabla^2 \varphi(x|\xi)]$$

where $\varphi(x|\xi)$ is the potential of a point charge located at x and viewed from ξ. Since we are using the s+ limit (just inside the boundary surface), we can say $\nabla^2 \varphi(x|\xi) = 0$ away from s and away from that boundary surface. So I guess you really can say $\{ \nabla^2 \partial_{\xi n} g(x|\xi) \} = 0$ at each point ξ .

Did I ever address this question in my Stak notes? I just took a quick look, and I don't see it anywhere. [Notes on the other stak thing are in Stak Support Ch 6 doc]

[Not sure how this works out, but I am sure that the BC integral is a homo solution.]

4. Peruse the u_I solution for a point source.

$$u_I(x) = \int_R dx'\, g(x|x')\, q(x') - \int_\sigma dS_\xi\, f(\xi)\, \partial_{\xi n} g(x|\xi) .$$

Now suppose q(x') is a point source located at x_1. Then the above solution becomes

$$u_I(x) = g(x|x_1) - \int_\sigma dS_\xi\, f(\xi)\, \partial_{\xi n} g(x|\xi) .$$

The second term *is still there*. Only if we specify f = 0 on the boundary, THEN this becomes problem II and the solution is then $u_{II}(x) = g(x|x_1)$.

5. Now let's try to convert this from Poisson world with φ, to magnetostatics world with A (not Helmholtz yet).

Here I just want to handle the transition from scalar φ to vector **A**.

$$\mathbf{A}_I(x) = \int_R d\xi\, g(x|\xi)\, \mathbf{J}(\xi) - \int_\sigma dS_\xi\, \mathbf{f}(\xi)\, \partial_{\xi n} g(x|\xi) . \qquad \text{// note second BC term}$$

which is a solution to this problem:

$-\nabla^2 \mathbf{A}_I(x) = \mathbf{J}(x)$	with potential f(ξ) specified on a boundary σ	Stak (6.80)
$-\nabla^2 g(x\|\xi) = \delta(x-\xi)$	with g(x|ξ) = 0 on that same boundary.	Stak (6.62) $g = u_{II}$

The g function is still a scalar, think in Cartesians of course. Then **f**(ξ) is **A** specified on surface σ. So maybe write these equations in this simpler notation

$$\mathbf{A}(x) = \int_R d\xi\, g(x|\xi)\, \mathbf{J}(\xi) - \int_\sigma dS_\xi\, \mathbf{A}(\xi)\, \partial_{\xi n} g(x|\xi)$$

which is a solution to this problem:

$-\nabla^2 \mathbf{A}(x) = \mathbf{J}(x)$	with potential A(ξ) specified on a boundary σ	Stak (6.80)
$-\nabla^2 g(x\|\xi) = \delta(x-\xi)$	with g(x|ξ) = 0 on that same boundary.	Stak (6.62) $g = u_{II}$

Now in $\nabla^2\,\mathbf{A}(x)$ the operator ∇^2 is the "vector Laplacian". I don't think Stak really got into the idea of a vector potential, so I am writing my own extension to Stak right here. So what is the nature of the Green function of a vector Laplacian? If we just stick with Cartesian coordinates, I think g is the same as the scalar g. Then we have

$$A_i(x) = \int_R d\xi\, g(x|\xi)\, J_i(\xi) - \int_\sigma dS_\xi\, A_i(\xi)\, \partial_{\xi n} g(x|\xi)$$

$-\nabla^2 A_i(x) = J_i(x)$ with potential A(ξ) specified on a boundary σ Stak (6.80)

$-\nabla^2 g(x|\xi) = \delta(x-\xi)$ with g(x|ξ) = 0 on that same boundary. Stak (6.62) $g = u_{II}$

Let's go with this and hope it is good enough. It is then the exact same g ! But I now have to think of the quantity δ(x-ξ) as a "point current component" and g(x|ξ) as the resulting vector potential component. Now what is the analogous "interpretation" of $\partial_{\xi n} g(x|\xi)$? It must be the "induced surface current" for the Green problem II where this phrase replaces "induced surface charge" for problem I.

So we put a "point current" δ(x-ξ) out in the middle of our region 1 and it induces a current on the boundary surface. But I need to find out what the new drawing is that goes with all of this?

In the Poisson case, I had a metal conductor with E = 0 inside it. What is the corresponding thing here? Recall E = -∇φ, so here we are going to have something = $-\nabla A_z$? I don't think this has a particular name. In general this thing is sort of $-\partial_j A_k$ and I do know that $B_i = \varepsilon_{ijk}\partial_j A_k$ so there is some connection with B here. Well, $\partial_j A_k$ is a rank 2 tensor I could call T_{jk} and then $B_i = \varepsilon_{ijk} T_{jk}$.

OK, this is too complicated to do all at once as a vector world, so

Ansatz: Let's assume for the moment that $\mathbf{A} = A_z \hat{\mathbf{z}}$ and for whatever reasons, $A_x = A_y = 0$. Maybe later we can do superposition of the three dimensions. Stay Cartesian always.

In this case, we have

$$A_z(x) = \int_R d\xi\, g(x|\xi)\, J_i(\xi) - \int_\sigma dS_\xi\, A_z(\xi)\, \partial_{\xi n} g(x|\xi)$$

$-\nabla^2 A_z(x) = J_z(x)$ with potential A(ξ) specified on a boundary σ Stak (6.80)

$-\nabla^2 g(x|\xi) = \delta(x-\xi)$ with g(x|ξ) = 0 on that same boundary. Stak (6.62) $g = u_{II}$

Then

$$\mathbf{B} = \text{curl}\,\mathbf{A} = \hat{\mathbf{x}}\,(\partial_y A_z - \partial_z A_y) + \hat{\mathbf{y}}\,(\partial_z A_x - \partial_x A_z) + (\partial_x A_y - \partial_y A_x)\,\hat{\mathbf{z}} \qquad \text{// each term cyclic x,y,z}$$

$$= \hat{\mathbf{x}}\,(\partial_y A_z) + \hat{\mathbf{y}}\,(\partial_x A_z)$$

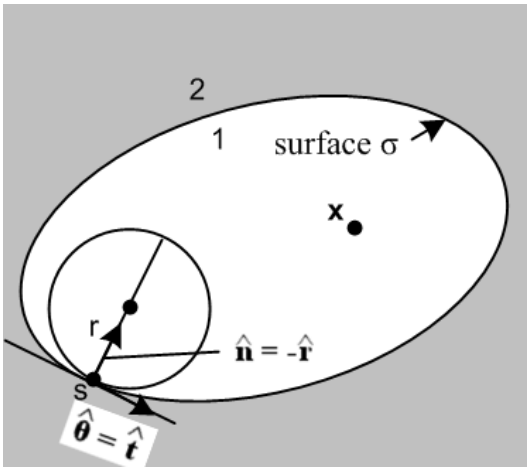
Write it again,

$\mathbf{B}(x,y,z) = \text{curl } \mathbf{A}(x,y,z) = \hat{\mathbf{x}}\,(\partial_y A_z(x,y,z)) + \hat{\mathbf{y}}\,(\partial_x A_z(x,y,z))$

Now, suppose we pick some point x on a surface which has some local curvature r, and then we find the origin and use a cylindrical coordinate system. Then $\hat{\boldsymbol{\theta}} = \hat{\mathbf{t}} = \hat{\mathbf{n}} = -\hat{\mathbf{r}}$

$$\mathbf{B} = \text{curl } \mathbf{A} = \hat{\mathbf{r}}\,[\, r^{-1}\partial_\theta A_z - \partial_z A_\theta] + \hat{\boldsymbol{\theta}}\,[\partial_z A_r - \partial_r A_z] + \hat{\mathbf{z}}\,[\, r^{-1}\partial_r(rA_\theta) - r^{-1}\partial_\theta A_r\,]$$

$$= \hat{\mathbf{r}}\,[\, r^{-1}\partial_\theta A_z] + \hat{\boldsymbol{\theta}}\,[-\partial_r A_z] = \hat{\mathbf{r}}\,B_r + \hat{\boldsymbol{\theta}}\,B_\theta \qquad B_r = r^{-1}\partial_\theta A_z \qquad B_\theta = \partial_r A_z$$



$\hat{z}$ points to the viewer

where now conductor I will assume is the gray 2 area. I am trying to stay as close as possible to the Poisson picture above. The fact is that I don't really know where I am going with this yet.

$$A_z(x) = \int_R d\xi\, g(x|\xi)\, J_i(\xi) - \int_\sigma dS_\xi\, A_z(\xi)\, \partial_{\xi n} g(x|\xi)$$

$-\nabla^2 A_z(x) = J_z(x)$	with potential A(ξ) specified on a boundary σ	Stak (6.80)
$-\nabla^2 g(x\|\xi) = \delta(x-\xi)$	with g(x|ξ) = 0 on that same boundary.	Stak (6.62) $g = u_{II}$

In this picture, think of a point current at location **x**. In the Green problem II, it results in an induced surface current at point s which is $K_z = -\partial_{\xi n} g(x|s)$.

Words: Suppose you have a "piece of current $J_z(\mathbf{x})d^3x$ at location **x** " which I call a point current. This creates a mag field H (think Biot Savart p 142 B&B) and at the point s, since H = curl J, this implies some current near the wall, and that is the induced current we are talking about here. In electrostatics you say that the point charge at x "attracts" opposite charge n(x) at the wall due to "Coulomb's Law", so here you have to say the point current at x "creates" an induced current at the wall due to the "Biot Savart Law". So maybe the gray is a conductor, and the point at x is a piece of a wire coming out of the plane of paper. Then there really is a real "free surface current" at the wall.

Question: Does this surface current have anything to do with the magnetization current J_m. ? Suppose neither medium is "magnetic" here, so there are no mag dipoles to line up so **M** = 0 and $\mathbf{J}_m$ = curl **M** = 0.

So in this case, we have $J_m = 0$, *but* we still have a real surface current K on the wall! The same in the Poisson electrostatics case. We may have no polarization charge on the wall, but we can still have induced charge on the wall if it is a conductor.

Reconsider. If both media are non-magnetic, a point current at x will produce an H at the boundary, but this H has to have a curl in order to create a free current at the wall, and perhaps that effect is not very strong. But at the boundary itself, if there is a μ difference, then you get a strong curl on B and this is associated with a surface mag current J_m at the wall, and this is probably a very strong effect. So I guess I am supporting the analogy with the Jackson problem: there, the point charge induces a polarization charge on the boundary. Here, the point current induces a magnetization current on the boundary. I might be able to firm up this stuff with the Scenarios from Disaster Repair doc #1 near the end. A point current comes out of the plane of paper, or we do the sheet of current doing the same.