**Cloying Problem** PhL 11.22.13

Written after the Disaster bug was found. I gave it a due diligence try but could not prove it, and it might not be true. I am not very good at delta functions on arbitrary surfaces.

I have a conjecture that I would like to see if I can prove. It is this.

**Theorem**: If you artificially include in your wave equation and its Helmholtz integral a term $-\mu_0 J_m$ where this accounts ONLY for that portion of $J_m$ (mag current) which lies on the boundary surface, *then* the principle integral will automatically handle both boundary conditions at the boundary for $A_z$.

I already know that adding such a term creates only homo solutions in the two regions which the boundary separates, so it is therefore perfectly "legal" to add such an artificial term.

This theorem works for the round wire with a uniform current.

In general language, here is what we have:

$$A_z(x) = \int_R dV' \, [\, \mu_2 J_{c2z}(x') + \mu_0 J_m(x') \,] \, E(x|x')$$

where E is the 2D propagator that goes with the wave equation.

**Big Kinematics Problem**

**Plan A**

Now I can write

$$J_m(x') = K_m \, \delta(???)$$

where somehow I want this delta function to limit things to the surface. I have always had a problem with stating this. For round wire it could be $\delta(r-a)$, and for elliptical somehow $\delta(x'^2/a^2 + y'^2/b^2 = c^2)$. The coordinates on the surface are called $\xi$ so perhaps we add an extra coordinate called $\xi_n$ which is in a direction perp to the surface at some point. Then $\delta^{(n)}(\mathbf{x}' - [\xi,\xi_n]) = \delta^{(n-1)}(x'_T - \xi)\, \delta(x'_n - \xi_n)$ where near a point of interest we write $\mathbf{x}' = (\mathbf{x}'_T, x'_n)$ where T means transverse to surface $\sigma$ at that point and n means normal at that point. So this then gives a meaning to $\delta^{(n-1)}(x'_T - \xi)$. So in the above case n = 2 and we have $\delta^{(1)}(x_T - \xi)$ where $x_T$ is a single coordinate perp to the surface. In the round wire case THIS is what becomes $\delta(r-a)$ which works at all boundary points, but I want to think of a general boundary. So the idea then is that, on that boundary

$$J_m(x') = K_m \, \delta(x'_T - \xi)$$

This pins a general point x' to have $x'_T = \omega$ and then $x'_n$ is not determined. Then consider

$$\int dV'\, J_m(x')\, E(x|x') = \int dS_\xi\, d\xi_n\, [K_m\ \delta(x'_T - \xi)]\ E(x|[\xi,\xi_n]) = K_m\, E(x|[\mathbf{x}_T',\xi])$$

where now the second argument of E has its transverse portion pinned to $\xi$ at a point in the integration.

There must be a better way to annotate this. I think it will help to have n = 3 to have at least something to visualize as we write things here. A 2D surface in 3D space may be written as

$$f_\sigma(x,y,z) = 0$$

Then what is the meaning of

$$\delta^{(1)}(\, f_\sigma(x,y,z)\,) \qquad ?$$

Only those points x for which $f_\sigma(x,y,z) = 0$ get a "hit". So maybe our surface current can be written

$$J_m(x') = K_m\ \delta^{(1)}(\, f_\sigma(x',y',z')\,) = K_m\ \delta^{(1)}(\, f_\sigma(\mathbf{x}'))$$

Now what happens when you put this into a full space integral?

$$f(\mathbf{x}) = \int dV'\, J_m(\mathbf{x}')\, E(\mathbf{x}|\mathbf{x}') = K_m \int dV'\ \delta^{(1)}(\, f_\sigma(\mathbf{x}'))\, E(\mathbf{x}|\mathbf{x}')$$

You must end up with an integral which is just over the surface

———

Perhaps a 2D surface σ is defined by

$$x' = x'(\xi)$$
$$y' = y'(\xi)$$

$$\mathbf{x}'_T = \mathbf{x}'_\sigma(\xi)$$

Then how do you pin something a 3D coordinate

$\delta^{(2)}(\mathbf{x}' - \mathbf{x}'_\sigma(\xi))$ = a full n dimensional δ, coordinate $\mathbf{x}'$ has n components.

Then we have

$$J_m(x') = K_m\, \delta(\mathbf{x}' - \mathbf{x}'_\sigma(\xi))$$

$$f(\mathbf{x}) = \int dV'\, J_m(x')\, E(x|x') = \int d^n x'\, K_m\, \delta(\mathbf{x}' - \mathbf{x}'_\sigma(\xi))\, E(\mathbf{x}|\mathbf{x}') =$$

———

**Plan B** . I am starting over.

Exercise 1. First, imagine an invertible transformation x' = x'(x) and x = x(x') in n dimensions say. Then if we define a Jacobian J by dx = J dx', we can write (in n dimensions)

$$1 = \int dx\ \delta(x-a) \quad = \int J\, dx'\ \delta[x(x') - a]$$

On the other hand, the same hit may be expressed as

$$1 = \int dx'\ \delta[x' - x'(a)]$$

Thus it would seem that

$$\delta^{(n)}[\mathbf{x}' - \mathbf{x}'(a)] = J\, \delta^{(n)}(\mathbf{x}-\mathbf{a})$$

This exercise relates an n dim delta to another n dim delta. But below we are going to have a 1 dim delta associated with an n dim space.

Exercise 2. Suppose we have an n-1 dimensional surface σ defined by f(**x**) = 0. What can we say about the 1D delta function δ[f(**x**)] which seems to "pin **x** to the surface σ " but could introduce some kind of scaling factor. We expect to see

$$\int dx\ \delta[f(\mathbf{x})]\ F(x) = \int_\sigma d\xi\ s(\xi)\ F(\xi)\ ?$$

where s(ξ) is that possible scaling factor.

Sub example. suppose surface is $x^2+ y^2+ z^2 = r^2$ so that f(**x**) = $x^2+ y^2+ z^2- r^2$. Then we would have

$\delta(f(\mathbf{x})) = = \delta(x^2+ y^2+ z^2- r^2 )$ and then we have

$$\int dx\ \delta(x^2+ y^2+ z^2- r^2 )\ F(x) = \int_\sigma d\xi\ s(\xi)\ F(\xi)$$

**Plan C** . Just write the surface integral that way, so start with

$$A_z(x) = \int dV'\ \mu_2 J_{c2z}(x')\ E(x|x') + \mu_0 \int d\xi\ K(\xi)\ ]\ E(x|\xi)$$

Ignore constants for now

$$\partial_n A_z(x) = \int dV'\ J_{c2z}(x')\ \partial_n \ln(|x-x'|^2) + \int_\sigma d\xi\ K(\xi)\ \partial_n \ln(|x-x'|^2)$$

I am pretty sure the required components come from both terms, not just the second term which is of the usual Stak form. So what can you possibly do next? Some identity? I have in mind that $\partial_{\mathbf{n}} = \partial_{\xi \mathbf{n}}$ where this is a normal direction to the surface σ

$$\partial_{\mathbf{n}} \ln(R^2) = R^{-2}\ \partial_{\mathbf{n}} R^2 = R^{-2}\ \partial_{\mathbf{n}}(|x-x'|^2) = R^{-2}\ \partial_{\mathbf{n}}(x^2+x'^2-2\mathbf{x}\bullet\mathbf{x}')$$

$\partial_{\mathbf{n}} x = \partial x/\partial\xi_{\mathbf{n}}$ = whatever that is. So

$$\partial_{\mathbf{n}}(x^2+x'^2-2x\bullet x') \quad = 2\mathbf{x}\bullet\partial\mathbf{x}/\partial\xi_{\mathbf{n}} - 2\mathbf{x}'\bullet\partial\mathbf{x}/\partial\xi_{\mathbf{n}} = 2\,(\mathbf{x}-\mathbf{x}')\bullet\partial\mathbf{x}/\partial\xi_{\mathbf{n}}$$

OK, then

$$\partial_{\mathbf{n}} \ln(R^2) = \frac{1}{R^2}\ 2\,(\mathbf{x}-\mathbf{x}')\bullet\partial\mathbf{x}/\partial\xi_{\mathbf{n}} =$$

And that first integral in $\partial_{\mathbf{n}} A_{\mathbf{z}}(x)$ [ close to surface, so $\xi_{\mathbf{n}}$ is a variable normal to surface nearby ]

$$\partial_{\mathbf{n}} A_{\mathbf{z}}(\mathbf{x}) = \int dV'\, J_{c2\mathbf{z}}(x')\,\frac{1}{R^2}\ 2\,(\mathbf{x}-\mathbf{x}')\bullet\partial\mathbf{x}/\partial\xi_{\mathbf{n}}$$

So we are at some x which is near the surface and $\xi_{\mathbf{n}}$ is an axis normal to the surface. I guess then

$$\partial_{\mathbf{n}} A_{\mathbf{z}}(\mathbf{x}) = 2\,\partial\mathbf{x}/\partial\xi_{\mathbf{n}} \bullet \int dV'\, J_{c2\mathbf{z}}(x')\,\frac{1}{R^2}\ \ (\mathbf{x}-\mathbf{x}')$$

But in this volume integral, all points are not really close to the surface, so $\xi_{\mathbf{n}}$ is ill defined for interior points perhaps.

Idea: Imagine some arbitrary surface shape. Pick some point inside. Then from that point draw 100 radial lines, all of which intersect the surface. Define a new surface by taking 99% the length of each line. In this way -- no, it does not work for hour glass shaped surface, for example. I don't really know how to create a "family of surfaces" which fill the volume. Only if we have a coordinate system do I know how to do that.

Well, go ahead anyway. Now take the limit that $\mathbf{x} = \mathbf{s} + d\xi_{\mathbf{n}}\mathbf{n}$ where s lies on the surface. Then

$$\partial_{\mathbf{n}} A_{\mathbf{z}}(\mathbf{s}+d\xi_{\mathbf{n}}\mathbf{n}) = 2\,(\partial\mathbf{x}/\partial\xi_{\mathbf{n}})(\mathbf{s})\ \bullet \int dV'\, J_{c2\mathbf{z}}(x')\,\frac{1}{|\,\mathbf{s}+d\xi_{\mathbf{n}}\mathbf{n}-\mathbf{x}'|^2}\ \ (\mathbf{s}+d\xi_{\mathbf{n}}\mathbf{n}-\mathbf{x}')$$

What would make this integral have different values for the two sides of the surface? by the way, I suppose the current $J_{c2\mathbf{z}}$ stops at this surface.

STOP. This problem is too hard for me, it is not well defined. Without a specific coordinate system, if we start with

$$A_z(x) = \int dV' \, [\, \mu_2 J_{c2z}(x') + \mu_0 J_m(x') \,] \, E(x|x')$$

I don't really see a simple way to compute $\partial_n A_z(x)$ by moving through the integral. There could be Stak type issues when you go approach a surface at least in the second term which is a surface integral.

**Conclusion**: I know this idea happens to work for a round wire, but I have no way to prove the idea for a wire of arbitrarily shaped cross section, especially if that shape does not belong to some known coordinate system such as elliptical coordinates. You cannot do "parts" (think of r and r' for the round wire:

$$R^2 = r^2 + r'^2 - 2rr'\cos x$$

$$\partial_r R^2 = 2(r - r'\cos x)$$

$$\partial_{r'} R^2 = 2(r' - r\cos x)$$

$$\partial_{r'} R^2 \neq -\partial_r R^2 \text{ just for example}$$

I don't even see a mechanism for having a discontinuity in $\partial_n A_z$ for the volume integral term.

The "other method" of adding homo terms as needed to match BC's just seems more reliable and teachable and supportable. Case closed.

Question: Suppose you have $A_z = V$ = constant all around the perimeter. Suppose you knew the exact Green's Function g. Then yes, you could write the famous Dirichlet result

$$A_z(x) = \int_R dV' \, [\, \mu_2 J_{c2z}(x')] \, g(x|x') + \int_R dS' \, \partial_n g \, V$$

Probably this is what I basically do for the two round conductors at the end of lines doc.

So yes, in this case, this second term $\int_R dS' \, \partial_n g \, V$ is creating the exact right homo solutions which make the boundary conditions work. AND, it is an integral over our surface of interest. But in my "formula", I don't have $\partial_n g$ in this integral, I have E.

So I am NOT going to put this notion into "lines", and therefore it should not even be mentioned, and this does imply an overhaul of Appendix B.