Paradox of the Day PhL 11.20.13

Resolution: The atoms for the 2D Laplace in polars are $e^{jn\theta}$ [$r^{\pm n}$] and A + B lnr. If situation is axially symmetric, then only atoms are A + B lnr. Thus, r^2 is not an atom in this case. Duh.

This is in lines Appendix B Version 2. Suppose we start with

$$\nabla^2_{2D}A_z(r) = -\mu_2 J_c(r) \quad r < a \qquad J_c(r) = J_c = I/(\pi a^2) = \text{uniform current density in wire}$$

so we then have

$$\nabla^2_{2D}A_z(r) = -\mu_2\, I/(\pi a^2) \qquad r<a$$

Think of this as just some constant source

$$\nabla^2_{2D}A_z(r) = K\theta(r<a)$$

The particular integral of this equation I think should be

$$A_z(r) = (1/2\pi) \int dV'\, K\theta(r<a) \ln(1/R) = (K/2\pi) \int_0^a dr'\, r' \int_{-\pi}^{\pi} d\theta' \ln(1/R)$$

since $(1/2\pi)\ln(1/R)$ is the 2D propagator. Rewrite as

$$A_z(r) = -(K/4\pi) \int_0^a dr'\, r' \int_{-\pi}^{\pi} d\theta' \ln(R^2)$$

From my App B picture I know that

$$R^2 = r^2 + r'^2 - 2rr'\cos(\theta-\theta')$$

I then show in App B that

$$\int_{-\pi}^{\pi} d\theta' \ln[r^2 + r'^2 - 2rr'\cos(\theta-\theta')] = 2\int_0^{\pi} d\theta' \ln[r^2 + r'^2 - 2rr'\cos(x)]$$

$$= 2\,Q(r',r) = 4\pi \begin{cases} \ln(r') & r' > r \\ \ln(r) & r' < r \end{cases}$$

Thus, we get

$$A_z(r) = -(K/4\pi) \int_0^a dr'\, r'\, 4\pi \begin{cases} \ln(r') & r' > r \\ \ln(r) & r' < r \end{cases} = -K \int_0^a dr'\, r' \begin{cases} \ln(r') & r' > r \\ \ln(r) & r' < r \end{cases}$$

$= -K \int_0^a dr' r' [\ln(r')\theta(r'>r) + \ln(r)\theta(r'<r) [$

$= -K \int_0^a \theta(r'>r)\, dr' r' \ln(r') - K \ln(r) \int_0^a \theta(r'<r)\, dr' r'$

$= -K \int_r^a dr' r' \ln(r') - K \ln(r) \int_0^r dr' r'$

$= -K [\int_r^a dr' r' \ln(r') + \ln(r) \int_0^r dr' r'] = -K * joe$

Let's hand this to our friend Maple

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joe := Int(x*log(x),x = r..a) + log(r)*Int(x,x=0..r);
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$$joe := \int_r^a x \ln(x)\, dx + \ln(r) \int_0^r x\, dx$$

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value(joe);
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$$\frac{1}{2} a^2 \ln(a) - \frac{1}{4} a^2 + \frac{1}{4} r^2$$

Thus we get a solution

$A_z(r) = -K (1/4) [2a^2 \ln a + (r^2 - a^2)]$

Now here is the paradox: This solution has the form $A_z(r) = \alpha + \beta r^2$. Both terms are atoms for the 2D Laplace equation, and therefore

$\nabla^2_{2D} A_z(r) = \nabla^2_{2D} [\alpha + \beta r^2] = 0$

But the paradox is that the starting equation was

$\nabla^2_{2D} A_z(r) = K\theta(r<a)$

We can do this manually since

$\nabla^2_{2D} A_z(r) = r^{-1} \partial_r (r\, \partial_r\, A_z(r)) = r^{-1} \partial_r (r\, \partial_r\, [\alpha + \beta r^2])$

$= r^{-1} \partial_r (r\, \partial_r\, [\beta r^2]) = \beta\, r^{-1} \partial_r (r\, \partial_r\, r^2)]) = \beta\, r^{-1} \partial_r (r\, 2r)]) = 2\beta\, r^{-1} \partial_r (r^2) = 2\beta r^{-1} (2r) = 4\beta$

Oops! Huh? Well OK, for n = 0 the only atoms are C + D log r, my bad, Paradox resolved.