

DC Internal Inductance for a Conductor of Rectangular Cross Section

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Abstract—In this paper, the dc internal inductance for conductors with rectangular cross section is investigated. Using a quasistatic Green's function, the magnetic fields inside a rectangular conductor are derived. These magnetic field expressions are used to formulate the internal inductance of the conductor. We present numerical results and closed-form expressions for the dc internal inductance for this geometry. Comparisons to various expressions found in the literature are made, and the inaccuracies of these other results are presented and discussed. Finally, we discuss some of the subtleties associated with the inductance of a single current path, in which we examine why the total and external inductances of a single infinitely long conductor have no physical meaning, whereas the internal inductance does.

Index Terms—DC internal inductance, interconnect, printed circuit boards (PCB), rectangular conductor, signal integrity, transmission lines.

I. INTRODUCTION

IT IS WELL known [1] that the inductance per unit length of a circular cylindrical conducting wire is dependent on frequency. At dc, its inductance includes a contribution from the magnetic energy stored within the conductor itself, in addition to that stored in the field outside the wire. As frequency increases, the skin effect begins to reduce the internal field, until in the limit of infinite frequency, the part of the inductance due to this field (the *internal inductance*) vanishes.

A wide array of planar transmission line structures (microstrip line, stripline, coplanar waveguides (CPW), coplanar strips (CPS), etc.) used for interconnects and high-speed printed circuit boards (PCBs) have signal traces with rectangular rather than circular cross sections. In the design of these types of transmission line structures, it is important to consider the waveform distortion. This is especially important in signal integrity considerations (see [2]–[6]). In such analyses, frequency-dependent models are used to investigate high-speed digital waveforms that propagate along lossy and dispersive transmission lines. The frequency dependencies of the conductor resistance and inductance caused by the skin effect are particularly important [7]–[27]. In these dispersive line models, knowledge of the dc and infinite frequency limits of resistance and inductance are needed to obtain the correct functional forms of the frequency dependence for transmission line parameters.

Of interest in this paper is the 2-D rectangular cross-sectional conductor illustrated in Fig. 1. The conductor is infinitely long in the z direction and is centered at the origin (i.e., $y = 0$ and $x = 0$). The conductor is assumed to have a thickness t and a width w . The dc internal resistance (per unit length or *pul*) for such a rectangular conductor is well known and given by

$$R_{dc} = \frac{1}{\sigma wt} \quad (1)$$

where σ is the conductivity of the conductor. The dc inductance of this conductor is less well known, and although a number of papers have given expressions for this structure [28]–[32], they are not accurate, nor do they agree with one another. Moreover, in contrast with the case of a circular wire, the less symmetrical rectangular conductor's change in inductance with frequency cannot be exclusively attributed to the reduction in internal inductance. As frequency increases, the fields external to the rectangular conductor will change to some extent, meaning that the entire inductance must be considered for practical applications, not just the internal part. We will treat the change in the inductance with frequency (from dc to high-frequency limit) for rectangular conductors in a future paper. In this current paper, we concentrate on the internal dc inductance L_i .

Here, we will present accurate results for the dc internal inductance of a rectangular cross-sectional conductor. In obtaining these results, we first derive expressions for the magnetic fields inside a rectangular conductor that are obtained from a quasistatic Green's function approach. These expressions are used to calculate the stored energy (and, in turn, the internal inductance) of this geometry. We present two approximate expressions for the dc internal inductance, one that is valid if $t/w < 0.01$, and a second one valid for $0.01 < t/w < 1$. We will need to discuss results only for $t/w \leq 1$, because of the symmetry of the cross section.

The paper is organized as follows: after the introduction, Section II presents the Green's function approach and illustrates how it is used to obtain the magnetic fields inside the conductor. This section also lays out the formulation of the dc internal inductance. In a section, we present an approximation for the magnetic fields for small t/w that is used to obtain a closed form for the internal inductance in this limit. For arbitrary values of t/w , the internal inductance requires a numerical evaluation of an integral of the magnetic fields, which is discussed in Section III. In this section, we also present an approximate expression for the dc internal inductance for an arbitrary value of t/w . Inaccuracies in other results found in the literature are also discussed. Section VI summarizes the results presented here. Finally, in the Appendix, we discuss some of the subtleties associated with the inductance of a single current path, in which

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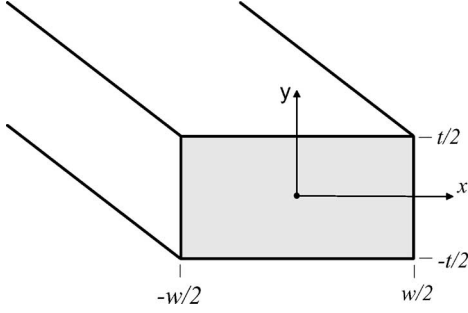


Fig. 1. Conductor with a rectangular cross section.

we examine why the total and external inductance of a single infinitely long conductor have no physical meaning, whereas the internal inductance does.

II. FORMULATION

The internal inductance of a conductor of arbitrary shape is given by [1]

$$L_i = \frac{1}{I^2} \int \bar{B} \cdot \bar{H} \, dV \quad (2)$$

where B and H are the magnetic field and magnetic field intensity inside the conductor, and the integral is carried out over the entire conductor volume. If the conductor is straight, has length l and a uniform cross section along the z -axis, then B and H become independent of z in the limit as $l \rightarrow \infty$, and the internal inductance per unit length can be expressed as

$$\frac{L_i}{l} = \frac{1}{\mu I^2} \int \bar{B} \cdot \bar{B} \, dS \quad (3)$$

where the integral is now carried out over the cross section of the conductor, and it has been assumed that μ is constant throughout the conductor.

One has to be careful with the interpretation of inductances of an infinitely long single conductor, so a few comments are needed about the inductance. In the Appendix, it is shown that the internal inductance for a single isolated infinite conductor does have meaning, even though its external inductance is undefined. The problem arises from the logarithmic divergence of the magnetic potential A when $l \rightarrow \infty$, as discussed in the Appendix. It is shown in the Appendix that (3) is finite.

For an arbitrarily shaped straight conductor with current flowing along the z -axis, the B -fields can be obtained from the z -component of the magnetic vector potential A , where it can be shown that [1]

$$B_x = \frac{\partial}{\partial y} A_z \quad \text{and} \quad B_y = -\frac{\partial}{\partial x} A_z. \quad (4)$$

The magnetic vector potential for a long current-carrying conductor can be expressed as (see the Appendix)

$$A_z = -\frac{\mu}{4\pi} \int J \ln[(x-x')^2 + (y-y')^2] \, dx' dy' + C \quad (5)$$

where J is the current density in the conductor, and C is a constant. From (4), we see that the derivatives of this constant give zero contribution to the B fields.

For the rectangular cross-section conductor shown in Fig. 1 with a uniform dc current density given by $J = I/(wt)$, the magnetic vector potential is expressed as

$$A_z = -\frac{\mu I}{4\pi w t} \int_{-t/2}^{t/2} \int_{-w/2}^{w/2} \ln[(x-x')^2 + (y-y')^2] \, dx' dy' + C. \quad (6)$$

Strutt [33], [34] evaluated this integral to obtain the magnetic vector potential (to within a constant) for the rectangular conductor. Strutt's results can also be found in [35] and [36]. His result for A_z could be substituted into (4) to obtain expressions for the B field. An alternative approach is to first recognize that the derivatives in (4) are in terms of x and y , while the integral in (6) is over y' and x' . As such, the derivatives with respect to x and y can be moved inside the integral and carried out analytically before evaluating the integrals. This allows one to obtain the B fields directly by evaluating the following integrals:

$$B_x = \frac{\mu I}{2\pi w t} \int_{-t/2}^{t/2} \int_{-w/2}^{w/2} \frac{y' - y}{(x' - x)^2 + (y' - y)^2} \, dx' dy' \quad (7)$$

and

$$B_y = -\frac{\mu I}{2\pi w t} \int_{-t/2}^{t/2} \int_{-w/2}^{w/2} \frac{x' - x}{(x' - x)^2 + (y' - y)^2} \, dx' dy'. \quad (8)$$

Note that the constant C is eliminated.

These two double integrals are fairly straightforward to evaluate, and leaving out the details, it can be shown that the x -component of the B field is given by

$$B_x = \frac{\mu I}{2\pi w t} W_1 \quad (9)$$

and the y -component by

$$B_y = -\frac{\mu I}{2\pi w t} W_2 \quad (10)$$

where

$$\begin{aligned} W_1 = & \left(\frac{w+2x}{4} \right) \ln \left[\frac{(w/2+x)^2 + (t/2-y)^2}{(w/2+x)^2 + (t/2+y)^2} \right] \\ & + \left(\frac{w-2x}{4} \right) \ln \left[\frac{(w/2-x)^2 + (t/2-y)^2}{(w/2-x)^2 + (t/2+y)^2} \right] \\ & + (t/2-y) \left[\tan^{-1} \left(\frac{w-2x}{t-2y} \right) + \tan^{-1} \left(\frac{w+2x}{t-2y} \right) \right] \\ & - (t/2+y) \left[\tan^{-1} \left(\frac{w-2x}{t+2y} \right) + \tan^{-1} \left(\frac{w+2x}{t+2y} \right) \right] \end{aligned} \quad (11)$$

and

$$\begin{aligned} W_2 = & \left(\frac{t+2y}{4} \right) \ln \left[\frac{(w/2-x)^2 + (t/2+y)^2}{(w/2+x)^2 + (t/2+y)^2} \right] \\ & + \left(\frac{t-2y}{4} \right) \ln \left[\frac{(w/2-x)^2 + (t/2-y)^2}{(w/2+x)^2 + (t/2-y)^2} \right] \end{aligned}$$

$$\begin{aligned}
& + (w/2 - x) \left[\tan^{-1} \left(\frac{t - 2y}{w - 2x} \right) + \tan^{-1} \left(\frac{t + 2y}{w - 2x} \right) \right] \\
& - (w/2 + x) \left[\tan^{-1} \left(\frac{t - 2y}{w + 2x} \right) + \tan^{-1} \left(\frac{t + 2y}{w + 2x} \right) \right].
\end{aligned} \quad (12)$$

This approach was used by Hague [35], but his results are incorrect.

With the x and y components of the B fields in hand, (3) can be used to obtain the following for the dc internal inductance per unit length of the rectangular conductor

$$\frac{L_i}{l} = \frac{\mu}{(2\pi wt)^2} \int_{-t/2}^{t/2} \int_{-w/2}^{w/2} [W_1^2 + W_2^2] dx dy. \quad (13)$$

Unfortunately, these integrals cannot be evaluated in closed form. The problem is that the integrand contains terms such as $[\tan^{-1}(f(x, y)/g(x, y))]^2$ and $\ln^2[v(x, y)/w(x, y)]$. These types of integrals result in functions known as polylogarithms [37], which cannot, in general, be expressed in terms of elementary functions. More importantly, the cross terms result in integrals that cannot be evaluated even in terms of standard special functions. In Section III, we present a numerical evaluation of this integral. Along with this numerical evaluation, we will also give approximate formulas for L_i/l for arbitrary values of t/w . However, for small t/w , a closed-form result can be obtained. This is the topic of the next section.

A. Internal Inductance for Small t/w

From (9)–(12), it can be shown that for small t/w , the fields can be expressed as

$$B_x = -\frac{\mu I}{4\pi w} W_3 \quad (14)$$

while the y -component is expressed as

$$B_y = \frac{\mu I}{4\pi w} W_4 \quad (15)$$

where

$$\begin{aligned}
W_3 = & \frac{4y\pi}{t} + \left(1 - \frac{2y}{t}\right) \left[\tan^{-1} \left(\frac{t - 2y}{w - 2x} \right) + \tan^{-1} \left(\frac{t - 2y}{w + 2x} \right) \right] \\
& - \left(1 + \frac{2y}{t}\right) \left[\tan^{-1} \left(\frac{t + 2y}{w - 2x} \right) + \tan^{-1} \left(\frac{t + 2y}{w + 2x} \right) \right]
\end{aligned} \quad (16)$$

and

$$W_4 = \ln \left(\frac{(w/2 - x)^2 + y^2}{(w/2 + x)^2 + y^2} \right). \quad (17)$$

If these fields are squared and substituted in (3), the internal inductance for small t/w is given by

$$\frac{L_i}{l} = \frac{\mu}{(4\pi w)^2} \int_{-t/2}^{t/2} \int_{-w/2}^{w/2} (W_3^2 + W_4^2) dx dy. \quad (18)$$

Unfortunately, we have the same problem as earlier; that is, the squared arctangents and logarithms result in integrals that cannot

be evaluated in closed form. However, by making further small-argument approximations for the logarithms and arctangents in (16) and (17), the integral can be evaluated in closed form. We start with W_3 . It can be shown that by use of a Taylor series expansion of the arctangents for small $(t/2 \pm y)/(w/2 \pm x)$, W_3 can be approximated to first order by

$$W_3 \approx \frac{2y\pi}{t/2} - 4y \left[\frac{1}{w/2 - x} + \frac{1}{w/2 + x} \right]. \quad (19)$$

The first term dominates, and thus

$$\int_{-t/2}^{t/2} \int_{-w/2}^{w/2} W_3^2 dx dy \approx \frac{4}{3} \pi^2 wt. \quad (20)$$

For small $(y)/(a \pm x)$, the logarithm in W_4 can be approximated to first order as

$$W_4 \approx 2 \left[\ln \left(\frac{w/2 - x}{w/2 + x} \right) \right] + \frac{y^2}{(w/2 - x)^2} - \frac{y^2}{(w/2 + x)^2}. \quad (21)$$

The first term dominates once again; thus

$$\int_{-t/2}^{t/2} \int_{-w/2}^{w/2} W_4^2 dx dy \approx 4t \int_{-w/2}^{w/2} \left[\ln \left(\frac{w/2 - x}{w/2 + x} \right) \right]^2 dx. \quad (22)$$

This integral is evaluated to be

$$\int_{-w/2}^{w/2} \left[\ln \left(\frac{w/2 - x}{w/2 + x} \right) \right]^2 dx = \frac{w}{3} \pi^2 \quad (23)$$

and so we have

$$\int_{-t/2}^t \int_{-w/2}^{w/2} W_4^2 dx dy \approx \frac{4}{3} \pi^2 wt. \quad (24)$$

Substituting (20) and (24) into (18), the dc internal inductance for small t/w can be approximated by

$$\frac{L_i}{l} \approx \frac{\mu t}{6 w}. \quad (25)$$

It is interesting to note that for the small t/w ratio, the B_x and B_y fields contribute identical amounts to the stored energy in the conductor. In the next section, this approximate expression for small t/w is compared with a numerical evaluation of the integral in (13).

III. DC INTERNAL INDUCTANCE: RESULTS AND COMPARISONS

The functions W_1^2 and W_2^2 are well-behaved functions, and thus, a numerical evaluation of the integration given in (13) is straightforward. This numerical integration can then be used to obtain L_i/l for any value of t/w . Figs. 2 and 3 show the results for the dc internal inductance L_i/l obtained from this numerical integration. The results in Fig. 3 are on a log scale in order to see more clearly the behavior for small values of t/w . The results are obtained assuming $\mu = \mu_0$. In both these figures, we show results only for $t/w \leq 1$. The rectangular cross section is a symmetric geometry; thus, the internal inductance is symmetric about $t/w = 1$ [in the sense that $L_i(t/w) = L_i(w/t)$], which is where the inductance reaches its maximum. As a comparison, the dc internal inductance of a circular wire (which is

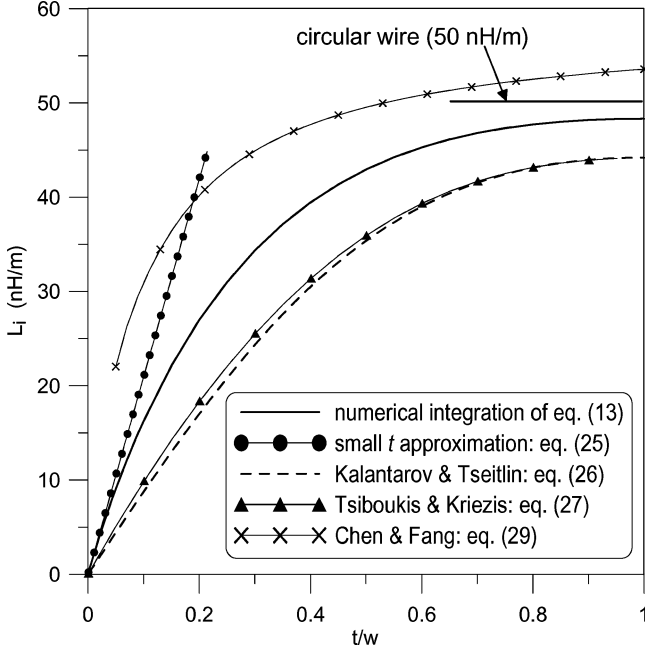


Fig. 2. DC internal inductance of a rectangular conductor as a function of t/w .

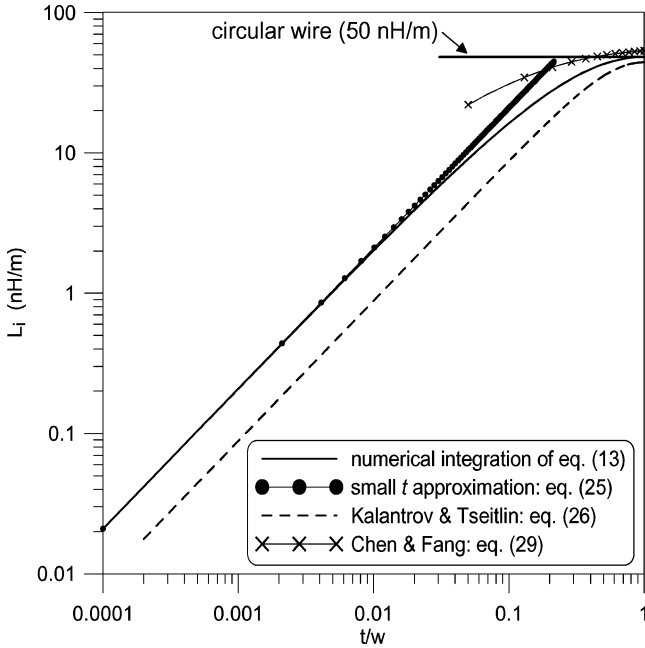


Fig. 3. Log plot of the dc internal inductance of a rectangular conductor as a function of t/w .

$\mu_0/8\pi = 50$ nH/m) is shown in the figures, while for a square conductor, the internal inductance is found to be 48.3 nH/m. In these figures, we also show the results obtained from the thin-strip approximation, $t/w \ll 1$ (25). From these figures, we see that for $t/w < 0.01$, the expression in (25) is indistinguishable from that of the numerical integration of (13).

There are only a few results in the literature for the dc internal inductance of a rectangular cross-section conductor; see [28]–[32]. Antonini *et al.* [31] present results of the frequency-dependent internal resistance and inductance obtained from a full numerical approach of subdividing the conductor cross section. In their paper, they investigate the high-frequency dependence; however, they also present a dc result for a square cross section. They state that the value is on the order of 48 nH/m. Our results give an internal inductance for a square cross section of 48.3 nH/m.

An approximate formula for the internal inductance can be found in [28]. Using a variational approach, Kalantarov and Tseitlin [28] derived the following expression for the dc internal inductance:

$$\frac{L_{i,KT}}{l} = \frac{0.221\mu}{\pi} \frac{wt}{w^2 + t^2}. \quad (26)$$

This expression is plotted in Figs. 2 and 3. From this comparison, we see that this expression is in error by about 8.5% for $t/w = 1$. The error increases quickly as t/w decreases, rising to 82% at $t/w = 0.1$.

Tsiboukis and Kriezis [29] also used a variational approach along with the Rayleigh–Ritz method to obtain the following approximate formula:

$$\frac{L_{i,TK}}{l} = \frac{0.209\mu}{\pi} \sum_{n=1,3,\dots}^{\infty} \sum_{m=1,3,\dots}^{\infty} \frac{wt}{n^2 m^2 (n^2 w^2 + m^2 t^2)}. \quad (27)$$

This expression is somewhat cumbersome because it requires an infinite summation. However, each summation converges after ten terms (i.e., the values are indistinguishable for ten terms when compared to the values for ten thousand terms in the summations). When converged, the results from this expression give results very similar to those of Kalantarov and Tseitlin, as seen in Fig. 2. The results from (27) in this figure were obtained using ten thousand terms in each summation. Hence, the error of (27) is essentially the same as that of Kalantarov and Tseitlin's expression.

Krakovski and Morawska [30] analyzed the skin effect of thin conductors (i.e., $t/w \ll 1$). In their analysis, they obtained the following for the dc internal inductance:

$$\frac{L_{i,KM}}{l} = \frac{\mu w}{6 t}. \quad (28)$$

This expression is obviously incorrect, since it predicts a non-physical result of increasing L_i as t decreases. The problems with the derivation in [30] are twofold. First of all, taking the limit as frequency goes to zero in (9) of [30] is problematic, and results in very large field values for small t . In fact, we showed earlier that for small t/w , B_y varies as $1/w$ and not as $1/t$, as predicted in [30]. Second, in [30], only the H_y component was used in calculating the stored energy. From (14) and (15), we see that both fields are present, and from (20) and (24), it is seen that for low aspect ratios t/w , both the x and y magnetic field components contribute the same amount toward the stored energy.

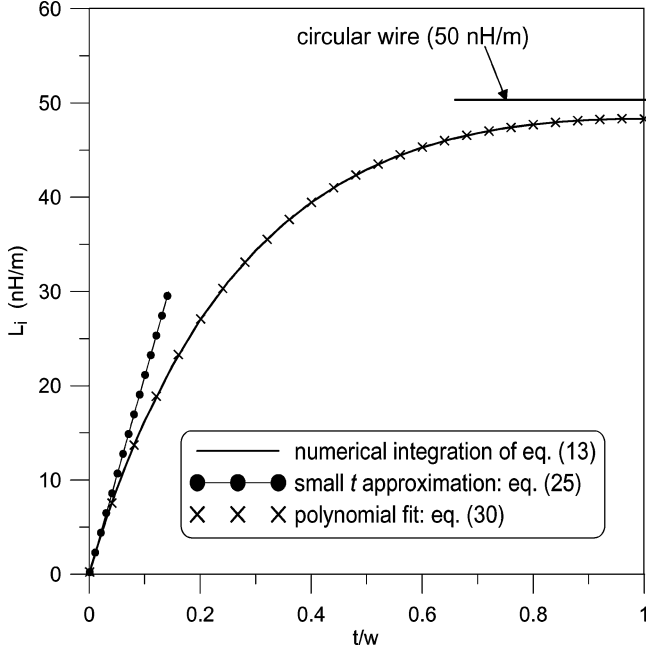


Fig. 4. Comparison of proposed approximate formulas for the dc internal inductance of a rectangular conductor as a function of t/w .

Finally, Chen and Fang [32] present the following polynomial approximate formula for the dc internal inductance:

$$\begin{aligned} \frac{L_{i,CF}}{l} = \frac{\mu}{8\pi} [& 1.07143 - 0.08608(w/t - 1) \\ & + 0.00553(w/t - 1)^2 - 1.99 \\ & \times 10^{-4}(w/t - 1)^3 + 2.863 \times 10^{-6}(w/t - 1)^4]. \end{aligned} \quad (29)$$

The claim in [32] is that this expression is valid for $1 \leq w/t \leq 20$ (or $0.05 \leq t/w \leq 1$). The expression is plotted in Figs. 2 and 3. The first thing we notice is that for $t/w = 1$, this expression gives a value of 53.6 nH/m, which is larger than both our result and that of Antonini *et al.* [31]. Indeed, the value predicted from (29) is also larger than for that for a circular wire (50 nH/m). Notice also that the expression has significant error for all values of t/w .

While our results in Figs. 2 and 3 give accurate results for the dc internal inductance, they are not in closed form. By assuming that $t/w \ll 1$, an accurate closed-form result was obtained for $t/w < 0.01$ in (25). A closed-form expression valid for $0.01 < t/w < 1$ can also be obtained by approximating the numerical results in Figs. 2 and 3 with a polynomial fit. Such a fit gives the following expression for the dc internal inductance in units of nH/m:

$$\begin{aligned} \frac{L_{i,fit}}{l} [nH/m] = & 0.045 + 200.23(t/w) - 443.01(t/w)^2 \\ & + 724.18(t/w)^3 - 823.70(t/w)^4 \\ & + 537.40(t/w)^5 - 146.845(t/w)^6, \\ \text{for } & 0.01 < t/w < 1. \end{aligned} \quad (30)$$

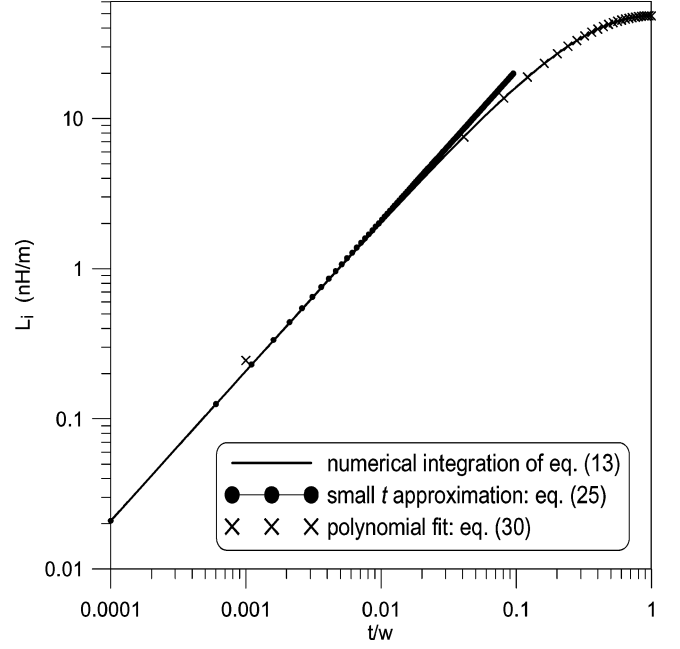


Fig. 5. Log plot of the comparison of proposed approximate formulas for the dc internal inductance of a rectangular conductor as a function of t/w .

Figs. 4 and 5 compare this expression to the results obtained from the numerical evaluation of the integral in (13). In fact, (30) exhibits a relative error of no larger than 0.1% for $0.005 < t/w < 1$.

Therefore, the dc internal inductance is well approximated for all values of t/w by the following expressions:

$$\frac{L_{dc,i}}{l} [nH/m] = \begin{cases} \frac{L_{i,fit}}{l}, & \text{for } 0.01 < t/w \leq 1 \\ \frac{400\pi}{6} \frac{t}{w}, & \text{for } t/w \leq 0.01 \end{cases} \quad (31)$$

where $L_{i,fit}/l$ is given in (30). From these comparisons, we see that these two expressions are indistinguishable from the numerical evaluation of the integral of (13) for all values of t/w .

IV. CONCLUSION

In this paper, we investigated the dc internal inductance for conductors with rectangular cross sections. We have presented numerical results and closed-form expressions for the dc internal inductance for this geometry that are valid for any value of t/w . We gave two different expressions, one valid for $t/w < 0.01$, and a second for $0.01 < t/w < 1$. Values from these two expressions are indistinguishable from numerical results. Comparisons to other expressions found in the literature were made, and the inaccuracies of these other results were presented and discussed. In this paper, we also obtained expressions for the magnetic fields associated with a constant current flowing in the rectangular conductor. These expressions are valid both inside and outside the conductor. Finally, in the Appendix, we discuss why the total and external inductance of a single infinitely

long conductor has no physical meaning, whereas the internal inductance does.

APPENDIX INDUCTANCE OF A SINGLE CURRENT PATH

As stated earlier, the internal inductance can be expressed in terms of the magnetic field (B) and field intensity (H). If we use (3) and recall that the B field can be expressed in term of the magnetic vector potential A as

$$\bar{B} = \nabla \times \bar{A} \quad (32)$$

the internal inductance can be expressed as

$$\frac{L_i}{l} = \frac{1}{I^2} \int \bar{B} \cdot \bar{H} dS = \frac{1}{I^2} \int \nabla \times \bar{A} \cdot \bar{H} dS. \quad (33)$$

With a few vector identities, we can show that

$$\begin{aligned} \frac{L_i}{l} &= \frac{1}{I^2} \int \nabla \times \bar{A} \cdot \bar{H} dS \\ &= \int [(\nabla \times \bar{H}) \cdot \bar{A} - \nabla \cdot (\bar{H} \times \bar{A})] dS \\ &= \int \bar{J} \cdot \bar{A} dS - \oint (\bar{H} \times \bar{A}) \cdot \bar{a}_n dl. \end{aligned} \quad (34)$$

The first term on the right-hand side of this expression can be “loosely” interpreted as the total inductance for a single conductor, while the second term can be “loosely” interpreted as the external inductance for a single conductor. We use the word “loosely” because for an infinitely long single conductor, these two terms are both singular (logarithmically infinite), but we will see that these two singularities cancel one another and give a finite and unique value for the internal inductance L_i/l for a single conductor.

The difficulties arise from $\bar{A}$. Starting with the static free-space Green’s function, it can be shown that the magnetic vector potential for a single conductor, whose length l is large compared to the cross-sectional dimensions, is given by Silvester [38] (we have corrected a sign error in his expression; see, for example, [39])

$$A_z = A_{z,(x,y)} + A_{z,C} \quad (35)$$

where $A_{z,(x,y)}$ is the part of A_z containing the x and y dependence, expressed as

$$A_{z,(x,y)} = -\frac{\mu I}{4\pi S_c} \int_x \int_y \ln [(x' - x)^2 + (y' - y)^2] dy' dx' \quad (36)$$

while $A_{z,C}$ is a constant independent of x and y , given by

$$A_{z,C} = \frac{\mu I}{2\pi} \ln(2l). \quad (37)$$

S_c is the cross-sectional area of the conductor. The singularity in A_z arises from $A_{z,C}$, and we see that as $l \rightarrow \infty$, $A_{z,C} \rightarrow \infty$. This singularity results in a singularity in the first and second terms of (34) (the total and external inductance of a single conductor, respectively). However, the singularities cancel when determining the internal inductance for a single conductor, i.e., when combining the external and total inductance. This is shown

by the following. From (34), the total inductance of the single conductor is expressed as

$$\begin{aligned} L_{\text{total}} &= \int \bar{J} \cdot \bar{A} dS = \int \bar{J} \cdot \bar{A}_{(x,y)} dS + \int \bar{J} \cdot \bar{A}_C dS \\ &= \int \bar{J} \cdot \bar{A}_{(x,y)} dS + A_{z,C} JS_c. \end{aligned} \quad (38)$$

We see that $L_{\text{total}} \rightarrow \infty$ as $l \rightarrow \infty$ (arising from $A_{z,C}$) and L_{total} has no meaning for an infinitely long single conductor. From (34), the external inductance is

$$\begin{aligned} L_{ex} &= \oint (\bar{H} \times \bar{A}) \cdot \bar{a}_n dl = \oint (\bar{H} \times \bar{A}_{(x,y)}) \cdot \bar{a}_n dl \\ &\quad - A_{z,C} \oint (\bar{H} \times \bar{a}_z) \cdot \bar{a}_n dl \\ &= \oint (\bar{H} \times \bar{A}_{(x,y)}) \cdot \bar{a}_n dl \\ &\quad + A_{z,C} \oint \bar{H} \cdot \bar{a}_l dl \\ &= \oint (\bar{H} \times \bar{A}_{(x,y)}) \cdot \bar{a}_n dl + A_{z,C} JS_c. \end{aligned} \quad (39)$$

We see that $L_{ex} \rightarrow \infty$ as $l \rightarrow \infty$ (arising once again from $A_{z,C}$), and L_{ex} has no meaning for an infinitely long single conductor. On the other hand, if (38) and (39) are substituted into (34), the $A_{z,C}$ terms cancel and the internal inductance is expressed as

$$\frac{L_i}{l} = \int \bar{J} \cdot \bar{A}_{(x,y)} dS - \oint (\bar{H} \times \bar{A}_{(x,y)}) \cdot \bar{a}_n dl \quad (40)$$

which is finite because the the $\ln(l)$ terms are no longer present and $A_{(x,y)}$ is finite. Recall that since $A_{z,C}$ is a constant in x and y , it does not appear explicitly in $\bar{H} = \frac{1}{\mu} \nabla \times \bar{A}$. Thus, the internal inductance of an infinitely long conductor has a unique finite value.

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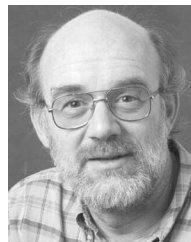
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