

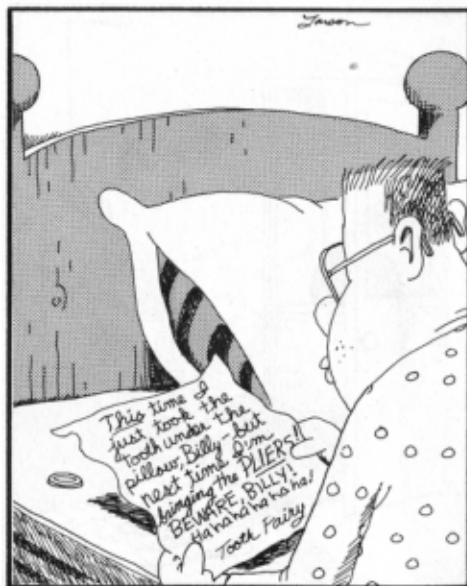
Capacitance

EE 141 Lecture Notes
Topic 15

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Motivation



Capacitance of an Isolated Conductor

Consider a (perfect) conductor that is removed from any other body (i.e., an isolated conductor). Because of the mutual repulsion between similar charges, energy must be expended to charge the conductor. Accordingly, the magnitude of its potential V increases as charge is added, the magnitude of the change in potential being proportional to the amount of charge added as well as upon the geometrical shape of the conductor body.

From Eq. (10) of Topic 14, the electrostatic energy of the charged body is given by

$$U_e = \frac{1}{2} \iint_S \varrho_s(\mathbf{r}) V(\mathbf{r}) d^2r = \frac{1}{2} QV, \quad (1)$$

where $Q = \iint_S \varrho_s(\mathbf{r}) d^2r$ is the net charge on the conductor surface and where V is its constant potential.

Capacitance of an Isolated Conductor

The ratio Q/V is defined as the **capacitance** of the isolated conductor

$$\boxed{C \equiv \frac{Q}{V}} \quad \text{Coul/Volt} \equiv \text{Farad (F)} \quad (2)$$

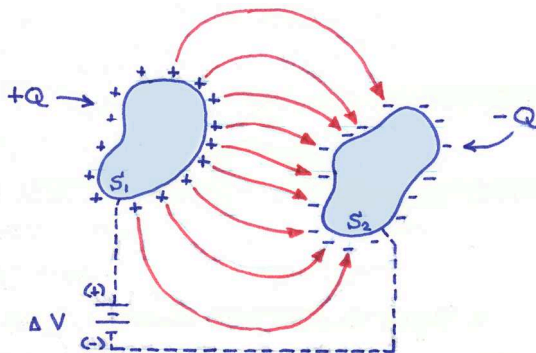
in terms of which its **stored electrostatic energy** is given by

$$U_e = \frac{1}{2}CV^2 = \frac{Q^2}{2C}. \quad (3)$$

Physically, the **capacitance** of an isolated body is the electric charge that must be added per unit increase in electrostatic potential on the body.

Capacitance of a Multi-Conductor System

Consider a generalized **capacitor** consisting of two (perfectly) conducting bodies separated in space by dielectric media with permittivities ϵ_j and brought to a static charge state with net charge $+Q$ on body 1 and $-Q$ on body 2, as illustrated.



Such a capacitor possesses the following properties:

Capacitance of a Multi-Conductor System

- Free charges $\pm Q$ reside on the conductor surfaces, resulting in a surface charge density $\rho_{sj}(\mathbf{r})$ on each body ($j = 1, 2$) such that

$$\pm Q = \oint_{S_j} \rho_{sj}(\mathbf{r}) d^2r, \quad (4)$$

the plus-sign applying for $j = 1$ and the minus-sign for $j = 2$.

- Gauss' law shows that the $\mathbf{E}$ -field lines originate normally from the surface S_1 of the positively charged body and terminate normally on the surface S_2 of the negatively charged body, with

$$\oint_S \mathbf{D}(\mathbf{r}) \cdot \hat{\mathbf{n}} d^2r = \pm Q \quad (5)$$

for any surface S enclosing either S_1 (the plus sign applies) or S_2 (the minus sign applies).

Capacitance of a Multi-Conductor System

- As a consequence of the perpendicularity of $\mathbf{E}$ at each conductor surface, they are **equipotential surfaces**, where $V(\mathbf{r}) = V_1$ on $\mathcal{S}_1$ and $V(\mathbf{r}) = V_2$ on $\mathcal{S}_2$.
- A single-valued **potential difference** $\Delta V = V_1 - V_2$ then exists between the two conducting bodies, given by

$$\Delta V = - \int_{\infty}^{\mathcal{S}_1} \mathbf{E}(\mathbf{r}) \cdot d\vec{\ell} + \int_{\infty}^{\mathcal{S}_2} \mathbf{E}(\mathbf{r}) \cdot d\vec{\ell} = - \int_{\mathcal{S}_2}^{\mathcal{S}_1} \mathbf{E}(\mathbf{r}) \cdot d\vec{\ell}. \quad (6)$$

In order to make ΔV positive, the **potential reference** $\mathcal{S}_2$ is taken on the negatively charged conductor body.

Capacitance of a Multi-Conductor System

For a **linear capacitor**, the potential difference ΔV is proportional to the charge Q , so that

$$Q = C\Delta V, \quad (7)$$

where C is the capacitance of the two-body system. From Eqs. (5) and (6), one obtains

$$C = \frac{Q}{\Delta V} = - \frac{\oint_{S_1} \mathbf{D}(\mathbf{r}) \cdot \hat{\mathbf{n}} d^2r}{\int_{S_2}^{S_1} \mathbf{E}(\mathbf{r}) \cdot d\vec{\ell}} \quad (8)$$

Electrostatic Energy & Capacitance

The voltage difference due to a charge Q on a linear capacitor with capacitance C is given by $\Delta V = Q/C$. The amount of work required to transfer an additional incremental amount of charge dQ from S_2 to S_1 is then given by

$$dW_e = \Delta V dQ = \frac{1}{C} Q dQ. \quad (9)$$

Beginning from an uncharged capacitor and continuing to transfer charge until a final charge Q is reached on S_1 , the total work expended is given by

$$W_e = \frac{1}{C} \int_0^Q Q dQ = \frac{Q^2}{2C} = \frac{1}{2} C (\Delta V)^2, \quad (10)$$

in agreement with Eq. (3) for the potential energy of an isolated conductor.

Electrostatic Energy & Capacitance

The capacitance of a two-conductor system is then seen to be given in terms of its electrostatic energy U_e by either of the two equations

$$C = \frac{2U_e}{(\Delta V)^2} = \frac{1}{(\Delta V)^2} \iiint_{\mathcal{V}} \mathbf{D}(\mathbf{r}) \cdot \mathbf{E}(\mathbf{r}) d^3r, \quad (11)$$

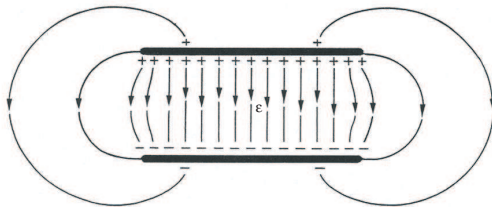
or

$$C = \frac{Q^2}{2U_e} = \frac{Q^2}{\iiint_{\mathcal{V}} \mathbf{D}(\mathbf{r}) \cdot \mathbf{E}(\mathbf{r}) d^3r}, \quad (12)$$

where the integration domain $\mathcal{V}$ contains all of the charged bodies comprising the multi-conductor capacitor.

Example: Parallel Plate Capacitor

Consider determining the capacitance of a parallel plate capacitor comprised of two parallel conducting plates, each of surface area A separated by a distance d filled with a dielectric of permittivity ϵ .



With an applied voltage difference V between the two plates, a charge $+Q$ accumulates on the surface of the upper plate and a charge $-Q$ accumulates on the surface of the lower plate, as illustrated. If the plate separation d is much smaller than the plate dimensions, then the effects of **fringing** may be neglected and the surface charge may be assumed to be uniformly distributed over each capacitor plate with density $\rho_s = \pm Q/A$.

Example: Parallel Plate Capacitor

The electrostatic field is then, to a good approximation, confined between the plates and linearly directed from the positive to the negative plate. Choosing the z -axis along this direction directed from the negative to the positive plate, one then has that

$$\mathbf{E} = -\hat{\mathbf{1}}_z E.$$

From the boundary condition given in Eq. (7) of Topic 13,

$$E = \frac{\rho_s}{\epsilon} = \frac{Q}{\epsilon A}.$$

In addition, from Eq. (6), the voltage difference between the two plates is given by

$$V = - \int_0^d \mathbf{E} \cdot \hat{\mathbf{1}}_z dz = E \int_0^d \hat{\mathbf{1}}_z \cdot \hat{\mathbf{1}}_z dz = Ed.$$

Example: Parallel Plate Capacitor

The **capacitance** is then given by

$$C \equiv \frac{Q}{V} = \frac{\epsilon A}{d} \quad (13)$$

From Eq. (3), the **stored electrostatic energy** in a parallel plate capacitor is given by

$$U_e = \frac{1}{2} CV^2 = \frac{1}{2} \left(\frac{\epsilon A}{d} \right) (Ed)^2 = \frac{1}{2} \epsilon E^2 Ad, \quad (14)$$

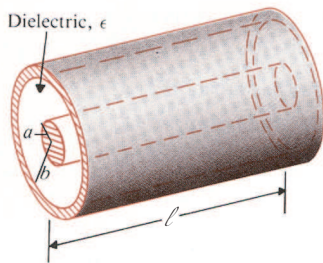
where Ad is the volume of the capacitor.

The **breakdown voltage** $V = V_{br}$ of the insulating material between the plates occurs at the **dielectric strength** $E = E_{ds}$ of the material. For quartz, $E_{ds} = 30 \text{ MV/m}$ so that the breakdown voltage when $d = 1 \text{ cm}$ is given by

$$V_{br} = E_{ds}d = (30 \times 10^6 \text{ V/m})(1 \times 10^{-2} \text{ m}) = 3 \times 10^5 \text{ V}.$$

Example: Capacitance of a Coaxial Line

Consider a length ℓ of an infinitely long coaxial line with inner radius a and outer radius b filled with a dielectric material with permittivity ϵ .



With an applied voltage difference V between the inner and outer conductors, charges $+Q/\ell$ and $-Q/\ell$ will be uniformly distributed per unit length along the outer and inner surfaces, respectively. The corresponding surface charge densities are then $\rho_s = +\frac{Q}{2\pi b\ell}$ on the outer conductor and $\rho_s = -\frac{Q}{2\pi a\ell}$ on the inner conductor surface.

Example: Capacitance of a Coaxial Line

The electrostatic field is then radially directed inwards and is given by (see Topic 6)

$$\mathbf{E}(r) = -\hat{\mathbf{1}}_r \frac{Q/\ell}{2\pi\epsilon r},$$

for $a < r < b$ and is zero otherwise.

The potential difference V between the outer and inner conductors is then given by

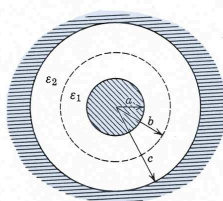
$$V = - \int_a^b \mathbf{E}(r) \cdot \hat{\mathbf{1}}_r dr = \frac{Q/\ell}{2\pi\epsilon} \int_a^b \frac{dr}{r} = \frac{Q/\ell}{2\pi\epsilon} \ln \left(\frac{b}{a} \right).$$

The capacitance C is then given by $C \equiv \frac{Q}{V} = \frac{2\pi\epsilon\ell}{\ln(b/a)}$ and the **capacitance per unit length** of the coaxial line is

$$\boxed{C' \equiv \frac{C}{\ell} = \frac{2\pi\epsilon}{\ln(b/a)}} \quad (15)$$

Problems

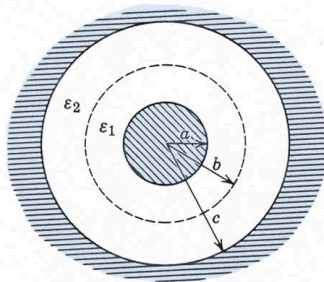
Problem 16. Determine the stored electrostatic energy per unit length U_e/ℓ of a coaxial line of inner radius a carrying charge $+Q/\ell$ and outer radius b carrying charge $-Q/\ell$ filled with a uniform dielectric with permittivity ϵ .



Problem 17. (a). Apply Gauss' law and the appropriate boundary conditions to determine the electric field in the annular region $a < r < c$ between the oppositely charged inner and outer conductors of a coaxial line filled with a dielectric material with permittivity ϵ_1 for $a < r < b$ and permittivity ϵ_2 for $b < r < c$. (b) Determine the capacitance per unit unit length $C' \equiv C/\ell$ of this line.

Problems

Problem 18. (a). Apply Gauss' law and the appropriate boundary conditions to determine the electric field in the spherical shell $a < r < c$ between the oppositely charged concentric inner and outer conducting spheres filled with a dielectric material with permittivity ϵ_1 for $a < r < b$ and permittivity ϵ_2 for $b < r < c$. (b) Determine the capacitance C of this spherical capacitor.



Capacitance & Energy of Multi-Conductor Systems

The total energy of the electrostatic field produced by a system of n charged ideal conductors embedded in a spatially homogeneous simple dielectric medium is given by [see eq. (15) of Topic 14]

$$\begin{aligned}U_e &= -\frac{\epsilon}{2} \iiint \mathbf{E} \cdot \nabla V d^3r \\&= -\frac{\epsilon}{2} \left[\iiint \nabla \cdot (V\mathbf{E}) d^3r - \iiint V \nabla \cdot \mathbf{E} d^3r \right] \\&= -\frac{\epsilon}{2} \iiint \nabla \cdot (V\mathbf{E}) d^3r,\end{aligned}$$

the second integral vanishing because $\nabla \cdot \mathbf{E} = 0$ throughout the region $\mathcal{V}$ external to the conductor bodies.

Capacitance & Energy of Multi-Conductor Systems

The divergence theorem transforms the remaining integral into a sum of surface integrals over each surface S_j of the entire system of conductors which spatially bound the electrostatic field plus an additional integral over an infinitely remote surface. This latter integral vanishes if the charged conductor system is situated within a finite region of space so that $\mathbf{E} \sim R^{-2}$ and $V \sim R^{-1}$ as $R \rightarrow \infty$. With the subscript j denoting the j^{th} conductor with constant potential V_j , the above expression becomes

$$U_e = \frac{\epsilon}{2} \sum_{j=1}^n V_j \oint_{S_j} \mathbf{E} \cdot \hat{\mathbf{n}} da,$$

where $\hat{\mathbf{n}}$ here denotes the unit outward normal to the conductor surface, directed from the conductor body into the field region.¹

¹Notice that this is opposite to the convention used in the divergence theorem, as reflected in the change of sign in the above equation.

Capacitance & Energy of Multi-Conductor Systems

Because $Q_j = -\epsilon \oint_{S_j} \mathbf{E} \cdot \hat{\mathbf{n}} da$, one finally obtains

$$U_e = \frac{1}{2} \sum_{j=1}^n Q_j V_j \quad (16)$$

which is analogous to the expression given in Eq. (8) of Topic 14 for the electrostatic potential energy of a system of point charges.

The charges Q_j and potentials V_j of the conductor bodies cannot both be arbitrarily prescribed and consequently must be related in some fashion. Because the field equations are linear & homogeneous, these relations should be linear. The potential of the j^{th} conductor is then directly proportional to the charge Q_k of the k^{th} conductor, with

$$V_j^{(k)} = p_{jk} Q_k, \quad j = 1, 2, \dots, n,$$

where the coefficients p_{jk} depend only upon the geometry of the conductor system.

Capacitance & Energy of Multi-Conductor Systems

By superposition, the potential on the j^{th} conductor due to the n charged conductors in the system is given by $V_j = \sum_{k=1}^n V_j^{(k)}$, so that

$$V_j = \sum_{k=1}^n p_{jk} Q_k \quad (17)$$

for $j = 1, 2, \dots, n$. The coefficients p_{jk} , which depend only upon the geometry of the multi-conductor system, are called *coefficients of potential*, where p_{jk} is the potential of the j^{th} conductor per unit charge on the k^{th} conductor. With this result, Eq. (16) becomes

$$U_e = \frac{1}{2} \sum_{j=1}^n \sum_{k=1}^n p_{jk} Q_j Q_k \quad (18)$$

The electrostatic potential energy of a system of charged conductors is a quadratic function of the charges on the conductors comprising the system.

Capacitance & Energy of Multi-Conductor Systems

Theorem

The coefficients of potential p_{jk} for a multi-conductor system satisfy the three fundamental properties:

$$\begin{aligned} p_{jk} &= p_{kj}, \\ p_{jk} &\geq 0, \\ p_{jj} &\geq p_{jk}, \quad \forall k. \end{aligned} \tag{19}$$

Capacitance & Energy of Multi-Conductor Systems

The relation (17) is a set of n linear equations giving the potentials V_j on each of the n conductors in terms of the charges Q_j residing on them. This set may be inverted to yield a set of equations giving Q_j in terms of V_j as

$$Q_j = \sum_{k=1}^n c_{jk} V_k \quad (20)$$

for $j = 1, 2, \dots, n$. The coefficients c_{jj} are called the *coefficients of capacitance* or the *capacity coefficients* whereas the coefficients c_{jk} with $j \neq k$ are called the *electrostatic induction coefficients*.

The *capacitance of a conductor* is then seen to be given by the total charge on the conductor when it is maintained at unit potential, all other conductors in the system being held at zero potential. In that case $Q_j = c_{jj} V_j$ and

$$c_{jj} = \frac{Q_j}{V_j}. \quad (21)$$

Capacitance & Energy of Multi-Conductor Systems

The coefficients of capacitance and induction form a matrix $\underline{\mathcal{C}} = (c_{jk})$, with the coefficients of capacitance forming the diagonal and the coefficients of inductance the off-diagonal elements, that is the inverse of the matrix $\underline{\mathcal{P}} = (p_{jk})$ formed from the coefficients of potential p_{jk} , so that

$$\underline{\mathcal{C}} = \underline{\mathcal{P}}^{-1}. \quad (22)$$

Theorem

The coefficients of capacitance and induction c_{jk} for a multi-conductor system satisfy the three fundamental properties:

$$\begin{aligned} c_{jk} &= c_{kj}, \\ c_{jj} &> 0, \\ c_{jk} &\leq 0, \quad j \neq k. \end{aligned} \quad (23)$$

Capacitance & Energy of Multi-Conductor Systems

Substitution of Eq. (20) into Eq. (16) for the electrostatic potential energy yields

$$U_e = \frac{1}{2} \sum_{j=1}^n \sum_{k=1}^n c_{jk} V_j V_k \quad (24)$$

The electrostatic potential energy of a system of charged conductors is thus seen to be a quadratic function of either the charges or the potentials on the various conductors comprising the system.

Capacitance & Energy of Multi-Conductor Systems

If two ideal conductors S_1 and S_2 form a capacitor, application of (17) to that arrangement yields

$$\begin{aligned}V_1 &= p_{11}(-Q) + p_{12}Q + V_E, \\V_2 &= p_{21}(-Q) + p_{22}Q + V_E,\end{aligned}$$

with $+Q$ on S_2 and $-Q$ on S_1 , where V_E is the common potential due to any external charges. The potential difference $\Delta V = V_2 - V_1$ is then given by

$$\Delta V = (p_{11} + p_{22} - 2p_{12})Q, \quad (25)$$

where the potential reference is taken on the negatively charged conductor in order to make ΔV non-negative. The *capacitance* of the capacitor is then given by [cf. Eq. (2)]

$$\boxed{C \equiv \frac{Q}{\Delta V} = (p_{11} + p_{22} - 2p_{12})^{-1}} \quad (F) \quad (26)$$

Capacitance & Energy of Multi-Conductor Systems

The mutual capacitance of a two conductor system can also be expressed in terms of the capacity and induction coefficients c_{jk} . From Eq. (18) the electrostatic potential energy of a two conductor capacitor is given by $U_e = \frac{1}{2}Q^2(p_{11} + p_{22} - 2p_{12})$. Because $\mathcal{P} = \mathcal{C}^{-1}$, then

$$\begin{pmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} \\ c_{12} & c_{22} \end{pmatrix}^{-1} = \frac{1}{c_{11}c_{22} - c_{12}^2} \begin{pmatrix} c_{22} & -c_{12} \\ -c_{12} & c_{11} \end{pmatrix},$$

so that

$$U_e = \frac{1}{2}Q^2 \frac{1}{c_{11}c_{22} - c_{12}^2} (c_{11} + c_{22} + 2c_{12}).$$

Comparison of this expression with that given in Eq. (3) then yields

$$\boxed{C = \frac{c_{11}c_{22} - c_{12}^2}{c_{11} + c_{22} + 2c_{12}}} \quad (27)$$

for the mutual capacitance of a two conductor system.

Capacitance & Energy of Multi-Conductor Systems

Consider now determining the change in the electrostatic energy of a system of conductors that is caused by an infinitesimal change in either their charges or their potentials.

Beginning with Eq. (16) of Topic 14, the variation of the total electrostatic energy of a system of n charged conductors is given by

$$\delta U_e = \frac{1}{2} \epsilon \iiint \delta(\mathbf{E} \cdot \mathbf{E}) d^3r = \epsilon \iiint \mathbf{E} \cdot \delta \mathbf{E} d^3r. \quad (28)$$

Upon setting $\mathbf{E} = -\nabla V$ and using the fact that $\nabla \cdot \mathbf{E} = 0$ in the dielectric so that $\nabla \cdot \delta \mathbf{E} = 0$, the above result becomes

$$\delta U_e = -\epsilon \iiint \nabla \cdot (V \delta \mathbf{E}) d^3r = \epsilon \sum_{j=1}^n V_j \oint_{S_j} \delta \mathbf{E} \cdot \hat{\mathbf{n}} d^2r$$

after application of the divergence theorem, where $\hat{\mathbf{n}}$ denotes the outward unit normal vector to the conductor surface S_j .

Capacitance & Energy of Multi-Conductor Systems

The variation in charge on the j^{th} conductor is obtained from Gauss' law as $\delta Q_j = \epsilon \oint_{S_j} \delta \mathbf{E} \cdot \hat{\mathbf{n}} d^2r$, so that

$$\delta U_e = \sum_{j=1}^n V_j \delta Q_j. \quad (29)$$

This expression then gives the *change in electrostatic energy due to a change in the charges on the conductors*. This is simply the work required to bring a set of n infinitesimal charges δQ_j from infinity to the various conductor bodies in the system in the presence of the (fixed) potential V_j .

Capacitance & Energy of Multi-Conductor Systems

The expression (28) for the change in electrostatic energy can also be written as $\delta U_e = -\epsilon \iiint \mathbf{E} \cdot \nabla(\delta V) d^3r = -\epsilon \iiint \nabla \cdot (\mathbf{E} \delta V) d^3r$.

Because infinitesimal changes in the potentials, just like the potentials themselves, are constant over the surface of each conductor, this expression becomes

$$\delta U_e = \epsilon \sum_{j=1}^n \delta V_j \oint_{S_j} \mathbf{E} \cdot \hat{\mathbf{n}} d^2r$$

after application of the divergence theorem. Each surface integral here is recognized as the charge $Q_j = \epsilon \oint_{S_j} \mathbf{E} \cdot \hat{\mathbf{n}} d^2r$ on the corresponding conductor, so that

$$\delta U_e = \sum_{j=1}^n Q_j \delta V_j, \quad (30)$$

which expresses *the change in electrostatic energy in terms of the change in the potentials of the conductor bodies.*

Capacitance & Energy of Multi-Conductor Systems

The relations given in Eqs. (29) & (30) show that, by differentiating the electrostatic energy U_e of a system of charged conductors with respect to the charges on the conductors, the potentials on the individual conductors are obtained as

$$V_j = \frac{\partial U_e}{\partial Q_j}, \quad (31)$$

whereas the derivatives of U_e with respect to the potentials gives the charges on the conductors as

$$Q_j = \frac{\partial U_e}{\partial V_j}. \quad (32)$$

The symmetry relation $c_{jk} = c_{kj}$ for the coefficients of capacitance then follows from the fact that $\partial^2 U_e / \partial V_j \partial V_k = \partial^2 U_e / \partial V_k \partial V_j$. The remaining properties in that theorem follow from the positive-definiteness of the quadratic form given in Eq. (24).

Capacitance & Energy of Multi-Conductor Systems

The electrostatic force that acts between charged bodies can be obtained through a consideration of the change in the total electrostatic energy of the system under a small virtual displacement.

As an illustration, consider the force per unit area acting on the surface $\mathcal{S}$ of an ideal conductor carrying surface charge density $\rho_s(\mathbf{r})$. An element of surface charge $\rho_s(\mathbf{r})da$ experiences an electrostatic force due to the electrostatic field of all the other charges in the system. Because the $\mathbf{E}$ -field is normal to the surface of a perfect conductor, this force is perpendicular to the conductor surface $\mathcal{S}$, and because the element of charge $\rho_s(\mathbf{r})da$ is bound to the conductor by internal forces, the force acting on it is directly transferred to the body of the conductor. In the immediate neighborhood of the conductor surface the electrostatic energy density is given by

$$u_e(\mathbf{r}) = \frac{\epsilon}{2} |\mathbf{E}(\mathbf{r})|^2 = \frac{1}{2\epsilon} \rho_s^2(\mathbf{r}). \quad (33)$$

Capacitance & Energy of Multi-Conductor Systems

An infinitesimally small virtual displacement $\Delta\zeta$ of an elemental area Δa of the conductor surface will then result in a decrease in the electrostatic energy by an amount given by the product of the energy density $u_e(\mathbf{r})$ at that point and the excluded volume $\Delta a\Delta\zeta$, so that $\Delta U_e = -\frac{1}{2\epsilon}\rho_s^2(\mathbf{r})\Delta a\Delta\zeta$. Because the magnitude of the force $\mathbf{F}(\mathbf{r})$ is given by $\Delta U_e/\Delta\zeta$, this result means that there is an outward force per unit area equal in magnitude to

$$\frac{dF(\mathbf{r})}{da} = \frac{1}{2\epsilon}\rho_s^2(\mathbf{r}) = u_e(\mathbf{r}) \quad (34)$$

at the surface of the conductor.

Capacitance & Energy of Multi-Conductor Systems

By Gauss' law, the total $\mathbf{E}$ -field flux emerging from the surface element da with surface charge density $\varrho_s(\mathbf{r})$ is given by $\frac{1}{\epsilon}\varrho_s(\mathbf{r})da$, half of it directed into the body and half directed out of the body of the conductor. The electrostatic field intensity due to the local surface charge density alone is then comprised of two parts directed along $\pm\hat{\mathbf{n}}$ with equal magnitude $\frac{1}{2\epsilon}\varrho_s(\mathbf{r})$.

Because the electric field intensity at an exterior point infinitesimally close to the surface of the conductor is given by $\mathbf{E}(\mathbf{r}) = \hat{\mathbf{n}}\varrho_s(\mathbf{r})/\epsilon$, it is then seen that the element of surface charge $\varrho_s(\mathbf{r})da$ at that point produces exactly half of the total field external to that point.

The electric field intensity acting on the element of surface charge $\varrho_s(\mathbf{r})da$ due to all of the other charges in the system is then given by $\mathbf{E}_0(\mathbf{r}) = \hat{\mathbf{n}}\varrho_s(\mathbf{r})/2\epsilon$.

Capacitance & Energy of Multi-Conductor Systems

The force $d\mathbf{F}(\mathbf{r})$ acting on the element of surface charge $\varrho_s(\mathbf{r})da$, and consequently acting on the surface element da of the conductor itself, is then given by $d\mathbf{F} = (\varrho_s da)\mathbf{E}_0$, so that

$$\frac{d\mathbf{F}(\mathbf{r})}{da} = \frac{1}{2\epsilon}\varrho_s^2(\mathbf{r})\hat{\mathbf{n}} = u_e(\mathbf{r})\hat{\mathbf{n}}, \quad (35)$$

in agreement with the expression given in Eq. (34), where $\hat{\mathbf{n}}$ is the outward unit normal vector to the conductor surface at the point $\mathbf{r} \in \mathcal{S}$. Hence, the electrostatic force on a conductor tends to pull the conductor into the field; that is, *an electrostatic field exerts a negative pressure on a conductor with magnitude equal to the energy density in the field.*

Capacitance & Energy of Multi-Conductor Systems

The total force acting on a conductor body is then obtained by integrating the expression for the force per unit area given in Eq. (35) over the entire surface $\mathcal{S}$ of the conductor body as

$$\mathbf{F} = \frac{1}{2\epsilon} \oint_{\mathcal{S}} \varrho_s^2(\mathbf{r}) \hat{\mathbf{n}} da = \frac{1}{2} \oint_{\mathcal{S}} \mathbf{D}(\mathbf{r}) \cdot \mathbf{E}(\mathbf{r}) \hat{\mathbf{n}} da. \quad (36)$$

Capacitance & Energy of Multi-Conductor Systems

Consider now an isolated electrostatic system (comprised of conductors, dielectrics, and point charges) when one of its parts undergoes a differential displacement $d\mathbf{r}$ under the influence of the electrostatic forces acting upon it. The mechanical work done by the system is then given by

$$dW_m = \mathbf{F} \cdot d\mathbf{r} = F_x dx + F_y dy + F_z dz. \quad (37)$$

Because the system is isolated, this work is done at the expense of the electrostatic energy U_e , so that (*conservation of energy*)

$$dU_e + dW_m = 0. \quad (38)$$

Capacitance & Energy of Multi-Conductor Systems

Eqs. (37)–(38) then show that

$$\boxed{F_x = - \left(\frac{\partial U_e}{\partial x} \right)_Q} \quad (39)$$

with analogous expressions for F_y and F_z . The subscript Q indicates that the total charge of the system remains constant during the displacement.

Capacitance & Energy of Multi-Conductor Systems

If the body under consideration is constrained to rotate about an axis, then Eq. (37) is replaced by

$$dW_m = \vec{\tau} \cdot d\vec{\theta}, \quad (40)$$

where $\vec{\tau} = (\tau_1, \tau_2, \tau_3)$ is the *electrical torque* & $d\vec{\theta} = (d\theta_1, d\theta_2, d\theta_3)$ is the differential angular displacement. Eqs. (38) & (40) then show that

$$\boxed{\tau_1 = - \left(\frac{\partial U_e}{\partial \theta_1} \right)_Q} \quad (41)$$

with analogous expressions for τ_2 and τ_3 .

Capacitance & Energy of Multi-Conductor Systems

Consider next an isolated electrostatic system in which all of the free charge resides on the surfaces of the conductors in the system which are maintained at fixed potentials by means of an external energy source (e.g., by batteries). If one of its parts undergoes a differential displacement $d\mathbf{r}$ under the influence of the electrostatic forces acting upon it, Eq. (37) still holds but conservation of energy now requires that [cf. Eq. (38)]

$$dU_e + dW_m = dW_{\text{ext}}, \quad (42)$$

where dW_{ext} is the energy supplied by the external source.

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From Eq. (16) which gives the electrostatic energy in terms of the charges q_k and potentials V_k on a system of charged conductors as $U_e = \frac{1}{2} \sum_k V_k q_k$, if some part of the system is displaced while the potentials V_k are held fixed, then

$$dU_e = \sum_k V_k dq_k. \quad (43)$$

The energy dW_{ext} supplied by the external sources is the work required to move each of the charge increments dq_k from zero potential to the potential V_k of the k^{th} conductor, so that

$$dW_{\text{ext}} = \sum_k V_k dq_k, \quad (44)$$

and consequently

$$dW_{\text{ext}} = 2dU_e. \quad (45)$$

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Substitution of this result in Eq. (42) to eliminate dW_{ext} and combining the result with Eq. (37) then gives

$$dU_e = F_x dx + F_y dy + F_z dz, \quad (46)$$

so that

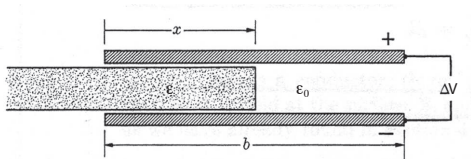
$$\boxed{F_x = \left(\frac{\partial U_e}{\partial x} \right)_V} \quad (47)$$

with analogous expressions for F_y and F_z . The subscript V indicates that all potentials of the system are maintained constant during the displacement $d\mathbf{r}$.

In a similar manner, analogous expressions are obtained for the electric torque, viz. $\tau_1 = (\partial U_e / \partial \theta_1)_V$, etc.

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As an example, consider a parallel plate capacitor where each plate has width w and length b with plate separation d filled with a dielectric block of permittivity ϵ . Let the capacitor plates be maintained at the constant potential difference ΔV .



If the dielectric block is withdrawn from the capacitor along the direction of the b -dimension until the length x remains between the plates, as illustrated, determine the force tending to pull the dielectric block back into place.

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From Eq. (15) of Topic 14, the energy of the capacitor system is given by

$$U_e = \frac{1}{2} \iiint_V \epsilon E^2(\mathbf{r}) d^3r$$

With fringing effects at the edge of the capacitor plates neglected so that $E = \Delta V/d$ between the capacitor plates and is zero outside, the above integration gives

$$U_e = \frac{\epsilon}{2} \left(\frac{\Delta V}{d} \right)^2 dwx + \frac{\epsilon_0}{2} \left(\frac{\Delta V}{d} \right)^2 dw(b-x).$$

The force pulling the dielectric block back into place is then obtained using Eq. (47) as

$$F_x = \left(\frac{\partial U_e}{\partial x} \right)_V = \frac{w}{2d} (\epsilon - \epsilon_0) (\Delta V)^2$$

in the direction of increasing x .