

Lecture Notes on
Linear Partial Differential Equations
(PDE 1)

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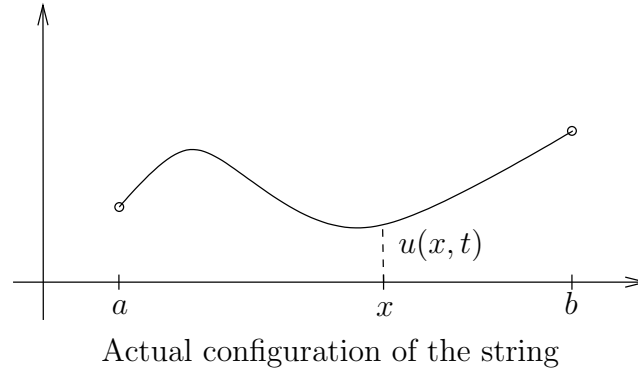
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1 The wave equation as mathematical model for the vibrating string and the vibrating membrane.

1.1 Potential energy of the linear elastic string

We want to formulate mathematical equations, which allow to compute the vibrations in time of an elastic string, which is fixed at both ends. To this we first compute the potential energy stored in the string at time t . This requires to make an assumption for the elastic material properties of the string. Let

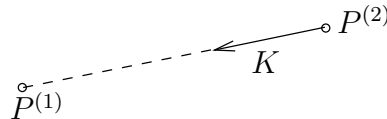


$-\infty < a < b < \infty$ be given numbers. Imagine first that the string is linearly stretched along the x -axis with ends fixed at $(a, 0)$ and $(b, 0)$ and is at rest. We call this the reference configuration of the string. Now consider an actual configuration, where the string is displaced from this reference configuration.

Hypothesis: Let $P^{(1)}, P^{(2)}$ be material points of the string, whose positions are $(x_1, 0), (x_2, 0)$ in the reference configuration and $(x^{(1)}, y^{(1)}), (x^{(2)}, y^{(2)})$ in the actual configuration. Assume that in the actual configuration the string is linearly and uniformly stretched between $P^{(1)}$ and $P^{(2)}$. Then the force K acting on $P^{(2)}$ is

$$K = \kappa \frac{|P^{(1)} - P^{(2)}|}{|x_1 - x_2|} \frac{P^{(1)} - P^{(2)}}{|P^{(1)} - P^{(2)}|} = \kappa \frac{P^{(1)} - P^{(2)}}{|x_1 - x_2|},$$

with a material constant $\kappa > 0$. Here we identified $P^{(1)}, P^{(2)}$ with the positions in the actual configuration. (Linearly elastic material behavior.)

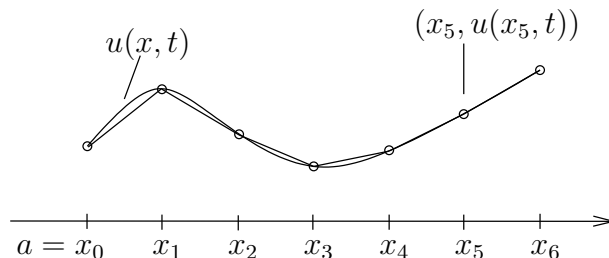


Note that by this law of force a variation of the actual positions of the points $P^{(1)}, P^{(2)}$ in direction orthogonal to the x -axis does not alter the component of

the force K parallel of the x -axis, and a variation of the positions parallel to this axis does not alter the force component orthogonal to this axis. Therefore the movements of the material points of the string in directions parallel and orthogonal to the x -axis are not coupled. We thus can and shall assume in the following that the material points of the string move only in the direction orthogonal to the x -axis. This implies that at time $t \geq 0$ the string can be represented by the graph of a function

$$x \mapsto u(x, t) : [a, b] \rightarrow \mathbb{R}.$$

To compute the potential energy we approximate the graph of this function by a piecewise linear function:



With $h = \frac{b-a}{n}$ let

$$x_i = ih + a, \quad i = 0, 1, \dots, n$$

be the x -coordinates of the node points of the polygonal arc. We first determine the potential energy stored in the polygonal arc. To this end we successively deform the string from the reference configuration to the polygon and compute the work done in every step. In the first step we move all material points of the string vertically and in parallel from the x -axis to the horizontal line passing through the point $(x_0, u(x_0, t))$. No work is done in this step. We then fix the endpoint of the string at $(x_0, u(x_0, t))$ and move all points on the line segment $\{(x, u(x_0, t)) \mid x_1 \leq x \leq b\}$ vertically and in parallel to the line segment $\{(x, u(x_1, t)) \mid x_1 \leq x \leq b\}$. During this movement an amount of energy V_1 is stored in the straight line segment above the interval $[x_0, x_1]$, which is equal to the work done in moving the material point at the position $(x_1, u(x_0, t))$ along a vertical path to the position $(x_1, u(x_1, t))$ against the vertical component K_2 of the elastic force K in this straight line segment. A parametrization of this path is

$$s \mapsto P(s) = (x_1, s) : [u(x_0, t), u(x_1, t)] \rightarrow \mathbb{R}^2.$$

Since in the reference configuration the position of the point $P(s) = (x_1, s)$ is $(x_1, 0)$ and the position of $(x_0, u(x_0, t))$ is $(x_0, 0)$, our hypothesis implies that the elastic force $K(s)$ at the point $P(s)$ is given by

$$K(s) = (K_1(s), K_2(s)) = \kappa \frac{(x_0, u(x_0, t)) - (x_1, s)}{x_1 - x_0},$$

whence

$$K_2(s) = -\kappa \frac{s - u(x_0, t)}{x_1 - x_0}.$$

We thus have

$$\begin{aligned} V_1 &= - \int_{u(x_0, t)}^{u(x_1, t)} K_2(s) ds = \frac{\kappa}{x_1 - x_0} \int_{u(x_0, t)}^{u(x_1, t)} (s - u(x_0, t)) ds \\ &= \frac{\kappa}{x_1 - x_0} \frac{1}{2} (s - u(x_0, t))^2 \Big|_{s=u(x_0, t)}^{s=u(x_1, t)} = \frac{\kappa}{2} \frac{(u(x_1, t) - u(x_0, t))^2}{x_1 - x_0} \\ &= \frac{\kappa}{2} \left(\frac{u(x_1, t) - u(x_0, t)}{x_1 - x_0} \right)^2 (x_1 - x_0) = \frac{\kappa}{2} \left(\frac{\partial}{\partial x} u(x_1^*, t) \right)^2 h, \end{aligned}$$

where x_1^* is a point between x_0 and x_1 . Here we used the mean value theorem. We proceed in the same way and obtain for the elastic energy V_i stored in the straight line segment above $[x_{i-1}, x_i]$ that

$$V_i = \frac{\kappa}{2} \left(\frac{\partial}{\partial x} u(x_i^*, t) \right)^2 h.$$

For the total energy $V^{(h)}(t)$ of the polygonal arc we thus have

$$V^{(h)}(t) = \sum_{i=1}^n V_i(t) = \frac{\kappa}{2} \sum_{i=1}^n \left(\frac{\partial}{\partial x} u(x_i^*, t) \right)^2 h.$$

For $h \rightarrow 0$ the polygon converges to the string. Therefore one defines the potential energy $V(t)$ of the string at time t by

$$V(t) = \lim_{h \rightarrow 0} V^{(h)}(t).$$

On the other hand, $\sum_{i=1}^n \left(\frac{\partial}{\partial x} u(x_i^*, t) \right)^2 h$ is a Riemann sum. If $x \mapsto \frac{\partial}{\partial x} u(x, t)$ is continuous we thus obtain by Riemann integration theory that

$$\lim_{n \rightarrow \infty} \frac{\kappa}{2} \sum_{i=1}^n \left(\frac{\partial}{\partial x} u(x_i^*, t) \right)^2 h = \frac{\kappa}{2} \int_a^b \left(\frac{\partial}{\partial x} u(x, t) \right)^2 dx.$$

Therefore we conclude that the **stored energy of the string at time t** is

$$V(t) = \frac{\kappa}{2} \int_a^b \left(u_x(x, t) \right)^2 dx.$$

1.2 The Hamiltonian principle

The velocity of the material point $(x, u(x, t))$ of the string at time t in the direction orthogonal to the x -axis is $\frac{d}{dt}u(x, t)$. Therefore the kinetic energy $E(t)$ of the string at time t is

$$E(t) = \int_a^b \frac{1}{2} \rho(x) (u_t(x, t))^2 dx,$$

where $\rho(x)$ is the mass of the string per unit length.

To formulate Hamilton's principle I use the following notations: For a continuously differentiable function $v : [a, b] \times [0, T] \rightarrow \mathbb{R}$ let

$$\begin{aligned} V_v(t) &= \int_a^b \frac{\kappa}{2} (v_x(x, t))^2 dx \\ E_v(t) &= \int_a^b \frac{\rho(x)}{2} (v_t(x, t))^2 dx. \end{aligned}$$

Hamilton's principle: Let $T > 0$, let the movement of the string be given by the continuously differentiable function

$$u : [a, b] \times [0, T] \rightarrow \mathbb{R},$$

and let $w : [a, b] \times [0, T] \rightarrow \mathbb{R}$ be a continuously differentiable function satisfying

$$w(x, 0) = w(x, T) = w(a, t) = w(b, t) = 0 \quad (1.1)$$

for all $a \leq x \leq b$ and all $0 \leq t \leq T$. Let s denote real numbers. Hamilton's principle states that the movement is such that

$$\frac{d}{ds} \int_0^T E_{u+sw}(t) - V_{u+sw}(t) dt \Big|_{s=0} = 0. \quad (1.2)$$

Remark. If $|s|$ is a small number, then

$$v(x, t) = u(x, t) + sw(x, t)$$

is a small perturbation of the movement of the string, which because of (1.1) does not change the boundary, initial and final values. Therefore Hamilton's principle states that the material points of the string move such that the integral

$$\int_0^T E(t) - V(t) dt$$

is stationary when the movement of the string is perturbed such that the initial, final and boundary values are not changed.

The equation (1.2) can be used to derive an equation for the movement of the string. For, (1.2) yields

$$\begin{aligned}
0 &= \frac{d}{ds} \int_0^T \int_a^b \frac{\rho(x)}{2} (u_t(x, t) + sw_t(x, t))^2 - \frac{\kappa}{2} (u_x(x, t) + sw_x(x, t))^2 dx dt \Big|_{s=0} \\
&= \int_0^T \int_a^b \rho(x) (u_t(x, t) + sw_t(x, t)) w_t(x, t) \\
&\quad - \kappa (u_x(x, t) + sw_x(x, t)) w_x(x, t) dx dt \Big|_{s=0} \\
&= \int_0^T \int_a^b (\rho(x) u_t(x, t) w_t(x, t) - \kappa u_x(x, t) w_x(x, t)) dx dt =: I.
\end{aligned}$$

If u is two times continuously differentiable, then the last integral can be transformed using partial integration. Since w vanishes at the boundary of the rectangle $[a, b] \times [0, T]$ we obtain

$$0 = I = - \int_0^T \int_a^b (\rho(x) u_{tt}(x, t) - \kappa u_{xx}(x, t)) w(x, t) dx dt.$$

This must hold for all continuously differentiable functions w vanishing at the boundary. If $\rho(x) u_{tt}(x, t) - \kappa u_{xx}(x, t)$ is continuous, this can only hold if

$$\rho(x) u_{tt}(x, t) - \kappa u_{xx}(x, t) = 0 \tag{1.3}$$

for all $(x, t) \in [a, b] \times [0, T]$.

1.3 Initial-boundary value problems for the one-dimensional wave equation

Since T is an arbitrary chosen positive number, we conclude that the vibrating string must satisfy the equation (1.3) in the whole domain $[a, b] \times [0, \infty)$. We thus have

$$\rho(x) u_{tt}(x, t) = \kappa u_{xx}(x, t), \quad (x, t) \in [a, b] \times [0, \infty).$$

This is a linear partial differential equation of second order for u , the wave equation in one space dimension. Since the ends of the string at $x = a$ or $x = b$ can be fixed or can be subjected to arbitrarily given motions, and since at time $t = 0$ the material points of the string can be displaced arbitrarily and can be submitted to arbitrarily given velocities, one wants to solve the following initial-boundary value problem to determine the motion of the string:

$$\begin{aligned}
&\rho(x) u_{tt}(x, t) = \kappa u_{xx}(x, t), & (x, t) \in [a, b] \times [0, \infty), \\
\text{(BD)} \quad &u(a, t) = u^{(a)}(t), \quad u(b, t) = u^{(b)}(t), & t \in [0, \infty), \\
\text{(IC)} \quad &u(x, 0) = u^{(0)}(x), \quad u_t(x, 0) = u^{(1)}(x), & x \in [a, b],
\end{aligned}$$

with given functions $u^{(a)}, u^{(b)} : [0, \infty) \rightarrow \mathbb{R}$, $u^{(0)}, u^{(1)} : [a, b] \rightarrow \mathbb{R}$. This is the **Dirichlet initial-boundary value problem** for the wave equation. The **Neumann initial-boundary value problem** is obtained if instead of the values $u(a, t)$ and $u(b, t)$ the values $u_x(a, t)$ and $u_x(b, t)$ for the x derivatives are prescribed:

$$\begin{aligned} \rho(x)u_{tt}(x, t) &= \kappa u_{xx}(x, t), & (x, t) &\in [a, b] \times [0, \infty), \\ \text{(BC)} \quad u_x(a, t) &= v^{(a)}(t), \quad u_x(b, t) = v^{(b)}(t), & t &\in [0, \infty), \\ \text{(IC)} \quad u(x, 0) &= u^{(0)}(x), \quad u_t(x, 0) = u^{(1)}(x), & x &\in [a, b]. \end{aligned}$$

If $a = -\infty$ and $b = \infty$ and no boundary conditions are posed, then one speaks of the **Cauchy problem**:

$$\begin{aligned} \rho(x)u_{tt}(x, t) &= \kappa u_{xx}(x, t), & (x, t) &\in (-\infty, \infty) \times [0, \infty), \\ \text{(IC)} \quad u(x, 0) &= u^{(0)}(x), \quad u_t(x, 0) = u^{(1)}(x), & x &\in (-\infty, \infty). \end{aligned}$$

1.4 Initial-boundary value problems for the wave equation in higher space dimensions

Consider an elastic membrane, which at the boundary is fixed to a wire forming a closed loop. The projection $\overline{\Omega}$ of the membrane to the plane $\mathbb{R}^2$ is a closed bounded set, the interior of which is Ω . We assume that the boundary $\partial\Omega$ is continuously differentiable, and that the wire is given by the graph of a continuously differentiable function $\mu : \partial\Omega \rightarrow \mathbb{R}$. Let $u(x, t) \in \mathbb{R}$ be the height of the membrane above the point $x \in \overline{\Omega}$ at time $t \geq 0$. Thus, at time t the membrane is represented by the graph of the function

$$x \mapsto u(x, t) : \overline{\Omega} \rightarrow \mathbb{R}.$$

Since the membrane is attached at the boundary to the wire, we have the Dirichlet boundary condition

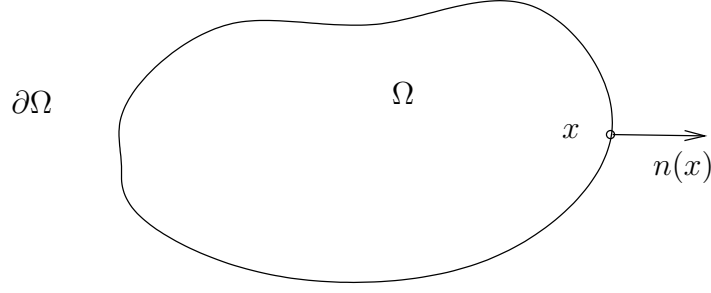
$$u(x, t) = \mu(x), \quad x \in \partial\Omega, \quad t \geq 0.$$

To determine a partial differential equation for the function u we again apply Hamilton's principle. We first need to make assumptions for the elastic properties of the membrane, or equivalently for the form of the potential energy stored in the membrane. Generalizing the one-dimensional potential energy we assume here that the potential energy $V_u(t)$ of the membrane at time t is given by

$$V_u(t) = \frac{\kappa}{2} \int_{\Omega} |\nabla_x u(x, t)|^2 dx,$$

with the gradient

$$\nabla_x u(x, t) = \begin{pmatrix} \frac{\partial}{\partial x_1} u(x, t) \\ \frac{\partial}{\partial x_2} u(x, t) \end{pmatrix}.$$



The exterior unit normal vector $n(x)$

The kinetic energy of the membrane is

$$E_u(t) = \int_{\Omega} \frac{\rho(x)}{2} (u_t(x, t))^2 dx.$$

If $T > 0$ and if

$$w : \overline{\Omega} \times [0, T] \rightarrow \mathbb{R}$$

is continuously differentiable with

$$\begin{aligned} w(x, t) &= 0, \quad (x, t) \in \partial\Omega \times [0, T] \\ w(x, 0) &= w(x, T) = 0, \quad x \in \Omega, \end{aligned}$$

then Hamilton's principle yields

$$\begin{aligned} 0 &= \frac{d}{ds} \int_0^T E_{u+sw}(t) - V_{u+sw}(t) dt \Big|_{s=0} \\ &= \frac{d}{ds} \int_0^T \int_{\Omega} \frac{\rho(x)}{2} (u_t(x, t) + sw_t(x, t))^2 - \frac{\kappa}{2} |\nabla_x u(x, t) + s \nabla_x w(x, t)|^2 dx dt \Big|_{s=0} \\ &= \int_0^T \int_{\Omega} (\rho(x) u_t(x, t) w_t(x, t) - \kappa \nabla_x u(x, t) \cdot \nabla_x w(x, t)) dx dt. \end{aligned}$$

If u is two times continuously differentiable then the first Green's formula yields

$$\begin{aligned} 0 &= - \int_0^T \int_{\Omega} (\rho(x) u_{tt}(x, t) - \kappa \Delta_x u(x, t)) w(x, t) dx dt \\ &\quad + \int_{\Omega} \rho(x) u_t(x, T) w(x, T) - \rho(x) u_t(x, 0) w(x, 0) dx \\ &\quad - \int_0^T \int_{\partial\Omega} \kappa \left(\frac{\partial}{\partial n_x} u(x, t) \right) w(x, t) d\sigma_x dt \\ &= - \int_0^T \int_{\Omega} (\rho(x) u_{tt}(x, t) - \kappa \Delta_x u(x, t)) w(x, t) dx dt, \end{aligned} \tag{1.4}$$

with the Laplace operator

$$\Delta_x u(x, t) = \sum_{i=1}^2 \frac{\partial^2}{\partial x_i^2} u(x, t)$$

and the normal derivative

$$\frac{\partial}{\partial n} u(x, t) = n(x) \cdot \nabla_x u(x, t),$$

where $n(x) \in \mathbb{R}^2$ is the unit normal vector to the boundary $\partial\Omega$ at $x \in \partial\Omega$ pointing into the exterior $\mathbb{R}^2 \setminus \overline{\Omega}$ of $\overline{\Omega}$. (1.4) must be satisfied for all w with the stated properties. This is only possible if the bracketed expression in the integrand on the right hand side vanishes identically, whence u must satisfy

$$\rho(x)u_{tt}(x, t) = \kappa\Delta_x u(x, t), \quad (x, t) \in \overline{\Omega} \times [0, T].$$

This is the wave equation in two space dimensions. Since T is arbitrary it follows that u must satisfy the wave equation for all $(x, t) \in \overline{\Omega} \times [0, \infty)$. We already noted that u must satisfy the Dirichlet boundary condition. Therefore u must be a solution of the Dirichlet initial-boundary value problem, which we immediately formulate for the n -dimensional wave equation. Thus, for $n \in \mathbb{N}$ let

$$\Delta_x u(x, t) = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2} u(x, t), \quad x = (x_1, \dots, x_n) \in \mathbb{R}^n$$

be the n -dimensional Laplace operator. With this operator the inhomogeneous Dirichlet initial-boundary value problem in a domain $\Omega \subseteq \mathbb{R}^n$ is

$$\begin{aligned} \rho(x)u_{tt}(x, t) &= \kappa\Delta_x u(x, t) + f(x, t), & (x, t) \in \overline{\Omega} \times [0, \infty), \\ \text{(BC)} \quad u(x, t) &= \mu(x, t), & (x, t) \in \partial\Omega \times [0, \infty), \\ \text{(IC)} \quad u(x, 0) &= u^{(0)}(x), \quad u_t(x, 0) = u^{(1)}(x), & x \in \overline{\Omega}, \end{aligned}$$

with given functions $f : \overline{\Omega} \times [0, \infty) \rightarrow \mathbb{R}$, $\mu : \partial\Omega \times [0, \infty) \rightarrow \mathbb{R}$, $u^{(0)}, u^{(1)} : \overline{\Omega} \rightarrow \mathbb{R}$.

The vibrations of the membrane can be determined by solving this problem for $n = 2$. Physically, f is a surface force acting on the membrane, for example the gravitational force.

The Neumann initial-boundary value problem for the wave equation in $\Omega \subseteq \mathbb{R}^n$ is

$$\begin{aligned} \rho(x)u_{tt}(x, t) &= \kappa\Delta_x u(x, t) + f(x, t), & (x, t) \in \overline{\Omega} \times [0, \infty), \\ \text{(BC)} \quad \frac{\partial}{\partial n} u(x, t) &= \nu(x, t), & (x, t) \in \partial\Omega \times [0, \infty), \\ \text{(IC)} \quad u(x, 0) &= u^{(0)}(x), \quad u_t(x, 0) = u^{(1)}(x), & x \in \overline{\Omega}, \end{aligned}$$

and the Cauchy problem is

$$\begin{aligned} \rho(x)u_{tt}(x,t) &= \kappa\Delta_x u(x,t) + f(x,t), & (x,t) \in \mathbb{R}^n \times [0,\infty), \\ \text{(IC)} \quad u(x,0) &= u^{(0)}(x), \quad u_t(x,0) = u^{(1)}(x), & x \in \mathbb{R}^n. \end{aligned}$$

2 The Helmholtz equation obtained by reduction of the wave equation

2.1 Separation of variables and boundary value problems for the Helmholtz equation

Let $n \geq 1$ be an integer, and let $\Omega \subseteq \mathbb{R}^n$ be an open set. Define

$$c(x) = \sqrt{\frac{\kappa}{\rho(x)}} > 0.$$

With this notation the homogeneous wave equation becomes

$$u_{tt}(x, t) = c(x)^2 \Delta_x u(x, t), \quad (x, t) \in \bar{\Omega} \times [0, \infty). \quad (2.1)$$

Complex valued solution. Up to now we considered solutions of (2.1) with values in the real numbers. One defines complex valued solutions $u : \Omega \times [0, \infty) \rightarrow \mathbb{C}$ with $u = u_1 + iu_2$, $u_1, u_2 : \Omega \times [0, \infty) \rightarrow \mathbb{R}$, by setting $\Delta_x u(x, t) = \Delta_x u_1(x, t) + i\Delta_x u_2(x, t)$ and $\partial_t^2 u(x, t) = \partial_t^2 u_1(x, t) + i\partial_t^2 u_2(x, t)$. Of course, since $c(x)^2$ in (2.1) is real, in this special case a complex valued function is a solution, if both the real part u_1 and the imaginary part u_2 solve the wave equation. Though complex valued solutions seem to be more complicated than real valued solutions, it turns out that allowing complex valued solutions elucidates the situation considerably. Henceforth we consider complex valued solutions.

Separation of variables. To find a solution of the wave equation (2.1) it suggests itself to try the product ansatz

$$u(x, t) = w(t) v(x).$$

Insertion into the wave equation yields

$$w_{tt}(t) v(x) = c(x)^2 w(t) \Delta v(x),$$

hence

$$\frac{w_{tt}(t)}{w(t)} = \frac{c(x)^2 \Delta v(x)}{v(x)}.$$

This equation must hold for all $x \in \Omega$ and all $t \in [0, \infty)$. Since the left hand side only depends on t and the right hand side on x , this is only possible if the fractions on both sides have a constant value $-\lambda \in \mathbb{R}$. Thus,

$$w_{tt}(t) + \lambda w(t) = 0, \quad t \in [0, \infty) \quad (2.2)$$

$$c(x)^2 \Delta v(x) + \lambda v(x) = 0, \quad x \in \bar{\Omega}. \quad (2.3)$$

The first equation is a linear, homogeneous ordinary differential equation of second order. The second equation is called Helmholtz equation or reduced wave

equation, a linear second order partial differential equation. More precisely, these names are usually reserved to the equations obtained for $c = 1$, but we use them also in the case when the coefficient $c(x)^2 > 0$ is variable. With $\lambda = 0$ one obtains the potential equation

$$\Delta v(x) = 0, \quad x \in \overline{\Omega}.$$

For $\lambda \neq 0$ the general solution of (2.2) is

$$\begin{aligned} w(t) &= C_1 e^{\sqrt{-\lambda}t} + C_2 e^{-\sqrt{-\lambda}t} \\ &= C_1 e^{\operatorname{Re} \sqrt{-\lambda}t} \left(\cos(\operatorname{Im} \sqrt{-\lambda}t) + i \sin(\operatorname{Im} \sqrt{-\lambda}t) \right) \\ &\quad + C_2 e^{-\operatorname{Re} \sqrt{-\lambda}t} \left(\cos(\operatorname{Im} \sqrt{-\lambda}t) - i \sin(\operatorname{Im} \sqrt{-\lambda}t) \right), \end{aligned}$$

whereas for $\lambda = 0$ the general solution is given by

$$w(t) = C_1 t + C_2.$$

By choosing the constant λ suitably we can thus construct solutions of the wave equation with special behavior in time. For example, if $\lambda > 0$ and if v is a solution of the Helmholtz equation to this λ , then

$$u(x, t) = (C_1 \cos(\sqrt{\lambda}t) + C_2 \sin(\sqrt{\lambda}t)) v(x)$$

is a solution representing an undamped oscillation with an amplitude $|v(x)|$ depending on the position x . If $\lambda < 0$ then

$$u(x, t) = C_1 e^{\sqrt{-\lambda}t} v(x)$$

is a solution, which increases exponentially in time, and

$$u(x, t) = C_2 e^{-\sqrt{-\lambda}t} v(x)$$

is an exponentially decreasing solution.

The method to solve the wave equation with the product ansatz $u(x, t) = w(t)v(x)$ is called method of separation of variables. Of course, with this ansatz the Dirichlet boundary condition

$$w(t) v(x) = u(x, t) = \mu(x, t), \quad (x, t) \in \partial\Omega \times [0, \infty)$$

can only be solved if also the given boundary data μ are of the form

$$\mu(x, t) = w(t) \gamma(x)$$

with a function $\gamma : \partial\Omega \rightarrow \mathbb{R}$. In this case $u(x, t) = w(t) v(x)$ solves the wave equation and the Dirichlet boundary condition if w solves (2.2) and v solves the Dirichlet boundary value problem

$$\begin{aligned} c(x)^2 \Delta v(x) + \lambda v(x) &= 0, & x \in \overline{\Omega}, \\ v(x) &= \gamma(x), & x \in \partial\Omega \end{aligned}$$

for the Helmholtz equation. Also, $u(x, t) = w(t)v(x)$ solves the wave equation and the Neumann boundary condition

$$\frac{\partial}{\partial n} u(x, t) = \nu(x, t) = w(t) \hat{\gamma}(t), \quad (x, t) \in \partial\Omega \times [0, \infty),$$

if v satisfies the Neumann boundary value problem

$$\begin{aligned} c(x)^2 \Delta v(x) + \lambda v(x) &= 0, & x \in \bar{\Omega}, \\ \frac{\partial}{\partial n} v(x) &= \hat{\gamma}(x), & x \in \partial\Omega \end{aligned}$$

for the Helmholtz equation.

2.2 Linear partial differential equations of order m

More general solutions of the wave equation can be obtained by adding two solutions $u_1(x, t) = w_1(t)v_1(x)$ and $u_2(x, t) = w_2(t)v_2(x)$ of the wave equation constructed with the method of separation of variables, for example by choosing different constants λ_1 and λ_2 . More precisely, any linear combination

$$a_1 u_1(x, t) + \dots + a_m u_m(x, t), \quad a_j \in \mathbb{C},$$

of solutions $u_i(x, t)$ of the wave equation is itself a solution of the wave equation. Even infinite series of solutions of the wave equation can yield new solutions. This is shown by the following

Theorem 2.1 *Let $\{u_m\}_{m=1}^{\infty}$ be a sequence of two times continuously differentiable solutions of the wave equation (2.1) in the domain $\bar{\Omega} \times [0, \infty)$. If the function series*

$$\sum_{m=1}^{\infty} \frac{\partial^{j_1+\dots+j_n+k}}{\partial x_1^{j_1} \dots \partial x_n^{j_n} \partial t^k} u_m(x_1, \dots, x_n, t), \quad k + j_1 + \dots + j_n \leq 2,$$

converge uniformly in every compact subset of $\bar{\Omega} \times [0, \infty)$, then

$$u(x, t) = \sum_{m=1}^{\infty} u_m(x, t)$$

is a two times differentiable solution of the wave equation. in $\bar{\Omega} \times [0, \infty)$.

The **proof** follows from the well known result of calculus, that under the assumptions of the theorem the function u is two-times continuously differentiable. We leave the proof to the reader.

Every linear combination of solutions of the wave equation is a solution since the unknown function u and its derivatives appear only linearly in the wave

equation. Partial differential equations with this property are called linear. To define precisely the notion of a linear partial differential equation I introduce the following notations:

For a multi-index $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$ and for $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ let

$$\begin{aligned} |\alpha| &= \alpha_1 + \dots + \alpha_n \quad (\text{length of the multi-index}), \\ D^\alpha v(x) &= \frac{\partial^{|\alpha|}}{\partial^{\alpha_1} x_1 \dots \partial^{\alpha_n} x_n} v(x_1, \dots, x_n), \\ \alpha! &= \alpha_1! \alpha_2! \dots \alpha_n!. \end{aligned}$$

Definition 2.2 Let $m \in \mathbb{N}$ be a given number and let $x = (x_1, \dots, x_n)$ denote points in $\mathbb{R}^n$. The expression

$$\sum_{\substack{\alpha \in \mathbb{N}_0^n \\ |\alpha| \leq m}} a_\alpha(x) D^\alpha v(x) = f(x)$$

is called linear partial differential equation for the function v with given (real or complex valued) coefficient functions a_α and given right hand side f . This equation is called of order m at the point x if at least one of the coefficient functions a_α with $|\alpha| = m$ does not vanish at x . The partial differential equation is called homogeneous if $f \equiv 0$.

Partial differential equations are grouped into various classes comprising equations with similar properties. Most important are the classes of elliptic, parabolic and hyperbolic equations. The Helmholtz equation is the prototype of a linear elliptic equation, the wave equation is the prototype of a linear hyperbolic equation, the heat equation

$$u_t(x, t) = c(x) \Delta_x u(x, t), \quad (x, t) \in \overline{\Omega} \times [0, \infty),$$

is the prototype of a linear parabolic equation. The precise definitions of elliptic and hyperbolic equations are given in Sections 8.1 and 10.1.

Well posed problem. A boundary value problem, initial value problem or initial-boundary value problem is called well posed, if it has the following three properties:

1. a solution of the problem exists,
2. the solution is unique,
3. the solution depends continuously on the right hand side, on the boundary data and on the initial data.

The meaning of continuous dependence has to be made precise in the context of the particular problem studied. These lecture notes are mostly devoted to the study of questions of well posedness of linear elliptic partial differential equations. Only in Section 10 we return to the wave equation, where we show how the solution theory for the Helmholtz equation, and more generally, for elliptic equations developed in Sections 8 and 9 can be used to solve initial-boundary value problems for the wave equation and for more general hyperbolic equations.

3 Tools from functional analysis. Weak solutions of one dimensional boundary value problems.

3.1 The Hilbert space $L^2(\Omega, \mathbb{C})$

Let $\Omega \subseteq \mathbb{R}^n$ be a nonempty, open or closed set. $L^2(\Omega) = L^2(\Omega, \mathbb{C})$ is the space of all quadratically integrable functions:

$$L^2(\Omega) = \{f : \Omega \rightarrow \mathbb{C} \mid \int_{\Omega} |f(x)|^2 dx < \infty\}.$$

We show that $L^2(\Omega)$ is a vector space:

Theorem 3.1 (Cauchy-Schwarz inequality) *Let $f, g \in L^2(\Omega)$. Then the product $f \cdot \bar{g}$ is integrable and satisfies*

$$\left| \int_{\Omega} f(x) \overline{g(x)} dx \right| \leq \left(\int_{\Omega} |f(x)|^2 dx \right)^{1/2} \left(\int_{\Omega} |g(x)|^2 dx \right)^{1/2}.$$

Proof: Let $a, b \geq 0$. From $0 \leq (a - b)^2 = a^2 - 2ab + b^2$ we infer that $ab \leq \frac{1}{2}(a^2 + b^2)$. Setting

$$a = \frac{|f(x)|}{\left(\int_{\Omega} |f(x)|^2 dx \right)^{1/2}}, \quad b = \frac{|g(x)|}{\left(\int_{\Omega} |g(x)|^2 dx \right)^{1/2}},$$

we conclude that

$$\frac{|f(x)\bar{g}(x)|}{\left(\int_{\Omega} |f(x)|^2 dx \right)^{1/2} \left(\int_{\Omega} |g(x)|^2 dx \right)^{1/2}} \leq \frac{|f(x)|^2}{2 \int_{\Omega} |f(x)|^2 dx} + \frac{|g(x)|^2}{2 \int_{\Omega} |g(x)|^2 dx}.$$

Since the right hand side is integrable we see from this inequality that $f \cdot \bar{g}$ is integrable and that

$$\frac{\int_{\Omega} |f(x)\bar{g}(x)| dx}{\left(\int_{\Omega} |f(x)|^2 dx \right)^{1/2} \left(\int_{\Omega} |g(x)|^2 dx \right)^{1/2}} \leq \frac{\int_{\Omega} |f(x)|^2 dx}{2 \int_{\Omega} |f(x)|^2 dx} + \frac{\int_{\Omega} |g(x)|^2 dx}{2 \int_{\Omega} |g(x)|^2 dx} = 1.$$

This shows that the Cauchy-Schwarz inequality holds.

Corollary 3.2 (Minkowski inequality) *Let $f, g \in L^2(\Omega)$. Then $f + g \in L^2(\Omega)$ and*

$$\left(\int_{\Omega} |f(x) + g(x)|^2 dx \right)^{1/2} \leq \left(\int_{\Omega} |f(x)|^2 dx \right)^{1/2} + \left(\int_{\Omega} |g(x)|^2 dx \right)^{1/2}.$$

Proof: The Cauchy-Schwarz inequality implies

$$\begin{aligned}
\int_{\Omega} |f(x) + g(x)|^2 dx &= \int_{\Omega} (f(x) + g(x))(\overline{f(x) + g(x)}) dx \\
&= \int_{\Omega} |f(x)|^2 + g(x)\overline{f(x)} + f(x)\overline{g(x)} + |g(x)|^2 dx \\
&= \int_{\Omega} |f(x)|^2 dx + 2 \operatorname{Re} \int_{\Omega} f(x)\overline{g(x)} dx + \int_{\Omega} |g(x)|^2 dx, \\
&\leq \int_{\Omega} |f(x)|^2 dx + 2 \left(\int_{\Omega} |f(x)|^2 dx \right)^{1/2} \left(\int_{\Omega} |g(x)|^2 dx \right)^{1/2} + \int_{\Omega} |g(x)|^2 dx \\
&= \left(\left(\int_{\Omega} |f(x)|^2 dx \right)^{1/2} + \left(\int_{\Omega} |g(x)|^2 dx \right)^{1/2} \right)^2.
\end{aligned}$$

This implies Minkowski's inequality.

For $f, g \in L^2(\Omega)$ let

$$\begin{aligned}
\|f\| &= \|f\|_{\Omega} = \left(\int_{\Omega} |f(x)|^2 dx \right)^{1/2}, \\
(f, g) &= (f, g)_{\Omega} = \int_{\Omega} f(x) \overline{g(x)} dx.
\end{aligned}$$

Corollary 3.3 $L^2(\Omega)$ is a vector space, $\|f\|$ is a norm and (f, g) is a scalar product on this vector space with

$$\|f\| = (f, f)^{1/2}.$$

Therefore $L^2(\Omega)$ is a pre-Hilbert space.

Theorem 3.4 (of Fischer-Riesz.) $L^2(\Omega)$ is a Hilbert space, i.e. the pre-Hilbert space $L^2(\Omega)$ is complete with respect to the norm $\|f\|$.

The **proof** can be found in the book "Lineare Funktionalanalysis" of H.W. Alt, Springer Verlag Berlin, 1999, pp. 49, 50.

3.2 The Riesz representation theorem and the projection theorem

Let X be an abstract Hilbert space over $\mathbb{C}$ with the scalar product (u, v) and the norm $\|u\| = (u, u)^{1/2}$. Let $F : X \rightarrow \mathbb{C}$ be a continuous linear functional (linear mapping). F is continuous if and only if F is bounded, i.e. if a constant C exists such that

$$|F(u)| \leq C\|u\|$$

for all $u \in X$. Define the mapping $J_F : X \rightarrow \mathbb{R}$ by

$$J_F(u) = \frac{1}{2} \|u\|^2 - \operatorname{Re} F(u),$$

for all $u \in X$.

Theorem 3.5 *Let Y be a closed subspace of X . Then $u \in Y$ satisfies*

$$J_F(u) = \min_{v \in Y} J_F(v)$$

if and only if for all $v \in Y$

$$(v, u) = F(v).$$

Proof: Let $J_F(u) = \min_{v \in Y} J_F(v)$. Then for all $v \in Y$ the function

$$\lambda \mapsto J_F(u + \lambda v) : \mathbb{R} \rightarrow \mathbb{R}$$

has the minimum at $\lambda = 0$, hence

$$\begin{aligned} 0 &= \frac{d}{d\lambda} J_F(u + \lambda v)|_{\lambda=0} = \frac{d}{d\lambda} \left(\frac{1}{2} (u + \lambda v, u + \lambda v) - \operatorname{Re} F(u + \lambda v) \right) \\ &= \frac{d}{d\lambda} \left(\frac{1}{2} (u, u) + \lambda \operatorname{Re} (v, u) + \lambda^2 \frac{1}{2} (v, v) - \operatorname{Re} F(u) - \lambda \operatorname{Re} F(v) \right) \\ &= \left(\operatorname{Re} (v, u) + \lambda (v, v) - \operatorname{Re} F(v) \right)|_{\lambda=0} \\ &= \operatorname{Re} (v, u) - \operatorname{Re} F(v). \end{aligned}$$

Therefore we have

$$\operatorname{Re} (v, u) = \operatorname{Re} F(v)$$

for all $v \in Y$. Thus, we also have for $v \in Y$

$$\operatorname{Im} (v, u) = -\operatorname{Re} i(v, u) = -\operatorname{Re} (iv, u) = -\operatorname{Re} F(iv) = -\operatorname{Re} i F(v) = \operatorname{Im} F(v).$$

Together it follows for all $v \in Y$

$$(v, u) = \operatorname{Re} (v, u) + i \operatorname{Im} (v, u) = \operatorname{Re} F(v) + i \operatorname{Im} F(v) = F(v).$$

Assume next that

$$(v, u) = F(v)$$

for all $v \in Y$. We have for all $v \in Y$

$$\begin{aligned} J_F(u + v) &= \frac{1}{2} (u + v, u + v) - \operatorname{Re} F(u + v) \\ &= \frac{1}{2} (u, u) - \operatorname{Re} F(u) + \frac{1}{2} (v, v) \geq J_F(u), \end{aligned}$$

whence

$$J_F(u) = \min_{v \in Y} J_F(u + v) = \min_{w \in Y} J_F(w).$$

Theorem 3.6 *The mapping J_F assumes the minimum on the subspace Y at a unique $u \in Y$.*

Proof: We use the parallelogram equality

$$\|u + v\|^2 + \|u - v\|^2 = 2\|u\|^2 + 2\|v\|^2,$$

which holds for all $u, v \in X$. Note also that for all $a, b \geq 0$ and all $\varepsilon > 0$

$$0 \leq \left(\sqrt{\varepsilon} a - \frac{1}{\sqrt{\varepsilon}} b \right)^2 = \varepsilon a^2 - 2ab + \frac{1}{\varepsilon} b^2,$$

whence

$$ab \leq \frac{\varepsilon}{2} a^2 + \frac{1}{2\varepsilon} b^2.$$

Therefore we have with $\varepsilon = \frac{1}{2}$

$$\begin{aligned} J_F(u) &= \frac{1}{2}\|u\|^2 - \operatorname{Re} F(u) \geq \frac{1}{2}\|u\|^2 - |F(u)| \\ &\geq \frac{1}{2}\|u\|^2 - C\|u\| \geq \frac{1}{2}\|u\|^2 - \frac{1}{2\varepsilon}C^2 - \frac{\varepsilon}{2}\|u\|^2 \\ &= \frac{1}{4}\|u\|^2 - C^2 \geq -C^2. \end{aligned}$$

Consequently the infimum of J_F exists on Y and satisfies

$$d = \inf_{v \in Y} J_F(v) \geq -C^2.$$

Choose a sequence $\{u_n\}_n \subseteq Y$ such that

$$\lim_{n \rightarrow \infty} J_F(u_n) = d.$$

The parallelogram equality yields

$$\begin{aligned} \|u_m - u_n\|^2 &= 2\|u_m\|^2 + 2\|u_n\|^2 - 4\left\|\frac{1}{2}(u_m + u_n)\right\|^2 \\ &= 4\left(\frac{1}{2}\|u_m\|^2 - \operatorname{Re} F(u_m) + \frac{1}{2}\|u_n\|^2 - \operatorname{Re} F(u_n)\right) \\ &\quad - 8\left(\frac{1}{2}\left\|\frac{1}{2}(u_m + u_n)\right\|^2 - \operatorname{Re} F\left(\frac{1}{2}(u_m + u_n)\right)\right) \\ &= 4J_F(u_m) + 4J_F(u_n) - 8J_F\left(\frac{1}{2}(u_m + u_n)\right) \\ &\leq 4J_F(u_m) + 4J_F(u_n) - 8d \rightarrow 0, \end{aligned}$$

for $m, n \rightarrow \infty$. Consequently, $\{u_n\}_n$ is a Cauchy sequence and has a limit u . Since Y is closed, u belongs to Y . From the Cauchy-Schwarz inequality $|(v, w)| \leq \|v\|\|w\|$ it follows that the mapping $w \mapsto \|w\|^2 : X \rightarrow \mathbb{R}$ is continuous, hence J_F is continuous. We thus obtain

$$\inf_{v \in Y} J_F(v) = \lim_{n \rightarrow \infty} J_F(u_n) = J_F(u).$$

Therefore u is the minimum of J_F on Y . To see that the minimum is unique, let u and v be two minima on Y . The calculation above yields

$$\|u - v\|^2 = 4J_F(u) + 4J_F(v) - 8J_F\left(\frac{1}{2}(u + v)\right) \leq 4d + 4d - 8d = 0,$$

whence $u = v$. This completes the proof.

Corollary 3.7 (i) **(Riesz representation theorem)** *To every bounded linear mapping $F : X \rightarrow \mathbb{C}$ there is a unique $u \in X$ such that*

$$(v, u) = F(v)$$

for all $v \in X$.

(ii) **Projection theorem** *Let Y be a closed subspace of X . To every $v \in X$ there is a unique $u \in Y$ such that*

$$\|v - u\| = \min_{w \in Y} \|v - w\|.$$

u is the unique element in Y which satisfies

$$(v - u, w) = 0, \tag{3.1}$$

for all $w \in Y$.

Proof: (i) For the subspace in Theorems 3.5 and 3.6 choose $Y = X$, let u be the minimum of J_F and apply Theorem 3.5.

(ii) Define the bounded linear functional $F : X \rightarrow \mathbb{C}$ by

$$F(w) = (w, v).$$

By Theorem 3.6 the mapping J_F has a unique minimum u on Y . Since

$$\begin{aligned} J_F(w) &= \frac{1}{2}\|w\|^2 - \operatorname{Re} F(w) \\ &= \left(\frac{1}{2}\|w\|^2 - \operatorname{Re}(w, v) + \frac{1}{2}\|v\|^2 \right) - \frac{1}{2}\|v\|^2 = \frac{1}{2}\|v - w\|^2 - \frac{1}{2}\|v\|^2, \end{aligned}$$

u is also the unique minimum of $w \mapsto \|v - w\|$ on Y . By Theorem 3.5, $u \in Y$ is the unique element satisfying $(w, u) = F(w) = (w, v)$ for all $w \in Y$. This implies (3.1).

Remark 3.8 The space X' of bounded linear functionals on X is called dual space of X . The Riesz representation theorem shows that for the Hilbert space X there is a mapping $T : X' \rightarrow X$, which assigns to every $F \in X'$ a unique element $TF \in X$, which allows to represent F by the mapping $(\cdot, TF)$. We see immediately that T is injective. It is also surjective: To see this, consider the linear mapping $G : X \rightarrow \mathbb{C}$ defined by $G(v) = (v, u)$. The Cauchy-Schwarz inequality implies $|G(v)| \leq \|u\| \|v\|$. Hence G is a bounded linear functional with $u = TG$. This shows that X' is isomorphic to X .

3.3 Complete orthonormal systems

Definition 3.9 Let $\{v_m\}_{m=1}^{\infty}$ be a sequence in a Hilbert space X .

- (i) If $(v_m, v_\ell) = 0$ for $m \neq \ell$ and $\|v_m\| = 1$ for all m , then $\{v_m\}_m$ is called a (countable) orthonormal system in X .
- (ii) The orthonormal system $\{v_m\}_m$ is called complete if the linear subspace

$$\text{span} \{v_m\}_m = \left\{ \sum_{m=1}^k a_m v_m \mid k \in \mathbb{N}, a_1, \dots, a_k \in \mathbb{C} \right\}$$

is dense in X .

Theorem 3.10 Let $\{v_m\}_m$ be an orthonormal system. Equivalent are

- (i) $\{v_m\}_m$ is complete.
- (ii) For all $f \in X$ the series $\sum_{m=1}^{\infty} (f, v_m) v_m$ converges to f in X :

$$f = \sum_{m=1}^{\infty} (f, v_m) v_m,$$

i.e.

$$\lim_{k \rightarrow \infty} \left\| f - \sum_{m=1}^k (f, v_m) v_m \right\| = 0.$$

- (iii) (**Parseval identity**) For all $f \in X$ we have

$$\|f\|^2 = \sum_{m=1}^{\infty} |(f, v_m)|^2.$$

For a **proof** cf. pp. 274, 275 of the book of Alt. $\sum_{m=1}^{\infty} (f, v_m) v_m$ is called Fourier series of f and (f, v_m) is the m -th Fourier coefficient.

Theorem 3.11 An orthonormal system $\{v_m\}_m$ is complete if and only if for all $f \in X$, $f \neq 0$, there is $v_k \in \{v_m\}_m$ such that

$$(f, v_k) \neq 0.$$

Proof. Let $V = \text{span}\{v_m\}_m$. It is obvious that there is $w \in \overline{V}$ with $(f, w) \neq 0$ if and only if there is $v_k \in \{v_m\}_m$ with $(f, v_k) \neq 0$. Therefore it suffices to show that $\overline{V} = X$ if and only if to all $f \in X$ with $f \neq 0$ there is $w \in \overline{V}$ such that $(f, w) \neq 0$.

Now, if $\overline{V} = X$ then for all $f \in X$, $f \neq 0$, choose $w = f$. This yields $(f, w) = (f, f) > 0$. On the other hand, if $\overline{V} \neq X$ choose $g \in X \setminus \overline{V}$. Since $\overline{V}$ is a closed subspace it follows by Corollary 3.7 (projection theorem) that there is $g_0 \in \overline{V}$ such that $f = g - g_0 \neq 0$ satisfies $(f, w) = 0$ for all $w \in \overline{V}$. Hence, the statement of the theorem follows.

Example 3.12 For $m \in \mathbb{Z}$ let $v_m : (0, 2\pi) \rightarrow \mathbb{C}$ be defined by

$$v_m(x) = \frac{1}{\sqrt{2\pi}} e^{imx}.$$

$\{v_m\}_{m=-\infty}^{\infty}$ is a complete orthonormal system in $L^2(0, 2\pi)$.

Proof. $\{v_m\}_m$ is an orthonormal system, since

$$(v_\ell, v_m) = \int_0^{2\pi} v_\ell(x) \overline{v_m(x)} dx = \frac{1}{2\pi} \int_0^{2\pi} e^{i(\ell-m)x} dx = \delta_{\ell m},$$

with the Kronecker symbol

$$\delta_{\ell m} = \begin{cases} 1, & \ell = m, \\ 0, & \text{otherwise.} \end{cases}$$

To show that the orthonormal system is complete, we need

Theorem 3.13 (Fejér) Let $g : \mathbb{R} \rightarrow \mathbb{C}$ be continuous and 2π -periodic. For $k, m, n \in \mathbb{Z}$, $m \geq 0$, $n \geq 1$ define

$$\begin{aligned} a_k &= \frac{1}{2\pi} \int_0^{2\pi} g(x) e^{-ikx} dx = \frac{1}{\sqrt{2\pi}} (g, v_k), \\ s_m(x) &= \sum_{k=-m}^m a_k e^{ikx}, \\ \sigma_n(x) &= \frac{1}{n} (s_0(x) + \dots + s_{n-1}(x)). \end{aligned}$$

Then the sequence $\{\sigma_n\}_{n=1}^{\infty}$ converges to g uniformly on $[0, 2\pi]$.

With this theorem we can prove that $\{v_m\}_m$ is complete: Let $f \in L^2(0, 2\pi)$ and $\varepsilon > 0$ be given arbitrarily. By a well known result from Lebesgue integration theory, the set of continuous functions on $[0, 2\pi]$ vanishing at $x = 0$ and $x = 2\pi$ is dense in $L^2(0, 2\pi)$. We can therefore choose such a function g with

$$\|f - g\| < \varepsilon.$$

Since g vanishes at the boundary points of the interval $[0, 2\pi]$, it follows that the 2π -periodic extension of g to $\mathbb{R}$ is continuous. By the Theorem of Fejér it thus follows that there is $n \in \mathbb{N}$ with

$$\sup_{0 \leq x \leq 2\pi} |g(x) - \sigma_n(x)| < \varepsilon.$$

Thus

$$\begin{aligned} \|f - \sigma_n\| &\leq \|f - g\| + \|g - \sigma_n\| \\ &\leq \varepsilon + \left(\int_0^{2\pi} |g(x) - \sigma_n(x)|^2 dx \right)^{1/2} \leq \varepsilon(1 + \sqrt{2\pi}). \end{aligned}$$

Since σ_n is a linear combination of functions from $\{v_m\}_{m=-\infty}^{\infty}$, we conclude from this estimate that $\text{span}\{v_m\}_{m=-\infty}^{\infty}$ is dense in $L^2(0, 2\pi)$. Consequently the orthonormal system is complete.

Remark 3.14 Since e^{imx} is 2π -periodic, the family $\{\frac{1}{\sqrt{2\pi}}e^{imx}\}_{m=-\infty}^{\infty}$ is obviously a complete orthonormal system on every interval $(a, 2\pi + a)$ obtained by translation of the interval $(0, 2\pi)$ by $a \in \mathbb{R}$. This remark holds also for the orthonormal system of the next example, which is often considered on the interval $(-\pi, \pi)$.

Example 3.15 A complete orthonormal system in $L^2(0, 2\pi)$ of real functions is given by

$$\left\{ \frac{1}{\sqrt{\pi}} \cos(mx), \frac{1}{\sqrt{\pi}} \sin(mx) \mid m = 0, 1, 2, \dots \right\}.$$

Proof: A well known computation shows that this system is orthonormal. To prove completeness it suffices to remark that for the functions v_m from Example 3.12

$$v_m(x) = \frac{1}{\sqrt{2\pi}} \cos(mx) + i \frac{1}{\sqrt{2\pi}} \sin(mx).$$

Hence, the linear span of this system is equal to the dense subspace $\text{span}\{v_m\}_m$.

3.4 Eigenfunctions of the Dirichlet boundary value problem in $\mathbb{R}^1$.

The Helmholtz equation in $\mathbb{R}^1$ is an ordinary differential equation. Therefore the solution of the boundary value problems to the Helmholtz equation is considerably simpler in one space dimension than in higher dimensions. Nevertheless, the solution properties of the one dimensional and higher dimensional problems are similar. Since it is helpful to know these properties when studying higher dimensional problems, we investigate in this section the one-dimensional problem. Thus, let $\Omega = (a, b)$, let $\gamma_a, \gamma_b \in \mathbb{C}$ and $\lambda \in \mathbb{C}$. We search a two times continuously differentiable solution $u : [a, b] \rightarrow \mathbb{C}$ of

$$\begin{aligned} u''(x) + \lambda u(x) &= 0, & x &\in [a, b], \\ u(a) = \gamma_a, \quad u(b) &= \gamma_b. \end{aligned}$$

For $\lambda = 0$ the general solution of the ordinary differential equation is

$$u(x) = C_1 x + C_2, \quad C_1, C_2 \in \mathbb{C}.$$

The boundary conditions yield the linear system

$$\begin{aligned} C_1 a + C_2 &= \gamma_a, \\ C_1 b + C_2 &= \gamma_b. \end{aligned}$$

It follows that for $\lambda = 0$ the boundary value problem has a unique solution given by

$$u(x) = \frac{\gamma_a - \gamma_b}{a - b}x + \frac{1}{a - b}(a\gamma_b - b\gamma_a).$$

For $\lambda \neq 0$ the general solution of the ordinary differential equation is

$$u(x) = C_1 e^{\sqrt{-\lambda}x} + C_2 e^{-\sqrt{-\lambda}x}$$

with $C_1, C_2 \in \mathbb{C}$. The boundary conditions imply

$$\begin{aligned} C_1 e^{\sqrt{-\lambda}a} + C_2 e^{-\sqrt{-\lambda}a} &= \gamma_a \\ C_1 e^{\sqrt{-\lambda}b} + C_2 e^{-\sqrt{-\lambda}b} &= \gamma_b. \end{aligned}$$

This is a linear system of equations for C_1 and C_2 with the coefficient matrix

$$A = \begin{pmatrix} e^{\sqrt{-\lambda}a} & e^{-\sqrt{-\lambda}a} \\ e^{\sqrt{-\lambda}b} & e^{-\sqrt{-\lambda}b} \end{pmatrix}.$$

Therefore the boundary value problem is uniquely solvable for all γ_a, γ_b if and only if $\det A \neq 0$. Now

$$\det A = e^{\sqrt{-\lambda}a} e^{-\sqrt{-\lambda}b} - e^{\sqrt{-\lambda}b} e^{-\sqrt{-\lambda}a} = e^{\sqrt{-\lambda}(a-b)}(1 - e^{2\sqrt{-\lambda}(b-a)}).$$

Thus, $\det A = 0$ if and only if

$$2\sqrt{-\lambda}(b - a) = 2\pi im, \quad m \in \mathbb{Z},$$

which is equivalent to

$$\lambda = \lambda_m = \left(\frac{\pi m}{b - a}\right)^2.$$

Together we obtain

Theorem 3.16 (i) *The boundary value problem*

$$\begin{aligned} u''(x) + \lambda u(x) &= 0, \quad a \leq x \leq b \\ u(a) = \gamma_a, \quad u(b) &= \gamma_b \end{aligned}$$

is uniquely solvable for all $\gamma_a, \gamma_b \in \mathbb{C}$ if $\lambda \neq \lambda_m$ for all $m \in \mathbb{N}$, where

$$\lambda_m = \left(\frac{\pi m}{b - a}\right)^2, \quad m \in \mathbb{N}.$$

In particular, $u = 0$ is the only solution to the homogeneous boundary value problem ($\gamma_a = \gamma_b = 0$).

(ii) *If there is $m \in \mathbb{N}$ such that $\lambda = \lambda_m$, then the boundary value problem is not solvable for all γ_a, γ_b , and the solution is not unique. In particular, for every $C \neq 0$ the function*

$$u_m(x) = C \sin(\sqrt{\lambda_m}(x - a)) = C \sin\left(\frac{\pi m}{b - a}(x - a)\right)$$

is a nonzero solution of the homogeneous boundary value problem.

For the **proof** it only remains to show that u_m solves the homogeneous boundary value problem. Yet, obviously

$$u_m(a) = 0, \quad u_m(b) = C \sin(\pi m) = 0.$$

Definition 3.17 The numbers $\lambda_m = \left(\frac{\pi m}{b-a}\right)^2$, $m \in \mathbb{N}$, are called eigenvalues of the boundary value problem

$$\begin{aligned} u''(x) + \lambda u(x) &= 0, \\ u(a) = \gamma_a, \quad u(b) &= \gamma_b. \end{aligned}$$

Every nonvanishing solution of this boundary value problem with $\lambda = \lambda_m$ and $\gamma_a = \gamma_b = 0$ is called eigenfunction to the eigenvalue λ_m .

Theorem 3.18 *Let*

$$u_m(x) = \sqrt{\frac{2}{b-a}} \sin\left(\frac{\pi m}{b-a}(x-a)\right).$$

$\{u_m\}_{m=1}^\infty$ is a complete orthonormal system in $L^2([a, b])$ of eigenfunctions to the Dirichlet boundary value problem.

Proof. Above we showed that u_m is an eigenfunction for the Dirichlet boundary value problem, and a simple computation yields that $\{u_m\}_m$ is orthonormal. To prove completeness, we scale and translate u_m to define the odd function $w_m : [-\pi, \pi] \rightarrow \mathbb{C}$ by

$$w_m(x) = \begin{cases} u_m\left(\frac{b-a}{\pi}x + a\right) = \sqrt{\frac{2}{b-a}} \sin(mx), & 0 \leq x \leq \pi, \\ -w_m(-x) = \sqrt{\frac{2}{b-a}} \sin(mx), & -\pi \leq x \leq 0. \end{cases}$$

By Remark 3.14 and Example 3.15, $\text{span}\{w_m\}_m$ is dense in the space

$$\{f \in L^2(-\pi, \pi) \mid f(x) = -f(-x)\},$$

since for odd functions the Fourier coefficients of the cosine functions vanish. From this we conclude immediately that $\text{span}\{u_m\}_m$ is dense in $L^2(a, b)$.

This result suggests to construct solutions of the Dirichlet boundary value problem

$$\begin{aligned} u''(x) + \lambda u(x) &= f(x), & a \leq x \leq b, \\ u(a) = u(b) &= 0 \end{aligned}$$

with a given function $f \in L^2([a, b])$ as follows:

Let $\{\lambda_m\}_m$ be the sequence of eigenvalues to the Dirichlet boundary value problem and assume that $\lambda \neq \lambda_m$ for all m . With the complete orthonormal system $\{u_m\}_m$ of eigenfunctions consider the series

$$\sum_{m=1}^{\infty} \frac{1}{\lambda - \lambda_m} (f, u_m) u_m.$$

This series converges in $L^2([a, b])$. To see this, note that

$$\left\| \sum_{m=k}^{\ell} \frac{1}{\lambda - \lambda_m} (f, u_m) u_m \right\|^2 = \sum_{m,j=k}^{\ell} \frac{(f, u_m)}{\lambda - \lambda_m} \frac{\overline{(f, u_j)}}{\lambda - \lambda_j} (u_m, u_j) = \sum_{m=k}^{\ell} \left| \frac{(f, u_m)}{\lambda - \lambda_m} \right|^2,$$

hence the series is a Cauchy sequence, and therefore converges, if and only if $\sum_{m=1}^{\infty} \left| \frac{(f, u_m)}{\lambda - \lambda_m} \right|^2 < \infty$. Now, $\lambda_m = \left(\frac{\pi m}{b-a} \right)^2 \rightarrow \infty$ for $m \rightarrow \infty$ implies that there is a constant $C > 0$ such that

$$\left| \frac{1}{\lambda - \lambda_m} \right| \leq \frac{C}{m^2}$$

for all $m \in \mathbb{N}$. Thus,

$$\sum_{m=1}^{\infty} \left| \frac{(f, u_m)}{\lambda - \lambda_m} \right|^2 \leq \sum_{m=1}^{\infty} \frac{C^2}{m^4} |(f, u_m)|^2 \leq C^2 \sum_{m=1}^{\infty} |(f, u_m)|^2 < \infty,$$

since the Fourier series $\sum_{m=1}^{\infty} (f, u_m) u_m$ converges to f in $L^2([a, b])$. Consequently, the series $\sum_{m=1}^{\infty} \frac{1}{\lambda - \lambda_m} (f, u_m) u_m$ converges. Denote the limit function by u :

$$u = \sum_{m=1}^{\infty} \frac{1}{\lambda - \lambda_m} (f, u_m) u_m.$$

We want to show that u is a solution of the inhomogeneous boundary value problem. To this end note that if u is two-times differentiable and if the derivatives can be interchanged with the summation sign it follows that

$$\begin{aligned} u'' + \lambda u &= \frac{d^2}{dx^2} \sum_{m=1}^{\infty} \frac{(f, u_m)}{\lambda - \lambda_m} u_m + \lambda \sum_{m=1}^{\infty} \frac{(f, u_m)}{\lambda - \lambda_m} u_m \\ &= \sum_{m=1}^{\infty} \frac{(f, u_m)}{\lambda - \lambda_m} (u_m'' + \lambda u_m) = \sum_{m=1}^{\infty} \frac{(f, u_m)}{\lambda - \lambda_m} (\lambda - \lambda_m) u_m \\ &= \sum_{m=1}^{\infty} (f, u_m) u_m = f. \end{aligned}$$

Moreover, if in addition the series $\sum_{m=1}^{\infty} (f, u_m) u_m(x)$ converges for all $x \in [a, b]$ to $u(x)$, then

$$u(a) = \sum_{m=1}^{\infty} \frac{(f, u_m)}{\lambda - \lambda_m} u_m(a) = 0, \quad u(b) = \sum_{m=1}^{\infty} \frac{(f, u_m)}{\lambda - \lambda_m} u_m(b) = 0,$$

because of $u_m(a) = u_m(b) = 0$. Thus, under the assumed properties of the series $\sum_{m=1}^{\infty} \frac{(f, u_m)}{\lambda - \lambda_m} u_m$ the limit function u is a solution of the inhomogeneous boundary value problem.

However, in general these assumptions are not satisfied for $f \in L^2([a, b])$. Namely, a precise investigation shows that the boundary value problem is solvable in the classical sense only if f satisfies certain regularity properties, for example if f is continuous. Yet, if the boundary value problem has a classical solution, then it coincides with the function u given by the series. From there the idea originates to generalize the notion of a solution of the boundary value problem and to define weak solutions. The weak solution has the property to coincide with the classical solution if it exists. I introduce weak solutions in the following.

3.5 Weak derivatives

First I define weak derivatives. I need the following standard notations:

Definition 3.19 (i) Let $\Omega \subseteq \mathbb{R}^n$ be open. For $m \in \mathbb{N}_0 \cup \{\infty\}$ let

$$C_m(\Omega) = C_m(\Omega, \mathbb{C}) = \{f : \Omega \rightarrow \mathbb{C} \mid D^\alpha f \text{ exists and is continuous for all } \alpha \in \mathbb{N}_0^n \text{ such that } |\alpha| \leq m\}$$

be the space of all m -times continuously differentiable functions. One also writes $C(\Omega) = C_0(\Omega)$.

(ii) $C_m^*(\Omega) = \{f \in C_m(\Omega) \mid D^\alpha f \in L^2(\Omega) \text{ for all } |\alpha| \leq m\}$,

(iii) $C_m(\overline{\Omega}) = \{f \in C_m(\Omega) \mid D^\alpha f \text{ can be extended continuously up to the boundary}\}$.

(iv) For $f \in C(\mathbb{R}^n)$ let $\text{supp } f = \overline{\{x \in \mathbb{R}^n \mid f(x) \neq 0\}}$ be the support of f .

(v) $\mathring{C}_\infty(\Omega) = \{\varphi \in C_\infty(\mathbb{R}^n) \mid \text{supp } \varphi \text{ is a compact subset of } \Omega\}$.

Of course $C_m(\Omega)$, $C_m^*(\Omega)$, $C_m(\overline{\Omega})$ and $\mathring{C}_\infty(\Omega)$ are vector spaces.

Theorem 3.20 The space $\mathring{C}_\infty(\Omega)$ is a dense subset of $L^2(\Omega)$, i.e. $\overline{\mathring{C}_\infty(\Omega)} = L^2(\Omega)$.

A **proof** can be found in the book of H.W. Alt, pp. 74, 75.

Definition 3.21 Let $v \in L^2(\Omega)$ and $\alpha \in \mathbb{N}_0^n$. If there is a function $w \in L^2(\Omega)$ such that

$$(-1)^{|\alpha|} (v, D^\alpha \varphi) = (w, \varphi)$$

for all $\varphi \in \mathring{C}_\infty(\Omega)$, then w is called the α -th weak derivative of v .

Theorem 3.22 (i) *The α -th weak derivative is uniquely determined.*
(ii) *For $v \in C_m^*(\Omega)$ and $|\alpha| \leq m$ the α -th weak derivative coincides with the classical derivative $D^\alpha v$.*

Proof. (i) Let w_1 and w_2 be weak α -th derivatives of $v \in L^2(\Omega)$. Then, for all $\varphi \in \mathring{C}_\infty(\Omega)$

$$(w_1, \varphi) = (-1)^{|\alpha|} (v, D^\alpha \varphi) = (w_2, \varphi),$$

hence $(w_1 - w_2, \varphi) = 0$. Since $\overline{\mathring{C}_\infty(\Omega)} = L^2(\Omega)$, there is a sequence $\{\varphi_m\}_m \subseteq \mathring{C}_\infty(\Omega)$ such that $\lim_{m \rightarrow \infty} \|(w_1 - w_2) - \varphi_m\| = 0$. Thus

$$\begin{aligned} (w_1 - w_2, w_1 - w_2) &= \lim_{m \rightarrow \infty} [(w_1 - w_2, (w_1 - w_2) - \varphi_m) + (w_1 - w_2, \varphi_m)] \\ &\leq \lim_{m \rightarrow \infty} \|w_1 - w_2\| \|(w_1 - w_2) - \varphi_m\| = 0, \end{aligned}$$

whence $w_1 = w_2$. Here I used Cauchy-Schwarz' inequality. Therefore v has at most one weak derivative.

(ii) To $\varphi \in \mathring{C}_\infty(\Omega)$ there is a neighborhood of the boundary $\partial\Omega$ where φ vanishes. Thus, for $v \in C_m^*(\Omega)$ it follows by partial integration

$$\begin{aligned} (-1)^{|\alpha|} (v, D^\alpha \varphi) &= (-1)^{|\alpha|} \int_\Omega v(x) D^\alpha \overline{\varphi(x)} dx \\ &= \int_\Omega D^\alpha v(x) \overline{\varphi(x)} dx = (D^\alpha v, \varphi). \end{aligned}$$

Consequently, $D^\alpha v \in L^2(\Omega)$ is the weak derivative of v .

Because of this theorem one uses the notation $D^\alpha v$ also for weak derivatives of v . Confusion is not possible, since the weak derivative is equal to the classical derivative, if the latter exists.

Examples (a) Let $\Omega = (-1, 1)$ and let $v \in L^2((-1, 1))$ be defined by $v(x) = |x|$. This function has the weak derivative

$$v'(x) = \begin{cases} -1, & -1 < x < 0 \\ 1, & 0 \leq x < 1. \end{cases}$$

For, if $\varphi \in \mathring{C}_\infty((-1, 1))$ then

$$\begin{aligned} -(v, \varphi') &= - \int_{-1}^1 v(x) \overline{\varphi'(x)} dx = \int_{-1}^0 x \overline{\varphi'(x)} dx - \int_0^1 x \overline{\varphi'(x)} dx \\ &= - \int_{-1}^0 \overline{\varphi} dx + \int_0^1 \overline{\varphi} dx = \int_{-1}^1 v'(x) \overline{\varphi(x)} dx = (v', \varphi). \end{aligned}$$

(b) v does not have a second weak derivative. For, if $v'' \in L^2(\Omega)$ is the second weak derivative then for all $\varphi \in \mathring{C}_\infty(\Omega)$

$$\begin{aligned} (v'', \varphi) &= (v, \varphi'') = -(v', \varphi') = \int_{-1}^0 \bar{\varphi}'(x) dx - \int_0^1 \bar{\varphi}'(x) dx \\ &= \bar{\varphi}(0) - \bar{\varphi}(-1) - \bar{\varphi}(1) + \bar{\varphi}(0) = 2\bar{\varphi}(0). \end{aligned}$$

Now choose $\varphi \in \mathring{C}_\infty((-1, 1))$ with $\varphi(0) \neq 0$ and define φ_ℓ by $\varphi_\ell(x) = \varphi(\ell x)$, for $\ell \in \mathbb{N}$. Then $\varphi_\ell \in \mathring{C}_\infty((-1, 1))$, and by the preceding equation

$$\begin{aligned} 2|\varphi(0)| &= 2|\varphi_\ell(0)| = 2 \lim_{\ell \rightarrow \infty} |\varphi_\ell(0)| = \lim_{\ell \rightarrow \infty} |(v'', \varphi_\ell)| \\ &\leq \lim_{\ell \rightarrow \infty} \|v''\| \|\varphi_\ell\| = \|v''\| \lim_{\ell \rightarrow \infty} \left(\int_{-1}^1 |\varphi(\ell x)|^2 dx \right)^{1/2} \\ &= \|v''\| \lim_{\ell \rightarrow \infty} \left(\int_{-\infty}^{\infty} |\varphi(y)|^2 \frac{1}{\ell} dy \right)^{1/2} = 0. \end{aligned}$$

This contradicts $\varphi(0) \neq 0$, hence v cannot have the second weak derivative v'' .

3.6 Sobolev spaces

Definition 3.23 For an open set $\Omega \subseteq \mathbb{R}^n$ and $m \in \mathbb{N}_0$ let

$$H_m(\Omega) = \{v \in L^2(\Omega) \mid \text{the weak derivative } D^\alpha v \text{ exists for all } |\alpha| \leq m\}.$$

$H_m(\Omega)$ is called Sobolev space. For $u, v \in H_m(\Omega)$ we define

$$(u, v)_m = (u, v)_{m, \Omega} = \sum_{|\alpha| \leq m} (D^\alpha u, D^\alpha v)_\Omega, \quad \|u\|_m = \|u\|_{m, \Omega} = (u, u)_m^{1/2}.$$

$H_m(\Omega)$ is a vector space. We even have:

Theorem 3.24 $H_m(\Omega)$ is a Hilbert space with the scalar product $(u, v)_m$ and the norm $\|u\|_m$.

Proof. It is immediately seen that $(u, v)_m$ has the properties of a scalar product. Therefore it remains to show that $H_m(\Omega)$ is complete. Thus, let $\{u_\ell\}_{\ell=1}^\infty$ be a Cauchy sequence in $H_m(\Omega)$. Since

$$\|u_\ell - u_k\|_m^2 = (u_\ell - u_k, u_\ell - u_k)_m = \sum_{|\alpha| \leq m} \|D^\alpha u_\ell - D^\alpha u_k\|^2,$$

it follows that $\{D^\alpha u_\ell\}_\ell$ is a Cauchy sequence in $L^2(\Omega)$ for $|\alpha| \leq m$. Because $L^2(\Omega)$ is complete, $\{D^\alpha u_\ell\}_\ell$ has a limit function $u^{(\alpha)} \in L^2(\Omega)$. I write $u = u^{(0)}$

and show that $u^{(\alpha)} = D^\alpha u$ for all $0 < |\alpha| \leq m$. To this end let $\varphi \in \mathring{C}_\infty(\Omega)$. Then

$$(-1)^{|\alpha|}(u, D^\alpha \varphi) = \lim_{\ell \rightarrow \infty} (-1)^{|\alpha|}(u_\ell, D^\alpha \varphi) = \lim_{\ell \rightarrow \infty} (D^\alpha u_\ell, \varphi) = (u^{(\alpha)}, \varphi).$$

This implies $u^{(\alpha)} = D^\alpha u$. Consequently, $u \in H_m(\Omega)$ and $\|u - u_\ell\|_m \rightarrow 0$ for $\ell \rightarrow \infty$, whence $H_m(\Omega)$ is complete.

Theorem 3.25 (i) $C_m^*(\Omega)$ is dense in $H_m(\Omega)$:

$$H_m(\Omega) = \overline{C_m^*(\Omega)}.$$

(ii) If Ω has Lipschitz boundary, then $C_m(\overline{\Omega})$ is dense in $H_m(\Omega)$:

$$H_m(\Omega) = \overline{C_m(\overline{\Omega})}.$$

A **proof** of this theorem can be found for example in the book of Alt, pp. 108-109, and also in my lecture notes: H.-D. Alber, Variationsrechnung und Sobolevräume, p. 33 and Chapter 3¹.

Definition 3.26 Let $\Omega \subseteq \mathbb{R}^n$ be an open set. The closure of the linear subspace $\mathring{C}_\infty(\Omega)$ in $H_m(\Omega)$ is denoted by $\mathring{H}_m(\Omega)$.

$\mathring{H}_m(\Omega)$ is a closed linear subspace of $H_m(\Omega)$, hence $\mathring{H}_m(\Omega)$ is complete as a closed subspace of the complete space $H_m(\Omega)$. Therefore $\mathring{H}_m(\Omega)$ is a Hilbert space with the scalar product $(u, v)_m$ and the norm $\|u\|_m$. In general $\mathring{H}_m(\Omega)$ is a proper subspace of $H_m(\Omega)$. This subspace consists of all functions of $H_m(\Omega)$, which in a generalized sense vanish on the boundary $\partial\Omega$.

Another important property of Sobolev functions is that if $m > \frac{n}{2}$, then $u \in H_m(\Omega)$ is continuous and all weak derivatives $D^\alpha u$ with $|\alpha| < m - \frac{n}{2}$ are classical, hence $H_m(\Omega) \subseteq C_{[m-\frac{n}{2}]}^*(\Omega)$, where $[r]$ denotes the largest integer not greater than r . This property is called Sobolev imbedding theorem.

The investigation of these properties of Sobolev functions is an extended topic. Fortunately, in this introductory course we almost exclusively need those properties of Sobolev functions which immediately follow from the definitions of the Sobolev spaces given above. Yet, to familiarize the reader with Sobolev spaces we prove now two of these properties in the case $\Omega \subseteq \mathbb{R}^1$:

Theorem 3.27 Let $\Omega = (a, b) \subseteq \mathbb{R}$ be an open interval and $u, v \in H_1((a, b))$. Then

$$|u(y) - u(x)| \leq \|u'\|_{(a,b)} |y - x|^{1/2}, \quad (3.2)$$

$$|u(x)| \leq r^{1/2} \|u'\|_{(a,b)} + r^{-1/2} \|u\|_{(a,b)}, \quad (3.3)$$

$$(u', v)_{(a,b)} + (u, v')_{(a,b)} = u(b)v(b) - u(a)v(a), \quad (3.4)$$

for almost all $x, y \in (a, b)$ and for all $0 < r \leq b - a$.

¹www.mathematik.tu-darmstadt.de/ags/ag6/Skripten/Skripten_Alber/Vorlesungen.html

Remark 3.28 This means that there is a set $M \subseteq (a, b)$ with $\text{meas}((a, b) \setminus M) = 0$, which consequently is dense in $[a, b]$, such that (3.2) and (3.3) hold for all $x, y \in M$. By (3.2), u is Hölder continuous on M with exponent $\frac{1}{2}$. Hence, u is uniformly continuous on M and can be modified on ∂M , such that the modified function $\tilde{u}$ is Hölder continuous on all of $\bar{M} = M \cup \partial M = [a, b]$. There can be no other continuous function in the equivalence class of u . Therefore we can single out this continuous function and identify the equivalence class with $\tilde{u}$. With this identification every $u \in H_1((a, b))$ belongs to the space $C_{1/2}([a, b])$ of Hölder continuous functions with exponent $\frac{1}{2}$, and $H_1((a, b))$ is embedded in this space. In (3.4) we use this identification, so $u(a), u(b), v(a), v(b)$ are the values of the continuous representatives. (3.4) shows that partial integration is allowed for weak derivatives.

Proof. Choose a sequence $\{u_\ell\}_\ell \subseteq C_1([a, b])$ such that $\|u - u_\ell\|_{1, (a, b)} \rightarrow 0$ for $\ell \rightarrow \infty$. Then $\{u_\ell\}_\ell$ converges in $L^2((a, b))$ to u . Thus, by a well known theorem from Lebesgue integration theory we can select a subsequence $\{u_{\ell_k}\}_k$ such that

$$\lim_{k \rightarrow \infty} u_{\ell_k}(x) = u(x)$$

for almost all $x \in [a, b]$. Let $x < y$ be two points with this property and let $\varepsilon > 0$. Then there is k_0 such that

$$|u(x) - u_{\ell_k}(x)| < \varepsilon, \quad |u(y) - u_{\ell_k}(y)| < \varepsilon$$

for $k \geq k_0$. The fundamental theorem of calculus yields for $k \geq k_0$

$$\begin{aligned} |u(y) - u(x)| &\leq |u(y) - u_{\ell_k}(y)| + |u_{\ell_k}(y) - u_{\ell_k}(x)| + |u(x) - u_{\ell_k}(x)| \\ &\leq 2\varepsilon + \left| \int_x^y u'_{\ell_k}(z) dz \right| \leq 2\varepsilon + \left(\int_x^y dz \right)^{1/2} \left(\int_x^y |u'_{\ell_k}(z)|^2 dz \right)^{1/2} \\ &\leq 2\varepsilon + |y - x|^{1/2} \|u'_{\ell_k}\|_{(a, b)} \\ &\leq 2\varepsilon + |y - x|^{1/2} (\|u'\|_{(a, b)} + \|u'_{\ell_k} - u'\|_{(a, b)}) . \end{aligned}$$

Because of $\|u'_{\ell_k} - u'\|_{(a, b)} \leq \|u_{\ell_k} - u\|_{1, (a, b)} < \varepsilon$ for $k \geq k_1$ with k_1 sufficiently large, we deduce from this inequality by choosing $k \geq \max(k_0, k_1)$ that

$$|u(y) - u(x)| \leq \varepsilon(2 + (b - a)^{1/2}) + \|u'\|_{(a, b)} |y - x|^{1/2}.$$

Since $\varepsilon > 0$ was arbitrary, (3.2) follows.

To prove (3.3), let (c, d) with $x \in (c, d) \subseteq (a, b)$ be an interval of finite length. We integrate (3.2) with respect to y from c to d and obtain

$$\begin{aligned} |u(x)|(d - c) &\leq \|u'\|_{(a, b)} \int_c^d |x - y|^{1/2} dx + \int_c^d |u(y)| dy \\ &\leq \|u'\|_{(a, b)} (d - c)^{3/2} + (d - c)^{1/2} \left(\int_c^d |u(y)|^2 dy \right). \end{aligned}$$

Division by $(d - c)$ yields

$$|u(x)| \leq (d - c)^{1/2} \|u'\|_{(a,b)} + (d - c)^{-1/2} \|u\|_{(a,b)}.$$

This implies (3.3) with $r = d - c$.

To prove (3.4) we can assume that u and v are continuous. Choose sequences $\{u_\ell\}_\ell, \{v_\ell\}_\ell \subseteq C_1([a, b])$ such that $\|u - u_\ell\|_{1,(a,b)} \rightarrow 0, \|v - v_\ell\|_{1,(a,b)} \rightarrow 0$ for $\ell \rightarrow \infty$. From (3.3) we obtain

$$\lim_{\ell \rightarrow \infty} |u(x) - u_\ell(x)| \leq \lim_{\ell \rightarrow \infty} (r^{1/2} \|u' - u'_\ell\| + r^{-1/2} \|u - u_\ell\|) = 0$$

for almost all $x \in [a, b]$. This relation shows that $\{u_\ell\}_\ell$ and $\{v_\ell\}_\ell$ converge uniformly on M to u and v , respectively, with the set M defined in Remark 3.28. Since u, v, u_ℓ, v_ℓ are continuous, we infer that the function sequences converge pointwise everywhere to u and v . In particular, we have $\lim_{\ell \rightarrow \infty} u_\ell(a) = u(a)$, $\lim_{\ell \rightarrow \infty} u_\ell(b) = u(b)$, and similarly $\lim_{\ell \rightarrow \infty} v_\ell(a) = v(a)$, $\lim_{\ell \rightarrow \infty} v_\ell(b) = v(b)$. Using the continuity of the scalar product we obtain by partial integration

$$\begin{aligned} (u', v) + (u, v') &= \lim_{\ell \rightarrow \infty} ((u'_\ell, v_\ell) + (u_\ell, v'_\ell)) \\ &= \lim_{\ell \rightarrow \infty} (u_\ell(b) v_\ell(b) - u_\ell(a) v_\ell(a)) = u(b)v(b) - u(a)v(a). \end{aligned}$$

■

Lemma 3.29 *The orthogonal space*

$$\mathring{H}_1^\perp((a, b)) = \{u \in H_1((a, b)) \mid (u, v)_1 = 0 \text{ for all } v \in \mathring{H}_1((a, b))\}$$

is given by

$$\mathring{H}_1^\perp((a, b)) = \{C_1 e^x + C_2 e^{-x} \mid C_1, C_2 \in \mathbb{C}\}.$$

Hence, the orthogonal space is of dimension 2.

Proof. u belongs to $\mathring{H}_1^\perp((a, b))$ if and only if for all $v \in \mathring{H}_1((a, b))$

$$(u', v') = -(u, v).$$

Since $\mathring{C}_\infty((a, b)) \subseteq \mathring{H}_1((a, b))$, this equation holds if and only if u has a second weak derivative which satisfies

$$u'' = u.$$

All solutions of this ordinary differential equation are of the form $u(x) = C_1 e^x + C_2 e^{-x}$ with arbitrary constants $C_1, C_2 \in \mathbb{C}$. ■

Theorem 3.30 $u \in \mathring{H}_1((a, b))$ if and only if $u \in H_1((a, b))$ and $u(a) = u(b) = 0$.

Proof. Let $u \in \mathring{H}_1((a, b))$. By definition of $\mathring{H}_1((a, b))$ there is a sequence $\{u_\ell\}_\ell \subseteq \mathring{C}_\infty((a, b))$ with $\|u - u_\ell\|_1 \rightarrow 0$ for $\ell \rightarrow \infty$. We apply (3.3) to the difference $u - u_\ell$ and note that $u_\ell(a) = 0$ to obtain

$$|u(a)| \leq r^{1/2} \|u' - u'_\ell\|_{(a, b)} + r^{-1/2} \|u - u_\ell\|_{(a, b)} \rightarrow 0,$$

for $\ell \rightarrow \infty$, whence $u(a) = 0$. In the same way we conclude that $u(b) = 0$. To prove the converse let $u \in H_1((a, b))$ satisfy $u(a) = u(b) = 0$. By Lemma 3.29 there is a unique $v \in \mathring{H}_1((a, b))$ and $C_1, C_2 \in \mathbb{C}$ such that

$$u(x) = v(x) + C_1 e^x + C_2 e^{-x}.$$

Since $v(a) = v(b) = 0$, we obtain from this equation by setting $x = a$ and $x = b$ that

$$\begin{aligned} C_1 e^a + C_2 e^{-a} &= 0 \\ C_1 e^b + C_2 e^{-b} &= 0. \end{aligned}$$

This is a system of two linear equations for C_1 and C_2 with determinant of the coefficient matrix $e^a e^{-b} - e^b e^{-a} = e^{a-b}(1 - e^{2(b-a)}) \neq 0$, since $b - a > 0$. Consequently we have $C_1 = C_2 = 0$. Thus, $u = v \in \mathring{H}_1((a, b))$. ■

3.7 Weak solution of the Dirichlet boundary value problem to the Helmholtz equation

We begin with the definition of weak solutions of the Helmholtz equation and of weak solutions to the homogeneous Dirichlet boundary value problem to this equation in n -dimensional space:

Definition 3.31 (i) Let $\Omega \subseteq \mathbb{R}^n$ be a nonempty open set, let $\lambda \in \mathbb{C}$ and assume that $f \in L^2(\Omega, \mathbb{C})$.

(i) A function $u \in \mathring{H}_1(\Omega, \mathbb{C})$ is called weak solution of the partial differential equation

$$\Delta u(x) + \lambda u(x) = f(x) \tag{3.5}$$

in Ω , if for all $\varphi \in \mathring{C}_\infty(\Omega, \mathbb{C})$ the equation

$$-(\nabla u, \nabla \varphi) + \lambda(u, \varphi) = (f, \varphi) \tag{3.6}$$

holds, where

$$(\nabla u, \nabla \varphi) = \int_{\Omega} \nabla u(x) \cdot \nabla \overline{\varphi(x)} \, dx = \sum_{i=1}^3 \int_{\Omega} \frac{\partial}{\partial x_i} u(x) \frac{\partial}{\partial x_i} \overline{\varphi(x)} \, dx.$$

(ii) A weak solution of the homogeneous Dirichlet boundary value problem

$$\begin{aligned}\Delta u(x) + \lambda u(x) &= f(x), & x \in \Omega, \\ u|_{\partial\Omega} &= 0,\end{aligned}$$

is by definition a weak solution u of the partial differential equation (3.5) belonging to $\mathring{H}_1(\Omega, \mathbb{C})$.

Formally the equation (3.6) is obtained by multiplication of both sides of the equation $\Delta u + \lambda u = f$ by φ , integration and application of the first Green's formula. The advantage is that weak solutions need to have only first derivatives and not second.

Every classical solution is also a weak solution, but not vice versa. However, if a weak solution belongs to $C_2^*(\Omega)$, then it is also a classical solution.

Again we restrict ourselves to $n = 1$ and assume that $\Omega = (a, b)$ is a bounded open interval. In this case $u \in \mathring{H}_1((a, b))$ is a weak solution if

$$-(u', \varphi') + \lambda(u, \varphi) = (f, \varphi)$$

for all $\varphi \in \mathring{C}_\infty((a, b))$.

Theorem 3.32 *Let $\{\lambda_m\}_m$ be the eigenvalues of the Dirichlet boundary value problem in the bounded interval $(a, b) \subseteq \mathbb{R}$, and let $\{u_m\}_m$ be a complete orthonormal system of eigenfunctions. Assume that $\lambda \neq \lambda_m$ for all m . Then the Dirichlet boundary value problem*

$$\begin{aligned}u''(x) + \lambda u(x) &= f(x), & a < x < b, \\ u(a) = u(b) &= 0,\end{aligned}$$

has a unique weak solution to every $f \in L^2((a, b))$, which is given by

$$u = \sum_{m=1}^{\infty} \frac{1}{\lambda - \lambda_m} (f, u_m) u_m.$$

Proof. At first it must be shown that u belongs to the space $\mathring{H}_1((a, b))$. Since the eigenfunction

$$u_m(x) = \sqrt{\frac{2}{b-a}} \sin\left(\frac{\pi m}{b-a}(x-a)\right)$$

satisfies $u_m(a) = u_m(b) = 0$, we infer from Theorem 3.30 that $u_m \in \mathring{H}_1((a, b))$. To prove that $u \in \mathring{H}_1((a, b))$ it therefore suffices to show that $\sum_{m=1}^{\infty} \frac{1}{\lambda - \lambda_m} (f, u_m) u_m$ converges in the norm of $H_1((a, b))$. Since we already proved in Section 3.4 that $\sum_{m=1}^{\infty} \frac{(f, u_m)}{\lambda - \lambda_m} u_m$ converges in $L^2((a, b))$, it suffices to

verify that also $\sum_{m=1}^{\infty} \frac{(f, u_m)}{\lambda - \lambda_m} u'_m$ converges in $L^2((a, b))$. Proceeding as in Section 3.4 we compute

$$\begin{aligned} \left\| \sum_{m=\ell}^k \frac{(f, u_m)}{\lambda - \lambda_m} u'_m \right\|^2 &= \left(\sum_{m=\ell}^k \frac{(f, u_m)}{\lambda - \lambda_m} u'_m, \sum_{j=\ell}^k \frac{(f, u_j)}{\lambda - \lambda_j} u'_j \right) \\ &= \sum_{m=\ell}^k \sum_{j=\ell}^k \frac{(f, u_m)}{\lambda - \lambda_m} \overline{\frac{(f, u_j)}{\lambda - \lambda_j}} (u'_m, u'_j). \end{aligned}$$

Since

$$(u'_m, u'_j) = -(u''_m, u_j) = (\lambda_m u_m, u_j) = \begin{cases} \lambda_m, & m = j \\ 0, & m \neq j, \end{cases}$$

we conclude

$$\left\| \sum_{m=\ell}^k \frac{(f, u_m)}{\lambda - \lambda_m} u'_m \right\|^2 = \sum_{m=\ell}^k \left| \frac{(f, u_m)}{\lambda - \lambda_m} \right|^2 \lambda_m \leq C \sum_{m=\ell}^k |(f, u_m)|^2,$$

with the constant $C = \sup_{m \in \mathbb{N}} \frac{\lambda_m}{|\lambda - \lambda_m|^2} < \infty$. This inequality and the equation $\sum_{m=1}^{\infty} |(f, u_m)|^2 = \|f\|^2 < \infty$ together imply that the series $\sum_{m=1}^{\infty} \frac{(f, u_m)}{\lambda - \lambda_m} u'_m$ satisfies the Cauchy convergence criterion, hence it converges in the complete space $L^2((a, b))$. This proves that $u \in \mathring{H}_1((a, b))$ and that

$$u' = \sum_{m=1}^{\infty} \frac{(f, u_m)}{\lambda - \lambda_m} u'_m.$$

In the next step of the proof we use this equation. Namely, for $\varphi \in \mathring{C}_{\infty}((a, b))$ we have

$$\begin{aligned} -(u', \varphi') &= - \sum_{m=1}^{\infty} \frac{(f, u_m)}{\lambda - \lambda_m} (u'_m, \varphi') = \sum_{m=1}^{\infty} \frac{(f, u_m)}{\lambda - \lambda_m} (u''_m, \varphi) \\ &= - \sum_{m=1}^{\infty} \frac{(f, u_m)}{\lambda - \lambda_m} (\lambda_m u_m, \varphi) = \sum_{m=1}^{\infty} \frac{(\lambda - \lambda_m) - \lambda}{\lambda - \lambda_m} ((f, u_m) u_m, \varphi) \\ &= -\lambda(u, \varphi) + \left(\sum_{m=1}^{\infty} (f, u_m) u_m, \varphi \right) = -\lambda(u, \varphi) + (f, \varphi). \end{aligned}$$

Consequently, u is a weak solution.

It remains to show that u is the only weak solution. Assume that $v \in \mathring{H}_1((a, b))$ is a second weak solution. Then for every $\varphi \in \mathring{C}_{\infty}((a, b))$

$$-(u' - v', \varphi') + \lambda(u - v, \varphi) = (f - f, \varphi) = 0.$$

Since every eigenfunction u_m belongs to $\mathring{H}_1((a, b))$, we can choose a sequence $\{\varphi_k\}_k \subseteq \mathring{C}_\infty((a, b))$ such that $\|u_m - \varphi_k\|_1 \rightarrow 0$ for $k \rightarrow \infty$, by definition of $\mathring{H}_1((a, b))$, and obtain from this equation and from the continuity of the scalar product that

$$-(u' - v', u'_m) + \lambda(u - v, u_m) = \lim_{k \rightarrow \infty} [-(u' - v', \varphi'_k) + \lambda(u - v, \varphi_k)] = 0.$$

Since $u(a) = v(a) = u(b) = v(b) = 0$, we obtain from the partial integration formula (3.4) that

$$\begin{aligned} \lambda(u - v, u_m) &= (u' - v', u'_m) = -(u - v, u''_m) = (u - v, \lambda_m u_m) \\ &= \lambda_m(u - v, u_m). \end{aligned}$$

Since by assumption $\lambda \neq \lambda_m$ it follows from this equation that $(u - v, u_m) = 0$ for all m . Because the orthonormal system $\{u_m\}_m$ is complete, we infer from Theorem 3.11 that $u - v = 0$, whence $u = v$. ■

For boundary value problems to the Helmholtz equation $\Delta u + \lambda u = f$ in higher dimensions a result holds, which is completely analogous to the result for the boundary value problem to the ordinary differential equation $u'' + \lambda u = f$ discussed here. This will be shown in Sections 8 and 9. However, in the following investigations of higher dimensional problems we first study classical solutions and return to weak solutions only later.

4 Boundary value problems in circular domains. Bessel functions

4.1 The Laplace operator in polar coordinates

Let

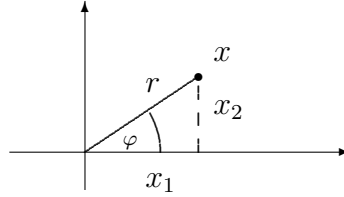
$$\Omega = \{x \in \mathbb{R}^2 \mid R_1 < |x| < R_2\}$$

with $0 \leq R_1 < R_2 \leq \infty$, or let

$$\Omega = \{x \in \mathbb{R}^2 \mid |x| < R\}.$$

To find solutions of $\Delta u(x) + \lambda u(x) = 0$ in Ω we want to use polar coordinates (r, φ) in $\mathbb{R}^2$ and apply separation of variables. To this end we must determine the form of the Laplace operator in polar coordinates. Thus, let $x = (x_1, x_2)$ and

$$\begin{aligned} r &= r(x) = \sqrt{x_1^2 + x_2^2} = |x| \\ \varphi &= \varphi(x) = \arctan \frac{x_2}{x_1}. \end{aligned}$$



Then

$$\begin{aligned} \frac{\partial}{\partial x_1} &= \frac{\partial r}{\partial x_1} \frac{\partial}{\partial r} + \frac{\partial \varphi}{\partial x_1} \frac{\partial}{\partial \varphi} = \frac{x_1}{|x|} \frac{\partial}{\partial r} - \frac{1}{1 + (\frac{x_2}{x_1})^2} \frac{x_2}{x_1^2} \frac{\partial}{\partial \varphi} = \frac{x_1}{|x|} \frac{\partial}{\partial r} - \frac{x_2}{|x|^2} \frac{\partial}{\partial \varphi}, \\ \frac{\partial}{\partial x_2} &= \frac{x_2}{|x|} \frac{\partial}{\partial r} + \frac{x_1}{|x|^2} \frac{\partial}{\partial \varphi}. \end{aligned}$$

Also,

$$\begin{aligned} \frac{\partial^2}{\partial x_1^2} &= \frac{\partial}{\partial x_1} \left(\frac{x_1}{|x|} \frac{\partial}{\partial r} - \frac{x_2}{|x|^2} \frac{\partial}{\partial \varphi} \right) \\ &= \left(\frac{1}{|x|} - \frac{x_1^2}{|x|^3} \right) \frac{\partial}{\partial r} + \frac{x_1}{|x|} \left(\frac{x_1}{|x|} \frac{\partial^2}{\partial r^2} - \frac{x_2}{|x|^2} \frac{\partial^2}{\partial r \partial \varphi} \right) \\ &\quad + 2 \frac{x_1 x_2}{|x|^4} \frac{\partial}{\partial \varphi} - \frac{x_2}{|x|^2} \left(\frac{x_1}{|x|} \frac{\partial^2}{\partial \varphi \partial r} - \frac{x_2}{|x|^2} \frac{\partial^2}{\partial \varphi^2} \right) \\ &= \frac{x_1^2}{|x|^2} \frac{\partial^2}{\partial r^2} + \left(\frac{1}{|x|} - \frac{x_1^2}{|x|^3} \right) \frac{\partial}{\partial r} + 2 \frac{x_1 x_2}{|x|^4} \frac{\partial}{\partial \varphi} \\ &\quad - 2 \frac{x_1 x_2}{|x|^3} \frac{\partial^2}{\partial \varphi \partial r} + \frac{x_2^2}{|x|^4} \frac{\partial^2}{\partial \varphi^2}, \end{aligned}$$

$$\begin{aligned}\frac{\partial^2}{\partial x_2^2} &= \frac{x_2^2}{|x|^2} \frac{\partial^2}{\partial r^2} + \left(\frac{1}{|x|} - \frac{x_2^2}{|x|^3} \right) \frac{\partial}{\partial r} - 2 \frac{x_1 x_2}{|x|^4} \frac{\partial}{\partial \varphi} \\ &\quad + 2 \frac{x_1 x_2}{|x|^3} \frac{\partial^2}{\partial \varphi \partial r} + \frac{x_1^2}{|x|^4} \frac{\partial^2}{\partial \varphi^2}.\end{aligned}$$

Thus, if $u(x) = \tilde{u}(r(x), \varphi(x))$ then

$$\begin{aligned}\Delta_x u(x) &= \sum_{i=1}^2 \frac{\partial^2}{\partial x_i^2} \tilde{u}(r(x), \varphi(x)) \\ &= \frac{x_1^2 + x_2^2}{|x|^2} \frac{\partial^2}{\partial r^2} \tilde{u}(r(x), \varphi(x)) + \left(\frac{2}{|x|} - \frac{x_1^2 + x_2^2}{|x|^3} \right) \frac{\partial}{\partial r} \tilde{u}(r(x), \varphi(x)) \\ &\quad + \frac{x_2^2 + x_1^2}{|x|^4} \frac{\partial^2}{\partial \varphi^2} \tilde{u}(r(x), \varphi(x)) \\ &= \frac{\partial^2}{\partial r^2} \tilde{u}(r(x), \varphi(x)) + \frac{1}{r(x)} \frac{\partial}{\partial r} \tilde{u}(r(x), \varphi(x)) + \frac{1}{r(x)^2} \frac{\partial^2}{\partial \varphi^2} \tilde{u}(r(x), \varphi(x)).\end{aligned}$$

Consequently

$$\Delta_{(r,\varphi)} = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \varphi^2}.$$

We next expand u in a Fourier series with respect to φ on every circle $|x| = r$ with $R_1 < r < R_2$. Thus, assume that $u \in C^2(\Omega)$ is a solution of $\Delta u + \lambda u = 0$. As usual, we drop the tilde and use the notation

$$u(x) = u(r, \varphi).$$

Since $\left\{ \frac{1}{\sqrt{2\pi}} e^{im\varphi} \right\}_{m=-\infty}^{\infty}$ is a complete orthonormal system in $L^2([0, 2\pi], \mathbb{C})$ we obtain

$$u(r, \varphi) = \sum_{m=-\infty}^{\infty} u_m(r) e^{im\varphi},$$

with

$$u_m(r) = \frac{1}{\sqrt{2\pi}} \left(u(r, \cdot), \frac{1}{\sqrt{2\pi}} e^{im\varphi} \right)_{[0, 2\pi]} = \frac{1}{2\pi} \int_0^{2\pi} u(r, \varphi) e^{-im\varphi} d\varphi.$$

If we can interchange partial derivatives up to order 2 with the summation sign, we obtain

$$\begin{aligned}0 &= (\Delta + \lambda)u(x) = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \varphi^2} \right) u(r, \varphi) + \lambda u(r, \varphi) \\ &= \sum_{m=-\infty}^{\infty} \left[\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} \right) u_m(r) + \left(\lambda - \frac{m^2}{r^2} \right) u_m(r) \right] e^{im\varphi}.\end{aligned}$$

Fix r . The Fourier series vanishes identically for all $0 \leq \varphi < 2\pi$ only if all coefficients vanish. Thus

$$\frac{d^2}{dr^2} u_m(r) + \frac{1}{r} \frac{d}{dr} u_m(r) + \left(\lambda - \frac{m^2}{r^2} \right) u_m(r) = 0,$$

for all $R_1 < r < R_2$ and all $m \in \mathbb{Z}$. This is a linear ordinary differential equation for u_m of second order.

4.2 Solution of the potential equation in circular domains.

We first consider the case $\lambda = 0$. In this case the general solution of this differential equation is

$$\begin{aligned} u_0(r) &= C_{01} + C_{02} \ln r \\ u_m(r) &= C_{m1} r^m + C_{m2} r^{-m}, \quad m \neq 0. \end{aligned}$$

Thus, the general solution of the potential equation

$$\Delta u(x) = 0$$

in a circular domain $\Omega = \{x \in \mathbb{R}^2 \mid R_1 < x < R_2\}$ is

$$u(x) = u(r, \varphi) = C_{01} + C_{02} \ln r + \sum_{\substack{m=-\infty \\ m \neq 0}}^{\infty} (C_{m1} r^m + C_{m2} r^{-m}) e^{im\varphi}$$

with arbitrary constants $C_{m1}, C_{m2} \in \mathbb{C}$. These coefficients must be determined from boundary conditions and possibly from conditions at infinity (radiation conditions).

Example 4.1 Let $\Omega = \{x \in \mathbb{R}^2 \mid |x| < R\}$ be a ball with center at 0. We want to solve the Dirichlet boundary value problem

$$\begin{aligned} \Delta u(x) &= 0, & \text{in } \Omega, \\ u(x) &= u^{(b)}(x), & x \in \partial\Omega. \end{aligned}$$

We try to find a classical solution $u \in C^2(\Omega)$. This requires that

$$C_{m1} r^m + C_{m2} r^{-m} = u_m(r) = \frac{1}{2\pi} \int_0^{2\pi} u(r, \varphi) e^{-im\varphi} d\varphi$$

and

$$C_{01} + C_{02} \ln r = u_0(r) = \frac{1}{2\pi} \int_0^{2\pi} u(r, \varphi) d\varphi$$

must be bounded at $x = 0$, hence $C_{m2} = 0$ for $m \geq 0$ and $C_{m1} = 0$ for $m < 0$. Thus

$$u(r, \varphi) = C_{01} + \sum_{m=1}^{\infty} r^m (C_{m1} e^{im\varphi} + C_{-m2} e^{-im\varphi}).$$

The Fourier series expansion of the boundary data is

$$u^{(b)}(\varphi) = \sum_{m=-\infty}^{\infty} a_m e^{im\varphi}.$$

From $u(R, \varphi) = u^{(b)}(\varphi)$ and from the uniqueness of the Fourier expansion we therefore obtain

$$a_m = R^m C_{m1}, \quad m \geq 0, \quad a_m = R^{|m|} C_{m2}, \quad m < 0.$$

Theorem 4.2 *Let $\Omega = B_R(0)$ and let $u^{(b)} \in L^2(\partial\Omega, \mathbb{C})$. Then*

$$u(x) = u(r, \varphi) = \sum_{m=-\infty}^{\infty} a_m \left(\frac{r}{R}\right)^{|m|} e^{im\varphi} \quad (4.1)$$

is the unique solution $u \in C_{\infty}(\Omega)$ of

$$\begin{aligned} \Delta u(x) &= 0, & x \in \Omega \\ \lim_{r \rightarrow R} u(r, \varphi) &= u^{(b)}(\varphi), & 0 \leq \varphi < 2\pi, \end{aligned}$$

where

$$a_m = \frac{1}{2\pi} \int_0^{2\pi} u^{(b)} e^{-im\varphi} d\varphi$$

and where the limit is understood in the L^2 -sense:

$$\lim_{r \rightarrow R} \|u(r, \cdot) - u^{(b)}\|_{[0, 2\pi]} = 0. \quad (4.2)$$

Proof: Let $0 < r_1 < R$. From $2\pi \sum_{m=-\infty}^{\infty} |a_m|^2 = \|u^{(b)}\|_{[0, 2\pi]}^2$ we obtain $|a_m|^2 \leq C$ for all m . Thus, for all $\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}_0^2$ and all $0 \leq r \leq r_1$, $0 \leq \varphi < 2\pi$ we conclude

$$\begin{aligned} & \sum_{m=-\infty}^{\infty} \left| \frac{\partial^{\alpha_1}}{\partial r^{\alpha_1}} \frac{\partial^{\alpha_2}}{\partial \varphi^{\alpha_2}} a_m \left(\frac{r}{R}\right)^{|m|} e^{im\varphi} \right| \\ & \leq \sum_{|m| \geq \alpha_1} |a_m| R^{-\alpha_1} |m|^{\alpha_1} \left(\frac{r}{R}\right)^{|m| - \alpha_1} |m|^{\alpha_2} \leq 2\sqrt{C} R^{-\alpha_1} \sum_{k=\alpha_1}^{\infty} k^{|\alpha|} \left(\frac{r_1}{R}\right)^{k - \alpha_1} \\ & = 2\sqrt{C} R^{-\alpha_1} \sum_{k=0}^{\infty} (k + \alpha_1)^{|\alpha|} \left(\frac{r_1}{R}\right)^k < \infty, \end{aligned}$$

which shows that the series $\sum_{m=-\infty}^{\infty} D_{(r,\varphi)}^{\alpha} a_m \left(\frac{r}{R}\right)^{|m|} e^{im\varphi}$ converges uniformly in every closed ball $\overline{B_{r_1}(0)} = \{r \leq r_1, 0 \leq \varphi < 2\pi\}$. From calculus we thus obtain that the classical derivative $D^{\alpha}u$ of the function u defined in (4.1) exists in all of $\Omega = B_R(0)$ and can be computed under the summation sign. This yields $u \in C_{\infty}(\Omega)$. Moreover, it shows that our assumption made in the construction of u is satisfied, whence $\Delta u = 0$ in Ω .

To verify equation (4.2) let $\varepsilon > 0$ and choose m_0 large enough such that $2\pi \sum_{|m| \geq m_0} |a_m|^2 < \varepsilon$. Then

$$\begin{aligned} \lim_{r \rightarrow R} \|u(r, \cdot) - u^{(b)}\|_{[0, 2\pi]}^2 &= \lim_{r \rightarrow R} \int_0^{2\pi} \left| \sum_{m=-\infty}^{\infty} a_m \left(\left(\frac{r}{R}\right)^{|m|} - 1\right) e^{im\varphi} \right|^2 d\varphi \\ &= \lim_{r \rightarrow R} 2\pi \sum_{m=-\infty}^{\infty} \left| a_m \left(\left(\frac{r}{R}\right)^{|m|} - 1\right) \right|^2 \\ &\leq 2\pi \lim_{r \rightarrow R} \sum_{|m| < m_0} |a_m|^2 \left(1 - \left(\frac{r}{R}\right)^{|m|}\right)^2 \\ &\quad + 2\pi \lim_{r \rightarrow R} \sum_{|m| \geq m_0} |a_m|^2 < \varepsilon. \end{aligned}$$

This proves (4.2), since $\varepsilon > 0$ was chosen arbitrarily.

Example 4.3 Let $\Omega = \{x \in \mathbb{R}^2 \mid |x| > R\}$ be an exterior domain. We want to find a solution of the Dirichlet boundary value problem

$$\begin{aligned} \Delta u(x) &= 0, \quad \text{in } \Omega, \\ u(x) &= u^{(b)}(x), \quad x \in \partial\Omega. \end{aligned}$$

In this case we cannot conclude that half of the coefficients in the expansion

$$u(x) = C_{01} + C_{02} \ln r + \sum_{\substack{m=-\infty \\ m \neq 0}}^{\infty} (C_{m1} r^m + C_{m2} r^{-m}) e^{im\varphi}$$

must vanish, and the boundary condition is not enough to determine all coefficients uniquely. Therefore the solution of the problem is not unique. To get a unique solution one must pose suitable conditions for the asymptotic behavior of u at infinity. Normally one requires that the solution is bounded:

$$|u(x)| \leq C, \quad x \in \Omega.$$

As above we then obtain $C_{m2} = 0$ for all $m \leq 0$ and $C_{m1} = 0$ for $m > 0$. Thus

$$u(r, \varphi) = C_{01} + \sum_{m=1}^{\infty} r^{-m} (C_{m2} e^{im\varphi} + C_{-m1} e^{-im\varphi}).$$

From the Fourier expansion of the boundary data

$$u^{(b)}(\varphi) = \sum_{m=-\infty}^{\infty} a_m e^{im\varphi}$$

we then conclude

$$u(x) = u(r, \varphi) = \sum_{m=-\infty}^{\infty} a_m \left(\frac{r}{R}\right)^{-|m|} e^{im\varphi}.$$

4.3 Bessel's differential equation. Solution of the Helmholtz equation in circular domains.

To solve the Helmholtz equation $\Delta u + \lambda u = 0$ in circular domains for $\lambda \neq 0$, first consider Bessel's differential equation

$$\frac{d^2}{dx^2} w(x) + \frac{1}{x} \frac{d}{dx} w(x) + \left(1 - \frac{\nu^2}{x^2}\right) w(x) = 0.$$

Here $\nu \in \mathbb{C}$ is a constant and $x \in \mathbb{C}$. This equation cannot be solved by elementary functions. Instead, the solutions are the Bessel- and Neumann functions. These functions belong to a set of functions called special functions of mathematical physics. The Bessel function or cylinder function of order $\nu \in \mathbb{C}$, $\nu \neq -1, -2, -3, \dots$, is

$$J_\nu(x) = \left(\frac{x}{2}\right)^\nu \sum_{k=0}^{\infty} \frac{(-1)^k}{k! \Gamma(\nu + k + 1)} \left(\frac{x}{2}\right)^{2k},$$

where Γ is the Gamma function. The power series converges for all $x \in \mathbb{C}$. If ν is not a nonnegative integer, then the term $\left(\frac{x}{2}\right)^\nu$ and therefore also the function J_ν are only defined on the set $\mathbb{C} \setminus (-\infty, 0]$. To be precise, $\left(\frac{x}{2}\right)^\nu$ and J_ν are defined on a Riemannian manifold. If ν is equal to a nonnegative integer m , then the formula for the Bessel function becomes

$$J_m(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k! (m+k)!} \left(\frac{x}{2}\right)^{2k+m},$$

where we used the equation

$$\Gamma(\ell + 1) = \ell!,$$

which holds for integers $\ell \geq 0$. Therefore J_m is represented by a power series converging on all of $\mathbb{C}$. Hence, J_m is an entire function.

Since Bessel's equation is a linear differential equation of second order there must exist other solutions of Bessel's equation which are linearly independent of J_ν . In fact, if ν is not an integer, then one sees immediately that also $J_{-\nu}$ is

a solution of Bessel's equation, which is linearly independent of J_ν . Hence, also the Neumann function

$$N_\nu(x) = \frac{J_\nu(x) \cos(\nu\pi) - J_{-\nu}(x)}{\sin(\nu\pi)}$$

is a solution of Bessel's equation linearly independent of J_ν . If $\nu = m$ is an integer this formula cannot be used to define N_m , since the denominator vanishes. Instead, in this case the Neumann function is

$$N_m(x) = \lim_{\nu \rightarrow m} N_\nu(x).$$

A series expression for N_m is given in the appendix. The general solution of Bessel's differential equation therefore is

$$w(x) = C_1 J_\nu(x) + C_2 N_\nu(x)$$

with arbitrary constants $C_1, C_2 \in \mathbb{C}$. If $\nu = m$ is a nonnegative integer the function $J_\nu(x)$ is infinitely differentiable at $x = 0$, whereas in the appendix it is shown that $\lim_{x \rightarrow 0} |N_m(x)| = \infty$.

We can now solve the Helmholtz equation in circular domains. For, using Bessel's equation we immediately see that if $\lambda \in \mathbb{C}$ with $\lambda \neq 0$ then

$$u_m(r) = C_1 J_m(\sqrt{\lambda}r) + C_2 N_m(\sqrt{\lambda}r)$$

satisfies

$$\frac{d^2}{dr^2} u_m(r) + \frac{1}{r} \frac{d}{dr} u_m(r) + \left(\lambda - \frac{m^2}{r^2}\right) u_m(r) = 0.$$

Remembering the results of 4.1 we therefore see that a solution of the Helmholtz equation $\Delta u + \lambda u = 0$ in circular domains must be of the form

$$u(x) = u(r, \varphi) = \sum_{m=-\infty}^{\infty} \left(C_{m1} J_{|m|}(\sqrt{\lambda}r) + C_{m2} N_{|m|}(\sqrt{\lambda}r) \right) e^{im\varphi}. \quad (4.3)$$

The constants C_{m1}, C_{m2} must be determined from the boundary and radiation conditions.

Example 4.4 Let $\Omega = B_R(0)$ be a ball, let $\lambda \in \mathbb{C}$, $\lambda \neq 0$ and assume that $u^{(b)} \in L^2(\partial\Omega)$. We want to solve

$$\begin{aligned} \Delta u(x) + \lambda u(x) &= 0, & x \in \Omega \\ u(x) &= u^{(b)}(x), & x \in \partial\Omega. \end{aligned}$$

Since u must be two times continuously differentiable at $x = 0$, it follows that in the expansion (4.3) of u we must have $C_{m2} = 0$ for all $m \in \mathbb{Z}$, since $J_{|m|}$ is regular and $N_{|m|}$ is singular at $r = 0$. Thus,

$$u(r, \varphi) = \sum_{m=-\infty}^{\infty} C_{m1} J_{|m|}(\sqrt{\lambda}r) e^{im\varphi}.$$

Let

$$u^{(b)}(\varphi) = \sum_{m=-\infty}^{\infty} a_m e^{im\varphi}$$

be the Fourier series of $u^{(b)}$. Since

$$u(R, \varphi) = \sum_{m=-\infty}^{\infty} C_{m1} J_{|m|}(\sqrt{\lambda}R) e^{im\varphi} = u^{(b)}(\varphi) = \sum_{m=-\infty}^{\infty} a_m e^{im\varphi},$$

the uniqueness of the Fourier expansion implies

$$a_m = C_{m1} J_{|m|}(\sqrt{\lambda}R).$$

Theorem 4.5 (i) *Let $\lambda \in \mathbb{C}$. If $\lambda \neq 0$ assume that $J_m(\sqrt{\lambda}R) \neq 0$ for all $m \in \mathbb{N}_0$. Then the Dirichlet boundary value problem*

$$\begin{aligned} \Delta u(x) + \lambda u(x) &= 0, \quad x \in B_R(0) \\ \lim_{r \rightarrow R} u(r, \varphi) &= u^{(b)}(R) \quad (\text{in the sense of } L^2) \end{aligned}$$

has a unique solution $u \in C_\infty(B_R(0))$ for all $u^{(b)} \in L^2(\partial B_R(0))$. This solution is given by

$$u(x) = u(r, \varphi) = \begin{cases} \sum_{m=-\infty}^{\infty} \frac{a_m}{J_{|m|}(\sqrt{\lambda}R)} J_{|m|}(\sqrt{\lambda}r) e^{im\varphi}, & \lambda \neq 0 \\ \sum_{m=-\infty}^{\infty} a_m \left(\frac{r}{R}\right)^{|m|} e^{im\varphi}, & \lambda = 0. \end{cases}$$

In particular, the only solution to homogeneous boundary data $u^{(b)} = 0$ is $u = 0$.

(ii) *Assume that $\lambda \neq 0$ and that*

$$\Lambda_\lambda = \{m \in \mathbb{N}_0 \mid J_m(\sqrt{\lambda}R) = 0\}$$

is not empty. Then the Dirichlet boundary value problem is only solvable if in the Fourier expansion

$$u^{(b)}(\varphi) = \sum_{m=-\infty}^{\infty} a_m e^{im\varphi}$$

of the boundary data we have $a_m = a_{-m} = 0$ for all $m \in \Lambda_\lambda$. On the other hand, if u is a solution of the homogeneous Dirichlet boundary value problem ($u^{(b)} = 0$), then the Fourier expansion is of the form

$$u(r, \varphi) = \sum_{m \in \Lambda_\lambda} J_m(\sqrt{\lambda}r)(C_m e^{im\varphi} + C_{-m} e^{-im\varphi}).$$

Moreover, any function with such a Fourier expansion where only finitely many C_m differ from zero is a solution of the homogeneous boundary value problem. Hence λ is an eigenvalue.

Statement (i) of this theorem is proved as in the case of $\lambda = 0$ using estimates for the Bessel functions J_m . We omit this proof. Statement (ii) is obvious from the Fourier expansion of the solution discussed above.

If u_1 and u_2 are eigenfunctions to the eigenvalue λ of the Dirichlet problem then also $C_1 u_1 + C_2 u_2$ is an eigenfunction, if this function is not zero. Therefore the set of eigenfunctions together with the zero function forms a vector space V_λ , the eigenspace of λ . The dimension of the eigenspace is called the geometric multiplicity of λ .

In the next theorem we show that for every λ the set Λ_λ is finite. The preceding theorem then implies that u is an eigenfunction to $\lambda \neq 0$ if and only if

$$u(r, \varphi) = \sum_{m \in \Lambda_\lambda} J_m(\sqrt{\lambda}r)(C_m e^{im\varphi} + C_{-m} e^{-im\varphi}),$$

hence

$$\dim V_\lambda \leq 2|\Lambda_\lambda|,$$

where $|\Lambda_\lambda|$ denotes the number of elements of Λ_λ . Since the functions $e^{im\varphi}$ and $e^{ik\varphi}$ are linearly independent for $m \neq k$, it follows that

$$\{J_m(\sqrt{\lambda}r)e^{im\varphi}, J_m(\sqrt{\lambda}r)e^{-im\varphi}\}_{m \in \Lambda_\lambda}$$

is a linearly independent set of functions. Hence it is a basis of V_λ . Therefore

$$\dim V_\lambda = 2|\Lambda_\lambda|.$$

Theorem 4.6 (i) Assume that $m \in \mathbb{N}_0$ and that $y \in \mathbb{C} \setminus \{0\}$ is a zero of J_m . Then y is real and satisfies

$$y^2 > m^2.$$

(ii) The zeros of J_m do not have an accumulation point in $\mathbb{C}$. Hence, the set of zeros is countable.

Proof: (ii) For $m \in \mathbb{N}_0$ the Bessel function J_m is entire. Therefore, if the zeros would accumulate in $\mathbb{C}$ we would have $J_m \equiv 0$. Consequently the zeros do not have an accumulation point.

(i) Let

$$u_m(r) = J_m(yr).$$

Then u_m satisfies

$$u_m(1) = 0$$

and

$$\frac{d^2}{dr^2} u_m(r) + \frac{1}{r} \frac{d}{dr} u_m(r) + \left(y^2 - \frac{m^2}{r^2}\right) u_m(r) = 0.$$

Multiply this equation by r and observe that

$$r \frac{d^2}{dr^2} u_m(r) + \frac{d}{dr} u_m(r) = \frac{d}{dr} \left(r \frac{d}{dr} u_m(r) \right).$$

Thus

$$\frac{d}{dr} \left(r \frac{d}{dr} u_m(r) \right) + \left(y^2 - \frac{m^2}{r^2} \right) r u_m(r) = 0.$$

We multiply this equation by $\overline{u_m(r)}$ and integrate:

$$\int_0^1 \frac{d}{dr} \left(r \frac{d}{dr} u_m(r) \right) \overline{u_m(r)} + \left(y^2 - \frac{m^2}{r^2} \right) r |u_m(r)|^2 dr = 0.$$

Partial integration yields

$$\begin{aligned} & - \int_0^1 r \frac{d}{dr} u_m(r) \frac{d}{dr} \overline{u_m(r)} dr + \int_0^1 \left(y^2 - \frac{m^2}{r^2} \right) r |u_m(r)|^2 dr \\ & = r \frac{d}{dr} u_m(r) \overline{u_m(r)} \Big|_{r=0}^{r=1} = 0. \end{aligned}$$

Since $\frac{d}{dr} u_m(r) \frac{d}{dr} \overline{u_m(r)} = \frac{d}{dr} u_m(r) \overline{\frac{d}{dr} u_m(r)} = \left| \frac{d}{dr} u_m(r) \right|^2$, it follows

$$\int_0^1 \left(- \left| \frac{d}{dr} u_m(r) \right|^2 + \left(y^2 - \frac{m^2}{r^2} \right) |u_m(r)|^2 \right) r dr = 0. \quad (4.4)$$

Since the imaginary part of this integral is

$$(\operatorname{Im} y^2) \int_0^1 |u_m(r)|^2 r dr = 0,$$

and since $\int_0^1 |u_m(r)|^2 r dr > 0$, it follows that $\operatorname{Im} y^2 = 0$, hence $y^2 \in \mathbb{R}$. Moreover, we must have

$$y^2 > m^2.$$

For, otherwise

$$\begin{aligned} & \int_0^1 \left(-\left| \frac{d}{dr} u_m(r) \right|^2 + \left(y^2 - \frac{m^2}{r^2} \right) |u_m(r)|^2 \right) r dr \\ & \leq \int_0^1 \left(-\left| \frac{d}{dr} u_m(r) \right|^2 + \left(m^2 - \frac{m^2}{r^2} \right) |u_m(r)|^2 \right) r dr < 0, \end{aligned}$$

which contradicts (4.4). The proof is complete.

Corollary 4.7 *The set of eigenvalues Σ of the Dirichlet boundary value problem in $B_R(0)$ is contained in the positive real axis. If $\lambda \in \Sigma$ then*

$$\Lambda_\lambda \subseteq \{0, 1, \dots, [R\sqrt{\lambda}]\},$$

where $[R\sqrt{\lambda}]$ denotes the largest integer not greater than $R\sqrt{\lambda}$. Thus, the geometric multiplicity of λ is not greater than $2([R\sqrt{\lambda}] + 1)$. The set Σ does not have an accumulation point, hence it is a countable set.

Proof: We already showed that $u = 0$ is the only solution of

$$\begin{aligned} \Delta u(x) &= 0, & x \in B_R(0) \\ u(x) &= 0, & |x| = R, \end{aligned}$$

hence $\lambda = 0$ is not an eigenvalue. Also we showed that $\lambda \in \mathbb{C} \setminus \{0\}$ is an eigenvalue if and only if

$$\Lambda_\lambda = \{m \in \mathbb{N}_0 \mid J_m(\sqrt{\lambda}R) = 0\} \neq \emptyset.$$

Consequently, λ is an eigenvalue if and only if there is $m \in \mathbb{N}_0$ such that $\sqrt{\lambda}R$ is a zero of J_m . Therefore, if $\{y_i^{(m)}\}_{i=1}^\infty \subseteq \mathbb{R} \setminus \{0\}$ is the (countable) set of non-vanishing zeros of J_m , it follows

$$\Sigma = \left\{ \left(\frac{y_i^{(m)}}{R} \right)^2 \mid m \in \mathbb{N}_0, i \in \mathbb{N} \right\} \subseteq (0, \infty).$$

This is a countable set. All non-vanishing zeros of J_m satisfy $(y_i^{(m)})^2 > m^2$, hence

$$(y_i^{(m)})^2 \in (m^2, \infty), \quad i = 1, 2, \dots,$$

which implies that every interval $(0, s]$ only contains zeros of those finitely many Bessel functions J_m with $m^2 < s$. Since the zeros of a Bessel function do not accumulate, the set of zeros of J_m in $(0, s]$ is finite, hence $\Sigma \cap (0, s]$ is finite. Consequently, Σ does not have an accumulation point.

If $m \in \Lambda_\lambda$, then $y = \sqrt{\lambda}R$ is a zero of J_m , hence $y^2 > m^2$ implies $\lambda R^2 = y^2 > m^2$, and therefore

$$\Lambda_\lambda \subseteq \{0, 1, \dots, [R\sqrt{\lambda}]\}.$$

It follows that the geometric multiplicity $\dim V_\lambda$ of λ satisfies

$$\dim V_\lambda = 2|\Lambda_\lambda| \leq 2([R\sqrt{\lambda}] + 1).$$

The proof is complete.

Definition 4.8 The set Σ of eigenvalues is called the spectrum of the Dirichlet problem.

We have not yet answered the question whether $\Sigma \neq \emptyset$, i.e. whether eigenvalues exist. This problem will be investigated later in full generality. It will be shown that in fact there exist countably infinitely many eigenvalues and that one can choose a complete orthonormal system in $L^2(\Omega, \mathbb{C})$ consisting of eigenfunctions. An easy corollary of this result is that every Bessel function J_m with $m \in \mathbb{N}_0$ has countably infinitely many nonnegative zeros. Thus the situation is completely analogous to the situation in one space dimension.

5 Maximum principle, subsolutions, Perron's method

In this section we only consider real valued solutions of the Helmholtz equation. Of course, the results can also be applied to complex valued solutions by considering the real and imaginary parts separately.

5.1 Maximum principle

Theorem 5.1 *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set, let $g : \Omega \rightarrow \mathbb{R}, f : \Omega \rightarrow \mathbb{R}$ and assume that for the function $u \in C(\bar{\Omega}, \mathbb{R})$ the partial derivatives $\frac{\partial u}{\partial x_i}, \frac{\partial^2 u}{\partial x_i^2}$ exist in Ω for $i = 1, \dots, n$ and that u satisfies*

$$\Delta u(x) - g(x)u(x) = f(x), \quad x \in \Omega.$$

(i) *If $g \geq 0$ in Ω , then for all $x \in \Omega$*

$$\begin{aligned} u(x) &\leq \max(0, \max_{y \in \partial\Omega} u(y)), & \text{if } f \geq 0 \text{ in } \Omega, \\ u(x) &\geq \min(0, \min_{y \in \partial\Omega} u(y)), & \text{if } f \leq 0 \text{ in } \Omega. \end{aligned}$$

(weak maximum principle)

(ii) *If $g > 0$ in Ω , then for all $x \in \Omega$*

$$\begin{aligned} u(x) \leq 0 \quad \text{or} \quad u(x) &< \max_{y \in \partial\Omega} u(y), & \text{if } f \geq 0 \text{ in } \Omega, \\ u(x) \geq 0 \quad \text{or} \quad u(x) &> \min_{y \in \partial\Omega} u(y), & \text{if } f \leq 0 \text{ in } \Omega. \end{aligned}$$

(strong maximum principle)

Proof: (i) We first consider the case $f \geq 0$. Assume that the statement is false. Then there is $x_0 \in \Omega$ such that $u(x_0) > 0$ and

$$\max_{x \in \bar{\Omega}} u(x) = u(x_0) > \max_{y \in \partial\Omega} u(y).$$

Define $v : \bar{\Omega} \rightarrow \mathbb{R}$ by

$$v(x) = u(x) + \varepsilon|x|^2, \quad x \in \bar{\Omega},$$

where $\varepsilon > 0$ is chosen small enough such that

$$\begin{aligned} v(x_0) = u(x_0) + \varepsilon|x_0|^2 &> \max_{y \in \partial\Omega} (u(y) + \varepsilon|y|^2) = \max_{y \in \partial\Omega} v(y), \\ \max_{y \in \bar{\Omega}} \varepsilon|y|^2 &< u(x_0). \end{aligned}$$

v is continuous on the compact set $\overline{\Omega}$ and therefore assumes its maximum in a point $z \in \overline{\Omega}$. By the choice of ε we have $z \notin \partial\Omega$ and

$$u(z) = v(z) - \varepsilon|z|^2 > v(x_0) - u(x_0) = \varepsilon|x_0|^2 \geq 0.$$

Thus, the maximum z belongs to the open set Ω , which implies that

$$\frac{\partial v}{\partial x_i}(z) = 0, \quad \frac{\partial^2 v}{\partial x_i^2}(z) \leq 0, \quad i = 1, \dots, n,$$

whence

$$\Delta v(z) = \sum_{i=1}^n \frac{\partial^2 v}{\partial x_i^2}(z) \leq 0.$$

On the other hand

$$\begin{aligned} \Delta v(z) &= \Delta u(z) + \Delta(\varepsilon|x|^2)|_{x=z} = g(z)u(z) + \varepsilon \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2} x_i^2 + f(z) \\ &= g(z)u(z) + 2n\varepsilon + f(z) > 0, \end{aligned}$$

because of $g(z) \geq 0, u(z) > 0, f(z) \geq 0$ and $\varepsilon > 0$. This is a contradiction, hence

$$\max_{x \in \overline{\Omega}} u(x) = u(x_0) \leq \max(0, \max_{y \in \partial\Omega} u(y)).$$

If $f \leq 0$ define $w = -u$. Then $\Delta w(x) - g(x)w(x) = -f(x) \geq 0$ for all $x \in \Omega$, hence

$$\begin{aligned} -\min_{x \in \overline{\Omega}} u(x) &= \max_{x \in \overline{\Omega}} w(x) \leq \max(0, \max_{y \in \partial\Omega} w(y)) \\ &= \max(0, -\min_{y \in \partial\Omega} u(y)) = -\min(0, \min_{y \in \partial\Omega} u(y)), \end{aligned}$$

which implies the statement for the minimum.

(ii) Let $g > 0$ and $f \geq 0$, and assume that the statement for the maximum is false. Then there is $x_0 \in \Omega$ such that $u(x_0) > 0$ and

$$u(x_0) = \max_{x \in \overline{\Omega}} u(x) \geq \max_{y \in \partial\Omega} u(y).$$

Consequently, x_0 is a local maximum of u in the open set Ω , hence

$$\frac{\partial u}{\partial x_i}(x_0) = 0, \quad \frac{\partial^2 u}{\partial x_i^2}(x_0) \leq 0, \quad i = 1, \dots, n,$$

whence

$$0 \geq \Delta u(x_0) = g(x_0)u(x_0) + f(x_0) > 0,$$

which is a contradiction, and the statement for the maximum must be true. The statement for the minimum is proved by considering $-u$. ■

We note some consequences of the maximum principle:

Corollary 5.2 Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set, let $g : \Omega \rightarrow \mathbb{R}_0^+$, $f : \Omega \rightarrow \mathbb{R}$ be given. Assume that $u, v \in C(\overline{\Omega}, \mathbb{R})$ and that for $w = u$ and $w = v$ the partial derivatives $\frac{\partial w}{\partial x_i}$, $\frac{\partial^2 w}{\partial x_i^2}$ exist in Ω for $i = 1, \dots, n$ and the equation

$$\Delta w(x) - g(x)w(x) = f(x), \quad x \in \Omega$$

holds. Then the following statements are true:

(i) If $u(y) \geq v(y)$ for all $y \in \partial\Omega$ then $u(x) \geq v(x)$ for all $x \in \overline{\Omega}$.

(ii) For all $x \in \Omega$,

$$|u(x) - v(x)| \leq \max_{y \in \partial\Omega} |u(y) - v(y)|.$$

(iii) Let $g > 0$ and $f = 0$ in Ω . If u assumes the maximum in Ω , then $u \leq 0$. If u assumes the minimum in Ω , then $u \geq 0$.

Proof: (i) $w = u - v$ satisfies $\Delta w - gw = 0$ in Ω . Hence, by the weak maximum principle

$$0 = \min(0, \min_{y \in \partial\Omega} w(y)) \leq w(x) = u(x) - v(x), \quad x \in \Omega.$$

(ii) Again we apply the weak maximum principle to $w = u - v$ and obtain

$$\begin{aligned} \min_{y \in \partial\Omega} (-|w(y)|) &\leq \min(0, \min_{y \in \partial\Omega} w(y)) \leq w(x) \\ &\leq \max(0, \max_{y \in \partial\Omega} w(y)) \leq \max_{y \in \partial\Omega} |w(y)|. \end{aligned}$$

Thus

$$w(x) \leq \max_{y \in \partial\Omega} |w(y)|, \quad -w(x) \leq -\min_{y \in \partial\Omega} (-|w(y)|) = \max_{y \in \partial\Omega} |w(y)|,$$

whence $|w(x)| \leq \max_{y \in \partial\Omega} |w(y)|$.

(iii) Let $x_0 \in \Omega$ and assume that $u(x_0) = \max_{x \in \overline{\Omega}} u(x)$. By the strong maximum principle this can only be if $u(x_0) \leq 0$. The statement for the minimum is proved in the same way. ■

Corollary 5.3 (Uniqueness) Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set, let $g : \Omega \rightarrow \mathbb{R}_0^+$, $f : \Omega \rightarrow \mathbb{R}$ and let the functions $u, v \in C(\overline{\Omega}, \mathbb{R})$ have the differentiability properties stated in Corollary 5.2. Assume that $u|_{\partial\Omega} = v|_{\partial\Omega}$ and that u and v both satisfy the differential equation

$$\Delta w(x) - g(x)w(x) = f(x), \quad x \in \Omega.$$

Then $u = v$.

Proof: The preceding corollary yields

$$|u(x) - v(x)| \leq \max_{y \in \partial\Omega} |u(y) - v(y)| = 0, \quad x \in \Omega.$$

■

5.2 Consequences of the maximum principle for the Helmholtz equation in $\mathbb{R}^2$

We showed that the Dirichlet problem for the Helmholtz equation in a ball B in $\mathbb{R}^2$ has a solution, which is infinitely differentiable in the interior of B and satisfies the boundary condition in the L^2 -sense. However, up to now we do not know whether the solution is continuous on $\overline{B}$ if the boundary data $u^{(b)}$ are continuous. The maximum principle can be used to show that this is in fact true. As preparation we need the following result:

Theorem 5.4 *Let $B_R(0) \subseteq \mathbb{R}^2$ and let $\lambda \leq 0$. Let $u_m^{(b)}, u^{(b)} \in L^2(\partial B)$ and let u_m, u be the solutions of*

$$\begin{aligned}\Delta v(x) + \lambda v(x) &= 0, \quad x \in B_R(0) \\ \lim_{r \nearrow R} \|v(r, \cdot) - v^{(b)}\|_{[0, 2\pi]} &= 0\end{aligned}$$

to the data $v^{(b)} = u_m^{(b)}, v^{(b)} = u^{(b)}$. If

$$\lim_{m \rightarrow \infty} \|u_m^{(b)} - u^{(b)}\|_{[0, 2\pi]} = 0,$$

then in every ball $\overline{B_{\hat{r}}(0)}$ with $\hat{r} < R$ the sequence $\{u_m\}_{m=1}^\infty$ converges uniformly to u .

Proof: Since $\{\frac{e^{in\varphi}}{\sqrt{2\pi}}\}_{n \in \mathbb{N}}$ is a complete orthonormal system in $L^2((0, 2\pi))$, we have for the Fourier expansions

$$u_m^{(b)}(\varphi) = \sum_{n=-\infty}^{\infty} a_n^{(m)} e^{in\varphi}, \quad u^{(b)}(\varphi) = \sum_{n=-\infty}^{\infty} a_n e^{in\varphi}$$

by Theorem 3.10 that

$$2\pi \sum_{n=-\infty}^{\infty} |a_n^{(m)} - a_n|^2 = \sum_{n=-\infty}^{\infty} |\sqrt{2\pi}(a_n^{(m)} - a_n)|^2 = \|u_m^{(b)} - u^{(b)}\|_{[0, 2\pi]}^2 \rightarrow 0,$$

for $m \rightarrow \infty$. We first consider the case $\lambda = 0$. Then

$$\begin{aligned}u_m(r, \varphi) &= \sum_{n=-\infty}^{\infty} a_n^{(m)} \left(\frac{r}{R}\right)^{|n|} e^{in\varphi} \\ u(r, \varphi) &= \sum_{n=-\infty}^{\infty} a_n \left(\frac{r}{R}\right)^{|n|} e^{in\varphi}.\end{aligned}$$

Let $\varepsilon > 0$ and choose m_0 such that $\|u_m^{(b)} - u^{(b)}\| < \varepsilon$ for all $m \geq m_0$. Cauchy-Schwarz inequality yields for $r \leq \hat{r}$, $0 \leq \varphi < 2\pi$ and $m \geq m_0$ that

$$\begin{aligned} |u_m(r, \varphi) - u(r, \varphi)| &\leq \sum_{n=-\infty}^{\infty} \left| (a_n^{(m)} - a_n) \left(\frac{r}{R} \right)^{|n|} \right| \\ &\leq \left(\sum_{n=-\infty}^{\infty} |a_n^{(m)} - a_n|^2 \right)^{1/2} \left(\sum_{n=-\infty}^{\infty} \left(\frac{r}{R} \right)^{2|n|} \right)^{1/2} \\ &\leq \frac{1}{\sqrt{2\pi}} \|u_m^{(b)} - u^{(b)}\| \left(\frac{2}{1 - \left(\frac{r}{R} \right)^2} \right)^{1/2} \leq \frac{1}{\sqrt{\pi}} \left(1 - \left(\frac{\hat{r}}{R} \right)^2 \right)^{-1/2} \varepsilon. \end{aligned} \quad (5.1)$$

Since $\varepsilon > 0$ was arbitrary, it follows that u_m converges uniformly to u in $\overline{B_{\hat{r}}(0)}$.

To prove the statement for $\lambda < 0$ we use that by Theorem 4.5 the representations

$$\begin{aligned} u_m(r, \varphi) &= \sum_{n=-\infty}^{\infty} \frac{a_n^{(m)}}{J_{|n|}(\sqrt{\lambda}R)} J_{|n|}(\sqrt{\lambda}r) e^{in\varphi}, \\ u(r, \varphi) &= \sum_{n=-\infty}^{\infty} \frac{a_n}{J_{|n|}(\sqrt{\lambda}R)} J_{|n|}(\sqrt{\lambda}r) e^{in\varphi}. \end{aligned}$$

hold for u_m and u . With $x = \sqrt{-\lambda}r$ and $y = \sqrt{-\lambda}R$ we thus obtain

$$|u_m(r, \varphi) - u(r, \varphi)| \leq \sum_{n=-\infty}^{\infty} |a_n^{(m)} - a_n| \frac{|J_{|n|}(ix)|}{|J_{|n|}(iy)|}. \quad (5.2)$$

The fraction $\frac{|J_{|n|}(ix)|}{|J_{|n|}(iy)|}$ can be estimated using Lemma A.1 in the appendix. For, $0 \leq r \leq R$ and $-\lambda > 0$ imply $x, y \in \mathbb{R}$ and $0 \leq x \leq y$. Therefore the assumptions of the lemma are satisfied. The estimate from that lemma and (5.2) together yield

$$|u_m(r, \varphi) - u(r, \varphi)| \leq |a_0^{(m)} - a_0| \frac{|J_0(ix)|}{|J_0(iy)|} + \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} |a_n^{(m)} - a_n| \left(\frac{r}{R} \right)^{|n|}.$$

With this estimate we can proceed as above and obtain that (5.1) also holds for $\lambda < 0$ with the right hand side multiplied by $C = \max \{1, \max_{0 \leq x \leq y} \{ \frac{|J_0(ix)|}{|J_0(iy)|} \} \}$, which shows that u_m converges uniformly to u in $\overline{B_{\hat{r}}(0)}$ also in this case. ■

Theorem 5.5 *Let $B \subseteq \mathbb{R}^2$ be a bounded open ball and let $\lambda \leq 0$. Then for every $u^{(b)} \in C(\partial B, \mathbb{R})$ there is a unique solution $u \in C(\overline{B}, \mathbb{R}) \cap C_{\infty}(B, \mathbb{R})$ of the Dirichlet problem*

$$\begin{aligned} \Delta u(x) + \lambda u(x) &= 0, \quad x \in B \\ u|_{\partial B} &= u^{(b)}. \end{aligned}$$

Proof: Without restriction of generality we can assume that $B = B_R(0)$ with $R > 0$. The uniqueness follows from Corollary 5.3. To prove that a continuous solution exists, let

$$u^{(b)} = \sum_{m=-\infty}^{\infty} a_m e^{im\varphi}$$

be the Fourier expansion of $u^{(b)}$. For $m \geq 0$ and $n \geq 1$ let

$$\begin{aligned} s_m(\varphi) &= \sum_{k=-m}^m a_k e^{ik\varphi}, \\ \sigma_n(\varphi) &= \frac{1}{n} (s_0(\varphi) + \dots + s_{n-1}(\varphi)). \end{aligned}$$

Since $\varphi \mapsto u^{(b)}(\varphi) : [0, 2\pi) \rightarrow \mathbb{R}$ is continuous and can be extended to a continuous, periodic function on $\mathbb{R}$, it follows from Theorem 3.13, that the sequence $\{\sigma_n\}_{n=1}^{\infty}$ converges uniformly on $[0, 2\pi)$ to $u^{(b)}$.

Let

$$v_m(r, \varphi) = \begin{cases} \sum_{k=-m}^m \frac{a_k}{J_{|k|}(\sqrt{\lambda}R)} J_{|k|}(\sqrt{\lambda}r) e^{ik\varphi}, & \lambda < 0 \\ \sum_{k=-m}^m a_k \left(\frac{r}{R}\right)^{|k|} e^{ik\varphi}, & \lambda = 0. \end{cases} \quad (5.3)$$

By the results of Section 4, every term in these finite sums is an infinitely differentiable solution of

$$\Delta v(x) + \lambda v(x) = 0$$

in all of $\mathbb{R}^2$, hence v_m is an infinitely differentiable solution of this equation in all of $\mathbb{R}^2$. We remark that if $u^{(b)}$ is real then

$$\bar{a}_k = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} u^{(b)}(\varphi) e^{-ik\varphi} d\varphi = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} u^{(b)}(\varphi) e^{ik\varphi} d\varphi = a_{-k},$$

whence $a_k e^{ik\varphi} + a_{-k} e^{-ik\varphi} = 2\operatorname{Re}(a_k e^{ik\varphi})$ is real. Moreover, it is seen from the power series expansion of $J_{|k|}$ that

$$\frac{J_{|k|}(\sqrt{\lambda}r)}{J_{|k|}(\sqrt{\lambda}R)} \in \mathbb{R},$$

since $\lambda < 0$, hence $\sqrt{\lambda}$ is imaginary. From (5.3) we consequently see that v_m has real values. For $n \geq 1$ we set

$$\nu_n(r, \varphi) = \frac{1}{n} (v_0(r, \varphi) + \dots + v_{n-1}(r, \varphi)).$$

The function $\nu_n \in C_\infty(\mathbb{R}^2, \mathbb{R})$ is a solution of the Dirichlet boundary value problem

$$\begin{aligned}\Delta \nu_n(x) + \lambda \nu_n(x) &= 0, & x \in B_R(0) \\ \nu_n(x) &= \sigma_n(x), & |x| = R.\end{aligned}$$

Therefore the maximum principle yields

$$|\nu_n(x) - \nu_\ell(x)| \leq \max_{|y|=R} |\sigma_n(y) - \sigma_\ell(y)|,$$

for all $x \in \overline{B_R(0)}$. Since $\{\sigma_n\}_n$ converges uniformly to $u^{(b)}$, it follows from this estimate that $\{\nu_n\}_{n=1}^\infty$ converges in $\overline{B_R(0)}$ to a limit function $u \in C(\overline{B_R(0)})$. The limit function satisfies

$$u(R, \varphi) = u^{(b)}, \quad 0 \leq \varphi < 2\pi.$$

Moreover, since $\{\sigma_n\}_{n=1}^\infty$ converges uniformly to $u^{(b)}$, it also converges in $L^2(\partial B_R(0))$ to $u^{(b)}$. From the preceding theorem we thus conclude that the sequence $\{\nu_n\}_{n=1}^\infty$ converges pointwise to the solution $\hat{u}$ of

$$\begin{aligned}\Delta \hat{u}(x) + \lambda \hat{u}(x) &= 0, & x \in B_R(0) \\ \lim_{r \rightarrow R} \|\hat{u}(r, \cdot) - u^{(b)}\|_{[0, 2\pi]} &= 0,\end{aligned}$$

which satisfies $\hat{u} \in C_\infty(B_R(0))$. Since the pointwise limit coincides with the uniform limit, we obtain that $u = \hat{u}$, hence u belongs to $C(\overline{B_R(0)}) \cap C_\infty(B_R(0))$ and is a solution of the Helmholtz equation in $B_R(0)$. The proof is complete.

We next show that solutions of the Helmholtz equation have the **mean value property**:

Theorem 5.6 *Let $B_R(0) \subseteq \mathbb{R}^2$, let $\lambda \leq 0$ and let $u \in C(\overline{B_R(0)}, \mathbb{R}) \cap C_\infty(B_R(0), \mathbb{R})$ solve*

$$\Delta u(x) + \lambda u(x) = 0, \quad x \in B_R(0).$$

Then

$$u(0) = \frac{1}{2\pi R J_0(\sqrt{\lambda} R)} \int_{|x|=R} u(x) ds.$$

Since $J_0(0) = 1$, this formula becomes for $\lambda = 0$

$$u(0) = \frac{1}{2\pi R} \int_{|x|=R} u(x) ds.$$

Proof: Let $\lambda < 0$. From the preceding investigations we know that if

$$u(R, \varphi) = \sum_{m=-\infty}^{\infty} a_m e^{im\varphi},$$

then

$$u(r, \varphi) = \sum_{m=-\infty}^{\infty} \frac{a_m}{J_{|m|}(\sqrt{\lambda}R)} J_{|m|}(\sqrt{\lambda}r) e^{im\varphi}, \quad 0 < r \leq R.$$

The power series expansion of J_m shows that

$$J_m(0) = \begin{cases} 1, & m = 0 \\ 0, & m \in \mathbb{N}, \end{cases}$$

hence

$$u(0) = \frac{a_0}{J_0(\sqrt{\lambda}R)} = \frac{1}{2\pi J_0(\sqrt{\lambda}R)} \int_0^{2\pi} u(R, \varphi) d\varphi = \frac{1}{2\pi R J_0(\sqrt{\lambda}R)} \int_{|x|=R} u(x) ds.$$

To prove the statement for $\lambda = 0$ we proceed in the same way, using the Fourier expansion of u . ■

Corollary 5.7 *Let $\Omega \subseteq \mathbb{R}^2$ be a open, connected set, let $\lambda \leq 0$ and let $u \in C_\infty(\Omega, \mathbb{R})$ be a solution of*

$$\Delta u(x) + \lambda u(x) = 0, \quad x \in \Omega.$$

Assume that $x_0 \in \Omega$ exists such that $u(x_0) = 0$. Then, if $u \geq 0$ or $u \leq 0$ in Ω it follows that $u = 0$ in Ω .

Proof: Let

$$M = \{x \in \Omega \mid u(x) = 0\}.$$

By assumption, M is not empty since $x_0 \in \Omega$. We prove that M is closed and open in Ω , which implies $M = \Omega$, since Ω is connected.

Since u is continuous, M is obviously closed. To verify that M is open, let $y \in M$. Since Ω is open there exists a ball $B_R(y)$ with center y contained in Ω . By the mean value property we have for all $0 < r < R$

$$\frac{1}{2\pi r} \int_{|x-y|=r} u(x) ds_x = J_0(\sqrt{\lambda}r) u(y) = 0.$$

If $u \geq 0$ or $u \leq 0$ in Ω this can only be if $u(x) = 0$ for all x with $|x-y| = r$. This holds for all $0 < r < R$, hence $u(x) = 0$ for all $x \in B_R(y)$. Thus, $B_R(y) \subseteq M$, hence M is open. The proof is complete. ■

Corollary 5.8 *Let Ω be a bounded, open, connected set and let $u \in C(\overline{\Omega}, \mathbb{R})$ be a solution of*

$$\Delta u(x) + \lambda u(x) = 0, \quad x \in \Omega.$$

(i) If $\lambda < 0$, then $u = 0$ or, for all $x \in \Omega$,

$$\min(0, \min_{y \in \partial\Omega} u(y)) < u(x) < \max(0, \max_{y \in \partial\Omega} u(y)).$$

(ii) If $\lambda = 0$, then $u = \text{const}$ or, for all $x \in \Omega$,

$$\min_{y \in \partial\Omega} u(y) < u(x) < \max_{y \in \partial\Omega} u(y).$$

Proof: (i) Combine the strong maximum principle with the preceding result.
(ii) Assume that there is $x_0 \in \Omega$ such that $u(x_0) = \max_{y \in \bar{\Omega}} u(y)$. Then the function $v \in C(\bar{\Omega}) \cap C_\infty(\Omega)$ defined by $v(x) = u(x) - u(x_0)$ satisfies $\Delta v = 0$, $v \leq 0$ and $v(x_0) = 0$. Since x_0 is an interior point of Ω , Corollary 5.7 implies that $v = 0$, hence $u = \text{const} = u(x_0)$. In the same way it follows that if there is $x_1 \in \Omega$ such that $u(x_1) = \min_{y \in \bar{\Omega}} u(y)$, then $u = \text{const} = u(x_1)$. Statement (ii) follows from these results. ■

5.3 Subsolutions, supersolutions, comparison

Up to now we only know how to solve boundary value problems in circular domains. Our goal is to develop a method to solve boundary value problems in very general domains $\Omega \subseteq \mathbb{R}^2$. To this end we need subsolutions and supersolutions, which we define and discuss in this section.

Thus, let $\Omega \subseteq \mathbb{R}^2$ be a bounded open domain and let $\lambda \leq 0$.

Definition 5.9 A function $v \in C(\bar{\Omega}, \mathbb{R})$ is called subsolution (supersolution) of the equation $\Delta u + \lambda u = 0$ in $\bar{\Omega}$, if to every open ball B with $\bar{B} \subseteq \Omega$ the uniquely determined solution $u \in C(\bar{B})$ of

$$\begin{aligned} \Delta u(x) + \lambda u(x) &= 0, \quad x \in B \\ u|_{\partial B} &= v|_{\partial B}, \end{aligned}$$

satisfies

$$v(x) \leq u(x), \quad (v(x) \geq u(x)),$$

for all $x \in B$.

Remark 5.10 In the case $\lambda = 0$ a subsolution is also called a subharmonic function, since a solution u of $\Delta u(x) = 0$ is called a harmonic function.

Theorem 5.11 Let $v_1, \dots, v_m$ be subsolutions of $\Delta u + \lambda u = 0$. Then

$$v = \max(v_1, \dots, v_m)$$

is a subsolution.

Proof: Let $v^{(2)} = \max(v_1, v_2)$, let $\bar{B} \subseteq \Omega$ be a closed ball and let $u_1, u_2, u^{(2)} \in C(\bar{B})$ be solutions of

$$\Delta w + \lambda w = 0$$

in B satisfying

$$u_1|_{\partial B} = v_1|_{\partial B}, \quad u_2|_{\partial B} = v_2|_{\partial B}, \quad u^{(2)}|_{\partial B} = v^{(2)}|_{\partial B}.$$

For $y \in \partial B$ we have

$$u^{(2)}(y) = v^{(2)}(y) \geq v_i(y) = u_i(y), \quad i = 1, 2.$$

Using the maximum principle and noting that v_1, v_2 are subsolutions we thus obtain for $x \in \overline{B}$

$$\begin{aligned} v_1(x) &\leq u_1(x) \leq u^{(2)}(x) \\ v_2(x) &\leq u_2(x) \leq u^{(2)}(x), \end{aligned}$$

which yields

$$v^{(2)}(x) = \max(v_1(x), v_2(x)) \leq u^{(2)}(x).$$

This shows that $v^{(2)}$ is a subsolution. Since

$$v^{(k)} = \max(v_1, \dots, v_k) = \max(v_k, \max(v_1, \dots, v_{k-1})) = \max(v_k, v^{(k-1)}),$$

the statement follows by induction.

Theorem 5.12 *Let v be a subsolution of $\Delta u + \lambda u = 0$ in Ω , let B be an open ball with $\overline{B} \subseteq \Omega$ and let $u \in C(\overline{B}) \cap C_\infty(B)$ be the solution of*

$$\begin{aligned} \Delta u(x) + \lambda u(x) &= 0, \quad x \in B \\ u|_{\partial B} &= v|_{\partial B}. \end{aligned}$$

Then the function $\tilde{v} : \Omega \rightarrow \mathbb{R}$,

$$\tilde{v}(x) = \begin{cases} v(x), & x \notin B \\ u(x), & x \in B \end{cases}$$

is a subsolution.

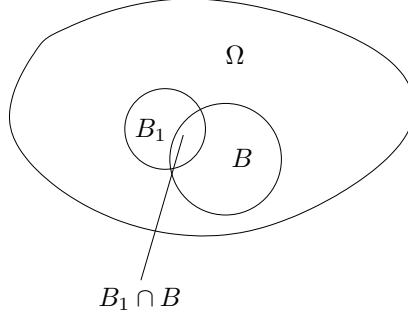
Proof: We have $\tilde{v} \geq v$ since v is a subsolution. Now let B_1 be an open ball with $\overline{B_1} \subseteq \Omega$ and let $\tilde{u} \in C(\overline{B_1})$ be the solution of

$$\begin{aligned} \Delta \tilde{u}(x) + \lambda \tilde{u}(x) &= 0, \quad x \in B_1 \\ \tilde{u}|_{\partial B_1} &= \tilde{v}|_{\partial B_1}. \end{aligned}$$

We are finished if we can show that $\tilde{u} \geq \tilde{v}$ in B_1 . By definition of $\tilde{v}$ this holds if

$$\tilde{u}(x) \geq v(x), \quad x \in B_1 \setminus B \tag{5.4}$$

$$\tilde{u}(x) \geq u(x), \quad x \in B_1 \cap B. \tag{5.5}$$



Ball B_1 in arbitrary position

To prove these relations, let $\hat{u} \in C(\overline{B_1})$ be the solution of

$$\begin{aligned}\Delta \hat{u}(x) + \lambda \hat{u}(x) &= 0, \quad x \in B_1 \\ \hat{u}|_{\partial B_1} &= v|_{\partial B_1}.\end{aligned}$$

Since v is a subsolution, it follows that $\hat{u} \geq v$ in $\overline{B_1}$. Also, $\tilde{u}$ and $\hat{u}$ solve the Helmholtz equation in B_1 and satisfy

$$\tilde{u}|_{\partial B_1} = \tilde{v}|_{\partial B_1} \geq v|_{\partial B_1} = \hat{u}|_{\partial B_1},$$

hence the maximum principle yields $\tilde{u} \geq \hat{u} \geq v$ in $\overline{B_1}$, which proves (5.4).

To verify (5.5), note that $\partial(B_1 \cap B) = (\overline{B_1} \cap \partial B) \cup (\overline{B} \cap \partial B_1)$ and that

$$\begin{aligned}\tilde{u}|_{\overline{B} \cap \partial B_1} &= \tilde{v}|_{\overline{B} \cap \partial B_1} = u|_{\overline{B} \cap \partial B_1} \\ \tilde{u}|_{\overline{B_1} \cap \partial B} &\geq v|_{\overline{B_1} \cap \partial B} = u|_{\overline{B_1} \cap \partial B},\end{aligned}$$

where we used (5.4) to get the last relation. Since both $\tilde{u}$ and u satisfy the Helmholtz equation in $B_1 \cap B$, it follows from these relations and from the maximum principle that $\tilde{u} \geq u$ in $B_1 \cap B$. This is (5.5). The proof is complete.

Theorem 5.13 (Comparison) *Let $\lambda \leq 0$ and let $v \in C(\overline{\Omega}, \mathbb{R})$ be a subsolution, $w \in C(\overline{\Omega}, \mathbb{R})$ be a supersolution with $v|_{\partial\Omega} \leq w|_{\partial\Omega}$. Then $v \leq w$ in $\overline{\Omega}$.*

Proof: Assume that there is $x_0 \in \Omega$ such that $v(x_0) > w(x_0)$. Then since $h = v - w$ is less or equal to zero on $\partial\Omega$, it follows that h assumes the positive maximum at a point $z \in \Omega$. We can assume that z is a boundary point of the closed set $M = \{x \in \Omega \mid h(x) = \max_{y \in \overline{\Omega}} h(y)\} \subset \Omega$. It follows that every neighborhood of z contain points, where h assumes values smaller than $h(z)$. Therefore we can choose a ball B with center z and with $\overline{B} \subseteq \Omega$ such that ∂B contains such a point. We thus have

$$h(z) > 0, \quad h(z) \geq \max_{x \in \partial B} h(x), \quad h(z) > \min_{x \in \partial B} h(x). \quad (5.6)$$

Now let $\hat{v}, \hat{w}$ be the solutions of

$$\begin{aligned}\Delta \hat{v}(x) + \lambda \hat{v}(x) &= 0 \text{ in } B, & \hat{v}|_{\partial B} &= v|_{\partial B}, \\ \Delta \hat{w}(x) + \lambda \hat{w}(x) &= 0 \text{ in } B, & \hat{w}|_{\partial B} &= w|_{\partial B}.\end{aligned}$$

Then $\hat{v} \geq v$ and $w \geq \hat{w}$ in B . Therefore the function

$$u = \hat{v} - \hat{w}$$

satisfies $u = \hat{v} - \hat{w} \geq v - w = h$ in B , hence

$$u(z) \geq h(z) > 0, \tag{5.7}$$

and

$$u|_{\partial B} = \hat{v}|_{\partial B} - \hat{w}|_{\partial B} = v|_{\partial B} - w|_{\partial B} = h|_{\partial B}.$$

This equation and (5.6), (5.7) imply

$$u(z) \geq \max(0, \max_{y \in \partial B} u(y)), \quad u(z) > \min_{y \in \partial B} u(y). \tag{5.8}$$

Since $\Delta u(x) + \lambda u(x) = 0$ in B , it follows in the case $\lambda < 0$ from the first of the inequalities (5.8) and from Corollary 5.8(i) that $u = 0$, which contradicts (5.7). In the case $\lambda = 0$ it follows from the first of the inequalities (5.8) and from Corollary 5.8(ii) that $u = \text{const}$, which contradicts the second of the inequalities (5.8). Consequently, in both cases we must have $v \leq w$ in $\overline{\Omega}$. ■

Theorem 5.14 *Let $\lambda \leq 0$ and let $w \in C(\overline{\Omega}) \cap C_\infty(\Omega)$ be a solution of the potential equation $\Delta w = 0$. If w is non-negative, then w is a supersolution, if w is non-positive, then w is a subsolution of the equation $\Delta u + \lambda u = 0$.*

Proof: For a ball B with $\overline{B} \subseteq \Omega$ let u be the solution of

$$\begin{aligned}\Delta u(x) + \lambda u(x) &= 0, & x &\in B \\ u|_{\partial B} &= w|_{\partial B}.\end{aligned}$$

Since $\Delta w = 0$, the function $h = w - u$ satisfies

$$\Delta h(x) + \lambda h(x) = \lambda w(x) \begin{cases} \leq 0, & \text{if } w \geq 0, \\ \geq 0, & \text{if } w \leq 0. \end{cases}$$

Since also $h|_{\partial B} = 0$, we conclude from the maximum principle in the first case that $h \geq 0$, hence $u \leq w$, which shows that w is a supersolution. In the second case the maximum principle yields $h \leq 0$, hence $u \geq w$, which implies that w is a subsolution. ■

Corollary 5.15 (Maximum principle for sub- and supersolutions) *Let $\lambda \leq 0$.*

(i) Any non-negative constant function is a supersolution and any non-positive constant function is a subsolution of $\Delta u + \lambda u = 0$.

(ii) If v is a subsolution and w is a supersolution, then

$$\begin{aligned} v(x) &\leq \max \left(0, \max_{y \in \partial\Omega} v(y) \right), \\ w(x) &\geq \min \left(0, \min_{y \in \partial\Omega} w(y) \right). \end{aligned}$$

Proof: (i) A non-negative constant function is a supersolution and a non-positive constant function is a subsolution, since they satisfy the potential equation.

(ii) For a subsolution v define $\hat{v} : \Omega \rightarrow \mathbb{R}$ by

$$\hat{v}(x) = \text{const} = \max \left(0, \max_{y \in \partial\Omega} v(y) \right), \quad x \in \Omega.$$

Then $\hat{v} \geq 0$ is a supersolution satisfying $v|_{\partial\Omega} \leq \hat{v}|_{\partial\Omega}$, whence $v \leq \hat{v}$, by Theorem 5.13. Similarly, for a supersolution w define $\hat{w} : \Omega \rightarrow \mathbb{R}$ by

$$\hat{w}(x) = \text{const} = \min \left(0, \min_{y \in \partial\Omega} w(y) \right), \quad x \in \Omega.$$

Then $\hat{w} \leq 0$ is a subsolution satisfying $\hat{w}|_{\partial\Omega} \leq w|_{\partial\Omega}$, whence $\hat{w} \leq w$. ■

5.4 Perron's method

For a bounded open set $\Omega \subseteq \mathbb{R}^2$, for $\lambda \leq 0$ and for a function $f \in C(\partial\Omega, \mathbb{R})$ define

$$\mathcal{S}_f = \{v \in C(\overline{\Omega}) \mid v \text{ is a subsolution of } \Delta u + \lambda u = 0 \text{ with } v|_{\partial\Omega} \leq f\}.$$

Note that by the preceding corollary every $v \in \mathcal{S}_f$ satisfies

$$v(x) \leq \max \left(0, \max_{y \in \partial\Omega} f(y) \right).$$

Theorem 5.16 (Oskar Perron (1880 – 1975)) *If $\mathcal{S}_f \neq \emptyset$ then*

$$u_f(x) = \sup_{v \in \mathcal{S}_f} v(x), \quad x \in \Omega,$$

satisfies $u_f \in C_\infty(\Omega)$ and

$$\Delta u_f(x) + \lambda u_f(x) = 0, \quad x \in \Omega.$$

Proof: The proof is in two steps. In the first step we construct in a neighborhood of an arbitrary $y \in \Omega$ a solution u of the Helmholtz equation with $u(y) = u_f(y)$. In the second step we show that $u = u_f$ in this neighborhood, hence u_f is a solution of the Helmholtz equation in this neighborhood. This proves the theorem, since y was arbitrary.

I.) Let $y \in \Omega$ and choose a sequence $\{v_m\}_{m=1}^\infty \subseteq \mathcal{S}_f$ with $\lim_{m \rightarrow \infty} v_m(y) = u_f(y)$. We can assume that

$$v_1 \leq v_2 \leq v_3 \leq \dots \quad (5.9)$$

Otherwise we consider the sequence $\{\bar{v}_m\}_{m=1}^\infty$ defined by

$$\bar{v}_m = \max(v_1, \dots, v_m).$$

This is a monotonically increasing sequence of subsolutions with $\bar{v}_m \in \mathcal{S}_f$ and $\bar{v}_m \geq v_m$, hence $\lim_{m \rightarrow \infty} \bar{v}_m(y) = u_f(y)$. Thus, let (5.9) be satisfied.

We choose an open bounded ball B with $y \in B$ and $\bar{B} \subseteq \Omega$. Consider the subsolution

$$w_m(x) = \begin{cases} v_m(x), & x \in \Omega \setminus B, \\ u_m(x), & x \in B, \end{cases}$$

where $u_m \in C(\bar{B}) \cap C_\infty(B)$ is the solution of

$$\Delta u_m(x) + \lambda u_m(x) = 0, \quad x \in B$$

$$u_m|_{\partial B} = v_m|_{\partial B}.$$

Then $w_m \in \mathcal{S}_f$ satisfies $v_m \leq w_m$. Moreover, since $u_m|_{\partial B} = v_m|_{\partial B}$ it follows that $\{u_m|_{\partial B}\}_m$ is monotonically increasing, hence the maximum principle implies that $\{u_m\}_m$ and therefore also $\{w_m\}_m$ is monotonically increasing. Because the last sequence is bounded above by the function u_f , it follows that $\{w_m\}_m$ converges pointwise everywhere on $\bar{\Omega}$ to a limit function. Let u be the restriction of this limit function to $\bar{B}$. The function u is the pointwise limit of $\{u_m\}_m$ and satisfies

$$u \leq u_f, \quad u(y) = u_f(y) \quad (5.10)$$

$$\Delta u(x) + \lambda u(x) = 0, \quad x \in B. \quad (5.11)$$

(5.10) follows from

$$u_f(y) = \lim_{m \rightarrow \infty} v_m(y) \leq \lim_{m \rightarrow \infty} w_m(y) = \lim_{m \rightarrow \infty} u_m(y) \leq u_f(y).$$

To see (5.11) note that $(u(x) - u_m(x))^2$ decreases pointwise monotonically to zero. The monotone convergence theorem of Beppo Levi therefore yields

$$\lim_{m \rightarrow \infty} \|u - u_m\|_{\partial B}^2 = \lim_{m \rightarrow \infty} \int_{\partial B} (u(x) - u_m(x))^2 dx = 0.$$

Since u_m satisfies the Helmholtz equation in B , it thus follows from Theorem 5.4 that $\{u_m\}_m$ converges uniformly in every compact subset of B to the solution of the Helmholtz equation with Dirichlet boundary data given by $u|_{\partial B}$. This solution equals u , since the pointwise and uniform limit functions coincide, whence (5.11) holds.

II.) To verify that $u|_B = u_f|_B$ let $y_1 \in B$ and choose as above a monotonically increasing sequence $\{v'_m\}_m \subseteq \mathcal{S}_f$ satisfying $\lim_{m \rightarrow \infty} v'_m(y_1) = u_f(y_1)$. We can assume that

$$v'_m \geq w_m, \quad (5.12)$$

since we otherwise replace v'_m by $\max(w_m, v'_m)$. With the sequence $\{v'_m\}_m$ we construct in the same way as above a solution $u' \in C(\bar{B})$ of the Helmholtz equation in the Ball B satisfying $u'(y_1) = u_f(y_1)$. Because u is the pointwise limit of $\{w_m\}$ and because $u' \geq v'_m$ for every m , it follows from (5.12) that

$$u' \geq u,$$

which implies that $u_f(y) = u(y) \leq u'(y) \leq u_f(y)$, whence $u'(y) = u(y)$. Therefore $u' - u$ is a solution of the Helmholtz equation in B satisfying $(u' - u) \geq 0$ and $(u' - u)(y) = 0$. Since y is an interior point of B , we conclude from Corollary 5.7 that $u' - u = 0$ in B , thence

$$u_f(y_1) = u'(y_1) = u(y_1).$$

Since $y_1 \in B$ was arbitrary, we obtain $u_f|_B = u|_B$, hence u_f is a solution of the Helmholtz equation in the neighborhood $\bar{B}$ of y . Since $y \in \Omega$ was arbitrary, we conclude that u_f is a solution of the Helmholtz equation in Ω , as asserted by the theorem. $\blacksquare$

5.5 Boundary value problems, regular points

Corollary 5.17 *Let the assumptions of Theorem 5.16 be satisfied. If $w \in C(\bar{\Omega}, \mathbb{R})$ is a supersolution with $f \leq w|_{\partial\Omega}$, then the solution u constructed in this theorem satisfies*

$$u(x) \leq w(x), \quad x \in \Omega.$$

Proof: Any subsolution $v \in \mathcal{S}_f$ satisfies $v|_{\partial\Omega} = f \leq w|_{\partial\Omega}$, hence $v \leq w$ on $\bar{\Omega}$, by comparison. Consequently

$$u(x) = \sup_{v \in \mathcal{S}_f} v(x) \leq w(x), \quad x \in \Omega.$$

Corollary 5.18 *Let $\Omega \subseteq \mathbb{R}^2$ be a bounded open set, let $\lambda \leq 0$ and let $f \in C(\partial\Omega, \mathbb{R})$. If there is a subsolution $v \in C(\bar{\Omega})$ and a supersolution $w \in C(\bar{\Omega})$ satisfying*

$$v|_{\partial\Omega} = f = w|_{\partial\Omega},$$

then there is a unique solution $u \in C(\overline{\Omega}) \cap C_\infty(\Omega)$ of

$$\begin{aligned}\Delta u(x) + \lambda u(x) &= 0, & x \in \Omega \\ u|_{\partial\Omega} &= f.\end{aligned}$$

The solution satisfies $v \leq u \leq w$ on $\overline{\Omega}$.

Proof: By assumption the set $\mathcal{S}_f$ of all subsolutions $\hat{v}$ satisfying $\hat{v}|_{\partial\Omega} \leq f$ contains the function v , hence is nonempty. Consequently, by Theorem 5.16 there is a solution $u \in C_\infty(\Omega)$ of

$$\Delta u(x) + \lambda u(x) = 0, \quad x \in \Omega,$$

which by Corollary 5.17 satisfies $v \leq u \leq w$ on Ω . Extend u from Ω to a function on $\overline{\Omega}$ by defining

$$u(x) = f(x), \quad x \in \partial\Omega.$$

To see that the extended function satisfies $u \in C(\overline{\Omega})$, let $x \in \partial\Omega$. Since $v, w \in C(\overline{\Omega})$ satisfy $v|_{\partial\Omega} = w|_{\partial\Omega} = f$, we obtain

$$f(x) = \lim_{\substack{y \rightarrow x \\ y \in \overline{\Omega}}} v(y) \leq \lim_{\substack{y \rightarrow x \\ y \in \overline{\Omega}}} u(y) \leq \lim_{\substack{y \rightarrow x \\ y \in \overline{\Omega}}} w(y) = f(x),$$

whence

$$\lim_{\substack{y \rightarrow x \\ y \in \overline{\Omega}}} u(y) = f(x).$$

Consequently, $u \in C(\overline{\Omega})$. Uniqueness of the solution follows from Corollary 5.3.

Example 5.19 Let $\lambda < 0$, let $a, b > 0$, let

$$\Omega = \{x = (x_1, x_2) \in \mathbb{R}^2 \mid |x_1| < a, |x_2| < b\}$$

and let $f \in C(\partial\Omega)$ be defined by

$$f(x) = c$$

with a constant $c > 0$. Then there is a unique solution $u \in C(\overline{\Omega}) \cap C_\infty(\Omega)$ of

$$\begin{aligned}\Delta u(x) + \lambda u(x) &= 0, & x \in \Omega, \\ u|_{\partial\Omega} &= f.\end{aligned}$$

For, $w(x) = c$ is a supersolution with $w|_{\partial\Omega} = f$, cf. Corollary 5.15. To construct a subsolution v with $v|_{\partial\Omega} = f$, note that

$$v_1(x_1, x_2) = \frac{c}{e^{\sqrt{-\lambda}a} + e^{-\sqrt{-\lambda}a}} \left(e^{\sqrt{-\lambda}x_1} + e^{-\sqrt{-\lambda}x_1} \right)$$

is a solution, hence a subsolution with

$$v_1(x_1, x_2) \begin{cases} = c, & |x_1| = a, & |x_2| \leq b \\ \leq c, & |x_1| \leq a_1, & |x_2| = b. \end{cases}$$

Also

$$v_2(x_1, x_2) = \frac{c}{e^{\sqrt{-\lambda}b} + e^{-\sqrt{-\lambda}b}} \left(e^{\sqrt{-\lambda}x_2} + e^{-\sqrt{-\lambda}x_2} \right)$$

is a solution, hence a subsolution with

$$v_2(x_1, x_2) \begin{cases} \leq c, & |x_1| = a, & |x_2| \leq b \\ = c, & |x_1| \leq a_1, & |x_2| = b. \end{cases}$$

Consequently

$$v(x) = \max(v_1(x), v_2(x))$$

is a subsolution with $v|_{\partial\Omega} = f$. Corollary 5.18 thus implies that the boundary value problem has a unique solution.

For an arbitrary domain and for arbitrarily given boundary data it is difficult to find sub- and supersolutions with boundary values equal to the given data. In the following we show that it suffices to find sub- and supersolutions, which satisfy the boundary condition locally.

Definition 5.20 Let $\Omega \subseteq \mathbb{R}^2$ be a bounded open set. We call $x_0 \in \partial\Omega$ a regular boundary point, if to every $\xi \in \mathbb{R}$, $\delta > 0$ and $r > |\xi|$ there is a supersolution $w \in C(\overline{\Omega})$ and a subsolution $v \in C(\overline{\Omega})$ satisfying $w(x_0) = v(x_0) = \xi$ and

$$\begin{aligned} w(x) &\geq \begin{cases} \xi, & x \in \overline{\Omega} \cap B_\delta(x_0), \\ r, & x \in \overline{\Omega} \setminus B_\delta(x_0), \end{cases} \\ v(x) &\leq \begin{cases} \xi, & x \in \overline{\Omega} \cap B_\delta(x_0), \\ -r, & x \in \overline{\Omega} \setminus B_\delta(x_0). \end{cases} \end{aligned}$$

Theorem 5.21 Let $\Omega \subseteq \mathbb{R}^2$ be a bounded open domain, let $\lambda \leq 0$ and let $f \in C(\partial\Omega, \mathbb{R})$. If $x_0 \in \partial\Omega$ is a regular point, then the solution u of

$$\Delta u(x) + \lambda u(x) = 0, \quad x \in \Omega$$

constructed by Perron's method is continuous at x_0 and satisfies

$$u(x_0) = f(x_0).$$

Proof: Let $\varepsilon > 0$. Then there is $\delta > 0$ such that

$$|f(x) - f(x_0)| < \varepsilon$$

for all $x \in \partial\Omega$ with $|x - x_0| < \delta$. By assumption there is a supersolution w and a subsolution v satisfying $w(x_0) = f(x_0) + \varepsilon$, $v(x_0) = f(x_0) - \varepsilon$ and

$$\begin{aligned} w(x) &\geq w(x_0) = f(x_0) + \varepsilon, & \text{for } |x - x_0| < \delta, \\ w(x) &\geq \sup_{y \in \partial\Omega} |f(y)|, & \text{for } |x - x_0| \geq \delta, \\ v(x) &\leq v(x_0) = f(x_0) - \varepsilon, & \text{for } |x - x_0| < \delta, \\ v(x) &\leq -\sup_{y \in \partial\Omega} |f(y)|, & \text{for } |x - x_0| \geq \delta. \end{aligned}$$

These conditions imply

$$\begin{aligned} w(x) &\geq f(x_0) + \varepsilon \geq f(x), & x \in \partial\Omega \cap B_\delta(x_0), \\ w(x) &\geq \sup_{y \in \partial\Omega} |f(y)| \geq f(x), & x \in \partial\Omega \setminus B_\delta(x_0), \\ v(x) &\leq f(x_0) - \varepsilon \leq f(x), & x \in \partial\Omega \cap B_\delta(x_0), \\ v(x) &\leq -\sup_{y \in \partial\Omega} |f(y)| \leq f(x), & x \in \partial\Omega \setminus B_\delta(x_0), \end{aligned}$$

hence $v|_{\partial\Omega} \leq f \leq w|_{\partial\Omega}$. Therefore we have $v \in \mathcal{S}_f$, hence $\mathcal{S}_f \neq \emptyset$. This implies that the solution u exists and satisfies

$$v(x) \leq u(x) \leq w(x),$$

for all $x \in \overline{\Omega}$. Thus,

$$\begin{aligned} \limsup_{\substack{x \rightarrow x_0 \\ x \in \overline{\Omega}}} u(x) &\leq \limsup_{\substack{x \rightarrow x_0 \\ x \in \overline{\Omega}}} w(x) = \lim_{\substack{x \rightarrow x_0 \\ x \in \overline{\Omega}}} w(x) = f(x_0) + \varepsilon \\ \liminf_{\substack{x \rightarrow x_0 \\ x \in \overline{\Omega}}} u(x) &\geq \liminf_{\substack{x \rightarrow x_0 \\ x \in \overline{\Omega}}} v(x) = \lim_{\substack{x \rightarrow x_0 \\ x \in \overline{\Omega}}} v(x) = f(x_0) - \varepsilon. \end{aligned}$$

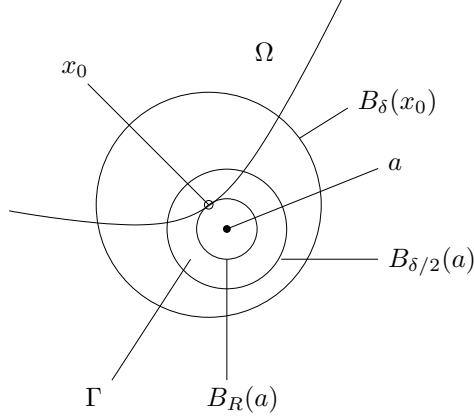
Since $\varepsilon > 0$ was arbitrary, we infer that

$$\limsup_{\substack{x \rightarrow x_0 \\ x \in \overline{\Omega}}} u(x) = \liminf_{\substack{x \rightarrow x_0 \\ x \in \overline{\Omega}}} u(x) = f(x_0),$$

which implies

$$\lim_{\substack{x \rightarrow x_0 \\ x \in \overline{\Omega}}} u(x) = f(x_0).$$

Therefore u is continuous at x_0 .



Ball with center a intersecting $\partial\Omega$ at x_0

Corollary 5.22 *Let $\lambda \leq 0$, let $\Omega \subseteq \mathbb{R}^2$ be a bounded open set and let $f \in C(\partial\Omega, \mathbb{R})$. If every point of $\partial\Omega$ is regular, then there is a unique solution $u \in C(\overline{\Omega}, \mathbb{R})$ of*

$$\begin{aligned} \Delta u(x) + \lambda u(x) &= 0, & x \in \Omega \\ u|_{\partial\Omega} &= f. \end{aligned}$$

This result follows immediately from the preceding theorem. It remains to find a criterion for regular boundary points.

Theorem 5.23 *Let $\Omega \subseteq \mathbb{R}^2$ be a bounded open set and let $x_0 \in \partial\Omega$. Assume that $\lambda \leq 0$. If there is an open ball $B \subseteq \mathbb{R}^2 \setminus \Omega$ such that*

$$\overline{B} \cap \overline{\Omega} = \{x_0\},$$

then x_0 is a regular boundary point.

Proof: Let $\xi \in \mathbb{R}$, $r > |\xi|$ and $\delta > 0$ be given. We must construct suitable super- and subsolutions. We first assume that $\xi \leq 0$. Let $B = B_R(a)$ be the ball with

$$\overline{B_R(a)} \cap \overline{\Omega} = \{x_0\}.$$

Without restriction of generality we can assume that $R < \frac{\delta}{2}$. Otherwise we shrink B until this estimate holds. This estimate for R and the equation $\overline{B_R(a)} \cap \overline{\Omega} = \{x_0\}$ imply

$$B_R(a) \subseteq B_{\delta/2}(a) \subseteq B_\delta(x_0).$$

For simplicity we write

$$\Gamma = B_{\delta/2}(a) \setminus \overline{B_R(a)}.$$

I.) To construct a supersolution we let $u \in C(\overline{\Gamma}) \cap C_\infty(\Gamma)$ be the solution of

$$\begin{aligned} \Delta u(x) + \lambda u(x) &= 0, & x \in \Gamma, \\ u(x) &= r, & x \in \partial B_{\delta/2}(a), \\ u(x) &= \xi, & x \in \partial B_R(a). \end{aligned}$$

This solution is given by

$$u(x) = \begin{cases} C_1 J_0(\sqrt{\lambda}|x-a|) + C_2 N_0(\sqrt{\lambda}|x-a|), & \text{for } \lambda < 0, \\ C_1 + C_2 \ln|x-a|, & \text{for } \lambda = 0, \end{cases}$$

with suitable constants C_1 and C_2 , which can be determined from the boundary conditions. We can use this formula to extend u to the region $\mathbb{R}^2 \setminus B_R(a)$, which contains Ω as a subset. The extended function satisfies the Helmholtz equation in the whole domain of definition. Therefore, for $y \in \mathbb{R}^2$ with $|y-a| > \delta/2$ the radial symmetry of u and the maximum principle applied to the region $\Gamma_y = B_{|y-a|}(a) \setminus \overline{B_R(a)}$ yield for all $x \in \Gamma_y$ that

$$\min(0, \xi, u(y)) = \min(0, \min_{\partial\Gamma_y} u) \leq u(x) \leq \max(0, \max_{\partial\Gamma_y} u) = \max(0, \xi, u(y)). \quad (5.13)$$

We can insert $x \in \partial B_{\delta/2}(a) \subseteq \Gamma_y$ into this inequality. Since u has the value r on $\partial B_{\delta/2}(a)$, the second inequality in (5.13) can only hold if

$$u(y) \geq r, \quad \text{for all } y \in \mathbb{R}^2 \setminus \partial B_{\delta/2}(a). \quad (5.14)$$

This implies $\min(0, \xi, u(y)) = \xi \leq 0$, whence, the first inequality in (5.13) yields

$$u(x) \geq \xi, \quad \text{for all } x \in \mathbb{R}^2 \setminus B_R(a). \quad (5.15)$$

Now define $w = u|_{\overline{\Omega}}$. Then w is a solution of the Helmholtz equation, hence it is a supersolution. Moreover, $x_0 \in \partial B_R(a)$ implies

$$w(x_0) = u(x_0) = \xi.$$

If we note that $\overline{\Omega} \subseteq \mathbb{R}^2 \setminus B_R(a)$ and $\overline{\Omega} \setminus B_\delta(x_0) \subseteq \mathbb{R}^2 \setminus B_{\delta/2}(a)$, we obtain from (5.14) and (5.15) that

$$w \geq \xi \text{ on } \overline{\Omega}, \quad \text{and} \quad w \geq r \text{ on } \overline{\Omega} \setminus B_\delta(x_0).$$

Therefore w satisfies all conditions required from the supersolution in Definition 5.20.

II.) To construct a subsolution v let $u \in C(\overline{\Gamma}) \cap C_\infty(\Gamma)$ be the solution of

$$\begin{aligned} \Delta u(x) &= 0, & x \in \Gamma, \\ u(x) &= -r, & x \in \partial B_{\delta/2}(a), \\ u(x) &= \xi, & x \in \partial B_R(a). \end{aligned}$$

This solution has the form

$$u(x) = C_1 + C_2 \ln|x-a|$$

with constants C_1, C_2 uniquely determined by the boundary conditions. We use this formula to extend u to the region $\mathbb{R}^2 \setminus B_R(a)$. The extended function satisfies the potential equation in the whole domain of definition. Choose $y \in \mathbb{R}^2$ with $|y - a| > \delta/2$. Since u is radially symmetric with respect to a , it has the constant value $u(y)$ on the circle $\partial B_{|y-a|}(a)$, whence Corollary 5.8(ii) implies for all $x \in \Gamma_y$ that

$$\min(\xi, u(y)) = \min_{\partial\Gamma_y} u < u(x) < \max_{\partial\Gamma_y} u = \max(\xi, u(y)). \quad (5.16)$$

This inequality must hold for $x \in \partial B_{\delta/2}(a) \subseteq \Gamma_y$. For such x we have $u(x) = -r$. The first inequality in (5.16) can therefore only hold if $u(y) < -r$. This implies $\max(\xi, u(y)) = \xi$, so the second inequality in (5.16) yields that $u(x) < \xi$ for all $x \in \Gamma_y$. Since y was an arbitrary point outside of the ball $\overline{B_{\delta/2}(a)}$, we obtain

$$u \leq \xi \text{ on } \mathbb{R}^2 \setminus B_R(a), \quad u \leq -r, \text{ on } \mathbb{R}^2 \setminus B_{\delta/2}(a).$$

Since $\xi \leq 0$ by assumption, it follows that $u \leq 0$. Consequently $v = u|_{\overline{\Omega}}$ is a non-positive solution of the potential equation, hence v is a subsolution, by Theorem 5.14. Since $x_0 \in \partial B_R(a)$, we have $v(x_0) = \xi$. From the inequalities above it is immediately seen that v satisfies all conditions required in Definition 5.20 from the subsolution.

III.) It remains to construct a supersolution and a subsolution in the case $\xi > 0$. To this end let $\hat{w}$ and $\hat{v}$ be the super- and subsolution to the value $-\xi$ constructed in the preceding part of the proof. Since the negative of a supersolution is a subsolution and the negative of a subsolution is a supersolution, it follows that

$$w = -\hat{v}, \quad v = -\hat{w}$$

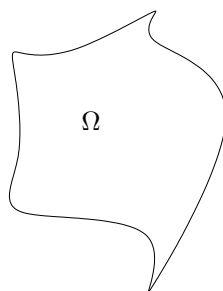
are a supersolution and a subsolution, respectively, satisfying the estimates required for a regular point. Consequently x_0 is a regular point. $\blacksquare$

Example 5.24 Let $\Omega \subseteq \mathbb{R}^2$ be a bounded, open and convex set. Then to every point $x \in \partial\Omega$ there is a ball B such that $\overline{\Omega} \cap \overline{B} = \{x\}$, hence every boundary point is regular. Therefore the Dirichlet boundary value problem

$$\begin{aligned} \Delta u(x) + \lambda u(x) &= 0, & x \in \Omega \\ u|_{\partial\Omega} &= f, \end{aligned}$$

with $\lambda \leq 0$ has a unique solution $u \in C(\overline{\Omega}) \cap C_\infty(\Omega)$ to every $f \in C(\partial\Omega)$.

Example 5.25 Let $\Gamma \subseteq \partial\Omega$ be a finite subset such that $\partial\Omega$ is two times continuously differentiable at every point of $\partial\Omega \setminus \Gamma$. Assume that through every point y of Γ a straight line ℓ is passing such that Ω is locally on one side of ℓ at y . Then to every point $x \in \partial\Omega$ there is a ball B with $\overline{B} \cap \overline{\Omega} = \{x\}$, hence every point of $\partial\Omega$ is regular, and the Dirichlet problem can be uniquely solved. In particular, the Dirichlet problem can be uniquely solved if $\partial\Omega \in C_2$.



A domain satisfying the conditions of Example 5.25

6 Fundamental solution, Green's function

6.1 Convolution integrals

Theorem 6.1 *Let $1 \leq p < \infty$, $\varphi \in L^1(\mathbb{R}^n, \mathbb{C})$ and $f \in L^p(\mathbb{R}^n, \mathbb{C})$. Then for almost all $x \in \mathbb{R}^n$ the integral*

$$F(x) = \int_{\mathbb{R}^n} \varphi(x-y)f(y)dy = \int_{\mathbb{R}^n} \varphi(y)f(x-y)dy$$

exists. The function F defined by this integral belongs to $L^p(\mathbb{R}^n, \mathbb{C})$ and satisfies

$$\|F\|_{L^p(\mathbb{R}^n)} \leq \|\varphi\|_{L^1(\mathbb{R}^n)} \|f\|_{L^p(\mathbb{R}^n)}.$$

Here

$$\|u\|_{L^p(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} |u(x)|^p dx \right)^{1/p}$$

denotes the norm in the Banach space $L^p(\mathbb{R}^n, \mathbb{C})$.

Proof: We have

$$\begin{aligned} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |\varphi(y)| |f(x-y)|^p dx dy &= \int_{\mathbb{R}^n} |\varphi(y)| \int_{\mathbb{R}^n} |f(x-y)|^p dx dy \\ &= \int_{\mathbb{R}^n} |\varphi(y)| dy \int_{\mathbb{R}^n} |f(x)|^p dx = \|\varphi\|_{L^1(\mathbb{R}^n)} \|f\|_{L^p(\mathbb{R}^n)}^p. \end{aligned} \quad (6.1)$$

Thus, Tonelli's theorem yields $((x, y) \mapsto |\varphi(y)| |f(x-y)|^p) \in L^1(\mathbb{R}^n \times \mathbb{R}^n)$, whence Fubini's theorem implies that $\int_{\mathbb{R}^n} |\varphi(y)| |f(x-y)|^p dy$ exists for almost all $x \in \mathbb{R}^n$, consequently $y \mapsto |\varphi(y)|^{1/p} |f(x-y)| \in L^p(\mathbb{R}^n)$ for almost all x . For $p = 1$ we therefore get from (6.1)

$$\int_{\mathbb{R}^n} |F(x)| dx \leq \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |\varphi(y)| |f(x-y)| dy dx \leq \|\varphi\|_{L^1(\mathbb{R}^n)} \|f\|_{L^1(\mathbb{R}^n)}.$$

This completes the proof for $p = 1$. For $1 < p < \infty$ let $1 < q < \infty$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$. Then Hölder's inequality yields

$$\begin{aligned} \left(\int_{\mathbb{R}^n} |\varphi(x-y)f(y)| dy \right)^p &= \left(\int_{\mathbb{R}^n} |\varphi(y)f(x-y)| dy \right)^p \\ &\leq \left(\int_{\mathbb{R}^n} |\varphi(y)|^{\frac{1}{q}} |\varphi(y)|^{\frac{1}{p}} |f(x-y)| dy \right)^p \leq \|\varphi\|_1^{\frac{p}{q}} \left(\int_{\mathbb{R}^n} |\varphi(y)| |f(x-y)|^p dy \right). \end{aligned}$$

The right hand side is bounded for almost all x . It thus follows for these x that $y \mapsto \varphi(x-y)f(y) \in L^1(\mathbb{R}^n)$. Furthermore, the last inequality and (6.1) together imply

$$\begin{aligned} \int_{\mathbb{R}^n} |F(x)|^p dx &= \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} \varphi(x-y)f(y) dy \right|^p dx \\ &\leq \|\varphi\|_{L^1(\mathbb{R}^n)}^{\frac{p}{q}} \|\varphi\|_{L^1(\mathbb{R}^n)} \|f\|_{L^p(\mathbb{R}^n)}^p = (\|\varphi\|_{L^1(\mathbb{R}^n)} \|f\|_{L^p(\mathbb{R}^n)})^p. \end{aligned}$$

Remark: Often one uses the notation

$$F(x) = \int_{\mathbb{R}^n} \varphi(x-y)f(y)dy = (\varphi * f)(x).$$

The operator $*$ is called convolution. With this notation the inequality just proved is

$$\|\varphi * f\|_{L^p(\mathbb{R}^n)} \leq \|\varphi\|_{L^1(\mathbb{R}^n)} \|f\|_{L^p(\mathbb{R}^n)}.$$

Consequently $f \mapsto \varphi * f : L^p(\mathbb{R}^n) \rightarrow L^p(\mathbb{R}^n)$ is a linear and continuous mapping with norm not greater than $\|\varphi\|_{L^1(\mathbb{R}^n)}$

6.2 Fundamental solution

Definition 6.2 Let $\lambda \in \mathbb{C}$. The fundamental solution F of the Helmholtz equation $\Delta u + \lambda u = 0$ in $\mathbb{R}^n$, $n = 2, 3$, is defined as follows:

(i) Let $n = 3$. For $x \in \mathbb{R}^3$ with $x \neq 0$ set

$$F(x) = \frac{e^{i\sqrt{\lambda}|x|}}{4\pi|x|}.$$

(ii) Let $n = 2$. For $x \in \mathbb{R}^2$ with $x \neq 0$ set

$$F(x) = \begin{cases} -\frac{1}{4}N_0(\sqrt{\lambda}|x|), & \lambda \neq 0, \\ -\frac{1}{2\pi} \ln |x|, & \lambda = 0, \end{cases}$$

where N_0 is Neumann's function of order 0. For the square root we take the branch satisfying $0 \leq \arg \sqrt{\lambda} < \pi$.

In the following we mainly study the fundamental solution in $\mathbb{R}^3$. Analogous results hold for the fundamental solution in $\mathbb{R}^2$.

Lemma 6.3 *The three-dimensional fundamental solution F is infinitely differentiable in $\mathbb{R}^3 \setminus \{0\}$ and satisfies*

$$\Delta F(x) + \lambda F(x) = 0, \quad x \neq 0.$$

Proof: It is obvious that F is infinitely differentiable in $\mathbb{R}^3 \setminus \{0\}$. To show that the Helmholtz equation is satisfied let $r = r(x) = |x|$. Then

$$\begin{aligned} (\Delta + \lambda)F(x) &= (\Delta + \lambda) \frac{e^{i\sqrt{\lambda}r(x)}}{4\pi r(x)} \\ &= \left(\sum_{i=1}^3 \left[\left(\frac{\partial r}{\partial x_i} \right)^2 \frac{\partial^2}{\partial r^2} + \frac{\partial^2 r}{\partial x_i^2} \frac{\partial}{\partial r} \right] + \lambda \right) \frac{e^{i\sqrt{\lambda}r}}{4\pi r} \\ &= \left(\sum_{i=1}^3 \left[\left(\frac{x_i}{|x|} \right)^2 \frac{\partial^2}{\partial r^2} + \left(\frac{1}{|x|} - \frac{x_i^2}{|x|^3} \right) \frac{\partial}{\partial r} \right] + \lambda \right) \frac{e^{i\sqrt{\lambda}r}}{4\pi r} \\ &= \left[\frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} + \lambda \right] \frac{e^{i\sqrt{\lambda}r}}{4\pi r} = 0. \end{aligned}$$

Since $\frac{\partial}{\partial x_i} F(x) = (i\sqrt{\lambda} - \frac{1}{|x|}) \frac{e^{i\sqrt{\lambda}|x|}}{4\pi|x|} \frac{x_i}{|x|}$, it follows that to every $R > 0$ there exist constants C_1, C_2 such that ■

$$|F(x)| \leq \frac{C_1}{|x|}, \quad \left| \frac{\partial}{\partial x_i} F(x) \right| \leq \frac{C_2}{|x|^2}$$

for all x with $0 < |x| \leq R$, hence

$$F, \frac{\partial}{\partial x_i} F \in L^1(B_R(0), \mathbb{C}),$$

where $B_R(0) = \{x \in \mathbb{R}^3 \mid |x| < R\}$. For $\lambda \in \mathbb{C} \setminus [0, \infty)$ it follows that

$$\operatorname{Re} i\sqrt{\lambda} < 0,$$

whence $|F(x)|$ and $\left| \frac{\partial}{\partial x_i} F(x) \right|$ decay exponentially for $|x| \rightarrow \infty$. This implies that

$$F, \frac{\partial}{\partial x_i} F \in L^1(\mathbb{R}^3, \mathbb{C}).$$

Theorem 6.4 Assume that $\lambda \in \mathbb{C} \setminus [0, \infty)$. For $f \in L^2(\mathbb{R}^3, \mathbb{C})$ set

$$u(x) = - \int_{\mathbb{R}^3} F(x-y)f(y)dy = -(F * f)(x), \quad x \in \mathbb{R}^3.$$

Then $u \in H_1(\mathbb{R}^3, \mathbb{C})$ is a weak solution of the equation

$$\Delta u + \lambda u = f. \tag{6.2}$$

Proof: To prove that $u = -F * f \in H_1(\mathbb{R}^3)$ note that since $F \in L^1(\mathbb{R}^3)$, $\frac{\partial}{\partial x_i} F \in L^1(\mathbb{R}^3)$ and $f \in L^2(\mathbb{R}^3)$, we conclude from Theorem 6.1 that

$$u = -F * f \in L^2(\mathbb{R}^3), \quad v_i = -\left(\frac{\partial}{\partial x_i} F \right) * f \in L^2(\mathbb{R}^3), \quad \text{for } i = 1, 2, 3.$$

Consequently, to prove that $u \in H_1(\mathbb{R}^3)$ it suffices to show that v_i is the weak derivative of u . To verify this let $\varphi \in \dot{C}_\infty(\mathbb{R}^3)$. Then

$$\begin{aligned}
(u, \varphi_{x_i}) &= - \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} F(x-y) f(y) dy \frac{\partial}{\partial x_i} \overline{\varphi(x)} dx \\
&= - \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} F(x-y) \frac{\partial}{\partial x_i} \overline{\varphi(x)} dx f(y) dy \\
&= - \int_{\mathbb{R}^3} \lim_{r \rightarrow 0} \int_{\mathbb{R}^3 \setminus B_r(y)} F(x-y) \frac{\partial}{\partial x_i} \overline{\varphi(x)} dx f(y) dy \\
&= \int_{\mathbb{R}^3} \lim_{r \rightarrow 0} \left(\int_{\mathbb{R}^3 \setminus B_r(y)} \frac{\partial}{\partial x_i} F(x-y) \overline{\varphi(x)} dx - \int_{\partial B_r(y)} n_i(x) F(x-y) \overline{\varphi(x)} dS_x \right) f(y) dy \\
&= \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{\partial}{\partial x_i} F(x-y) \overline{\varphi(x)} dx f(y) dy \\
&= \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{\partial}{\partial x_i} F(x-y) f(y) dy \overline{\varphi(x)} dx = -(v_i, \varphi).
\end{aligned}$$

This proves that $v_i = \frac{\partial}{\partial x_i} u$, $i = 1, 2, 3$, hence $u \in H_1(\mathbb{R}^3)$. In the computation above we used Gauß' theorem. $n(x) = (n_1(x), n_2(x), n_3(x))$ denotes the interior unit normal vector to $\partial B_r(y)$. We also used that

$$\begin{aligned}
\lim_{r \rightarrow 0} \left| \int_{\partial B_r(y)} n_i(x) F(x-y) \overline{\varphi(x)} dS_x \right| &= \lim_{r \rightarrow 0} \left| \int_{\partial B_r(y)} n_i(x) \frac{e^{i\sqrt{\lambda}r}}{4\pi r} \overline{\varphi(x)} dS_x \right| \\
&\leq \limsup_{r \rightarrow 0} |\varphi| \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{1}{r} dS_x = \sup_{\mathbb{R}^3} |\varphi| \lim_{r \rightarrow 0} r = 0.
\end{aligned}$$

To prove that u is a weak solution of (6.2) in $\mathbb{R}^3$ we must by Definition 3.31 show that

$$-(\nabla u, \nabla \varphi) + \lambda(u, \varphi) = (f, \varphi) \quad (6.3)$$

holds for all $\varphi \in \dot{C}_\infty(\mathbb{R}^3, \mathbb{C})$. To prove this we use the first Green's formula and proceed similarly. We compute

$$\begin{aligned}
&-(\nabla u, \nabla \varphi) + \lambda(u, \varphi) \\
&= \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} (\nabla_x F(x-y) \cdot \nabla_x \overline{\varphi(x)} - \lambda F(x-y) \overline{\varphi(x)}) dx f(y) dy \\
&= \int_{\mathbb{R}^3} \lim_{r \rightarrow 0} \int_{\mathbb{R}^3 \setminus B_r(y)} \left(\nabla_x F(x-y) \cdot \nabla_x \overline{\varphi(x)} - \lambda F(x-y) \overline{\varphi(x)} \right) dx f(y) dy \\
&= \int_{\mathbb{R}^3} \lim_{r \rightarrow 0} \left(\int_{\mathbb{R}^3 \setminus B_r(y)} (-\Delta_x - \lambda) F(x-y) \overline{\varphi(x)} dx \right. \\
&\quad \left. + \int_{\partial B_r(y)} \frac{\partial}{\partial n_x} F(x-y) \overline{\varphi(x)} dS_x \right) f(y) dy \\
&= - \int_{\mathbb{R}^3} \lim_{r \rightarrow 0} \int_{\partial B_r(y)} \frac{\partial}{\partial r} \frac{e^{i\sqrt{\lambda}r}}{4\pi r} \overline{\varphi(x)} dS_x f(y) dy.
\end{aligned} \quad (6.4)$$

Now

$$\begin{aligned}
& -\lim_{r \rightarrow 0} \int_{|x-y|=r} \frac{\partial}{\partial r} \frac{e^{i\sqrt{\lambda}r}}{4\pi r} \overline{\varphi(x)} dS_x \\
& = -\lim_{r \rightarrow 0} \frac{e^{i\sqrt{\lambda}r}}{4\pi r} \left(i\sqrt{\lambda} - \frac{1}{r} \right) \int_{|x-y|=r} \overline{\varphi(y)} + (\overline{\varphi(x)} - \overline{\varphi(y)}) dS_x = \overline{\varphi(y)},
\end{aligned} \tag{6.5}$$

since

$$\begin{aligned}
& \lim_{r \rightarrow 0} \left| \frac{e^{i\sqrt{\lambda}r}}{4\pi r} \left(i\sqrt{\lambda} - \frac{1}{r} \right) \int_{|x-y|=r} \overline{\varphi(x) - \varphi(y)} dS_x \right| \\
& \leq \lim_{r \rightarrow 0} \sup_{x \in \partial B_r(y)} |\varphi(x) - \varphi(y)| \frac{1}{4\pi r} (|\sqrt{\lambda}| + \frac{1}{r}) 4\pi r^2 = 0.
\end{aligned} \tag{6.6}$$

Here we used the continuity of φ . We combine (6.5) with (6.4) and obtain (6.3). Consequently, u is a weak solution. $\blacksquare$

Remark 6.5 If f is more regular then

$$u(x) = - \int_{\mathbb{R}^3} F(x-y) f(y) dy$$

is not only a weak solution, but also a classical solution of $\Delta u + \lambda u = 0$. For, a weak solution u is a classical solution if $u \in C_2(\mathbb{R}^3)$. This regularity of u is obtained for example if $f \in C_1(\mathbb{R}^3)$ and

$$|f(x)|, \left| \frac{\partial}{\partial x_1} f(x) \right|, \dots, \left| \frac{\partial}{\partial x_3} f(x) \right| \leq C$$

for all $x \in \mathbb{R}^3$, with a suitable constant C . To see this, suppose that f satisfies these conditions and that $\operatorname{Re} i\sqrt{\lambda} < 0$. Then

$$\begin{aligned}
\frac{\partial}{\partial x_i} u(x) &= - \int_{\mathbb{R}^3} \frac{\partial}{\partial x_i} F(x-y) f(y) dy \\
&= \int_{\mathbb{R}^3} \frac{\partial}{\partial y_i} F(x-y) f(y) dy = - \int_{\mathbb{R}^3} F(x-y) \frac{\partial}{\partial y_i} f(y) dy
\end{aligned}$$

and

$$\frac{\partial^2}{\partial x_j \partial x_i} u(x) = - \int_{\mathbb{R}^3} \frac{\partial}{\partial x_j} F(x-y) \frac{\partial}{\partial y_i} f(y) dy,$$

and some technical considerations show that these derivatives exist in the classical sense.

6.3 Green's function

Convolution with the fundamental solution yields a solution of the Helmholtz equation in the whole space $\mathbb{R}^3$, but it does not yield the solution of a boundary value problem, since the boundary condition will not be satisfied in general. To find a replacement for the fundamental solution in case of a boundary value problem assume that $\lambda \in \mathbb{C}$ and that the boundary value problem

$$\begin{aligned}\Delta u(x) + \lambda u(x) &= f(x), & x \in \Omega, \\ u(x) &= g(x), & x \in \partial\Omega,\end{aligned}$$

has a solution u in the domain $\Omega \subseteq \mathbb{R}^3$. Assuming that the solution is regular enough such that Green's formula can be applied we obtain with the fundamental solution F

$$\begin{aligned}\int_{\Omega} F(x-y)f(y) dy &= \int_{\Omega} F(x-y)(\Delta_y + \lambda)u(y) dy \\ &= \lim_{r \rightarrow 0} \int_{\Omega \setminus B_r(x)} F(x-y)(\Delta_y + \lambda)u(y) dy \\ &= \lim_{r \rightarrow 0} \left(\int_{\Omega \setminus B_r(x)} (\Delta_y + \lambda)F(x-y)u(y) dy \right. \\ &\quad + \int_{\partial\Omega} F(x-y) \frac{\partial}{\partial n_y} u(y) - \frac{\partial}{\partial n_y} F(x-y)u(y) dS_y \\ &\quad \left. + \int_{\partial B_r(x)} F(x-y) \frac{\partial}{\partial n_y} u(y) - \frac{\partial}{\partial n_y} F(x-y)u(y) dS_y \right) \\ &= \int_{\partial\Omega} F(x-y) \frac{\partial}{\partial n_y} u(y) - \frac{\partial}{\partial n_y} F(x-y)u(y) dS_y - u(x),\end{aligned} \tag{6.7}$$

since $(\Delta_y + \lambda)F(x-y) = 0$ in the domain $\Omega \setminus B_r(x)$, and since

$$\begin{aligned}\lim_{r \rightarrow 0} \int_{\partial B_r(x)} F(x-y) \frac{\partial}{\partial n_y} u(y) dS_y &= 0 \\ - \lim_{r \rightarrow 0} \int_{\partial B_r(x)} \frac{\partial}{\partial n_y} F(x-y)u(y) dS_y &= -u(x),\end{aligned}$$

which is proved as in (6.5), (6.6). From (6.7) we thus obtain

$$u(x) = - \int_{\Omega} F(x-y)f(y) dy + \int_{\partial\Omega} F(x-y) \frac{\partial}{\partial n_y} u(y) - \frac{\partial}{\partial n_y} F(x-y)u(y) dS_y.$$

This is a representation formula for the solution in terms of the boundary values $u|_{\partial\Omega}$ and $\frac{\partial u}{\partial n}$ of the solution.

Since $u|_{\partial\Omega}$ is equal to g , we can insert $g(y)$ for $u(y)$ in the boundary integral on the right hand side of the representation formula. However, the normal

derivative $\partial u/\partial n$ appearing in the boundary integral is unknown and can only be determined by solving the boundary value problem. Yet, one gets a representation formula which does not contain the unknown normal derivative if one replaces the fundamental solution F by the *Green's function for the Dirichlet boundary value problem*

$$G(x, y) = F(x - y) + w(x, y),$$

where $w : \Omega \times \bar{\Omega} \rightarrow \mathbb{C}$ is defined as follows: For $x \in \Omega$ let $v : \bar{\Omega} \rightarrow \mathbb{C}$ be the solution of

$$\begin{aligned} \Delta v(y) + \lambda v(y) &= 0, \quad \text{for } y \in \Omega \\ v(y) &= -F(x - y), \quad \text{for } y \in \partial\Omega. \end{aligned}$$

Then set

$$w(x, y) = v(y).$$

The function G satisfies for every $\varphi \in C(\Omega)$ and $x \in \Omega$

- (i) $G(x, y) = 0, \quad y \in \partial\Omega,$
- (ii) $(\Delta_y + \lambda)G(x, y) = 0, \quad y \in \Omega, \quad x \neq y,$
- (iii) $\lim_{r \rightarrow 0} \int_{\partial B_r(x)} \varphi(y) G(x, y) dS_y = 0,$
- (iv) $\begin{aligned} &\lim_{r \rightarrow 0} \int_{\partial B_r(x)} \varphi(y) \frac{\partial}{\partial n_y} G(x, y) dS_y \\ &= \lim_{r \rightarrow 0} \int_{\partial B_r(x)} \varphi(y) \frac{\partial}{\partial n_y} F(x - y) dS_y + \lim_{r \rightarrow 0} \int_{\partial B_r(x)} \varphi(y) \frac{\partial}{\partial n_y} w(x, y) dS_y \\ &= \varphi(x). \end{aligned}$

(n_y is the interior normal vector to $\partial B_r(x)$.)

(ii) – (iv) are precisely the properties needed in the computation (6.7). Therefore the same computation yields for G instead of F and for the solution of the Dirichlet boundary value problem, using that $G(x, y) = 0$ for $y \in \partial\Omega$,

$$u(x) = - \int_{\Omega} G(x, y) f(y) dy - \int_{\partial\Omega} \frac{\partial}{\partial n_y} G(x, y) g(y) dS_y. \quad (6.8)$$

Of course, the determination of G requires to solve the Dirichlet boundary value problem in Ω . Therefore G can only be constructed if it is known in advance that the Dirichlet boundary value problem has a solution. The Green's function cannot be used to answer the questions of existence and uniqueness of boundary value problems. But if G can be determined explicitly it offers a means to represent, to compute and to study properties of the solution.

It is also possible to define the *Green's function for the Neumann boundary value problem*

$$\begin{aligned}\Delta u(x) + \lambda u(x) &= f(x), & x \in \Omega, \\ \frac{\partial}{\partial n} u(x) &= g(x), & x \in \partial\Omega,\end{aligned}$$

where $\lambda \in \mathbb{C}$ is a given constant. The Green's function for this problem is

$$G(x, y) = F(x - y) + w(x, y),$$

where $w : \Omega \times \overline{\Omega} \rightarrow \mathbb{C}$ is defined as follows: For $x \in \Omega$ let $v : \overline{\Omega} \rightarrow \mathbb{C}$ be the solution of

$$\begin{aligned}\Delta v(y) + \lambda v(y) &= 0, & \text{for } y \in \Omega \\ \frac{\partial}{\partial n_y} v(y) &= -\frac{\partial}{\partial n_y} F(x - y), & \text{for } y \in \partial\Omega.\end{aligned}$$

Then set

$$w(x, y) = v(y).$$

The function G satisfies

$$\frac{\partial}{\partial n_y} G(x, y) = 0, \quad y \in \partial\Omega,$$

and we obtain the representation formula for the solution of the Neumann boundary value problem

$$u(x) = - \int_{\Omega} G(x, y) f(y) dy + \int_{\partial\Omega} G(x, y) g(y) dS_y.$$

6.4 The Green's function for the potential equation in a ball. Poisson's representation formula.

It is possible to determine the Green's function explicitly in some cases. Here we derive the Green's function for a ball $B_R(0)$ in $\mathbb{R}^n$ with $n = 2$ and $n = 3$.

To this end let $R > 0$ and consider the Kelvin transformation $K : \mathbb{R}^n \setminus \{0\} \rightarrow \mathbb{R}^n \setminus \{0\}$ defined by

$$K(x) = \left(\frac{R}{|x|} \right)^2 x.$$

(Reflection at the sphere with radius R .)

Lemma 6.6 For $x, y \in \mathbb{R}^n$ with $0 < |x| < R$, $|y| = R$ we have

$$|y - K(x)| = \frac{R}{|x|} |y - x|.$$

Proof: We have

$$\begin{aligned}
|y - K(x)|^2 &= |K(x)|^2 + |y|^2 - 2y \cdot K(x) = \\
&= \left(\frac{R}{|x|}\right)^4 |x|^2 + R^2 - 2\left(\frac{R}{|x|}\right)^2 x \cdot y \\
&= \left(\frac{R}{|x|}\right)^2 R^2 + \left(\frac{R}{|x|}\right)^2 |x|^2 - 2\left(\frac{R}{|x|}\right)^2 x \cdot y \\
&= \left(\frac{R}{|x|}\right)^2 (|y|^2 + |x|^2 - 2x \cdot y) = \left(\frac{R}{|x|}\right)^2 |y - x|^2.
\end{aligned}$$

Theorem 6.7 *The Green's function to the Dirichlet problem for the potential equation in the ball $B_R(0) \subseteq \mathbb{R}^3$ is*

$$G(x, y) = \frac{1}{4\pi|x - y|} + w(x, y)$$

with

$$w(x, y) = \begin{cases} -\frac{R}{4\pi|x|} \frac{1}{|y - \left(\frac{R}{|x|}\right)^2 x|}, & 0 < |x| < R \\ -\frac{1}{4\pi R}, & x = 0. \end{cases}$$

Proof: For $0 < |x| < R$ we have $\left|\left(\frac{R}{|x|}\right)^2 x\right| = R \frac{R}{|x|} > R$. Thus, for all $x \in B_R(0)$

$$y \mapsto w(x, y) \in C_\infty(\overline{B_R(0)}),$$

and

$$\Delta_y w(x, y) = 0.$$

Also, for $0 < |x| < R$ and $y \in \partial B_R(0)$ we have

$$w(x, y) = -\frac{1}{4\pi \frac{|x|}{R} |y - K(x)|} = -\frac{1}{4\pi|x - y|} = -F(x - y).$$

Clearly, for $x = 0$ and $y \in \partial B_R(0)$

$$w(x, y) = -\frac{1}{4\pi R} = -\frac{1}{4\pi|y|} = -F(x - y).$$

Consequently, w satisfies

$$\begin{aligned}
\Delta_y w(x, y) &= 0, & (x, y) &\in B_R(0) \times B_R(0) \\
w(x, y) &= -F(x - y), & (x, y) &\in B_R(0) \times \partial B_R(0),
\end{aligned}$$

hence

$$G(x, y) = F(x - y) + w(x, y)$$

is the Green's function. ■

Corollary 6.8 Let $B_R(0) \subseteq \mathbb{R}^3$ and let $u \in C_1(\overline{B_R(0)}) \cap C_2(B_R(0))$ be a solution of the Dirichlet problem

$$\begin{aligned}\Delta u(x) &= 0, & x \in B_R(0), \\ u(x) &= f(x), & x \in \partial B_R(0).\end{aligned}$$

Then this solution is given by the Poisson representation formula

$$u(x) = \frac{1}{4\pi R} \int_{|y|=R} \frac{R^2 - |x|^2}{|x - y|^3} f(y) dS_y.$$

Proof: If $u \in C_1(\overline{B_R(0)}) \cap C_2(B_R(0))$, then the derivation of the representation formula (6.8) is valid, hence

$$u(x) = - \int_{\partial B_R(0)} \frac{\partial}{\partial n_y} G(x, y) f(y) dS_y. \quad (6.9)$$

Now for $0 < |x| < R$ and $|y| = R$

$$\begin{aligned}\frac{\partial}{\partial n_y} G(x, y) &= \frac{y}{|y|} \cdot \nabla_y \left(\frac{1}{4\pi|x - y|} - \frac{R}{4\pi|x|} \frac{1}{|y - (\frac{R}{|x|})^2 x|} \right) \\ &= -\frac{1}{4\pi|x - y|^2} \frac{(y - x)}{|x - y|} \cdot \frac{y}{|y|} + \frac{R}{4\pi|x|} \frac{1}{|y - (\frac{R}{|x|})^2 x|^2} \frac{y - (\frac{R}{|x|})^2 x}{|y - (\frac{R}{|x|})^2 x|} \cdot \frac{y}{|y|} \\ &= \frac{-1}{4\pi|x - y|^3} (y - x) \cdot \frac{y}{|y|} + \frac{|x|^2}{4\pi R^2} \frac{1}{|x - y|^3} \left(y - (\frac{R}{|x|})^2 x \right) \cdot \frac{y}{|y|} \\ &= \frac{1}{4\pi|x - y|^3} \left(-(y - x) \cdot \frac{y}{|y|} + (\frac{|x|}{R})^2 \left(y - (\frac{R}{|x|})^2 x \right) \cdot \frac{y}{|y|} \right) \\ &= \frac{1}{4\pi|x - y|^3} (-|y| + \frac{|x|^2}{R^2} |y|) = -\frac{1}{4\pi R} \frac{R^2 - |x|^2}{|x - y|^3}.\end{aligned}$$

Insertion into (6.9) yields the formula claimed in the lemma. ■

For $\Omega \subseteq \mathbb{R}^2$ the Green's function to the Dirichlet problem for the potential equation is defined by

$$G(x, y) = -\frac{1}{2\pi} \ln |x - y| + w(x, y),$$

where for every $x \in \Omega$

$$\begin{aligned}\Delta_y w(x, y) &= 0, & y \in \Omega \\ w(x, y) &= \frac{1}{2\pi} \ln |x - y|, & y \in \partial\Omega.\end{aligned}$$

By the same method as in the three-dimensional case one obtains:

Theorem 6.9 (i) *The Green's function for the Dirichlet problem to the potential equation in the circle $B_R(0) \subseteq \mathbb{R}^2$ is*

$$G(x, y) = -\frac{1}{2\pi} \ln \frac{R|x-y|}{|x| \left| y - \left(\frac{R}{|x|}\right)^2 x \right|}.$$

(ii) *Let $u \in C_1(\overline{B_R(0)}) \cap C_2(B_R(0))$ be a solution of*

$$\Delta u(x) = 0, \quad x \in B_R(0), \quad (6.10)$$

$$u(x) = f(x), \quad x \in \partial B_R(0). \quad (6.11)$$

Then u is given by the Poisson representation formula

$$\begin{aligned} u(x) &= \frac{1}{2\pi R} \int_{|y|=R} \frac{R^2 - |x|^2}{|x-y|^2} f(y) ds_y \\ &= \frac{1}{2\pi} \int_0^{2\pi} \frac{(R^2 - \rho^2)}{R^2 - 2R\rho \cos(\varphi - \vartheta) + \rho^2} f(\vartheta) d\vartheta, \end{aligned}$$

where $x = (\rho, \varphi)$, in polar coordinates.

Remark 6.10 Under the assumptions of Theorem 6.9 the boundary data f are continuously differentiable. However, by Theorem 5.5 we know that the Dirichlet problem (6.10), (6.11) in $\mathbb{R}^2$ has a unique solution for every continuous function f . In fact, using Theorem 5.4 it is not difficult to prove by approximation of a given continuous function f by continuously differentiable functions that the Poisson representation formula also holds if f is only continuous.

Up to now we have not shown that the Dirichlet problem for the potential equation in a ball $B_R(0) \subseteq \mathbb{R}^3$ has a solution. Of course, the integral

$$u(x) = \frac{1}{4\pi R} \int_{|y|=R} \frac{R^2 - |x|^2}{|x-y|^3} f(y) dS_y$$

in the Poisson representation formula in $\mathbb{R}^3$ exists for every function $f \in C(\partial B_R(0))$. Therefore one surmises that as in $\mathbb{R}^2$ the solution of the Dirichlet problem in a three-dimensional ball exists and is given by this integral formula if the boundary data are continuous. This is true and can be proved directly by showing that the function u given by the formula is twice continuously differentiable in $B_R(0)$ and satisfies

$$\begin{aligned} \Delta u(x) &= 0, \quad x \in B_R(0), \\ \lim_{\substack{x \rightarrow z \\ x \in B_R(0)}} u(x) &= f(z), \quad \text{for all } z \in \partial B_R(0). \end{aligned}$$

The proof of the first assertion is obvious, but the second assertion is difficult to verify, since the denominator of the integrand in the Poisson representation formula tends to zero if x converges to a boundary point. We do not analyse this boundary behavior here, but investigate a similar integral in Section 7, where we prove a general existence result for the Dirichlet problem in bounded domains in $\mathbb{R}^3$.

6.5 Green's function for the half space

As last example we determine the Green's function for the Dirichlet problem in the half space $H = \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_3 > 0\}$. We have

$$\partial H = \{(x_1, x_2, 0) \mid (x_1, x_2) \in \mathbb{R}^2\} \cong \mathbb{R}^2.$$

To $f \in \mathring{C}(\mathbb{R}^2)$ and $\lambda \in \mathbb{C}$ one wants to find a solution $u \in C_2(H) \cap C(\overline{H})$ of

$$\begin{aligned} \Delta u(x) + \lambda u(x) &= 0, & x \in H \\ u(x', 0) &= f(x'), & x' = (x_1, x_2) \in \mathbb{R}^2. \end{aligned}$$

Lemma 6.11 *The Green's function for the Dirichlet problem in the half space H is given by*

$$G(x, y) = \frac{1}{4\pi} \frac{e^{i\sqrt{\lambda}|x-y|}}{|x-y|} - \frac{1}{4\pi} \frac{e^{i\sqrt{\lambda}|\hat{x}-y|}}{|\hat{x}-y|}$$

where $x = (x_1, x_2, x_3) \in H$, $\hat{x} = (x_1, x_2, -x_3)$, $y \in \overline{H}$ with $x \neq y$.

Proof: Obviously we have

$$(\Delta_y + \lambda) \frac{1}{4\pi} \frac{e^{i\sqrt{\lambda}|\hat{x}-y|}}{|\hat{x}-y|} = 0, \quad x, y \in H, \quad x \neq y.$$

For $y = (y_1, y_2, 0) \in \partial H$ and $x \in H$ we obtain

$$|\hat{x} - y| = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + (-x_3)^2} = |x - y|,$$

hence

$$G(x, y) = \frac{1}{4\pi} \frac{e^{i\sqrt{\lambda}|x-y|}}{|x-y|} - \frac{1}{4\pi} \frac{e^{i\sqrt{\lambda}|x-y|}}{|x-y|} = 0.$$

■

Corollary 6.12 *Let $\lambda \in \mathbb{C} \setminus [0, \infty)$ and let $u \in C_1(\overline{H}) \cap C_2(H)$ be a solution of*

$$\begin{aligned} \Delta u(x) + \lambda u(x) &= 0, & x \in H \\ u|_{\partial H} &= f, \\ |u(x)|, |\nabla u(x)| &\leq C, & x \in H. \end{aligned}$$

Then for $x \in H$

$$u(x) = -\frac{1}{2\pi} \int_{\partial H} \frac{\partial}{\partial n_y} \frac{e^{i\sqrt{\lambda}|x-y|}}{|x-y|} f(y) dy = +\frac{1}{2\pi} \int_{\partial H} \frac{\partial}{\partial y_3} \frac{e^{i\sqrt{\lambda}|x-y|}}{|x-y|} f(y) dy.$$

Remark 6.13 $|u(x)|, |\nabla u(x)| \leq C$ is a condition for the behavior of $u(x)$ and $\nabla u(x)$ if $|x| \rightarrow \infty$.

Proof: Let $x \in H$ and $y \in \overline{H}$ with $|y| \geq 2|x|$. Then

$$|\hat{x} - y| \geq |x - y| \geq |y| - |x| \geq \frac{1}{2}|y|.$$

Thus, since $\operatorname{Re} i\sqrt{\lambda} < 0$, for $x, y \in H$ with $|y| \geq \max(1, 2|x|)$

$$|G(x, y)| \leq \frac{2}{4\pi} \frac{2}{|y|} e^{\frac{1}{2}\operatorname{Re} i\sqrt{\lambda}|y|} \leq C_1 e^{-c|y|},$$

with a suitable constant $C_1 > 0$ and $c = -\frac{1}{2}\operatorname{Re} i\sqrt{\lambda} > 0$. Similarly, for $|y| \geq \max(1, 2|x|)$

$$|\nabla_y G(x, y)| \leq C_2 e^{-c|y|}.$$

Let $\Omega_R = \{x \in H \mid |x| < R\}$. Then u has the representation

$$\begin{aligned} u(x) &= - \int_{\partial\Omega_R} \frac{\partial}{\partial n_y} G(x, y) u(y) - G(x, y) \frac{\partial}{\partial n_y} u(y) dS_y \\ &= - \int_{\substack{y \in \partial H \\ |y| < R}} \frac{\partial}{\partial n_y} G(x, y) u(y) dS_y \\ &\quad - \int_{\substack{|y|=R \\ y_3 > 0}} \frac{\partial}{\partial n_y} G(x, y) u(y) - G(x, y) \frac{\partial}{\partial n_y} u(y) dS_y. \end{aligned}$$

This formula holds for all $R > 0$. Thus,

$$\begin{aligned} u(x) &= \lim_{R \rightarrow \infty} u(x) = - \lim_{R \rightarrow \infty} \int_{\substack{y \in \partial H \\ |y| < R}} \frac{\partial}{\partial n_y} G(x, y) u(y) dS_y \\ &\quad - \lim_{R \rightarrow \infty} \int_{\substack{|y|=R \\ y_3 > 0}} \frac{\partial}{\partial n_y} G(x, y) u(y) - G(x, y) \frac{\partial}{\partial n_y} u(y) dS_y \\ &= - \int_{y \in \partial H} \frac{\partial}{\partial n_y} G(x, y) f(y) dS_y. \end{aligned} \tag{6.12}$$

Here we use that

$$\begin{aligned} &\lim_{R \rightarrow \infty} \left| \int_{\substack{|y|=R \\ y_3 > 0}} \frac{\partial}{\partial n_y} G(x, y) u(y) - G(x, y) \frac{\partial}{\partial n_y} u(y) dS_y \right| \\ &\leq \lim_{R \rightarrow \infty} \int_{\substack{|y|=R \\ y_3 > 0}} (C_2 e^{-cR} C + C_1 e^{-cR} C) dS_y = 0. \end{aligned}$$

Now, for $y = (y_1, y_2, 0) \in \partial H$

$$\begin{aligned} \frac{\partial}{\partial n_y} |\hat{x} - y| &= - \frac{\partial}{\partial y_3} |\hat{x} - y| = - \frac{y_3 + x_3}{|\hat{x} - y|} \Big|_{y_3=0} \\ &= - \frac{x_3}{|x - y|} = \frac{\partial}{\partial y_3} |x - y| = - \frac{\partial}{\partial n_y} |x - y|. \end{aligned}$$

Consequently, for $y \in \partial H$

$$\begin{aligned}
\frac{\partial}{\partial n_y} G(x, y) &= \frac{\partial}{\partial n_y} \frac{e^{i\sqrt{\lambda}|x-y|}}{4\pi|x-y|} - \frac{\partial}{\partial n_y} \frac{e^{i\sqrt{\lambda}|\hat{x}-y|}}{4\pi|\hat{x}-y|} \\
&= \frac{\partial}{\partial n_y} \frac{e^{i\sqrt{\lambda}|x-y|}}{4\pi|x-y|} + \frac{\partial}{\partial n_y} \frac{e^{i\sqrt{\lambda}|x-y|}}{4\pi|x-y|} \\
&= \frac{1}{2\pi} \frac{\partial}{\partial n_y} \frac{e^{i\sqrt{\lambda}|x-y|}}{4\pi|x-y|}.
\end{aligned}$$

Insertion of this expression into the representation formula (6.12) yields the statement of the corollary.

7 Integral equation method

7.1 The boundary integral equations

Let $\Omega \subseteq \mathbb{R}^3$ be a bounded open set. In this section we study the Dirichlet and Neumann boundary value problems for the Helmholtz equation in Ω and in the complement $\mathbb{R}^3 \setminus \overline{\Omega}$ and show that all four problems can be solved uniquely, if the boundary $\partial\Omega$ is sufficiently smooth. The Green's function method cannot be used for this, since for such general domains the Green's functions cannot be determined explicitly. Instead, we use the method of boundary integral equations. To explain the method consider the Dirichlet problem

$$\Delta u(x) + \lambda u(x) = 0, \quad x \in \Omega \quad (7.1)$$

$$u(x) = f(x), \quad x \in \partial\Omega. \quad (7.2)$$

If the boundary $\partial\Omega$ is smooth, then in sufficiently small neighborhoods of every point $x_0 \in \partial\Omega$ the boundary will be almost planar. This suggests to represent the solution of (7.1), (7.2) by using the representation formula derived in Corollary 6.12 for the Dirichlet problem in the half space, for which the boundary is a plane. We thus try to find the solution u in the form

$$u(x) = \frac{1}{2\pi} \int_{\partial\Omega} \frac{\partial}{\partial n_y} \frac{e^{i\sqrt{\lambda}|x-y|}}{|x-y|} v(y) dS_y. \quad (7.3)$$

with a suitable function $v \in C(\partial\Omega, \mathbb{C})$. In this equation u is called *double layer potential*, v is called the *boundary layer*. As will be seen, the double layer potential u satisfies the Helmholtz equation

$$\Delta u(x) + \lambda u(x) = 0$$

for every $x \in \mathbb{R}^3 \setminus \partial\Omega$. To see how v must be chosen, note that if Ω is equal to the half space H and if u is the solution of the Dirichlet problem in H to the boundary condition $u|_{\partial H} = f$, then we know from Corollary 6.12 that u is given by (7.3) with the choice $v = -f$. This means that for every boundary point x_0 we have

$$\lim_{\substack{x \rightarrow x_0 \\ x \in \Omega}} u(x) = f(x_0) = -v(x_0). \quad (7.4)$$

One cannot expect that u given by (7.3) satisfies this simple limit relation also for curved boundaries. Instead, we shall show that for general boundaries $\partial\Omega$ a correction term appears on the right hand side of (7.4). This correction term is given by the *jump relation*

$$\lim_{\substack{x \rightarrow x_0 \\ x \in \Omega}} u(x) = -v(x_0) + \frac{1}{2\pi} \int_{\partial\Omega} \frac{\partial}{\partial n_y} \frac{e^{i\sqrt{\lambda}|x_0-y|}}{|x_0-y|} v(y) dS_y, \quad x_0 \in \partial\Omega. \quad (7.5)$$

From this jump relation we see that u is a solution of the Dirichlet boundary value problem (7.1), (7.2), if the boundary layer v satisfies

$$-v(x) + \frac{1}{2\pi} \int_{\partial\Omega} \frac{\partial}{\partial n_y} \frac{e^{i\sqrt{\lambda}|x-y|}}{|x-y|} v(y) dS_y = f(x), \quad (7.6)$$

for all $x \in \partial\Omega$. This is an integral equation for the unknown function $v \in C(\partial\Omega, \mathbb{C})$ with the given right hand side $f \in C(\partial\Omega, \mathbb{C})$. If a solution v of this integral equation can be determined to a given function f , then the double layer potential u defined in (7.3) with v as the boundary layer is a solution of the Dirichlet boundary value problem to the boundary data f . Thus, if this integral equation is solvable for every $f \in C(\partial\Omega, \mathbb{C})$, the Dirichlet boundary value problem is solvable for all continuous boundary data. Therefore we must study under what conditions the boundary integral equation is solvable.

For $x \in \partial\Omega$ we write

$$(Kv)(x) = \frac{1}{2\pi} \int_{\partial\Omega} \frac{\partial}{\partial n_y} \frac{e^{i\sqrt{\lambda}|x-y|}}{|x-y|} v(y) dS_y. \quad (7.7)$$

With this notation the integral equation (7.6) can be written in the short form

$$(-I + K)v = f, \quad (7.8)$$

where I denotes the identity operator. The solution of the Dirichlet boundary value problem for the Helmholtz equation is thus reduced to the determination of the inverse of the linear operator $-I + K$.

To solve the Neumann boundary value problem

$$\Delta u(x) + \lambda u(x) = 0, \quad x \in \Omega, \quad (7.9)$$

$$\frac{\partial}{\partial n} u(x) = f(x), \quad x \in \partial\Omega, \quad (7.10)$$

we represent the solution by a *single layer potential*

$$u(x) = \frac{1}{2\pi} \int_{\partial\Omega} \frac{e^{i\sqrt{\lambda}|x-y|}}{|x-y|} v(y) dS_y, \quad x \in \Omega. \quad (7.11)$$

The single layer potential satisfies the Helmholtz equation

$$\Delta u(x) + \lambda u(x) = 0$$

for all $x \in \mathbb{R}^3 \setminus \partial\Omega$ and is continuous across the boundary $\partial\Omega$, but the normal derivative satisfies a jump relation. To state this relation, we define for $x \in \partial\Omega$

$$\frac{\partial}{\partial n} u(x+) = \lim_{\substack{s \rightarrow 0 \\ s > 0}} \frac{\partial}{\partial s} u(x + sn_x), \quad \frac{\partial}{\partial n} u(x-) = \lim_{\substack{s \rightarrow 0 \\ s < 0}} \frac{\partial}{\partial s} u(x + sn_x). \quad (7.12)$$

The jump relation is

$$\frac{\partial}{\partial n}u(x\pm) = \mp v(x) + \frac{1}{2\pi} \int_{\partial\Omega} \frac{\partial}{\partial n_x} \frac{e^{i\sqrt{\lambda}|x-y|}}{|x-y|} v(y) dS_y, \quad x \in \partial\Omega, \quad (7.13)$$

where $\frac{\partial}{\partial n_x}$ denotes the derivative with respect to the variable x in the direction of the exterior normal vector n_x at $x \in \partial\Omega$. Note that $\frac{\partial}{\partial n}u(x-)$ is the limit of the normal derivative from the interior of Ω . Therefore the Neumann boundary condition (7.10) is satisfied if the boundary layer v satisfies the integral equation

$$v + K'v = f,$$

with the operator K' defined by

$$(K'v)(x) = \frac{1}{2\pi} \int_{\partial\Omega} \frac{\partial}{\partial n_x} \frac{e^{i\sqrt{\lambda}|x-y|}}{|x-y|} v(y) dS_y. \quad (7.14)$$

Therefore the Neumann problem is solvable if the operator $(I + K') : C(\partial\Omega, \mathbb{C}) \rightarrow C(\partial\Omega, \mathbb{C})$ is invertible.

To determine whether the inverse $(-I + K)^{-1}$ exists we must first study the operator K . It is not obvious that the operator K is well defined, since the denominator of the integral kernel in the expression on the right hand side of (7.7) vanishes at $y = x$; the integrand might thus have a non-integrable singularity. In Section 7.2 we therefore study this integral and show that it exists for every $v \in C(\partial\Omega, \mathbb{C})$; we also show that the mapping $x \mapsto (Kv)(x) : \partial\Omega \rightarrow \mathbb{C}$ is continuous, which implies that

$$K : C(\partial\Omega, \mathbb{C}) \rightarrow C(\partial\Omega, \mathbb{C})$$

is a linear operator. At the end of Section 7.2 we prove the jump relation (7.5). In Section 7.3 we study the single layer potential and verify the jump relation (7.13). In Section 7.4 we shortly review the functional analytic theory of compact operators. With this theory we study in Sections 7.5 and 7.6 the invertibility of the mappings $(I + K')$ and $(-I + K)$ and use the results to solve the Neumann and Dirichlet boundary value problems.

7.2 Properties of the double layer potential

In this section we investigate the behavior of the double layer potential when x lies in $\mathbb{R}^3 \setminus \overline{\Omega}$ but is close to $\partial\Omega$ and when x belongs to $\partial\Omega$. In these investigations we use a technical lemma, which we prove first. Throughout Section 7 we assume that $\Omega \subseteq \mathbb{R}^3$ is a bounded domain with $\partial\Omega \in C_2$ and that $v \in C(\partial\Omega, \mathbb{C})$.

We need some notations and definitions: Let $x_0 \in \partial\Omega$ be an arbitrarily chosen point. We can choose the x_1, x_2, x_3 -coordinate system such that the x_1, x_2 -plane is tangential to $\partial\Omega$ at x_0 and such that the exterior normal vector

n_{x_0} points into the direction of the negative x_3 -axis. For $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ we write $x' = (x_1, x_2)$. By this choice of the coordinate system we have

$$x = (x', x_3), \quad x_0 = (x'_0, 0).$$

Since $\partial\Omega \in C_2$, a parametrization of a neighborhood $V_R = V_R(x_0) \subseteq \partial\Omega$ of x_0 in $\partial\Omega$ is given by

$$y' \mapsto (y', \varphi(y')) : B'_R \subseteq \mathbb{R}^2 \rightarrow V_R,$$

where $B'_R = \{y' \in \mathbb{R}^2 \mid |y' - x'_0| < R\}$, and where $\varphi \in C_2(B'_R, \mathbb{R})$ satisfies

$$|\varphi(y')| \leq M|y' - x'_0|^2, \quad |\nabla\varphi(y')| \leq M'|y' - x'_0|,$$

with suitable constants $M, M' > 0$. For $y \in V_R$ we thus have

$$y = (y', \varphi(y')), \quad y \in V_R.$$

The exterior unit normal vector at $y \in V_R$ is given by

$$n_y = \frac{1}{\sqrt{1 + |\nabla\varphi(y')|^2}} \begin{pmatrix} \nabla\varphi(y') \\ -1 \end{pmatrix} \in \mathbb{R}^3.$$

The constants R, M and M' depend on x_0 , but since $\partial\Omega$ is of class C_2 , they can be chosen independently of $x_0 \in \partial\Omega$.

After these preparations we can formulate the technical lemma:

Lemma 7.1 *Let $\lambda \in \mathbb{C}$. There is a function $g : \{(x, y) \mid x \in \mathbb{R}^3, x' = x'_0, y \in V_R\} \rightarrow \mathbb{C}$ and constants C, c' , which can be chosen independent of $x_0 \in \partial\Omega$, such that*

$$|g(x, y)| \leq Ce^{c'|x-y|}|x-y|^2$$

and such that for every $v \in C(V_R, \mathbb{C})$ and for every $x \in \mathbb{R}^3$ with $x' = x'_0$

$$\int_{V_R} \frac{\partial}{\partial n_y} \frac{e^{i\sqrt{\lambda}|x-y|}}{|x-y|} v(y) dS_y = - \int_{B'_R} \frac{x_3 h(|x-y|)}{|x-y|^3} v(y) dy' + \int_{V_R} \frac{g(x, y)}{|x-y|^3} v(y) dS_y,$$

where $h(r) = e^{i\sqrt{\lambda}r} (1 - i\sqrt{\lambda}r)$.

Proof: We have at $y = (y', \varphi(y')) \in V_R$

$$\begin{aligned} & \sqrt{1 + |\nabla\varphi|^2} \frac{\partial}{\partial n_y} \frac{e^{i\sqrt{\lambda}|x-y|}}{|x-y|} \\ &= \frac{e^{i\sqrt{\lambda}|x-y|}}{|x-y|} \left(-\frac{1}{|x-y|} + i\sqrt{\lambda} \right) \frac{y-x}{|y-x|} \cdot \begin{pmatrix} \nabla\varphi \\ -1 \end{pmatrix} \\ &= \frac{e^{i\sqrt{\lambda}|x-y|} (i\sqrt{\lambda}|x-y| - 1)}{|x-y|^3} (\nabla\varphi \cdot (y' - x') - (\varphi(y') - x_3)) \\ &= -h(|x-y|) \frac{x_3 + \nabla\varphi \cdot (y' - x') - \varphi(y')}{|x-y|^3} \\ &= \frac{-x_3 h(|x-y|) + g_1(x, y)}{|x-y|^3}, \end{aligned}$$

with

$$g_1(x, y) = -h(|x - y|)(\nabla\varphi(y') \cdot (y' - x') - \varphi(y')). \quad (7.15)$$

Thus,

$$\begin{aligned} & \int_{V_R} \frac{\partial}{\partial n_y} \frac{e^{i\sqrt{\lambda}|x-y|}}{|x-y|} v(y) dS_y \\ &= \int_{B'_R} \frac{\partial}{\partial n_y} \frac{e^{i\sqrt{\lambda}|x-y|}}{|x-y|} v(y) \sqrt{1 + |\nabla\varphi(y')|^2} dy' \\ &= - \int_{B'_R} \frac{x_3 h(|x-y|)}{|x-y|^3} v(y) dy' + \int_{B'_R} \frac{g_1(x, y)}{|x-y|^3} v(y) dy' \\ &= - \int_{B'_R} \frac{x_3 h(|x-y|)}{|x-y|^3} v(y) dy' + \int_{V_R} \frac{g(x, y)}{|x-y|^3} v(y) dy, \end{aligned}$$

with $g(x, y) = \frac{g_1(x, y)}{\sqrt{1 + |\nabla\varphi(y')|^2}}$. From (7.15) we obtain

$$|g(x, y)| \leq |g_1(x, y)| \leq |h(|x-y|)| (M'|y' - x'|^2 + M|y' - x'|^2) \leq Ce^{c'|x-y|} |x-y|^2.$$

■

If we now choose $x = x_0 \in \partial\Omega$, hence $x_3 = 0$, we obtain from Lemma 7.1 that

$$\begin{aligned} (Kv)(x) &= \frac{1}{2\pi} \int_{\partial\Omega} \frac{\partial}{\partial n_y} \frac{e^{i\sqrt{\lambda}|x-y|}}{|x-y|} v(y) dS_y \\ &= \frac{1}{2\pi} \int_{V_R} \frac{g(x, y)}{|x-y|^3} v(y) dS_y + \frac{1}{2\pi} \int_{\partial\Omega \setminus V_R} \frac{\partial}{\partial n_y} \frac{e^{i\sqrt{\lambda}|x-y|}}{|x-y|} v(y) dS_y. \end{aligned}$$

Since

$$\left| \frac{g(x, y)}{|x-y|^3} \right| \leq \frac{Ce^{c'|x-y|} |x-y|^2}{|x-y|^3} \leq \frac{Ce^{c'|x-y|}}{|x-y|},$$

it follows that the integral in the double layer potential exists for every $x \in \partial\Omega$, hence it defines a function $Kv : \partial\Omega \rightarrow \mathbb{C}$. The next theorem shows that this function is Hölder continuous.

Theorem 7.2 (i) (Hölder continuity) *There is a constant M such that for all $v \in C(\partial\Omega, \mathbb{C})$ and $x^{(1)}, x^{(2)} \in \partial\Omega$*

$$|(Kv)(x^{(1)}) - (Kv)(x^{(2)})| \leq M |x^{(1)} - x^{(2)}|^{\frac{1}{4}} \|v\|_{\infty}.$$

Hence, the function Kv defined by the double layer potential is Hölder continuous with exponent $1/4$. Here we use the norm $\|v\|_{\infty} = \sup_{x \in \partial\Omega} |v(x)|$.

(ii) *The linear operator $v \mapsto Kv : C(\partial\Omega) \rightarrow C(\partial\Omega)$ is bounded.*

Proof: (i) With $\tau \geq 2|x^{(1)} - x^{(2)}|$ let

$$V_\tau = \{x \in \partial\Omega \mid |x - x^{(1)}| < \tau\}.$$

Then $x^{(2)} \in V_\tau$ with

$$\text{dist}(x^{(2)}, \partial\Omega \setminus V_\tau) = \inf_{y \in \partial\Omega \setminus V_\tau} |x^{(2)} - y| \geq \frac{\tau}{2}. \quad (7.16)$$

By Lemma 7.1 we have

$$\begin{aligned} & (Kv)(x^{(1)}) - (Kv)(x^{(2)}) \\ &= \frac{1}{2\pi} \int_{\partial\Omega \setminus V_\tau} \left(\frac{\partial}{\partial n_y} \frac{e^{i\sqrt{\lambda}|x^{(1)}-y|}}{|x^{(1)}-y|} - \frac{\partial}{\partial n_y} \frac{e^{i\sqrt{\lambda}|x^{(2)}-y|}}{|x^{(2)}-y|} \right) v(y) dy \\ &+ \frac{1}{2\pi} \int_{V_\tau} \frac{g(x^{(1)}, y)}{|x^{(1)}-y|^3} v(y) dS_y - \frac{1}{2\pi} \int_{V_\tau} \frac{g(x^{(2)}, y)}{|x^{(2)}-y|^3} v(y) dS_y \\ &= I_1 + I_{21} + I_{22}. \end{aligned} \quad (7.17)$$

Since the constants C, c' in Lemma 7.1 are independent of $x_0 \in \partial\Omega$, we have that

$$\left| \frac{g(x^{(i)}, y)}{|x^{(i)}-y|^3} \right| \leq \frac{Ce^{c'|x^{(i)}-y|}}{|x^{(i)}-y|} \leq \frac{C'}{|x^{(i)}-y|},$$

for all $x^{(i)}, y \in \partial\Omega$, whence

$$\begin{aligned} |I_{2i}| &\leq \frac{1}{2\pi} \int_{V_\tau} \frac{C'}{|x^{(i)}-y|} \|v\|_\infty dS_y \\ &\leq \frac{1}{2\pi} C' \|v\|_\infty \int_{V_\tau} \frac{1}{|x^{(i)}-y|} dS_y \leq C_1 \|v\|_\infty \tau, \end{aligned} \quad (7.18)$$

with a suitable constant C_1 only depending on $\partial\Omega$. The mean value theorem yields for $y \in \Omega \setminus V_\tau$ that

$$\begin{aligned} & \left| \frac{\partial}{\partial n_y} \frac{e^{i\sqrt{\lambda}|x^{(1)}-y|}}{|x^{(1)}-y|} - \frac{\partial}{\partial n_y} \frac{e^{i\sqrt{\lambda}|x^{(2)}-y|}}{|x^{(2)}-y|} \right| \\ &\leq \left| (x^{(1)} - x^{(2)}) \cdot \nabla_x \left(\frac{\partial}{\partial n_y} \frac{e^{i\sqrt{\lambda}|x-y|}}{|x-y|} \right) \right|_{x=x^*} \\ &\leq |x^{(1)} - x^{(2)}| \frac{C_2}{|x^* - y|^3} \leq \frac{2^3 C_2}{\tau^3} |x^{(1)} - x^{(2)}|, \end{aligned}$$

with x^* on the line segment connecting $x^{(1)}$ and $x^{(2)}$ and with a constant C_2 only depending on $\partial\Omega$. In the last step we used that (7.16) implies $|y - x^*| \geq \tau/2$ for all $y \in \partial\Omega \setminus V_\tau$. Thus

$$|I_1| \leq \frac{1}{2\pi} \int_{\partial\Omega} \frac{2^3 C_2}{\tau^3} |x^{(1)} - x^{(2)}| \|v\|_\infty dS_y \leq C_3 \|v\|_\infty \frac{1}{\tau^3} |x^{(1)} - x^{(2)}|, \quad (7.19)$$

with a constant C_3 only depending on $\partial\Omega$. Choose $\tau = |x^{(1)} - x^{(2)}|^{1/4}$. Then

$$\tau = |x^{(1)} - x^{(2)}|^{1/4} = \frac{1}{|x^{(1)} - x^{(2)}|^{3/4}} |x^{(1)} - x^{(2)}| \geq 2|x^{(1)} - x^{(2)}|,$$

for $|x^{(1)} - x^{(2)}| \leq (\frac{1}{2})^{3/4}$. For such $x^{(1)}, x^{(2)}$ the relations (7.17) – (7.19) are valid. Together these relations yield

$$\begin{aligned} |(Kv)(x^{(1)}) - (Kv)(x^{(2)})| &\leq (2C_1\tau + C_3 \frac{1}{\tau^3} |x^{(1)} - x^{(2)}|) \|v\|_\infty \\ &= (2C_1 + C_3) |x^{(1)} - x^{(2)}|^{1/4} \|v\|_\infty. \end{aligned}$$

This proves (i).

To prove (ii) note that for $x \in \partial\Omega$

$$|(Kv)(x)| = \left| \frac{1}{2\pi} \int_{\partial\Omega} \frac{\partial}{\partial n_y} \frac{e^{i\sqrt{\lambda}|x-y|}}{|x-y|} v(y) dS_y \right| \leq \frac{1}{2\pi} \int_{\partial\Omega} \left| \frac{\partial}{\partial n_y} \frac{e^{i\sqrt{\lambda}|x-y|}}{|x-y|} \right| dS_y \|v\|_\infty,$$

which implies $\|Kv\|_\infty \leq C\|v\|_\infty$ with $C = \frac{1}{2\pi} \int_{\partial\Omega} \left| \frac{\partial}{\partial n_y} \frac{e^{i\sqrt{\lambda}|x-y|}}{|x-y|} \right| dS_y$. ■

Next we prove the jump relation (7.5):

Theorem 7.3 (Jump relations) *Let $\Omega \subseteq \mathbb{R}^3$ be a bounded domain with $\partial\Omega \in C_2$, and let $n_y \in \mathbb{R}^3$ be the exterior unit normal vector to $\partial\Omega$ at $y \in \partial\Omega$. Assume that $\lambda \in \mathbb{C}$ and $v \in C(\partial\Omega, \mathbb{C})$. For $x \in \mathbb{R}^3 \setminus \partial\Omega$ set*

$$w(x) = \frac{1}{2\pi} \int_{\partial\Omega} \frac{\partial}{\partial n_y} \frac{e^{i\sqrt{\lambda}|x-y|}}{|x-y|} v(y) dS_y.$$

(i) *Then $w \in C_\infty(\mathbb{R}^3 \setminus \partial\Omega)$ satisfies*

$$\Delta w(x) + \lambda w(x) = 0, \quad x \in \mathbb{R}^3 \setminus \partial\Omega.$$

(ii) *For $x_0 \in \partial\Omega$ we have the jump relations*

$$\begin{aligned} \lim_{\substack{x \rightarrow x_0 \\ x \in \Omega}} w(x) &= -v(x_0) + \frac{1}{2\pi} \int_{\partial\Omega} \frac{\partial}{\partial n_y} \frac{e^{i\sqrt{\lambda}|x_0-y|}}{|x_0-y|} v(y) dS_y \\ \lim_{\substack{x \rightarrow x_0 \\ x \in \mathbb{R}^3 \setminus \Omega}} w(x) &= v(x_0) + \frac{1}{2\pi} \int_{\partial\Omega} \frac{\partial}{\partial n_y} \frac{e^{i\sqrt{\lambda}|x_0-y|}}{|x_0-y|} v(y) dS_y. \end{aligned}$$

Proof: (i) Since for all $\alpha \in \mathbb{N}_0^3$

$$\left((x, y) \mapsto \frac{\partial^{|\alpha|}}{\partial x^\alpha} \frac{\partial}{\partial n_y} \frac{e^{i\sqrt{\lambda}|x-y|}}{|x-y|} \right) \in C((\mathbb{R}^3 \setminus \partial\Omega) \times \partial\Omega, \mathbb{C}),$$

it follows as usual that $w \in C_\infty(\mathbb{R}^3 \setminus \partial\Omega, \mathbb{C})$ with

$$\frac{\partial^{|\alpha|}}{\partial x^\alpha} w(x) = \frac{1}{2\pi} \int_{\partial\Omega} \frac{\partial}{\partial n_y} \frac{\partial^{|\alpha|}}{\partial x^\alpha} \frac{e^{i\sqrt{\lambda}|x-y|}}{|x-y|} v(y) dS_y$$

for $x \in \mathbb{R}^3 \setminus \partial\Omega$. In particular, this implies

$$(\Delta + \lambda)w(x) = \frac{1}{2\pi} \int_{\partial\Omega} \frac{\partial}{\partial n_y} (\Delta_x + \lambda) \frac{e^{i\sqrt{\lambda}|x-y|}}{|x-y|} v(y) dS_y = 0,$$

since $\frac{1}{4\pi} \frac{e^{i\sqrt{\lambda}|x-y|}}{|x-y|}$ is the fundamental solution of the Helmholtz equation. This proves (i).

To prove (ii) assume that $x \in \Omega$ is a point on the line normal to $\partial\Omega$ at x_0 , hence $x = (x', x_3)$, $x_3 > 0$, $x_0 = (x'_0, 0)$ and $x' = x'_0$. Lemma 7.1 yields

$$w(x) + v(x_0) - \frac{1}{2\pi} \int_{\partial\Omega} \frac{\partial}{\partial n_y} \frac{e^{i\sqrt{\lambda}|x_0-y|}}{|x_0-y|} v(y) dS_y = I_1 + I_2 + I_3, \quad (7.20)$$

with

$$\begin{aligned} I_1 &= v(x_0) - \frac{1}{2\pi} \int_{B'_R} \frac{x_3}{|x-y|^3} h(|x-y|) v(y) dz, \\ I_2 &= \frac{1}{2\pi} \int_{V_R} \left(\frac{g(x, y)}{|x-y|^3} - \frac{g(x_0, y)}{|x_0-y|^3} \right) v(y) dz, \\ I_3 &= \frac{1}{2\pi} \int_{\partial\Omega \setminus V_R} \left(\frac{\partial}{\partial n_y} \frac{e^{i\sqrt{\lambda}|x-y|}}{|x-y|} - \frac{\partial}{\partial n_y} \frac{e^{i\sqrt{\lambda}|x_0-y|}}{|x_0-y|} \right) v(y) dS_y. \end{aligned}$$

In the following lemmas we derive estimates for $|I_1|, |I_2|, |I_3|$.

Lemma 7.4 *To every $\varepsilon > 0$ there is $\delta_1 = \delta_1(R) > 0$ such that*

$$|I_3| < \varepsilon$$

for all $0 < x_3 < \delta_1$.

Proof: For $x_3 \rightarrow 0$ it follows that $x \rightarrow x_0$, hence the integrand of I_3 tends to zero uniformly in $\partial\Omega \setminus V_R$. This yields the statement.

Lemma 7.5 *To $\varepsilon > 0$ there is $R_1 > 0$ such that for all $R \leq R_1$ and all $1 \geq x_3 > 0$*

$$|I_2| < \varepsilon.$$

Proof: By Lemma 7.1 we have

$$|I_2| \leq \frac{\|v\|_\infty}{2\pi} \int_{V_R} \frac{Ce^{c'|x-y|}}{|x-y|} + \frac{Ce^{c'|x_0-y|}}{|x_0-y|} dS_y \leq \frac{C'\|v\|_\infty}{\pi} \int_{V_R} \frac{1}{|x_0-y|} dS_y < \varepsilon,$$

for $R \leq R_1$ with R_1 sufficiently small. ■

Lemma 7.6 For $y = (z, \varphi(z))$ with $z \in B_R$ we have

$$\begin{aligned} \left| \frac{1}{|x-y|^3} - \frac{1}{|x-(z,0)|^3} \right| &\leq 3M(1+MR)^3 \frac{1}{|x-(z,0)|^2} \\ &\leq 3M(1+MR)^3 \frac{(R^2+x_3^2)^{\frac{1}{2}}}{|x-(z,0)|^3}. \end{aligned}$$

Proof: Set $r_1 = |x-(z,0)|$, $r_2 = |x-y|$, $r = |x'-z|$. Since

$$x-y = x-(z,0) - (0, \varphi(z)),$$

the inverse triangle inequality yields

$$|r_1 - r_2| \leq |(0, \varphi(z))| \leq Mr^2,$$

and the triangle inequality implies

$$\begin{aligned} \frac{r_2}{r_1} &\leq \frac{r_1 + |\varphi(z)|}{r_1} \leq 1 + \frac{|\varphi(z)|}{r} \leq 1 + Mr, \\ \frac{r}{r_2} &\leq \frac{r_1}{r_2} \leq \frac{r_2 + |\varphi(z)|}{r_2} \leq 1 + \frac{|\varphi(z)|}{r} \leq 1 + Mr. \end{aligned}$$

Thus,

$$\begin{aligned} \left| \frac{1}{r_2^3} - \frac{1}{r_1^3} \right| &\leq \left| \frac{(r_1 - r_2)(r_1^2 + r_1 r_2 + r_2^2)}{r_1^3 r_2^3} \right| \leq \frac{Mr^2}{r_2^2} \left(\frac{r_1}{r_2} + 1 + \frac{r_2}{r_1} \right) \frac{1}{r_1^2} \\ &\leq M(1+Mr)^2(3+2Mr) \frac{1}{r_1^2} \leq 3M(1+MR)^3 \frac{1}{r_1^2}. \end{aligned}$$

The statement follows from this inequality and from $r_1^2 = r^2 + x_3^2 \leq R^2 + x_3^2$.

Lemma 7.7

$$\frac{1}{2\pi} \int_{B'_R} \frac{x_3}{|x-(z,0)|^3} dz = 1 - \frac{x_3}{\sqrt{x_3^2 + R^2}} \leq 1.$$

Proof: We use polar coordinates (r, θ) with the origin at $x'_0 = x'$ and note again that $|x-(z,0)|^2 = r_1^2 = x_3^2 + r^2$ to conclude

$$\begin{aligned} \frac{1}{2\pi} \int_{B'_R} \frac{x_3}{|x-(z,0)|^3} dz &= \frac{1}{2\pi} \int_0^{2\pi} \int_0^R \frac{x_3}{\sqrt{x_3^2 + r^2}^3} r dr d\theta \\ &= -x_3 \frac{1}{\sqrt{x_3^2 + r^2}} \Big|_{r=0}^{r=R} = 1 - \frac{x_3}{\sqrt{x_3^2 + R^2}}. \end{aligned}$$

Lemma 7.8 *To $\varepsilon > 0$ there is a constant $R_2 > 0$ such that for all $0 < R \leq R_2$ there is $\delta_2 = \delta_2(R) > 0$ such that*

$$|I_1| \leq 3\varepsilon$$

holds for all $x_3 < \delta_2$.

Proof: Note that $h(0) = 1$, which means that the continuous function

$$(z, x_3) \rightarrow v(x_0) - h(|(x', x_3) - (z, \varphi(z))|)v(z, \varphi(z)) = v(x_0) - h(|x - y|)v(y)$$

has a zero at $(z, x_3) = (x', 0) = x_0$. Consequently there are $R_2 > 0, \delta_2 > 0$ such that for $|x' - z| < R_2$ and $0 < x_3 < \delta_2$

$$|v(x_0) - h(|x - y|)v(y)| < \varepsilon.$$

For $R \leq R_2$ and $x_3 < \delta_2$ we thus obtain from Lemma 7.6 and 7.7 that

$$\begin{aligned} |I_1| &= \left| v(x_0) \left(\frac{1}{2\pi} \int_{B'_R} \frac{x_3}{|x - (z, 0)|^3} dz + \frac{x_3}{\sqrt{x_3^2 + R^2}} \right) \right. \\ &\quad \left. - \frac{1}{2\pi} \int_{B'_R} \frac{x_3}{|x - y|^3} h(|x - y|)v(y) dz \right| \\ &= \left| \frac{1}{2\pi} \int_{B'_R} \frac{x_3}{|x - (z, 0)|^3} (v(x_0) - h(|x - y|)v(y)) dz \right. \\ &\quad \left. + \frac{1}{2\pi} \int_{B'_R} \left(\frac{x_3}{|x - (z, 0)|^3} - \frac{x_3}{|x - y|^3} \right) h(|x - y|)v(y) dz + v(x_0) \frac{x_3}{\sqrt{x_3^2 + R^2}} \right| \\ &\leq \varepsilon \frac{1}{2\pi} \int_{B'_R} \frac{x_3}{|x - (z, 0)|^3} dz \\ &\quad + \|hv\|_\infty 3M(1 + MR)^3 (R^2 + x_3^2)^{\frac{1}{2}} \frac{1}{2\pi} \int_{B'_R} \frac{x_3}{|x - (z, 0)|^3} dz \\ &\quad + |v(x_0)| \frac{x_3}{\sqrt{x_3^2 + R^2}} \tag{7.21} \\ &\leq \varepsilon + \|hv\|_\infty 3M(1 + MR)^3 R \sqrt{1 + \left(\frac{\delta_2}{R}\right)^2} + |v(x_0)| \frac{\delta_2}{R}, \end{aligned}$$

where we used the notation

$$\|hv\|_\infty = \sup_{\substack{y \in \partial\Omega \\ x \in \Omega}} |h(|x - y|)v(y)|.$$

Now choose $R_2 > 0$ small enough such that $\|hv\|_\infty 3M(1 + MR_2)^3 R_2(1 + \varepsilon^2)^{\frac{1}{2}} < \varepsilon$. Subsequently, to $0 < R < R_2$ choose $\delta_2 = \delta_2(R)$ small enough such that $\frac{\delta_2}{R}(1 + |v(x_0)|) < \varepsilon$. From (7.21) we then obtain $|I_1| < 3\varepsilon$ for $R \leq R_2$ and $0 < x_3 < \delta_2$. $\blacksquare$

End of the proof of the Theorem 7.3: To $\varepsilon > 0$ set

$$R = \min\{R_1, R_2\}, \quad \delta = \min\{\delta_1(R), \delta_2(R)\}, \quad (7.22)$$

where $R_1, R_2, \delta_1, \delta_2$ are the numbers given in Lemmas 7.4, 7.5 and 7.8. From these lemmas and from (7.20) we then obtain for all $0 < x_3 < \delta$ and for

$$\hat{v}(x_0) = -v(x_0) + \frac{1}{2\pi} \int_{\partial\Omega} \frac{\partial}{\partial n_y} \frac{e^{i\sqrt{\lambda}|x_0-y|}}{|x_0-y|} v(y) dS_y$$

that

$$|w(x) - \hat{v}(x_0)| \leq |I_1| + |I_2| + |I_3| < 5\varepsilon,$$

whence

$$\lim_{\substack{x_3 \rightarrow 0 \\ x_3 > 0}} w(x', x_3) = \lim_{\substack{s \rightarrow 0 \\ s > 0}} w(x_0 - sn_{x_0}) = \hat{v}(x_0). \quad (7.23)$$

It remains to show that this limit also holds if x approaches x_0 not along the line normal to the boundary. To this end we note that the limit (7.23) is uniform with respect to $x_0 \in \partial\Omega$, since the numbers $R_1, R_2, \delta_1, \delta_2$ in (7.22) can be chosen independently of x_0 , which is seen by examination of the proof. To $\varepsilon > 0$ we can therefore choose $\delta_3 > 0$ such that $|w(z - sn_z) - \hat{v}(z)| < \varepsilon$, for all $z \in \partial\Omega$ and all $0 < s < \delta_3$. By Theorem 7.2 we have that $\hat{v} \in C(\partial\Omega)$. Consequently there is $\delta_4 > 0$ such that $|\hat{v}(z) - \hat{v}(x_0)| < \varepsilon$ for all $z \in \partial\Omega$ with $|z - x_0| < \delta_4$. By these estimates we obtain for all points $z - sn_z$ from the neighborhood

$$\mathcal{U}(x_0) = \{z - sn_z \mid z \in \partial\Omega, |z - x_0| < \delta_4, 0 < s < \delta_3\} \cap \Omega$$

of x_0 in Ω that

$$|w(z - sn_z) - \hat{v}(x_0)| \leq |w(z - sn_z) - \hat{v}(z)| + |\hat{v}(z) - \hat{v}(x_0)| < 2\varepsilon.$$

This means that

$$\lim_{\substack{x \rightarrow x_0 \\ x \in \Omega}} w(x) = \hat{v}(x_0),$$

which proves the first jump relation in statement (ii). The second jump relation is proved analogously. ■

7.3 Properties of the single layer potential

We next study the single layer potential (7.11). We assume that $\Omega \subseteq \mathbb{R}^3$ is a bounded open set with $\partial\Omega \in C_2$ and that $\lambda \in \mathbb{C}$. Just as in the previous section it can be shown that to $v \in C(\partial\Omega, \mathbb{C})$ there are constants C, c' such that for all $x, y \in \partial\Omega$ the estimate

$$\left| \frac{\partial}{\partial n_x} \frac{e^{i\sqrt{\lambda}|x-y|}}{|x-y|} \right| \leq \frac{C e^{c'|x-y|}}{|x-y|}$$

holds. It follows that for every $v \in C(\partial\Omega, \mathbb{C})$ and all $x \in \partial\Omega$ the integral in (7.14) exists. (7.14) thus defines a function $(K'v) : \partial\Omega \rightarrow \mathbb{C}$.

Theorem 7.9 (i) *There is a constant M such that for all $v \in C(\partial\Omega, \mathbb{C})$ and $x^{(1)}, x^{(2)} \in \partial\Omega$*

$$|(K'v)(x^{(1)}) - (K'v)(x^{(2)})| \leq M|x^{(1)} - x^{(2)}|^{\frac{1}{4}} \|v\|_{\infty}.$$

(ii) *The linear operator $v \mapsto K'v : C(\partial\Omega) \rightarrow C(\partial\Omega)$ is bounded.*

The **proof** is similar to the proof of Theorem 7.2. Therefore we omit it.

Theorem 7.10 *Assume that $v \in C(\partial\Omega, \mathbb{C})$. For $x \in \mathbb{R}^3 \setminus \partial\Omega$ set*

$$\omega(x) = \frac{1}{2\pi} \int_{\partial\Omega} \frac{e^{i\sqrt{\lambda}|x-y|}}{|x-y|} v(y) dS_y.$$

Then

(i) *ω belongs to $C(\mathbb{R}^3, \mathbb{C}) \cap C_{\infty}(\mathbb{R}^3 \setminus \partial\Omega, \mathbb{C})$ and satisfies*

$$(\Delta + \lambda)\omega(x) = 0, \quad x \in \mathbb{R}^3 \setminus \partial\Omega.$$

(ii) *At $x \in \partial\Omega$ the one sided derivatives $\frac{\partial\omega}{\partial n}(x\pm)$ defined in (7.12) exist and satisfy*

$$\frac{\partial\omega}{\partial n}(x\pm) = \mp v(x) + \frac{1}{2\pi} \int_{\partial\Omega} \frac{\partial}{\partial n_x} \frac{e^{i\sqrt{\lambda}|x-y|}}{|x-y|} v(y) dS_y.$$

To prove this theorem we need a lemma.

Lemma 7.11 *For $x \in \partial\Omega$ and $s \in \mathbb{R}$ let $x_s = x + sn_x$. Then*

$$\begin{aligned} \lim_{s \rightarrow 0} \int_{\partial\Omega} \left(\frac{\partial}{\partial s} \frac{e^{i\sqrt{\lambda}|x_s-y|}}{|x_s-y|} + \frac{\partial}{\partial n_y} \frac{e^{i\sqrt{\lambda}|x_s-y|}}{|x_s-y|} \right) v(y) dS_y \\ = \int_{\partial\Omega} \left(\frac{\partial}{\partial n_x} \frac{e^{i\sqrt{\lambda}|x-y|}}{|x-y|} + \frac{\partial}{\partial n_y} \frac{e^{i\sqrt{\lambda}|x-y|}}{|x-y|} \right) v(y) dS_y. \end{aligned} \quad (7.24)$$

Proof: Since

$$\lim_{s \rightarrow 0} \frac{\partial}{\partial s} \frac{e^{i\sqrt{\lambda}|x_s-y|}}{|x_s-y|} = \frac{\partial}{\partial n_x} \frac{e^{i\sqrt{\lambda}|x-y|}}{|x-y|},$$

for all $y \in \partial\Omega \setminus \{x\}$, it suffices to show that the limit can be interchanged with the integral. To verify this we construct a majorant for the integrand on the left hand side of (7.24), which is independent of s . The result is then implied by Lebesgue's integration theorem.

Note first that to every r_0 there is a constant c_1 such that $|\frac{\partial}{\partial r} \frac{e^{i\sqrt{\lambda}r}}{r}| \leq \frac{c_1}{r^2}$ for all $0 < r \leq r_0$. Together with this estimate it thus follows for all $s \in \mathbb{R}$ and all $y \in \partial\Omega$ with $y \neq x$ that

$$\begin{aligned} \left| \frac{\partial}{\partial s} \frac{e^{i\sqrt{\lambda}|x+sn_x-y|}}{|x+sn_x-y|} + \frac{\partial}{\partial n_y} \frac{e^{i\sqrt{\lambda}|x_s-y|}}{|x_s-y|} \right| \\ = \left| \left[\frac{\partial}{\partial r} \frac{e^{i\sqrt{\lambda}r}}{r} \right]_{r=|x_s-y|} \frac{(x_s - y) \cdot (n_x - n_y)}{|x_s - y|} \right| \leq \frac{c_1}{|x_s - y|^2} |n_x - n_y|. \end{aligned} \quad (7.25)$$

Since $\partial\Omega$ is bounded and of class C_2 , there is a constant c_2 such that for all $y \in \partial\Omega$

$$|n_x - n_y| \leq c_2|x - y|. \quad (7.26)$$

We choose $s_0 > 0$ small enough such that the line segment $\{x + sn_x \mid |s| \leq s_0\}$ intersects $\partial\Omega$ only in the point x . It then follows by standard considerations that there is a constant $c_3 > 0$ such that for all $|s| \leq s_0$ we have

$$|x - y| \leq c_3|x_s - y|.$$

From this estimate and from (7.25), (7.26) we conclude that

$$\left| \frac{\partial}{\partial s} \frac{e^{i\sqrt{\lambda}|x+sn_x-y|}}{|x+sn_x-y|} + \frac{\partial}{\partial n_y} \frac{e^{i\sqrt{\lambda}|x_s-y|}}{|x_s-y|} \right| \leq \frac{c_1 c_2 |x-y|}{c_3^{-1}|x-y|^2} = \frac{C}{|x-y|},$$

with the constant $C = c_1 c_2 c_3$ independent of s . The function $\frac{C}{|x-y|}$ is integrable over the two-dimensional manifold $\partial\Omega$, hence it is a majorant for the integrand on the left hand side of (7.24). $\blacksquare$

Proof of Theorem 7.10: The proof of (i) is standard and we omit it. In the proof of (ii) we restrict ourselves to the verification of the formula for $\frac{\partial\omega}{\partial n}(x-)$. The other formula is proved in the same way.

From Lemma 7.11 and from the jump relations in Theorem 7.3(ii) we conclude that

$$\begin{aligned} \frac{\partial\omega}{\partial n}(x-) &= \lim_{\substack{s \rightarrow 0 \\ s < 0}} \frac{1}{2\pi} \int_{\partial\Omega} \frac{\partial}{\partial s} \frac{e^{i\sqrt{\lambda}|x_s-y|}}{|x_s-y|} v(y) dS_y \\ &= \lim_{\substack{s \rightarrow 0 \\ s < 0}} \frac{1}{2\pi} \int_{\partial\Omega} \left(\frac{\partial}{\partial s} \frac{e^{i\sqrt{\lambda}|x_s-y|}}{|x_s-y|} + \frac{\partial}{\partial n_y} \frac{e^{i\sqrt{\lambda}|x_s-y|}}{|x_s-y|} \right) v(y) dS_y \\ &\quad - \lim_{\substack{s \rightarrow 0 \\ s < 0}} \frac{1}{2\pi} \int_{\partial\Omega} \frac{\partial}{\partial n_y} \frac{e^{i\sqrt{\lambda}|x_s-y|}}{|x_s-y|} v(y) dS_y \\ &= \frac{1}{2\pi} \int_{\partial\Omega} \left(\frac{\partial}{\partial n_x} \frac{e^{i\sqrt{\lambda}|x-y|}}{|x-y|} + \frac{\partial}{\partial n_y} \frac{e^{i\sqrt{\lambda}|x-y|}}{|x-y|} \right) v(y) dS_y \\ &\quad + v(x) - \frac{1}{2\pi} \int_{\partial\Omega} \frac{\partial}{\partial n_y} \frac{e^{i\sqrt{\lambda}|x-y|}}{|x-y|} v(y) dS_y \\ &= v(x) + \frac{1}{2\pi} \int_{\partial\Omega} \frac{\partial}{\partial n_x} \frac{e^{i\sqrt{\lambda}|x-y|}}{|x-y|} v(y) dS_y. \end{aligned}$$

$\blacksquare$

7.4 Compact operators on a Banach space

To study the solvability of the boundary integral equations we use a result from functional analysis, which we present here. In the following X denotes a Banach space with norm $\|\cdot\|$.

Definition 7.12 A linear operator $T : X \rightarrow X$ is called bounded if there is a constant C such that

$$\|Tx\| \leq C\|x\|$$

for all $x \in X$.

Theorem 7.13 A linear operator $T : X \rightarrow X$ is bounded if and only if it is continuous.

Proof: If T is continuous at 0 it follows that there is $\delta > 0$ such that

$$\|Tx\| \leq 1$$

for all $x \in X$ with $\|x\| \leq \delta$. Since for every $y \in X$, $y \neq 0$ we have

$$\|\delta \frac{y}{\|y\|}\| = \delta \frac{\|y\|}{\|y\|} = \delta,$$

it follows that

$$\|Ty\| = \|T(\frac{\|y\|}{\delta} \delta \frac{y}{\|y\|})\| = \frac{\|y\|}{\delta} \|T(\delta \frac{y}{\|y\|})\| \leq \frac{1}{\delta} \|y\|.$$

This proves that T is bounded.

On the other hand, assume that T satisfies

$$\|Tx\| \leq C\|x\|$$

for all $x \in X$. Let $y \in X$, $\varepsilon > 0$, and set $\delta = \frac{\varepsilon}{C}$. Then for all $z \in X$ with $\|z - y\| \leq \delta$ it follows

$$\|T(z) - T(y)\| = \|T(z - y)\| \leq C\|z - y\| \leq \varepsilon,$$

hence T is continuous at y . Since y was arbitrary, T is continuous on X . ■

Definition 7.14 A linear operator $T : X \rightarrow X$ is called compact if to every bounded sequence $\{x_n\}_n \subseteq X$ the sequence of images $\{Tx_n\}_n$ has a subsequence, which converges in X .

Lemma 7.15 A compact operator is bounded.

Proof: If the compact operator T would not be bounded then there would exist a sequence $\{x_n\}_n \subseteq X$ with $\|x_n\| = 1$ and $\|Tx_n\| \geq n$, for all $n \in \mathbb{N}$. The sequence $\{Tx_n\}_n$ would not have a convergent subsequence, hence T is not compact. ■

Remember that for a linear operator $T : X \rightarrow X$ a number $\lambda \in \mathbb{C}$ with the property that there is $x \in X$, $x \neq 0$ satisfying $Tx - \lambda x = 0$ is called eigenvalue of T . The element x is called eigenvector. The set

$$E = \{x \in X \mid Tx - \lambda x = 0\}$$

is a linear subspace of X called eigenspace of the eigenvalue λ . The dimension of E is called the multiplicity of λ .

Definition 7.16 Let $T : X \rightarrow X$ be a bounded operator. The resolvent set $\rho(T)$ of T consists of all points $\lambda \in \mathbb{C}$, which are not eigenvalues and for which the operator $(T - \lambda I) : X \rightarrow X$ is surjective. Here I is the identity. The complement $\Sigma(T) = \mathbb{C} \setminus \rho(T)$ is called spectrum of T .

Clearly, $\lambda \in \rho(T)$ if and only if $T - \lambda I$ is injective and surjective. Hence λ belongs to the resolvent set if and only if $(T - \lambda I)^{-1}$ exists.

Theorem 7.17 Let $T : X \rightarrow X$ be compact. $\Sigma(T)$ is a countable set with no accumulation point different from zero. Each nonzero $\lambda \in \Sigma(T)$ is an eigenvalue of T with finite multiplicity. If X has infinite dimension, then 0 belongs to $\Sigma(T)$.

I only give part of the proof. The complete proof can be found for example in the book of Alt, pp. 363.

Proof: I.) First I show that the eigenvalues of T do not accumulate at a point $\lambda \neq 0$. Otherwise there would exist a sequence $\{\lambda_n\}_n$ of distinct eigenvalues of T with eigenvectors x_n such that $0 \neq \lambda_n \rightarrow \lambda \neq 0$. Let M_n be the subspace spanned by the n vectors $x_1, \dots, x_n$. The space M_n is invariant under T ; for if $x \in M_n$ then $x = c_1x_1 + \dots + c_nx_n$, hence

$$Tx = T(c_1x_1 + \dots + c_nx_n) = \lambda_1c_1x_1 + \dots + \lambda_nc_nx_n \in M_n,$$

thus $T(M_n) \subseteq M_n$.

Since eigenvectors to distinct eigenvalues are linearly independent, the vectors $x_1, x_2, \dots$ are linearly independent. Therefore M_{n-1} is a proper subspace of M_n and there is $y_n \in M_n$ such that $\|y_n\| = 1$ and $\text{dist}(y_n, M_{n-1}) = 1$. This holds since M_n is isomorphic to $\mathbb{R}^n$. With the sequence $\{y_n\}_n$ thus defined I show that $\{\lambda_n^{-1}Ty_n\}_n$ contains no Cauchy sequence, contradicting the assumption that T is compact. (Note that $\{\lambda_n^{-1}y_n\}_n$ is a bounded sequence.) We have for $m < n$

$$\lambda_n^{-1}Ty_n - \lambda_m^{-1}Ty_m = y_n - (\lambda_m^{-1}Ty_m - \lambda_n^{-1}(T - \lambda_n)y_n)$$

where the second term on the right belongs to M_{n-1} because $y_m \in M_{n-1}$, M_{n-1} is invariant under T and $(T - \lambda_n)y_n \in M_{n-1}$. Since $\text{dist}(y_n, M_{n-1}) = 1$, it follows that each element of the sequence $\{\lambda_n^{-1}Ty_n\}_n$ has distance ≥ 1 from any other one, showing that no subsequence of this sequence can be convergent.

II.) If there would be an eigenvalue $\lambda \neq 0$ of infinite multiplicity we could derive a contradiction by exactly the same arguments, defining λ_n by $\lambda_n = \lambda$ for all n and choosing for $\{x_n\}_n$ a sequence of linearly independent eigenvectors to λ .

It remains to show that if $\lambda \neq 0$ is not an eigenvalue it belongs to $\rho(T)$, hence the range $R(T - \lambda I)$ is equal to X . To this end it is shown that $R(T - \lambda I)$ is closed and that there is no nontrivial complementary space. For the details I refer to the book of Alt.

This theorem shows that if T is a compact operator, then for given $h \in X$ the equation $(\pm I + T)x = h$ is solvable if ∓ 1 is not an eigenvalue of T .

7.5 Solution of the Neumann problem

I use Theorem 7.17 to show that the integral equations $(I + K')v = f$ and $(-I + K')v = f$ can be solved, which implies that the interior and exterior Neumann boundary value problems have solutions. To this end I show that K' is a compact operator on the Banach space $C(\partial\Omega, \mathbb{C})$. The norm on this Banach space is

$$\|v\|_\infty = \sup_{x \in \partial\Omega} |v(x)|.$$

Definition 7.18 Let $\Gamma \subseteq \mathbb{R}^m$. A sequence $\{v_n\}_n$ of functions $v_n : \Gamma \rightarrow \mathbb{C}$ is called uniformly equicontinuous if to every $\varepsilon > 0$ there is $\delta > 0$ such that

$$|v_n(x) - v_n(y)| < \varepsilon$$

for all $n \in \mathbb{N}$ and all $x, y \in \Gamma$ with $|x - y| < \delta$.

Theorem 7.19 (Arzela-Ascoli)² Let $\{v_n\}_n$ be a bounded, uniformly equicontinuous sequence of functions on Γ . Then there is a uniformly convergent subsequence $\{v_{n_\ell}\}_\ell$.

Corollary 7.20 The operator $K' : C(\partial\Omega, \mathbb{C}) \rightarrow C(\partial\Omega, \mathbb{C})$ is compact.

Proof: Let $\{v_n\}_n$ be a bounded sequence in $C(\partial\Omega, \mathbb{C})$. By Theorem 7.9(ii) the operator K' is bounded, which implies that also the sequence $\{K'v_n\}_n$ is bounded in $C(\partial\Omega, \mathbb{C})$. Let $\varepsilon > 0$, set $C = \sup \|v_n\|_\infty$ and let $M > 0$ be the constant from Theorem 7.9(i). This theorem implies for all $x_1, x_2 \in \partial\Omega$ with

$$|x_1 - x_2| < \delta = \left(\frac{\varepsilon}{CM}\right)^4$$

that

$$|(K'v_n)(x_1) - (K'v_n)(x_2)| \leq M|x_1 - x_2|^{1/4} \|v_n\|_\infty \leq MC\left(\frac{\varepsilon}{CM}\right) = \varepsilon.$$

Thus, $\{K'v_n\}_n$ is a uniformly equicontinuous sequence. Therefore all the assumptions of Theorem 7.19 are satisfied for this sequence, from which we conclude that it has a subsequence converging with respect to the norm $\|\cdot\|_\infty$ of $C(\partial\Omega, \mathbb{C})$. This means that K' is compact. ■

Lemma 7.21 Let $\lambda \in \mathbb{C} \setminus [0, \infty)$. Then 1 and -1 are no eigenvalues of K' .

²Cesare Arzelà (1847–1912), Giulio Ascoli (1843–1896).

Proof: We prove first that -1 is not an eigenvalue. To this end it suffices to show if $v \in C(\partial\Omega, \mathbb{C})$ satisfies

$$(I + K')v = 0, \quad (7.27)$$

then $v = 0$, since this implies that the kernel of $K' - (-1)I$ is equal to $\{0\}$. To verify that v vanishes, we define for $x \in \mathbb{R}^3$ the single layer potential

$$u(x) = \frac{1}{2\pi} \int_{\partial\Omega} \frac{e^{i\sqrt{\lambda}|x-y|}}{|x-y|} v(y) dS_y. \quad (7.28)$$

Equation (7.27) and the jump relations from Theorem 7.10(ii) imply that u solves the boundary value problem

$$\begin{aligned} \Delta u(x) + \lambda u(x) &= 0, & x \in \Omega, \\ \frac{\partial}{\partial n} u(x) &= 0, & x \in \partial\Omega. \end{aligned}$$

The first Green's formula yields

$$\begin{aligned} 0 &= \int_{\partial\Omega} \frac{\partial}{\partial n} u(x) \overline{u(x)} dS = \int_{\Omega} \Delta u(x) \overline{u(x)} + \nabla u(x) \cdot \overline{\nabla u(x)} dS \\ &= \int_{\Omega} (-\lambda |u(x)|^2 + |\nabla u(x)|^2) dx \\ &= -i \operatorname{Im} \lambda \int_{\Omega} |u(x)|^2 dx + \int_{\Omega} |\nabla u(x)|^2 - \operatorname{Re} \lambda |u(x)|^2 dx. \end{aligned}$$

Since $\operatorname{Im} \lambda \neq 0$ or $\operatorname{Re} \lambda < 0$ it follows from this equation that $u \equiv 0$ in Ω . Since the single layer potential u is continuous on $\mathbb{R}^3$ it thus follows that u is also a solution of the boundary value problem

$$\begin{aligned} \Delta u(x) + \lambda u(x) &= 0, & x \in \mathbb{R}^3 \setminus \overline{\Omega} \\ u(x) &= 0, & x \in \partial(\mathbb{R}^3 \setminus \overline{\Omega}). \end{aligned}$$

To apply the Green's formula in $\mathbb{R}^3 \setminus \overline{\Omega}$ note that for $\lambda \in \mathbb{C} \setminus [0, \infty)$ we have by our choice of the square root that $\operatorname{Re} i\sqrt{\lambda} < 0$, whence

$$|u(x)| \leq \frac{1}{2\pi} \int_{\partial\Omega} \frac{e^{\operatorname{Re} i\sqrt{\lambda}|x-y|}}{|x-y|} |v(y)| dS_y \leq \frac{e^{\operatorname{Re} i\sqrt{\lambda} \operatorname{dist}(x, \partial\Omega)}}{\operatorname{dist}(x, \partial\Omega)} \frac{1}{2\pi} \int_{\partial\Omega} |v(y)| dS_y.$$

Therefore $|u(x)|$ decreases exponentially for $|x| \rightarrow \infty$. Because

$$\nabla u(x) = \frac{1}{2\pi} \int_{\partial\Omega} \nabla_x \frac{e^{i\sqrt{\lambda}|x-y|}}{|x-y|} v(y) dS_y,$$

it follows in the same way that also $|\nabla u(x)|$ decreases exponentially for $|x| \rightarrow \infty$. Therefore the first Green's formula yields

$$\begin{aligned} & \int_{\mathbb{R}^3 \setminus \overline{\Omega}} (-\lambda |u(x)|^2 + |\nabla u(x)|^2) dx \\ &= \lim_{R \rightarrow \infty} \int_{\substack{\mathbb{R}^3 \setminus \overline{\Omega} \\ |x| < R}} (-\lambda |u(x)|^2 + |\nabla u(x)|^2) dx \\ &= \lim_{R \rightarrow \infty} \left(\int_{\partial\Omega} \frac{\partial}{\partial n} u(x) \overline{u(x)} dS + \int_{|x|=R} \frac{\partial}{\partial n} u(x) \overline{u(x)} dS \right) = 0. \end{aligned}$$

As above it follows from this equation that $u = 0$ in $\mathbb{R}^3 \setminus \overline{\Omega}$, whence $u \equiv 0$ in $\mathbb{R}^3$. Again using the jump relations from Theorem 7.10, we now conclude for all $x \in \partial\Omega$ that

$$\begin{aligned} 0 &= \frac{\partial u}{\partial n_x}(x+) - \frac{\partial u}{\partial n_x}(x-) \\ &= -v(x) + \frac{1}{2\pi} \int_{\partial\Omega} \frac{\partial}{\partial n_x} \frac{e^{i\sqrt{\lambda}|x-y|}}{|x-y|} v(y) dS_y \\ &\quad - v(x) - \frac{1}{2\pi} \int_{\partial\Omega} \frac{\partial}{\partial n_x} \frac{e^{i\sqrt{\lambda}|x-y|}}{|x-y|} v(y) dS_y = -2v(x), \end{aligned}$$

hence $v = 0$. Therefore -1 is not an eigenvalue.

To prove that 1 is not an eigenvalue we assume that $v \in C(\partial\Omega, \mathbb{C})$ satisfies

$$(-I + K')v = 0.$$

We insert v into (7.28). The jump relations for single layer potentials then imply that u solves the boundary value problem

$$\begin{aligned} \Delta u(x) + \lambda u(x) &= 0, \quad x \in \mathbb{R}^3 \setminus \overline{\Omega}, \\ \frac{\partial}{\partial n} u(x) &= 0, \quad x \in \partial(\mathbb{R}^3 \setminus \overline{\Omega}). \end{aligned}$$

Proceeding as above we conclude from this that $u = 0$ in $\mathbb{R}^3 \setminus \overline{\Omega}$ and in Ω , from which we infer by the jump relations that $v = 0$. Therefore 1 is not an eigenvalue. $\blacksquare$

Corollary 7.22 *Let $\Omega \subseteq \mathbb{R}^3$ be a bounded open set with $\partial\Omega \in C_2$. Suppose that $\lambda \in \mathbb{C} \setminus [0, \infty)$. Then the interior Neumann boundary value problem*

$$\begin{aligned} \Delta u(x) + \lambda u(x) &= 0, \quad x \in \Omega, \\ \frac{\partial}{\partial n} u(x-) &= f(x), \quad x \in \partial\Omega \end{aligned}$$

and the exterior Neumann boundary value problem

$$\begin{aligned}\Delta u(x) + \lambda u(x) &= 0, \quad x \in \mathbb{R}^3 \setminus \overline{\Omega}, \\ \frac{\partial}{\partial n} u(x) &= f(x), \quad x \in \partial(\mathbb{R}^3 \setminus \overline{\Omega}) \\ |u(x)|, |\nabla u(x)| &= O(e^{\operatorname{Re} i\sqrt{\lambda}|x|}), \quad |x| \rightarrow \infty\end{aligned}\tag{7.29}$$

have unique solutions for all $f \in C(\partial\Omega, \mathbb{C}) = C(\partial(\mathbb{R}^3 \setminus \overline{\Omega}), \mathbb{C})$. The solutions are given by the single layer potentials

$$u(x) = \frac{1}{2\pi} \int_{\partial\Omega} \frac{e^{i\sqrt{\lambda}|x-y|}}{|x-y|} v(y) dS_y, \quad x \in \Omega, \tag{7.30}$$

where v satisfies the integral equation $(I + K')v = f$ for the interior problem and $(-I + K')v = f$ for the exterior problem.

Proof: Since by Corollary 7.20 the operator K' is compact and since by Lemma 7.21 the number -1 is not an eigenvalue of this operator, it follows from Theorem 7.17 that -1 belongs to the resolvent set of K' , whence the mapping $(I + K') : C(\partial\Omega, \mathbb{C}) \rightarrow C(\partial\Omega, \mathbb{C})$ is invertible. Consequently, the boundary integral equation $(I + K')v = f$ has a unique solution $v \in C(\partial\Omega, \mathbb{C})$. With this v as boundary layer the single layer potential u from (7.30) is a solution of the interior Neumann boundary value problem. To prove that the solution is unique let $\hat{u}$ be another solution of the same problem. Then $w = u - \hat{u}$ satisfies

$$\begin{aligned}\Delta w(x) + \lambda w(x) &= 0, \quad x \in \Omega \\ \frac{\partial}{\partial n} w(x) &= 0, \quad x \in \partial\Omega,\end{aligned}$$

hence the first Green's formula yields

$$\begin{aligned}0 &= \int_{\partial\Omega} \frac{\partial}{\partial n} w(x) \overline{w(x)} dS = \int_{\Omega} \Delta w(x) \overline{w(x)} + |\nabla w(x)|^2 dx \\ &= \int_{\Omega} (-\lambda |w(x)|^2 + |\nabla w(x)|^2) dx \\ &= -i \operatorname{Im} \lambda \int_{\Omega} |w(x)|^2 dx + \int_{\Omega} |\nabla w(x)|^2 - \operatorname{Re} \lambda |w(x)|^2 dx.\end{aligned}$$

This implies $w = 0$, hence $u = \hat{u}$. Therefore the solution is unique.

A solution u of the exterior Neumann boundary value problem is obtained if we insert the unique solution v of the boundary integral equation $(-I + K')v = f$ into (7.30). As in the proof of Lemma 7.21 we see that u defined in this way satisfies the radiation condition (7.29). To prove uniqueness of the solution suppose that $\hat{u}$ is a second solution. We apply the first Green's formula to $w = u - \hat{u}$ in the exterior domain $\mathbb{R}^3 \setminus \overline{\Omega}$ as in the proof of Lemma 7.21, noting that $w(x)$ decreases exponentially for $|x| \rightarrow \infty$ because of the radiation condition (7.29) satisfied by u and $\hat{u}$, and conclude as above that $w = 0$. ■

7.6 Solution of the Dirichlet problem

To solve the interior and exterior Dirichlet problems we must show that the boundary integral equations $(-I + K)v = f$ and $(I + K)v = f$ are solvable.

Theorem 7.23 *The operator $K : C(\partial\Omega, \mathbb{C}) \rightarrow C(\partial\Omega, \mathbb{C})$ is compact.*

This theorem is proved in the same way as Corollary 7.20 using Theorem 7.2 instead of Theorem 7.9.

Lemma 7.24 *Let $\lambda \in \mathbb{C} \setminus [0, \infty)$. Then 1 and -1 are no eigenvalues of K .*

Proof: For $u, v \in C(\partial\Omega)$ we write

$$\langle u, v \rangle_{\partial\Omega} = \int_{\partial\Omega} u(x)v(x)dS_x.$$

By interchanging the order of integration we obtain

$$\begin{aligned} \langle K'u, v \rangle_{\partial\Omega} &= \int_{\partial\Omega} \frac{1}{2\pi} \int_{\partial\Omega} \frac{\partial}{\partial n_x} \frac{e^{i\sqrt{\lambda}|x-y|}}{|x-y|} u(y)dS_y v(x)dS_x \\ &= \int_{\partial\Omega} u(y) \frac{1}{2\pi} \int_{\partial\Omega} \frac{\partial}{\partial n_x} \frac{e^{i\sqrt{\lambda}|x-y|}}{|x-y|} v(x)dS_x dS_y \\ &= \langle u, Kv \rangle_{\partial\Omega}. \end{aligned} \tag{7.31}$$

Now let $\mu = 1$ or $\mu = -1$ and assume that $v \in C(\partial\Omega)$ satisfies

$$(-\mu I + K)v = 0.$$

From (7.31) we conclude for all $u \in C(\partial\Omega)$ that

$$\langle (-\mu I + K')u, v \rangle_{\partial\Omega} = \langle u, (-\mu I + K)v \rangle_{\partial\Omega} = 0.$$

Since by Lemma 7.21 neither 1 nor -1 is an eigenvalue of the compact operator K' , the mapping $(-\mu I + K') : C(\partial\Omega) \rightarrow C(\partial\Omega)$ is surjective. Therefore there is $u \in C(\partial\Omega)$ such that $(-\mu I + K')u = \bar{v}$. Thus,

$$\int_{\partial\Omega} |v(x)|^2 dS_x = \langle \bar{v}, v \rangle_{\partial\Omega} = \langle (-\mu I + K')u, v \rangle_{\partial\Omega} = 0.$$

Consequently v must be equal to zero and cannot be an eigenfunction. This implies that μ is not an eigenvalue of K . ■

Corollary 7.25 *Let $\Omega \subseteq \mathbb{R}^3$ be a bounded open set with $\partial\Omega \in C_2$. Suppose that $\lambda \in \mathbb{C} \setminus [0, \infty)$. Then the interior Dirichlet boundary value problem*

$$\begin{aligned} \Delta u(x) + \lambda u(x) &= 0, & x \in \Omega \\ u(x) &= f(x), & x \in \partial\Omega, \end{aligned}$$

and the exterior Dirichlet boundary value problem

$$\begin{aligned}\Delta u(x) + \lambda u(x) &= 0, & x \in \mathbb{R}^3 \setminus \Omega \\ u(x) &= f(x), & x \in \partial(\mathbb{R}^3 \setminus \bar{\Omega}), \\ |u(x)|, |\nabla u(x)| &= O(e^{\operatorname{Re} i\sqrt{\lambda}|x|}), & |x| \rightarrow \infty\end{aligned}$$

have unique solutions for all $f \in C(\partial\Omega, \mathbb{C})$. The solutions are given by the double layer potentials

$$u(x) = \frac{1}{2\pi} \int_{\partial\Omega} \frac{\partial}{\partial n_y} \frac{e^{i\sqrt{\lambda}|x-y|}}{|x-y|} v(y) dS_y,$$

where v satisfies the integral equation $(-I + K)v = f$ for the interior problem and $(I + K)v = f$ for the exterior problem.

This corollary is proved as the corresponding result for the Neumann problem.

8 Hilbert space methods

8.1 Elliptic differential operators, weak solutions

Let $\Omega \subseteq \mathbb{R}^n$ be an open set and let

$$Lu(x) = \sum_{\substack{|\alpha| \leq 1 \\ |\beta| \leq 1}} D^\alpha (a_{\alpha\beta}(x) D^\beta u(x)), \quad x \in \Omega$$

be a linear differential operator of second order with coefficient functions

$$a_{\alpha\beta} : \Omega \rightarrow \mathbb{C}, \quad \alpha, \beta \in \mathbb{N}_0^n, \quad |\alpha|, |\beta| \leq 1.$$

The sum

$$\sum_{|\alpha+\beta|=2} a_{\alpha\beta}(x) D^{\alpha+\beta} u(x)$$

is called principle part of this operator.

Definition 8.1 (i) The operator L is called elliptic if for all $\xi \in \mathbb{R}^n$, $\xi \neq 0$ and all $x \in \Omega$

$$\sum_{|\alpha+\beta|=2} a_{\alpha\beta}(x) \xi^{\alpha+\beta} \neq 0.$$

(ii) L is called strongly elliptic if to every $x \in \Omega$ there is $\eta > 0$ such that for all $\xi \in \mathbb{R}^n$

$$\operatorname{Re} \left(\sum_{|\alpha+\beta|=2} a_{\alpha\beta}(x) \xi^{\alpha+\beta} \right) \geq \eta |\xi|^2.$$

η is the ellipticity constant.

(iii) L is uniformly strongly elliptic if L is strongly elliptic with an ellipticity constant which can be chosen independent of $x \in \Omega$.

Example: Choose

$$a_{\alpha\beta}(x) = \begin{cases} 1, & \text{if } \alpha = \beta, |\alpha| = 1 \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$\sum_{\substack{|\alpha| \leq 1 \\ |\beta| \leq 1}} D^\alpha (a_{\alpha\beta}(x) D^\beta u(x)) = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2} u(x) = \Delta u(x).$$

For this operator we have

$$\sum_{|\alpha|=|\beta|=1} a_{\alpha\beta}(x) \xi^{\alpha+\beta} = \sum_{i=1}^n \xi_i^2 = |\xi|^2,$$

consequently Δ is uniformly strongly elliptic with ellipticity constant $\eta = 1$.

In the following I assume that $a_{\alpha\beta} : \Omega \rightarrow \mathbb{R}$ is measurable and bounded for all multi-indices $\alpha, \beta \in \mathbb{N}_0^n$ with $|\alpha|, |\beta| \leq 1$. The operator

$$Lu = \sum_{\substack{|\alpha| \leq 1 \\ |\beta| \leq 1}} D^\alpha (a_{\alpha\beta} D^\beta u)$$

is in divergence form. For such operators the Definition 3.31 of weak solutions for the Helmholtz equation can be generalized immediately:

Definition 8.2 Let $f \in L^2(\Omega, \mathbb{C})$ and $\lambda \in \mathbb{C}$.

(i) The function $u \in H_1(\Omega, \mathbb{C})$ is called weak solution of the partial differential equation

$$\sum_{\substack{|\alpha| \leq 1 \\ |\beta| \leq 1}} D^\alpha (a_{\alpha\beta}(x) D^\beta u(x)) + \lambda u(x) = f(x)$$

in Ω , if for all $\varphi \in \mathring{C}_\infty(\Omega, \mathbb{C})$ the equation

$$\sum_{\substack{|\alpha| \leq 1 \\ |\beta| \leq 1}} (-1)^{|\alpha|} \int_{\Omega} a_{\alpha\beta}(x) D^\beta u(x) D^\alpha \overline{\varphi(x)} dx + \lambda \int_{\Omega} u(x) \overline{\varphi(x)} dx = \int_{\Omega} f(x) \overline{\varphi(x)} dx$$

holds.

(ii) Let $g \in H_1(\Omega, \mathbb{C})$. Then $u \in H_1(\Omega, \mathbb{C})$ is called weak solution of the Dirichlet boundary value problem

$$\begin{aligned} \sum_{\substack{|\alpha| \leq 1 \\ |\beta| \leq 1}} D^\alpha (a_{\alpha\beta}(x) D^\beta u(x)) + \lambda u(x) &= f(x), \quad x \in \Omega \\ u|_{\partial\Omega} &= g|_{\partial\Omega}, \end{aligned}$$

if u is a weak solution of the partial differential equation and if $u - g \in \mathring{H}_1(\Omega, \mathbb{C})$.

In the following I write for $u, v \in H_1(\Omega, \mathbb{C})$

$$B(u, v) = \int_{\Omega} \sum_{\substack{|\alpha| \leq 1 \\ |\beta| \leq 1}} (-1)^{|\alpha|} a_{\alpha\beta}(x) D^\beta u(x) D^\alpha \overline{v(x)} dx. \quad (8.1)$$

With this definition it follows that $u \in H_1(\Omega, \mathbb{C})$ is a weak solution of

$$\sum_{\substack{|\alpha| \leq 1 \\ |\beta| \leq 1}} D^\alpha (a_{\alpha\beta} D^\beta u) + \lambda u = f$$

if and only if

$$B(u, \varphi) + \lambda(u, \varphi)_\Omega = (f, \varphi)_\Omega \quad (8.2)$$

for all $\varphi \in \mathring{C}_\infty(\Omega)$.

For the Laplace operator $L = \Delta$ we have

$$B(u, v) = -(\nabla u, \nabla v)_\Omega.$$

Insertion of this expression into (8.2) shows that for the Helmholtz equation Definition 3.31 of weak solutions coincides with Definition 8.2.

8.2 Coercivity of sesquilinear forms to elliptic operators

Definition 8.3 Let X be a vector space over $\mathbb{C}$ with norm $\|u\|$, and let $(u, v) \mapsto [u, v] : X \times X \rightarrow \mathbb{C}$ be a mapping. This mapping is called

1. a sesquilinear form, if

$$[\lambda u + \mu v, w] = \lambda[u, w] + \mu[v, w], \quad [u, \lambda v + \mu w] = \bar{\lambda}[u, v] + \bar{\mu}[u, w],$$

2. symmetric, if $[u, v] = \overline{[v, u]}$,

3. bounded, if $|[u, v]| \leq K\|u\| \|v\|$,

4. strictly coercive, if there is $c > 0$ such that $[u, u] \geq c\|u\|^2$ for all $u \in X$.

The simplest example of a symmetric, bounded, strictly coercive sesquilinear form is the scalar product (u, v) on a Hilbert space. The mapping $(u, v) \mapsto B(u, v) : H_1(\Omega, \mathbb{C}) \times H_1(\Omega, \mathbb{C}) \rightarrow \mathbb{C}$ defined in (8.1) is linear in the first argument and antilinear in the second argument, hence B and of course also $-B$ are sesquilinear forms. In this section we study the coercivity of $-B$, which is a slightly weaker property than strict coercivity. In the formulation of the respective result we write for $u \in H_1(\Omega, \mathbb{C})$

$$|u|_{1,\Omega} = (\nabla u, \nabla u)_\Omega^{1/2}.$$

With this notation one has

$$\|u\|_{1,\Omega}^2 = \|u\|_\Omega^2 + |u|_{1,\Omega}^2.$$

Theorem 8.4 Let $a_{\alpha\beta} : \Omega \rightarrow \mathbb{C}$ be bounded measurable with

$$\begin{aligned} a_{\alpha\beta}(x) &= (-1)^{|\alpha+\beta|} \overline{a_{\beta\alpha}(x)}, & \text{if } |\alpha| + |\beta| \leq 1, \\ a_{\alpha\beta}(x) &= a_{\beta\alpha}(x) \in \mathbb{R}, & \text{if } |\alpha| = |\beta| = 1, \end{aligned}$$

and assume that

$$Lu = \sum_{|\alpha|, |\beta| \leq 1} D^\alpha (a_{\alpha\beta} D^\beta u)$$

is uniformly strongly elliptic with ellipticity constant $\eta > 0$. Then $\hat{B}(u, v) = -B(u, v)$ is a symmetric and bounded sesquilinear form on $H_1(\Omega, \mathbb{C})$, which satisfies

$$\hat{B}(u, u) \geq c_1 |u|_{1,\Omega}^2 - c_2 \|u\|_\Omega^2, \quad \text{for all } u \in H_1(\Omega, \mathbb{C}), \quad (8.3)$$

where

$$c_1 = \frac{\eta}{2}, \quad c_2 = \frac{K^2}{2\eta} + K, \quad K = \sum_{|\alpha+\beta| \leq 1} \|a_{\alpha\beta}\|_\infty.$$

Definition 8.5 A sesquilinear form $\hat{B}$ satisfying (8.3) with suitable constants $c_1 > 0$ and $c_2 \geq 0$ is called coercive on $H_1(\Omega, \mathbb{C})$.

Remark 8.6 Since $a_{\alpha\beta} \in \mathbb{R}$ for $|\alpha| = |\beta| = 1$ the condition of strong ellipticity is

$$\sum_{|\alpha|=|\beta|=1} a_{\alpha\beta} \xi^{\alpha+\beta} \geq \eta |\xi|^2.$$

Proof of the theorem: For $u, v \in H_1(\Omega, \mathbb{C})$ we have

$$\begin{aligned} B(u, v) &= \sum_{|\alpha|, |\beta| \leq 1} (-1)^{|\alpha|} (a_{\alpha\beta} D^\beta u, D^\alpha v) \\ &= \sum_{|\alpha|, |\beta| \leq 1} (-1)^{|\alpha|+|\alpha|+|\beta|} (\overline{a_{\beta\alpha}} D^\beta u, D^\alpha v) \\ &= \sum_{|\alpha|, |\beta| \leq 1} (-1)^{|\beta|} (D^\beta u, a_{\beta\alpha} D^\alpha v) = \overline{B(v, u)}. \end{aligned}$$

Thus, B is symmetric. Also we have

$$|B(u, v)| \leq \sum_{|\alpha|, |\beta| \leq 1} \|a_{\alpha\beta}\|_\infty |(D^\beta u, D^\alpha v)| \leq C \|u\|_{1,\Omega} \|v\|_{1,\Omega}.$$

Thus B is bounded.

To see that $-B$ is coercive define

$$B'(u, v) = - \sum_{|\alpha|=|\beta|=1} (a_{\alpha\beta} D^\beta u, D^\alpha v).$$

The above calculation shows that B' is symmetric. Since $a_{\alpha\beta}(x) \in \mathbb{R}$ for $|\alpha| = |\beta| = 1$ we thus obtain for real valued functions u, v that

$$B'(u, v) = \overline{B'(v, u)} = B'(v, u).$$

Thus, if $u \in H_1(\Omega, \mathbb{C})$ and $u_1 = \operatorname{Re} u$, $u_2 = \operatorname{Im} u$, it follows

$$\begin{aligned}
-B'(u, u) &= -B'(u_1 + iu_2, u_1 + iu_2) = -B'(u_1, u_1) - B'(u_2, u_2) \\
&\quad -iB'(u_2, u_1) + iB'(u_1, u_2) = -B'(u_1, u_1) - B'(u_2, u_2) \\
&= \int_{\Omega} \sum_{\substack{|\alpha|=1 \\ |\beta|=1}} a_{\alpha\beta}(x) (D^{\beta}u_1(x)D^{\alpha}u_1(x) + D^{\beta}u_2(x)D^{\alpha}u_2(x)) dx \\
&= \int_{\Omega} \left(\sum_{\substack{|\alpha|=1 \\ |\beta|=1}} a_{\alpha\beta}(x) (\nabla u_1(x))^{\alpha+\beta} + \sum_{\substack{|\alpha|=1 \\ |\beta|=1}} a_{\alpha\beta}(x) (\nabla u_2(x))^{\alpha+\beta} \right) dx \\
&\geq \int_{\Omega} \eta |\nabla u_1(x)|^2 + \eta |\nabla u_2(x)|^2 dx = \eta |u|_{1,\Omega}^2.
\end{aligned}$$

This together with $ab \leq \frac{\varepsilon}{2} a^2 + \frac{1}{2\varepsilon} b^2$ yields

$$\begin{aligned}
-B(u, u) &= -B'(u, u) - \sum_{|\alpha+\beta|=1} (-1)^{|\alpha|} (a_{\alpha\beta} D^{\beta} u, D^{\alpha} u) - (a_{00} u, u) \\
&\geq \eta |u|_{1,\Omega}^2 - \sum_{|\alpha+\beta|=1} \|a_{\alpha\beta}\|_{\infty} |u|_{1,\Omega} \|u\|_{0,\Omega} - \|a_{00}\|_{\infty} \|u\|_{0,\Omega}^2 \\
&\geq \eta |u|_{1,\Omega}^2 - \frac{\varepsilon}{2} |u|_{1,\Omega}^2 - \frac{1}{2\varepsilon} K^2 \|u\|_{0,\Omega}^2 - K \|u\|_{0,\Omega}^2.
\end{aligned}$$

Choosing $\varepsilon = \eta$ shows that $\hat{B} = -B$ is coercive with the constant c_2 given in the theorem. The proof is complete.

8.3 Existence of weak solutions to elliptic equations

The coercivity of the sesquilinear form $-B$ allows to prove that boundary value problems to elliptic operators have weak solutions. To show this we reformulate Definition 8.2 of weak solutions slightly.

Assume that the operator $L = \sum_{\substack{|\alpha| \leq 1 \\ |\beta| \leq 1}} D^{\alpha} (a_{\alpha\beta} D^{\beta})$ satisfies the assumption of Theorem 8.4, let $\lambda \in \mathbb{R}$ and let $f \in L^2(\Omega, \mathbb{C})$. By Definition 8.2 b.) the function $u \in \mathring{H}_1(\Omega)$ is a weak solution of the homogeneous Dirichlet boundary value problem

$$Lu + \lambda u = f \quad \text{in } \Omega, \tag{8.4}$$

$$u|_{\partial\Omega} = 0, \tag{8.5}$$

if for all $v \in \mathring{C}_{\infty}(\Omega, \mathbb{C})$

$$B(u, v) + \lambda(u, v)_{\Omega} = (f, v)_{\Omega} \tag{8.6}$$

holds. The sesquilinear form B is bounded on $\mathring{H}_1(\Omega, \mathbb{C})$, hence it is continuous in both arguments. Therefore, since $\mathring{C}_\infty(\Omega, \mathbb{C})$ is dense in $\mathring{H}_1(\Omega, \mathbb{C})$, equation (8.6) holds for all $v \in \mathring{C}_\infty(\Omega, \mathbb{C})$ if and only if it holds for all $v \in \mathring{H}_1(\Omega, \mathbb{C})$. Using that λ is real, we conclude that $u \in \mathring{H}_1(\Omega, \mathbb{C})$ is a weak solution of the homogeneous Dirichlet boundary value problem (8.4), (8.5) if and only if

$$B(v, u) + \lambda(v, u)_\Omega = (v, f)_\Omega \quad (8.7)$$

holds for all $v \in \mathring{H}_1(\Omega, \mathbb{C})$. We have thus reduced the problem of the existence of weak solutions to an abstract problem for symmetric sesquilinear forms B on the Hilbert space $\mathring{H}_1(\Omega)$. Accordingly, the existence proof is based on the coercivity of B and on the following easy result:

Lemma 8.7 *Let $[u, v]$ be a symmetric, bounded, strictly coercive sesquilinear form on a Banach space X over $\mathbb{C}$. Then $[u, v]$ is a scalar product on X . The associated norm $|u| = [u, u]^{1/2}$ is equivalent to the norm $\|u\|$. The space X is complete with respect to the norm $|u|$, whence X is a Hilbert space with the scalar product $[u, v]$.*

Proof: Obviously every symmetric, strictly coercive sesquilinear form is a scalar product. From the boundedness and the strict coercivity we obtain

$$c\|u\|^2 \leq [u, u] = |u|^2 \leq K\|u\|^2, \quad (8.8)$$

which means that $\|\cdot\|$ and $|\cdot|$ are equivalent norms. If $\{u_n\}_{n=1}^\infty$ is a Cauchy sequence with respect to the norm $|\cdot|$, then (8.8) implies that $\{u_n\}_{n=1}^\infty$ is also a Cauchy sequence with respect to the norm $\|\cdot\|$. Since X is complete with respect to this norm, there is a limit element $u \in X$ of this Cauchy sequence. From (8.8) we obtain

$$\lim_{n \rightarrow \infty} |u - u_n| \leq \lim_{n \rightarrow \infty} K^{\frac{1}{2}} \|u - u_n\| = 0,$$

hence u is also the limit of $\{u_n\}_{n=1}^\infty$ with respect to the norm $|\cdot|$. Therefore X is complete with respect to this norm.

Corollary 8.8 *Let $\Omega \subseteq \mathbb{R}^n$ be an open set, let*

$$L = \sum_{|\alpha|, |\beta| \leq 1} D^\alpha (a_{\alpha\beta} D^\beta)$$

satisfy the assumptions of Theorem 8.4, and let $\lambda < -c_2$ with

$$c_2 = \frac{K^2}{2\eta} + K, \quad K = \sum_{|\alpha|+|\beta| \leq 1} \|a_{\alpha\beta}\|_\infty,$$

where $\eta > 0$ is the ellipticity constant of L . Then the homogeneous Dirichlet boundary value problem (8.4), (8.5) has a unique weak solution $u \in \mathring{H}_1(\Omega, \mathbb{C})$ for all $f \in L^2(\Omega, \mathbb{C})$. This solution satisfies

$$\|u\|_{1,\Omega} \leq \max\left(\frac{2}{\eta}, \frac{1}{-\lambda - c_2}\right) \|f\|_{\Omega}.$$

Proof: Define the sesquilinear form $[u, v]$ on $\mathring{H}_1(\Omega, \mathbb{C}) \times \mathring{H}_1(\Omega, \mathbb{C})$ by

$$[u, v] = -B(u, v) - \lambda(u, v)_{\Omega}. \quad (8.9)$$

Theorem 8.4 implies that this sesquilinear form is symmetric, bounded and satisfies for $u \in \mathring{H}_1(\Omega, \mathbb{C})$

$$[u, u] = -B(u, u) - \lambda(u, u)_{\Omega} \geq \frac{\eta}{2} |u|_{1,\Omega}^2 - (c_2 + \lambda) \|u\|_{\Omega}^2 \geq c \|u\|_{1,\Omega}^2, \quad (8.10)$$

with $c = \min(\frac{\eta}{2}, -\lambda - c_2) > 0$. Thus, $[u, v]$ is strictly coercive. Consequently, by Lemma 8.7 this sesquilinear form is a scalar product on $\mathring{H}_1(\Omega, \mathbb{C})$ with norm

$$|u|^2 = [u, u] = -B(u, u) - \lambda(u, u)_{\Omega}.$$

Moreover, the linear form $h : \mathring{H}_1(\Omega) \rightarrow \mathbb{C}$ defined by

$$h(v) = -(v, f)_{\Omega}$$

is bounded because (8.10) yields

$$|h(v)| \leq \|f\|_{\Omega} \|v\|_{\Omega} \leq \|f\|_{\Omega} \|v\|_{1,\Omega} \leq \|f\|_{\Omega} c^{-\frac{1}{2}} |v|.$$

The Riesz representation theorem (Corollary 3.7) thus implies that there is a unique function $u \in \mathring{H}_1(\Omega, \mathbb{C})$ satisfying

$$[v, u] = h(v)$$

for all $v \in \mathring{H}_1(\Omega, \mathbb{C})$. By definition of $[u, v]$ this equation is equivalent to (8.7). Consequently, u is the unique weak solution of the boundary value problem. This solution satisfies

$$c \|u\|_{1,\Omega}^2 \leq [u, u] = h(u) \leq \|f\|_{\Omega} \|u\|_{1,\Omega},$$

hence $\|u\|_{1,\Omega} \leq \frac{1}{c} \|f\|_{\Omega}$. This proves the corollary.

Let $f \in L^2(\Omega)$ and $g \in H_1(\Omega)$. By definition, $u \in H_1(\Omega)$ is a weak solution of the inhomogeneous Dirichlet boundary value problem

$$Lu + \lambda u = f, \quad (8.11)$$

$$u|_{\partial\Omega} = g|_{\partial\Omega}, \quad (8.12)$$

if $w = u - g \in \mathring{H}_1(\Omega)$ and

$$B(v, u) + \lambda(v, u)_\Omega = (v, f)_\Omega$$

for all $v \in \mathring{H}_1(\Omega)$. This implies that w satisfies

$$B(v, w) + \lambda(v, w)_\Omega = (v, f - \lambda g)_\Omega - B(v, g)$$

for all $v \in \mathring{H}_1(\Omega)$. On the other hand, if $w \in \mathring{H}_1(\Omega)$ satisfies this equation for all $v \in \mathring{H}_1(\Omega)$, then $u = w + g$ is a weak solution of the problem (8.11), (8.12).

Corollary 8.9 *Let the assumptions of Corollary 8.8 be satisfied. Then for all $\lambda < -c_2$, all $f \in L^2(\Omega)$ and $g \in H_1(\Omega)$ there is a unique weak solution $u \in H_1(\Omega)$ of the inhomogeneous Dirichlet boundary value problem (8.11), (8.12).*

Proof: Let the linear form $h : \mathring{H}_1(\Omega) \rightarrow \mathbb{C}$ be defined by

$$h(v) = (v, \lambda g - f) + B(v, g).$$

The function $u = g + w$ is a weak solution of the Dirichlet boundary value problem if and only if $w \in \mathring{H}_1(\Omega)$ satisfies

$$[v, w] = h(v)$$

for all $v \in \mathring{H}_1(\Omega)$, where $[v, w]$ is the sesquilinear form defined in (8.9). From the boundedness of B we have

$$\begin{aligned} |h(v)| &\leq \|\lambda g - f\|_\Omega \|v\|_\Omega + K\|g\|_{1,\Omega} \|v\|_{1,\Omega} \\ &\leq (\|\lambda g - f\|_\Omega + K\|g\|_{1,\Omega}) \|v\|_{1,\Omega} \leq (\|\lambda g - f\|_\Omega + K\|g\|_{1,\Omega}) c^{-\frac{1}{2}} \|u\|, \end{aligned}$$

where in the last step we used (8.10). Therefore h is a bounded linear form on the Hilbert space $\mathring{H}_1(\Omega)$ equipped with the scalar product $[u, v]$. Consequently, by Corollary 3.7 applied to this Hilbert space there is a unique solution $w \in \mathring{H}_1(\Omega)$.

Example 8.10 The operator $L = \Delta$ does not have lower order terms, hence $c_2 = 0$. Therefore there is a unique weak solution of

$$\begin{aligned} \Delta u + \lambda u &= f, \\ u|_{\partial\Omega} &= g|_{\partial\Omega} \end{aligned}$$

for all $\lambda < 0$, $f \in L^2(\Omega)$, $g \in H_1(\Omega)$.

9 Eigenvalue problems, spectral theory

9.1 The Friedrichs' extension of the operator L

Let $L = \sum_{|\alpha|, |\beta| \leq 1} D^\alpha a_{\alpha\beta} D^\beta$ with $a_{\alpha\beta} : \Omega \rightarrow \mathbb{C}$ bounded measurable, and let $f \in L^2(\Omega)$. By definition $u \in H_1(\Omega, \mathbb{C})$ is a weak solution of $Lu = f$ if and only if

$$B(u, \varphi) = (f, \varphi)_\Omega$$

for all $\varphi \in \mathring{C}_\infty(\Omega)$. In this point of view $L = \sum D^\alpha a_{\alpha\beta} D^\beta$ is merely a symbolic expression. Yet, we can attach a precise meaning to L and define it as an operator on the Hilbert space $L^2(\Omega, \mathbb{C})$ as follows: The domain of definition $D(L)$ of L is given by

$$D(L) = \left\{ u \in H_1(\Omega) \mid \exists_{f \in L^2(\Omega)} \forall_{\varphi \in \mathring{C}_\infty(\Omega)} : B(u, \varphi) = (f, \varphi)_\Omega \right\}.$$

It is immediately seen that $D(L)$ is a linear subspace of $H_1(\Omega)$. Note that if $u \in D(L)$ then the function $f \in L^2(\Omega)$ satisfying $B(u, \varphi) = (f, \varphi)_\Omega$ for all $\varphi \in \mathring{C}_\infty(\Omega)$, which exists by definition, is unique. For, if $g \in L^2(\Omega)$ is a second such function then

$$(f, \varphi)_\Omega = B(u, \varphi) = (g, \varphi)_\Omega,$$

whence $(f - g, \varphi)_\Omega = 0$ for all $\varphi \in \mathring{C}_\infty(\Omega)$. Since $\mathring{C}_\infty(\Omega)$ is dense in $L^2(\Omega)$, this equation implies $f = g$. Therefore for $u \in D(L)$ we can define

$$Lu := f.$$

This defines a linear operator $L : D(L) \subseteq L^2(\Omega) \rightarrow L^2(\Omega)$. For this operator the equation $Lu = f$ holds if and only if u is a weak solution of this equation in the above sense.

We obtain an operator L_D “adapted” to the homogeneous Dirichlet problem if we restrict this operator to the set $\mathring{H}_1(\Omega) \cap D(L)$:

$$L_D = L|_{\mathring{H}_1(\Omega) \cap D(L)}.$$

This operator has the following property: For $f \in L^2(\Omega)$ the equation

$$L_D u = f$$

holds if and only if u is a solution of

$$\begin{aligned} \sum_{\substack{|\alpha| \leq 1 \\ |\beta| \leq 1}} D^\alpha (a_{\alpha\beta} D^\beta u) &= f, \\ u|_{\partial\Omega} &= 0 \end{aligned}$$

in the weak sense.

If in particular $a_{\alpha\beta} \in C_1(\Omega)$ for all $|\alpha|, |\beta| \leq 1$, then for $u \in \mathring{C}_\infty(\Omega)$ the expression $\sum_{|\alpha|, |\beta| \leq 1} D^\alpha(a_{\alpha\beta} D^\beta u)$ can be computed in the classical sense. By partial integration it thus follows for $u, \varphi \in \mathring{C}_\infty(\Omega)$ that

$$B(u, \varphi) = \left(\sum_{|\alpha|, |\beta| \leq 1} D^\alpha(a_{\alpha\beta} D^\beta u), \varphi \right)_\Omega.$$

Since $f = \sum D^\alpha(a_{\alpha\beta} D^\beta u) \in L^2(\Omega)$ and since $\mathring{C}_\infty(\Omega) \subseteq \mathring{H}_1(\Omega) \subseteq H_1(\Omega)$, it follows by definition of L and L_D that

$$Lu = L_D u = \sum_{|\alpha|, |\beta| \leq 1} D^\alpha(a_{\alpha\beta} D^\beta)u$$

for $u \in \mathring{C}_\infty(\Omega)$. Consequently, on $\mathring{C}_\infty(\Omega)$ the operators L and L_D coincide with the classical differential operator $\sum_{|\alpha|, |\beta| \leq 1} D^\alpha(a_{\alpha\beta} D^\beta)$, both are extensions of this operator. L_D is called the Friedrichs' extension of $\sum_{|\alpha|, |\beta| \leq 1} D^\alpha(a_{\alpha\beta} D^\beta)$ on $\mathring{C}_\infty(\Omega)$.

Corollary 8.8 implies that for $\lambda < -c_2$ and $f \in L^2(\Omega)$ there is a unique solution u of

$$-L_D u - \lambda u = f,$$

and this solution satisfies

$$\|u\|_\Omega \leq \|u\|_{1,\Omega} \leq \max\left(\frac{2}{\eta}, \frac{1}{-\lambda - c_2}\right) \|f\|_\Omega.$$

This means that the inverse operator

$$(-L_D - \lambda)^{-1} : L^2(\Omega) \rightarrow D(L) \subseteq L^2(\Omega)$$

exists, and that this operator satisfies for all $f \in L^2(\Omega)$

$$\|u\|_\Omega = \|(-L_D - \lambda)^{-1} f\|_\Omega \leq \max\left(\frac{2}{\eta}, \frac{1}{-\lambda - c_2}\right) \|f\|_\Omega.$$

Therefore this operator is bounded. Consequently

$$(-\infty, -c_2) \subseteq \rho(-L_D).$$

Thus, the spectrum $\Sigma(-L_D)$ belongs to the complement of $(-\infty, -c_2)$ in $\mathbb{C}$. In the following the spectrum will be determined precisely.

9.2 Existence of eigenvalues in bounded domains

The results of this section are based on the following fundamental result:

Theorem 9.1 (Rellich selection theorem.) *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open subset. Every bounded sequence in $\mathring{H}_1(\Omega)$ has a subsequence, which converges in the norm of $L^2(\Omega)$.*

We omit the proof. It can be found for example in the books of Alt and Leis.

In this section we always assume that $a_{\alpha\beta} : \Omega \rightarrow \mathbb{C}$ are bounded measurable with

$$\begin{aligned} a_{\alpha\beta}(x) &= (-1)^{|\alpha+\beta|} \overline{a_{\beta\alpha}(x)}, & \text{if } |\alpha| + |\beta| \leq 1, \\ a_{\alpha\beta}(x) &= a_{\beta\alpha} \in \mathbb{R}, & \text{if } |\alpha| = |\beta| = 1, \end{aligned}$$

and that

$$Lu(x) = \sum_{|\alpha|, |\beta| \leq 1} D^\alpha (a_{\alpha\beta}(x) D^\beta u(x)),$$

is a uniformly elliptic operator with ellipticity constant η .

Let $\lambda \in \mathbb{C}$ be an eigenvalue and $u \in D(L_D)$ be an eigenfunction of $-L_D$, hence

$$(-L_D - \lambda)u = 0.$$

By definition this holds if and only if $u \in \mathring{H}_1(\Omega)$ satisfies

$$B(u, \varphi) + \lambda(u, \varphi)_\Omega = 0$$

for all $\varphi \in \mathring{C}_\infty(\Omega)$. This is equivalent to

$$B(u, v) + \lambda(u, v)_\Omega = 0$$

for all $v \in \mathring{H}_1(\Omega)$.

Lemma 9.2 *Every eigenvalue of $-L_D$ is real. Eigenfunctions u_1 and u_2 to distinct eigenvalues are orthogonal:*

$$(u_1, u_2)_\Omega = B(u_1, u_2) = 0.$$

Proof: Let $u \in \mathring{H}_1(\Omega)$ be an eigenfunction to the eigenvalue λ . Then

$$B(u, u) + \lambda(u, u)_\Omega = 0.$$

The symmetry of B implies $B(u, u) \in \mathbb{R}$, hence

$$\lambda = -\frac{B(u, u)}{\|u\|_\Omega^2} \in \mathbb{R}.$$

If u_1 and u_2 are eigenfunctions to the distinct eigenvalues λ_1, λ_2 , then

$$\lambda_1(u_1, u_2)_\Omega = -B(u_1, u_2) = -\overline{B(u_2, u_1)} = \lambda_2 \overline{(u_2, u_1)_\Omega} = \lambda_2(u_1, u_2)_\Omega,$$

hence $(\lambda_1 - \lambda_2)(u_1, u_2)_\Omega = 0$. Since $\lambda_1 - \lambda_2 \neq 0$, this yields $(u_1, u_2)_\Omega = 0$, whence

$$B(u_1, u_2) = -\lambda_1(u_1, u_2)_\Omega = 0.$$

The proof is complete.

Let M be a finite dimensional linear subspace of $\mathring{H}_1(\Omega)$ spanned by eigenfunctions of $-L_D$. We denote by $M^\perp$ the linear space of all functions in $\mathring{H}_1(\Omega)$, which are orthogonal to M with respect to the scalar product (u, v) . Since $|(u, v)| \leq \|u\|_\Omega \|v\|_\Omega \leq \|u\|_1 \|v\|_1$, the scalar product (u, v) is continuous with respect to the norm $\|u\|_1$, whence $M^\perp$ is closed. We allow $M = \emptyset$, in which case $M^\perp = \mathring{H}_1(\Omega)$.

Theorem 9.3 *If $u \in M^\perp$ with $\|u\|_\Omega = 1$ exists such that*

$$-B(u, u) = \min_{\substack{v \in M^\perp \\ \|v\|_\Omega = 1}} (-B(v, v)),$$

then $\lambda = -B(u, u)$ is an eigenvalue of $-L_D$ and u is an eigenfunction to this eigenvalue.

Proof: Note that $\|\frac{v}{\|v\|_\Omega}\|_\Omega = 1$ for all $v \neq 0$, hence

$$\min_{\substack{v \in M^\perp \\ \|v\|_\Omega = 1}} (-B(v, v)) = \min_{v \in M^\perp} \left(-B\left(\frac{v}{\|v\|_\Omega}, \frac{v}{\|v\|_\Omega}\right) \right) = \min_{v \in M^\perp} \left(-\frac{B(v, v)}{\|v\|_\Omega^2} \right).$$

It follows that $-\frac{B(u, u)}{\|u\|_\Omega^2} = -B(u, u) \leq -\frac{B(v, v)}{\|v\|_\Omega^2}$ for all v , hence $\lambda \mapsto -\frac{B(u + \lambda v, u + \lambda v)}{\|u + \lambda v\|_\Omega^2} : \mathbb{R} \rightarrow \mathbb{R}$ has the minimum at $\lambda = 0$ for all $v \in M^\perp$. Thus

$$\begin{aligned} 0 &= \frac{d}{d\lambda} \frac{B(u + \lambda v, u + \lambda v)}{\|u + \lambda v\|_\Omega^2} \Big|_{\lambda=0} \\ &= \frac{d}{d\lambda} \frac{B(u, u) + \lambda 2 \operatorname{Re} B(u, v) + \lambda^2 B(v, v)}{\|u\|_\Omega^2 + \lambda 2 \operatorname{Re}(u, v) + \lambda^2 \|v\|_\Omega^2} \Big|_{\lambda=0} \\ &= \frac{2 \operatorname{Re} B(u, v) \|u\|_\Omega^2 - B(u, u) 2 \operatorname{Re}(u, v)}{\|u\|_\Omega^2} = 2 \operatorname{Re}(B(u, v) + \lambda(u, v)). \end{aligned}$$

Since $M^\perp$ is a linear space it follows that $iv \in M^\perp$ if $v \in M^\perp$. Thus

$$\begin{aligned} \operatorname{Im}(B(u, v) + \lambda(u, v)) &= \operatorname{Re}(-i B(u, v) - i \lambda(u, v)) \\ &= \operatorname{Re}(B(u, iv) + \lambda(u, iv)) = 0, \end{aligned}$$

hence

$$B(u, v) + \lambda(u, v) = 0 \quad (9.1)$$

for all $v \in M^\perp$. Let w be one of the finitely many eigenfunctions which span M , and let μ be the eigenvalue to w . Then

$$B(w, v) + \mu(w, v)_\Omega = 0$$

for all $v \in \mathring{H}_1(\Omega)$. Since $u \in M^\perp$ we have $(w, u)_\Omega = 0$, thus $B(w, u) = 0$, whence $B(u, v) = (u, v)_\Omega = 0$ for all $v \in M$. Together with (9.1) this implies $B(u, v) + \lambda(u, v) = 0$ for all $v \in \mathring{H}_1(\Omega)$, whence λ is an eigenvalue and u is an eigenfunction.

Theorem 9.4 *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set. The function $-B(v, v)$ assumes a minimum on the set $\{v \in M^\perp \mid \|v\|_\Omega = 1\}$, which is not smaller than $-c_2$, where $c_2 \geq 0$ is the constant from the coercivity estimate proved in Theorem 8.4. (The minimum is not unique.)*

Proof: It has been shown in Theorem 8.4 that $-B$ is coercive. This implies for all $u \in H_1(\Omega)$ with $\|u\|_\Omega = 1$ that

$$-B(u, u) \geq \frac{\eta}{2} |u|_{1,\Omega}^2 - c_2 \|u\|_\Omega^2 \geq -c_2,$$

consequently the infimum

$$\lambda = \inf_{\substack{v \in M^\perp \\ \|v\|_\Omega = 1}} (-B(v, v)) \geq -c_2$$

exists, and we can select a sequence $\{u_k\}_k \subseteq \{v \in M^\perp \mid \|v\|_\Omega = 1\}$ satisfying

$$\lim_{k \rightarrow \infty} -B(u_k, u_k) = \lambda.$$

The coercivity implies

$$\frac{\eta}{2} |u_k|_{1,\Omega}^2 \leq -B(u_k, u_k) + c_2 \rightarrow \lambda + c_2,$$

which yields

$$\|u_k\|_1 = (\|u_k\|_\Omega^2 + |u_k|_{1,\Omega}^2)^{1/2} \leq c$$

with a suitable constant C . Hence $\{u_k\}_k$ is bounded in $\mathring{H}_1(\Omega)$. In general, the sequence $\{u_k\}_k$ does not converge. However, we can select a convergent subsequence: Let $|u|^2 = -B(u, u)$. Since $B(u, v)$ is a sesquilinear form, the parallelogram equality holds:

$$|u + v|^2 + |u - v|^2 = 2|u|^2 + 2|v|^2.$$

Thus

$$\begin{aligned}
|u_\ell - u_k|^2 &= 2|u_\ell|^2 + 2|u_k|^2 - |u_\ell + u_k|^2 \\
&= |u_\ell|^2 + 2|u_k|^2 - \|u_\ell + u_k\|_\Omega^2 \left| \frac{u_\ell + u_k}{\|u_\ell + u_k\|_\Omega} \right|^2 \\
&\leq 2|u_\ell|^2 + 2|u_k|^2 - \lambda^2 \|u_\ell + u_k\|_\Omega^2.
\end{aligned} \tag{9.2}$$

Here we used that $\frac{u_\ell + u_k}{\|u_\ell + u_k\|_\Omega} \in \{v \in M^\perp \mid \|v\|_\Omega = 1\}$, whence $\left| \frac{u_\ell + u_k}{\|u_\ell + u_k\|_\Omega} \right| \geq \lambda$.

Since $\{u_k\}_k$ is bounded in $\mathring{H}_1(\Omega)$ and since Ω is bounded, there is a subsequence $\{u_{k_s}\}_s$ converging in $L^2(\Omega)$, by the Rellich selection theorem. Let $u \in L^2(\Omega)$ be the limit function. Denoting the subsequence by $\{u'_k\}_k$, for simplicity, we obtain from the continuity of the norm that $\|u\|_\Omega = 1$ and $\|u'_k + u'_\ell\|_\Omega \rightarrow \|2u\|_\Omega = 2$, for $k, \ell \rightarrow \infty$. The inequality (9.2) together with the coercivity of B thus yields for $k, \ell \rightarrow \infty$ that

$$\frac{\eta}{2} |u'_\ell - u'_k|_{1,\Omega}^2 \leq |u'_\ell - u'_k|^2 + c_2 \|u'_\ell - u'_k\|_\Omega^2 \rightarrow 2\lambda^2 + 2\lambda^2 - 4\lambda^2 = 0.$$

Consequently $\{u'_k\}_k$ converges in $\mathring{H}_1(\Omega)$ with limit function u , since the limits in $L^2(\Omega)$ and $\mathring{H}_1(\Omega)$ coincide. From the continuity of B on $\mathring{H}_1(\Omega) \times \mathring{H}_1(\Omega)$ we thus conclude

$$-B(u, u) = \lim_{k \rightarrow \infty} -B(u'_k, u'_k) = \lambda = \inf_{\substack{v \in M^\perp \\ \|v\|_\Omega = 1}} -B(v, v).$$

From the closedness of $M^\perp$ we conclude that $u \in \{v \in M^\perp \mid \|v\|_\Omega = 1\}$, hence u is a minimum of $-B(v, v)$ on this set.

Corollary 9.5 *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set and let $M \subseteq \mathring{H}_1(\Omega)$ be a finite dimensional linear space spanned by eigenfunctions of $-L_D$, or let $M = \emptyset$. Then there is an eigenvalue λ of $-L_D$ and an eigenfunction $u \in M^\perp$ to λ , which satisfy $\|u\|_\Omega = 1$ and*

$$\lambda = -B(u, u) = \min_{\substack{v \in M^\perp \\ \|v\|_\Omega = 1}} -B(v, v) \geq -c_2.$$

Proof: Combination of the preceding two theorems.

9.3 Spectral theorem and resolvent set

Also in this section we assume that $\Omega \subseteq \mathbb{R}^n$ is a bounded open set and that

$$Lu(x) = \sum_{|\alpha|, |\beta| \leq 1} D^\alpha (a_{\alpha\beta}(x) D^\beta u(x))$$

is a uniformly elliptic operator with bounded, measurable coefficient functions $a_{\alpha\beta} : \Omega \rightarrow \mathbb{C}$ satisfying

$$\begin{aligned} a_{\alpha\beta}(x) &= (-1)^{|\alpha+\beta|} \overline{a_{\beta\alpha}(x)}, & |\alpha + \beta| \leq 1 \\ a_{\alpha\beta}(x) &= a_{\beta\alpha}(x) \in \mathbb{R}, & |\alpha| = |\beta| = 1. \end{aligned}$$

Theorem 9.6 (Spectral theorem for $-L_D$) *There is a countably infinite sequence $\{\lambda_m\}_m \subseteq \mathbb{R}$ of eigenvalues of $-L_D$ satisfying*

$$-c_2 \leq \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_m \leq \dots \rightarrow \infty, \quad m \rightarrow \infty,$$

where the eigenvalues are repeated according to multiplicity. Moreover, there is a sequence $\{u_m\}_m \subseteq \mathring{H}_1(\Omega)$ of corresponding eigenfunctions, which form a complete orthonormal system in $L^2(\Omega)$.

Proof: We construct the sequences $\{\lambda_m\}_m$ and $\{u_m\}_m$ by induction: If $\lambda_1, \dots, \lambda_m$ and $u_1, \dots, u_m$ are already constructed, let M_m be the space spanned by $u_1, \dots, u_m$. Define λ_{m+1} and $u_{m+1} \in M_m^\perp$ to be the eigenvalue and eigenfunction of $-L_D$ satisfying $\|u_{m+1}\|_\Omega = 1$ and

$$\lambda_{m+1} = -B(u_{m+1}, u_{m+1}) = \min_{\substack{v \in M_m^\perp \\ \|v\|_\Omega = 1}} -B(v, v),$$

which exist according to Corollary 9.5. The corollary also yields $\lambda_m \geq -c_2$. Since $M_{m+1}^\perp \subseteq M_m^\perp$ it follows that

$$\lambda_{m+1} = \min_{\substack{v \in M_m^\perp \\ \|v\|_\Omega = 1}} -B(v, v) \leq \min_{\substack{v \in M_{m+1}^\perp \\ \|v\|_\Omega = 1}} -B(v, v) = \lambda_{m+2}.$$

Moreover, $\lambda_m \rightarrow \infty$ for $m \rightarrow \infty$. Otherwise there would exist $C > 0$ with $\lambda_m \leq C$ for all m . The coercivity of B yields

$$\begin{aligned} \frac{\eta}{2} |u_m|_{1,\Omega}^2 &\leq -B(u_m, u_m) + c_2 \|u_m\|_\Omega^2 = \lambda_m (u_m, u_m)_\Omega + c_2 \\ &= \lambda_m + c_2 \leq C + c_2, \end{aligned}$$

whence $\{u_m\}_m$ is bounded in $\mathring{H}_1(\Omega)$. By the Rellich selection theorem we could select a subsequence converging in $L^2(\Omega)$. However, such a subsequence does not exist, since $(u_\ell, u_m) = 0$ implies

$$\|u_\ell - u_m\|_\Omega^2 = \|u_\ell\|_\Omega^2 + \|u_m\|_\Omega^2 = 2,$$

whenever $\ell \neq m$. Therefore $\lambda_m \rightarrow \infty$.

By construction, $\{u_m\}_m$ is an orthonormal system in $L^2(\Omega)$. If it is not complete there is $f \in L^2(\Omega)$ different from zero such that

$$(u_m, f)_\Omega = 0$$

for all m .

In Theorem 8.4 we proved that the sesquilinear form $-B(u, v) + (c_2 + 1)(u, v)$ is strictly coercive, which implies that there is $w \in \mathring{H}_1(\Omega)$, $w \neq 0$, such that

$$-B(v, w) + (c_2 + 1)(v, w) = (v, f)$$

for all $v \in \mathring{H}_1(\Omega)$. For the eigenfunctions u_m we thus obtain

$$(\lambda_m + c_2 + 1)(u_m, w) = -B(u_m, w) + (c_2 + 1)(u_m, w) = (u_m, f) = 0.$$

Because $\lambda_m + c_2 + 1 \geq 1$, it follows that $(u_m, w) = 0$ for all m , thus $w \in M_k^\perp$ for all k . Setting $w' = \frac{w}{\|w\|_\Omega}$, we obtain for all k

$$\lambda_{k+1} = \min_{\substack{v \in M_k^\perp \\ \|v\|_\Omega = 1}} -B(v, v) \leq -B(w', w').$$

This contradicts $\lambda_k \rightarrow \infty$ for $k \rightarrow \infty$. Consequently the orthonormal system $\{u_m\}_m$ is complete in $L^2(\Omega)$.

Corollary 9.7 *Let $\{\lambda_m\}_m$ be the eigenvalues constructed in the preceding theorem and let $\{u_m\}_m$ be the complete orthonormal system of eigenfunctions.*

(a) *$u \in L^2(\Omega)$ belongs to $\mathring{H}_1(\Omega)$ if and only if $\sum_{m=1}^\infty |\lambda_m| |(u, u_m)|^2 < \infty$. In this case*

$$\begin{aligned} \lim_{\ell \rightarrow \infty} \|u - \sum_{m=1}^\ell (u, u_m)_\Omega u_m\|_{1, \Omega} &= 0, \\ \sum_{m=1}^\infty \lambda_m |(u, u_m)_\Omega|^2 &\leq B(u, u). \end{aligned}$$

(b) *$u \in L^2(\Omega)$ belongs to $D(L_D)$ if and only if $\sum_{m=1}^\infty |\lambda_m|^2 |(u, u_m)|^2 < \infty$. In this case*

$$-L_D u = \sum_{m=1}^\infty \lambda_m (u, u_m) u_m.$$

Proof: (a) For $u \in L^2(\Omega)$ and $k \leq \ell$ set

$$u_{k\ell} = \sum_{m=k}^\ell (u, u_m) u_m \in \mathring{H}_1(\Omega).$$

Choose $\lambda < -c_2$. Then $-B(v, w) - \lambda(v, w)$ is a strictly coercive sesquilinear form on $\mathring{H}_1(\Omega)$, hence

$$\begin{aligned} c\|u_{k,\ell}\|_{1,\Omega}^2 &\leq -B(u_{k\ell}, u_{k\ell}) - \lambda(u_{k\ell}, u_{k\ell}) \\ &= \sum_{m,s=k}^{\ell} (u, u_m) \overline{(u, u_s)} (-B(u_m, u_s) - \lambda(u_m, u_s)) \\ &= \sum_{m,s=k}^{\ell} (u, u_m) \overline{(u, u_s)} (\lambda_m - \lambda) (u_m, u_s) = \sum_{m=k}^{\ell} (\lambda_m - \lambda) |(u, u_m)|^2. \end{aligned}$$

From this inequality we conclude that if $\sum_{m=1}^{\infty} |\lambda_m| |(u, u_m)|^2 < \infty$, then $\{\sum_{m=1}^{\ell} (u, u_m) u_m\}_{\ell}$ converges in $\mathring{H}_1(\Omega)$. Since the limit in $\mathring{H}_1(\Omega)$ coincides with the limit u in $L^2(\Omega)$, we obtain $u \in \mathring{H}_1(\Omega)$.

On the other hand, if $u \in \mathring{H}_1(\Omega)$ we compute similarly

$$\begin{aligned} 0 &\leq -B(u - u_{1\ell}, u - u_{1\ell}) - \lambda(u - u_{1\ell}, u - u_{1\ell}) \\ &= -B(u, u) - \lambda(u, u) - \sum_{m=1}^k (\lambda_m - \lambda) |(u, u_m)|^2, \end{aligned}$$

hence

$$\sum_{m=1}^{\infty} \lambda_m |(u, u_m)|^2 \leq B(u, u) + \lambda \left((u, u) - \sum_{m=1}^{\infty} |(u, u_m)|^2 \right) = B(u, u).$$

(b) By definition, $u \in D(L_D)$ if and only if $u \in \mathring{H}_1(\Omega)$ and there is $f \in L^2(\Omega)$ such that

$$-B(v, u) = (v, f), \quad (9.3)$$

for all $v \in \mathring{H}_1(\Omega)$. In this case we have $-L_D u = f$. Since $\sum_{m=1}^{\infty} (v, u_m) u_m$ converges to $v \in \mathring{H}_1(\Omega)$ in $\mathring{H}_1(\Omega)$, we obtain from the continuity of B that

$$-B(v, u) = - \sum_{m=1}^{\infty} (v, u_m) B(u_m, u) = \sum_{m=1}^{\infty} (v, u_m) \lambda_m (u_m, u)$$

and

$$(v, f) = \sum_{m=1}^{\infty} (v, u_m) (u_m, f).$$

Thus (9.3) holds if and only if

$$\sum_{m=1}^{\infty} (\lambda_m (u_m, u) - (u_m, f)) (v, u_m) = 0$$

for all $v \in \mathring{H}_1(\Omega)$. Setting $v = u_k$ shows that (9.3) holds if and only if

$$\lambda_k(u_k, u) = (u_k, f), \quad \text{for all } k \in \mathbb{N}. \quad (9.4)$$

Thus, $u \in D(L_D)$ if and only if $u \in \mathring{H}_1(\Omega)$ and there is $f \in L^2(\Omega)$ satisfying (9.4).

If $u \in D(L_D)$ we conclude from (9.4) that

$$\sum_{m=1}^{\infty} |\lambda_m|^2 |(u, u_m)|^2 = \sum_{m=1}^{\infty} |(f, u_m)|^2 < \infty$$

and

$$-L_D u = f = \sum_{m=1}^{\infty} (f, u_m) u_m = \sum_{m=1}^{\infty} \lambda_m (u, u_m) u_m.$$

On the other hand, if $\sum_{m=1}^{\infty} |\lambda_m|^2 |(u, u_m)|^2 < \infty$ we conclude from the above that $u \in \mathring{H}_1(\Omega)$. Define a function $f \in L^2(\Omega)$ by $f = \sum_{m=1}^{\infty} \lambda_m (u, u_m) u_m$. Since this function satisfies (9.4), we infer that $u \in D(L_D)$.

Corollary 9.8 *To every $\lambda \in \mathbb{C} \setminus \{\lambda_m\}_m$ and every $f \in L^2(\Omega, \mathbb{C})$ there is a unique solution u of*

$$-L_D u - \lambda u = f$$

given by

$$u = \sum_{m=1}^{\infty} \frac{(f, u_m)}{\lambda_m - \lambda} u_m.$$

Consequently $\rho(-L_D) = \mathbb{C} \setminus \{\lambda_m\}_m$, $\Sigma(-L_D) = \{\lambda_m\}_m$.

Remark 9.9 This result means, of course, that the Dirichlet problem

$$\begin{aligned} \sum_{|\alpha|, |\beta| \leq 1} D^\alpha (a_{\alpha\beta} D^\beta u) + \lambda u &= f \\ u|_{\partial\Omega} &= 0 \end{aligned}$$

has a unique weak solution for all $\lambda \neq \lambda_m$ and all $f \in L^2(\Omega)$.

Proof: From $u = \sum_{m=1}^{\infty} \frac{(f, u_m)}{\lambda_m - \lambda} u_m$ it follows

$$(u, u_m) = \frac{(f, u_m)}{\lambda_m - \lambda},$$

hence

$$\sum_{m=1}^{\infty} |\lambda_m|^2 |(u, u_m)|^2 = \sum_{m=1}^{\infty} \left| \frac{\lambda_m}{\lambda_m - \lambda} \right|^2 |(f, u_m)|^2 \leq C \sum_{m=1}^{\infty} |(f, u_m)|^2 < \infty.$$

Corollary 9.7 thus shows that $u \in D(L_D)$ and

$$-L_D u - \lambda u = \sum_{m=1}^{\infty} (\lambda_m - \lambda)(u, u_m)u_m = \sum_{m=1}^{\infty} (f, u_m)u_m = f.$$

The solution is unique since $-L_D - \lambda$ is injective. For,

$$0 = -L_D v - \lambda v = \sum_{m=1}^{\infty} (\lambda_m - \lambda)(v, u_m)u_m$$

yields together with $\lambda_m - \lambda \neq 0$ that $(v, u_m) = 0$ for all m , hence $v = 0$.

10 Linear hyperbolic equations of second order

10.1 Hyperbolic differential operators

The wave equation $\frac{\partial^2}{\partial t^2} u(x, t) = c \Delta_x u(x, t)$ is a hyperbolic equation. We now show that the spectral theorem from Section 9 can be used to prove existence of solutions for the wave equation and other hyperbolic equations.

Let $Lu(x) = \sum_{\substack{|\alpha| \leq 1 \\ |\beta| \leq 1}} D^\alpha (a_{\alpha\beta}(x) D^\beta u(x))$ be a linear differential operator of second order. In the remainder I always assume that the coefficients of the principal part

$$L'u(x) = \sum_{\substack{|\alpha|=1 \\ |\beta|=1}} a_{\alpha\beta}(x) D^{\alpha+\beta} u(x)$$

are real valued functions:

$$a_{\alpha\beta}(x) \in \mathbb{R}, \quad \text{for } |\alpha| = |\beta| = 1.$$

One uses the set of zeros of the “principal symbol”

$$p(x, \xi) = \sum_{|\alpha|=|\beta|=1} a_{\alpha\beta}(x) \xi^{\alpha+\beta}, \quad \xi \in \mathbb{R}^n,$$

of the differential operator L to classify the operator. An operator, whose set of zeros only consists of $0 \in \mathbb{R}^n$ is elliptic.

A subset M of $\mathbb{R}^n$ is called conic with vertex at 0 if $\xi \in M$ implies $\mu\xi \in M$ for all $\mu \geq 0$. Since $p(x, \xi)$ is homogeneous of order 2 with respect to ξ , it follows that if ξ is a zero of $p(x, \xi)$, then $\mu\xi$ is a zero for all $\mu \in \mathbb{R}$, hence the set of zeros of $p(x, \xi)$ is a conic subset of $\mathbb{R}^n$ symmetric with respect to the vertex 0.

The operator L is called hyperbolic if the set of zeros of the principal symbol p is a double cone. This is made precise in the following

Definition 10.1 The operator L is hyperbolic at $x \in \mathbb{R}^n$, if there is a vector $\theta \neq 0$ such that every line in $\mathbb{R}^n$ parallel to θ , not passing through the origin, intersects the set $\{\xi \mid p(x, \xi) = 0\}$ in precisely two distinct points.

Example 10.2 Let $\sum_{|\alpha|, |\beta| \leq 1} D_x^\alpha (a_{\alpha\beta}(x) D_x^\beta)$, $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, be an elliptic operator satisfying

$$\sum_{|\alpha|=|\beta|=1} a_{\alpha\beta}(x) \xi^{\alpha+\beta} > 0, \quad \xi \in \mathbb{R}^n, \xi \neq 0.$$

Then

$$Lu(x, t) = \frac{\partial^2}{\partial t^2} u(x, t) - \sum_{\substack{|\alpha| \leq 1 \\ |\beta| \leq 1}} D_x^\alpha (a_{\alpha\beta}(x) D_x^\beta u(x, t))$$

is a hyperbolic operator. To see this note that with $x, \xi \in \mathbb{R}^n$, $t, \zeta \in \mathbb{R}$ the principal symbol is

$$p(x, t, \xi, \zeta) = \zeta^2 - \sum_{\substack{|\alpha|=1 \\ |\beta|=1}} a_{\alpha\beta}(x) \xi^{\alpha+\beta}.$$

Set $\theta = (0, \dots, 0, 1) \in \mathbb{R}^{n+1}$. Every line in $\mathbb{R}^{n+1}$ parallel to θ and not passing through the origin is of the form $\zeta \mapsto (\xi, \zeta)$ with $\xi \in \mathbb{R}^n$, $\xi \neq 0$. For such ξ the equation

$$p(x, t, \xi, \zeta) = \zeta^2 - \sum_{|\alpha|=|\beta|=1} a_{\alpha\beta}(x) \xi^{\alpha+\beta} = 0$$

has the two distinct solutions $\zeta = \pm \sqrt{\sum_{|\alpha|=|\beta|=1} a_{\alpha\beta}(x) \xi^{\alpha+\beta}}$. In particular, the d'Alembert operator

$$\partial_t^2 - c\Delta_x = \frac{\partial^2}{\partial t^2} - c \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$$

is hyperbolic for every constant $c > 0$. Therefore the wave equation

$$\frac{\partial^2}{\partial t^2} u(x, t) = c\Delta_x u(x, t)$$

is a hyperbolic equation.

10.2 Energy estimate for the wave equation, uniqueness of solutions

Let $\Omega \subseteq \mathbb{R}^n$ be an open set, let $f : \Omega \times [0, \infty) \rightarrow \mathbb{C}$ be a bounded continuous function.

Theorem 10.3 *Let $u : C_2(\Omega \times [0, \infty), \mathbb{C}) \cap C(\overline{\Omega} \times [0, \infty), \mathbb{C})$ be a solution of the initial-boundary value problem*

$$\begin{aligned} \frac{\partial^2}{\partial t^2} u(x, t) &= c\Delta_x u(x, t) + f(x, t), & (x, t) \in \Omega \times (0, \infty) \\ u(x, t) &= 0, & (x, t) \in \partial\Omega \times [0, \infty) \\ u(x, 0) &= u^{(0)}(x), \quad u_t(x, 0) = u^{(1)}(x), & x \in \overline{\Omega}. \end{aligned}$$

with a constant $c > 0$. Then u satisfies the energy estimate

$$E(u, t)^{1/2} \leq E(u, 0)^{1/2} + \int_0^t \|f(t)\|_{\Omega} dt,$$

where the energy $E(u, t)$ is defined by

$$E(u, t) = \int_{\Omega} \frac{1}{2} |u_t(x, t)|^2 + \frac{c}{2} |\nabla_x u(x, t)|^2 dx.$$

Proof: Since u is two times continuously differentiable we have

$$\begin{aligned}
\frac{d}{dt} E(u, t) &= \frac{d}{dt} \int_{\Omega} \frac{1}{2} |u_t(x, t)|^2 + \frac{c}{2} |\nabla_x u(x, t)|^2 dx \\
&= \int_{\Omega} \operatorname{Re} \left(u_{tt}(x, t) \overline{u_t(x, t)} + c \nabla_x u(x, t) \cdot \nabla_x \overline{u_t(x, t)} \right) dx \\
&= \int_{\Omega} \operatorname{Re} \left((u_{tt}(x, t) - c \Delta_x u(x, t)) \overline{u_t(x, t)} \right) dx \\
&= \operatorname{Re} \int_{\Omega} f(x, t) \overline{u_t(x, t)} dx \leq \|f(t)\|_{\Omega} \|u_t(t)\|_{\Omega} \leq 2 \|f(t)\|_{\Omega} E(u, t)^{1/2}.
\end{aligned}$$

Now

$$\frac{d}{dt} E(u, t) = \frac{d}{dt} (E(u, t)^{1/2})^2 = 2E(u, t)^{1/2} \frac{d}{dt} E(u, t)^{1/2}.$$

Combination of these relations yields

$$\frac{d}{dt} E(u, t)^{1/2} \leq \|f(t)\|_{\Omega}.$$

Integration yields the stated estimate.

Corollary 10.4 *The initial-boundary value problem*

$$\begin{aligned}
\frac{\partial^2}{\partial t^2} u(x, t) &= c \Delta_x u(x, t) + f(x, t) \\
u(x, t) &= 0, \quad (x, t) \in \partial\Omega \times [0, \infty) \\
u(x, 0) &= u^{(0)}(x), \quad u_t(x, 0) = u^{(1)}(x), \quad x \in \overline{\Omega}
\end{aligned}$$

has at most one solution $u \in C_2(\Omega \times [0, \infty), \mathbb{C}) \cap C(\overline{\Omega} \times [0, \infty), \mathbb{C})$.

Proof: Let u and v be two solutions. Then the difference $w = u - v$ satisfies

$$\begin{aligned}
\frac{\partial^2}{\partial t^2} w(x, t) &= c \Delta_x w(x, t) \\
w(x, t) &= 0, \quad (x, t) \in \partial\Omega \times [0, \infty) \\
w(x, 0) &= w_t(x, 0) = 0, \quad x \in \overline{\Omega}.
\end{aligned}$$

Form the energy estimate it thus follows

$$\int_{\Omega} \frac{1}{2} |w_t(x, t)|^2 + \frac{c}{2} |\nabla_x w(x, t)|^2 dx \leq E(w, 0) = 0.$$

Consequently $w_t(x, t) = 0$ for all $(x, t) \in \Omega \times [0, \infty)$ and $w(x, 0) = 0$ for all $x \in \Omega$, whence

$$w(x, t) = \int_0^t w_t(x, \tau) d\tau = 0,$$

for all $(x, t) \in \overline{\Omega} \times [0, \infty)$. Thus, $u = v$.

10.3 Existence of weak solutions of initial-boundary value problems to hyperbolic equations

In the following I consider hyperbolic differential operators of the form $\frac{\partial^2}{\partial t^2} - L$, where

$$L = \sum_{\substack{|\alpha| \leq 1 \\ |\beta| \leq 1}} D_x^\alpha (a_{\alpha\beta}(x) D_x^\beta)$$

is a uniformly strongly elliptic differential operator with bounded measurable coefficient functions $a_{\alpha\beta} : \Omega \rightarrow \mathbb{C}$ satisfying

$$\begin{aligned} a_{\alpha\beta}(x) &= (-1)^{|\alpha+\beta|} \overline{a_{\beta\alpha}(x)}, & |\alpha + \beta| \leq 1 \\ a_{\alpha\beta}(x) &= a_{\beta\alpha}(x) \in \mathbb{R}, & |\alpha| = |\beta| = 1. \end{aligned}$$

$\Omega \subseteq \mathbb{R}^n$ is a bounded open set, for $T > 0$

$$Z_T = \Omega \times (0, T).$$

denotes a cylindric subset of $\mathbb{R}^{n+1}$, and for $u : Z_T \rightarrow \mathbb{C}$ and $0 < t < T$ the function $u(t) : \Omega \rightarrow \mathbb{C}$ is defined by

$$(u(t))(x) = u(x, t).$$

The goal of this section is to show that the initial-boundary value problem

$$\begin{aligned} \frac{\partial^2}{\partial t^2} u(x, t) &= Lu(x, t) + f(x, t), & (x, t) \in Z_T \\ u(x, t) &= 0, & (x, t) \in \partial\Omega \times (0, \infty) \\ u(x, 0) &= u^{(0)}(x), \quad u_t(x, 0) = u^{(1)}(x), & x \in \Omega, \end{aligned}$$

has a weak solution. In order to give the definition of weak solutions inhomogeneous Sobolev spaces must be introduced:

Definition 10.5 For $T > 0$, $m \in \mathbb{N}$ let

$$H_m^{(t)}(Z_T, \mathbb{C}) = \{u \in L^2(Z_T, \mathbb{C}) \mid \frac{\partial^k}{\partial t^k} u \in L^2(Z_T, \mathbb{C}), k \leq m\}.$$

$H_m^{(t)}(Z_T)$ is a Hilbert space with the scalar product

$$(u, v)_m^{(t)} = \sum_{k=0}^m \left(\frac{\partial^k}{\partial t^k} u, \frac{\partial^k}{\partial t^k} v \right)_{Z_T}$$

and the norm $\|u\|_m^{(t)} = ((u, u)_m^{(t)})^{1/2}$.

Theorem 10.6 (Sobolev embedding theorem.) *Let $T > 0$ and $0 \leq \tau \leq T$. Then there is a unique continuous linear mapping $B_\tau : H_1^{(t)}(Z_T, \mathbb{C}) \rightarrow L^2(\Omega, \mathbb{C})$ satisfying*

$$(B_\tau u)(x) = u(x, \tau),$$

for all $u \in C(\Omega \times [0, T]) \cap H_1^{(t)}(Z_T, \mathbb{C})$.

A proof can be found in the book of Alt, p.249.

Definition 10.7 The function $B_\tau u$ is called the trace of the mapping $u \in H_1^{(t)}(Z_T)$ on $\Omega \times \{\tau\}$ and is denoted by $u|_{\Omega \times \{\tau\}}$.

As in Section 9 let $L_D : D(L_D) \subseteq L^2(\Omega) \rightarrow L^2(\Omega)$ denote the Friedrichs' extension of the operator $\sum_{|\alpha|, |\beta| \leq 1} D^\alpha (a_{\alpha\beta} D^\beta)$ in Ω .

Definition 10.8 Let $T > 0$, $f \in L^2(Z_T, \mathbb{C})$, $u^{(0)} \in D(L_D)$, $u^{(1)} \in \mathring{H}_1(\Omega, \mathbb{C})$. A function $u : Z_T \rightarrow \mathbb{C}$ is a weak solution of the Dirichlet initial-boundary value problem

$$\begin{aligned} \partial_t^2 u(x, t) &= Lu(x, t) + f(x, t), & (x, t) \in Z_T, \\ u(x, t) &= 0, & (x, t) \in \partial\Omega \times (0, T), \\ u(x, 0) &= u^{(0)}(x), \quad u_t(x, 0) = u^{(1)}(x), & x \in \Omega. \end{aligned} \quad (10.1)$$

if

1. $u \in H_2^{(t)}(Z_T)$,
2. $u(t) \in D(L_D)$, for almost all $t \in (0, T)$,
3. $\partial_t^2 u(t) = L_D u(t) + f(t)$, for almost all $t \in (0, T)$,
4. $u|_{\Omega \times \{0\}} = u^{(0)}$, $u_t|_{\Omega \times \{0\}} = u^{(1)}$.

Theorem 10.9 *Let $\Omega \subseteq \mathbb{R}^n$ be an open bounded set. To every $f \in H_1(Z_T, \mathbb{C})$, $u^{(0)} \in D(L_D)$ and $u^{(1)} \in \mathring{H}_1(\Omega, \mathbb{C})$ there is a weak solution of the Dirichlet initial-boundary value problem (10.1). This solution is given by*

$$u(x, t) = \sum_{m=1}^{\infty} \alpha_m(t) u_m(x),$$

where $\{u_m\}_{m=1}^{\infty}$ is the complete orthonormal system of eigenfunctions of the operator $-L_D$, and where $\alpha_m : [0, \infty) \rightarrow \mathbb{C}$ is the solution of the initial value problem

$$\begin{aligned} \frac{\partial^2}{\partial t^2} \alpha_m(t) + \lambda_m \alpha_m(t) &= (f(t), u_m)_\Omega \\ \alpha_m(0) &= (u^{(0)}, u_m)_\Omega \\ \frac{\partial}{\partial t} \alpha_m(0) &= (u^{(1)}, u_m)_\Omega. \end{aligned}$$

Here λ_m is the eigenvalue to u_m .

Clearly, this implies for $\lambda_m > 0$

$$\begin{aligned}\alpha_m(t) &= \cos(\sqrt{\lambda_m}t)(u^{(0)}, u_m)_\Omega + \frac{1}{\sqrt{\lambda_m}} \sin(\sqrt{\lambda_m}t)(u^{(1)}, u_m)_\Omega \\ &\quad + \int_0^t \frac{1}{\sqrt{\lambda_m}} \sin(\sqrt{\lambda_m}(t-\tau))(f(\tau), u_m)_\Omega d\tau.\end{aligned}$$

For $\lambda_m = 0$ we obtain

$$\alpha_m(t) = (u^{(0)}, u_m)_\Omega + (u^{(1)}, u_m)_\Omega + \int_0^t (t-\tau)(f(\tau), u_m)_\Omega d\tau,$$

and for $\lambda_m < 0$

$$\begin{aligned}\alpha_m(t) &= \cosh(\sqrt{-\lambda_m}t)(u^{(0)}, u_m)_\Omega + \frac{1}{\sqrt{-\lambda_m}} \sinh(\sqrt{-\lambda_m}t)(u^{(1)}, u_m)_\Omega \\ &\quad + \int_0^t \frac{1}{\sqrt{-\lambda_m}} \sinh(\sqrt{-\lambda_m}(t-\tau))(f(\tau), u_m)_\Omega d\tau.\end{aligned}$$

Since $-c_2 \leq \lambda_1 \leq \dots \leq \lambda_m \leq \dots \rightarrow \infty$ for $m \rightarrow \infty$ there are only finitely many eigenvalues $\lambda_m \leq 0$.

Proof: From the explicit expression for α_m given above we obtain for $s = 0, 1, 2$

$$\begin{aligned}|\partial_t^s \alpha_m(t)| &\leq C(t) \left(\sqrt{1 + |\lambda_m|} \right)^s \left(|(u^{(0)}, u_m)_\Omega| \right. \\ &\quad \left. + \frac{1}{\sqrt{1 + |\lambda_m|}} |(u^{(1)}, u_m)_\Omega| + \frac{1}{\sqrt{1 + |\lambda_m|}} \int_0^t |(f(\tau), u_m)_\Omega| d\tau \right) \\ &\quad + \delta_{s2} |(f(t), u_m)_\Omega|,\end{aligned}\tag{10.2}$$

where

$$\delta_{s2} = \begin{cases} 0, & s \neq 2 \\ 1, & s = 2, \end{cases}$$

and

$$C(t) = \begin{cases} \hat{C} e^{\sqrt{-\lambda_1}t}, & \text{if } \lambda_1 < 0 \\ \hat{C}(1+t), & \text{if } \lambda_1 = 0 \\ \hat{C}, & \text{if } \lambda_1 > 0, \end{cases}$$

with a suitable constant $\hat{C}$. Using the Cauchy-Schwarz inequality, which yields

$$|a + b + c + d|^2 \leq 4(a^2 + b^2 + c^2 + d^2)$$

and

$$\left(\int_0^t |(f(\tau), u_m)| d\tau \right)^2 \leq t \int_0^t |(f(\tau), u_m)|^2 d\tau,$$

we obtain from (10.2)

$$\begin{aligned} |\partial_t^s \alpha_m(t)|^2 &\leq 4C(t)^2(1 + |\lambda_m|)^s \left(|(u^{(0)}, u_m)_\Omega|^2 \right. \\ &\quad \left. + \frac{1}{1 + |\lambda_m|} |(u^{(1)}, u_m)_\Omega|^2 + \frac{t}{1 + |\lambda_m|} \int_0^t |(f(\tau), u_m)|^2 d\tau \right) \\ &\quad + 4\delta_{s2} |(f(t), u_m)_\Omega|^2. \end{aligned} \quad (10.3)$$

The assumptions $u^{(0)} \in D(L_D)$ and $u^{(1)} \in \dot{H}_1(\Omega)$ imply

$$\begin{aligned} \sum_{m=1}^{\infty} (1 + |\lambda_m|^2) |(u^{(0)}, u_m)_\Omega|^2 &< \infty, \\ \sum_{m=1}^{\infty} (1 + |\lambda_m|) |(u^{(1)}, u_m)_\Omega|^2 &< \infty, \end{aligned}$$

cf. Corollary 9.7. Moreover, from $f \in H_1(Z_T)$ it follows by the theorem of Fubini that $f(t) \in H_1(\Omega)$ for almost all $t > 0$. Thus, Corollary 9.7 (a) yields

$$\begin{aligned} \sum_{m=1}^{\infty} (1 + |\lambda_m|) \int_0^t |(f(\tau), u_m)_\Omega|^2 d\tau \\ \leq \int_0^t K_1 \left(B(f(\tau), f(\tau)) + \|f(\tau)\|_\Omega^2 \right) d\tau \\ \leq K_2 \int_0^t \|f(\tau)\|_{1,\Omega}^2 d\tau \leq K \|f\|_{1,Z_t}^2 < \infty, \end{aligned}$$

with suitable constants $K_1, K_2 > 0$. From these estimates and from (10.3) we obtain

$$\begin{aligned} \sum_{m=1}^{\infty} |\lambda_m|^2 |\alpha_m(t)|^2 &\leq C_1 C(t)^2 (1 + t), \\ \sum_{m=1}^{\infty} |\partial_t^s \alpha_m(t)|^2 &\leq C_1 C(t)^2 (1 + t) + 4\delta_{s2} \sum_{m=1}^{\infty} |(f(t), u_m)_\Omega|^2 \\ &= C_1 C(t)^2 (1 + t) + 4\delta_{s2} \|f(t)\|_\Omega^2. \end{aligned} \quad (10.4)$$

Thus, for $s = 0, 1, 2$

$$\begin{aligned} \int_0^T \sum_{m=1}^{\infty} |\partial_t^s \alpha_m(t)|^2 dt &\leq \int_0^T C_1 C(t)^2 (1 + t) + 4\delta_{s2} \|f(t)\|_\Omega^2 dt \\ &\leq C_1 C(T)^2 (1 + T) T + 4\delta_{s2} \|f\|_{Z_T}^2, \end{aligned} \quad (10.5)$$

where we used that $C(t)$ is an increasing function. Since $u_\ell = \sum_{m=1}^\ell \alpha_m u_m$ satisfies for $k \leq \ell$

$$\begin{aligned} \|\partial_t^s(u_\ell - u_k)\|_{Z_T}^2 &= (\partial_t^s(u_\ell - u_k), \partial_t^s(u_\ell - u_k))_{Z_T} = \sum_{m,j=k}^\ell (\partial_t^s \alpha_m u_m, \partial_t^s \alpha_j u_j)_{Z_T} \\ &= \sum_{m,j=k}^\ell \int_0^T \partial_t^s \alpha_m \partial_t^s \overline{\alpha_j} dt (u_m, u_j)_\Omega = \sum_{m=k}^\ell \int_0^T |\partial_t^s \alpha_m(t)|^2 dt, \end{aligned}$$

we obtain from (10.5) that $\{\partial_t^s u_\ell\}_\ell$ is a Cauchy sequence in $L^2(Z_T)$ with

$$\lim_{\ell \rightarrow \infty} \partial_t^s u_\ell = \sum_{m=1}^\infty \partial_t^s \alpha_m u_m,$$

for $s = 0, 1, 2$. This means that $\{u_\ell\}_\ell$ converges in the space $H_2^{(t)}(Z_T, \mathbb{C})$ and that the limit function $u \in H_2^{(t)}(Z_T, \mathbb{C})$ satisfies

$$\partial_t^s u = \sum_{m=1}^\infty \partial_t^s \alpha_m u_m. \quad (10.6)$$

Also, since $u(t) = \sum_{m=1}^\infty \alpha_m(t) u_m$ implies $(u(t), u_m)_\Omega = \alpha_m(t)$, we infer from (10.4) that

$$\sum_{m=1}^\infty |\lambda_m|^2 |(u(t), u_m)_\Omega|^2 < \infty,$$

whence $u(t) \in D(L_D)$ for all $t \geq 0$ and

$$-L_D u(t) = \sum_{m=1}^\infty \lambda_m (u(t), u_m)_\Omega u_m, \quad (10.7)$$

by Corollary 9.7. Summing up, we conclude from (10.6) and (10.7) that

$$\begin{aligned} \partial_t^2 u(t) &= \partial_t^2 \sum_{m=1}^\infty \alpha_m(t) u_m = \sum_{m=1}^\infty \partial_t^2 \alpha_m(t) u_m \\ &= \sum_{m=1}^\infty \left(-\lambda_m \alpha_m(t) + (f(t), u_m)_\Omega \right) u_m \\ &= -\sum_{m=1}^\infty \lambda_m (u(t), u_m)_\Omega u_m + \sum_{m=1}^\infty (f(t), u_m)_\Omega u_m \\ &= L_D u(t) + f(t). \end{aligned}$$

Thus, the first three conditions of Definition 10.8 are satisfied. To verify the last condition note that $\alpha_m(0) = (u^{(0)}, u_m)_\Omega$ and $\frac{\partial}{\partial t} \alpha_m(0) = (u^{(1)}, u_m)_\Omega$ yield

$$\begin{aligned} u|_{\Omega \times \{0\}} &= \sum_{m=1}^{\infty} \alpha_m(0) u_m = \sum_{m=1}^{\infty} (u^{(0)}, u_m)_\Omega u_m = u^{(0)}, \\ \partial_t u|_{\Omega \times \{0\}} &= \sum_{m=1}^{\infty} \partial_t \alpha_m(0) u_m = \sum_{m=1}^{\infty} (u^{(1)}, u_m)_\Omega u_m = u^{(1)}. \end{aligned}$$

This completes the proof.

A Appendix: Bessel and Neumann functions

Let m be a nonnegative integer. The Neumann function of order m is

$$\begin{aligned} N_m(x) = & -\frac{(\frac{1}{2}x)^{-m}}{\pi} \sum_{k=0}^{m-1} \frac{(m-k-1)!}{k!} (\frac{1}{4}x^2)^k \\ & + \frac{2}{\pi} \ln(\frac{1}{2}x) J_m(x) \\ & - \frac{(\frac{1}{2}x)^m}{\pi} \sum_{k=0}^{\infty} [\psi(k+1) + \psi(m+k+1)] \frac{(-\frac{1}{4}x^2)^k}{k!(m+k)!}, \end{aligned} \quad (\text{A.1})$$

where J_m is the Bessel function and where the ψ -function is defined by

$$\begin{aligned} \psi(1) &= -\gamma, \\ \psi(m) &= -\gamma + \sum_{k=1}^{m-1} k^{-1}, \quad m \geq 2. \end{aligned}$$

Here

$$\gamma = \lim_{m \rightarrow \infty} \left(1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{m} - \ln(m) \right) = 0.5772156649 \dots$$

denotes the Euler constant. In particular, for $m = 0$ we obtain

$$\begin{aligned} N_0(x) = & \frac{2}{\pi} (\ln(\frac{1}{2}x) + \gamma) J_0(x) \\ & + \frac{2}{\pi} \left(\frac{\frac{1}{4}x^2}{(1!)^2} - (1 + \frac{1}{2}) \frac{(\frac{1}{4}x^2)^2}{(2!)^2} + (1 + \frac{1}{2} + \frac{1}{3}) \frac{(\frac{1}{4}x^2)^3}{(3!)^2} - \dots \right). \end{aligned} \quad (\text{A.2})$$

To determine the asymptotic behavior of N_m at 0 we use the representation

$$J_m(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(m+k)!} \left(\frac{x}{2} \right)^{2k+m}$$

of the Bessel function, which implies

$$\begin{aligned} J_0(0) &= 1, \\ J_m(x) &= O(x^m), \quad x \rightarrow 0, \quad m \geq 1. \end{aligned}$$

From this relation and from (A.1) and (A.2) we thus obtain for $x \rightarrow 0$ that

$$\begin{aligned} N_0(x) &= \frac{2}{\pi} \ln(x) + O(1), \\ N_1(x) &= -\frac{2}{\pi} x^{-1} + O(x \ln(x)), \\ N_m(x) &= -\frac{2^m(m-1)!}{\pi} x^{-m} + O(x^{-m+2}), \quad m \geq 2. \end{aligned}$$

The following lemma is needed in the proof of Theorem 5.4.

Lemma A.1 *Let $m \in \mathbb{N}$. Then we have for $R \geq r \geq 0$ that*

$$\frac{|J_m(ir)|}{|J_m(iR)|} \leq e^{-\int_r^R (\frac{m^2}{s^2} + 1)^{1/2} ds} \leq \left(\frac{r}{R}\right)^m.$$

Proof: For $r = 0$ the statement holds since $J_m(0) = 0$. To prove the statement for $r > 0$ set $u_m(r) = J_m(ir)$. This function satisfies

$$u_m''(r) + \frac{1}{r}u_m'(r) + (-1 - \frac{m^2}{r^2})u_m(r) = 0$$

for positive r . Since $ru_m''(r) + u_m'(r) = (ru_m'(r))'$, this yields

$$(ru_m'(r))' - (1 + \frac{m^2}{r^2})ru_m(r) = 0. \quad (\text{A.3})$$

For any complex valued function v we have $(\frac{1}{2}|v|^2)' = (\frac{1}{2}v\bar{v})' = \text{Re } v'\bar{v}$. Multiplication of (A.3) with $\frac{1}{2}r\bar{u}_m'(r)$ therefore results in

$$\left(\frac{1}{2}|ru_m'(r)|^2\right)' - (1 + \frac{m^2}{r^2})\left(\frac{1}{2}|ru_m(r)|^2\right)' = 0,$$

hence

$$\frac{d}{dr} \left(\frac{1}{2}|ru_m'(r)|^2 - (1 + \frac{m^2}{r^2})\frac{1}{2}|ru_m(r)|^2 \right) = \frac{m^2}{r^3}|ru_m(r)|^2 \geq 0. \quad (\text{A.4})$$

For $m \in \mathbb{N}$ we have $u_m(0) = 0$ and $ru_m'(r)|_{r=0} = 0$. Using this, we obtain by integration of (A.4) over the interval $[0, r]$ that

$$|ru_m'(r)|^2 - (1 + \frac{m^2}{r^2})|ru_m(r)|^2 \geq 0.$$

Since u_m does not have a zero on the interval $(0, \infty)$, as we showed in Theorem 4.6, this implies that

$$\frac{|u_m'(r)|}{|u_m(r)|} \geq \sqrt{\frac{m^2}{r^2} + 1}.$$

The series representation of J_m shows that $\frac{u_m'(r)}{u_m(r)} = \frac{J_m'(ir)}{J_m(ir)}$ is real for real r . This implies $\left|\frac{u_m'(r)}{u_m(r)}\right| = \pm \frac{u_m'(r)}{u_m(r)}$ and $\frac{|u_m(r)|'}{|u_m(r)|} = \text{Re } \frac{u_m'(r)}{u_m(r)} = \frac{u_m'(r)}{u_m(r)}$. From the last inequality we therefore conclude that

$$\frac{|u_m(r)|'}{|u_m(r)|} = \frac{u_m'(r)}{u_m(r)} \geq \sqrt{\frac{m^2}{r^2} + 1}$$

holds for all $r > 0$ or that

$$\frac{|u_m(r)|'}{|u_m(r)|} = \frac{u_m'(r)}{u_m(r)} \leq -\sqrt{\frac{m^2}{r^2} + 1}$$

holds for all $r > 0$. The second inequality cannot be true, since it implies $|u_m(r)|' < 0$, which in view of $u_m(0) = 0$ is impossible. The first inequality is equivalent to

$$(\ln |u_m(r)|)' \geq \sqrt{\frac{m^2}{r^2} + 1}.$$

Integration of this inequality over the interval $[r, R]$ yields

$$\frac{|u_m(r)|}{|u_m(R)|} \leq e^{-\int_r^R (\frac{m^2}{s^2} + 1)^{1/2} ds}.$$

The lemma follows from $e^{-\int_r^R (\frac{m^2}{s^2} + 1)^{1/2} ds} \leq e^{-\int_r^R \frac{m}{s} ds} = e^{-m \ln(R/r)} = \left(\frac{r}{R}\right)^m$. ■