Chapter 4 rewrite PhL 5.11.14

Chapter 4: Transmission Line Equations

In this Chapter we use the potential integral expressions derived in Chapter 1 to derive the classic transmission line equations. We learn that most transmission line parameters are determined by a single geometric integral K. The approximations are clearly stated.

4.1 Computation of potential φ due to one conductor of a transmission line

Our starting point is the potential φ expression given in box (1.5.23) for the potential at some arbitrary point **x** in the dielectric *due to conductor* C_1 of a transmission line,

$$\varphi_1(\mathbf{x},\omega) = \frac{1}{4\pi\xi}\int_{C_1}\rho_1(x',y',z',\omega)\frac{e^{-j\beta R}}{R}\,dx'dy'dz' \quad . \qquad R = |\mathbf{x} - \mathbf{x}'| \qquad (4.1.1)$$

Here the point **x'** = (x',y',z') runs over the surface of C_1 and R is the distance between the observation point **x** in the dielectric and the integration point **x'**. Parameters β and ξ are for the dielectric.

Comments on ρ_1:

1. ρ_1 is the volume charge density associated with "surface charge" n_1 according to $\rho_1 dV' = n_1 dS'$.

2. ρ_1 is a distribution. For example, for a round wire of radius a we expect ρ_1 to be proportional to δ(r'-a) where $r' = \sqrt{x'^2+y'^2}$. Perhaps $\rho_1 = f(\theta')\delta(r'-a)$ where (r',θ',z') are cylindrical coordinates with axis at the round wire center.

3. Recall from Section 1.5 (c) and (1.5.17) the fact that there are two distinct areal charge distributions called n_c and n_s which are related by $n_c = (\xi/\varepsilon)n_s$. Here n_s is the *actual* surface charge distribution, whereas n_c is an *adjusted* charge density which is directly associated with the current I in the conductor and which accounts for possible leakage in the dielectric. Our n_1 and ρ_1 are associated with this n_c adjusted charge distribution, not with n_s. That is why the external factor in (4.1.1) is $1/4\pi\xi$ instead of $1/4\pi\varepsilon$.

Consider now this charge density $\rho_1(\mathbf{x})$. Following a standard methodology, we make the assumption that its functional form may be factored in the following manner,

$$\underset{C/m^3}{\rho_1(x,y,z)} = \underset{1/m^2}{\alpha_1(x,y)}\ \underset{C/m}{q_1(z)} \quad . \qquad (4.1.2)$$

The dimensions of the functions in this factorization are as indicated, so the charge goes with q_1. Moreover, without any loss of generality we select the relative scale of the two factors such that the integral of $\alpha_1(x,y)$ over a slice of conductor C_1 at any z is unity,

$$\int_{C_1} dx\, dy\, \alpha_1(x,y) = 1 \; . \qquad (4.1.3)$$

Therefore, we can interpret $q_1(z)$ as the total charge per unit length on C_1 at location z :

$$\int_{C_1} dx\, dy\, \rho_1(x,y,z) = q_1(z) \int_{C_1} dx\, dy\, \alpha_1(x,y) = q_1(z) \bullet 1 = q_1(z) \; .$$

Assume that $q_2(z)$ is the charge on the other conductor C_2 of a two-conductor transmission line. If $q_1(z) + q_2(z) \neq 0$, then we have a net charge per unit length and the transmission line is acting as a radiating antenna as well as a transmission line. From now on, we ignore this superposed radiation problem and assume that at each value of z, the net charge on both conductors is 0 -- the line is "balanced". This means that

$$q_2(z) = - q_1(z) \equiv -q(z) \; . \qquad (4.1.4)$$

To simplify notation, we now dispense with the subscript and denote $q_1(z) = q(z)$. However, we maintain the subscript on $\alpha_1(x,y)$ to emphasize that the two conductors can have completely different cross sectional shapes. The shape of the transverse distribution of charge on C_1 is determined by $\alpha_1(x,y)$, but the total charge is q(z) per unit length.

How can we justify assumption (4.1.2)? This is "separation of variables". The idea is that we assume it without any justification, and then we try to find a solution to our problem which is consistent with the assumption. All we really want is to find a solution to our basic differential equations with their boundary conditions, and any assumptions we make can be justified in the end once we have found a solution. On the other hand, if an assumption like (4.1.2) does not lead to a solution, then it must have been a bad assumption. We have seen earlier how the expected EM field pattern on a transmission line has a constant transverse "shape" and this certainly motivates the assumption (4.1.2).

Now insert (4.1.2) into (4.1.1) to get,

$$\varphi_1(x,y,z) = \frac{1}{4\pi\xi} \int_{-\infty}^{\infty} dz'\, q(z') \int_{C_1} dx'\, dy'\, \alpha_1(x',y') \frac{e^{-j\beta R}}{R} \qquad (4.1.5)$$

$$R^2 = (x-x')^2 + (y-y')^2 + (z-z')^2 \; .$$

4.2 Computation of potential φ due to both conductors of a transmission line

Let us now write the potential at an arbitrary point **x** in the dielectric due to *both* conductors C_1 and C_2 :

$$\varphi_{12}(\mathbf{x}) = \varphi_1(\mathbf{x}) + \varphi_2(\mathbf{x}) =$$

$$\frac{1}{4\pi\xi} \int_{-\infty}^{\infty} dz'\, q(z') \{ \int_{C_1} dx_1'\, dy_1'\, \alpha_1(x_1',y_1') \frac{e^{-j\beta R_1}}{R_1} - \int_{C_2} dx_2'\, dy_2'\, \alpha_2(x_2',y_2') \frac{e^{-j\beta R_2}}{R_2} \}$$

$$R_1^2 = (x-x_1')^2 + (y-y_1')^2 + (z-z')^2 = s_1^2 + (z-z')^2 \qquad s_1^2 = (x-x_1')^2 + (y-y_1')^2 \qquad (4.2.1)$$
$$R_2^2 = (x-x_2')^2 + (y-y_2')^2 + (z-z')^2 = s_2^2 + (z-z')^2 \qquad s_2^2 = (x-x_2')^2 + (y-y_2') \ .$$

The minus sign between the terms is due to (4.1.4). Each conductor has its own arbitrary transverse charge distribution α_i. The transverse integration variables on C_1 are $dx_1'\ dy_1'$, while those on C_2 are instead $dx_2'\ dy_2'$. In the last two lines we introduce certain transverse distances s_1 and s_2 as shown. The same dz' integration variable is used for both conductors. The following drawing shows an arbitrary dielectric point $\mathbf{x} = (x,y,z)$ located in the z = z plane. The point $\mathbf{x_1}' = (x_1',y_1',z')$ lies on C_1 at some point of the C_1 integration and similarly for $\mathbf{x_2}' = (x_2',y_2',z')$. The full distances R_1 and R_2 and the transverse distances s_1 and s_2 are shown.

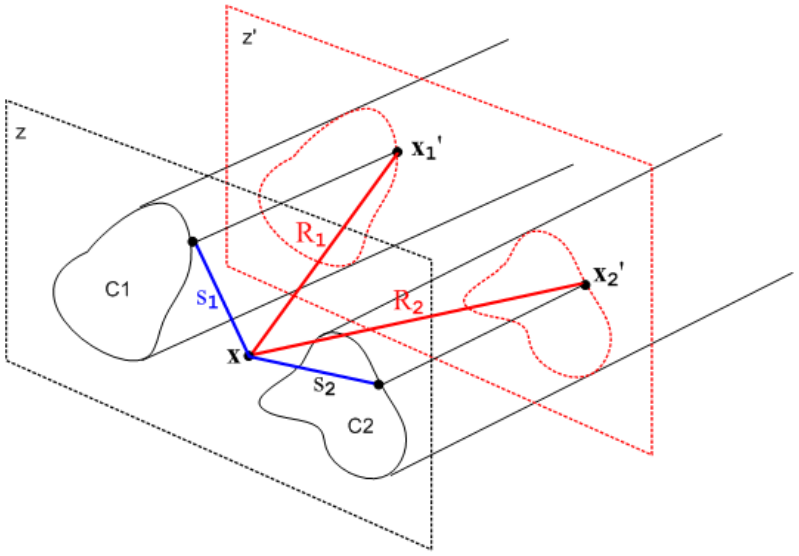


Fig 4.1

One can imagine an expression similar to (4.2.1) for a transmission line consisting of N conductors with some relationship among the $q_i(z)$ on each conductor, but we shall restrict our interest to N = 2.

4.3 The Transmission Line Limit

Consider again the potential at **x** due to both conductors shown in (4.2.1),

$$\varphi_{12}(\mathbf{x}) = \frac{1}{4\pi\xi} \int_{-\infty}^{\infty} dz'\, q(z')\{ \int_{C_1} dx_1'\, dy_1'\, \alpha_1(x_1',y_1') \frac{e^{-j\beta R_1}}{R_1} - \int_{C_2} dx_2'\, dy_2'\, \alpha_2(x_2',y_2') \frac{e^{-j\beta R_2}}{R_2} \} \qquad (4.3.1)$$

As the red-dashed z = z' plane shown in Fig 4.1 is pushed back far from the z = z plane, the vectors which are labeled by distances R_1 and R_2 become more aligned, and both R_1 and R_2 become larger. During the transverse integrations over $\mathbf{x_1}'$ and $\mathbf{x_2}'$, these R_i vectors then don't vary much. One could then replace the transverse charge density $\alpha_1(x_1',y_1')$ with a point charge at the "center of the conductor" and not make much difference in the $\mathbf{R_1}$ vector and its length R_1. In this situation, the {...} integrand of the above integral has this form

$$\{ \frac{e^{-j\beta R_1}}{R_1} - \frac{e^{-j\beta R_2}}{R_2} \} . \qquad \text{// when } |z-z'| \text{ is large} \qquad (4.3.2)$$

If we then expand the exponentials showing the first few terms, this becomes

$$\{ \frac{1-j\beta R_1+(j\beta)^2 R_1^2/2}{R_1} - \frac{1-j\beta R_2+(j\beta)^2 R_2^2/2}{R_2} \} = \{ [\frac{1}{R_1} - \frac{1}{R_2}] + [-j\beta + j\beta] + (j\beta)^2/2 [R_1-R_2] + ... \}$$

$$= \{ [\frac{1}{R_1} - \frac{1}{R_2}] - (\beta^2/2) [R_1-R_2] + \text{order}(\beta^3) \} \qquad (4.3.3)$$

Since $R_1 \approx R_2$ for large $|z-z'|$ as just discussed, both the leading term and the β^2 term are small in an absolute sense as long as β^2 is not huge. When $|z-z'|$ is large, both R_1 and R_2 are large and thus both $1/R_1$ and $1/R_2$ are small, and $[1/R_1 - 1/R_2]$ is smaller still due to cancellation between the terms.

So our first point is that, in the dz' integration, the main contribution to $\varphi_{12}(\mathbf{x})$ comes from regions of z' for which $|z-z'|$ is small.

Given then that the dz' integration in (4.3.1) is dominated by that part for which $|z-z'|$ is small, we can see that for this controlling integration region the size of distances R_1 and R_2 will be on the order of the transverse dimension of the transmission line, assuming that we select the point **x** somewhere between the two conductors. If we vaguely define the transmission line's transverse extent as distance D, then suppose we make the following assumption concerning β :

$$\beta D << 1 \qquad \text{"small } \beta\text{"} . \qquad (4.3.4)$$

In this case, we can replace $e^{-j\beta R_1} = 1$ and $e^{-j\beta R_2} = 1$ in the integration without significantly changing the result. Then as shown in (4.3.3) there will be a correction term that is order β^2 which we shall neglect, as well as higher terms of order β^n with n> 2.

Notice that the linear β term vanished exactly in our large $|z-z'|$ analysis. This linear term also vanishes in the full analysis since the α_i transverse charge functions are normalized to unity:

$$\{ \int_{C_1} dx_1' dy_1' \alpha_1(x_1',y_1') \frac{-j\beta R_1}{R_1} - \int_{C_2} dx_2' dy_2' \alpha_2(x_2',y_2') \frac{-j\beta R_2}{R_2} \}$$

$$= (-j\beta) \{ \int_{C_1} dx_1' dy_1' \alpha_1(x_1',y_1') - \int_{C_1} dx_1' dy_1' \alpha_1(x_1',y_1')) = (-j\beta) \{1 - 1\} - 0. \qquad (4.3.5)$$

Thus, by setting $\beta = 0$ in (4.3.1) we are ignoring corrections on the order of β^2 and higher, and if β is small, these corrections are very small.

The Helmholtz parameter β for the dielectric is $2\pi/\lambda$ where λ is the wavelength of a wave passing down the transmission line. Thus, our "small β" assumption stated above can also be written

$$\lambda >> D \qquad (4.3.6)$$

which says the wavelength is much longer than the size of the transmission line transverse dimensions. This assumption is called the Transmission Line Limit. If we operate within this limit, then (4.3.1) may be approximated as

$$\varphi_{12}(\mathbf{x}) = \frac{1}{4\pi\xi} \int_{-\infty}^{\infty} dz'\, q(z') \{ \int_{C_1} dx_1'\, dy_1'\, \alpha_1(x_1',y_1') \frac{1}{R_1} - \int_{C_2} dx_2'\, dy_2'\, \alpha_2(x_2',y_2') \frac{1}{R_2} \} \qquad (4.3.7)$$

We shall now use the small β assumption one more time. We assume that the linear charge density q(z') has the characteristics of a wave traveling down the transmission line (see also Chapter 5),

$$q(z) = q(0)\, e^{-j\beta z} \qquad // \; q(z,t) = q(0,0)\, e^{j(\omega t - \beta z)} \qquad (4.3.8)$$

so that

$$q'(z) = -j\beta\, q(z)$$
$$q''(z) = (-j\beta)^2 q(z) \quad \text{and so on.}$$

We can then write a Taylor expansion for charge density q(z') which appears in our integration,

$$\begin{aligned} q(z') &= q(z) + (z'-z)\, q'(z) + (1/2)\,(z'-z)^2\, q''(z) + \ldots \\ &= q(z) + (-j\beta)\, q(z)\,(z'-z) + (1/2)\,(-j\beta)^2\, q(z)\,(z'-z)^2 + \ldots \\ &= q(z)\,[\, 1 + (-j\beta)\,(z'-z) + (1/2)\,(-j\beta)^2 (z'-z)^2 + \ldots \,] \,. \end{aligned} \qquad (4.3.9)$$

Since both R_1 and R_2 are even functions of the quantity (z'-z), and since there is no other (z'-z) dependence in the (4.3.7) integrand, the (-jβ) term in (4.3.9) contributes nothing (this is also true more generally for (4.3.1)). Thus, if we assume small β, we can approximate $q(z') \approx q(z)$ where we are then ignoring a β^2 size term. Once again, if β is small, β^2 is very small so our error in replacing q(z') by q(z) is very small. We are only interested in the contributing region where |z-z'| is on the order of transverse dimension D, so the same $\beta D << 1$ is being assumed as earlier.

We arrive then at our final result for the potential at a point **x** between the conductors,

$$\varphi_{12}(\mathbf{x}) = \frac{1}{4\pi\xi}\, q(z) \int_{-\infty}^{\infty} dz' \{ \int_{C_1} dx_1'\, dy_1'\, \alpha_1(x_1',y_1') \frac{1}{R_1} - \int_{C_2} dx_2'\, dy_2'\, \alpha_2(x_2',y_2') \frac{1}{R_2} \} \qquad (4.3.10)$$

where we have thrown out terms of order β^2 and higher. In this transmission line limit approximation, our Helmholtz integral (4.3.1) has been reduced to essentially an electrostatics Coulomb integral where we just sum over the contribution of each piece of charge to the total potential. As noted earlier, q(z) has the normalization of n_c and not n_s as discussed in Section 1.5 (c) which explains why the leading factor is $\frac{1}{4\pi\xi}$ and not $\frac{1}{4\pi\varepsilon}$. This allows for the dielectric to have some conductance.

It should be noted that the integral of (4.3.10) converges due to the subtraction of the two terms which in turn results from the two conductors having opposite longitudinal charge densities. The individual terms in (4.3.10) do not converge and are in fact each logarithmically divergent in the sense

$$\int_{-\infty}^{\infty} dz' \, (1/z') = \infty \, .$$

4.4 General Calculation of V(z)

We now introduce two new points $\mathbf{x}_1$ and $\mathbf{x}_2$. The point $\mathbf{x}_1$ lies on C_1 in the z = z plane, while $\mathbf{x}_2$ lies on C_2 in this same plane. We then evaluate $\varphi_{12}(\mathbf{x})$ at $\mathbf{x} = \mathbf{x}_1$ and subtract from that $\varphi_{12}(\mathbf{x})$ at $\mathbf{x} = \mathbf{x}_2$ and in this way we obtain the potential difference between the surfaces of the two conductors at z = z. Recall,

Fact 2: $\varphi \approx$ constant on a conductor surface in the strong or extreme skin effect regimes within the Transmission Line Limit. *(3.7.4)*

Thus, assuming the small δ regime and treating $\varphi \approx$ constant as an equality, the potential difference will be independent of the locations of $\mathbf{x}_2$ and $\mathbf{x}_1$ as long as they are on their respective surfaces and both have z = z. For this reason, the potential difference is a function only of z. Thus we write, using two copies of (4.3.10),

$$V(z) \equiv \varphi_{12}(\mathbf{x}_1) - \varphi_{12}(\mathbf{x}_2)$$

$$= \frac{1}{4\pi\xi} \, q(z) \int_{-\infty}^{\infty} dz' \, \{ \int_{\mathbf{C}_1} dx_1' \, dy_1' \, \alpha_1(x_1',y_1') \frac{1}{R_{11}} - \int_{\mathbf{C}_2} dx_2' \, dy_2' \, \alpha_2(x_2',y_2') \frac{1}{R_{12}} \}$$

$$- \frac{1}{4\pi\xi} \, q(z) \int_{-\infty}^{\infty} dz' \, \{ \int_{\mathbf{C}_1} dx_1' \, dy_1' \, \alpha_1(x_1',y_1') \frac{1}{R_{21}} - \int_{\mathbf{C}_2} dx_2' \, dy_2' \, \alpha_2(x_2',y_2') \frac{1}{R_{22}} \} \quad (4.4.1)$$

where

$$\begin{aligned}
R_{11}^2 &= (x_1-x_1')^2 + (y_1-y_1')^2 + (z-z')^2 = s_{11}^2 + (z-z')^2 & \qquad s_{11}^2 &= (x_1-x_1')^2 + (y_1-y_1')^2 \\
R_{12}^2 &= (x_1-x_2')^2 + (y_1-y_2')^2 + (z-z')^2 = s_{12}^2 + (z-z')^2 & \qquad s_{12}^2 &= (x_1-x_2')^2 + (y_1-y_2')^2 \\
R_{22}^2 &= (x_2-x_2')^2 + (y_2-y_2')^2 + (z-z')^2 = s_{22}^2 + (z-z')^2 & \qquad s_{22}^2 &= (x_2-x_2')^2 + (y_2-y_2')^2 \\
R_{21}^2 &= (x_2-x_1')^2 + (y_2-y_1')^2 + (z-z')^2 = s_{21}^2 + (z-z')^2 & \qquad s_{21}^2 &= (x_2-x_1')^2 + (y_2-y_1')^2 \, . \qquad (4.4.2)
\end{aligned}$$

The vector $\mathbf{R}_{12}$ points from our new point $\mathbf{x}_1$ to an integration point $\mathbf{x}_2'$ on C_2. Here is a drawing of our new and more complicated situation:

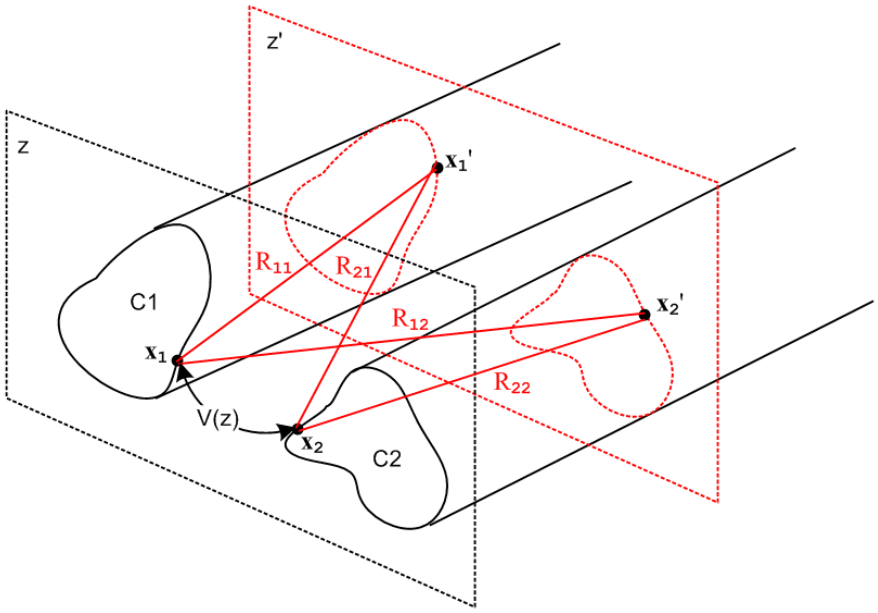


Fig 4.2

We next rearrange the four terms in (4.4.1) to get

$$V(z) = q(z)\frac{1}{4\pi\xi}\int_{-\infty}^{\infty} dz' \left\{ \int_{C_1} dx_1'\, dy_1'\, \alpha_1(x_1',y_1')\left(\frac{1}{R_{11}} - \frac{1}{R_{21}}\right) - \int_{C_2} dx_2'\, dy_2'\, \alpha_2(x_2',y_2')\left(\frac{1}{R_{12}} - \frac{1}{R_{22}}\right) \right\} . \tag{4.4.3}$$

It is now possible to carry out the dz' integrations. The integral of interest is the following,

$$\int_{-\infty}^{\infty} dx \left\{ \frac{1}{\sqrt{a^2+x^2}} - \frac{1}{\sqrt{b^2+x^2}} \right\} = \ln(b^2/a^2) . \tag{4.4.4}$$

Since this is quite important, we confirm with Maple,

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Int(1/sqrt(a^2+x^2) - 1/sqrt(b^2 +x^2), x=-infinity..infinity);
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$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{a^2+x^2}} - \frac{1}{\sqrt{b^2+x^2}}\, dx$$

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value(%);
```

$$-\ln(a^2) + \ln(b^2)$$

The separate integrals here are logarithmically divergent, but the combination converges. Thus,

$$\int_{-\infty}^{\infty} dz' \left(\frac{1}{R_{11}} - \frac{1}{R_{21}}\right) = \int_{-\infty}^{\infty} dz' \left(\frac{1}{\sqrt{s_{11}^2 + (z-z')^2}} - \frac{1}{\sqrt{s_{21}^2 + (z-z')^2}}\right) = \ln(s_{21}^2/s_{11}^2)$$

and

$$\int_{-\infty}^{\infty} dz' \left(\frac{1}{R_{12}} - \frac{1}{R_{22}}\right) = \ln(s_{22}^2/s_{12}^2) \tag{4.4.5}$$

so that

$$V(z) = q(z)\,\frac{1}{4\pi\xi}\;\{\;\int_{C_1} dx_1'\,dy_1'\,\alpha_1(x_1',y_1')\,\ln(s_{21}^2/s_{11}^2) - \int_{C_2} dx_2'\,dy_2'\,\alpha_2(x_2',y_2')\,\ln(s_{22}^2/s_{12}^2)\;\}$$

$$\begin{aligned} s_{21}^2 &= (x_2-x_1')^2 + (y_2-y_1')^2 \qquad & s_{22}^2 &= (x_2-x_2')^2 + (y_2-y_2')^2 \\ s_{11}^2 &= (x_1-x_1')^2 + (y_1-y_1')^2 \qquad & s_{12}^2 &= (x_1-x_2')^2 + (y_1-y_2')^2\;. \end{aligned} \tag{4.4.6}$$

The four transverse distances are shown in this figure,

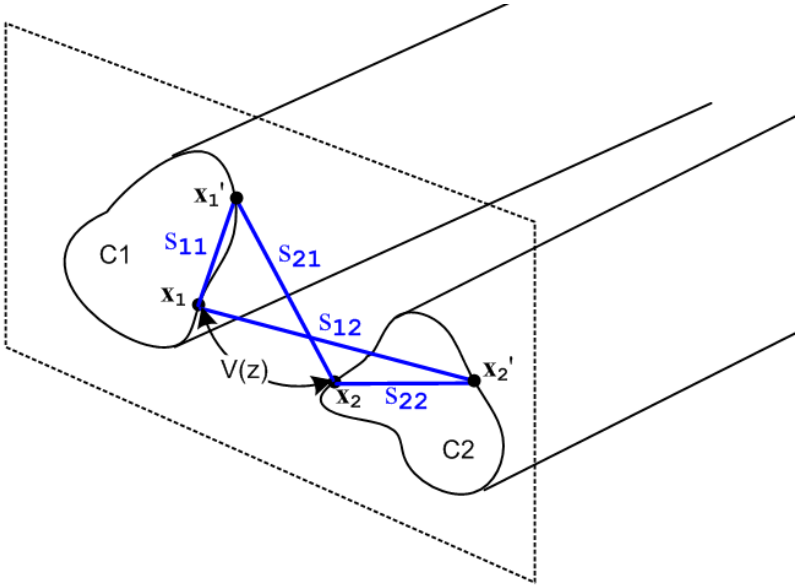


Fig 4.3

Equation (4.4.6) expresses the potential between the two transmission line conductors at some plane z in terms of the charge distributions on the conductors α_i. In general, these charge distributions are not known, so one cannot regard (4.4.6) as a general purpose silver bullet to solve transmission line problems. On the other hand, as we shall see, equation (4.4.6) is one of a group of equations which will allow us to express several different transmission line parameters in terms the same integral, and one then obtains a relation between these parameters.

For example, in analogy to what we did with a parallel plate capacitor in (1.5.19), we may define the complex capacitance C' per unit length of our transmission line using (4.4.6) as follows:

$$\frac{1}{C'} = \frac{V(z)}{q(z)} = \frac{1}{4\pi\xi}\;\{\;\int_{C_1} dx_1'\,dy_1'\,\alpha_1(x_1',y_1')\,\ln(s_{21}^2/s_{11}^2) - \int_{C_2} dx_2'\,dy_2'\,\alpha_2(x_2',y_2')\,\ln(s_{22}^2/s_{12}^2)\;\}$$

$$= \frac{1}{4\pi\xi}\;K \qquad \text{and} \qquad V(z) = q(z)\,\frac{1}{4\pi\xi}\;K \tag{4.4.7}$$

where K is a *dimensionless real number* obtained from a geometric integral of the normalized transverse charge distributions α_i (recall that α_i is has dimensions $1/m^2$ in (4.1.2)),

$$K \equiv \int_{C_1} dx_1' \, dy_1' \, \alpha_1(x_1',y_1') \ln(s_{21}^2/s_{11}^2) - \int_{C_2} dx_2' \, dy_2' \, \alpha_2(x_2',y_2') \ln(s_{22}^2/s_{12}^2) \quad . \tag{4.4.8}$$

Recall from (1.5.20) that (C', C and G are discussed further in Section 4.11 below)

$$C' = C + 1/(j\omega R) = C + G/(j\omega) \tag{4.4.9}$$

where conductance (per unit length) G is associated with the imaginary part of C'. We then have

$$4\pi\xi/K = C' = C + G/j\omega$$

or

$$4\pi(\varepsilon+\sigma/j\omega)/K = C + G/j\omega \qquad \text{// (1.5.1) for } \xi$$

so that

$C = 4\pi\varepsilon/K$ capacitance per unit length of the transmission line

$G = 4\pi\sigma/K$ conductance per unit length of the transmission line

so

$$C/G = \varepsilon/\sigma \quad . \tag{4.4.10}$$

Here G = 1/R' is the conductance across the dielectric *between* a unit length of the two conductors. This is unrelated to the longitudinal resistance R of the conductors themselves, though that parameter will arise later on in the form of surface impedance Z_s. We only have $G \neq 0$ if the dielectric has some conductance $\sigma \neq 0$.

Note from above and (1.2.8) that dim(C) = dim(ε) = farad/m and dim(G) = dim(σ) = mho/m .

We now quote several results that will be derived later in Section 4.11.

First we show below in (4.11.29) with (4.11.30) that the external inductance per unit length of our transmission line is *also* related to this same constant K,

$$L_e = (\mu/4\pi)K \quad . \tag{4.4.11}$$

Second, we show in (4.11.16) that the **characteristic impedance** of the transmission line is given by

$$Z_0 \equiv \frac{V(z)}{i(z)} = \sqrt{\frac{R + j\omega L}{G + j\omega C}} \quad . \tag{4.4.12}$$

At sufficiently large ω we can neglect the R and G terms to get this real value,

$$Z_0 = \sqrt{\frac{L}{C}} \quad . \qquad \text{// large } \omega \tag{4.4.13}$$

Third, we show in (4.11.36) that, for large ω, $L \to L_e$ so

$$Z_0 = \sqrt{\frac{L_e}{C}} = \sqrt{\frac{(\mu/4\pi)K}{4\pi\varepsilon/K}} = (1/4\pi)\, K \sqrt{\mu/\varepsilon} \quad = \quad (1/4\pi)\, K\, Z_m \tag{4.4.14}$$

where $Z_m = \sqrt{\mu/\varepsilon}$ is the "impedance of the dielectric medium" having μ and ε. Recall that for free space we had in (1.1.29)

$$Z_{fs} = \sqrt{\mu_0/\varepsilon_0} = 376.73032 \text{ ohms} \quad \Rightarrow \quad Z_{fs}/4\pi = 29.97948 \approx 30\,\Omega \ . \tag{4.4.15}$$

Typically one has $\mu = \mu_0$ so then (note that ε_{rel} and K are dimensionless),

$$Z_0 = (1/4\pi)\, K \sqrt{\mu_0/\varepsilon} = (1/4\pi)\, K \sqrt{\mu_0/\varepsilon_0}\sqrt{\varepsilon_0/\varepsilon} = (1/4\pi)\, K\, Z_{fs} \sqrt{\varepsilon_0/\varepsilon} = K\,(Z_{fs}/4\pi) / \sqrt{\varepsilon/\varepsilon_0}$$

or

$$Z_0 \approx (K/\sqrt{\varepsilon_{rel}})\, 30\Omega \ . \qquad \varepsilon_{rel} \equiv \varepsilon/\varepsilon_0 \ . \tag{4.4.16}$$

We then summarize the parameters of a transmission line in terms of dimensionless real integral K :

$C = 4\pi\varepsilon/K$	capacitance per unit length	(4.4.17)
$G = 4\pi\sigma/K$	transverse conductance per unit length	
$L_e = (\mu/4\pi)\, K$	external inductance per unit length	
$Z_0 \approx (K/\sqrt{\varepsilon_{rel}})\, 30\Omega$	characteristic impedance $\mu = \mu_0$, $\varepsilon_{rel} \equiv \varepsilon/\varepsilon_0$	
$R = \mathrm{Re}(Z_{s1}+Z_{s2})$	resistance of conductors, see (4.11.34)	
$L = L_e + (1/\omega)\, \mathrm{Im}(Z_{s1}+ Z_{s2})$	total inductance, see (4.11.34)	

Z_{si} = surface impedance of conductor i, see (2.4.1) and (4.11.10)

The last three items are *not* determined by integral K and we just mention them for completeness's sake. All these equations will be more fully developed in Section 4.11 below, but we jump ahead a bit in order to display two important examples.

4.5 Example: Transmission line with widely-spaced round wires of unequal diameters

Consider a transmission line made from two round wires of radii a_1 and a_2 and center line spacing b. In the case that $b >> a_1$ and a_2, the charge distribution on each round wire is symmetric about the wire and in this situation (a rare one admittedly) we *know* the two charge distributions:

$$\alpha_1(x,y) = \alpha_1(r,\theta) = \delta(r - a_1)/(2\pi a_1)$$
$$\alpha_2(x,y) = \alpha_2(r,\theta) = \delta(r - a_2)/(2\pi a_2) \ . \tag{4.5.1}$$

The $1/(2\pi a_1)$ factor is required so that the integral of α_1 is unity as required by (4.1.3),

$$\int_{C_1} dx\, dy\, \alpha_1(x,y) = \int_0^{2\pi} d\theta \int_0^{\infty} r dr\, \alpha_1(r,\theta) = \int_0^{2\pi} d\theta \int_0^{\infty} r dr\, \delta(r - a_1)/(2\pi a_1)$$

$$= \int_0^{2\pi} d\theta\, a_1/(2\pi a_1) = 2\pi\, a_1/(2\pi a_1) = 1 \ . \qquad (4.5.2)$$

Note: The reason the α_i are symmetric is that the two conductors are so far apart that each one is essentially "in isolation" and so the charge assumes an axially symmetric distribution. An analogy would be that for two point charges far apart, the E field close to either point charge is spherically symmetric because close to one charge the field of the other can be neglected.

Our task is then to compute the integral K shown in (4.4.8),

$$K = \int_0^{2\pi} d\theta_1 \int_0^{a_1} r_1 dr_1\ [\delta(r_1 - a_1)/(2\pi a_1)] \ln(s_{21}^2/s_{11}^2)$$
$$- \int_0^{2\pi} d\theta_2 \int_0^{a_2} r_2 dr_2\ [\delta(r_2 - a_2)/(2\pi a_2)] \ln(s_{22}^2/s_{12}^2) \ .$$

$$= \int_0^{2\pi} d\theta_1\ 1/(2\pi)\ \ln(s_{21}^2/s_{11}^2)$$
$$- \int_0^{2\pi} d\theta_2\ 1/(2\pi) \ln(s_{22}^2/s_{12}^2)$$

$$= 1/(2\pi)\ \{\ \int_0^{2\pi} d\theta_1\ [\ln(s_{21}^2) - \ln(s_{11}^2)]\ -\ \int_0^{2\pi} d\theta_2\ [\ln(s_{22}^2) - \ln(s_{12}^2)]\ \} \qquad (4.5.3)$$

where we then have four integrals to evaluate.

We shall choose our V(z) potential-determining reference points $\mathbf{x}_1$ and $\mathbf{x}_2$ as shown in this drawing,

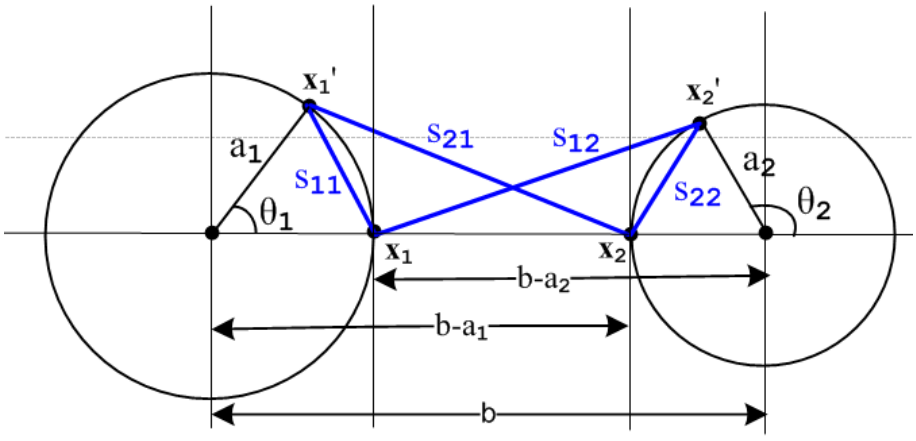


Fig 4.4

The four s_{ij} distances can be read off from the drawing using the law of cosines,

$$s_{21}^2 = a_1^2 + (b-a_1)^2 - 2\, a_1(b-a_1)\cos(\theta_1)$$

$$s_{11}^2 = a_1^2 + a_1^2 - 2\, a_1\, a_1 \cos(\theta_1) \;=\; 2a_1^2(1 - \cos(\theta_1))$$

$$s_{22}^2 = a_2^2 + a_2^2 - 2\, a_2\, a_2 \cos(\pi-\theta_2) \;=\; 2a_2^2(1 + \cos(\theta_2))$$

$$s_{12}^2 = a_2^2 + (b-a_2)^2 + 2\, a_2(b-a_2)\cos(\theta_2)\ . \qquad (4.5.4)$$

We then invoke the following integral from p 531 of GR7,

$$9.^8 \qquad \int_0^\pi \ln(a + b\cos x)\, dx = \pi \ln \frac{a + \sqrt{a^2 - b^2}}{2} \qquad [a \geq |b| > 0] \qquad \text{GW (322)(15)}$$

which we rewrite as

$$\int_0^{2\pi} d\theta \ln(A \pm B\cos\theta) = 2\pi \ln[(1/2)(A + \sqrt{A^2-B^2}\ )]\ . \qquad (4.5.5)$$

The four integrals are then easily evaluated:

$$\int_0^{2\pi} d\theta_1 \ln(s_{21}^2) \;=\; \int_0^{2\pi} d\theta_1 \ln([a_1^2 + (b-a_1)^2 - 2\, a_1(b-a_1)\cos(\theta_1)]$$

$$A = a_1^2 + (b-a_1)^2 \qquad\qquad B = 2\, a_1(b-a_1)$$

$$A^2-B^2 = [a_1^2 + (b-a_1)^2]^2 - 4\, a_1^2(b-a_1)^2 = [a_1^2 - (b-a_1)^2]^2 \;\Rightarrow \sqrt{A^2-B^2} = (b-a_1)^2 - a_1^2 > 0 \qquad b >> a_1$$

$$\Rightarrow \int_0^{2\pi} d\theta_1 \ln(s_{21}^2) = 2\pi \ln[(1/2)(\, a_1^2 + (b-a_1)^2 + (b-a_1)^2 - a_1^2\,) = 2\pi \ln[(b-a_1)^2]\ .$$

The fourth integral is the same with $1 \leftrightarrow 2$, and the different sign of the second term in s_{12}^2 makes no difference,

$$\int_0^{2\pi} d\theta_2 \ln(s_{12}^2) = 2\pi \ln[(b-a_2)^2]\ .$$

The second integral is

$$\int_0^{2\pi} d\theta_1 \ln(s_{11}^2) = \int_0^{2\pi} d\theta_1 \ln([2a_1^2(1 - \cos(\theta_1))] \qquad A = B = 2a_1^2\ ,\ A^2-B^2 = 0$$

$$= 2\pi \ln[(1/2)\, 2a_1^2] = 2\pi \ln(a_1^2)\ .$$

The third integral is similar giving

$$\int_0^{2\pi} d\theta_2 \ln(s_{22}^2) = 2\pi \ln(a_2^2) \ .$$

To summarize:

$$\int_0^{2\pi} d\theta_1 \ln(s_{21}^2) = 2\pi \ln[(b-a_1)^2]$$

$$\int_0^{2\pi} d\theta_1 \ln(s_{11}^2) = 2\pi \ln(a_1^2)$$

$$\int_0^{2\pi} d\theta_2 \ln(s_{22}^2) = 2\pi \ln(a_2^2)$$

$$\int_0^{2\pi} d\theta_2 \ln(s_{12}^2) = 2\pi \ln[(b-a_2)^2] \ . \qquad (4.5.6)$$

Then from (4.5.3) we find

$$K = 1/(2\pi) \{ \int_0^{2\pi} d\theta_1 [\ln(s_{21}^2) - \ln(s_{11}^2)] - \int_0^{2\pi} d\theta_2 [\ln(s_{22}^2) - \ln(s_{12}^2)] \}$$

$$= \ln[(b-a_1)^2] - \ln(a_1^2) - \ln(a_2^2) + \ln[(b-a_2)^2] = \ln\left[\frac{(b-a_1)^2(b-a_2)^2}{a_1^2 a_2^2}\right]$$

$$= 2 \ln\left[\frac{(b-a_1)(b-a_2)}{a_1 a_2}\right] = 2 \ln\left[\frac{b^2}{a_1 a_2}\right] \qquad \text{// since we assumed at the start that } b >> a_1, a_2$$

$$= 4 \ln(b/\sqrt{a_1 a_2}) \ . \qquad (4.5.7)$$

Therefore the transmission line parameters from (4.4.17) are,

$$K = \ln(b/\sqrt{a_1 a_2})$$

$$C = 4\pi\varepsilon/K = \pi\varepsilon / \ln(b/\sqrt{a_1 a_2})$$

$$G = 4\pi\sigma/K = \pi\sigma / \ln(b/\sqrt{a_1 a_2})$$

$$L_e = (\mu/4\pi) K = (\mu/\pi) \ln(b/\sqrt{a_1 a_2})$$

$$Z_0 = (K/\sqrt{\varepsilon_{rel}})\, 30\Omega = (1/\sqrt{\varepsilon_{rel}}) \ln(b/\sqrt{a_1 a_2})\ 120\Omega \ . \qquad (4.5.8)$$

Sometimes these formulas are written in terms of wire diameters $d_i = 2a_i$ in which case

$$K = 4 \ln[b/\sqrt{a_1 a_2}] = 4 \ln[2b/\sqrt{d_1 d_2}] = 2 \ln[4b^2/d_1 d_2] \ . \qquad (4.5.9)$$

Since we are assuming $b >> d_1, d_2$ we know that $x \equiv 2b^2/d_1 d_2 >> 1$. Therefore

$$ch^{-1}x = \ln[x + \sqrt{x^2-1}] \approx \ln(2x) \qquad \text{// an identity Siegel 8.56, then an approximation} \qquad (4.5.10)$$

so

$ch^{-1}(2b^2/d_1d_2) \approx \ln(4b^2/d_1d_2)$.

Then we can write K as

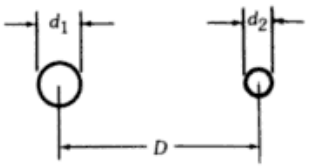
$$K = 2\ln[4b^2/d_1d_2] = 2\,ch^{-1}(2b^2/d_1d_2) \qquad (4.5.11)$$

and so

$$Z_0 = (K/\sqrt{\varepsilon_{rel}})\,30\Omega = (1/\sqrt{\varepsilon_{rel}})\,ch^{-1}(2b^2/d_1d_2)\,60\Omega\,. \qquad (4.5.12)$$

It is not easy to find expressions for C,G and L_e for the unequal radii geometry, but Z_0 does appear for example in Reference RDE page 29-23 where we find:

N. Balanced 2-wire—unequal diameters



$$Z_0 = (60/\epsilon^{1/2})\cosh^{-1}N$$

$$N = \tfrac{1}{2}[(4D^2/d_1d_2) - (d_1/d_2) - (d_2/d_1)]$$

Fig 4.5

with D = our b. For D >> d_1,d_1 this shows $N = 2D^2/(d_1d_2)$, and this then agrees with (4.5.12). This quoted result is in fact correct (with the two extra terms shown in N) even when D is not large. We shall derive this full result in Chapter 6, equation (6.3.12).

In the special case that $a_1 = a_2 \equiv a$ we get,

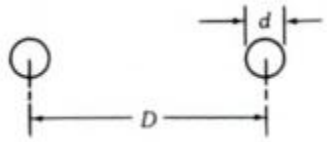
$$K = 4\ln(b/a)$$
$$C = 4\pi\varepsilon/K = \pi\varepsilon/\ln(b/a)$$
$$G = 4\pi\sigma/K = \pi\sigma/\ln(b/a)$$
$$L_e = (\mu/4\pi)\,K = (\mu/\pi)\ln(b/a)$$
$$Z_0 = (K/\sqrt{\varepsilon_{rel}})\,30\Omega = (1/\sqrt{\varepsilon_{rel}})\ln(b/a)\;120\Omega$$
$$= (1/\sqrt{\varepsilon_{rel}})\ln(2b/d)\;120\Omega \qquad d = 2a\,. \qquad (4.5.13)$$

The first three results agree with King TLT p17 (30b),

$$g \equiv \frac{\pi\sigma}{\ln(b/a)} \qquad c \equiv \frac{\pi\epsilon}{\ln(b/a)} \qquad l^e = \frac{\mu}{\pi}\ln\frac{b}{a} \qquad (30b)$$

The expression for Z_0 agrees with the RDE source quoted above,

D. Open two-wire line in air



$$Z_0 = 120 \cosh^{-1}(D/d)$$
$$\approx 276 \log_{10}(2D/d)$$
$$\approx 120 \log_e(2D/d)$$

Fig 4.6

where again D = b and for "air" $\varepsilon_{rel} = 1$.

Power Transmission Lines

Ignoring proximity effects of the ground and possible ground wires, one can consider a single phase power transmission line as fitting into this example. The first interesting number is skin depth. For aluminum at f = 60 Hz we find

$\sigma_{aluminum} = 3.7 \times 10^7$ mho/m // recall $\sigma_{copper} \approx 5.8 \times 10^7$ (annealed)
$\mu_0 = 4\pi \times 10^{-7}$ henry/m

$$\delta \equiv \sqrt{2/(\omega\mu\sigma)} \approx \sqrt{2/(2\pi f\mu_0\sigma)} = \sqrt{1/(\pi f\mu_0\sigma)}$$

```
delta := sqrt(1/(Pi*60*4*Pi*3.7)): evalf(%);
                                .01068178037
```

So $\delta \approx 1$ cm. Thus, skin effect could be significant for a very large diameter wire. Typically the individual strands of a 1500 amp cable are 1/6" in diameter or 0.2 cm in radius, so there is some slight non-uniformity in the current distribution. If the strands are not insulated one should think of this more in terms of the total cable diameter including all strand layers which might be 1".

Usually the requirement of low power loss requires that R be relatively small compared to ωL. The 1500A cable just noted has R = .02Ω per thousand feet. Similarly, the conductance G (mostly from insulator leakage) is very small compared to ωC. Thus, (4.4.12) leads to (4.4.16) stating $Z_0 \approx K\ 30\Omega$. If the full cable is 1" in diameter and the two lines are spaced 1 m apart, we can compute K from (4.5.13),

K = 4 ln(b/a) = 4 ln(1m/0.5") = 4 ln(39.37*2) = 17.5

so then from (4.4.16),

$Z_0 \approx K\ 30\Omega = 17.5 * 30\ \Omega = 524\Omega$

Rajput (p 554) claims power lines typically range from 400 to 600 Ω. See southwire.com for data on transmission line cables.

4.6 Example: A coaxial cable

A coaxial cable is the other transmission line where we know the surface charge distribution is that given by (4.5.1). The analysis of the previous section resulting in (4.5.3) is then unchanged, and we find that K is still given by (4.5.3),

$$K = 1/(2\pi)\ \{\ \int_0^{2\pi} d\theta_1\ [\ln(s_{21}^2) - \ln(s_{11}^2)]\ -\ \int_0^{2\pi} d\theta_2\ [\ln(s_{22}^2) - \ln(s_{12}^2)]\ \}\ . \qquad (4.5.3)$$

What is different is that we have a different picture describing the various s_{ij} distances. The new picture is this, where the cross section circles have radii $a_2 > a_1$:

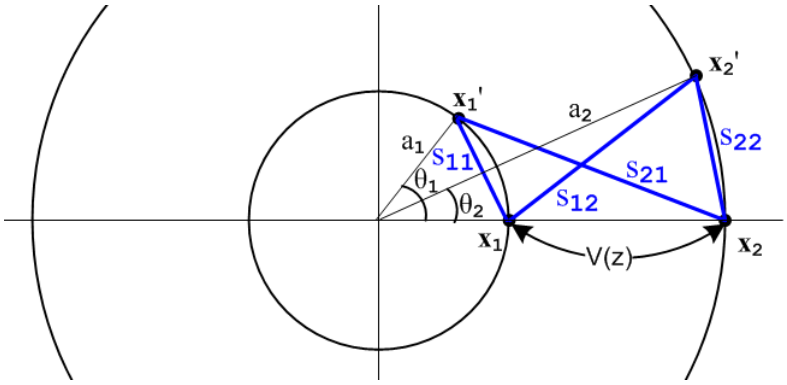


Fig 4.7

As we did in the previous section, we "read off" the s_{ij} expressions using the law of cosines:

$$s_{21}^2 = a_1^2 + a_2^2 - 2\,a_1a_2\cos(\theta_1)$$

$$s_{11}^2 = a_1^2 + a_1^2 - 2\,a_1\,a_1\cos(\theta_1) \;=\; 2a_1^2(1 - \cos(\theta_1))$$

$$s_{22}^2 = a_2^2 + a_2^2 - 2\,a_2\,a_2\cos(\theta_2) \;=\; 2a_2^2(1 - \cos(\theta_2))$$

$$s_{12}^2 = a_2^2 + a_1^2 - 2\,a_1a_2\cos(\theta_2)\ . \qquad (4.6.1)$$

Recalling,

$$\int_0^{2\pi} d\theta\ \ln(A \pm B\cos\theta) = 2\pi\ \ln[(1/2)(A + \sqrt{A^2-B^2}\ )]\ . \qquad (4.5.5)$$

we find,

$$\int_0^{2\pi} d\theta_1\ \ln(s_{21}^2) = \int_0^{2\pi} d\theta_1\ \ln[a_1^2 + a_2^2 - 2a_1a_2\cos(\theta_1)] \qquad A = a_1^2 + a_2^2 \quad B = 2a_1a_2$$

$A^2-B^2 = (a_1^2 + a_2^2)^2 - 4a_1^2a_2^2 = (a_1^2 - a_2^2)^2 \Rightarrow \sqrt{A^2-B^2} = (a_2^2 - a_1^2) > 0$ since $a_2 > a_1$

so

$$\int_0^{2\pi} d\theta_1 \ln(s_{21}^2) = 2\pi \ln[(1/2)(a_1^2 + a_2^2 + (a_2^2 - a_1^2)) = 2\pi\ln(a_2^2) \ .$$

Similarly

$$\int_0^{2\pi} d\theta_2 \ln(s_{12}^2) = 2\pi \ln[(1/2)(a_1^2 + a_2^2 + (a_2^2 - a_1^2)) = 2\pi\ln(a_2^2) \quad = \text{same as above} \ .$$

The other two integrals are,

$$\int_0^{2\pi} d\theta_1 \ln(s_{11}^2) = \int_0^{2\pi} d\theta_1 \ln[2a_1^2(1 - \cos(\theta_1))] \qquad A = B = 2a_1^2$$

$$= 2\pi \ln[(1/2)2a_1^2] = 2\pi\ln(a_1^2)$$

$$\int_0^{2\pi} d\theta_2 \ln(s_{22}^2) = \int_0^{2\pi} d\theta_1 \ln[2a_2^2(1 - \cos(\theta_2))] \qquad A = B = 2a_2^2$$

$$= 2\pi \ln[(1/2)2a_2^2] = 2\pi\ln(a_2^2) \ .$$

To summarize:

$$\int_0^{2\pi} d\theta_1 \ln(s_{21}^2) = 2\pi\ln(a_2^2)$$

$$\int_0^{2\pi} d\theta_1 \ln(s_{11}^2) = 2\pi \ln(a_1^2)$$

$$\int_0^{2\pi} d\theta_2 \ln(s_{22}^2) = 2\pi \ln(a_2^2)$$

$$\int_0^{2\pi} d\theta_2 \ln(s_{12}^2) = 2\pi \ln(a_2^2) \ . \qquad (4.6.2)$$

Then from (4.5.3) we find

$$K = 1/(2\pi) \{ \int_0^{2\pi} d\theta_1 [\ln(s_{21}^2) - \ln(s_{11}^2)] - \int_0^{2\pi} d\theta_2 [\ln(s_{22}^2) - \ln(s_{12}^2)] \}$$

$$= \ln(a_2^2) - \ln(a_1^2) - \ln(a_2^2) + \ln(a_2^2) = \ln(a_2^2/a_1^2) = 2 \ln(a_2/a_1) \ . \qquad (4.6.3)$$

The coaxial transmission line parameters are then given by,

$$\begin{aligned} C &= 4\pi\varepsilon/K && = 2\pi\varepsilon / \ln(a_2/a_1) \\ G &= 4\pi\sigma/K && = 2\pi\sigma / \ln(a_2/a_1) \\ L_e &= (\mu/4\pi)K && = (\mu/2\pi) \ln(a_2/a_1) \\ Z_0 &= (K/\sqrt{\varepsilon_{rel}}) \, 30\Omega && = (1/\sqrt{\varepsilon_{rel}}) \ln(a_2/a_1) \ 60\Omega \end{aligned} \qquad (4.6.4)$$

We verify the C and L_e parameters from http://en.wikipedia.org/wiki/Coaxial_cable ,

Fundamental electrical parameters [edit]

- Shunt capacitance per unit length, in farads per metre.[6]

$$\left(\frac{C}{h}\right) = \frac{2\pi\epsilon}{\ln(D/d)} = \frac{2\pi\epsilon_0\epsilon_r}{\ln(D/d)}$$

- Series inductance per unit length, in henrys per metre.

$$\left(\frac{L}{h}\right) = \frac{\mu}{2\pi}\ln(D/d) = \frac{\mu_0\mu_r}{2\pi}\ln(D/d)$$

Fig 4.8

To verify the Z_0 value, first recall that (the positive square root is implied here)

$$ch^{-1}x = \ln[x + \sqrt{x^2-1}\,] \qquad \text{// Spiegel identity 8.56, valid for } x \geq +1 \tag{4.6.5}$$

If we set $x \equiv \frac{1}{2}\left(\frac{a}{b} + \frac{b}{a}\right) = \frac{1}{2}\frac{a^2+b^2}{ab}$ and we assume $a > 0$ and $b > 0$, then

$$x^2 - 1 = \frac{1}{4}\frac{(a^2+b^2)^2}{a^2b^2} - 1 = \frac{1}{4}\left\{\frac{(a^2+b^2)^2 - 4a^2b^2}{a^2b^2}\right\} = \frac{1}{4}\frac{(a^2-b^2)^2}{a^2b^2}$$

$$\Rightarrow \sqrt{x^2-1} = \frac{1}{2}\frac{|b^2-a^2|}{ab} = \text{sign}(b-a)\,\frac{1}{2}\left(\frac{b}{a} - \frac{a}{b}\right)$$

$$\Rightarrow x + \sqrt{x^2-1} = \frac{1}{2}\left(\frac{a}{b} + \frac{b}{a}\right) + \text{sign}(b-a)\,\frac{1}{2}\left(\frac{b}{a} - \frac{a}{b}\right) = \begin{cases} b/a & b \geq a \\ a/b & a \geq b \end{cases}$$

$$\Rightarrow \ln\left[x + \sqrt{x^2-1}\,\right] = \ln\left[\begin{cases} b/a & b \geq a \\ a/b & a \geq b \end{cases}\right] = \text{sign}(b-a)\ln(b/a)\ .$$

Thus we have shown that (note that both sides are invariant under $a \leftrightarrow b$)

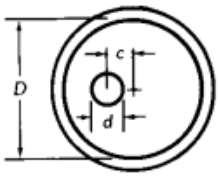
$$ch^{-1}\left[\frac{1}{2}\left(\frac{a}{b} + \frac{b}{a}\right)\right] = \text{sign}(b-a)\ln\frac{b}{a} \quad . \qquad a > 0 \text{ and } b > 0 \tag{4.6.6}$$

With this rather elaborate fact, and since we have $b > a$, we can rewrite Z_0 above as

$$Z_0 = (1/\sqrt{\varepsilon_{rel}})\,ch^{-1}\left[\frac{1}{2}\left(\frac{a}{b} + \frac{b}{a}\right)\right]\ 60\ \Omega \quad . \tag{4.6.7}$$

Again we quote from reference RDE page 29-24

U. Eccentric line



$$Z_0 = (60/\epsilon^{1/2}) \cosh^{-1} U$$

$$U = \tfrac{1}{2}\,[(D/d) + (d/D) - (4c^2/dD)]$$

Fig 4.9

In our centered case c = 0 so U = (1/2)(D/d+d/D) and we have agreement. The full off-center result is derived later in Chapter 6, equation (6.3.15).

Comment: In the examples of Sections 4.5 and 4.6, the current distributions in the involved round wires are axially symmetric. Therefore all the results of Chapter 2 apply. In particular, Chapter 2 calculates the surface impedance Z_s for a round wire in complete detail, including its limits for large and small ω. For example, at low frequency for a wire of radius a,

$$Z_s(\omega) = \frac{1}{\sigma\pi a^2} + j\omega\,\frac{\mu}{8\pi} = R_s + j\omega L_s \qquad \text{// low frequency limit} \qquad (2.4.12)$$

and one sees that R_s is the expected DC resistance and L_s is the internal impedance L_i as computed in Appendix C equation (C.3.5).

Having presented our two Examples, we now resume development of the transmission line equations.

4.7 Computation of A_z due to one conductor of a transmission line

In summary box (1.5.23) we state the following Helmholtz integral for the vector potential arising from currents in a set of conductors,

$$\mathbf{A}(\mathbf{x},\omega) = \frac{1}{4\pi}\ \Sigma_i \int \mu_i \mathbf{J}_i(\mathbf{x}',\omega)\,\frac{e^{-j\beta R}}{R}\,dV' \qquad (1.5.23)$$

where the sum Σ_i is over the conductors and μ_i is the permeability of conductor i.

In our transmission line context, and as discussed in Chapter 3, the dominant current is in the z (longitudinal) direction, while transverse currents are very small. For example, in the estimate of Section 3.7 (s) we found that $J_r/J_z < 1.4 \times 10^{-4}$ below 10 GHz and $J_r/J_z < 4.6 \times 10^{-8}$ at low frequency. Looking at the above Helmholtz solution to the Helmholtz equation, if we neglect these transverse currents, we are then in effect neglecting the transverse components of **A**, and this is what we shall do from now on:

Fact: The transverse components A_x and A_y can be neglected so that $\mathbf{A} = A_z\hat{\mathbf{z}}$. (Appendix M) (4.7.1)

Our starting point then is the following expression for the potential A_z at some arbitrary point **x** in the dielectric due to conductor C_1 of a transmission line,

$$A_{z1}(\mathbf{x},\omega) = \frac{\mu_1}{4\pi}\int_{C_1} J_{z1}(x',y',z',\omega)\frac{e^{-j\beta R}}{R}\,dx'dy'dz' \quad . \qquad R = |\mathbf{x} - \mathbf{x}'| \tag{4.7.2}$$

Here the point $\mathbf{x}' = (x',y',z')$ runs over the *volume* of C_1 and R is the distance between the observation point **x** in the dielectric and the integration point **x**'. Parameter β is for the dielectric while μ_1 is for the conductor.

In the analogous φ solution (4.1.1) everything has the same form as (4.7.2) but in (4.1.1) the charge density exists only on the conductor surface. Nevertheless, we represented that charge density as a volume density, and only later in examples set that volume density to a surface distribution. Thus, the parallel between the φ and the A_z analysis is very close, not surprising in light of (1.3.11). Another difference is that for φ the leading factor is $1/(4\pi\xi)$ where ξ was the complex dielectric constant of the dielectric. In (4.7.2) this factor is replaced by $(\mu_1/4\pi)$ where μ_1 is the magnetic permeability of the conductor.

We next make the same assumption of separation of variables to write

$$\underset{A/m^2}{J_{z1}(x,y,z)} = \underset{1/m^2}{b_1(x,y)}\ \underset{A}{i_1(z)} \tag{4.7.3}$$

where i_1 is scaled such that

$$\int_{C_1} dx\,dy\,b_1(x,y) = 1 \ . \tag{4.7.4}$$

Here $b_1(x,y)$ describes the distribution of the current density across the conductor C_1 cross section. At DC this density is a uniform constant, but at higher ω the density becomes non-uniform in two ways. First, it becomes concentrated away from the central region due to the skin effect. Second it is non-uniform in that it tends to concentrate on the portion of conductor C_1 which is closest to conductor C_2. In the corresponding equation $\rho_1(x,y,z) = \alpha_1(x,y)\,q_1(z)$ of (4.1.2), $\alpha_1(x,y)$ exists only on the conductor surface, and is generally non-uniform in the second sense noted above for $b_1(x,y)$.

As before, we can now interpret $i_1(z)$ as the *total current* in C_1 at z. Again assuming that there is no net superposed radiating antenna current, we have equal and opposite currents in the two conductors so the line is a balanced line, and then

$$i_2(z) = -\,i_1(z) = -i(z) \ . \tag{4.7.5}$$

This then leads to

$$A_{z1}(x,y,z) = \frac{\mu_1}{4\pi} \int_{-\infty}^{\infty} dz'\, i(z') \int_{C_1} dx'\, dy'\, b_1(x',y') \frac{e^{-j\beta R}}{R} \quad . \tag{4.7.6}$$

Comments regarding μ

This is a subtle subject and is not discussed in King's transmission line theory book.

If the conductor and dielectric have the same permeability so that $\mu_1 = \mu$, then there exists no "magnetic boundary" between the conductor and dielectric. The solution (4.7.2) is then smooth at this boundary, and so $A_{z1}(x,y,z)$ "naturally" satisfies these two boundary conditions,

$$\begin{aligned} &A_{z1}(\mathbf{x}+) = A_{z1}(\mathbf{x}-) \\ &(1/\mu)\partial_n A_{z1}(\mathbf{x}+) = (1/\mu_1)\, \partial_n A_{z1}(\mathbf{x}-) \end{aligned} \tag{4.7.7}$$

where $\mathbf{x}+$ is just outside the conductor surface and $\mathbf{x}-$ is just inside. The second equation here is just (1.1.46) in the case that there is no free surface current K_{free} flowing on the boundary, and indeed in our example at hand there is no such free surface current. Since we have assumed that $\mu = \mu_1$, this second boundary condition just says $\partial_n A_{z1}(\mathbf{x}+) = \partial_n A_{z1}(\mathbf{x}+)$. Since there is no magnetic boundary at the conductor/dielectric interface, the solution (4.7.2) is continuous and all its derivatives are also continuous at the boundary, since nothing special happens at that boundary. Thus, the Helmholtz integral solution provides the whole solution for A_{z1} since it meets both "boundary conditions" at this pseudo boundary.

If on the other hand we have $\mu_1 \neq \mu$, then there *is* a magnetic boundary between conductor and dielectric which we have to worry about. In this case, (4.7.2) cannot possibly satisfy the second boundary condition of (4.7.7) since, as already noted, the A_{z1} of (4.7.2) satisfies $\partial_n A_{z1}(\mathbf{x}+) = \partial_n A_{z1}(\mathbf{x}+)$. Thus, in this case (4.7.2) is not the full solution for A_{z1}. One must add a homogeneous Helmholtz equation solution to (4.7.2) in order to have a proper solution for A_{z1} that satisfies both equations in (4.7.7).

It *turns out* that the correct total A_{z1} solution can be generated by adding a certain fictitious surface current term to $\mu_1 J_{z1}$ in (4.7.2). Since such a surface current vanishes on both sides of the boundary between μ and μ_1, the Helmholtz solution due just to this surface current term is in fact a homogeneous solution to the Helmholtz equation in both the conductor and dielectric regions, away from that boundary. It turns out moreover that the correct fictitious surface current to add is in fact the magnetization surface current J_m which is created at the boundary between $\mu \neq \mu_1$. Adding this surface current is just a "trick" in order to generate the correct homogeneous adder solution so that the resulting total A_{z1} satisfies both boundary conditions in (4.7.7). Formally speaking, the $\mathbf{J}_i$ appearing in (1.5.4) and then J_{z1} in (4.7.2) should *not* include such magnetization currents since this **J** is really the **J** in Maxwell's equation curl **H** = $\partial_t \mathbf{D}$ + **J,** and this **J** does not include magnetization currents -- it includes only normal conduction currents.

In our current Chapter 4, we *want* (4.7.2) to represent the complete solution for A_{z1} and for that reason we must restrict our analysis to the situation where dielectric and all conductors have the same permeability which we shall just call μ. In practice, one normally has $\mu = \mu_1 = \mu_0$. In order to handle the more general case of $\mu \neq \mu_1$, we have to deal with the inhomogeneous adder solutions or equivalently

with the abovementioned fictitious surface current, and this complicates our analysis which is already quite complicated. So, for the moment, we now make the same assumption made by King and other authors:

Fact: From now on, conductors and dielectric must have the same permeability μ. (4.7.8)

After fully developing this special case, we shall then extend the theory in Section 4.12 to allow for $\mu \neq \mu_1$.

Appendix G shows for the round wire how the inhomogeneous adder solution is found and how it then causes the boundary conditions (4.7.7) to be met when $\mu_1 \neq \mu$.

Appendix B shows how the addition of a fictitious surface current term $\mu_0 J_m$ provides an alternate and simpler solution to the same problem of meeting boundary conditions (4.7.7) when $\mu_1 \neq \mu$. It then shows exactly how this works in the special case of a round wire.

Having now mentioned that the Helmholtz integral might not provide a total solution, the reader might fairly ask why it is that the Helmholtz integral solution $\varphi_1(\mathbf{x},\omega)$ of (4.1.1) provides a complete and viable solution to the φ Helmholtz equation, given that in general the conductor (ε_1) and dielectric (ε) have different ε values, so there should be an "electric boundary" where ε meets ε_1. The reason is that, according to (1.1.47), the boundary condition corresponding to the second line of (4.7.7) reads

$$[\varepsilon_1 E_n(\mathbf{x}+) - \varepsilon E_n(\mathbf{x}+)] = n_{free}(\mathbf{x}) \ .$$

Since we are neglecting transverse **A** components as stated in (4.7.1), and since our notation ∂_n indicates a normal conductor derivative which is transverse (to z), we have

$$\mathbf{E} = -\text{grad}\,\varphi - \partial_t \mathbf{A} \quad \Rightarrow \quad E_n = -\partial_n \varphi \tag{4.7.9}$$

so we have then this set of boundary conditions for φ_1,

$$\begin{aligned} &\varphi_1(\mathbf{x}+) = \varphi_1(\mathbf{x}-) \\ &[\varepsilon_1 \partial_n \varphi_1(\mathbf{x}+) - \varepsilon \partial_n \varphi_1(\mathbf{x}-)] = n_{free}(\mathbf{x}). \end{aligned} \tag{4.7.10}$$

These look a bit like (4.7.7) for A_{z1}. The big difference is that in this case there *does* exist a free surface charge n_{free} and it simply adjusts itself to make (4.7.10) be true. Thus, the Helmholtz integral (4.1.1) does in fact meet the required electrical boundary conditions without the need for a homogeneous solution adder term. A less formal way to state this is that, in the electrical case, we can regard the surface charge as in fact lying on the dielectric side of the boundary, and then the boundary is of no interest in our problem of analyzing fields in the dielectric.

4.8 Computation of potential A_z due to both conductors of a transmission line

Let us now write the potential at an arbitrary point $\mathbf{x}$ in the dielectric due to *both* conductors C_1 and C_2. We accept the requirement of (4.7.8) and require that all conductors have the same μ as the dielectric, so then $\mu_1 = \mu$ and $\mu_2 = \mu$. Then,

$$A_{z12}(\mathbf{x}) = A_{z1}(\mathbf{x}) + A_{z2}(\mathbf{x}) =$$

$$\frac{\mu}{4\pi} \int_{-\infty}^{\infty} dz'\, i(z') \left\{ \int_{C_1} dx_1'\, dy_1'\, b_1(x_1',y_1') \frac{e^{-j\beta R_1}}{R_1} - \int_{C_2} dx_2'\, dy_2'\, b_2(x_2',y_2') \frac{e^{-j\beta R_2}}{R_2} \right\}$$

$$R_1^2 = (x-x_1')^2 + (y-y_1')^2 + (z-z')^2 = s_1^2 + (z-z')^2 \qquad s_1^2 = (x-x_1')^2 + (y-y_1')^2 \qquad (4.8.1)$$
$$R_2^2 = (x-x_2')^2 + (y-y_2')^2 + (z-z')^2 = s_2^2 + (z-z')^2 \qquad s_2^2 = (x-x_2')^2 + (y-y_2') \ .$$

The picture going with the above equation is identical to Fig 4.1 below (4.2.1) except the points $\mathbf{x}_1'$ and $\mathbf{x}_2'$ can be in the interior of the conductors, not just on the boundary of the conductor.

4.9 Transmission Line Limit Revisited

In Section 4.3 we discussed the so-called transmission line limit of small β in the context of the scalar potential φ. We could (but won't) repeat the discussion verbatim here making the following substitutions:

$$q(z) \to i(z) \qquad \alpha_i \to b_i \qquad \varphi_{12} \to A_{z12} \qquad \frac{1}{4\pi\xi} \to \frac{\mu}{4\pi} \ .$$

The conclusion is that in the transmission line limit (small β, long wavelength $\lambda = 2\pi/\beta$) we may write

$$A_{z12}(\mathbf{x}) = \frac{\mu}{4\pi}\, i(z) \int_{-\infty}^{\infty} dz' \left\{ \int_{C_1} dx_1'\, dy_1'\, b_1(x_1',y_1') \frac{1}{R_1} - \int_{C_2} dx_2'\, dy_2'\, b_2(x_2',y_2') \frac{1}{R_2} \right\} \qquad (4.9.1)$$

which is analogous to (4.3.7).

4.10 General Calculation of W(z)

We now introduce two new points $\mathbf{x}_1$ and $\mathbf{x}_2$. The point $\mathbf{x}_1$ lies on C_1 in the z = z plane, while $\mathbf{x}_2$ lies on C_2 in this same plane. We then evaluate $A_{z12}(\mathbf{x})$ at $\mathbf{x} = \mathbf{x}_1$ and subtract from that $A_{z12}(\mathbf{x})$ at $\mathbf{x} = \mathbf{x}_2$ and in this way we obtain the A_z potential difference between the surfaces of the two conductors at z = z which we shall call W(z). Recall,

Fact 5: On each conductor boundary, $A_z \approx$ constant in the extreme or strong skin effect regimes. *(3.7.20)*

Thus, assuming the small δ regime and treating $A_z \approx$ constant as an equality, the A_z potential difference will be independent of the locations of $\mathbf{x}_2$ and $\mathbf{x}_1$ as long as they are on their respective surfaces and both have z = z. For this reason, the A_z potential difference is a function only of z. Thus we write, using two copies of (4.9.1),

$$W(z) \equiv A_{z12}(\mathbf{x}_1) - A_{z12}(\mathbf{x}_2)$$

$$= \frac{\mu}{4\pi}\ i(z)\ \int_{-\infty}^{\infty} dz' \{\ \int_{C_1} dx_1'\, dy_1'\, b_1(x_1',y_1') \frac{1}{R_{11}} - \int_{C_2} dx_2'\, dy_2'\, b_2(x_2',y_2') \frac{1}{R_{12}}\ \}$$

$$- \frac{\mu}{4\pi}\ i(z)\ \int_{-\infty}^{\infty} dz' \{\ \int_{C_1} dx_1'\, dy_1'\, b_1(x_1',y_1') \frac{1}{R_{21}} - \int_{C_2} dx_2'\, dy_2'\, b_2(x_2',y_2') \frac{1}{R_{22}}\ \} \quad (4.10.1)$$

where

$$\begin{aligned}
R_{11}^2 &= (x_1-x_1')^2 + (y_1-y_1')^2 + (z-z')^2 = s_{11}^2 + (z-z')^2 & s_{11}^2 &= (x_1-x_1')^2 + (y_1-y_1')^2 \\
R_{12}^2 &= (x_1-x_2')^2 + (y_1-y_2')^2 + (z-z')^2 = s_{12}^2 + (z-z')^2 & s_{12}^2 &= (x_1-x_2')^2 + (y_1-y_2')^2 \\
R_{22}^2 &= (x_2-x_2')^2 + (y_2-y_2')^2 + (z-z')^2 = s_{22}^2 + (z-z')^2 & s_{22}^2 &= (x_2-x_2')^2 + (y_2-y_2')^2 \\
R_{21}^2 &= (x_2-x_1')^2 + (y_2-y_1')^2 + (z-z')^2 = s_{21}^2 + (z-z')^2 & s_{21}^2 &= (x_2-x_1')^2 + (y_2-y_1')^2 \ . \quad (4.10.2)
\end{aligned}$$

The picture going with the above equation is identical to Fig 4.2 below (4.4.2) except, once again, the points $\mathbf{x}_1'$ and $\mathbf{x}_2'$ can be in the interior of the conductors, not just on the surface of the conductors. Also, we replace the figure's double arrow label V(z) with W(z). We then reorder the four terms to get

$$W(z) \quad (4.10.3)$$
$$= \frac{\mu}{4\pi}\ i(z) \int_{-\infty}^{\infty} dz' \{ \int_{C_1} dx_1'\, dy_1'\, b_1(x_1',y_1') \left(\frac{1}{R_{11}} - \frac{1}{R_{21}} \right) - \int_{C_2} dx_2'\, dy_2'\, b_2(x_2',y_2') \left(\frac{1}{R_{12}} - \frac{1}{R_{22}} \right) \} .$$

The dz' integrals are the same as those done in Section 4.4 and we then arrive at

$$W(z) = i(z) \frac{\mu}{4\pi} \{ \int_{C_1} dx_1'\, dy_1'\, b_1(x_1',y_1') \ln(s_{21}^2/s_{11}^2) - \int_{C_2} dx_2'\, dy_2'\, b_2(x_2',y_2') \ln(s_{22}^2/s_{12}^2) \}$$

$$\begin{aligned}
s_{21}^2 &= (x_2-x_1')^2 + (y_2-y_1')^2 & s_{22}^2 &= (x_2-x_2')^2 + (y_2-y_2')^2 \quad (4.10.4) \\
s_{11}^2 &= (x_1-x_1')^2 + (y_1-y_1')^2 & s_{12}^2 &= (x_1-x_2')^2 + (y_1-y_2')^2
\end{aligned}$$

which is analogous to (4.4.6) for V(z). The corresponding drawing is analogous to Fig 4.3 where, once again, the integration points $\mathbf{x}_1'$ and $\mathbf{x}_2'$ are inside the conductor :

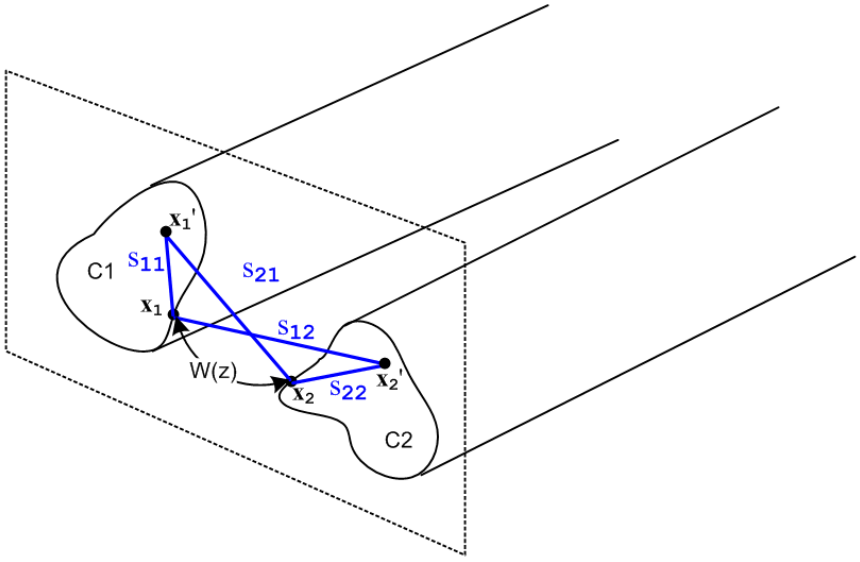


Fig 4.10

Equation (4.10.4) expresses the A_z potential between the two transmission line conductors at some plane z in terms of the current distributions b_i within the conductors.

Now, the Stokes theorem applied to **B** = curl **A** says

$$\text{curl } \mathbf{A} = \mathbf{B} \quad \Leftrightarrow \quad \oint \mathbf{A} \bullet d\mathbf{s} \; = \int_S \mathbf{B} \bullet d\mathbf{A} \; . \qquad (1.1.39)$$

Consider the red loop shown in this top view of the two transmission line conductors. The loop is intended to have a tiny width dz, and the top view obscures the fact that each conductor has an arbitrary cross section. The loop makes contact with the points $\mathbf{x_1}$ and $\mathbf{x_2}$ shown in the previous figure,

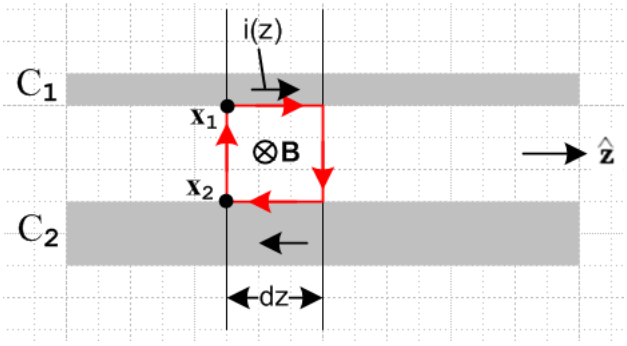


Fig 4.11

Since we neglect any transverse components of **A**, the Stokes theorem says

$$[A_{z1}(\text{top}) - A_{z2}(\text{bottom})]\, dz \; = [\text{ magnetic flux through red loop}] \; = \int_S \mathbf{B} \bullet d\mathbf{A} \; . \qquad (4.10.5)$$

If we regard the two short dz length conductor pieces as forming a tiny "inductor", closed on the ends by the vertical red lines, we can use this definition of inductance to compute the inductance of that inductor:

[magnetic flux through red loop] $= (L_e dz)\, i(z)$. (4.10.6)

Here $(L_e dz)$ is the inductance of our tiny loop, so L_e is the transmission line inductance per unit length. We know (as in Appendix C) that there will be magnetic flux inside the conductors as well as between them, and for that reason L_e as defined here only accounts for the "external" inductance of the transmission line, again see Appendix C.

Since $[A_{z1}(\text{top}) - A_{z2}(\text{bottom})] = W(z)$ according to (4.10.1), we may combine (4.10.5) and (4.10.6) to obtain

$$W(z) = L_e\, i(z) \,. \tag{4.10.7}$$

Therefore from (4.10.4) we have found that

$$L_e = \frac{W(z)}{i(z)} = \frac{\mu}{4\pi} K_L \tag{4.10.8}$$

where K_L is the following dimensionless real number,

$$K_L \equiv \int_{C_1} dx_1'\, dy_1'\, b_1(x_1',y_1') \ln(s_{21}^2/s_{11}^2) - \int_{C_2} dx_2'\, dy_2'\, b_2(x_2',y_2') \ln(s_{22}^2/s_{12}^2) \,. \tag{4.10.9}$$

This number is reminiscent of the number K obtained in Section 4.4,

$$K \equiv \int_{C_1} dx_1'\, dy_1'\, \alpha_1(x_1',y_1') \ln(s_{21}^2/s_{11}^2) - \int_{C_2} dx_2'\, dy_2'\, \alpha_2(x_2',y_2') \ln(s_{22}^2/s_{12}^2) \,. \tag{*4.4.8*}$$

In the next section it will be shown that these two dimensionless numbers are exactly the same.

4.11 The Classic Transmission Line Equations

The results of the previous sections of this chapter may be succinctly summarized as:

(4.11.1)

$$\frac{1}{C'} = \frac{V(z)}{q(z)} = \frac{1}{4\pi\xi} K \tag{*4.4.7*}$$

$$L_e = \frac{W(z)}{i(z)} = \frac{\mu}{4\pi} K_L \tag{*4.10.8*}$$

$$K \equiv \int_{C_1} dx_1'\, dy_1'\, \alpha_1(x_1',y_1') \ln(s_{21}^2/s_{11}^2) - \int_{C_2} dx_2'\, dy_2'\, \alpha_2(x_2',y_2') \ln(s_{22}^2/s_{12}^2) \tag{*4.4.8*}$$

$$K_L \equiv \int_{C_1} dx_1'\, dy_1'\, b_1(x_1',y_1') \ln(s_{21}^2/s_{11}^2) - \int_{C_2} dx_2'\, dy_2'\, b_2(x_2',y_2') \ln(s_{22}^2/s_{12}^2) \tag{*4.10.9*}$$

Notice that we have made no assumptions whatsoever about the cross-sectional shape of the transmission line. We have only assumed that the transverse dimensions are small compared to the wavelength λ that corresponds to β -- this was the transmission line limit.

There are several equations from Chapter 1 we shall now press into service:

$$\mathbf{E} = -\,\text{grad}\,\varphi - \partial_t \mathbf{A} \qquad \textit{(1.3.1)}$$
$$\text{div}\,\mathbf{A} = -\,\mu\varepsilon\,\partial_t\varphi - \mu\sigma\varphi\ . \qquad \text{// the King gauge} \qquad \textit{(1.3.18)}$$

In the frequency domain these become,

$$\mathbf{E} = -\,\text{grad}\,\varphi - j\omega\mathbf{A}$$
$$\text{div}\,\mathbf{A} = -\,j\,(\beta^2/\omega)\,\varphi\ . \qquad \text{// the King gauge, see (1.5.1) and (1.5.5) re } \beta^2 \qquad (4.11.2)$$

According to the Fact stated in (4.7.1), potential **A** has only component A_z, so these equations become

$$E_z(\mathbf{x}) = -\,\partial_z\varphi(\mathbf{x}) - j\omega A_z(\mathbf{x})$$
$$\partial_z A_z(\mathbf{x}) = -\,j\,(\beta^2/\omega)\,\varphi(\mathbf{x})\ . \qquad (4.11.3)$$

However, as was shown at the end of Step 1 below (3.7.8), the second line of (4.11.3) can only be justified in the strong or extreme skin effect regimes, and we continue then to assume our transmission line is operating at sufficiently high ω to be in the small δ regime.

The potentials in the above equations are those due to both conductors and were denoted as φ_{12} and A_{z12} in the previous sections. We then rewrite the above as

$$E_z(\mathbf{x}) = -\,\partial_z\varphi_{12}(\mathbf{x}) - j\omega A_{z12}(\mathbf{x})$$
$$\partial_z A_{z12}(\mathbf{x}) = -\,j\,(\beta^2/\omega)\varphi_{12}(\mathbf{x}) \qquad (4.11.4)$$

Recall now the conductor-surface-located points $\mathbf{x}_1$ and $\mathbf{x}_2$ as shown for example in Fig 4.10. If we evaluate each of the above equations at $\mathbf{x} = \mathbf{x}_1$ and then $\mathbf{x} = \mathbf{x}_2$ and then subtract, we get

$$E_z(\mathbf{x}_1) - E_z(\mathbf{x}_2) = -\partial_z[\varphi_{12}(\mathbf{x}_1) - \varphi_{12}(\mathbf{x}_2)] - j\omega[A_{z12}(\mathbf{x}_1) - A_{z12}(\mathbf{x}_2)]$$
$$\partial_z[A_{z12}(\mathbf{x}_1) - A_{z12}(\mathbf{x}_2)] = -\,j\,(\beta^2/\omega)[\varphi_{12}(\mathbf{x}_1) - \varphi_{12}(\mathbf{x}_2)]\ . \qquad (4.11.5)$$

Then using these definitions (again, we are assuming the strong or extreme skin depth regime)

$$V(z) \equiv \varphi_{12}(\mathbf{x}_1) - \varphi_{12}(\mathbf{x}_2) \qquad \textit{(4.4.1)}$$
$$W(z) \equiv A_{z12}(\mathbf{x}_1) - A_{z12}(\mathbf{x}_2) \qquad \textit{(4.10.1)}$$

we may rewrite (4.11.5) in this simple manner,

$$E_z(\mathbf{x}_1) - E_z(\mathbf{x}_2) = -\,\partial_z V - j\omega W \qquad (4.11.6a)$$
$$\partial_z W = -\,j\,(\beta^2/\omega)\,V\ . \qquad (4.11.6b)$$

The quantity $E_z(\mathbf{x_1})$ is the longitudinal electric field at point $\mathbf{x_1}$ on the surface of conductor C_1. It is related to the conductor's at-the-surface current density by $J_z(\mathbf{x_1}) = \sigma\, E_z(\mathbf{x_1})$. If the conductor were "perfect", we would have $\sigma = \infty$ and $E_z(\mathbf{x_1}) = 0$, but real conductors are not perfect. However, since we are assuming the strong or extreme skin effect all along here in our analysis, we do know that $E_z(\mathbf{x_1})$ and $E_z(\mathbf{x_2})$ are very small.

Our theory now has an inconsistency. At the start of Section 4.4 we replaced the approximate fact that $\varphi \approx$ constant on the conductor boundaries with φ = constant. Similarly at the start of Section 4.10 we replaced $A_z \approx$ constant on the conductor boundaries with A_z = constant. This allowed us to define V(z) and W(z) as if they depended only on coordinate z. We know that for a general transmission line operating at $\omega > 0$, the current density J_z inside the conductors will not be uniformly distributed. It will be larger in the conductor region closest to the other conductor. This "proximity effect" is discussed in Appendix P from an eddy current point of view, see Fig P.13 for an example. The J_z current non-uniformity can be very dramatic as for example in a transmission line having this cross section:



Fig 4.12

Since J_z is non-uniform in each conductor, so is E_z, and so we expect $E_z(\mathbf{x_1})$ to be a strong function of the point $\mathbf{x_1}$ on the perimeter of C_1, certainly in the above cross section example. Recall that E_z is continuous through the boundary, so $\mathbf{x_1}$ being on the boundary is unambiguous. This means that the left side of (4.11.6a) is a function of (x_1,y_1,z) and (x_2,y_2,z) whereas the right side in our theory is a function only of z. To remedy this inconsistency, we now have to think of V and W as having very slight dependence on $\mathbf{x_1}$ and $\mathbf{x_2}$ which we generally ignore, but which we must face up to in (4.11.6a). This is a manifestation of the fact that in reality $\varphi \approx$ constant and $A_z \approx$ constant on the boundaries (with $\approx$ and not =). In the extreme skin effect regime (think a very good conductor), the left side of (4.11.6a) can be a violent function of $\mathbf{x_1}$ and $\mathbf{x_2}$ as in the case of the above figure, but the left side is always very small, even where it is largest, and its variation can be accommodated by the right side of (4.11.6a) which is the difference of large valued functions which vary only slightly with $\mathbf{x_1}$ and $\mathbf{x_2}$. We now proceed in our development of the transmission line equations with this understanding of the functions V and W.

As shown in (2.4.1), $E_z(\mathbf{x_1})$ can be related to the total current in the conductor i(z) by a quantity known as the surface impedance Z_s, so

$$E_z(\mathbf{x_1}) = Z_{s1}(\mathbf{x_1})\ i_1(z) \qquad E_{z2}(\mathbf{x_2}) = Z_{s2}(\mathbf{x_2})\ i_2(z)\ . \tag{4.11.7}$$

The surface impedance of a perfect conductor is zero. Since $i_1(z) = -i_2(z) = i(z)$, we rewrite (4.11.6) as,

$$[Z_{s1}(\mathbf{x_1}) + Z_{s2}(\mathbf{x_2})]\ i(z) = -\,\partial_z V(z) - j\omega W(z) \tag{4.11.8a}$$

$$\partial_z W(z) = -\,j\,(\beta^2/\omega)\,V(z)\ . \tag{4.11.8b}$$

We just noted that $E_z(\mathbf{x}_1)$ may vary violently around the perimeter of conductor C_1, especially for a cross section like that shown above. This of course means that $Z_{s1}(\mathbf{x}_1)$ also varies violently on the perimeter, causing equation (4.11.8a) to have the same inconsistency as (4.11.6a).

We now perform a certain slight of hand. We back up to the first equation of (4.11.4) evaluated at $\mathbf{x} = \mathbf{x}_1$,

$$E_z(\mathbf{x}_1) = - \partial_z \varphi_{12}(\mathbf{x}_1) - j\omega A_{z12}(\mathbf{x}_1)$$

or

$$Z_{s1}(\mathbf{x}_1)\ i_1(z) = - \partial_z \varphi_{12}(\mathbf{x}_1) - j\omega A_{z12}(\mathbf{x}_1) \ . \qquad (4.11.9)$$

Multiply both sides by $J_z(\mathbf{x}_1)$, integrate over the boundary of C_1 (all at z = z) , then divide by the integral of $J_z(\mathbf{x}_1)$ over the boundary. The left side so obtained we define as the *constant* Z_{s1} :

$$Z_{s1} \equiv \frac{\int_{C1} ds\ Z_{s1}(\mathbf{x}_1)\ J_z(\mathbf{x}_1)}{\int_{C1} ds\ J_z(\mathbf{x}_1)} \qquad \varphi_{12}(C_1) \equiv \frac{\int_{C1} ds\ \varphi_{12}(\mathbf{x}_1)\ J_z(\mathbf{x}_1)}{\int_{C1} ds\ J_z(\mathbf{x}_1)} \qquad (4.11.10a)$$

while $\varphi_{12}(C_1)$ is the weighted mean value of φ_{12} over the boundary of C_1 and similarly for $A_{z12}(C_1)$. We do a similar average over C_2 to obtain Z_{s2} and $\varphi_{12}(C_2)$ and $A_{z12}(C_2)$,

$$Z_{s2} \equiv \frac{\int_{C2} ds\ Z_{s2}(\mathbf{x}_2)\ J_z(\mathbf{x}_2)}{\int_{C2} ds\ J_z(\mathbf{x}_2)} \qquad \varphi_{12}(C_2) \equiv \frac{\int_{C2} ds\ \varphi_{12}(\mathbf{x}_2)\ J_z(\mathbf{x}_2)}{\int_{C2} ds\ J_z(\mathbf{x}_2)} \ . \qquad (4.11.10b)$$

Equation (4.11.9) now becomes,

$$Z_{s1}\ i_1(z) = - \partial_z \varphi_{12}(C_1) - j\omega A_{z12}(C_1) \ . \qquad (4.11.11)$$

We then *slightly* redefine V and W as follows,

$$V(z) \equiv \varphi_{12}(C_1) - \varphi_{12}(C_2)$$
$$W(z) \equiv A_{z12}(C_1) - A_{z12}(C_2) \qquad (4.11.12)$$

Performing the same weighted average on (4.11.8b), we arrive at this adjusted version of (4.11.8),

$$[Z_{s1} + Z_{s2}]\ i(z) = - \partial_z V(z) - j\omega W(z)$$
$$\partial_z W(z) = - j\ (\beta^2/\omega)\ V(z) \ . \qquad (4.11.13)$$

where now Z_{s1} and Z_{s2} are the constants just defined and V(z) and W(z) are given by (4.11.12).

Comment: For transmission lines made of "thin wires", which means thin relative to their spacing, J_z is close to uniform over each conductor and the above averaging process is not necessary.

The circle now closes when we insert into (4.11.13) the fact that $W = L_e\ i(z)$ from (4.11.1),

$$(Z_{s1} + Z_{s2})\ i(z) = - \partial_z V - j\omega\ L_e\ i(z)$$
$$L_e\ \partial_z\ i(z) = - j\ (\beta^2/\omega)\ V$$

which we then rearrange as

$$\partial_z V(z) = - [\ Z_{s1} + Z_{s1} + j\omega L_e]\ i(z)$$
$$\partial_z\ i(z) = - [\ j\beta^2/(\omega L_e)]\ \ V(z)\ . \qquad (4.11.14)$$

These are the classic **transmission line equations**. They are usually written in this form:

$$\frac{\partial V(z)}{\partial z} = -\mathbf{z}\, i(z) \qquad \frac{\partial i(z)}{\partial z} = -\mathbf{y}\, V(z)$$

with

$$\mathbf{z} = R + j\omega L \qquad \mathbf{y} = G + j\omega C\ . \tag{4.11.15}$$

Note: We have been using bold notation only for vectors, and we now break that guideline by bolding these complex quantities **z** and **y**. Our purpose for this bolding is to distinguish them from Cartesian coordinates z and y which typically appear in the same problem. In King's books, all complex parameters are put in bold font, but we do this only for **z** and **y**.

Jumping the gun a bit, if we assume now a traveling-wave z dependence $e^{j(\omega t - \beta_d z)}$ for both V(z) and i(z), where β_d is the wave's wavenumber in the dielectric, then $\partial_z \rightarrow -j\beta_d$ and the transmission line equations become

$$-j\beta_d V(z) = -\mathbf{z}\, i(z) \qquad -j\beta_d i(z) = -\mathbf{y}\, V(z)$$

or

$$-j\beta_d = -\mathbf{z}\, i(z)/V(z) \qquad -j\beta_d = -\mathbf{y}\, V(z)/i(z)$$

so equating the two expressions gives

$$-\mathbf{z}\, i(z)/V(z) = -\mathbf{y}\, V(z)/i(z) \quad \Rightarrow \quad \mathbf{z}/\mathbf{y} = [V(z)/i(z)]^2$$

and we then have,

$$Z_0 \equiv V(z)/i(z) = \sqrt{\frac{\mathbf{z}}{\mathbf{y}}} = \sqrt{\frac{R + j\omega L}{G + j\omega C}} \tag{4.11.16}$$

where by definition Z_0 is the characteristic impedance of the transmission line.

Here, **z** and **y** are called the transmission line **impedance** and **admittance**, and the four numbers R,L,G,C are defined to be the appropriate real and imaginary parts. Comparing (4.11.14) and (4.11.15), we may therefore conclude that:

$$\mathbf{z} = R + j\omega L = Z_{s1} + Z_{s2} + j\omega L_e \tag{4.11.17}$$

$$\mathbf{y} = G + j\omega C = j\beta^2/(\omega L_e)\ . \tag{4.11.18}$$

The expression for **z** seems quite reasonable since $\omega L_e = X_L$ = inductive reactance, but the expression for **y** seems a bit unusual. This is because we still have more work to do.

There is one more equation we have not yet utilized. Recall from Chapter 1 the integral form of the equation of continuity, which in the frequency domain takes this form,

$$\text{div}\,\mathbf{J} = -j\omega\rho \quad \Leftrightarrow \quad -j\omega\left[\int_V \rho\, dV\right] = \int_S \mathbf{J} \bullet d\mathbf{S}\ . \tag{1.1.35}$$

We now apply this to a Gaussian box (blue) whose faces have the same shape as the conductor cross section but are slightly larger than that cross section so as to include the conductor surface charge :

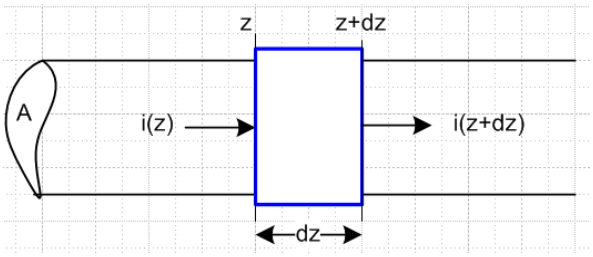


Fig 4.13

Ignoring transverse current out the sides of the box (since dz is tiny), we get

$-j\omega[q(z)dz] = i(z+dz) - i(z)$ = total current flowing out of the box

which then says

$$\partial_z\, i(z) = -j\omega q(z)\ . \qquad (4.11.19)$$

From summary box (4.11.1) recall that $q(z) = C'V(z)$ so we get

$$\partial_z\, i(z) = -\,[\,j\omega C'\,]\, V(z)\ . \qquad (4.11.20)$$

Comparison with the second equation of (4.11.14) we find the following identity,

$$-\,[\,j\beta^2/(\omega L_e)\,] \;=\; -\,[\,j\omega C'\,]$$

or

$$L_e C' = \beta^2/\omega^2 = \mu\xi \quad . \qquad \text{// see (1.5.1) regarding } \beta^2 \qquad (4.11.21)$$

Then we can write (4.11.18) as

$$\mathbf{y} = G + j\omega C = j\beta^2/(\omega L_e) \;=\; j\,(\beta^2/\omega^2)\,(\omega/L_e) = j\,(L_e C')\,(\omega/L_e) \;= j\omega C'\ . \qquad (4.11.22)$$

Thus, the line capacitance C is the real part of complex capacitance, $C = \mathrm{Re}(C')$, and $G = -\,\omega\,\mathrm{Im}(C')$.

Digression on the meaning of C'

Back in Section 1.5 (c) we discussed the fact that $n_c = (\xi/\varepsilon)\, n_s$ which relates actual surface charge n_s to the adjusted surface charge density n_c which allows for dielectric leakage. This relationship (1.5.17) was derived in two different ways. As noted in Comment 3 at the start of Section 4.1, and looking at (4.1.1) and (4.1.2), one sees that the linear charge density q(z) which appears in all our equations is in fact related to n_c and not n_s, so we momentarily shall refer to q(z) as $q_c(z)$. Then $q_c(z) = \int n_c\, dxdy$ = an integral over the conductor surface for length dz. The *actual* charge on the surface of this piece of conductor is $q_s(z) \equiv$

$\int n_s\, dxdy$ and therefore $q_c/q_s = n_c/n_a = (\xi/\varepsilon)$. The capacitance C per unit length of our transmission line is defined by $q_s = C\, V(z)$. The complex capacitance C', which includes the effect of dielectric leakage current, is defined by $q_c = C'\, V(z)$. Therefore

$$C'/C = q_c/q_s = (\xi/\varepsilon) \quad . \tag{4.11.23}$$

and so then from (4.11.22), (4.11.23) and (1.5.1),

$$\mathbf{y} = G + j\omega C = j\omega C' = j\omega(\xi/\varepsilon)C = j\omega\,(1 - j\sigma/\varepsilon\omega)C = j\omega C + (\sigma/\varepsilon)C \tag{4.11.24}$$

so that

$$G = (\sigma/\varepsilon)C \quad . \tag{4.11.25}$$

We saw an example of (4.11.23) in (1.5.19) for a parallel plate capacitor, and more generally in (4.4.10).

We may now rewrite the first equation in summary box (4.11.1) as

$$\frac{1}{C'} = \frac{1}{4\pi\xi}\, K \qquad \Rightarrow \qquad \frac{1}{C} = \frac{1}{4\pi\varepsilon}\, K \quad . \tag{4.11.26}$$

Next, combining (4.11.21) and (4.11.23) we find that

$$L_e C' = \mu\xi \tag{4.11.27}$$

$$L_e C = \mu\varepsilon = 1/v^2 \tag{4.11.28}$$

where v is the speed of light in the dielectric. Now the second equation in (4.11.1) says that

$$L_e = \frac{\mu}{4\pi}\, K_L \, . \tag{4.11.29}$$

Inserting (4.11.29) for L_e and (4.11.26) for C' into (4.11.27) gives

$$\left(\frac{\mu}{4\pi}\, K_L\right)(4\pi\xi/K) = \mu\xi$$

or

$$K_L = K \, . \tag{4.11.30}$$

This is a remarkable connection between our two seemingly unrelated constants K and K_L,

$$K \equiv \int_{C_1} dx_1'\, dy_1'\, \alpha_1(x_1',y_1') \ln(s_{21}^2/s_{11}^2) - \int_{C_2} dx_2'\, dy_2'\, \alpha_2(x_2',y_2') \ln(s_{22}^2/s_{12}^2) \qquad \textit{(4.4.8)}$$

$$K_L \equiv \int_{C_1} dx_1'\, dy_1'\, b_1(x_1',y_1') \ln(s_{21}^2/s_{11}^2) - \int_{C_2} dx_2'\, dy_2'\, b_2(x_2',y_2') \ln(s_{22}^2/s_{12}^2) \quad . \qquad \textit{(4.10.9)}$$

Since K involves a line integral of linear surface charge densities α_i whereas K_L involves a full cross sectional area integral of the current densities b_i, it seems unlikely these integrals would be equal, but they are equal.

An example of $K = K_L$

The equality even seems unlikely in a case with symmetric densities on round wires, so let's do a check using our Section 4.5 example with widely-space round wires of unequal diameters. The first thing we need is a new picture to display the "kinematics" of the K_L integral (since densities are symmetric, one should regard this picture as having b much larger than shown relative to a_1 and a_2),

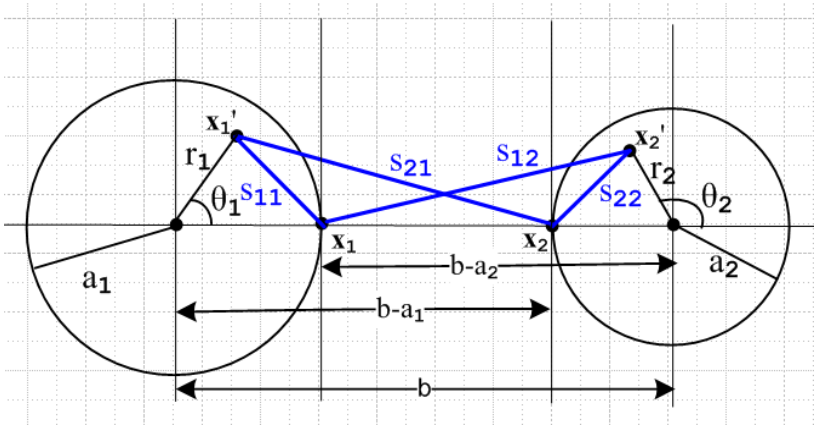


Fig 4.14

As before, we read off the four distances of interest using the law of cosines. The new distances are of course all different than they were before since $\mathbf{x}_1'$ and $\mathbf{x}_2'$ are now each integrated over their respective disks instead of the bounding circles.

$$s_{21}^2 = r_1^2 + (b-a_1)^2 - 2\, r_1(b-a_1)\cos(\theta_1)$$
$$s_{11}^2 = r_1^2 + a_1^2 - 2\, r_1\, a_1 \cos(\theta_1)$$
$$s_{22}^2 = r_2^2 + a_2^2 + 2\, r_2\, a_2 \cos(\theta_2)$$
$$s_{12}^2 = r_2^2 + (b-a_2)^2 + 2\, r_2(b-a_2)\cos(\theta_2)\ .$$

The integration rule is still

$$\int_0^{2\pi} d\theta\, \ln(A \pm B\cos\theta) = 2\pi \ln[(1/2)(A + \sqrt{A^2-B^2}\ )]\ . \qquad (4.5.5)$$

The first integral is:

$$\int_0^{2\pi} d\theta_1\, \ln(s_{21}^2) = \int_0^{2\pi} d\theta_1 \ln([r_1^2 + (b-a_1)^2 - 2\, r_1(b-a_1)\cos(\theta_1)]$$

$$A = r_1^2 + (b-a_1)^2 \qquad B = 2\, r_1(b-a_1)$$

$$A^2\text{-}B^2 = [r_1^2 + (b\text{-}a_1)^2]^2 - 4\, r_1^2(b\text{-}a_1)^2 = [r_1^2 - (b\text{-}a_1)^2]^2 \Rightarrow \sqrt{A^2\text{-}B^2} = (b\text{-}a_1)^2 - r_1^2 > 0 \qquad b >> a_1$$

$$\Rightarrow \int_0^{2\pi} d\theta_1 \ln(s_{21}^2) = 2\pi \ln[(1/2)(\, r_1^2 + (b\text{-}a_1)^2 + (b\text{-}a_1)^2 - r_1^2\,) = 2\pi \ln[(b\text{-}a_1)^2]$$

But this integral is the same as before! The s_{11}^2 integral is obtained from the above with $b\text{-}a_1 \rightarrow a_1$

$$\int_0^{2\pi} d\theta_1 \ln(s_{11}^2) = 2\pi \ln(a_1^2)$$

which is also the same as before. The other two integrals are found from $1 \rightarrow 2$. Our integral summary is then exactly the same as (4.5.6),

$$\int_0^{2\pi} d\theta_1 \ln(s_{21}^2) = 2\pi \ln[(b\text{-}a_1)^2]$$

$$\int_0^{2\pi} d\theta_1 \ln(s_{11}^2) = 2\pi \ln(a_1^2)$$

$$\int_0^{2\pi} d\theta_2 \ln(s_{22}^2) = 2\pi \ln(a_2^2)$$

$$\int_0^{2\pi} d\theta_2 \ln(s_{12}^2) = 2\pi \ln[(b\text{-}a_2)^2] \quad . \qquad (4.5.6)$$

We now assume that the current densities b_i each have radial symmetry ("widely spaced wires") ,

$$b_1(r_1,\theta_1) = b_1(r_1) \qquad (4.11.31)$$

where $b_1(r_1)$ is a completely arbitrary function, with the following normalization of (4.7.4),

$$\int_0^{2\pi} d\theta_1 \int_0^{a_1} r_1 dr_1\, b_1(r_1) = 1 \qquad \Rightarrow \qquad \int_0^{a_1} r_1 dr_1\, b_1(r_1) = 1/2\pi \quad . \qquad (4.11.32)$$

We now proceed to calculate the constant K_L

$$K_L = \int_0^{2\pi} d\theta_1 \int_0^{a_1} r_1 dr_1\, b_1(r_1) \ln(s_{21}^2/s_{11}^2) - \int_0^{2\pi} d\theta_2 \int r_2 dr_2\, b_2(r_2) \ln(s_{22}^2/s_{12}^2)$$

$$= \int_0^{a_1} r_1 dr_1\, b_1(r_1) \int_0^{2\pi} d\theta_1 \ln(s_{21}^2/s_{11}^2) - \int_0^{a_2} r_2 dr_2 b_2(r_2) \int_0^{2\pi} d\theta_2 \ln(s_{22}^2/s_{12}^2)$$

$$= 2\pi \int_0^{a_1} r_1 dr_1\, b_1(r_1) [\ln[(b\text{-}a_1)^2] - \ln(a_1^2)] - \int_0^{a_2} r_2 dr_2 b_2(r_2) [\ln[(b\text{-}a_2)^2] - \ln(a_2^2)]$$

$$= 2\pi [\ln[(b\text{-}a_1)^2/a_1^2] \int_0^{a_1} r_1 dr_1\, b_1(r_1) - 2\pi [\ln[(b\text{-}a_2)^2/a_2^2] \int_0^{a_2} r_2 dr_2\, b_2(r_2)$$

$$= [\ln[(b-a_1)^2/a_1^2] - [\ln[(b-a_2)^2/a_2^2]$$

$$= \ln\left[\frac{(b-a_1)^2(b-a_2)^2}{a_1^2 a_2^2}\right]$$

$$= K \qquad \text{as obtained in (4.5.7)} \qquad (4.11.33)$$

and we have then shown $K_L = K$ for this particular example. The key fact is that the $d\theta$ integrals *appear* to be functions of r_i , but the r_i^2 terms cancel and so the $d\theta$ integrals are independent of r_i.

We now summarize our results:

Classical Transmission Line Equations and Parameters (4.11.34)

$$K \equiv \int_{C_1} dx_1'\, dy_1'\, \alpha_1(x_1',y_1') \ln(s_{21}^2/s_{11}^2) - \int_{C_2} dx_2'\, dy_2'\, \alpha_2(x_2',y_2') \ln(s_{22}^2/s_{12}^2) \qquad (4.4.8)$$

$$K_L \equiv \int_{C_1} dx_1'\, dy_1'\, b_1(x_1',y_1') \ln(s_{21}^2/s_{11}^2) - \int_{C_2} dx_2'\, dy_2'\, b_2(x_2',y_2') \ln(s_{22}^2/s_{12}^2)\,, \qquad (4.10.9)$$

$K = K_L$ real and dimensionless *(4.11.30)*

$\frac{\partial V(z)}{\partial z} = -\mathbf{z}\, i(z)$ where $\mathbf{z} = R + j\omega L$ transmission line equations

$\frac{\partial i(z)}{\partial z} = -\mathbf{y} V(z)$ $\mathbf{y} = G + j\omega C$ *(4.11.15)*

$\mathbf{z} = Z_{s1} + Z_{s2} + j\omega L_e$ *(4.11.17)* $X_L \equiv \omega L_e$, $X_C \equiv 1/(\omega C)$

$\mathbf{y} = j\omega C' = j\omega C + (\sigma/\varepsilon)C$ *(4.11.24)* $G = (\sigma/\varepsilon)C$ *(4.11.25)*

$R = \mathrm{Re}(Z_{s1} + Z_{s2})$ $L = L_e + (1/\omega)\,\mathrm{Im}(Z_{s1} + Z_{s2})$

$L_e = \frac{\mu}{4\pi} K$ *(4.11.29) and (4.11.30)*

$C' = 4\pi\xi/K$ *(4.11.26)* $C' = (\xi/\varepsilon)C$ *(4.11.23)*

$C = 4\pi\varepsilon/K$ *(4.11.26)*

$G = 4\pi\sigma/K$ *(4.11.26) + (4.11.25)*

$L_e C' = \mu\xi$ *(4.11.27)*

$L_e C = \mu\varepsilon = 1/v^2$ *(4.11.28)*

$$Z_0 = \sqrt{\frac{R + j\omega L}{G + j\omega C}} = \sqrt{\frac{\mathbf{z}}{\mathbf{y}}} \qquad (4.11.16)$$

$$Z_0 \text{ (large } \omega) \approx \sqrt{\frac{L}{C}} \approx \sqrt{\frac{L_e}{C}} = (1/4\pi)\, K \sqrt{\mu/\varepsilon} = (K/4\pi)\, Z_m \qquad \text{// See comments below}$$

$\lambda >> D$ *(4.3.6)* assumed transmission line limit where $\beta = 2\pi/\lambda$

$\beta^2 = \mu\varepsilon\omega^2 - j\omega\mu\sigma = \omega^2\mu(\varepsilon - j\sigma/\omega) = \omega^2\mu\,\xi$ $\xi \equiv \varepsilon - j\sigma/\omega$. *(1.5.1)*

$F(z) = F(0)\, e^{-az}\, e^{-jbz}$ $a \equiv \mathrm{Re}(\sqrt{\mathbf{zy}}) = \mathrm{Re}[\sqrt{(R+j\omega L)(G+j\omega C)}]$ $b = \mathrm{Im}(\sqrt{\mathbf{zy}})$

see (5.3.6) attenuation phase

Comments:

1. In Chapter 2 we computed the surface impedance Z_s for a round wire (radius a, μ_c, σ_c) in the case of axially symmetric current and we found that, for large ω,

$$Z_s(\omega) \approx \frac{1}{\sigma_c(2\pi a)\delta}\ (1+j) \qquad (2.4.16)$$

$$\delta \equiv \sqrt{2/\omega\mu_c\sigma_c} \ = \text{skin depth} \qquad (2.2.20)$$

so that

$$Z_s(\omega) \approx \frac{1}{2\pi}\sqrt{\frac{\mu_c}{2a^2\sigma_c}}\ (1+j)\sqrt{\omega} \quad . \qquad (4.11.35)$$

Presumably the result will be $Z_s(\omega) \sim \sqrt{\omega}$ for any conductor cross section shape. Then

$$L = L_e + (1/\omega)\ \mathrm{Im}(Z_{s1} + Z_{s2}) \ = L_e + (\text{stuff})\ 1/\sqrt{\omega} \quad \rightarrow L_e \qquad \text{for large } \omega \qquad (4.11.36)$$

For this reason, the high frequency characteristic impedance Z_0 can be written as shown in (4.11.34).

2. Conductors have internal inductance L_i as well as external inductance L_e. In Appendix C.3 (a) we compute the low frequency internal inductance of a round wire to be $L_i = \mu_c/8\pi = (\mu_c/\mu_0) * 50$ nH/m . Our Chapter 4 transmission line development makes no mention of L_i. This can be traced to Figure 4.11 where only the external magnetic flux is involved. In fact, L_i is accounted for in the imaginary part of the surface impedance Z_s . For example, we found that for our round wire situation,

$$Z_s(\omega) = \frac{1}{\sigma_c\pi a^2} + j\omega\frac{\mu_c}{8\pi} = R_s + j\omega L_s \qquad \text{// low frequency limit} \qquad (2.4.12)$$

and here one sees that $L_s = \frac{\mu_c}{8\pi} = L_i$.

3. We have assumed that ε and μ are real. If not, the usual adjustments can be made in (4.11.34) for the interpretations of R,L,G and C. See for example (3.3.4) concerning σ being replaced by σ_{eff} if ε has an imaginary part.

4. Apart from the symmetric cases like the examples of Section 4.5 and 4.6, we do not *yet* have a way to compute K and the transmission line parameters since the charge and current distributions α_i and b_i are not known. This matter will be remedied in Chapter 5.

5. A **strip** transmission line of width w and separation s with s << w is the simplest example of the above summary:

$$E = V/s \qquad n = \varepsilon E = \varepsilon V/s \quad q = nw \quad C = q/V = \varepsilon w/s \ \Rightarrow K = 4\pi\varepsilon/C = 4\pi(s/w)$$

so

$C = 4\pi\varepsilon/K \quad = \varepsilon \text{ (w/s)} \qquad\qquad K = 4\pi \text{ (s/w)}$

$G = 4\pi\sigma/K \quad = \sigma \text{ (w/s)}$

$L_e = (\mu/4\pi)\, K \quad = \mu \text{ (s/w)}$

$Z_0 \approx (K/\sqrt{\varepsilon_{rel}})\, 30\Omega = 4\pi \text{ (s/w)} (1/\sqrt{\varepsilon_{rel}})\, 30\Omega = \text{(s/w)} (1/\sqrt{\varepsilon_{rel}})\, 377\Omega$

4.12 Modifications to account for $\mu \neq \mu_1 \neq \mu_2$.

These modifications only affect the A_z and W(z) part of this chapter, not the first six sections which are concerned with φ and V(z). So changes start with Section 4.7.

If the equality $\mu = \mu_1 = \mu_2$ assumed in Section 4.7 is broken, the result is that surface magnetization currents appear on one or both of the conductor surfaces and these cause an alteration of the theory. Thanks to the "J_m Theorem" proven in Appendix B, this alteration can be carried through with a very minimal impact, as we now show.

In Appendix B conductor magnetization surface currents are studied in some detail. The reader interested in how the magnetic modification is carried out would do well to read Appendix B at this point. A reader less interested can accept the Appendix B results and then learn below that basically nothing changes!

So imagine starting with $\mu = \mu_1 = \mu_2$ and then changing μ_1 and μ_2 to new values. The question is: how do the various parameters and equations of the theory change? The first modification arises in Section 4.7. As described in Appendix B.6, the modified version of (4.7.2) is this,

$$A_{z1}(\mathbf{x}) = \frac{\mu_1}{4\pi} \int_{C_1} [\, J_{z1}(\mathbf{x}') + \frac{\mu_0}{\mu_1} J_{zm1}(\mathbf{x}') \,] \frac{e^{-j\beta R}}{R} dx'dy'dz' \quad . \qquad R = |\mathbf{x} - \mathbf{x}'| \qquad (4.7.2)'$$

where J_{zm1} includes only the surface component of the magnetization current on conductor C_1. Appendix B.6 shows how this J_{zm1} adder term in effect adds a certain homogeneous solution to the particular solution (first term above) of the A_z Helmholtz equation such that the A_z boundary conditions are duly satisfied at the magnetic conductor C_1 boundary. According to (B.1.10), the surface current J_{zm1} when expressed in surface rather than volume notation is given by $K_z = -\,(\frac{\mu_1}{\mu_0} - \frac{\mu}{\mu_0})\, H_\theta$ and thus vanishes when $\mu_1 = \mu$, resulting in the unmodified version of (4.7.2).

We maintain the next two equations of Section 4.7 as is, involving separation of variables,

$$J_{z1}(x,y,z) = b_1(x,y)\, i_1(z)$$
$$A/m^2 \qquad 1/m^2 \quad A \qquad\qquad (4.7.3)$$

where i_1 is scaled such that

$$\int_{C_1} dx\, dy\, b_1(x,y) = 1 \; . \qquad\qquad (4.7.4)$$

This $i_1(z)$ is still the total conduction current in C_1. But we now add two new equations,

$$J_{z1m}(x,y,z) = b_{1m}(x,y)\, i_{1m}(z)$$
$$A/m^2 \qquad 1/m^2 \qquad A \tag{4.12.1}$$

where i_{1m} is scaled such that

$$\int_{C_1} dx\, dy\, b_{1m}(x,y) = 1\ . \tag{4.12.2}$$

It is understood here that $b_{1m}(x,y)$ is a distribution which is restricted to the surface of C_1, but we continue to write it as if it existed at all points in the cross section of C_1. The integration in (4.12.2) is of course meant to include this surface distribution.

We know from (B.1.11) and (B.1.12) that, for an arbitrarily shaped conductor C_1,

$$i_{1m}(z) \equiv -\left(\frac{\mu_1}{\mu_0} - \frac{\mu}{\mu_0}\right) i(z) \qquad [\ \mu_1 = \text{conductor } C_1,\ \mu = \text{dielectric }], \tag{4.12.3}$$

and that the current ratio is therefore given by,

$$f_{1m} \equiv i_{1m}(z)/\, i(z) = -\left(\frac{\mu_1}{\mu_0} - \frac{\mu}{\mu_0}\right). \tag{4.12.4}$$

With the above definitions, our modified (4.7.6) becomes

$$A_{z1}(x,y,z) = \frac{\mu_1}{4\pi} \int_{-\infty}^{\infty} dz'\, i(z') \int_{C_1} dx'\, dy' \left[\ b_1(x',y') + \frac{\mu_0}{\mu_1} f_{1m}\, b_{1m}(x',y')\ \right] \frac{e^{-j\beta R}}{R}$$

$$= \frac{\mu}{4\pi} \int_{-\infty}^{\infty} dz'\, i(z') \int_{C_1} dx'\, dy' \left[\ \frac{\mu_1}{\mu} b_1(x',y') + \frac{\mu_0}{\mu} f_{1m}\, b_{1m}(x',y')\ \right] \frac{e^{-j\beta R}}{R}\ . \tag{4.12.5}$$

This leads us to define a new effective transverse current density,

$$b'_1(x,y) \equiv \frac{\mu_1}{\mu} b_1(x',y') + \frac{\mu_0}{\mu} f_{1m}\, b_{1m}(x',y')$$

$$= \frac{\mu_1}{\mu} b_1(x',y') + \left[1 - \frac{\mu_1}{\mu}\right] b_{1m}(x',y')\ . \tag{4.12.6}$$

This new transverse density b'_1 is still normalized to unity, using (4.7.4) and (4.12.2) above,

$$\int_{C_1} dx\, dy\, b'_1(x,y) = \frac{\mu_1}{\mu} \int_{C_1} dx\, dy\, b_1(x,y) + [1-\frac{\mu_1}{\mu}] \int_{C_1} dx\, dy\, b'_{1m}(x,y)$$

$$= \frac{\mu_1}{\mu} * 1 + [1-\frac{\mu_1}{\mu}] * 1 = 1 \quad . \tag{4.12.7}$$

How does b'_1 differ from b_1? The difference is that b_1 does not include a surface current and b'_1 does. We can represent equation (4.12.6) in this symbolic graphic manner:



(4.12.6)

Thus, from (4.12.5) and (4.12.6) we have this new version of (4.7.6),

$$\mathbf{A_{z1}}(x,y,z) = \frac{\mu}{4\pi} \int_{-\infty}^{\infty} dz'\, i(z') \int_{C_1} dx'\, dy'\, b'_1(x,y)\, \frac{e^{-j\beta R}}{R} \quad . \tag{4.7.6)'}$$

The differences are that the leading factor is μ instead of μ_1, and b_1 is replaced by b'_1.

Moving into Section 4.8 we have this new version of (4.8.1),

$$\mathbf{A_{z12}}(\mathbf{x}) = \mathbf{A_{z1}}(\mathbf{x}) + \mathbf{A_{z2}}(\mathbf{x}) =$$

$$\frac{\mu}{4\pi} \int_{-\infty}^{\infty} dz'\, i(z') \{ \int_{C_1} dx_1'\, dy_1'\, b'_1(x_1',y_1') \frac{e^{-j\beta R_1}}{R_1} - \int_{C_2} dx_2'\, dy_2'\, b'_2(x_2',y_2') \frac{e^{-j\beta R_2}}{R_2} \} \tag{4.8.1)'}$$

which is identical to (4.8.1) except $b_i \rightarrow b'_i$. Then in the transmission line limit, we get this new version of (4.9.1),

$$\mathbf{A_{z12}}(\mathbf{x}) = \frac{\mu}{4\pi}\, i(z) \int_{-\infty}^{\infty} dz' \{ \int_{C_1} dx_1'\, dy_1'\, b'_1(x_1',y_1') \frac{1}{R_1} - \int_{C_2} dx_2'\, dy_2'\, b'_2(x_2',y_2') \frac{1}{R_2} \} \tag{4.9.1)'}$$

From this point onward, all equations are the same apart from $b_i \rightarrow b'_i$. Here are some of those equations after modification:

$$W(z) \equiv \mathbf{A_{z12}}(\mathbf{x_1}) - \mathbf{A_{z12}}(\mathbf{x_2}) \tag{4.10.1)'}$$

$$= \frac{\mu}{4\pi}\, i(z) \int_{-\infty}^{\infty} dz' \{ \int_{C_1} dx_1'\, dy_1'\, b'_1(x_1',y_1') \frac{1}{R_{11}} - \int_{C_2} dx_2'\, dy_2'\, b'_2(x_2',y_2') \frac{1}{R_{12}} \}$$

$$- \frac{\mu}{4\pi}\, i(z) \int_{-\infty}^{\infty} dz' \{ \int_{C_1} dx_1'\, dy_1'\, b'_1(x_1',y_1') \frac{1}{R_{21}} - \int_{C_2} dx_2'\, dy_2'\, b'_2(x_2',y_2') \frac{1}{R_{22}} \}$$

$$W(z) \qquad (4.10.3)'$$
$$= \frac{\mu}{4\pi}\, i(z) \int_{-\infty}^{\infty} dz' \{ \int_{C_1} dx_1' dy_1' b'_1(x_1',y_1')\left(\frac{1}{R_{11}} - \frac{1}{R_{21}}\right) - \int_{C_2} dx_2' dy_2' b'_2(x_2',y_2') \left(\frac{1}{R_{12}} - \frac{1}{R_{22}}\right) \} .$$

$$W(z) = i(z) \frac{\mu}{4\pi} \{ \int_{C_1} dx_1' dy_1' b'_1(x_1',y_1') \ln(s_{21}^2/s_{11}^2) - \int_{C_2} dx_2' dy_2' b'_2(x_2',y_2') \ln(s_{22}^2/s_{12}^2) \}$$

(4.10.4)'

$$K_L \equiv \int_{C_1} dx_1' dy_1' b'_1(x_1',y_1') \ln(s_{21}^2/s_{11}^2) - \int_{C_2} dx_2' dy_2' b'_2(x_2',y_2') \ln(s_{22}^2/s_{12}^2) \qquad (4.10.9)'$$

$$L_e = \frac{W(z)}{i(z)} = \frac{\mu}{4\pi} K_L \qquad \text{// no change} \qquad (4.10.8)$$

We then enter Section 4.11. The derivation of the transmission line equations (4.11.15) is *unaffected* by the above modifications; the only change is that the b'_i appear in the integral K_L in place of the b_i. The derivation of the fact that $K = K_L$ ending in (4.11.30) is also unchanged! This at first seems strange since K has not changed, but we have apparently altered K_L by the replacements $b_i \rightarrow b'_i$. But K_L is not an evaluation -- it is an integral equation relating K_L to the b'_i. In the self-consistent solution, the new functions (distributions) b'_i adjust themselves so that K_L does not change. K_L cannot change because (4.11.30) says it must remain equal to K which is determined by the electrostatic side of the problem. It is perhaps helpful to look at (4.11.29) which says $L_e = \frac{\mu}{4\pi} K_L$. We know that if the dielectric μ value does not change, the *external* inductance L_e of the transmission line cannot change so K_L stays fixed. Changing μ_1 and/or μ_2 away from the value μ will of course change the *internal* inductances of the conductors, and this is duly noted below in terms of surface impedances. As μ_1 is increased, the B field inside conductor C_1 increases (H stays the same) so the stored B field increases, and L_i increases.

Finally, if we look at the example associated with Fig 4.13, we still find explicitly that $K_L = K$ because the calculation leading to (4.11.33) is unchanged when b_i are replaced with b'_i, since the b'_i are still normalized to unity as shown in (4.12.7).

The happy bottom line is that all of summary box (4.11.34) is unchanged except $b_i \rightarrow b'_i$ in the K_L integral. The constant K can still be evaluated using the "capacitor problem" of Section 5.5 below and it is unaffected by conductors having $\mu_i \neq \mu$.

Having said this, let us now consider what happens to an operating transmission line which starts off with $\mu_1 = \mu_2 = \mu = \mu_0$ and we then gradually turn a magic "permeability knob" so that μ_1 gradually increases from μ_0 to some value $\mu_1 > \mu_0$. That is to say, we gradually cause conductor C_1 to become magnetic. The constant K (and therefore $K_L = K$) does not change at all. This K is determined by the potential φ part of the problem in Section 4.4 and does not even know about the magnetic modification. Thus, looking at (4.11.34), C', C, G and L_e do not change. In particular, L_e does not change because we have not altered μ of the dielectric. The following two items shown in box (4.11.34) *do* change :

$$R = \mathrm{Re}(Z_{s1} + Z_{s2}) \qquad L = L_e + (1/\omega)\, \mathrm{Im}(Z_{s1} + Z_{s2})$$

where Z_{si} is the surface impedance of conductor C_i. The non-L_e term in L can be interpreted as the internal inductance of the conductors. R and L change because Z_{s1} changes if we change μ_1. This is so because Z_{s1} is always a function of the skin depth δ_1, and $\delta_1 \equiv \sqrt{2/(\omega\mu_1\sigma_1)}$ from (2.2.20). In the special case that C_1 is a round wire of radius a_1 with an axially symmetric current distribution (such as the center wire of a coaxial cable), we showed in (2.4.11) that the surface impedance is given by

$$Z_{1s}(\omega) = \frac{+j\omega\mu_1}{2\pi a_1(\sqrt{2}/\delta_1)} \frac{\mathrm{ber}_0[\sqrt{2}(a_1/\delta_1)] + j\,\mathrm{bei}_0[\sqrt{2}(a_1/\delta_1)]}{\mathrm{ber}_0'[\sqrt{2}(a_1/\delta_1)] + j\,\mathrm{bei}_0'[\sqrt{2}(a_1/\delta_1)]}, \qquad (2.4.11)$$

so certainly this $Z_s(\omega)$ is a function of μ_1 both due to the leading constant and through the five occurrences of δ_1. Both the real and imaginary parts of $Z_{1s}(\omega)$ will change as μ_1 changes, so the transmission line parameters R and L both change. In the high frequency limit ,

$$Z_{1s}(\omega) \approx \frac{1}{\sigma_1(2\pi a_1)\delta_1}(1+j) \qquad \delta << 16a\ . \qquad (2.4.16)$$

so now the variation with μ_1 is through the single δ_1 factor shown. Again, both real and imaginary parts of $Z_{1s}(\omega)$ vary with μ_1.

Since R and L change as noted above, the transmission line characteristic impedance will also change,

$$Z_0 = \sqrt{\frac{R + j\omega L}{G + j\omega C}} = \sqrt{\frac{\mathbf{z}}{\mathbf{y}}} \qquad (K.11)$$

This means, for example, if we drive a semi-infinite transmission line with some fixed voltage V(z), the driving current i(z) will vary in amplitude and phase as we turn our "permeability knob" for conductor C_1. This is simply because $i(z) = V(z)/Z_0$.

So **the good news** is that the theory of Chapter 4 is easily extended to allow for magnetic conductors and or dielectric. Once again, the summary box (4.11.34) is unchanged when $\mu_1 = \mu_2 = \mu$ is broken except for the appearance of b'_i in the K_L integral, and except for the fact that Z_{s1} and Z_{s2} change as noted above, causing changes in R, L and Z_0. At very high frequency, one will have $Z_0 = \sqrt{(L_e/C)}$ and in this case Z_0 is not altered, see (4.11.36).