Working on Paradox #1 and Paradox #2 PhL 5.8.14

[Final review 5.4.14. I am still totally happy with the resolutions of these paradoxes, and it is all written up now in lines doc.]

As of May 8, 2014 I am totally happy with the resolution of paradox 1 outlined here. It is just one of those little things you check that is OK. Always looking for trouble!

5/12/14. Things here are all resolved in lines doc.

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Statement of the Paradox 1. This "paradox" is discussed in "the current asymmetry issue.doc". The paradox is the following: We know that for any pair of closely spaced conductors, J_z and hence $E_z = J_z/\sigma$ will be highly asymmetrical across each conductor cross section. E_z will be largest in the "active region" close to the other conductor. This asymmetry exists for all values of ω, even very low values. At one time I was unhappy about this being true at very low ω until I learned that the ω→0 limit of the transmission line theory has I = 0 so there is no current at all. But basically for any ω > 0, the asymmetry seems to be constant in some sense, though that claim needs work. At higher ω the skin effect pushes the asymmetry out to the boundary sheath only. Here then is the paradox: We have this known transmission line equation,

$$E_z = j\,\beta_d\,\varphi - j\omega A_z$$

and we know that φ is constant on the boundary, but E_z varies greatly as noted above. Looking at the equation w have

$$j\omega A_z = j\,\beta_d\,\varphi - E_z \qquad (*)$$

and this suggests that A_z also varies greatly on the boundary. At large ω, we were hoping that in the skin effect regime A_z would NOT vary greatly but would be nearly constant on the boundary in line with King's theory.

So the paradox #1 is this: how can we have A_z = constant on the boundary in the skin effect regime? This is required for the Chapter 4 King theory to be valid. That seems inconsistent with the equation just quoted, where E_z varies and φ = constant on the boundary.

Claimed Resolution of Paradox 1. My claimed resolution is this: E_z is very small since we are dealing with good conductors. So when we look at the above equation,

$$j\omega A_z = j\,\beta_d\,\varphi - E_z$$

the E_z term, though highly variable on the boundary, is always much smaller than the $j\,\beta_d\,\varphi$ term, and therefore the right side and hence A_z is fairly constant on the boundary.

Numerical Example of Paradox 1. I want to see some numbers to prove this resolution really is true -- an example. The example I seem to know the most about is the two cylinders transmission line. I would like to compute φ, E_z and A_z separately for this example, and then see if the above resolution is viable.

1. What is φ for this example?

Looking at Chapter 6, I see in (6.3.1) that

$$\varphi_t(\mathbf{x}) = \ln(s_2^2/s_1^2)$$

with some BC's, and I know that

$$\varphi(x,y,z) = \frac{1}{4\pi\xi}\; q(z)\;\varphi_t(x,y)$$

Unfortunately, I don't state $\varphi_t(\mathbf{x})$ more "specifically" in Ch 6, I concentrate there on K which controls the line parameters. But I think I do treat this subject in my bipolar.doc starting there with Section 10. If the two cylinders have ξ_1 and ξ_2 as labels, I find that

$$\varphi(\xi) = -\frac{\xi}{\xi_2-\xi_1}\,V \qquad (10.8)$$

and later I write this as

$$\varphi(x,y,z) = \frac{1}{4\pi\varepsilon}\; q(z)\;\varphi_t(x,y) \qquad \text{Ref [5] (5.1.1)}$$

and comparison with (10.19a) shows that (using also (4.5) above),

$$\varphi_t(x,y) = \ln\left[\frac{s_2^2}{s_1^2}\right] = \ln\left[\frac{(x-a)^2+y^2}{(x+a)^2+y^2}\right] = -2\xi \qquad (10.20)$$

and there we have things in (x,y) as well as ξ coordinates.

So I think I know all about "φ for this example". This is in the dielectric of course, and if I then approach the surface, I get

$$\varphi(\text{surface 1}) = -\frac{\xi_1}{\xi_2-\xi_1} V \qquad (10.8)$$

which is of course just some finite fraction of the applied voltage V.

Now "closely spaced conductors" means that $\xi_2 > 0$ is small, and $\xi_1 < 0$ is also small as in Fig (10.2) of bipolar doc. But this fact does not really affect the fraction much. For a symmetric case, the fraction will always be 1/2 no matter how closely the cylinders are spaced.

Thus, in terms of "scale", we might just say

$$|\varphi(\text{surface 1})| \approx V/2$$

and then we have a handle on one of our three quantities in (*).

But our equation is this:

$$E_z = j\,\beta_d\,\varphi - j\omega A_z$$

so what is β_d ? From "very low freq limit" non v2 we have

$$\beta_d = (\omega/v_d) \approx (1/3) \times 10^{-8}\,\omega$$

If we want $a/\delta = 10$, and since I will use a = 1mm = 1000 μ below, we have from lines doc table (2.3.9) for δ = 100 μ that f = 500 KHz let's say so $\omega = 2\pi f = 6*1/2*10^6 = 3 \times 10^6$ so then,

$$\beta_d = (1/3) \times 10^{-8} * \; 3 \times 10^6 = 10^{-2}\ m^{-1}$$

We then get

$$\beta_d\,\varphi = 10^{-2} * V/2 = (1/200)\ V \ \text{volts/m}$$

2. What is E_z for this example?

Appendix A of bipolar doc I think get's us started on this question. It shows that

$$n_1(\xi_1,\theta) = \frac{q}{2\pi}\,\frac{|\text{sh}\xi_1|}{\text{ch}\xi_1+\cos\theta}\ , \qquad \textit{(10.28)}$$

$$\eta_m \equiv N_m/N_0 = (-1)^m\, e^{-|m\xi_1|}\ . \qquad (A.12)$$

$$n_1(\xi_1,\theta) = (q/2\pi)[\,1 + 2\sum_{m=1}^{\infty}(-1)^m\, e^{-m|\xi_1|}\cos(m\theta)\,] \qquad (A.13)$$

so at least we know the charge distribution on the conductor surface (and of course it is strongly peaked in the active region between the conductors.

Now my plan at this point has been to use the above η_m values in lines doc (D.2.33) which says

$$E_z(r,m) = (1/4)\,\eta_m\, I\, R_{dc}\ (a\beta')\left[\frac{J_m(x)}{J_{m+1}(x_a)} - \frac{J_m(x)}{J_{m-1}(x_a)}\right]$$

$$\eta_m \equiv N_m/N_0 = (-1)^m\ e^{-|m\xi_1|}\ . \qquad \text{(A.12)}$$

so this seems to be a very explicit statement of the partial wave $E_z(r,m)$. I can write this then as

$$E_z(r,m) = (1/4)\,\eta_m\, I\, R_{dc}\ (a\beta')\, f_m \qquad f_m = \left[\frac{J_m(x)}{J_{m+1}(x_a)} - \frac{J_m(x)}{J_{m-1}(x_a)}\right]$$

where f_m appears in low frequency folder "very low frequency limit of Appendix D fields v.2.doc". There I take the low ω limit and find that

$$f_m = (2/a\beta)\,(m+1)(r/a)^m \qquad m > 0$$

$$f_0 = (4/a\beta) \qquad m = 0 \qquad \beta = e^{3\pi j/4}\sqrt{\sigma\mu}\ \ \omega^{1/2}$$

but I am not interested in that limit right now, I do know that

$$\beta'^2 = \beta^2 - \beta_d^2$$

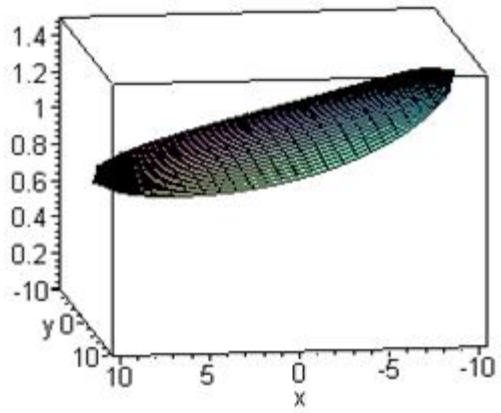
and these parameters are treated in detail in the non-v,2 version of the just-quoted doc:

$$\beta' \approx \sqrt{-j\sigma\mu\omega} = e^{3\pi j/4}\sqrt{\sigma\mu}\ \ \omega^{1/2} = \beta \qquad [\text{ this is still OK at any reasonable } \omega\ ]$$

Now where have I already computed $E_z(r,\theta)$ from these partial wave amplitudes?? I do it in the non v2 doc just mentioned and I get for low ω only,

$$E_z(r,\theta) = I\, R_{dc}\left[1 + \sum_{m=1}^{\infty}(-1)^m\, e^{-m|\xi_1|}\,(r/a)^m\,[(m+1)]\cos(m\theta)\right]$$

where β' has been dealt with and ends up being in the $(r/a)^m$ factor. Also, this is using the low-ω limit of f_m. I also plot this function $E_z(r,\theta)$,



where I used $\xi_1 = -2$ which in fact means things are fairly widely spaced. And so there is the famous picture of the E_z asymmetry if I have done things right! I remember being surprised at getting such a strong asymmetry at low ω.

The more general result from non-v2 is this [from "Computation of Az for two cylinders case..".]

$$E_z(r,\theta) = I R_{dc} [(1/4) x_a f_0 + (1/2) x_a \sum_{m=1}^{\infty} \eta_m f_m(x,x_a) \cos(m\theta)]$$

or

$$E_z(r,\theta) = I R_{dc} (1/4) x_a [f_0 + 2 \sum_{m=1}^{\infty} \eta_m f_m(x,x_a) \cos(m\theta)] \qquad (*)$$

where

$$\eta_m \equiv N_m/N_0 = (-1)^m e^{-|m\xi_1|}$$

$$f_m = [\frac{J_m(x)}{J_{m+1}(x_a)} - \frac{J_m(x)}{J_{m-1}(x_a)}] \qquad \text{all m} \qquad x = \beta' r \qquad x_a = \beta' a$$

I have a plot of this function in "compute Ez for 2cyl.mws" in the overhaul folder. There I enter just [...] portion of (*) and I add perhaps 20 terms (easy to vary).

What can we say about the "scale" of this E_z ? First of all, somewhere I have plotted f_m and have discussed the plots. Seems not to be in the low freq folder. I am having much trouble locating my doc having these plots! It has to be an mws file, so I will now look through every single such file in all my related folders.

bipolar folder: examined 10 mws files, not there

overhaul folder: YES, the plots are in "current asym studies.mws". So we have now the right folder. The doc discussion is in "the current asymmetry issue.doc". What I show is that as you vary m, the $E_z(r,m)$ curve is always vary similar.

I want now to do some new plotting of f_m just to see what it does in terms of scale. Recall that

$\beta = (j-1)/\delta$ from box (2.2.30)

Then

$$x_a = \beta a = (j-1)(a/\delta)$$

If we are in the skin effect regime, this x_a is "large" so maybe we can use some large-x limits on Bessel functions to see what to expect. Let's first rewrite the above as

$$\beta = e^{j3\pi/4}(\sqrt{2}/\delta) \qquad = (j-1)/\sqrt{2} \ * (\sqrt{2}/\delta) = (j-1)/\delta$$

Then

$$x_a = \beta a = e^{j3\pi/4}(\sqrt{2}\, a/\delta) \qquad \Rightarrow \qquad z_a = \sqrt{2}\, a/\delta = \text{large}$$

Then we can use from lines doc

$$J_m(e^{j3\pi/4}z) = M_m(z)\, e^{j\theta_m(z)} . \tag{2.3.3}$$

$$M_m(z) \approx \frac{\exp(z/\sqrt{2})}{\sqrt{2\pi z}} \left[1 - \frac{m-1}{8\sqrt{2}\, z} + O(1/z^2) \right]$$

$$\theta_m(z) \approx (z/\sqrt{2}) + (\pi/2)[m - 1/4] + \frac{m-1}{8\sqrt{2}\, z} + O(1/z^2) \ . \tag{2.3.5}$$

Let's try to express m in terms of these M things, were $x = e^{j3\pi/4}z$ and $x_a = e^{j3\pi/4}z_a$

$$\frac{J_m(x)}{J_{m+1}(x_a)} = M_m(z)\, e^{j\theta_m(z)} / [M_{m+1}(z_a)\, e^{j\theta_{m+1}(za)}] = \frac{M_m(z)}{M_{m+1}(z_a)}\, e^{j[\theta_m(z) - \theta_{m+1}(z)]}$$

If z is large, keep only the leading term in M and we then have [I am setting $x = x_a$ now]

$$\frac{J_m(x_a)}{J_{m+1}(x_a)} = \frac{M_m(z_a)}{M_{m+1}(z_a)}\, e^{j[\theta_m(za) - \theta_{m+1}(za)]} \approx e^{j[\theta_m(za) - \theta_{m+1}(za)]}$$

$$= \text{expj}\,[\{(z_a/\sqrt{2}) + (\pi/2)[m - 1/4]\} - \{(z_a/\sqrt{2}) + (\pi/2)[m+1 - 1/4]\}$$ ok

$$= \text{expj}\,[\{+(\pi/2)\,0\} - \{+(\pi/2)[1]\} = \text{expj}\,[\{-\pi/2\}] = -j$$

If we now consider the second term in f_m which is $-\frac{J_m(x)}{J_{m-1}(x_a)}$, we repeat the above, but there are two changes. The first is the leading - sign. The second is that $m+1 \rightarrow m-1$ so we will then find

$$\frac{J_m(x_a)}{J_{m-1}(x_a)} \approx -\,\text{expj}\,[\{+\pi/2\}] = -j$$

Thus, the two terms together give

$f_m \approx -2j$ in the limit (a/δ) is very large.

[I have verified this in Maple with an example,

```
f(m);
```

$$\frac{J\left(3, 20\left(-\frac{1}{2}\sqrt{2}+\frac{1}{2}j\sqrt{2}\right)\sqrt{2}\right)}{J\left(4, 20\left(-\frac{1}{2}\sqrt{2}+\frac{1}{2}j\sqrt{2}\right)\sqrt{2}\right)} - \frac{J\left(3, 20\left(-\frac{1}{2}\sqrt{2}+\frac{1}{2}j\sqrt{2}\right)\sqrt{2}\right)}{J\left(2, 20\left(-\frac{1}{2}\sqrt{2}+\frac{1}{2}j\sqrt{2}\right)\sqrt{2}\right)}$$

```
evalf(%);
```

$$-.03749640916 - 2.024727669\,j$$

so we really do seem to get -2j]

so we expect

$|f_m| \approx 2.$ at r = a, in the significant skin effect regime

Here are some Maple plots confirming this rough estimate from "plots of fm.mws" :

```
> restart; alias(I=I,j=sqrt(-1),J = BesselJ):
> f := J(m,x)/J(m+1,xa) - J(m,x)/J(m-1,xa);
```

$$f := \frac{J(m, x)}{J(m+1, xa)} - \frac{J(m, x)}{J(m-1, xa)}$$

```
> x := beta*r:
> xa := beta*a:
```

$\beta = e^{j3\pi/4}(\sqrt{2}/\delta)$ and $\beta^2 = -2j/\delta^2$. (2.2.21)

```
> beta := exp(j*3*Pi/4)*sqrt(2)/delta;
```

$$\beta := \frac{\left(-\frac{1}{2}\sqrt{2}+\frac{1}{2}j\sqrt{2}\right)\sqrt{2}}{\delta}$$

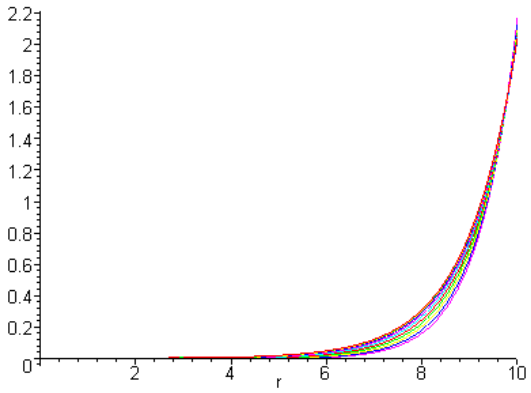
```
> a := 10;
```

$$a := 10$$

```
> delta := a/10;
```

$$\delta := 1$$

```
> plot([seq(abs(f),m=0..10)],r=0..a);
```



You see that the peak value at r = a on the right is about $| f_m | \approx 2$, and everywhere else we have a lesser magnitude.

Now go back to

$$E_z(r,\theta) = I R_{dc} [1 + (1/2) x_a \sum_{m=1}^{\infty} \eta_m f_m(x,x_a) \cos(m\theta)]$$

where

$$\eta_m \equiv N_m/N_0 = (-1)^m e^{-|m\xi_1|}$$

$$f_m = [\frac{J_m(x)}{J_{m+1}(x_a)} - \frac{J_m(x)}{J_{m-1}(x_a)}] \qquad x = \beta' r \qquad x_a = \beta' a$$

where we were asking about the "scale" of E_z. We have seen now something about the "scale" of f_m. So we then have

$x_a = \beta a = e^{j3\pi/4} (\sqrt{2}\, a/\delta)$ with magnitude on the order of maybe 14 for ratio $a/\delta = 10$.

$\eta_m \approx 1$ or less

$\cos(m\theta) \approx 1$ or less

$f_m \approx 2$ or less

Now if we assume a large ξ_1 meaning cylinders not closely spaced, we will have only the leading term in the series more or less! So I think we can fairly reach this conclusion which screamed out at the start:

$$\text{scale}(E_z) \approx I R_{dc.} = I * 1/(\pi a^2 \sigma)$$

Suppose I = 1 amp and a = 1 mm, then

```
I := 1:
sigma :=   5.81e7:
a := 1e-3:
Ez := evalf(I/(Pi*a^2*sigma));
```

$$Ez := .005478655527$$

so we have E_z = 5.5 mV/m, fine!

Notes added 5/13/14 while writing Section 6.5.

Assuming the result above is correct that $f_m = -2j$ in our large x limit, and assuming

$$J_z(r,\theta) = J_{dc}\,(\beta a/4)\,[\; f_0(r) + 2\,\Sigma_{m=1}^{\infty}\,(-1)^m\, e^{-m|\xi_1|}\, f_m(r)\cos(m\theta)\;] \qquad (6.5.15)$$

we find in this limit that

$$J_z(a,\theta) = J_{dc}\,(\beta a/4)(-2j)\,[\; 1+ 2\,\Sigma_{m=1}^{\infty}\,(-1)^m\, e^{-m|\xi_1|}\cos(m\theta)\;] \qquad (6.5.15)$$

$$= J_{dc}\,(\beta a/4)(-2j)\; n_1(\xi_1,\theta)(2\pi/q)$$

$$= J_{dc}\,(\beta a/4)(-2j)\,(2\pi/q)\; n_1(\xi_1,\theta)$$

$$= J_{dc}\,(\beta a)(-j)\,(\pi/q)\; n_1(\xi_1,\theta)$$

$$= (-j)(I/\pi a^2)(\beta a)\,(\pi/q)\; n_1(\xi_1,\theta)$$

$$= (-j)(I/a)(\beta)\,(1/q)\; n_1(\xi_1,\theta)$$

$$= (-j)(I/a)(\beta/q)\; n_1(\xi_1,\theta) \qquad\qquad q = 2\pi\varepsilon\,\frac{V}{\xi_2-\xi_1}$$

$$= (-j)(I/aV)(\beta/2\pi\varepsilon)\,(\xi_2 - \xi_1)\; n_1(\xi_1,\theta)$$

$$= (-j)(I/V)(\beta/2\pi\varepsilon a)\,(\xi_2 - \xi_1)\; n_1(\xi_1,\theta)$$

$$= (-j)(\beta/2\pi\varepsilon a Z_0)\,(\xi_2 - \xi_1)\; n_1(\xi_1,\theta)$$

Now what is Z_0 for large ω? Box in lines doc shows

$$Z_0 = (1/4\pi)\,K\sqrt{\mu/\varepsilon} \qquad\qquad 1/Z_0 = 4\pi\sqrt{\varepsilon/\mu}\;\;(1/K)$$

Then we have

$$J_z(a,\theta) = (-j)(\beta/2\pi\varepsilon a)[\; 4\pi\sqrt{\varepsilon/\mu}\;\;(1/K)]\,(\xi_2 - \xi_1)\; n_1(\xi_1,\theta)$$

But we know below (6.5.3) that $K = 2(\xi_2-\xi_1)$ so we then have

$$J_z(a,\theta) = (-j)(\beta/2\pi\varepsilon a)[\, 4\pi \sqrt{\varepsilon/\mu}\ \ (1/K)]\ (1/2)\ 2(\xi_2\ -\ \xi_1)\ n_1(\xi_1,\theta)$$

$$= (-j)(\beta/2\pi\varepsilon a)[\, 4\pi \sqrt{\varepsilon/\mu}\ ]\ (1/2)\ n_1(\xi_1,\theta)$$

$$= (-j)(\beta/\pi\varepsilon a)[\, \pi \sqrt{\varepsilon/\mu}\ ]\ n_1(\xi_1,\theta)$$

$$= (-j)(\beta/\varepsilon a)[\, \sqrt{\varepsilon/\mu}\ ]\ n_1(\xi_1,\theta)$$

$$= (-j)(\beta/a)[\, 1/\sqrt{\mu\varepsilon}\ ]\ n_1(\xi_1,\theta)$$

$$= (-j)(\beta/a)v_d\ n_1(\xi_1,\theta)$$

$$= (-j)\ (v_d/a)\ e^{j3\pi/4}\ (\sqrt{2}\ /\delta)\ n_1(\xi_1,\theta)$$

$$= (-j)\ e^{j3\pi/4}\ (v_d/a)\ (\sqrt{2}\ a/\delta)\ n_1(\xi_1,\theta)(1/a)$$

$$= e^{-j\pi/2}\ e^{j3\pi/4}\ (v_d/a)\ (\sqrt{2}\ a/\delta)\ n_1(\xi_1,\theta)(1/a)$$

$$= e^{-j2\pi/4}\ e^{j3\pi/4}\ (v_d/a)\ (\sqrt{2}\ a/\delta)\ n_1(\xi_1,\theta)(1/a)$$

$$= e^{+j\pi/4}\ (v_d/a)\ (\sqrt{2}\ a/\delta)\ (1/a)\ n_1(\xi_1,\theta)$$

and this is fairly close to my intuitive derivation of this fact !!! That result is this

$$J_z(a,\theta) =\ \ (\ v_d\ /a)\ [1/2 + (1+j)a/\delta\ ]\ (1/a)\ n(\theta)$$

and if I take large a/δ it becomes

$$J_z(a,\theta) =\ \ (\ v_d\ /a)\ [(1+j)a/\delta\ ]\ (1/a)\ n(\theta)$$

$$=\ \ (\ v_d\ /a)\ \sqrt{2}\ [(1/\sqrt{2}\ )(1+j)a/\delta\ ]\ (1/a)\ n(\theta)$$

$$=\ \ (\ v_d\ /a)\ \sqrt{2}\ [e^{j\pi/4}a/\delta\ ]\ (1/a)\ n(\theta)$$

$$=\ e^{j\pi/4}\ (\ v_d\ /a)\ [\sqrt{2}\ a/\delta\ ]\ (1/a)\ n(\theta)$$

and we are now in full agreement.

3. What do we know so far with our numerical example?

We are considering the equation,

$E_z = j\,\beta_d\,\varphi - j\omega A_z$

and we are in the skin effect regime where maybe $a/\delta = 10$. We have found that

scale(φ) = V/2

scale($\beta_d\,\varphi$) = V/200 = (1/2)V x 10^{-2} volts/m

scale(E_z) = 5 x 10^{-3} volts/m

Suppose our signal is V = 1 volt. Then we get

scale($\beta_d\,\varphi$) = V/200 = 0.5 x 10^{-2} = 5 x 10^{-3} volts/m

scale(E_z) = 5 x 10^{-3} volts/m

So this is throwing mud at my "claimed resolution of the problem". My claim was that E_z was very small compared to $\beta_d\,\varphi$, but here in my example they are about the same!!! Thus, as E_z does its variation around the boundary, ωA_z also does a large variation, blowing a giant hole in my argument.

But: I assumed V = 1 volt and I = 1 amp which implies that Z_0 = 1 ohm. But if I am talking a normal coaxial cable, I know that Z_0 = 50 ohms, say. For twin lead and using chapter 6 I have

$Z_0 = (K/\sqrt{\varepsilon_{rel}})\,30\Omega$ $\qquad\qquad$ $K = 2\,ch^{-1}[(b^2/2a^2) - 1]$

So let a = 1 mm as above, and assume we are moderately spaced so b = 3. Then

$K = 2\,ch^{-1}[9/2 - 1] = 2\,ch^{-1}(3.5) = 3.8$ and then

$Z_0 = 3.8 * 30 = 114$ ohms.

Remember that TV twinlead is 300 ohms. So for this case, let's say Z_0 = 100 ohms. Then if V = 1 volt, I am forced to take I = 10^{-2} amps and then

scale(E_z) = 5 x 10^{-3} volts/m * 10^{-2} = 5 x 10^{-5} volts/m

scale($\beta_d\,\varphi$) = V/200 = 0.5 x 10^{-2} = 5 x 10^{-3} volts/m

Now things are recovered a bit. The E_z term is now only 1/100th of the $\beta_d\,\varphi$ term, so then the variation in A_z might only be 1% which I can live with!

$E_z = j\,\beta_d\,\varphi - j\omega A_z$

Is there a more general way to show this result?

Well, here is one method. Suppose we are in extreme skin effect so A_z = constant on surfaces. In this case, I think the solutions for φ_t and A_{zt} are the same, since they have the same ODE and the same BC's. Here are some items from Chapter 5:

$$\varphi(x,y,z) = \frac{1}{4\pi\xi}\ q(z)\ \varphi_t(x,y) \qquad (5.1.1)$$

$$[\ \nabla_t^2 + (\beta^2 - k_\varphi^2)]\ \varphi_t(x,y) = 0 \qquad (5.1.8)$$

$$\varphi_t(C_1) = K_1 \qquad \varphi_t(C_2) = K_2 \qquad K_1 - K_2 = K \qquad (5.1.10)$$

$$A_z(x,y,z) = \frac{\mu}{4\pi}\ i(z)\ A_{zt}(x,y) \qquad (5.2.1)$$

$$[\ \nabla_t^2 + (\beta^2 - k_A^2)]\ A_{zt}(x,y) = 0 \qquad (5.2.8)$$

$$A_{zt}(C_1) = W_1 \qquad A_{zt}(C_2) = W_2 \qquad W_1 - W_2 = K \qquad (5.2.10)$$

For the "lossless" case where the Helm parm is zero, I think we find that

$$A_{zt}(x,y) = \varphi_t(x,y) \qquad \text{// both are dimensionless}$$

Then we must have

$$A_z(x,y,z)\frac{4\pi}{\mu i(z)} = \varphi(x,y,z)\frac{4\pi\varepsilon}{q(z)} = \frac{4\pi\varepsilon}{q(z)}\ \left[-\frac{\xi}{\xi_2-\xi_1} V \right]$$

or

$$A_z = \frac{4\pi\varepsilon}{q(z)}\ \frac{\mu\, i(z)}{4\pi}\ \left[-\frac{\xi}{\xi_2-\xi_1} V \right] = \mu\varepsilon\ \frac{i(z)}{q(z)}\ \left[-\frac{\xi}{\xi_2-\xi_1} V \right] = (1/v_d)^2\frac{i(z)}{q(z)}\ \left[-\frac{\xi}{\xi_2-\xi_1} V \right]$$

But $i(z) = q(z)\ v_d$ (draw a picture, both sides are Cou/sec) so this really says

$$A_z = (1/v_d)\left[-\frac{\xi}{\xi_2-\xi_1} V \right] = (1/v_d)\ \varphi$$

OK, now let's take a look at our famous little equation,

$$E_z = j\ \beta_d\ \varphi - j\omega A_z$$

$$= j\omega(\varphi/v_d) - j\omega A_z$$

$$= j\omega\ [(\varphi/v_d) - A_z\ ]$$

$$= j\omega\ [A_z - A_z\ ]$$

$$= 0$$

I found this simple result elsewhere. If you assume the *extreme* skin effect limit, then A_z really does hug the boundary, then you really do get $A_{zt}(x,y) = \varphi_t(x,y)$, and you really do get $E_z = 0$. Extreme skin effect means $\sigma \rightarrow \infty$ and that means $E_z = 0$. So when we are in *strong* skin effect, A_z is not quite given by what I write above, and E_z is not quite zero, and we then get something like our numerical example.

Paradox #2.

This is DIRECTLY RELATED to Paradox #1. When you write out the King gauge condition,

$$\text{div } \mathbf{A} = -\mu\varepsilon j\omega\varphi - \mu\sigma\varphi = -j\omega\mu(\varepsilon+\sigma/j\omega)\varphi = -j\omega\mu\xi\varphi = -j(\beta^2/\omega)\ \varphi \qquad \text{King gauge} \qquad (1.5.5)$$

in the dielectric, one way to express this is

$$\text{div } \mathbf{A} = -j\,\beta_d\,(\beta_d/\omega)\,\varphi = -j\,\beta_d\,(1/v_d)\,\varphi$$

Writing out the details, this says

$$(\partial_x A_x + \partial_y A_y) + (\partial_z A_z) = -j\,\beta_d(1/v_d)\,\varphi$$

Using the usual transmission line $\partial_z \rightarrow -j\beta_d$ this says

$$(\partial_x A_x + \partial_y A_y) - j\beta_d A_z = -j\,\beta_d\,(1/v_d)\,\varphi$$

or

$$(\partial_x A_x + \partial_y A_y) = j\beta_d\,[\,A_z - (\varphi/v_d)\,]$$

This is the same RHS we see in the E_z expression above. In the extreme skin effect limit, we expect as previously to get $A_z = (\varphi/v_d)$ and the result is then $(\partial_x A_x + \partial_y A_y) = 0$. This agrees with the fact that in this limit, all current flows in a pure skin, so there can be no transverse current perp z.

But for just a strong skin effect, we will have some small $(\partial_x A_x + \partial_y A_y) \neq 0$ just as we have some small $E_z \neq 0$ in the previous section.

Why is this a "paradox"? Well, only if you insist that $(\partial_x A_x + \partial_y A_y) \equiv 0$ and then you find that the King gauge forces you to have $A_z = (\varphi/v_d)$ and that was a paradox away from the strong skin limit (such as near DC) where we know A_z does not track the conductor boundary.