The Current Asymmetry Issue PhL 3.2.14

[Final review 5.14.14. Added summary below. (23 pages, long doc)]

Summary

In **Section 1** I quote the Appendix D E fields and examine values of β,β' and β_d.

In **Section 2** I make δ graphs of $E_z(r,m)/ E_z(a,m)$ which shows things about same for various m.

In **Section 3** I explain *why* all E_i field components have the same expo δ decay.

In **Section 4** I show that $|f_m| \approx 2$ for various m and so things don't depend much on m. Later in lines doc (6.5.22) I show that in fact $f_m = -2j$ in the extreme skin limit.

Section 5 is Paradox #1: η_m implies $E_z(a,\theta)$ on surface, so how can A_z be constant?

Section 6 is early realization that $B_r = 0$ on surface in skin regime and then A_z = constant

Section 7 is Paradox #2: King gauge says $A_z = (1/v_d)\,\varphi$ so how explain $A_z \neq$ constant at DC?

Section 8 continues Section 7 on $B_r = 0$ on surface in skin regime.

Section 9 continues the study of Paradox #2 with the King gauge.

Section 10 wonders whether φ = constant quasi-static is valid at all ω and questions low ω. This same questioning now exists in line doc Section 3.7 where I could only prove φ = constant for large ω.

Section 11 resolves paradox #1 by saying φ = constant and A_z = almost constant. I think now I would say that both are "almost" constant.

Section 12 resolves paradox #2 with same notion that A_z = almost constant.

In **Section 13** I show $A_z \neq$ constant for DC twin lead with picture. I start thinking here about $\omega \rightarrow 0$ of the theory. I float the idea of a separate A_z^{AC} and A_z^{DC} but reject it. I then take low ω limits of the Appendix D fields. I find that the m=0 does the right thing, but to my surprise m > 0 fields don't vanish. This is a problem since I wanted to see J_z have no asymmetry at DC! So now I have current asymmetry down to ω = 0 and this violates my eddy current ideas. so a big new Paradox. This led to a whole new folder of docs on this subject! My current resolution is that the general theory does not apply at low ω, and of course the escape hatch idea that I = 0 since $Z_0 = \infty$. But here is where I first discovered this problem.

Comments added in red for each section on 3.4.14. A lot is going on in this doc.

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1. Quoting the round wire E field partial wave results and β, β_d and β'

Here I quote the results and then study the meaning of the various β one might use in these formulas

Consider the E field solution of Appendix D, including the two assumed boundary conditions.

Second summary of the E field solutions : $R_{dc} = \frac{1}{\sigma\pi a^2}$ $\beta'^2 = \beta^2 - \beta_d^2$ (D.2.33)

$$E_z(r,m) = (1/4)\,\eta_m\, I\, R_{dc}\, (a\beta')\left[\frac{J_m(x)}{J_{m+1}(x_a)} - \frac{J_m(x)}{J_{m-1}(x_a)}\right] \qquad a = \text{radius} \qquad \eta_m \equiv \frac{N_m}{N_0}$$

$$E_r(r,m) = (j/4)\,\eta_m\, I\, R_{dc}\, (a\beta_d)\left[\frac{2mx^{-1}J_m(x)}{J_{m-1}(x_a)} + \frac{J_{m+1}(x)}{J_{m+1}(x_a)} - \frac{J_{m+1}(x)}{J_{m-1}(x_a)}\right] \qquad x = \beta' r$$

$$E_\varphi(r,m) = (1/4)\,\eta_m\, I\, R_{dc}\, (a\beta_d)\left[-\frac{2mx^{-1}J_m(x)}{J_{m-1}(x_a)} + \frac{J_{m+1}(x)}{J_{m+1}(x_a)} + \frac{J_{m+1}(x)}{J_{m-1}(x_a)}\right] \qquad x_a = \beta' a$$

What can be said about this partial wave solution in the Skin Effect Limit? In that limit, δ < (a/10), say.

Comments about β

First, recall

$$\beta = e^{j3\pi/4}(\sqrt{2}/\delta) \quad \text{and} \quad \beta^2 = -2j/\delta^2 \ . \qquad (2.2.21)$$

$$\beta^2 = \mu\varepsilon\omega^2 - j\omega\mu\sigma = \omega^2\mu(\varepsilon - j\sigma/\omega) \approx -j\omega\mu\sigma \ . \qquad (1.5.1)$$

In the conductor, we have assume in some sense that σ/ω >> ε. This is discussed below (2.2.2) and we conclude that for copper this approximation is valid up to 10^9 GHz, so that is really not an issue.

So, we then have

$$\beta = e^{j3\pi/4}(\sqrt{2}/\delta) \qquad \text{with } \delta < (a/10)$$

$$|\beta| = (\sqrt{2}/\delta) > 10\,(\sqrt{2}/a)$$

Notice that this is not particularly "huge" as I keep saying in lines doc, at least for this δ.

My interest now is the radial dependence of the above E field solution components. Suppose I just make this exact identification for the moment,

$$\beta = e^{j3\pi/4}\, 10\, (\sqrt{2}/a)$$

Comments about β_d

Now

$$\beta_d^2 = \omega^2 \mu_d (\varepsilon_d - j\sigma_d/\omega)$$

If we include loss tangent stuff, then

$$\varepsilon_d = \varepsilon_d' [1 - j \tan_L]$$

but an easier way to think of this is

$$\beta_d^2 = \omega^2 \mu_d (\varepsilon'_d - j\sigma_{d,eff}/\omega) \qquad \sigma_{d,eff} = \sigma_d + \omega\varepsilon'_d \tan_L$$

The only real adjustment here is to realize that although you may have $\sigma_d = 10^{-15}$, you will still have

$$\sigma_{d,eff} = \sigma_d + \omega\varepsilon'_d \tan_L \approx \omega\varepsilon'_d \tan_L \approx \omega * [2 * 10^{-11}] [2 * 10^{-4}] \approx 8\pi f 10^{-15}$$

So at f = 100GHz = 10^{11} we have

$$\sigma_{d,eff} = 25\, 10^{11}\, 10^{-15} = 25\, 10^{-4} = 0.25 * 10^{-2}$$

So this is much larger than 10^{-15}, but much smaller than copper's 10^7.

Now back to β_d :

$$\beta_d^2 = \omega^2 \mu_d (\varepsilon'_d - j[\sigma_d + \omega\varepsilon'_d \tan_L]/\omega) \approx \omega^2 \mu_d (\varepsilon'_d - j[\omega\varepsilon'_d \tan_L]/\omega)$$

$$= \omega^2 \mu_d (\varepsilon'_d - j\varepsilon'_d \tan_L) = \omega^2 \mu_d \varepsilon'_d (1 - j \tan_L)$$

Another way to say this is shown in (3.3.7)

$$\xi = \varepsilon'(1 - j \tan_L)$$

$$\beta^2 = \mu\varepsilon\omega^2 - j\omega\mu\sigma = \omega^2\mu (\varepsilon - j\sigma/\omega) = \omega^2\mu \xi = \omega^2\mu \varepsilon'(1 - j \tan_L)$$

So I am just making sure the result is correct. Here it is again:

$$\beta_d^2 = \omega^2 \mu_d \varepsilon'_d (1 - j \tan_L)$$

Now consider

$(1 - j \tan_L)^{1/2} \approx [1 - j (1/2)\tan_L]$ since $\tan_L \sim 10^{-4}$

Then we get

$\beta_d = \omega \sqrt{\mu_d \varepsilon'_d}\ [1 - j (1/2)\tan_L] = (\omega/v_d) [1 - j (1/2)\tan_L]$

and you might then say

$\beta_d = \beta_{d0}\ [1 - j (1/2)\tan_L]$ $\beta_{d0} = (\omega/v_d)$

The main observation is that β_d is mainly real with a small negative imaginary part. Also,

$|\beta_d| \approx |\beta_{d0}| = (\omega/v_d)$

Ratio of the two β's

To summarize our main results

$\beta = e^{j3\pi/4} (\sqrt{2}/\delta) \approx e^{j3\pi/4}\ 10 (\sqrt{2}/a)$ for $\delta = a/10$

$\beta = (j-1)(1/\delta)$

$\beta_d = \beta_{d0}\ [1 - j (1/2)\tan_L]$

and then

$|\beta| = (\sqrt{2}/\delta) \approx 10 (\sqrt{2}/a)$ for $\delta = a/10$

$|\beta_d| \approx (\omega/v_d) = \beta_{d0} = 2\pi/\lambda_d$

Now the Transmission Line Limit requires $\lambda_d > 10a$ lets say, so then

$|\beta_d| = 2\pi/\lambda_d < 2\pi/(10a)$

So here we have a "reasonable" skin effect situation with a "reasonable" TL limit. In this case

$|\beta|/|\beta_d| = 10 (\sqrt{2}/a) * 10a/(2\pi) = 100 (\sqrt{2}) * /(2\pi) = 100 / (\sqrt{2}\,\pi) = 22.5$

Now this is the situation in which I want to study the E fields in the box above.

Note added 3/6/14: I want to write this ratio in some other ways.

1. Here is one:

$$|\beta|/\,|\beta_d| \;=\; (\sqrt{2}\,/\delta)\,/\;\; 2\pi/\lambda_d \;=\; \frac{\sqrt{2}}{2\pi}\,(\lambda_d/\delta) \qquad\qquad \delta \equiv \sqrt{2/(\omega\mu\sigma)}$$

which you can see is very small for normal values of δ.

2. Next, write $\lambda = 2\pi v/\omega$ and $\delta \equiv \sqrt{2/(\omega\mu\sigma)}$ to get

$$|\beta|/\,|\beta_d| \;=\frac{\sqrt{2}}{2\pi}\,(\lambda_d/\delta) \;=\frac{\sqrt{2}}{2\pi}\;*\,2\pi v/\omega\;*\sqrt{\omega\mu\sigma}\,/\sqrt{2}\;=v/\omega\;\sqrt{\omega\mu\sigma}\;\;=\;v_d\,\sqrt{\mu\sigma}\;\;(1/\sqrt{\omega}\,)$$

3. Now we can write $\sqrt{\varepsilon\mu} = 1/v$ and then $\sqrt{\mu} = (1/\sqrt{\varepsilon}\,)(1/v)$ so we get another form

$$|\beta|/\,|\beta_d| \;=\; \sqrt{\sigma/\varepsilon\omega} \qquad \text{and} \qquad |\beta_d|/\,|\beta| \;=\sqrt{\varepsilon\omega/\sigma}$$

This says that at low ω the second ratio is small, and at ω = 0 is zero, but as ω increases, the ratio increases. I think it increases to about 10^{-3} for copper at 1000 GHz.

I think it is OK to approximate $\beta' \approx \beta$ given this factor of 22.5. Thus, the discussion of β_d above will play no role, once we know it is small enough to be ignored in this context.

2. Study of the Ratio $E_z(r,m)/\,E_z(a,m)$ and $E_r(r,m)/\,E_r(a,m)$

I show that you get about the same skin effect decay in all the partial waves both for E_z and E_r and obviously also for E_φ. So all components of the E field have the same skin effect.

So what do the Bessel functions look like? It is quite complicated. Start with E_z and write

$$E_z(r,m)/\,E_z(a,m) \;=\;[\frac{J_m(x)}{J_{m+1}(x_a)}\;-\;\frac{J_m(x)}{J_{m-1}(x_a)}\,]\;/\,[\frac{J_m(x_a)}{J_{m+1}(x_a)}\;-\;\frac{J_m(x_a)}{J_{m-1}(x_a)}\,] \qquad x = \beta r \quad x_a = \beta a$$

I am mainly interested in the abs value of this equation to see hopefully the skin effect at work. Rewrite as

$$\frac{E_z(r,m)}{E_z(a,m)} \;=\; \frac{\dfrac{J_m(x)}{J_{m+1}(x_a)}\;-\;\dfrac{J_m(x)}{J_{m-1}(x_a)}}{\dfrac{J_m(x_a)}{J_{m+1}(x_a)}\;-\;\dfrac{J_m(x_a)}{J_{m-1}(x_a)}} \;=\; \frac{J_m(x)}{J_m(x_a)}\;*\;\frac{\dfrac{1}{J_{m+1}(x_a)}\;-\;\dfrac{1}{J_{m-1}(x_a)}}{\dfrac{1}{J_{m+1}(x_a)}\;-\;\dfrac{1}{J_{m-1}(x_a)}} \;=\; \frac{J_m(x)}{J_m(x_a)}$$

So this ratio is simpler than I thought it was. In (2.2.22) we had the m = 0 version of this ratio!

$$E(r) = E(a)\,\frac{J_0(\beta r)}{J_0(\beta a)} \quad . \qquad\qquad (2.2.22)$$

So let's try a Maple plot.

```
restart; alias(J=BesselJ,I=I,j=sqrt(-1));
```

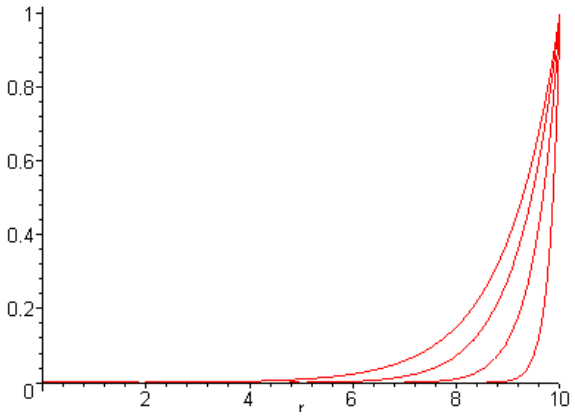
$$J, j$$

```
f := abs(J(m,x)/J(m,xa));
```

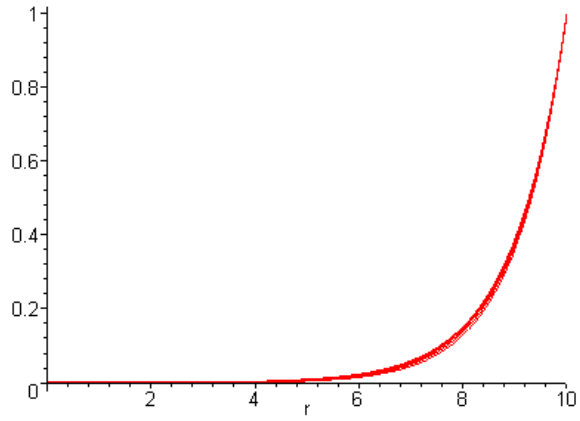
$$f := \left| \frac{J(m, x)}{J(m, xa)} \right|$$

```
beta := exp(3*Pi*j/4)*(sqrt(2)/delta):
delta := a/10:
a := 10:  xa := beta*a:  x := beta*r:
plot({seq(f(m),m=[0,1,2,3,4,5])}, r=0..a, color = red);

plot({seq(f(m),m=[0,10,20,50])}, r=0..a, color = red);
```




```
plot({seq(f(m),m=[0,1,2,3,4,5])}, r=0..a, color = red);
```



The values m = 0,1,2,3,4,5 are almost identical, but the corner moves slightly to the right as m increases. The first plot shows this more dramatically.

Fact: For E_z, all the partial waves exhibit the same characteristic skin effect decrease away from the surface, though higher m values give faster decays.

Let's now try this for E_r :

$$\frac{E_r(r,m)}{E_r(a,m)} = \frac{\frac{2mx^{-1}J_m(x)}{J_{m-1}(x_a)} + \frac{J_{m+1}(x)}{J_{m+1}(x_a)} - \frac{J_{m+1}(x)}{J_{m-1}(x_a)}}{\frac{2mx_a^{-1}J_m(x_a)}{J_{m-1}(x_a)} + \frac{J_{m+1}(x_a)}{J_{m+1}(x_a)} - \frac{J_{m+1}(x_a)}{J_{m-1}(x_a)}}$$

I don't see a simple way to reduce this. I wrote up the Maple and made the plots, and the two plots look exactly the same! Why is that? \

3. Why do all components have same skin effect?

I show that the reason for same skin effect of all components is that all have the same Bessel large argument behavior with same phase. I show that all square brackets in (D.2.33) above have same scale in skin effect, and therefore the relative size of the E field components are in fact determined by the leading factors and this is true for all m. Thus, the idea that E_r and E_φ are scaled down from E_z by (β_d/β) is valid in all partial waves m. Just confirming something I always presumed was true.

Let's consider a single Bessel function like $J_m(x)$ with $x = \beta r$ and $\beta = e^{j3\pi/4}\ 10\ (\sqrt{2}\ /a)$

For the range of r close to a, the argument is on the order of

$$x_a = \beta a = e^{j3\pi/4}\ 10\ (\sqrt{2}\ /a)\ a = e^{j3\pi/4}\ 10\ (\sqrt{2}\ ) = 14.1\ e^{j3\pi/4}$$

which perhaps is a "large argument" in this situation. Here is some math on that subject: [Ref?]

$$J_m(x) = (2/\pi x)^{1/2} \cos(x - m\pi/2 - \pi/4) \qquad \beta = (j-1)(1/\delta) \qquad x = \beta r = (j-1)(r/\delta)$$

"Near" r = a, x is "large" so we can use this large x expansion. then

$$\cos = (1/2)\exp[-j\ \{x - m\pi/2 - \pi/4\}] = (1/2)\exp[-j\ \{(j-1)(r/\delta) - m\pi/2 - \pi/4\}\ ]$$

$$= (1/2)\exp[+ (r/\delta) + j(r/\delta) + jm\pi/2 + j\pi/4\ ]$$

$$= (1/2)e^{+r/\delta}\ e^{j[(r/\delta) + m\pi/2\ +\ \pi/4]}$$

We ignore the other expo term since it blows down instead of up. So then

$$J_m(x) = (2/\pi x)^{1/2}\ (1/2)\ e^{+r/\delta}\ e^{j[(r/\delta) + m\pi/2\ +\ \pi/4]} \qquad \text{// assumes large x limit}$$

So this is the leading term for large x and it has weak m dependence.

Now the coefficient in the next lowest term is proportional to m^2 so very large m, that next term will play a role, and this explains why the E_z curves are different for large m values.

Now if you ignore the difference in the m's, we have

$$E_r(r,m) = \sim \left[\frac{2mx^{-1}J_m(x)}{J_{m-1}(x_a)} + \frac{J_{m+1}(x)}{J_{m+1}(x_a)} - \frac{J_{m+1}(x)}{J_{m-1}(x_a)} \right] \approx \frac{2mx^{-1}J_m(x)}{J_m(x_a)}$$

$$= \frac{J_m(x)}{J_m(x_a)} (2m/x)$$

and then

$$\frac{E_r(r,m)}{E_r(a,m)} = \frac{J_m(x)}{J_m(x_a)} (a/r)$$

so for $r \approx a$ this explains the similarity in the curves more or less.

I think I can assume that E_φ will have a similar decay pattern. We just change the sign of the first term. // Yes, curves look about the same.

Fact: All the partial waves (all m) and all components exhibit the same skin depth behavior.

Fact: The Relative Size between components is determined by the leading factors.

We see that the *square bracket quantities* in (D.2.33) all look about the same in the Skin Depth Limit as functions of r coming in from the surface, that is what I showed above. Therefore, the relative size of components is determined by those leading factors for any m. Restating:

Fact: For all m, E_r and E_φ are both smaller than E_z by amount $(\beta_d/\beta) \geq 22.5$ for Skin Depth regime. As shown in (D.6.1), this is also true for m = 0.

4. Comparison of fields for different m values

I show that the E field square bracket factors are more or less independent of m in terms of scale. Thus, the scale of contributions is really determined by the η_m moments.

Now go back to

$$E_z(r,m) = (1/4)\, \eta_m\, I\, R_{dc}\, (a\beta') \left[\frac{J_m(x)}{J_{m+1}(x_a)} - \frac{J_m(x)}{J_{m-1}(x_a)} \right]$$

If we ignore the η_m dependence here, and since curves versus r have same general shape, what can we say about $|E_z(a,m)|$ as a function of m ?

```
restart; alias(J=BesselJ,I=I,j=sqrt(-1));
g := abs(J(m,xa)/J(m+1,xa) -  J(m,xa)/J(m-1,xa));
beta := exp(3*Pi*j/4)*(sqrt(2)/delta):
delta := a/10:
a := 10:  xa := beta*a:  x := beta*r:
for m from 0 to 10 do
  print(m,evalf(g));
od;
```

$$g := \left| \frac{J(m, xa)}{J(m+1, xa)} - \frac{J(m, xa)}{J(m-1, xa)} \right|$$

0, 2.050495334
1, 2.050397612
2, 2.050276960
3, 2.050654074
4, 2.052404799
5, 2.056757669
6, 2.065262364
7, 2.079699243
8, 2.101907514
9, 2.133546745
10, 2.175850001

This says basically that the lowest 10 partial waves have about the same size at r = a, so you cannot in any way claim that somehow larger m waves are intrinsically weaker in our skin effect regime.

5. Paradox #1 Brewing

Paradox #1 is this: Using the η_m obtained from n(θ), I can *see* from the general discussion above (without doing the calculation in detail) that $E_z(r,\theta)$ is going to have "strong θ dependence" similar to n(θ). Since $E_z(\theta) = +j\,\beta_d\,\varphi(\theta) - j\omega A_z(\theta)$ and since φ is constant on the periphery, it suggests that A_z is therefore NOT constant on the periphery, and this then does not support the King approach in Ch 4 *even in the strong skin effect regime*. [I agree on 5/7/14.] At the time, I was really torn between (a) E_z is constant on periphery; (b) E_z strongly varies on periphery something like n(θ). ALSO, if A_z is not constant on the periphery, then $\mathbf{H} = H_\theta \hat{\boldsymbol{\theta}}$ is not true on the periphery, and I investigate this aspect in the next section. This seems to go against web pictures I have seen, for example.

I found in "details of Chap 6 exact" that for the two cylinder problem, the moments are

$$\eta_m \equiv N_m / N_0 = (-1)^m \exp(-|m|\xi_1)$$

and for $\xi_1 = 0.25$ I got these numbers

```
xi1 := 0.25;
```

$$\xi 1 := .25$$

```
for m from 0 to 10 do
  print("m= ",m,  "etam = ", (-1)^m*exp(-m*xi1));
od;
```

"m= ", 0, "etam = ", 1
"m= ", 1, "etam = ", -.7788007831
"m= ", 2, "etam = ", .6065306597
"m= ", 3, "etam = ", -.4723665527
"m= ", 4, "etam = ", .3678794412
"m= ", 5, "etam = ", -.2865047969
"m= ", 6, "etam = ", .2231301601
"m= ", 7, "etam = ", -.1737739435
"m= ", 8, "etam = ", .1353352832
"m= ", 9, "etam = ", -.1053992246
"m= ", 10, "etam = ", .08208499862

and I noted the alternating signs. The decay is rather slow, so there are lots of partial waves active, and I showed how this is needed to regenerate the surface charge function n(φ) in this case.

Now if I use these η_m values in my $E_z(r,m)$ expression,

$$E_z(r,m) = (1/4)\,\eta_m\, I\, R_{dc}\ (a\beta')\left[\frac{J_m(x)}{J_{m+1}(x_a)} - \frac{J_m(x)}{J_{m-1}(x_a)}\right]$$

I find that the η_m decay slowly in magnitude, while the [..] factor does not decay with m near r = a. I also showed that

$$n(\varphi)/N_0 = 1 + 2\,\Sigma_{m=1}^{\infty}\eta_m \cos(m\varphi)$$

$$= 1 + 2\,\Sigma_{m=1}^{\infty}(-1)^m \exp(-|m|\xi_1)\cos(m\varphi)$$

Therefore, near φ = 0, cos(mφ) = 1 and we get cancellation between the partial waves, and so I expect the sum of partial waves $E_z(a,\varphi)$ to be SMALL at φ = 0 due to these cancellations. On the other hand, at φ = π, all terms have the same sign, and I expect to have E_z = large.

<u>This case study *suggests* therefore that $E_z(a,\theta)$ varies going around the cross section perimeter</u> if we are in the skin effect regime (where I could claim that [...] does not vary much with m).

Now go back to our old friend,

$$\mathbf{E} = -\,\text{grad}\,\varphi - \partial_t \mathbf{A}\ . \qquad (1.3.1)$$

$$E_z = +j\,\beta_d\,\varphi - j\omega A_z$$

I still maintain φ = constant around the perimeter. In the skin effect limit, I *thought* I was going to obtain the result that A_z was also constant on the perimeter, and that would imply that E_z was constant on the perimeter. But in my case study, I find that E_z has large variation between φ = 0 and φ = π, so we do indeed have a Paradox here.

6. Review of H being tangent to the surface in skin regime:

The gist here is that my Stokes and Gauss box rules with skin effect say that on the periphery, $\mathbf{H} = H_\theta \hat{\boldsymbol{\theta}}$ and this seems to agree with the idea of A_z = constant on periphery. But previous section seems to say A_z is *not* constant because E_z is not constant, so confusion continues.

1. Draw Stokes loop around skin layer and use Ampere integral form to conclude that

$$H_t = H_\theta = J_z \delta = K_z$$

and this is the result everyone quotes. It is based on **H** = 0 inside skin layer Claim can ignore the short portions of the Stokes loop. [OK]

Note added 3/5/14. The above is based on curl $\mathbf{H} = \partial_t \mathbf{D} + \mathbf{J}$ and $\oint_C \mathbf{H} \bullet d\mathbf{s} = \int_S [\partial_t \mathbf{D} + \mathbf{J}] \bullet d\mathbf{S}$ for a rectangular thin loop which is "sideways to the surface" so that $d\mathbf{S} = dS\, \hat{\mathbf{z}}$ (always have a picture!!!). Thus, the displacement term involves $j\omega D_z = j\omega\varepsilon E_z$. As in lines doc (3.4.1), we know we can always ignore this term compared to the conduction current term $J_z = \sigma E_z$. The short sides of the loop pick up H_n but only contribute if $H_n = \infty$. Note that we are NOT claiming $H_n = 0$ in this step, only $H_t = K_z$.

2. Draw Gaussian box but for magnetic. Conclude that $H_n = 0$, again based on **H** = 0 inside skin layer. Claim can ignore thin side surfaces of the box. [true as $\delta \to 0$, or $E \equiv 0$ inside.]

3. The conclusion is that, in the extreme skin effect regime, the **H** field is tangent to the surface is $H_t = K_z$.

4. Therefore, there is an **H** field line which is an exact circle going around a round conductor which is operating in a transmission line in the skin limit. [In this limit, $\mathbf{E} \approx 0$, $H_n \approx 0$]

5. There is no claim being made that $|\mathbf{H}| = H_t$ is constant going around the perimeter !!!! [correct] If there is current asymmetry, then we expect this to vary, and our example above strongly suggests there IS a current asymmetry. [We expect $H_t(\theta) = K_z(\theta)$ and $K_z(\theta)$ generally varies with θ .]

6. Here is why I think $\mathbf{H} = H_\theta \hat{\boldsymbol{\theta}}$ on periphery means A_z = constant on periphery: [correct, see next]

Align a thin Stokes loop which is long in the z direction and such that the B flux through the loop is 0. The thin edges of the loop are along B field lines. Along the long loop edges, we must have $A_{z1} = A_{z2}$

from our **B** = curl**A** Stokes law. Thus, even though the B field strength might vary along the B field lines in this area, A_z is still a constant going along the field line. [correct]

Similarly, in electrostatics, if you move along a φ = constant line in 2D, the E field might vary as you move along the equipotential line. [correct analogy]

7. Paradox #2 with the King Gauge

King Gauge says $A_z(\mathbf{x})$ = constant * φ(**x**) if you ignore A_x and A_y. This conflicts with my DC plots of A_z contours which do not match those of φ at surface. So it must be that at low frequency, you *cannot* ignore A_x and A_y. I give a simple reason why this might be so.

The King gauge condition says (applying this in the dielectric right now)

$$\text{div } \mathbf{A} = -j(\beta_d^2/\omega)\varphi$$

If we ignore $\partial_x A_x$ and $\partial_y A_y$ then this says

$$\partial_z A_z = -j(\beta_d^2/\omega)\varphi$$

and if A_z has our usual wave form in z, this says

$$-j\beta_d A_z = -j(\beta_d^2/\omega)\varphi \qquad // \ \beta_{d0} \approx \beta_d = (\omega/v_d) \quad \text{ignoring loss tangent}$$

or

$$A_z(\mathbf{x}) = (\beta_d/\omega)\varphi = (1/v_d)\,\varphi(\mathbf{x})$$

where I suppose **x** can be any point in the dielectric, including right at the conductor surfaces.

This equation has several problems:

Paradox 2: This says $A_z(\mathbf{x})$ = constant * φ(**x**) . This says that at some very low frequency ω, since circle has constant φ, circle must also have constant A_z which I know is not true, having drawn the field lines [It is correct that the B field lines do not match the periphery circles in my DC plots!]

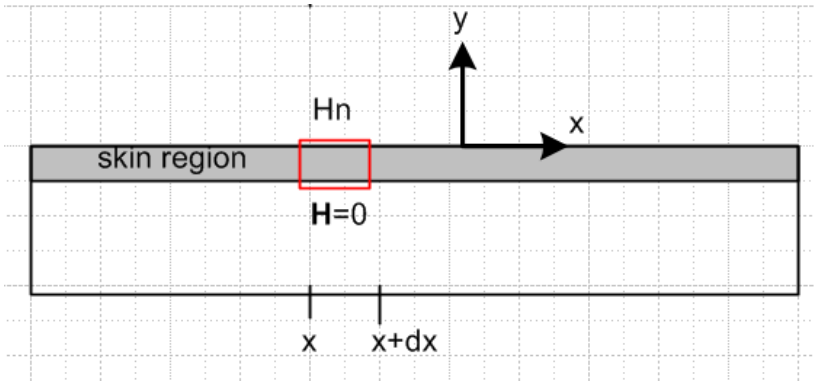
Possible way out: At low frequency where there is little or no skin effect, *there are* J_r and J_φ currents inside the conductor, and these do create A_x and A_y, and things vary with x and y, and the way out would be that at low frequency, you cannot ignore $\partial_x A_x + \partial_y A_y$. As you move into strong skin effect, the ratio β′/β gets small, and then those J_r and J_φ go away!

8. A Review of the Idea that $H_n = 0$ at a surface in the skin effect limit.

I show here that $H_n = 0$ for perfect conductor ($\delta \to 0$) , but not for an imperfect one.

I am thinking of a piece of a round conductor and imagine that along the surface, the B field weakens as one moves away from the hot spot between the conductors. One could at least imagine this happening. So

let's postulate that there exists an H field for which $H_z = 0$ (as apropo for a cross section like this), but might have some general H_x and H_y : [$\int_{box} \mathbf{H} \bullet d\mathbf{S} = 0$ since div $\mathbf{B} = 0$.]



This drawing shows a blowup of a piece of the round conductor surface in cross section. The red box has width dx and it has depth dz into the plane of paper. The height is skin depth δ. Assume now that there exists some H_x in the region of the box *which varies* with x. Then since div **H** = 0, Gauss's law applied to this box says (I add up all the outflows from the box) the total flux out of the box must be zero:

Let's look at all surfaces of our gaussian box. We assume no action in the plane of paper, so the near and far sides which involve H_z we just ignore assuming $H_z = 0$. In terms of left and right, consider the right face. There we can say (think of y axis pointing down for the moment and x pointing to the left!)

$$\int_{leftface} \mathbf{H} \bullet d\mathbf{S} = - \int_{-\delta}^{0} dydz\, H_x(x,y) = - dz\, H_x(x,y=0) \int_{-\delta}^{0} dy\, e^{+y/\delta}$$

$$= - dz\, H_x(x,y=0) \int_{0}^{\delta} dy'\, e^{-y/\delta} \qquad y' = -y$$

$$\approx - dz\, H_x(x,y=0) \int_{0}^{\infty} dy'\, e^{-y/\delta} = - dz\, H_x(x,y=0)\, \delta$$

$$\int_{rightface} \mathbf{H} \bullet d\mathbf{S} = + dz\, H_x(x+dx,y=0)\, \delta \qquad \text{// plus since face pointing right}$$

Meanwhile, the bottom face contributes nothing since $H_n = 0$ there and top face makes

$$\int_{topface} \mathbf{H} \bullet d\mathbf{S} = dzdx\, H_y(x,y=0)$$

We add the face contributions and set to 0 since div **B** = 0,

$$- dz\, H_x(x,y=0)\, \delta + dz\, H_x(x+dx,y=0)\, \delta + dzdx\, H_y(x,y=0) = 0$$

Divide by dz to get

$$- H_x(x,y=0)\,\delta + H_x(x+dx,y=0)\,\delta + dx\,H_y(x,y=0) = 0$$

Divide by dx to get

$$\delta\,[H_x(x+dx,y=0) - H_x(x,y=0)]/dx = - H_y(x,y=0)$$

or

$$\delta\,\partial_x H_x(x,y=0) = - H_y(x,y=0)$$

Now $H_y = H_n$ (normal) at the surface, so we conclude that

$$H_n(x,y=0) = - \delta\,\partial_x H_x(x,y=0)$$

Had we included the z-sides of the box, we would have gotten

$$\int_{frontface} \mathbf{H} \bullet d\mathbf{S} = + dx\,H_z(z+dz,y=0)\,\delta$$

$$\int_{backface} \mathbf{H} \bullet d\mathbf{S} = - dx\,H_z(z,y=0)\,\delta$$

Then div H = 0 would have said

$$- dz\,H_x(x,y=0)\,\delta + dz\,H_x(x+dx,y=0)\,\delta + dzdx\,H_y(x,y=0) + dx\,H_z(z+dz,y=0)\,\delta - dx\,H_z(z,y=0)\,\delta = 0$$

Divide by dz to get

$$- H_x(x,y=0)\,\delta + H_x(x+dx,y=0)\,\delta + dx\,H_y(x,y=0) + dx\,\delta\,[H_z(z+dz,y=0) - H_z(z,y=0)]/dz = 0$$

Then divide by dx to get

$$\delta\,[H_x(x+dx,y=0) - H_x(x,y=0)]/dx + H_y(x,y=0) + \delta\,[H_z(z+dz,y=0) - H_z(z,y=0)]/dz = 0$$

or

$$\delta\,\partial_x H_x(x,y=0) + H_y(x,y=0) + \delta\,\partial_z H_z(x,y=0) = 0$$

so

$$H_y(x,y=0) = - \delta\,\partial_x H_x(x,y=0) - \delta\,\partial_z H_z(x,y=0)$$

or

$$H_n(x,y=0) = - \delta\,[\partial_x H_x(x,y=0) + \partial_z H_z(x,y=0)]$$

Now, if we go to the extreme skin effect limit where $\delta \rightarrow 0$, THEN we really do get

$$H_n(x,y=0) = 0.$$

Conclusion: Even if the H field at the surface *varies* in the transverse directions, in the extreme skin effect limit $\delta \to 0$ and we have $H_n = 0$ just outside the surface .

This conclusion is not true of course as we approach the limit, only at the limit.

Now recall **Paradox #1** which consists of two logic threads. The first thread looks at the two cylinders case and explicitly calculates that E_z is not constant around the round conductor. The second thread is this:

(a) skin limit => H tangential => H lines match surface => A_z = constant on surface
(b) φ = constant on surface as assumed in capacitor problem
(c) $E_z = -\partial_z\varphi + \partial_t A_z \quad => \quad E_z = j\beta_d\varphi + j\omega A_z$

(a) + (b) + (c) => E_z = constant on surface => J_z is symmetrical all around the conductor

This paradox however has a **Resolution** as follows: If we have something like $\delta = a/10$, we are in the skin effect regime, but not in the extreme limit. In this case, E_z varies with θ violently for our two-cylinder problem, but is always very small in magnitude. Thus we have $j\omega A_z(\mathbf{x}) = - j\beta_d\varphi(\mathbf{x}) + E_z(\mathbf{x})$ and we conclude that $A_z(\mathbf{x})$ is very close to being constant on the surface, but not quite due to the small E_z. In terms of the Paradox statement above, what we are saying is that step (a) is not true with small but finite δ.

9. Review of Paradox #2 of the King Gauge

Here I am wondering if $\partial_x A_x + \partial_y A_y$ *not small* provides an escape hatch from this gauge paradox. I realize that I might be above to compute all of **A** for the two cylinder problem and this leads to a new doc on that subject. But there I find the calculation is too hard to do in a short time.

Paradox #2 seems more fundamental and scary. Recall that if we ignore the A_t contributions, we get

$$A_z(\mathbf{x}) = (1/v_d)\,\varphi(\mathbf{x}) \qquad (*)$$

In our capacitor two-conductor analysis we have $\varphi(\mathbf{x})$ constant around perimeter and we know that $A_z(\mathbf{x})$ is not constant on the perimeter, so (*) is blatantly violated in this example. [correct]

So consider with the two-cylinder prototype example,

$$\mathbf{E} = -\text{ grad }\varphi - \partial_t \mathbf{A} \ . \qquad (1.3.1)$$

$$E_x = -\partial_x\varphi - \partial_t A_x \qquad E_x = -\partial_x\varphi - j\omega A_x$$

In my capacitor problem, I have $\omega = 0$ and so this really does say $E_x = -\partial_x\varphi$ and that is how the electrostatics capacitor problem works. But for $\omega > 0$, we have the extra A_x term to worry about. We have

to argue that $\omega A_x \approx 0$ to justify the part of our capacitor problem transmission line solution which says that $\mathbf{E}_t = -\nabla_t \varphi$ (from which I get n(θ) etc etc). I do believe A_x really is small, but it is not zero because there are currents other than J_z . For example, inside both wires there is in fact a small J_x and then the King integral gives some small A_x. This is a "quasi-static" approximation somehow.

So here is the question: if A_x is very small, you would think $\partial_x A_x$ would also be small. Well I guess this has to be true:

The King gauge condition says,

$$\text{div } \mathbf{A} = -j(\beta^2/\omega)\, \varphi$$

Therefore

$$\partial_x A_x + \partial_y A_y + \partial_z A_z = -j(\beta^2/\omega)\, \varphi$$

$$\partial_x A_x + \partial_y A_y \;\; -j\beta_d\, A_z = -j(\beta^2/\omega)\, \varphi$$

$$(\partial_x A_x + \partial_y A_y)(\mathbf{x}) = -j(\beta^2/\omega)\, \varphi(\mathbf{x}) \;\; + -j\beta_d\, A_z(\mathbf{x})$$

So this is a "formula" for the quantity on the left that we are wonder if is "small". Since in our example we think we know all about $\varphi(\mathbf{x})$ So is there some way to check this equation for two-cylinders?

Question: How can we compute A_z for the two-cylinder problem in skin effect regime?

I know that

$$E_z = -\partial_z \varphi - \partial_t A_z$$

$$E_z = j\beta_d \varphi - j\omega\, A_z$$

$$-j\omega A_z(\mathbf{x}) = E_z(\mathbf{x}) - j\beta_d \varphi(\mathbf{x})$$

OK, I think I know both φ and E_z for this problem. I know φ everywhere for sure. I get the moments η_m and then I know E_z from my Appendix D formula for E_z. So maybe I really need to go "work through" this problem so I at least have one example to look at. It will all assume that $E_\varphi = 0$ at the surface, however, but with small A_x that is the same as φ = constant on surface ring.

10. Residual Question on Quasi Static

What happens to the quasi-static as ω increases from 0?

Consider the two-cylinder problem. I compute φ using quasi-static capacitor. I compute static $\mathbf{E}_t = -\nabla_t \varphi$ and from that I compute n(θ) and the moments η_m. But as we ramp up ω above ω = 0, we then have

$$\mathbf{E}_t = -\nabla_t\varphi - j\omega\mathbf{A}_t$$

It is incumbent upon "the writer" to show that as ω ramps up, one can neglect this extra term $-j\omega\mathbf{A}_t$. If one can show this, then the n(θ) distribution stays at its basic electrostatic form. I suppose it is possible that, as ω ramps up and we move away from electrostatics, φ itself changes away from its electrostatic functional form. I do think that, if we were to ramp up σ of the conductor to force more into the skin effect with ω = constant, then $\mathbf{A}_t \rightarrow 0$ because $\mathbf{J}_t \rightarrow 0$. I really do think King's stuff is valid for the skin effect or perfect conductor regime, but I have not yet assembled the pieces of this claim. [but this is now all written up in lines doc Section 3.7]

11. Resolution of Paradox #1 (added 3/5/14)

When this Paradox first arose, I was truly mystified, but in retrospect it is not so mysterious. My current view of things is that φ = constant exactly on the two cylinder surfaces, even for an imperfect conductor like copper, and even when one leaves the DC case and goes to ω > 0 perhaps up to the transmission line limit. Copper is a "good" conductor, not a "perfect" conductor. In this good conductor case, there is some E_z inside the conductor which is quite small in magnitude, though it does dramatically vary as a function of θ. But even at θ = π where it peaks, E_z is still small (since copper is so "good"). The way the paradox is resolved is this. Consider this equation evaluated at the conductor surface:

$$E_z = j\,\beta_d\,\varphi - j\omega A_z$$

I think of φ = constant and A_z = "almost constant". Think of ωA_z having magnitude 100 and E_z having magnitude 1. Then a 1% variation in A_z around the conductor perimeter allows for small E_z to be present and of course to vary with θ. This 1% effect also means that the **B** field is not exactly tangent to the conductor surface and misses being so by about 1%.

So one "take away" from this resolution is to keep in mind that A_z is not exactly constant on the surface, but for a "good conductor", it is very close to being constant, and then King's development is justified in terms of W(z) depending only on z. I of course have to edit lines doc to clarify that fact.
[this resolution sounds good on 5/7/14 and the next day I reviewed it in detail in a separate doc which is called "review of paradox 1.doc"].

12. Resolution of Paradox #2 (added 3/5/14)

Consider the King gauge condition again, which I write in a manner to isolate the transverse parts (we are in the dielectric here)

$$(\partial_x A_x + \partial_y A_y)(\mathbf{x}) = -j(\beta_d^2/\omega)\,\varphi(\mathbf{x}) - \partial_z A_z(\mathbf{x}) \;=\; -j(\beta_d^2/\omega)\,\varphi(\mathbf{x}) + j\beta_d A_z(\mathbf{x})$$

The "paradox #2" is that I always want to say LHS = 0 because $\mathbf{A}_t \approx 0$ and then the RHS tells us that if φ is constant on the conductor surface, then A_z must also be constant on the surface. The resolution here is related to the resolution of Paradox #1. For a "good" conductor, we know that although φ = constant, we have only that A_z = "close to constant", and the non-constancy of $A_z(x)$ then gives $(\partial_x A_x + \partial_y A_y)(\mathbf{x}) \neq 0$ going around the surface. So there is then no Paradox #2, nothing is being violated. The claim of course

is that since A_z is close to constant, the variation of $(\partial_x A_x + \partial_y A_y)(\mathbf{x})$ is small, but still exists, so one cannot willy-nilly just claim that $(\partial_x A_x + \partial_y A_y)(\mathbf{x}) \equiv 0$ as your approximation for a good conductor.

This resolution is consistent with another argument. In the *perfect* conductor limit where $\delta = 0$, we have only a surface current K_z on the conductor surface (skin + Debye if you like), and thus we have only the A_z component $\neq 0$ from the King integral: there are no transverse currents, so $\mathbf{A}_t \equiv 0$ and A_z is a perfect constant on the round conductor surface. But when we move to finite $\delta > 0$, there is some depth to the skin current and there are then transverse currents and transverse E fields, and this then leads to transverse $A_t \neq 0$ and this then leads to $(\partial_x A_x + \partial_y A_y)(\mathbf{x}) \neq 0$.

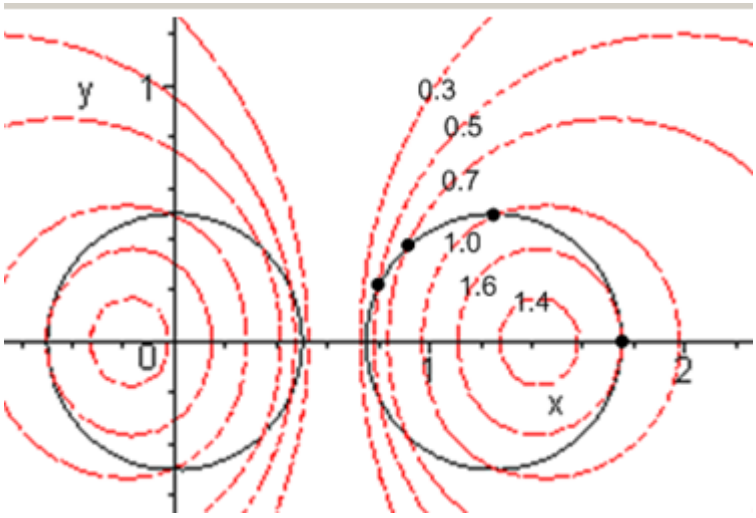
So both Paradox #1 and Paradox #2 are resolve in the same way: A_z = "almost constant" on the conductor surface for a "good" conductor.

13. But why is A_z largely non-constant on conductor at DC? Paradox 3.

In my two-cylinder plots of B I shows that the B lines are "way off" the conductor surfaces, so I presume that A_z in this case is "highly non-constant" on the surfaces. But since presumably φ = constant at very low frequency on the surfaces, and presumably E_z is very small. So Paradox 3: how can φ not vary at all, and yet A_z varies a lot, when I claim in Appendix M that we should have A_z = constant to 1 part in 10^4 ? [I don't claim that in Appendix M ! I claim that $A_x << A_z$. So no Paradox 3]

Hint #1. I presume that for ω near DC, $A_t <<<< A_z$ since there is no transverse current at all. So then you would think that $A_z(\mathbf{x}) = (1/v_d)\, \varphi(\mathbf{x})$ would be truer than ever! And we are very much in the TML.

Support: I added some detail to "two cylinder calculations" where I actually compute A_z directly from the Helm integral in the DC case. Here is a piece of the labeled plot I just now added: [the code here is in Az lines 1.mws as an implicitplot, hence ratty but OK, I quote this now in lines doc. See doc just mentioned for more details. 5.14.14]



For example, on the right black circle I show points with A_z = 1.6, 1.0, 0.7, 0.5 . This shows that A_z varies *substantially* on the black circle, just as you would think. This is not a 10^{-4} variation!

Fact: since this is DC, I know that $E_z = I/(\sigma\, \pi a_1^2)$ inside the conductor. Thus consider the following where we are "just outside" a conductor. We know that E_z is still $I/(\sigma\, \pi a_1^2)$ just outside since E_z is a tangential field at the boundary.

$E_z = -\partial_z\varphi - \partial_t A_z$

$E_z = j\beta_d\varphi - j\omega A_z$ $\qquad\qquad$ $\beta_d = \omega/v_d$

$E_z = j(\omega/v_d)\,\varphi - j\omega A_z$

Think now of ω being very small but not 0. We then have

$I/(\sigma \pi a_1^2) = j(\omega/v_d)\,\varphi - j\omega A_z$

How do I explain that as $\omega \rightarrow 0$, we get $I/(\sigma \pi a_1^2) = 0$ which is obviously wrong!

Here is one possible explanation. In the time domain write

$A_z(\mathbf{x},t) = A_z(\mathbf{x},\omega)e^{j\omega t} + A_z^{DC}(\mathbf{x})$ $\qquad$ where we allow a *constant* as part of the solution

$\partial_t A_z(\mathbf{x},t) = j\omega A_z(\mathbf{x},\omega)e^{j\omega t}$ $\qquad$ // constant does not appear here!

Now when we say

$A_z(\mathbf{x},\omega) = (1/v_d)\,\varphi(\mathbf{x},\omega)$

we are referring only to the non-constant part of $A_z(\mathbf{x},t)$! We could write this as

$A_z^{AC}(\mathbf{x},\omega) = (1/v_d)\,\varphi(\mathbf{x},\omega)$

and then this last equation can become true again to 10^{-4} accuracy! My plot shows $A_z^{DC}(\mathbf{x})$.

This explanation of course brings up a whole new raft of questions.

• I want to continue to believe that the quantity $\varphi(\mathbf{x},\omega)$ is independent of ω -- that is my Rock of Gibraltar. This then suggests that

$A_z^{AC}(\mathbf{x},\omega) = A_z^{AC}(\mathbf{x}) = (1/v_d)\,\varphi(\mathbf{x})$

so then A_z^{AC} is also independent of ω to our 10^{-4} accuracy. And it is pretty small since v_d is large.

• I have no direct way to compute $A_z^{AC}(\mathbf{x},\omega)$. The Helmholtz integral with constant J_z gives me the other term $A_z^{DC}(\mathbf{x})$. Perhaps I could compute $A_z^{AC}(\mathbf{x},\omega)$ by replacing J_z with $J_z = (J_z^{incip} - J_z^{DC})$. I think in this tiny difference you would have to worry about all partial waves. I think I know that

$$E_z(r,m) = (1/4)\,\eta_m\, I\, R_{dc}\, (a\beta')\left[\frac{J_m(x)}{J_{m+1}(x_a)} - \frac{J_m(x)}{J_{m-1}(x_a)}\right]$$

$$J_z(r,m) = \sigma\,(1/4)\,\eta_m\,I\,R_{dc}\,(a\beta')\left[\frac{J_m(x)}{J_{m+1}(x_a)} - \frac{J_m(x)}{J_{m-1}(x_a)}\right] \qquad x = r\beta' \qquad (*)$$

From App M, we still have $|\beta_d/\beta| = \sqrt{\varepsilon\omega/\sigma}$ and this is "always small" up to 1000 GHz, so

$$\beta' = \beta = (j-1)(1/\delta) \quad \text{with} \quad \delta \equiv \sqrt{2/(\omega\mu\sigma)} \qquad (M.8)$$

$$\beta_d = \beta_{d0}\,[1 - j\,(1/2)\tan_L] \approx \beta_{d0} = (\omega/v_d) = 2\pi/\lambda_d \qquad (M.7)$$

So then $x = \beta r = (j-1)(r/\delta)$ where δ is large at low ω so x is small.

DC limits of the Appendix D Fields

Question: What is the limit of $J_z(r,m)$ shown in (*) as $\omega \to 0$? All Bessel arguments are small and in general we have $x \to 0$.

10.7.3 $J_\nu(z) \sim (\frac{1}{2}z)^\nu / \Gamma(\nu+1), \quad \nu \neq -1, -2, -3, \ldots,$

therefore

$$J_m(x) = (x/2)^m / m! \qquad \text{for } m = 0,1,2,.....$$

But the rule

$$J_{-m}(x) = (-1)^m J_m(x)$$

says then that

$$J_{-m}(x) = (-1)^m J_m(x) = (-x/2)^m / m! \qquad \text{for } m = 0,1,2....$$

so be careful if negative m integers are encountered. Here I restrict to m = 0,1,2... :

$$f_m = \left[\frac{J_m(x)}{J_{m+1}(x_a)} - \frac{J_m(x)}{J_{m-1}(x_a)}\right] = \left[\frac{(x/2)^m / m!}{(x_a/2)^{m+1}/(m+1)!} - \frac{(x/2)^m / m!}{(x_a/2)^{m-1}/(m-1)!}\right]$$

$$= [\,(m+1)\,(x/2)^m\,(x_a/2)^{-m-1} - (1/m)\,(x/2)^m(x_a/2)^{-m+1}]$$

$$= (x/2)^m\,(x_a/2)^{-m}\,[\,(m+1)\,(x_a/2)^{-1} - (1/m)\,(x_a/2)\,]$$

$$= (x/x_a)^m\,[\,(m+1)\,(x_a/2)^{-1} - (1/m)\,(x_a/2)\,] = (x/x_a)^m\,[\,(m+1)\,(x_a/2)^{-1}] \text{ since } x_a \to 0$$

For m = 0 we get

$$\left[\frac{J_m(x)}{J_{m+1}(x_a)} - \frac{J_m(x)}{J_{m-1}(x_a)}\right] = 2\,\frac{J_m(x)}{J_{m+1}(x_a)} = \text{twice the first term above}$$

$$= 2(x/x_a)^0 \left[(0+1) (x_a/2)^{-1} \right] = 2 (x_a/2)^{-1} = 2 (2/x_a) = 4/x_a$$

So I think then that the low ω limit of J_z is this

$$J_z(r,m) = \sigma (1/4) \eta_m I R_{dc} (a\beta') (x/x_a)^m \left[(m+1) (x_a/2)^{-1} \right] \qquad m \neq 0$$

$$J_z(r,0) = \sigma (1/4) I R_{dc} (a\beta') (x/x_a)^0 \left[2 (x_a/2)^{-1} \right] \qquad m = 0$$

$$= \sigma (1/4) I \frac{1}{\sigma\pi a^2} a\beta' (4/x_a) = \sigma (1/4) I \frac{1}{\sigma\pi a^2} a\beta' (4/\beta' a)$$

$$= \sigma I \frac{1}{\sigma\pi a^2} = I / (\pi a^2)$$

so the m = 0 partial wave DOES approach the DC current. The higher partial wave currents?

$$J_z(r,m) = \sigma (1/4) \eta_m I R_{dc} (a\beta') (r/a)^m \left[(m+1) (\beta' a/2)^{-1} \right] \quad m \neq 0$$

$$= \sigma (1/4) \eta_m I R_{dc} (a\beta') (r/a)^m \left[(m+1) (2/a\beta') \right]$$

$$= \sigma (1/4) \eta_m I R_{dc} (r/a)^m \left[(m+1) (2) \right]$$

but $\beta' \rightarrow 0$ so we get

$$J_z(r,m) = \sigma (1/4) \eta_m I R_{dc} (r/a)^m \left[(m+1) (2) \right]$$

$$= \sigma (1/4) \eta_m I \frac{1}{\sigma\pi a^2} (r/a)^m \left[(m+1) (2) \right]$$

$$= (1/2) \eta_m I \frac{1}{\pi a^2} (r/a)^m (m+1)$$

This is a surprise to me. It suggests there is asymmetry at DC in the longitudinal current! I have checked the m=0 and general m results with a full pass.

Paradox and Confusion: Suppose the conductor has q = 0 on the surface so that $n(\theta) \equiv 0$. In this case

$$N_m = (q/2\pi) (-1)^m e^{-|m\xi_1|}$$

and I would argue that all the moments vanish if q = 0, $N_m = 0$. But lines doc says

$$N_0 = (\beta_d/2\pi\omega a) I = (1/2\pi a) I (\beta_d/\omega) = (1/2\pi a) I (1/v_d)$$

and this does NOT vanish at DC.

Very Low Frequency Wave going down two fat closely spaced round conductors.

From the above analysis, I seem to find this:

$J_z(r,0) = I / (\pi a^2)$

$J_z(r,m) = (1/2)\,\eta_m\, I\, \frac{1}{\pi a^2}\,(r/a)^m\,(m+1)$ $\qquad\qquad$ $\eta_m = (-1)^m\, e^{-|m\xi_1|}$

****

What is $J_z(r,-m)$?

$J_z(r,m) = \sigma\,(1/4)\,\eta_m\, I\, R_{dc}\,(a\beta')\, f_m$

From elsewhere I know that

$f_m \equiv \left[\frac{J_m(x)}{J_{m+1}(x_a)} - \frac{J_m(x)}{J_{m-1}(x_a)}\right]$ $\qquad\qquad$ $f_{-m} = f_m$

$\eta_m \equiv N_m / N_0 = (-1)^m \exp(-|m|\xi_1)$ $\qquad\qquad$ $\eta_{-m} = \eta_m$

So then

$J_z(r,-m) = \sigma\,(1/4)\,\eta_{-m}\, I\, R_{dc}\,(a\beta')\, f_{-m} = J_z(r,m)$

So for the mth partial wave we get

$J_z(r,\theta) = J_z(r,m)\, e^{jm\theta} + J_z(r,-m)\, e^{-jm\theta} = J_z(r,m)\, 2\cos(m\theta)$

$= \eta_m\, I\, \frac{1}{\pi a^2}\,(r/a)^m\,(m+1)\cos(m\theta)$ $\qquad\qquad$ in the DC limit

Maybe as $\omega \to 0$, the Helm for E_z becomes Laplace, and then these are atomic forms in polar coordinates.

From bipolar doc we have

$\eta_m \equiv N_m / N_0 = (-1)^m \exp(-|m|\xi_1)$

$n(\theta)/N_0 = 1 + 2\,\Sigma_{m=1}^{\infty}\eta_m \cos(m\theta) = 1 + 2\,\Sigma_{m=1}^{\infty}(-1)^m \exp(-|m|\xi_1)\cos(m\theta)$

So here is the z current density at $\omega = 0$ for the two-cylinders problem:

$$J_z(r,\theta) = I / (\pi a^2) + \Sigma_{m=1}^{\infty} \eta_m I \frac{1}{\pi a^2} (r/a)^m (m+1) \cos(m\theta)$$

$$= \frac{I}{\pi a^2} \left[1 + \Sigma_{m=1}^{\infty} \eta_m (m+1) (r/a)^m \cos(m\theta) \right]$$

$$= \frac{I}{\pi a^2} \left[1 + \Sigma_{m=1}^{\infty} (-1)^m (m+1) \exp(-|m|\xi_1) (r/a)^m \cos(m\theta) \right]$$

Like $n(\theta)$, there are alternating sign coefficients so the current is strongly peaked where $n(\theta)$ is peaked. In fact we can see that

$$J_z(a,\theta) = \frac{I}{\pi a^2} \left[1 + \Sigma_{m=1}^{\infty} (-1)^m (m+1) \exp(-|m|\xi_1) \cos(m\theta) \right]$$

$$n(\theta)/N_0 = 1 + 2 \Sigma_{m=1}^{\infty} (-1)^m \exp(-|m|\xi_1)\cos(m\theta)$$

Not quite the same however due to the (m+1) extra factor.

[at this point I did lots of edits on bipolar doc, including adding the η_m calculation as Appendix A]