Two Cylinder Calculations PhL 3.1.14

[Final review 5.14.14: all is well, and Fig 3.6b got its number labels repaired.]

Reviewed on 3/6/14, see red comments below.

Summary:

Section 1 is a tiny recap of the moment values η_m for two cylinders.

In **Section 2** I compute $E_r(a,m)$ from η_m and then just stop for some reason.

In **Section 3** I am back to a Two Adjacent Circles Paradox which is really The Z_s Paradox and this has all been settled, but I was learning that settlement right here.

In **Section 4** I compute A_z for two cylinders. I start with the Helm integral, but then I realize all I have to do is add up the results of Appendix B for each cylinder using superposition. That answer is stated below. I go ahead and plot this $A_z(x,y)$ as a plot3d in contour mode, look down from the top, and it gives exactly those B contours I got elsewhere. This sort of confirmed my idea that B field lines are A_z contours. I then did my implicitplot for $A_z(x,y)$ = various constants, and that gave me the famous labeled picture I used in lines doc now as Fig 3.6b! It is all right here including code (but numbers needed fixing)

1. Review of the Moments η_m 1

2. These moments then determine the $E_r(r=a-\varepsilon,m)$ via the charge pump BC. 2

3. Erroneous Development of the "two circles" Contradiction 3

4. Compute and Plot A_z for two cylinders (uniform current) 5

1. Review of the Moments η_m

Having written the bipolar paper, I am now in a position to "calculate everything" for a two cylinder transmission line. I have already calculated the "moments" in doc "details of Chap 6 exact" and found

$$\eta_m \equiv N_m / N_0 = (-1)^m \exp(-|m|\xi_1)$$

$$N_0 = (\beta_d / 2\pi\omega a)\, I \quad . \qquad \text{(D.2.31)}$$

and this is not affected by the ξ_2 label of the other conductor. I took $\xi_1 = 0.25$ which (if both are like that) is a fairly closely spaced twin-lead. The first 10 moments were

```
xi1 := 0.25;
```

$$\xi 1 := .25$$

```
for m from 0 to 10 do
  print("m= ",m,  "etam = ", (-1)^m*exp(-m*xi1));
od;
```

"m= ", 0, "etam = ", 1
"m= ", 1, "etam = ", -.7788007831
"m= ", 2, "etam = ", .6065306597
"m= ", 3, "etam = ", -.4723665527
"m= ", 4, "etam = ", .3678794412
"m= ", 5, "etam = ", -.2865047969
"m= ", 6, "etam = ", .2231301601
"m= ", 7, "etam = ", -.1737739435
"m= ", 8, "etam = ", .1353352832
"m= ", 9, "etam = ", -.1053992246
"m= ", 10, "etam = ", .08208499862

And I noted the alternating signs. This is the conductor on the right, and recall how θ is defined, so the terms are additive around θ = π.

2. These moments then determine the E_r(r=a-ε,m) via the charge pump BC.

Now, what are the E fields inside my round wire which has some radius a? Without any boundary conditions applied, I have

First summary of the E field solutions (D.2.21)

$$E_z(r,m) = -j\,(\beta'/\beta_d)\,\frac{K_m}{2}\,J_m(x) \qquad x = \beta' r \qquad \textit{(D.1.27)}$$

$$E_r(r,m) = a_m\,x^{-1}\,J_m(x) + \frac{K_m}{2}\,J_{m+1}(x) \qquad \beta'^2 = \beta^2 - \beta_d^2 \qquad \textit{(D.2.11)}$$

$$jE_\varphi(r,m) = -a_m\,x^{-1}\,J_m(x) + \left(\frac{K_m}{2} + \frac{a_m}{m}\right) J_{m+1}(x) \qquad \textit{(D.2.15)}$$

where I don't know the a_m and K_m coefficients, they could be anything for each mode.

If I believe my own "charge pumping condition", then lines doc says

$$E_r(r=a-\varepsilon,m) = (j\omega/\sigma)\,N_m \quad . \quad [\text{ still valid on 3/6/14 }] \qquad (D.2.25)$$

This says

$$E_r(r=a-\varepsilon,m) = (\beta_d/2\pi\omega a)\, I\, (j\omega/\sigma)\, (-1)^m \exp(-|m|\xi_1)$$

$$= (\beta_d/2\pi a)\, I\, (j/\sigma)\, (-1)^m \exp(-|m|\xi_1)$$

This condition is only one boundary condition and it is this from lines doc

$$a_m\, x_a^{-1}\, J_m(x_a) + \frac{K_m}{2}\, J_{m+1}(x_a) = (j\omega/\sigma)\, N_m \qquad (1)$$

The big problem now is finding a second boundary condition!!! Maybe I will find something new here! Apart from m = 0, all the Bessels vanish at r = 0, so we don't get anything new there.

How about this idea:

$$\int J_z\, dA = I$$

Well, only the m = 0 contributes, and I have already done that. Too bad.

3. Erroneous Development of the "two circles" Contradiction

But, in order to get the moments, I had to solve the capacitor problem, and in that problem I assumed that $E_\varphi = 0$ at the surface. [if you set $A_x = A_y = 0$; what was really assumed was φ = constant] This does not mean that $E_\varphi = 0$ inside, keep that in mind, so it is only on the surface that we have these rings. This then does imply that $E_z(r=a,\theta)$ will be constant all around! [huh? Where does this claim come from? See Note below.]That does not necessarily mean that the current density is uniform inside, but around the *surface* you must have J_z be the same going all around, and that then allows Z_s to be independent of θ. [but this is completely wrong as following note comments.]

Note added 3/6/14: With the "two circles separated by dz" idea, each circle has constant φ and so all the way around there is a constant dφ between the two circles. If you think that $E_z = -d\varphi/dz$, then you get a constant E_z and J_z around the perimeter. Another way to say this is $d\varphi/dz = -j\beta_d\varphi$ and so φ = constant says that dφ/dz = constant, and IF $E_z = -d\varphi/dz$, then E_z = constant. BUT in fact $E_z = -d\varphi/dz - j\omega A_z$. The first term is constant around the circle, yes, but the second term A_z is not constant on the circle except for the "perfect conductor". In this limit, A_z = constant and $E_z \equiv 0$ all the way around -- a constant. Remember that a very small percentage variation of A_z on the circle results in a large variation in E_z because E_z is a tiny difference between two large terms in $E_z = -d\varphi/dz - j\omega A_z$.

In the case of the gapped "slotline", you would solve the capacitor problem, and you would assume that φ was constant on the rectangular perimeter in order to do that. There would be charge density on all surfaces of both conductors, just not very strong far from the slot. But the equivalent of E_φ would have to be zero on the surface in order to get those constant-potential cross sections you assumed for the capacitor problem. So you could measure ΔV anywhere on a conductor's perimeter for some dz and get the same thing. [same wrong idea as above]

So this might alleviate my $Z_s(\theta)$ problem. [but no, $Z_s(\theta)$ really does vary strongly with θ !]

I said at one point that if you add up all the moments, you will find that Z_s depends on θ, but that must be wrong! [no, it is right!]

$$Z_s(\varphi) = (1/I) \sum_{m=-\infty}^{\infty} E_z(a,m)\, e^{jm\varphi} \qquad \text{// (D.1.3a)}$$

$$= (1/4)\, R_{dc} \sum_{m=-\infty}^{\infty} \eta_m \left[\frac{x_a J_m(x_a)}{J_{m+1}(x_a)} - \frac{x_a J_m(x_a)}{J_{m-1}(x_a)}\right] e^{jm\varphi} \qquad \text{// (D.2.33)}$$

When you add all this up, you will get a result with no φ dependence. How can that possibly happen? [it can't] If this is a Fourier Series expansion, the only way to get $Z_s(\varphi)$ = constant is this

$$E_z(a,m) = 0 \qquad \text{for all } m \neq 0$$

But looking at (D.2.33) that says

$$(1/4)\, \eta_m\, I\, R_{dc}\, (a\beta') \left[\frac{J_m(x_a)}{J_{m+1}(x_a)} - \frac{J_m(x_a)}{J_{m-1}(x_a)}\right] = 0$$

which requires that

$$\left[\frac{J_m(x_a)}{J_{m+1}(x_a)} - \frac{J_m(x_a)}{J_{m-1}(x_a)}\right] = 0$$

but this just is not true! It would require basically that

$$J_{m+1}(x_a) = J_{m-1}(x_a)$$

Even simpler, we know that

$$E_z(r,m) = -j\,(\beta'/\beta_d)\,\frac{K_m}{2}\,J_m(x)$$

so you would need

$$\frac{K_m}{2}\,J_m(x_a) = 0$$

Status: I have some kind of contradiction built into the solution in Appendix D. But I thought Appendix D was all done exactly, full Maxwell equations and so on. Maybe the ansatz of wave is bad?

[Appendix D is OK, there is no built in contradiction. The contradiction arises from rigidly assuming that A_z is exactly constant on the conductor perimeter along with φ being constant. The confusion is understandable, since I assumed just this fact about A_z in Chapter 4 !]

What about E_φ at r= a?

$$E_\varphi(r,m) = (1/4)\,\eta_m\, I\, R_{dc}\ (a\beta_d)\ [\ -\frac{2mx^{-1}J_m(x)}{J_{m-1}(x_a)} + \frac{J_{m+1}(x)}{J_{m+1}(x_a)} + \frac{J_{m+1}(x)}{J_{m-1}(x_a)}\ ]$$

$$E_\varphi(a,m) = (1/4)\,\eta_m\, I\, R_{dc}\ (a\beta_d)\ [\ -\frac{2mx_a^{-1}J_m(x_a)}{J_{m-1}(x_a)} + \frac{J_{m+1}(x_a)}{J_{m+1}(x_a)} + \frac{J_{m+1}(x_a)}{J_{m-1}(x_a)}\ ]$$

Maple shows this really is identically 0, so we do have $E_\varphi(a,m) = 0$ for all m. [well it has to be since I assumed this as a boundary condition to determine the constants a_n and K_n] Therefore, any cross section circle has φ = constant. Oops, I forgot about A_z!!! [about time!]

$$\mathbf{E} = -\text{ grad }\varphi - \partial_t \mathbf{A}$$

$$E_z = -\partial_z\varphi - j\omega A_z$$

$$E_z = j\beta_d\varphi - j\omega A_z$$

Since $A_z \neq$ constant on the circle in general, E_z is also not constant, so Z_s is also not constant! [right]

Mystery resolved: Since $A_z(C) \neq$ constant (see my B field line plot), although φ(C) = constant, you cannot conclude that $E_z(C)$ = constant, and therefore you should find $Z_s \neq$ constant, and that is exactly what I found above! [correct]

4. Compute and Plot A_z for two cylinders (uniform current)

Here for DC I compute A_z for two cylinders. I start with the Helm integral, but then I realize all I have to do is add up the results of Appendix B for each cylinder using superposition. That answer is stated below. I go ahead and plot this $A_z(x,y)$ as a plot3d in contour mode, look down from the top, and it gives exactly those B contours I got elsewhere. This sort of confirmed my idea that B field lines are A_z contours. I then did my implicitplot for $A_z(x,y)$ = various constants, and that gave me the famous labeled picture I used in lines doc now as Fig 3.6b! It is all right here including code.

Do I have an expression for A_z for the two cylinders at low frequency? Well I do have this,

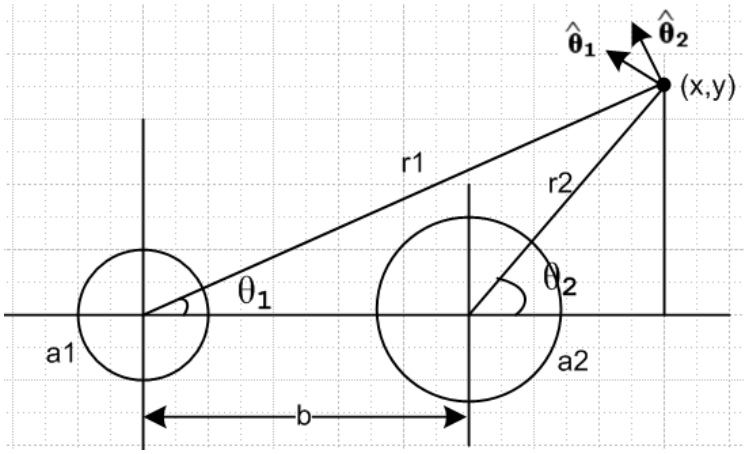
$$A_{z12}(\mathbf{x}) = \frac{\mu}{4\pi}\ i(z)\ \int_{-\infty}^{\infty} dz'\{\ \int_{C_1} dx_1'\, dy_1'\, b_1(x_1',y_1')\frac{1}{R_1} - \int_{C_2} dx_2'\, dy_2'\, b_2(x_2',y_2')\frac{1}{R_2}\ \} \qquad (4.9.1)$$

But I have never done this integration for uniform current density (that I can remember). Then in Section 5.2 I process this a bit,

$$A_z(x,y,z) = \frac{\mu}{4\pi}\, i(z)\, A_{zt}(x,y)$$

$$[\,\nabla_t^2 + (\beta^2 - k_A^2)]\, A_{zt}(x,y) = 0 \qquad (5.2.8)$$

If I think of losses as low this is just Laplace, but my boundary conditions are then no good and there is nothing I can do with this thing! [that is, I cannot find A_z by solving the ODE as I did for φ.] The only way is a brute force calculation. Let's go back to this picture



$$b_1(x_1',y_1') = (I/\pi a_1^2)\ \theta(r_1<a_1)$$

$$b_2(x_2',y_2') = (I/\pi a_2^2)\ \theta(r_2<a_2)$$

$$A_{z12}(\mathbf{x}) = \frac{\mu}{4\pi}\, i(z) \int_{-\infty}^{\infty} dz' \{ \int_{C_1} dx_1'\, dy_1'\, b_1(x_1',y_1') \frac{1}{R_1} - \int_{C_2} dx_2'\, dy_2'\, b_2(x_2',y_2') \frac{1}{R_2} \} \qquad (4.9.1)$$

Now write

$$\int_{C_1} dx_1'\, dy_1' = \int_{C_1} r_1 dr_1 d\theta_1 \qquad \text{and similarly, so}$$

$$A_{z12}(\mathbf{x}) = \frac{\mu}{4\pi}\, i(z) \int_{-\infty}^{\infty} dz' \{ \int_{C_1} r_1'dr_1'd\theta_1\, b_1(r_1') \frac{1}{R_1} - \int_{C_2} r_1'dr_1'd\theta_1'\, b_2(r_2') \frac{1}{R_2} \}$$

Now you have to imagine a source point inside the a_1 circle at some r_1' and θ_1'. Then

$$R_1 = |\, x,y \;-\; r_1',\theta_1'| \;= |\, r_1,\theta_1 \;-\; r_1',\theta_1'|$$

$$R_1^2 = r_1^2 + r_1'^2 - 2r_1r_1' \cos(\theta_1' - \theta_1) + (z')^2 \qquad \text{assume picture is at } z' = 0$$

So at this point we install the uniform current expression and continue:

$$A_{z12}(\mathbf{x}) = \frac{\mu}{4\pi}\ i(z)\ \int_{-\infty}^{\infty} dz' *$$

$$\{\ \int_{C_1}\ r_1'dr_1'd\theta_1\ (I/\pi a_1^2)\ \theta(r_1<a_1)\ \frac{1}{R_1}\ \ -\ \int_{C_2}\ r_1'dr_1'd\theta_1'\ (I/\pi a_2^2)\ \theta(r_2<a_2)\ \frac{1}{R_2}\ \}$$

$$A_{z12}(\mathbf{x}) = \frac{\mu}{4\pi}\ i(z)\ \int_{-\infty}^{\infty} dz' *$$

$$\{\ (I/\pi a_1^2)\ \int_{C_1}\ r_1'dr_1'd\theta_1\ \theta(r_1<a_1)\ \frac{1}{R_1}\ \ -(I/\pi a_2^2)\ \ \int_{C_2}\ r_1'dr_1'd\theta_1'\ \theta(r_2<a_2)\ \frac{1}{R_2}\ \}$$

$$A_{z12}(\mathbf{x}) = \frac{\mu}{4\pi}\ i(z)\ \int_{-\infty}^{\infty} dz' *$$

$$\{\ (I/\pi a_1^2)\ \int_0^{a_1}\ r_1'dr_1'\ \int_0^{2\pi} d\theta'_1\ \frac{1}{R_1} -\ (I/\pi a_2^2)\ \ \int_0^{a_2}\ r_2'dr_2'\ \int_0^{2\pi} d\theta'_2\ \frac{1}{R_2}\ \}$$

$$R_1^2 = r_1^2 + r_1'^2 + (z')^2 - 2r_1r_1'\cos(\theta_1' - \theta_1) = s_1^2 + z'^2$$
$$R_2^2 = r_2^2 + r_2'^2 + (z')^2 - 2r_2r_2'\cos(\theta_2' - \theta_2) = s_2^2 + z'^2$$

Following Appendix J, at this point I do the dz' integration with a cutoff

$$\int_{-\infty}^{\infty} \frac{dz'}{\sqrt{s^2 + z'^2}} = -2\ln(s/\Lambda)$$

So then I have

$$A_{z12}(\mathbf{x}) = \frac{\mu}{4\pi}\ i(z)\ (-1)$$

$$\{\ (I/\pi a_1^2)\ \int_0^{a_1}\ r_1'dr_1'\ \int_0^{2\pi} d\theta'_1\ \ln(s_1^2) -\ (I/\pi a_2^2)\ \ \int_0^{a_2}\ r_2'dr_2'\ \int_0^{2\pi} d\theta'_2\ \ln(s_2^2)\ \}$$

So

$$-A_{z12}(\mathbf{x}) = \frac{\mu}{4\pi}(I/\pi a_1^2)\ \ \int_0^{a_1}\ r_1'dr_1'\ \int_0^{2\pi} d\theta'_1\ \ln(s_1^2)\ \ -\ \ \text{second term}$$

$$= \frac{\mu I}{4\pi^2 a_1^2}\ \int_0^{a_1}\ r_1'dr_1'\ \int_0^{2\pi} d\theta'_1\ \ln(s_1^2)\ \ -\ \ \text{second term}$$

where now

$$s_1^2 = r_1^2 + r_1'^2\ - 2r_1r_1'\cos(x)$$
$$s_2^2 = r_2^2 + r_2'^2\ - 2r_2r_2'\cos(x)$$

I have already computed the first term in (B.7.3) where I found that

$$A_{z12}{}^{(c)}(r_1>a_1) = -\frac{\mu I}{2\pi}\ln r_1 \qquad \text{(B.7.7)}$$

$$A_{z12}{}^{(c)}(r_1<a_1) = \frac{\mu I}{2\pi a_1{}^2}\ \{\ (a_1{}^2-r_1{}^2)/2 - a_1{}^2\ln a_1\ \}$$

This is just the first of my two terms. I can write

$$A_{z12}(x,y) = -\frac{\mu I}{2\pi}\ln r_1\ \theta(r_1>a_1) + \frac{\mu I}{2\pi a_1{}^2}\ \{\ (a_1{}^2-r_1{}^2)/2 - a_1{}^2\ln a_1\ \}\ \theta(r_1<a_1)$$
$$+\frac{\mu I}{2\pi}\ln r_2\ \theta(r_2>a_2) - \frac{\mu I}{2\pi a_2{}^2}\ \{\ (a_2{}^2-r_2{}^2)/2 - a_2{}^2\ln a_2\ \}\ \theta(r_2<a_2)$$

So I think this is the final answer, and it is fairly messy.

$$r_1{}^2 = x^2 + y^2 \qquad\qquad r_2{}^2 = (x-b)^2 + y^2$$

But this is the first time I have written down this explicit result. [how about a Maple plot! See below]

Can I do a numeric dsolve of $A_{z12}(x,y) = 3$? It happens that I have no derivatives in my equations. I will try this in Az lines 1.mws. Rewrite

$$\frac{2\pi}{\mu I}A_{z12}(x,y) = \underset{t1}{-\ln r_1\ \theta(r_1>a_1)} + \underset{t3}{\{\ (1-r_1{}^2/a_1{}^2)/2 - \ln a_1\ \}\ \theta(r_1<a_1)}$$
$$+\underset{t2}{\ln r_2\ \theta(r_2>a_2)} - \underset{t4}{\{\ (1-r_2{}^2/a_2{}^2)/2 - \ln a_2\ \}/\ \theta(r_2<a_2)} \equiv f(x,y)$$

$$r_1{}^2 = x^2 + y^2 \qquad\qquad r_2{}^2 = (x-b)^2 + y^2$$

I just want Maple to deal with something like f(x,y) = 3, where there are no derivatives, but dsolve always wants an initial condition. [see "Az lines 1.mws"]

OK, if I just do a plot3d contour plot of f(x,y) I get this nice thing

Maple program is: Az lines 1.mws

```
restart; alias(I=I, j=sqrt(-1)): with(plots):with(plottools):
t1 := - ln(r1)*Heaviside(r1-a1);
```

$$t1 := -\ln(r1)\,\text{Heaviside}(r1 - a1)$$

```
t2 := + ln(r2)*Heaviside(r2-a2);
```

$$t2 := \ln(r2)\,\text{Heaviside}(r2 - a2)$$

```
t3 := ((1-r1^2/a1^2)/2 - ln(a1))*Heaviside(a1-r1);
```

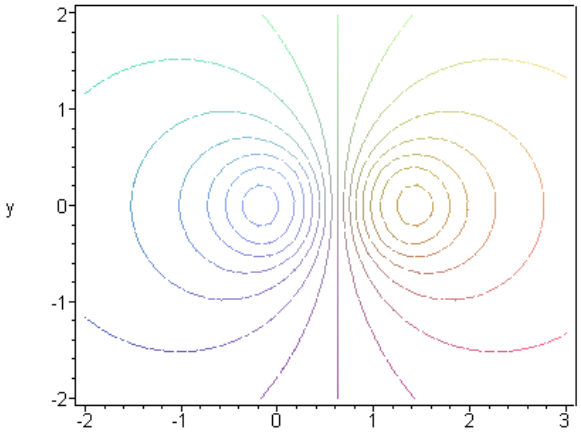
$$t3 := \left(\frac{1}{2} - \frac{1}{2}\frac{r1^2}{a1^2} - \ln(a1)\right)\text{Heaviside}(-r1 + a1)$$

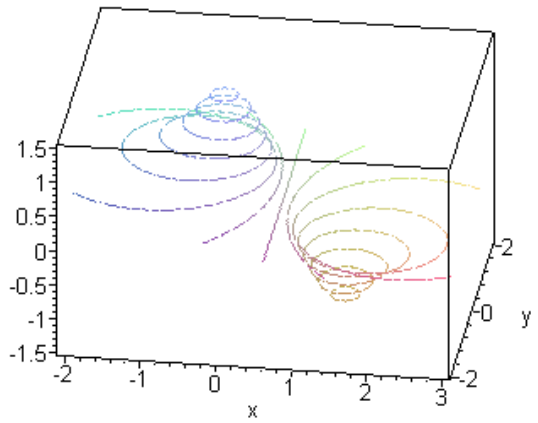
```
t4 := -((1-r2^2/a2^2)/2 - ln(a2))*Heaviside(a2-r2);
```

$$t4 := -\left(\frac{1}{2} - \frac{1}{2}\frac{r2^2}{a2^2} - \ln(a2)\right)\text{Heaviside}(-r2 + a2)$$

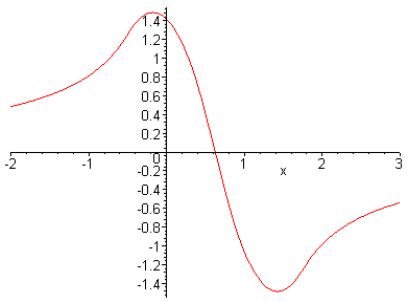
```
f := t1+t2+t3+t4:
r1 := sqrt(x^2+y^2):
r2 := sqrt((x-b)^2+y^2):
b := 1.25: a1 := .5: a2 := .5:
f_ := unapply(f,x,y);

plot3d(f_(x,y),x=-2..3,y=-2..2, grid = [40,40], style = contour, scaling =
constrained, axes = boxed);
```



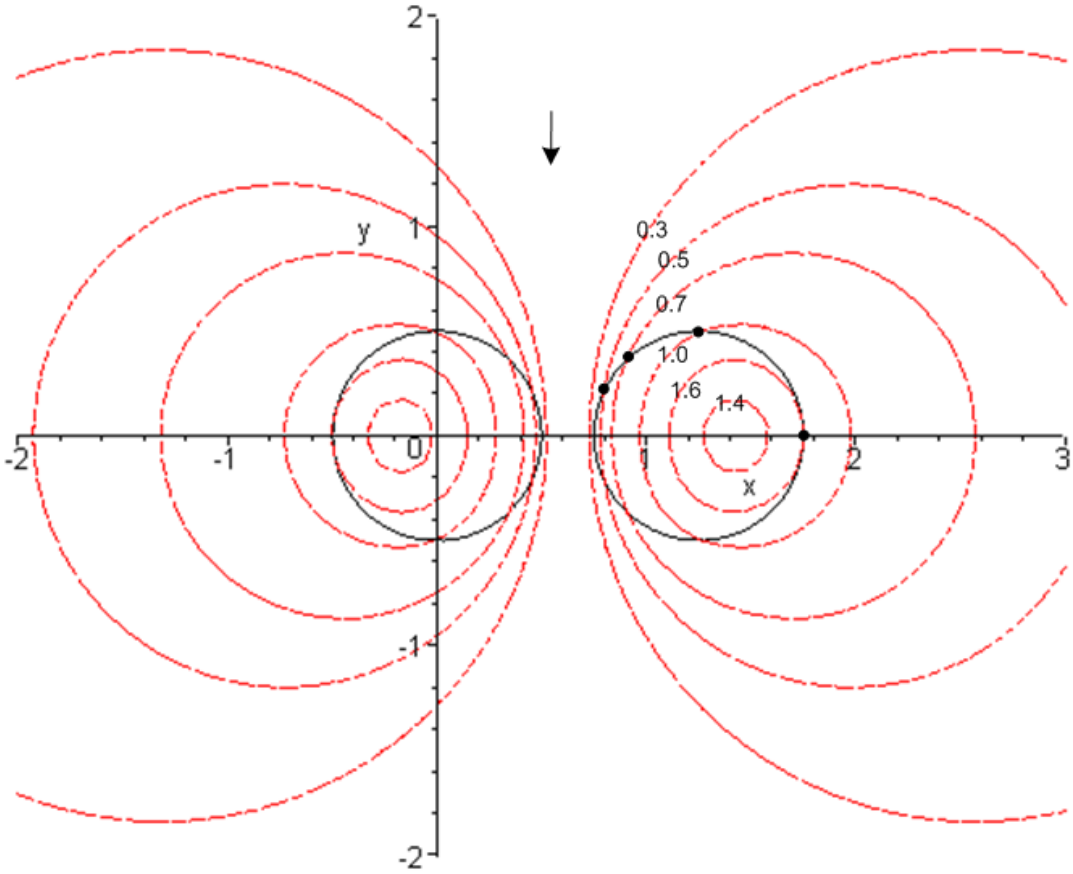


and sure enough it looks JUST LIKE my B field line picture, as I hoped it would. Here is a cross section at y = 0 to show that nothing singular happens at the max and min points,



I have this now also as an implicitplot,

```
p := implicitplot({seq(f_(x,y) = 1/N,N =
[0.7,0.8,1,1.5,2,3,-0.7,-0.8,-1,-1.5,-2,-3])},x=-2..3,y=-2..2, scaling =
constrained,grid = [50,50], view = [-2..3,-2..2]):
c1 := circle([0,0],a1,color=black):
c2 := circle([b,0],a2,color=black):
display(p,c1,c2);
```



[The numbers in this picture are wrong, I edited Az lines 1.mws to make things right and the right numbers now appear in lines doc as Fig 3.6b. .]

I am now quite satisfied that lines of constant A_{z12} are the same as the B field lines [as shown in "the field line mapping problem.doc". And you see explicitly above that A_z is not constant on the conductor surfaces, just as in the B field plots. For example, on the right black circle I show points with A_z = 1.6, 1.0, 0.7, 0.5 . This shows that A_z varies *substantially* on the black circle, just as you would think.

Note Added 3/6/14. This is a typical result at DC -- the B field lines and the A_z equipotentials do NOT align with the boundaries, and you know this will also be true for low frequencies certainly, so here is real-world evidence that the "W(z) Hypothesis" of Chap 4 is not generally correct and needs some restrictions (perhaps to strong skin depth limit).