Calculation of an integral PhL 3.4.14

This integral appears in my attempt to compute A_z for the two cylinder problem (in the extreme skin effect limit) using the King Helmholtz integral [see "computation of A_z for two cylinders case.doc" Section 10]. As written, I cannot find this integral in GR7 (which never has both sinθ and cosθ together for trig integrals), and Maple could only do it for small values of m. Here I convert it to a contour integral after doing parts integration and I show that one could in fact do the integral with some amount of energy and Maple coding! The sample Maple code for doing this is located in "Maple does contour integral.mws". I can see that the result is going to be very complicated, so this is probably not a good way to compute A_z. But it is good to know that Maple can in effect do any contour integral using its "residue" call, and I just added an example to my Maple User's Guide.

The integral in question is this

$$I(m) = \int_0^{2\pi} d\theta \cos(m\theta) \ln [A + B\sin\theta + C\cos\theta]$$

where m is an integer. I can write

$$\cos(m\theta) = (1/m)\partial_\theta \sin(m\theta)$$

and do parts integration. The parts vanish since sin(mθ) vanishes and we get then

$$I(m) = \int_0^{2\pi} d\theta \sin(m\theta) [A + B\sin\theta + C\cos\theta]^{-1} (B\cos\theta - C\sin\theta)$$

Now write everything in terms of $e^{i\theta}$:

$$I(m) = \int_0^{2\pi} d\theta \{(1/2)(e^{jm\theta} - e^{-jm\theta})\}[A + B\{(1/2)(e^{j\theta} - e^{-j\theta})\} + C\{(1/2)(e^{j\theta} + e^{-j\theta})\}]^{-1}$$
$$(\{ (B\{(1/2)(e^{j\theta} + e^{-j\theta})\} - C\{(1/2)(e^{j\theta} - e^{-j\theta})\})$$

or

$$I(m) = (1/2) \int_0^{2\pi} d\theta (e^{jm\theta} - e^{-jm\theta})[2A + B(e^{j\theta} - e^{-j\theta}) + C (e^{j\theta} + e^{-j\theta})]^{-1}$$
$$((B\{e^{j\theta} + e^{-j\theta})\} - C(e^{j\theta} - e^{-j\theta}))$$

Now set $z = e^{j\theta}$ so that $dz = jzd\theta$ and then we have

$$I(m) = -j (1/2) \oint dz/z (z^m - z^{-m})[2A + B(z - z^{-1}) + C (z + z^{-1})]^{-1} [(B\{z + z^{-1})\} - C(z - z^{-1})]$$

Multiply top and bottom by z to get

$$I(m) = -j (1/2) \oint dz/z (z^m - z^{-m})[2Az + B(z^2 - 1) + C (z^2 + 1)]^{-1} [(B\{z^2 + 1)\} - C(z^2 - 1)]$$

This has the general form a = B-C b = B+C

$$I(m) = -j\,(1/2) \oint dz/z\,(z^m - z^{-m})[2Az + B(z^2 - 1) + C(z^2 + 1)]^{-1}\ [\,(B\{z^2 + 1)\} - C(z^2 - 1)\,]$$

$$= -j\,(1/2) \oint dz\,\frac{z^{m-1}\,(az^2+b)}{bz^2+2Az - a} + [\text{ same with } m \rightarrow -m\ ]$$

This is a completely doable contour integral showing two three possible pole locations.

Can Maple help with contour integration? Here is some text from a book on Maple

```
> exp(a*z)/(1+exp(z));
```

$$\frac{e^{(a\,z)}}{1+e^{z}}$$

```
> solve(denom(%), z); # the solution in the strip
```

$$\pi\,I$$

```
> residue(%%, z=%);
```

$$-e^{(a\,\pi\,I)}$$

```
> (1-exp(2*Pi*a*I))*integral = 2*Pi*I*%;
```

$$(1 - e^{(2\,I\,\pi\,a)})\,integral = -2\,I\,\pi\,e^{(a\,\pi\,I)}$$

```
> (lhs/rhs)(%) = 1;
```

$$\frac{\frac{1}{2}\,I\,(1 - e^{(2\,I\,\pi\,a)})\,integral}{\pi\,e^{(a\,\pi\,I)}} = 1$$

```
> simplify(%);
```

$$\frac{integral\,\sin(a\,\pi)}{\pi} = 1$$

```
> Integrate(exp(a*x)/(1+exp(x)),
>   x=-infinity..infinity) = solve(%, integral);
```

$$\int_{-\infty}^{\infty} \frac{e^{(a\,x)}}{1+e^{x}}\,dx = \frac{\pi}{\sin(a\,\pi)}$$

It shows how in a very simple case you can use the solve and residue calls to assist the evaluation, and it also shows a few other Maple tricks which are new to me like lhs and rhs.

I wrote some test code to do the above integral

$$\oint dz \frac{z^{m-1}(az^2+b)}{bz^2+2Az - a}$$

and it works just fine, though I have to do each m separately. For each m, then, I can get an analytic expression for this integral

$$I(m) = \int_0^{2\pi} d\theta \cos(m\theta) \ln [A + B\sin\theta + C\cos\theta]$$

and therefore I can get a finite sum approximation for $A_z(r,\theta)$ for the two cylinders problem!