This was my 2nd effort at averaging repair, I got rid of the J_z weighting, and this was installed some time perhaps around June 16. (give or take a week).

Averaging Repair. Our theory now has an inconsistency which needs to be fixed.

We know that for a general transmission line operating at $\omega > 0$, the current density J_z inside the conductors will not be uniformly distributed. It will be larger in the conductor region closest to the other conductor. This "proximity effect" is discussed in Appendix P from an eddy current point of view, see Fig P.13 for an example. The J_z current non-uniformity can be very dramatic as for example in a transmission line having this cross section, where J_z will be large near the gap and small far from the gap:

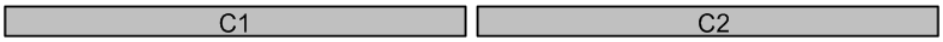


Fig 4.12

Since J_z is non-uniform in each conductor, so is E_z, and so we expect $E_z(\mathbf{x}_1)$ to be a strong function of the point $\mathbf{x}_1$ on the perimeter of C_1, certainly for the above cross section example. This means that the left side of (4.11.6a) is a function of $\mathbf{x}_1 = (x_1,y_1,z)$ and $\mathbf{x}_2 = (x_2,y_2,z)$ whereas the right side in our theory is a function only of z. To remedy this inconsistency, we now have to think of V and W as having very slight dependence on $\mathbf{x}_1$ and $\mathbf{x}_2$ which we generally ignore, but which we must face up to in (4.11.6a). In reality we have $V(\mathbf{x}_1,\mathbf{x}_2)$ and $W(\mathbf{x}_1,\mathbf{x}_2)$. This is a manifestation of the fact that in reality $\varphi \approx$ constant and $A_z \approx$ constant on the boundaries (with $\approx$ and not $=$). In the extreme skin effect regime (think a very good conductor), the left side of (4.11.6a) can be a violent function of $\mathbf{x}_1$ and $\mathbf{x}_2$ as in the case of the above figure, but the left side is always very small, even where it is largest, and its variation can be accommodated by the right side of (4.11.6a) which is the difference of large-valued functions which vary only slightly with $\mathbf{x}_1$ and $\mathbf{x}_2$. So first rewrite (4.11.6a) as

$$E_z(\mathbf{x}_1) - E_z(\mathbf{x}_2) = - \partial_z V(\mathbf{x}_1,\mathbf{x}_2) - j\omega W(\mathbf{x}_1,\mathbf{x}_2) \qquad (4.11.6a)'$$

Backing up another step, we write out of (4.11.4a) for the two perimeter points $\mathbf{x}_1$ and $\mathbf{x}_2$,

$$(1/\sigma)J_z(\mathbf{x}_1) = E_z(\mathbf{x}_1) = - \partial_z\varphi_{12}(\mathbf{x}_1) - j\omega A_{z12}(\mathbf{x}_1) \qquad \mathbf{x}_1 \text{ on perimeter of } C_1$$
$$(1/\sigma)J_z(\mathbf{x}_2) = E_z(\mathbf{x}_2) = - \partial_z\varphi_{12}(\mathbf{x}_2) - j\omega A_{z12}(\mathbf{x}_2) \qquad \mathbf{x}_2 \text{ on perimeter of } C_2 \qquad (4.11.4a)'$$

Calling the perimeter distances of the conductors P_1 and P_2, we then average each of these equations around its appropriate perimeter. Apply $(1/P_1) \int_{C1} ds_1$ to the first equation and $(1/P_2) \int_{C1} ds_2$ to the second to get [ds_1 is a distance element along the perimeter of C_1] ,

$$(1/\sigma)\langle J_z(\mathbf{x}_1)\rangle_{C1} = \langle E_z(\mathbf{x}_1)\rangle_{C1} = - \partial_z\langle\varphi_{12}(\mathbf{x}_1)\rangle_{C1} - j\omega\langle A_{z12}(\mathbf{x}_1)\rangle_{C1}$$
$$(1/\sigma)\langle J_z(\mathbf{x}_2)\rangle_{C2} = \langle E_z(\mathbf{x}_2)\rangle_{C2} = - \partial_z\langle\varphi_{12}(\mathbf{x}_2)\rangle_{C2} - j\omega\langle A_{z12}(\mathbf{x}_2)\rangle_{C2} \ .$$

Subtract the second line from the first to get,

$$[\langle E_z(\mathbf{x}_1)\rangle_{C1} - \langle E_z(\mathbf{x}_2)\rangle_{C2}]$$
$$= \quad -\partial_z[\langle \varphi_{12}(\mathbf{x}_1)\rangle_{C1} - \langle \varphi_{12}(\mathbf{x}_2)\rangle_{C2}] \; - j\omega\,[\langle A_{z12}(\mathbf{x}_1)\rangle_{C1} - \langle A_{z12}(\mathbf{x}_2)\rangle_{C2}] \;.$$

We now *redefine* V and W to be the averages appearing in these equations, along with E_{z1} and E_{z2} :

$$E_{z1}(z) \equiv \langle E_z(\mathbf{x}_1)\rangle_{C1} = (1/P_1) \int_{C1} ds_1\, E_z(\mathbf{x}_1)$$

$$E_{z2}(z) \equiv \langle E_z(\mathbf{x}_2)\rangle_{C2} = (1/P_2) \int_{C2} ds_2\, E_z(\mathbf{x}_2)$$

$$V(z) \equiv \langle \varphi_{12}(\mathbf{x}_1)\rangle_{C1} - \langle \varphi_{12}(\mathbf{x}_2)\rangle_{C2} = \langle V(\mathbf{x}_1,\mathbf{x}_2)\rangle_{C1,C2}$$
$$W(z) \equiv \langle A_{z12}(\mathbf{x}_1)\rangle_{C1} - \langle A_{z12}(\mathbf{x}_2)\rangle_{C2} = \langle W(\mathbf{x}_1,\mathbf{x}_2)\rangle_{C1,C2} \qquad (4.11.7)$$

with this result

$$[E_{z1}(z) - E_{z2}(z)] = -\partial_z V(z) - j\omega W(z) \qquad (4.11.8)$$

Meanwhile, the surface impedances on C_1 and C_2 are defined by (see C.2.1) ,

$$E_{z1}(\mathbf{x}_1) = Z_{s1}(\mathbf{x}_1)\; i_1(z)$$
$$E_{z2}(\mathbf{x}_2) = Z_{s2}(\mathbf{x}_2)\; i_2(z) \qquad (4.11.9)$$

which we average in the same way to obtain

$$E_{z1}(z) = Z_{s1}\, i_1(z) \qquad Z_{s1} \equiv (1/P_1) \int_{C1} ds_1\, Z_{s1}(\mathbf{x}_1)$$

$$E_{z2}(z) = Z_{s2}\, i_2(z) \qquad Z_{s2} \equiv (1/P_2) \int_{C2} ds_2\, Z_{s2}(\mathbf{x}_2) \qquad (4.11.10)$$

Then using $i(z) = i_1(z) = -i_2(z)$ we may rewrite (4.11.8) and (4.11.6b) as

$$[Z_{s1} + Z_{s2}]\, i(z) = -\partial_z V(z) - j\omega W(z)$$
$$\partial_z W(z) = -j\,(\beta^2/\omega)\, V(z) \qquad (4.11.11)$$

where the second equation above is the $\langle\ \rangle_{C1,C2}$ average of (4.11.6b).

Continuing this repair effort, we back up to box (4.11.1) and write

$$V(\mathbf{x}_1,\mathbf{x}_2) = q(z) / C'(\mathbf{x}_1,\mathbf{x}_2) = q(z)\,[\frac{1}{4\pi\xi}\; K(\mathbf{x}_1,\mathbf{x}_2)\,]$$

$$W(\mathbf{x}_1,\mathbf{x}_2) = i(z)\, L_e(\mathbf{x}_1,\mathbf{x}_2) = i(z)\,[\frac{\mu}{4\pi}\; K_L(\mathbf{x}_1,\mathbf{x}_2)] \qquad (4.11.12)$$

which we average in the same way to get

$$V(z) = q(z)\,\frac{1}{C'} \qquad\qquad W(z) = i(z)\,L_e$$

$$\frac{1}{C'} \;\equiv\; (1/P_1)\int_{C1} ds_1\,(1/P_2)\int_{C2} ds_2\;\frac{1}{C'(\mathbf{x}_1,\mathbf{x}_2)} \;=\; \left\langle \frac{1}{C'(\mathbf{x}_1,\mathbf{x}_2)} \right\rangle_{C1,C2}$$

$$L_e \;\equiv\; (1/P_1)\int_{C1} ds_1\,(1/P_2)\int_{C2} ds_2\;L_e(\mathbf{x}_1,\mathbf{x}_2) \;=\; \langle L_e(\mathbf{x}_1,\mathbf{x}_2)\rangle_{C1,C2}\;. \qquad (4.11.13)$$

The "constants" K and K_L in (4.11.1) are similarly replaced with their $\langle\rangle_{C1,C2}$ averages.

Inserting the equations on the first line of (4.11.13) into (4.11.11) then gives

$$(Z_{s1} + Z_{s2})\,i(z) = -\,\partial_z V(z) - j\omega\,L_e\;i(z)$$
$$L_e\;\partial_z\,i(z) = -\,j\,(\beta^2/\omega)\,V(z)$$

which we then rearrange as

$$\partial_z V(z) = -\,[\,Z_{s1} + Z_{s1} + j\omega L_e]\,i(z)$$
$$\partial_z\,i(z) = -\,[\,j\beta^2/(\omega L_e)]\;V(z)\;. \qquad (4.11.14)$$

These are the classical **transmission line equations**. They are usually written in this form:

$$\frac{\partial V(z)}{\partial z} = -\mathbf{z}\, i(z) \qquad \frac{\partial i(z)}{\partial z} = -\mathbf{y}\, V(z)$$

with

$$\mathbf{z} = R + j\omega L \qquad \mathbf{y} = G + j\omega C\ . \tag{4.11.15}$$

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Now let's try to compute Z_{s1} for the two cylinder geometry! From Bipolar I know that

$$\mathbf{E} = (V/a)\,(ch\xi - cosu)\,\frac{1}{\xi_2 + |\xi_1|}\,\hat{\xi}$$

but this is the transverse field in the dielectric, not what I want! Where do I talk about E_z and n(θ) and tracking? How about (6.5.19) which says

$$E_z(a,\theta) = (-j\omega/\sigma)\,(\beta/\beta_d)\,n(\theta)$$

$$n_1(\xi_1,\theta) = (q/2\pi)\Big[\,1 + 2\sum_{m=1}^{\infty}(-1)^m\, e^{-m|\xi_1|}\cos(m\theta)\,\Big]\ . \qquad \textit{Bipolar (A.13)} \tag{6.5.5}$$

$$n(\theta) = n(\theta)/a$$

Then we seem to have

$$E_z(a,\theta) = (-j\omega/\sigma)\,(\beta/\beta_d)(1/a)\,(q/2\pi)\Big[\,1 + 2\sum_{m=1}^{\infty}(-1)^m\, e^{-m|\xi_1|}\cos(m\theta)\,\Big]$$

$$E_z(a,\theta)_{max} = (-j\omega/\sigma)\,(\beta/\beta_d)(1/a)\,(q/2\pi)\Big[\,1 + 2\sum_{m=1}^{\infty} e^{-m|\xi_1|}\,\Big]$$

$$\langle E_z(a,\theta)\rangle = (1/2\pi)\int d\theta\, E_z(a,\theta) = (-j\omega/\sigma)\,(\beta/\beta_d)(1/a)\,(q/2\pi)[1]$$

$$P = 2\pi a$$

Then we get (my formula)

$$p = (2\pi a) * \langle E_z(a,\theta)\rangle / E_z(a,\theta)_{max}$$

$$= (2\pi a)\ 1/\Big[\,1 + 2\sum_{m=1}^{\infty} e^{-m|\xi_1|}\,\Big]$$

$$\xi_1 = -\,sh^{-1}(a/a) = -\,sh^{-1}\Big[\frac{(1/2)\sqrt{b^2-4a^2}}{a}\Big]$$

$$sh\,\xi_1 = -\,(a/a) = -\frac{(1/2)\sqrt{b^2-4a^2}}{a}$$

Now back up

$$1 + 2 \sum_{m=1}^{\infty} e^{-m|\xi_1|}$$

$1 + \Sigma_{m=1}^{\infty} x^m = 1 + x + x^2 + \quad = 1/(1-x)$

$\Sigma_{m=1}^{\infty} x^m = 1/(1-x) - 1 = x / (1-x)$ Maple confirms

$1 + 2\,\Sigma_{m=1}^{\infty} x^m = 1 + 2x/(1-x) = (1+x)/(1-x)$ Maple confirms, this is > 1

Let $x = e^{-\xi}$ so then have

$$1 + 2 \sum_{m=1}^{\infty} e^{-m\xi} = \frac{1+e^{-\xi}}{1-e^{-\xi}} = \frac{e^{\xi/2}+e^{-\xi/2}}{e^{\xi/2}-e^{-\xi/2}} = \frac{ch(\xi/2)}{sh(\xi/2)} = \coth(\xi/2) = \frac{ch\xi+1}{sh\xi} = \frac{sh\xi}{ch\xi-1}$$

We then seem to have

$$p = (2\pi a)\ 1/\left[\ 1 + 2 \sum_{m=1}^{\infty} e^{-m\xi_2}\right] = (2\pi a)\frac{sh\xi_2}{ch\xi_2+1}$$

Check limit first. As $\xi \to \infty$, get $p = 2\pi a$ which is correct, good! Now assume that $\xi_1 = -\xi_2$ and $\xi_2 > 0$ so we then have

$b = a(\coth\xi_2 - \coth\xi_1) = 2a\coth\xi_2 = 2a\, ch\xi_2/sh\xi_2$

$sh\xi_2 = (a/a)$

so $b = 2a\, ch\xi_2/((a/a)) = 2a\, ch\xi_2$ $ch\xi_2 = b/(2a)$

This seems to simple, but I see in (11.2) of bipolar

$b = x_{c2} - x_{c1} = a/th\xi_2 - a/th\xi_1 = 2a/th\xi_2 = 2a\coth\xi_2$ agrees with above

So OK we then have

$$\frac{sh\xi_2}{ch\xi_2+1} = \frac{(a/a)}{b/(2a)+1} = \frac{a}{b/2+a}$$

$a = (1/2)\sqrt{b^2-4a^2}$ from below (6.5.7)

so we get

$$p = (2\pi a)\frac{sh\xi_2}{ch\xi_2+1} = (2\pi a)\frac{a}{b/2+a} = (2\pi a)\frac{(1/2)\sqrt{b^2-4a^2}}{b/2+a} = (2\pi a)\frac{\sqrt{b^2-4a^2}}{b+2a}$$

$$= (2\pi a)\frac{\sqrt{b-2a}}{\sqrt{b+2a}} = (2\pi a)\frac{\sqrt{1-2a/b}}{\sqrt{1+2a/b}} = (2\pi a)\left[\frac{1-2a/b}{1+2a/b}\right]^{1/2}$$

If $b >> a$, this becomes

$$p = (2\pi a)\left[1- 4a/b\right]^{1/2} \approx (2\pi a)(1 - 2a/b)$$

$$p = 2\pi a\sqrt{1 - (2a/b)^2} \approx 2\pi a$$

which is certainly very similar but not the same.

What happens if I square my ratio to get something more power oriented, so then

$$p_1 \equiv (Z_{s1}/Z_{s1,max})^2 P_1$$

Then the above would say

$$p = (2\pi a) = (2\pi a)\left[\frac{sh\xi_2}{ch\xi_2+1}\right]^2 = (2\pi a)\frac{1-2a/b}{1+2a/b}$$

and this also disagrees with the King quoted result.

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Reader Exercise. First, recall these high-frequency results for a transmission line consisting of two round conductors,

$$E_z(a,\theta) = (-j\omega/\sigma)(\beta/\beta_d)\, n(\theta) \qquad \text{// just below the surface} \qquad (6.5.19)$$

$$n(\theta) = (q/2\pi)(1/a)\left[1 + 2\sum_{m=1}^{\infty}(-1)^m e^{-m\xi_2}\cos(m\theta)\right] \qquad (6.5.5)$$

where we use $\xi_2 > 0$ for the right conductor in Fig. 6.7 and the fact that $n = n/a$. Thus, the longitudinal E field on the right-side conductor's boundary is given by,

$$E_z(a,\theta) = (-j\omega/\sigma)(\beta/\beta_d)(1/a)(q/2\pi)\left[1 + 2\sum_{m=1}^{\infty}(-1)^m e^{-m\xi_2}\cos(m\theta)\right] .$$

(a) Using our model definition of the effective active perimeter p given in (4.11.10),

$$p \equiv \frac{Z_s}{Z_{smax}} P = \frac{\langle E_z \rangle_c}{E_{zmax}} P$$

show that

$$p = (2\pi a) \frac{1}{1 + 2\Sigma_{m=1}^{\infty} e^{-m\xi_2}} = (2\pi a) \frac{sh\xi_2}{ch\xi_2 + 1}$$

[Hint: Show that $1 + 2 \Sigma_{m=1}^{\infty} x^m = (1+x)/(1-x) = \coth(\xi_2/2)$ where $x = e^{-\xi_2}$.]

(b) For cylindrical conductors both of radius a, using (6.5.6) that

$sh\xi_2 = (d/a)$
$b = 2d\, ch\xi_2/sh\xi_2$

show that $ch\xi_2 = b/(2a)$ and therefore that

$$p = (2\pi a) \frac{\sqrt{b-2a}}{\sqrt{b+2a}} = (2\pi a) \frac{\sqrt{1-2a/b}}{\sqrt{1+2a/b}} .$$

This is similar to (but different from) the result we quoted from King,

$$p = 2\pi a \sqrt{1 - (2a/b)^2} . \qquad a = \text{wire radius, } b = \text{center line separation} \qquad (2.5.2)$$

King in turn quotes the result from this reference which we have not been able to examine,

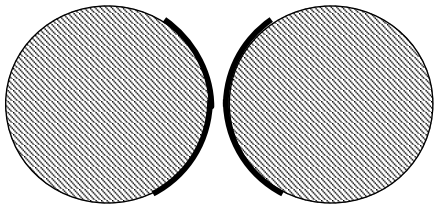
J.R. Carson, "Wave Propagation over Parallel Wires: The Proximity effect", *Philosophical Magazine*, volume IXLI, June 1921, pages 607–633.

King does not specify how his active perimeter region is defined. What he says (TLT p 30) is that the surface impedance is given with radius a replaced by an effective radius $a_{eff} = a \sqrt{1 - (2a/b)^2}$.

For wide-spaced conductors, both formulas give $p = 2\pi a$ as we would expect.

When the conductors almost touch so $b \rightarrow 2a$, both formulas give $p \rightarrow 0$ again as we would expect.

For $b >> a$, our result gives $p = (2\pi a)(1-2a/b)$ while King's gives $p = (2\pi a)(1-2a^2/b^2)$.



Fat twinlead *Fig 2.16*