**Computation of $A_t$ for "some situation"** PhL 3.6.14

[ 5/14/14 final review: Everything in this doc now appears in Appendix M and Section 3.7, so no need to ever read this doc again. Here is where I first developed the theory. ]

This appendix contains two big items: Early Appendix M proof and φ = constant proof.

In this doc I make arguments for why $3.7A_x << A_z$ , and then why $A_z \approx (\beta_d/\omega)\varphi$, and finally why the "capacitor model" for transmission line theory is justified. I add right now the point that $A_z \approx (\beta_d/\omega)\varphi$ implies that $A_z$ = constant on conductor surfaces, and that in turn requires the strong skin effect limit. See red notes below. Section 1 below now appears in **Appendix M** (not yet installed).

Added 5/12/14. Section 3 below I now realize was my original writeup of the set of Steps needed to justify φ = constant on the conductor surfaces. This is now part of lines doc Section 3.7.


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## 1. Show $A_t$ is small: taking a wild shot with the m=0 round wires case

After pondering a few methods, I try using the Helmholtz integral to estimate the $A_{x,y}$ size and I end up with some typical integral forms. I then assume wide spaced so only m = 0 contributes and I can then use the App D E fields for m=0. This puts some Bessel $J_1$ function in my integrals, but I never use this result. I go back to having $J_x$ inside the integrals and then I make my "strong cancellation" argument. My conclusion is then (1) $A_t$ is small because $J_t$ are small to begin with; (2) $A_t$ are still smaller due to cancellations in the integration. This all exists now in Appendix M, not yet installed.

I want to compute $\mathbf{A_t}$ for some simple situation in order to show that it can be neglected in this equation,

$$\mathbf{E_t} = -\nabla_t\varphi - j\omega\mathbf{A_t}$$

What methods are available? There is an ODE system you could solve ( here x = t = transverse comp)

$$(\nabla^2 + \beta_d^2)A_x = -\Sigma_{i=1}^{2}\ \mu_i J_x$$

but I am unsure of the BC's. It seems better to use the Helmholtz integral

$$\mathbf{A}(\mathbf{x},\omega) = \frac{1}{4\pi}\ \Sigma_i \int \mu_i \mathbf{J}_i(\mathbf{x}',\omega)\frac{e^{-j\beta R}}{R}\,dV' \qquad R = |\mathbf{x} - \mathbf{x}'| \quad . \qquad (1.5.9)$$

For small β we quickly simplify this to say

$$A_{x1}(\mathbf{x},\omega) = \frac{\mu}{4\pi} \int J_{xi}(\mathbf{x}',\omega)\,\frac{1}{R}\,dV' \qquad R = |\mathbf{x} - \mathbf{x}'| \quad . \tag{1.5.9}$$

Then we add the two conductors to get

$$A_x(\mathbf{x},\omega) = \frac{\mu}{4\pi}\ \{\ \int_{C_1} J_{x1}(\mathbf{x}',\omega)\,\frac{1}{R_1}\,dz'\ dx'\ dy' \ -\ \int_{C_2} J_{x2}(\mathbf{x}',\omega)\,\frac{1}{R_2}\,dz'\ dx'\ dy'\ \}$$

Now what is the next step? For longitudinal we made the ansatz that

$$\begin{aligned} &J_{z1}(x,y,z) = b_1(x,y)\ i_1(z) \\ &\ A/m^2 \qquad 1/m^2 \quad A \end{aligned} \tag{4.7.3}$$

but that makes no sense at all for the transverse components. We can instead take

$$J_{x1}(\mathbf{x}',\omega) \ = e^{-j\beta dz}\ J_{x1}(x',y',\omega)$$

but then again we are in the small $\beta_d$ regime so we ignore the phase.

We can then do the dz' integration in the usual way using

$$\int_{-\infty}^{\infty} \frac{dz'}{\sqrt{s^2 + z'^2}} \ = \ -2\ln(s/\Lambda)$$

and set $\Lambda = 1$ to get

$$A_x(\mathbf{x},\omega) = -\frac{\mu}{4\pi}\ \{\ \int_{C_1} J_{x1}(\mathbf{x}',\omega)\,\ln(s_1^2)\,dx'\ dy' \ -\ \int_{C_2} J_{x2}(\mathbf{x}',\omega)\,\ln(s_2^2)\,dx'\ dy'\ \}$$

$$A_y(\mathbf{x},\omega) = -\frac{\mu}{4\pi}\ \{\ \int_{C_1} J_{y1}(\mathbf{x}',\omega)\,\ln(s_1^2)\,dx'\ dy' \ -\ \int_{C_2} J_{y2}(\mathbf{x}',\omega)\,\ln(s_2^2)\,dx'\ dy'\ \}$$

where

$$\begin{aligned} s_1^2 &= (x-x'_1)^2 + (y-y'_1)^2 \\ s_2^2 &= (x-x'_2)^2 + (y-y'_2)^2 \end{aligned}$$

Rewrite as

$$A_x(\mathbf{x},\omega) = -\frac{\mu}{4\pi}\,\sigma\ \{\ \int_{C_1} E_{x1}(\mathbf{x}',\omega)\,\ln(s_1^2)\,dx_1'\ dy_1' \ -\ \int_{C_2} E_{x2}(\mathbf{x}',\omega)\,\ln(s_2^2)\,dx_1'\ dy_1'\ \}$$

$$A_y(\mathbf{x},\omega) = -\frac{\mu}{4\pi}\ \sigma\{\ \int_{C_1} E_{y1}(\mathbf{x}',\omega)\,\ln(s_1^2)\,dx_2'\ dy_2' \ -\ \int_{C_2} E_{y2}(\mathbf{x}',\omega)\,\ln(s_2^2)\,dx_2'\ dy_2'\ \}$$

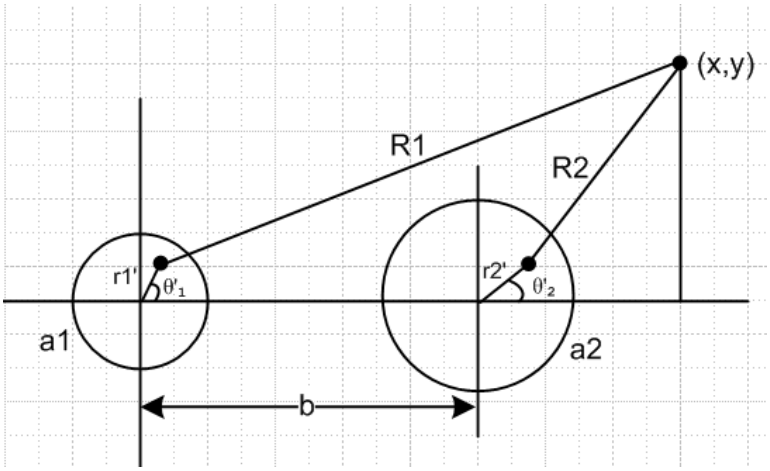
In the wide twinlead simplest case maybe we can say m = 0 in each wire where we know

$$E_z(r,0) = -j(\beta'/\beta_d)\frac{K_0}{2} J_0(x) \qquad \text{// large} \qquad a_0 = 0 \qquad x = \beta' r \qquad \beta'^2 = \beta^2 - \beta_d^2$$

$$E_r(r,0) = \frac{K_0}{2} J_1(x) \ . \qquad \text{// small} \qquad \frac{K_0}{2} = (j/2)(a\beta_d) I R_{dc} \frac{1}{J_1(x_a)}$$

$$jE_\theta(r,0) = 0 \qquad R_{dc} = \frac{1}{\sigma \pi a^2}$$

Here is a picture,




In conductor 1 we have

$$E_{x1} = E_r(r'_1,0)\cos\theta'_1 = \frac{K_0}{2} J_1(\beta' r'_1)\cos\theta'_1$$

$$E_{y1} = E_r(r'_1,0)\sin\theta'_1 = \frac{K_0}{2} J_1(\beta' r'_1)\sin\theta'_1$$

In conductor 2 we have

$$E_{x2} = E_r(r'_2,0)\cos\theta'_2 = \frac{K_0}{2} J_1(\beta' r'_2)\cos\theta'_2$$

$$E_{y2} = E_r(r'_2,0)\sin\theta'_2 = \frac{K_0}{2} J_1(\beta' r'_2)\sin\theta'_2$$

Now

$$\cos\theta'_1 = x'_1/r'_1$$
$$\sin\theta'_1 = y'_1/r'_1$$

$$\cos\theta'_2 = (x'_2 - b)/r'_2$$
$$\sin\theta'_2 = y'_2/r'_2$$

So then

$$E_{x1} = \frac{K_0}{2} J_1(\beta' r'_1)\, x'_1/r'_1$$

$$E_{y1} = \frac{K_0}{2} J_1(\beta' r'_1)\, y'_1/r'_1$$

$$E_{x2} = \frac{K_0}{2} J_1(\beta' r'_2)\, (x'_2-b)/r'_2$$

$$E_{y2} = \frac{K_0}{2} J_1(\beta' r'_2)\, y'_2/r'_2$$

$$s_1^2 = (x-x'_1)^2 + (y-y'_1)^2$$
$$s_2^2 = (x-x'_2)^2 + (y-y'_2)^2$$

Then we have

$$A_x(\mathbf{x},\omega) = -\frac{\mu}{4\pi}\sigma \{ \int_{C_1} J_1(\beta' r'_1)\, x'_1/r'_1 \ln(s_1^2)\, dx_1'\, dy_1' - \int_{C_2} J_1(\beta' r'_2)\, (x'_2-b)/r'_2 \ln(s_2^2)\, dx_1'\, dy_1' \}$$

$$A_y(\mathbf{x},\omega) = -\frac{\mu}{4\pi}\sigma \{ \int_{C_1} J_1(\beta' r'_1)\, y'_1/r'_1 \ln(s_1^2)\, dx_2'\, dy_2' - \int_{C_2} J_1(\beta' r'_2)\, y'_2/r'_2 \ln(s_2^2)\, dx_2'\, dy_2' \}$$

Here is what one of these four integrals looks like:

$$\int dx'_1\, dy'_1\, \theta(x'^2_1 + y'^2_1 \le a_1)\, J_1(\beta'\sqrt{x'^2_1 + y'^2_1})\, (x'_1/r'_1)\, (\ln[(x-x'_1)^2 + (y-y'_1)^2]$$

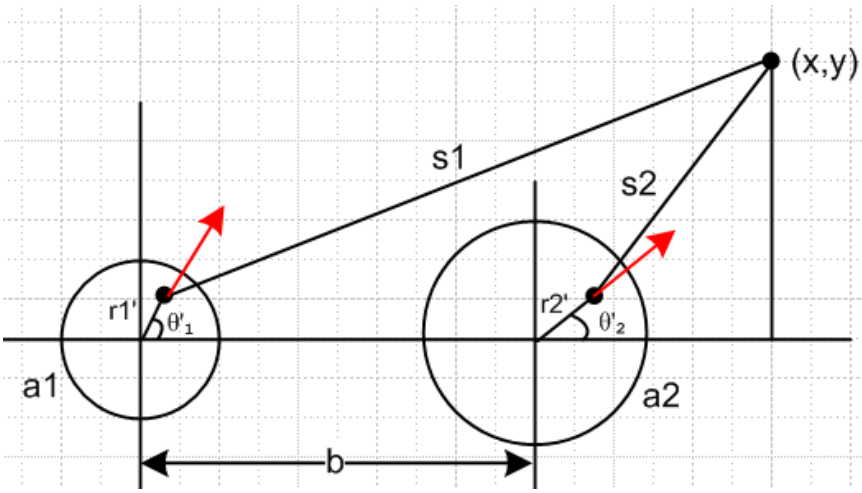
If I change to r,θ this becomes

$$a_1 \int dr_1 \int d\theta_1\, J_1(\beta' r_1) \cos\theta'_1\, \ln[(x-r'_1\cos\theta_1)^2 + (y-r'_1\sin\theta_1)^2]$$

$$= a_1 \int dr_1\, J_1(\beta' r_1) \int d\theta_1 \cos\theta'_1\, \ln[x^2 + y^2 + r'^2_1 - 2xr'_1\cos\theta_1 - 2yr'_1\sin\theta_1]$$

Now I know the θ integral can be done because I ran into this same integral the last time I tried to do this calculation! See separate doc called "a certain integral...". The result is a bit messy, and this is just for the lowest moment m = 0.

But look again at the above picture:




Here the red arrows show the red radial $J_r$ currents (for m = 0 there are no $J_\theta$ currents). And here is one of our integrals

$$A_x(\mathbf{x},\omega) = -\frac{\mu}{4\pi}\ \{\ \int_{C_1} J_{x1}(\mathbf{x}',\omega)\ln(s_1^2)\,dx'\,dy' \ -\ \int_{C_2} J_{x2}(\mathbf{x}',\omega)\ln(s_2^2)\,dx'\,dy'\ \}$$

I argue there is a HUGE cancellation effect because for each positive contribution there is a nearly offsetting negative one from the source point inverted through the origin. It is a little different because the angle to (x,y) is not the same and $R_1$ is not the same, but $\ln(s_1^2)$ is a very slow variation for a slightly different $s_1$.

IN the higher partial waves we have $E_r$ and $E_\theta$ but the same idea is valid, you are going to get lots of cancellation in the integration.

Maybe I can find a simpler parallel plate capacitor example or something like that where I can actually compute $A_t$. Barring that, here is a conclusion.

Conclusion: Although I could not do the integrals explicitly to show it, I can see that in the two-cylinders case, in any partial wave one is going to have $\mathbf{A}_t$ be very small. This is so for two reasons:

(1) the transverse currents are small to begin with.
(2) there is massive cancellation which takes place during the integration.

**2. How can one relate $A_z$ to something we know?**

Here I argue two ways that $A_z \approx (\beta_d/\omega)\varphi$ which "connects $A_z$ to something in the capacitor world". I then apply my previous section notion that $A_x << A_z$ to conclude $A_x << (\beta_d/\omega)\varphi$ . I then examine $E_x = -\partial_x\varphi - j\omega A_x$ . I estimate $\partial_x\varphi \sim (1/D)\varphi$ and then I show $\omega A_x << (2\pi/D)\varphi$ which then implies that $E_x \approx -\partial_x\varphi$ for $\omega > 0$ and this justifies the "capacitor approach" to transmission lines, and also the notion that $E_t = 0$ on the conductor surfaces. I try with some examples to verify that $\partial_x\varphi \sim (1/D)\varphi$ (mixed success).

I have now written Appendix M which writes up the previous section pretty well, I think, but is unable to say just "how small" $A_t$ really is, because I cannot easily do the integrals. Here is an alternate idea:

1. Make the *ansatz* that $(\partial_x A_x+\partial_y A_y) << \partial_z A_z$ . There is some dim logic to this ansatz since we have argued above that $\mathbf{A_t}$ is "small" . With this ansatz, we can write the King gauge condition,

$$\text{div } \mathbf{A} = - \mu\varepsilon j\omega\varphi - \mu\sigma\varphi = -j\omega\mu(\varepsilon+\sigma/j\omega)\varphi = -j\omega\mu\xi\varphi = -j(\beta^2/\omega)\ \varphi \qquad \text{King gauge} \qquad (1.5.5)$$

as :

$$\partial_z A_z \approx -j(\beta_d^2/\omega)\ \varphi$$

$$-j\beta_d A_z \approx -j(\beta_d^2/\omega)\ \varphi$$

$$A_z \approx (\beta_d/\omega)\ \varphi \qquad\qquad v_d = \omega/k = \omega/\beta_d$$

$$A_z \approx (1/v_d)\ \varphi\ .$$

I have gone this path several times before. For example, in "computation of $A_z$ for two cylinders.." I did it this way starting with a different equation,

$$E_z = -\partial_z\varphi - \partial_t A_z$$
$$E_z = j\beta_d\varphi - j\omega\ A_z$$
$$-j\omega A_z(\mathbf{x}) = E_z(\mathbf{x}) - j\beta_d\varphi(\mathbf{x})$$

Then ansatz that $E_z$ can be neglected in the dielectric so then

$$-j\omega A_z(\mathbf{x}) \approx -j\beta_d\varphi(\mathbf{x})$$
$$\omega A_z(\mathbf{x}) \approx \beta_d\varphi(\mathbf{x})$$
$$A_z = (\beta_d/\omega)\ \varphi$$

and there we have it again.

Main point: this is about the only connection I have between $A_z$ and anything else other than an intractable Helmholtz integral.

Next, since we have the general feeling that $A_x << A_z$, say, we can write

$$A_x << (\beta_d/\omega)\ \varphi \qquad \text{or perhaps even} \qquad A_x < 10^{-3}\ (\beta_d/\omega)\ \varphi$$

and now we have our first-ever connection between the size of $A_x$ and something from the "capacitor" world.

Next, I have been trying to deal with this question. In electrostatics,

$E_x = -\partial_x \varphi$

whereas in a transmission line we have

$E_x = -\partial_x \varphi - j\omega A_x$

In the worst way I want to claim that the second term on the right can be neglected compared to $-\partial_x \varphi$ and that would be the basis as maintaining the DC capacitor result as quasi-static into the transmission line world. Now suppose we dimly say that

$\partial_x \varphi \sim (1/D)\varphi$ (2) below in next section

where D is some vague transverse dimension of the line. Then I want to show that

$(1/D)\varphi >> \omega A_x$

or

$\omega A_x << (1/D)\varphi$ // want to demonstrate this]

But what I know from above is that

$\omega A_x << \beta_d \varphi$

or

$\omega A_x << (2\pi/\lambda) \varphi$ // what I think I know

Is this going to fly, or bite me in the butt? We start

$\lambda >> D$

$(1/\lambda) << (1/D)$

I know that

$\omega A_x << 2\pi\varphi (1/\lambda)$

But from just above

$(1/\lambda) << (1/D)$

therefore

$\omega A_x << 2\pi\varphi (1/\lambda) << 2\pi\varphi (1/D)$

and the conclusion is that

$\omega A_x << (2\pi/D) \varphi$

which is more or less what I wanted to demonstrate!

Now go back to this dim idea

$\partial_{\mathbf{x}}\varphi \sim (1/D)\varphi$

One way to argue this is to ride only the y = 0 "center line" between the two conductors which are spaced by distance D, say. Then

$$|\partial_{\mathbf{x}}\varphi| \equiv \frac{\Delta\varphi}{\Delta x} = \frac{V}{D} \sim (1/D)\,|\varphi| \qquad \text{where } \varphi \text{ is at some typical point in the capacitor near a plate!}$$

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This example did not work too well but I keep it here:

Example: For the two cylinders I found that

$$\varphi(x,y) = \frac{V}{2(\xi_2-\xi_1)}\ \ln\left[\frac{(x-a)^2+y^2}{(x+a)^2+y^2}\right]$$

```
f := ln(((x-a)^2 + y^2)/((x+a)^2 + y^2)) ;
```

$$f := \ln\left(\frac{(x-a)^2+y^2}{(x+a)^2+y^2}\right)$$

```
g := diff(f,x): simplify(%);
```

$$4\,\frac{a\,(x^2-a^2-y^2)}{(x^2-2\,x\,a+a^2+y^2)\,(x^2+2\,x\,a+a^2+y^2)}$$

Now replace x and y by dimensionless variables X = x/a and Y = y/a, and the above becomes

```
x := a * X;
```

$$x := a\,X$$

```
y := a * Y;
```

$$y := a\,Y$$

```
g: simplify(%);
```

$$4\,\frac{X^2-1-Y^2}{a\,(X^2-2\,X+1+Y^2)\,(X^2+2\,X+1+Y^2)}$$

```
f: simplify(%);
```

$$\ln\left(\frac{X^2-2\,X+1+Y^2}{X^2+2\,X+1+Y^2}\right)$$

In the region of strong fields for our capacitor, these X and Y coordinates are on the order of unity (ballpark), so the polynomial ratio is on the order of unity and we then have

$\varphi \approx$ order of unity $= \ln$ (ratio)

$\partial_x\varphi \approx (4/a)$ * order of unity

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Try this a different way. We know for the two-cylinder cap that

$$\varphi(\xi) = -\frac{\xi}{\xi_2-\xi_1} V \qquad (10.8)$$

And then

$$\partial_\xi\varphi = -\frac{1}{\xi_2-\xi_1} V$$

Then

$$\partial_x\varphi = \partial\varphi/\partial_\xi * \partial\xi/\partial x = \partial\varphi/\partial_\xi \; / \; \partial x/\partial\xi$$

But I know that

$$x = a\,sh\xi/(ch\xi - cosu)$$

```
x := a*sinh(xi)/(cosh(xi)-cos(u));
```

$$x := \frac{a \sinh(\xi)}{\cosh(\xi) - \cos(u)}$$

```
diff(x,xi): simplify(%);
```

$$-\frac{a\,(\cosh(\xi)\cos(u) - 1)}{\cosh(\xi)^2 - 2\cosh(\xi)\cos(u) + \cos(u)^2}$$

and then

$$\partial x/\partial\xi = -a\,(ch\xi cosu - 1)/(ch\xi - cosu)^2$$

and then

$$\partial_x\varphi = \partial\varphi/\partial_\xi \; / \; \partial x/\partial\xi = (1/a)\,\frac{1}{\xi_2-\xi_1} V * (ch\xi - cosu)^2 /(ch\xi cosu - 1)$$

$$= -(1/a)[-\frac{1}{\xi_2-\xi_1} V] * (ch\xi - cosu)^2 /(ch\xi cosu - 1)$$

$$= -(1/a)[-\frac{\xi}{\xi_2-\xi_1} V]\;(1/\xi) * (ch\xi - cosu)^2 /(ch\xi cosu - 1)$$

$= - (1/a)[\varphi] \ (1/\xi) \ * (ch\xi - cosu)^2 /(ch\xi cosu-1)$

and then

$\partial_x\varphi = \varphi * [ - (1/a) (1/\xi) * (ch\xi - cosu)^2 /(ch\xi cosu-1) ]$

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**3. Restate previous section: Justification of the Capacitor Model of Transmission Line**

Here I recap the logic of the previous section.

1.By one of two methods, we argue first that

$$A_z \approx (\beta_d/\omega) \varphi \qquad (1)$$

2. We then argue that for φ evaluated near one of the conductors in the active region between the conductors,

$$|\partial_x\varphi| \approx \frac{\Delta\varphi}{\Delta x} \approx \frac{V}{D} \approx (1/D) |\varphi| \qquad D = \text{general transverse dimension} \qquad (2)$$

3. We argue next, based on previous sections, that,

$|A_x| << |A_z|$ // from previous sections

$=> |A_x| << (\beta_d/\omega) |\varphi|$ // from (1) above

in particular, $|A_x| < 10^{-3} (\beta_d/\omega) |\varphi|$ from f = 0 Hz to 1000 GHz

which we rewrite slightly,

$\omega|A_x| << \beta_d |\varphi|$

or

$$|\omega A_x| << (2\pi/\lambda) |\varphi| = 2\pi |\varphi| (1/\lambda) \qquad (3)$$

4. Now consider

$\lambda >> D$

$(1/\lambda) << (1/D)$

$$2\pi |\varphi| (1/\lambda) << 2\pi |\varphi| (1/D) \ . \qquad (4)$$

Thus from (3) and (4) and then finally (2)

$$|\omega A_{\mathbf{x}}| << 2\pi\,|\varphi|\,(1/\lambda) \;<< 2\pi\,|\varphi|\,(1/D) \;\approx 2\pi\,|\partial_{\mathbf{x}}\varphi| \,.$$

From this we conclude, ignoring the $2\pi$ factor,

$$|\omega A_{\mathbf{x}}| << |\partial_{\mathbf{x}}\varphi| \qquad (5)$$

5. Finally, consider

$$E_{\mathbf{x}} = -\partial_{\mathbf{x}}\varphi \; - j\omega A_{\mathbf{x}}$$

Based on (5) , we can ignore the last term and we get

$$E_{\mathbf{x}} \approx \; -\partial_{\mathbf{x}}\varphi \qquad (6)$$

6. Making this same argument for $\partial_{\mathbf{y}}\varphi$, we end up with

$$\mathbf{E}_{\mathbf{t}} \approx - \nabla_{\mathbf{t}}\, \varphi$$

which says that the transverse electric field $\mathbf{E}_{\mathbf{t}}$ for a transmission line operating at frequency $\omega > 0$, but operating in the transmission line limit of $\lambda >> D$, is the same as the electrostatic electric field $\mathbf{E}_{\mathbf{t}}$ obtained from the $\omega = 0$ DC capacitor problem. This is the basis of our "electro-quasi-static" approach to the transmission line analysis.