Continuity at conductor surface PhL 3.4.14

[5.14.14 final review: Basically this is a test and vindication of the charge pump boundary condition that is so important to everything I do. It shows you can ignore "Debye currents" since they are really already included in the bulk volume current.]

This doc examines the connection between surface charge n and current densities J_i under the conductor surface using the continuity equation. Case 1 with the deep box gives a result not too useful. Case 2 develops the charge pump condition *ignoring* Debye surface currents. Case 3 includes these currents and shows that they don't affect the charge pump condition because they are negligible. Case 4 is nada.

Continuity Analysis #1 : The deep gaussian box idea 1
Continuity Analysis #2 : Thin gaussian box at surface (ignoring Debye surface currents) 4
Continuity Analysis #3 : Thin gaussian box at surface (including Debye surface currents) 4
Continuity Analysis #4 : Gaussian box inside the skin layer 8

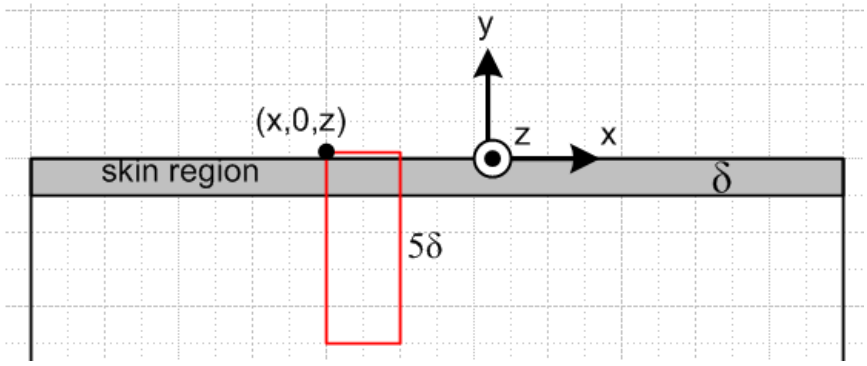
Continuity Analysis #1 : The deep gaussian box idea

Here δ is normal skin depth. The conclusion here for the "deep box" shown in the figure is this:

$$-j\omega\, n(x,0,z) = [\partial_x K_x(x,0,z) + \partial_z K_z(x,0,z)]/(1+j)$$

where J_y plays no role since the box is deep. This is probably correct for the deep box, but I don't think it proves anything one way or the other.

First of all, I want a top quality picture so I can see what I am doing:



The red thing is a Gaussian box with three pairs of faces. Assume there is some current **J** in the skin region and that all components drop off with δ. We take the box to extend 5δ

$$\text{div}\,\mathbf{J} = -j\omega\rho$$

$\partial_{\mathbf{x}} J_{\mathbf{x}} + \partial_{\mathbf{y}} J_{\mathbf{y}} + \partial_{\mathbf{z}} J_{\mathbf{z}} = -j\omega\rho$

$$\int \mathbf{J} \bullet d\mathbf{S} = -j\omega \int \rho \, dV$$

Question 1: What is the contribution to $\int \mathbf{J} \bullet d\mathbf{S}$ from the top and bottom faces of the box?

The area of the faces is dx dz. So we get [arguments are always (x,y,z)]

top = dx dz $J_{\mathbf{y}}(x,\varepsilon,z)$ bottom = - dx dz $J_{\mathbf{y}}(x,-5\delta,z)$

Now, the top is just above the surface and above the surface charge layer, and we assume the dielectric is non-conducting, so then $J_{\mathbf{y}}(x,\varepsilon,z) = 0$. Meanwhile, the bottom is so far from the surface, $J_{\mathbf{y}}(x,-5\delta,z) = 0$. Thus, both these faces make no contribution !

Answer to Question 1 = 0

Question 2: What is the contribution to $\int \mathbf{J} \bullet d\mathbf{S}$ from the left and right faces of the box?

$$\text{right} = dz \int_{\mathbf{0}}^{\mathbf{-5\delta}} dy \, J_{\mathbf{x}}(x+dx,y,z) = dz\, \delta\, J_{\mathbf{x}}(x+dx,0,z)$$

At this point, we model $J_{\mathbf{x}}$ as in (2.1.8) so that $J_{\mathbf{x}}(y) = J_{\mathbf{x}}(0)e^{-(j+1)y/\delta}$ so then

$$\int_{\mathbf{0}}^{\mathbf{5\delta}} dy' \, J_{\mathbf{x}}(x+dx,-y',z) = \int_{\mathbf{0}}^{\mathbf{5\delta}} dy' \, J_{\mathbf{x}}(x+dx,0,z) e^{-(j+1)y'/\delta}$$

$$\approx J_{\mathbf{x}}(x+dx,0,z) \int_{\mathbf{0}}^{\infty} dy' \, e^{-(j+1)y'/\delta} = [\delta/(j+1)]\, J_{\mathbf{x}}(x+dx,0,z)$$

So then

$$\text{right} = dz\ [\delta/(j+1)]\, J_{\mathbf{x}}(x+dx,0,z)$$

$$\text{left} = -\, dz\ [\delta/(j+1)]\, J_{\mathbf{x}}(x,0,z)$$

$$\text{right} + \text{left} = dz\ [\delta/(j+1)]\, J_{\mathbf{x}}(x+dx,0,z) - dz\ [\delta/(j+1)]\, J_{\mathbf{x}}(x,0,z) = dz\ [\delta/(j+1)]\, dx\, \partial_{\mathbf{x}} J_{\mathbf{x}}(x,0,z)$$

Answer to Question 2 = dz [δ/(j+1)] dx $\partial_{\mathbf{x}} J_{\mathbf{x}}(x,0,z)$

Question 3: What is the contribution to $\int \mathbf{J} \bullet d\mathbf{S}$ from the front and back faces of the box?

Repeating question 2 where we change x to z, we get

Answer to Question 3 = dx [δ/(j+1)] dz $\partial_z J_z(x,0,z)$

Question 4: What is $\int \rho\, dV$?

Answer: $\int \rho\, dV = n\, dx\, dz$ // since surface charge on surface.

Then "continuity" says this:

$0 + dz\ [\delta/(j+1)]\ dx\ \partial_x J_x(x,0,z) + dx\ [\delta/(j+1)]\ dz \partial_z J_z(x,0,z) = -j\omega\, n\, dx\, dz$

or

$[\delta/(j+1)]\ \partial_x J_x(x,0,z) + [\delta/(j+1)]\ \partial_z J_z(x,0,z) = -j\omega\, n$

so that

$-j\omega\, n(x,0,z) = [\delta/(j+1)]\ [\partial_x J_x(x,0,z) + \partial_z J_z(x,0,z)]$.

This then gives the ridiculous result that as $\delta \rightarrow 0$, the surface charge vanishes. I don't like that result. One problem is that 5δ is a pretty "deep box" and I know that there is strong phase variation of the transverse currents on the side walls which I have not incorporated. I assumed only a magnitude variation. So I guess my conclusion is that this result is not to be trusted and is in fact wrong.

Notes added 3/6/14.

1. First, if you include the phase effect, the integral becomes δ/(1+j) instead of δ and so the nature of the result does not really change. [5.20.14: I have edited this fix into all of the above]

2. I think the way to understand the above equation is to think of $K_z = \delta J_z$ and as you take $\delta \rightarrow 0$, K_z stays constant, but $J_z \rightarrow \infty$. So making these two changes, the above would say

$-j\omega\, n(x,0,z) = [\partial_x K_x(x,0,z) + \partial_z K_z(x,0,z)]/(1+j)$ // $\delta \rightarrow 0$

In the tail end of this limit, then, the surface currents remain finite and n remains finite and we are then not forced to the wrong conclusion that n = 0 for a perfect conductor. The reason $J_z \rightarrow \infty$ is that the conductor is still constrained to carry some current i(z) and if you force it out to a skin sheath, J_z gets larger without limit.

3. What happens if we express things in terms of E?

$-j\omega\, n(x,0,z) = [\delta/(j+1)]\ [\partial_x J_x(x,0,z) + \partial_z J_z(x,0,z)]$.

$-j\omega\, n(x,0,z) = [\delta/(j+1)]\ \sigma\ [\partial_x E_x(x,0,z) + \partial_z E_z(x,0,z)]$. $\qquad \delta \equiv \sqrt{2/(\omega\mu\sigma)}$

Write $\sigma = 2/(\omega\mu\delta^2)$ and then $\delta\sigma = 2/(\omega\mu\delta)$ and then you get

$$-j\omega\, n(x,0,z) = (2/\omega\mu)\,\delta^{-1}\,[\partial_x E_x(x,0,z) + \partial_z E_z(x,0,z)]/(j+1)$$

How do we interpret THIS form then? Well, we know that $E_i \to 0$ in the perfect conductor limit $\delta \to 0$. Write for a round wire this approximation (wide spaced twin lead)

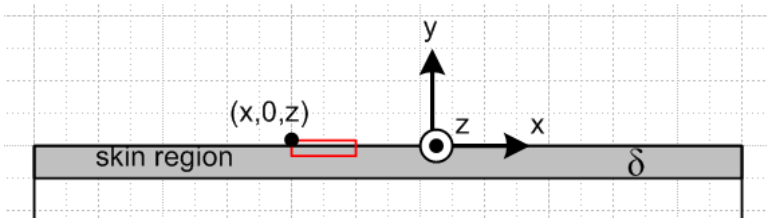
$$J_z = (I/2\pi a\delta) \quad \Rightarrow \quad E_z = J_z/\sigma = (I/2\pi a)(1/\sigma\delta) = (I/2\pi a)\,\omega\mu\delta/2 = (I\omega\mu/4\pi a)\,\delta$$

In this little model then we see explicitly that $E_z \to 0$.

Continuity Analysis #2 : Thin gaussian box at surface (ignoring Debye surface currents)

Here, with a thin box, ignoring Debye surface currents, I obtain the usual CPBC used in Appendix D.

Now we use a different box,



The sides of the box approach 0 and so make no contribution because the transverse current is not infinite anywhere. Then for this box we get only a "bottom" contribution which is

$$\int \mathbf{J} \bullet d\mathbf{S} = \text{bottom} = -\,J_y(x,-\varepsilon,z)\,dx\,dz$$

and then continuity says $\delta \quad \lambda_D$

$$-j\omega\, n\, dx\, dz = -\,J_y(x,-\varepsilon,z)\,dx\,dz$$

or

$$j\omega\, n = J_y(x,-\varepsilon,z)$$

where ε is tiny, so this is the normal current "just below the surface". I have then replicated my "radial charge pumping boundary condition" from lines doc, which was:

$$J_r(r=a-\varepsilon,\varphi) = j\omega\, n(\varphi) \quad . \qquad (D.2.23)$$

Continuity Analysis #3 : Thin gaussian box at surface (including Debye surface currents)

Here we have a thin box that always catches the surface currents, giving the worrisome full boundary condition that

$$-j\omega\, n = -\,J_y(x,-\varepsilon,z) + \partial_x K_x(x,0,z) + \partial_z K_z(x,0,z) \qquad K_x = \delta_{debye} J_x$$

I then use Ohm's law J= σE only for transverse fields at the surface to rewrite this as

$$(-j\omega/\sigma)\ n = -E_y(x,-\varepsilon,z) + \delta_{debye}\left[\partial_x E_x(x,0,z) - j\,(2\pi/\lambda)\,E_z(x,0,z)\right] \qquad (*)$$

I then assume that $\partial_x E_x \sim D E_x$ to get

$$(-j\omega/\sigma)\ n \approx -E_y(x,-\varepsilon,z) + (\delta_{debye}/D)\,E_x(x,0,z) - j\,(2\pi)\,(\delta_{debye}/\lambda)\,E_z(x,0,z)] \qquad (*)$$

I then argue that both ratios are incredibly small so that last two terms may be neglected to give

$$(-j\omega/\sigma)\ n \approx -E_y(x,-\varepsilon,z)$$

and then the CPBC is reconstituted with the conclusion that you can neglect Debye surface currents.

Suppose there existed "Debye surface currents" K_x and K_z. Then no matter how thin I make the gaussian box in Analysis #2, I always capture these currents. Then:

$$\left(\int \mathbf{J} \bullet d\mathbf{S}\right)_{rightside} = dz\,K_x(x+dx)$$

$$\left(\int \mathbf{J} \bullet d\mathbf{S}\right)_{leftside} = -\,dz\,K_x(x)$$

So these two faces then give contribution

$$dz\,K_x(x+dx) - dz\,K_x(x) = dz\,dx\,\partial_x K_x(x,0,z)$$

The front and back faces will contribute

$$dx\,dz\partial_z K_z(x,0,z)$$

Then we have

$$-j\omega\ n\,dx\,dz = -J_y(x,-\varepsilon,z)dx\,dz + dz\,dx\,\partial_x K_x(x,0,z) + dx\,dz\partial_z K_z(x,0,z)$$

or

$$-j\omega\ n = -J_y(x,-\varepsilon,z) + \partial_x K_x(x,0,z) + \partial_z K_z(x,0,z)$$

and this is then a "modified" charge pumping condition which now has to include the surface currents.

Now if I take the point of view that the Debye stuff has some thickness δ_{debye}, I can try to "squeeze out" the surface currents by making the gaussian red box super thin. But then I also squeeze out the charge, so that does not fly.

Comments: In the model with some δ_{debye} thickness we might say that

$K_x = \delta_{debye} J_x$ where J_x is the volume current inside the Debye layer

and you might also say that

$J_x = \sigma E_x$

just as you would anywhere else inside the conductor. Then maybe we have

$-j\omega\, n = - J_y(x,-\varepsilon,z) + \delta_{debye}\partial_x J_x(x,0,z) + \delta_{debye}\partial_z J_z(x,0,z)$

$-j\omega\, n = - \sigma E_y(x,-\varepsilon,z) + \delta_{debye}\, \sigma\, \partial_x E_x(x,0,z) + \delta_{debye}\sigma\, \partial_z E_z(x,0,z)$ // replace J_i by E_i transverse

$(-j\omega/\sigma)\, n = - E_y(x,-\varepsilon,z) + \delta_{debye}\, [\, \partial_x E_x(x,0,z) + \partial_z E_z(x,0,z)]$ // divide thru by σ

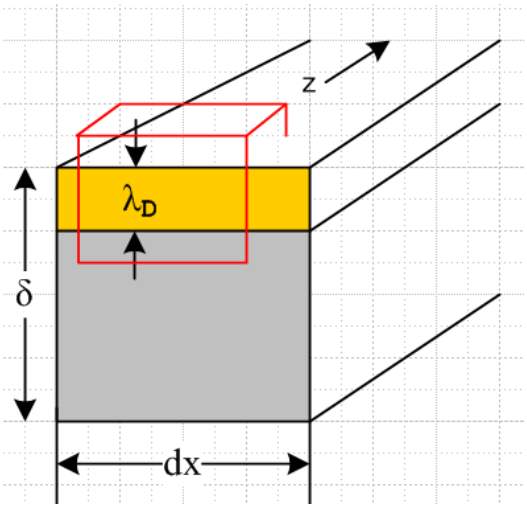
$(-j\omega/\sigma)\, n = - E_y(x,-\varepsilon,z) + \delta_{debye}\, [\, \partial_x E_x(x,0,z) - j\, \beta_d E_z(x,0,z)]$ // $\partial_z \rightarrow -j\, \beta_d$

$(-j\omega/\sigma)\, n = - E_y(x,-\varepsilon,z) + \delta_{debye}\, [\, \partial_x E_x(x,0,z) - j\, (2\pi/\lambda)\, E_z(x,0,z)]$

$(-j\omega/\sigma)\, n \approx - E_y(x,-\varepsilon,z) + (\delta_{debye}/D)\, E_x(x,0,z) - j\, (2\pi)\, (\delta_{debye}/\lambda)\, E_z(x,0,z)]$ (*)

You would certainly claim that (δ_{debye} / λ) was awesomely small and that last term is 0, even though perhaps E_z is the largest of the three E field components. And maybe you could claim that $\partial_x E_x$ is somehow on the order of D E_x where D is some transverse dimension of the transmission line or of the conductor, and then (δ_{debye} / D) is super small as well. This offers a possible way to "get rid of" these Debye current terms. This argument then makes K_x and K_z go away again .

Added 5.20.14. Here is a slightly different argument. Consider this picture where the Debye surface charge layer is greatly exaggerated in thickness. This is the layer that carries the Debye surface current as well. This picture shows a tiny slice of width dx of a piece of the conductor cross section at its surface.



The total current in the z direction is the sum of that current in the Debye layer plus the current in the skin effect layer below the Debye layer, so we write

$$I^{(D)} = J_z^{(D)}\, \lambda_D\, dx \;\; = \sigma_D\, E_z^{(D)}\, \lambda_D\, dx$$

$$I^{(\delta-D)} = J_z^{(\delta-D)} (\delta-\lambda_D)\, dx \;\; = \sigma\, E_z^{(\delta-D)} (\delta-\lambda_D)\, dx$$

where we allow the Debye layer to have conductivity σ_D that differs slightly from the bulk σ. Now we use the fact that E_z, being a transverse field, is continuous at the lower Debye layer boundary, according to (1.1.41), so then

$$E_z^{(D)} = E_z^{(\delta-D)} = E_z$$

Consider then the ratio of the currents carried in the two layers,

$$\frac{I^{(\delta-D)}}{I^{(D)}} \;=\; \frac{\sigma_D\, E_z\, \lambda_D\, dx}{\sigma\, E_z (\delta-\lambda_D)\, dx} \;= \frac{\sigma_D}{\sigma}\;\frac{\lambda_D}{\delta-\lambda_D} \;\approx\; 1 * \frac{\lambda_D}{\delta}$$

As noted at the end of Appendix E, even at 100 GHz the ratio $\frac{\lambda_D}{\delta}$ = 1/4000 for copper. Our conclusion then is that we may completely ignore the Debye layer from a current point of view because the current it carries is swamped by the current carried in the bulk conductor beneath the Debye layer.

OK, that may be true, but the "thin gaussian box" is shown in red, and it includes only the Debye layer current. Let's go back then to

$$(-j\omega/\sigma)\;\; n \;\approx\;\; - E_y(x,-\varepsilon,z) \;+ (\lambda_D/D)\, E_x(x,0,z) - j\,(2\pi)\,(\lambda_D/\lambda)\, E_z(x,0,z)] \qquad (*)$$

Here we regard E_y as measured just below the Debye layer, while E_x and E_z are perhaps in the middle of the Debye layer. Since both E_x and E_y are transverse fields, we can regard all three field components as being evaluated just below the Debye layer. Since these fields are all inside a good conductor, they are all small, though we expect $E_z >> E_x$ and E_y. Now consider the ratios for Belden 8281

$$(\lambda_D/D) \;\approx\; 10^{-10}\text{m}/\,(400 \times 10^{-6}\text{m}) \;\approx\; 10^{-10}/\,(0.4\text{x}10^{-3}) \;\approx 2.5 \times 10^{-7}$$

$$(\lambda_D/\lambda) \;=(\lambda_D/D)(D/\lambda) \approx\; 2.5 \times 10^{-7}\;\; (D/\lambda) \;<< 2.5 \times 10^{-7}$$

since in the transmission line limit $(D/\lambda) << 1$. Based on these ratios, it seems reasonable to drop the last two surface current terms in (*) to arrive at the usual boundary condition

$$(-j\omega/\sigma)\;\; n \;\approx\;\; - E_y(x,-\varepsilon,z)$$

[I hope the presence of the charges in the Debye later does not heavily affect E_x or E_z there. That is sort of a plasma issue that I don't want to think about. You would think E_y was most affected. I expect them only to affect the normal E_y in the Debye later.

Note: My elaborate argument about the ratio of currents supports the idea that the amount of current carried in the Debye later is much smaller than in the adjacent bulk skin effect layer. However, this argument does not seem to assist in obtaining the desired simple boundary condition. Going back to

$$-j\omega n = -J_y(x,-\varepsilon,z) + \partial_x K_x(x,0,z) + \partial_z K_z(x,0,z) \qquad K_x = \lambda_D J_x$$

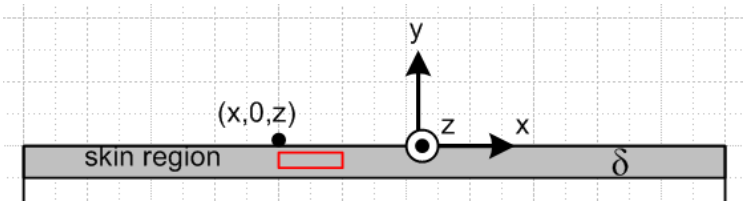
it is true that if we just ignore the surface currents and set $K_x = K_y = 0$, we do get the desired result, but just staring at this equation does not show why you can do that.

Note added 3/6/14. Let's compare Analysis #3 to Analysis #1. In Case 1, as $\delta \to 0$ we had $J_z \to \infty$ and $E_z \to 0$. The reason was that J_z has to carry the whole conductor current I. In Case 3 things are different. In Case 3, the Debye surface currents K_z does not have to carry all of I since most of I is carried by the normal skin sheath. Thus, in case 3 we have E_z does not change if you were to take $\delta_{debye} \to 0$. This E_z is just the same E_z which lives in the normal skin sheath. That is why when we take $\delta_{debye} \to 0$ in equation (*) above, only the first term $-E_y(x,-\varepsilon,z)$ remains. In contrast, when we took $\delta \to 0$ in Case 1, it was K_z that remained constant and then we avoid the disaster of $n \to 0$.

Continuity Analysis #4 : Gaussian box inside the skin layer

Not useful.

Here we put a box tiny in all directions inside the conductor



Since now div **J** = 0, we can just write

$$\partial_x J_x(x,-\varepsilon,z) + \partial_y J_y(x,-\varepsilon,z) + \partial_z J_z(x,-\varepsilon,z) = 0$$

I am not sure this helps anywhere.