**Old averaging repair from Section 4.11 (rejected)** 6.6.14

Our theory now has an inconsistency which needs repair.

We know that for a general transmission line operating at $\omega > 0$, the current density $J_z$ inside the conductors will not be uniformly distributed. It will be larger in the conductor region closest to the other conductor. This "proximity effect" is discussed in Appendix P from an eddy current point of view, see Fig P.13 for an example. The $J_z$ current non-uniformity can be very dramatic as for example in a transmission line having this cross section, where $J_z$ will be large near the gap and small far from the gap:

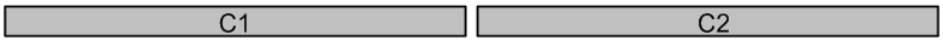


Fig 4.12

Since $J_z$ is non-uniform in each conductor, so is $E_z$, and so we expect $E_z(\mathbf{x}_1)$ to be a strong function of the point $\mathbf{x}_1$ on the perimeter of $C_1$, certainly in the above cross section example. This means that the left side of (4.11.6a) is a function of $\mathbf{x}_1 = (x_1,y_1,z)$ and $\mathbf{x}_2 = (x_2,y_2,z)$ whereas the right side in our theory is a function only of z. To remedy this inconsistency, we now have to think of V and W as having very slight dependence on $\mathbf{x}_1$ and $\mathbf{x}_2$ which we generally ignore, but which we must face up to in (4.11.6a). In reality we have $V(\mathbf{x}_1,\mathbf{x}_2)$ and $W(\mathbf{x}_1,\mathbf{x}_2)$. This is a manifestation of the fact that in reality $\varphi \approx$ constant and $A_z \approx$ constant on the boundaries (with $\approx$ and not $=$ ). In the extreme skin effect regime (think a very good conductor), the left side of (4.11.6a) can be a violent function of $\mathbf{x}_1$ and $\mathbf{x}_2$ as in the case of the above figure, but the left side is always very small, even where it is largest, and its variation can be accommodated by the right side of (4.11.6a) which is the difference of large-valued functions which vary only slightly with $\mathbf{x}_1$ and $\mathbf{x}_2$. We now proceed in our development of the transmission line equations with this understanding of the functions V and W.

As shown in (2.4.1), $E_z(\mathbf{x}_1)$ can be related to the total current in the conductor i(z) by a quantity known as the surface impedance $Z_s$, so

$$E_z(\mathbf{x}_1) = Z_{s1}(\mathbf{x}_1)\ i_1(z) \qquad E_{z2}(\mathbf{x}_2) = Z_{s2}(\mathbf{x}_2)\ i_2(z)\ . \tag{4.11.7}$$

The surface impedance of a perfect conductor is zero. Since $i_1(z) = -i_2(z) = i(z)$, we rewrite (4.11.6) as,

$$[Z_{s1}(\mathbf{x}_1) + Z_{s2}(\mathbf{x}_2)]\ i(z) = -\partial_z V - j\omega W \tag{4.11.8a}$$
$$\partial_z W = -j\,(\beta^2/\omega)\ V\ . \tag{4.11.8b}$$

We just noted that $E_z(\mathbf{x}_1)$ may vary violently around the perimeter of conductor $C_1$, especially for a cross section like that shown above. This of course means that $Z_{s1}(\mathbf{x}_1)$ also varies violently on the perimeter, causing equation (4.11.8a) to have the same inconsistency as (4.11.6a).

We now perform a certain slight of hand. We back up to the first equation of (4.11.4) evaluated at $\mathbf{x} = \mathbf{x}_1$,

$$E_z(\mathbf{x}_1) = -\partial_z\varphi_{12}(\mathbf{x}_1) - j\omega A_{z12}(\mathbf{x}_1)$$

or

$$Z_{s1}(\mathbf{x}_1)\ i_1(z) = -\partial_z \varphi_{12}(\mathbf{x}_1) - j\omega A_{z12}(\mathbf{x}_1)\ . \tag{4.11.9}$$

Recall now (4.7.3),

$$J_{z1}(x,y,z) = b_1(x,y)\ i_1(z)\ . \tag{4.7.3}$$

We shall now average both sides of (4.11.9) over the perimeter of $C_1$ weighting by $J_{z1}$ at the surface, which we intuitively regard as a measure of the relative significance of a piece of the $C_1$ boundary. Thus, define

$$Z_{s1} \equiv \frac{\int_{C1}\ ds_1\ Z_{s1}(\mathbf{x}_1)\ J_{z1}(\mathbf{x}_1)}{\int_{C1}\ ds_1\ J_{z1}(\mathbf{x}_1)} = \frac{\int_{C1}\ ds_1\ Z_{s1}(\mathbf{x}_1)\ b_1(\mathbf{x}_1)}{\int_{C1}\ ds_1\ b_1(\mathbf{x}_1)} = \langle Z_{s1}(\mathbf{x}_1)\rangle_{C1}$$

$$\varphi_{12}(z) \equiv \frac{\int_{C1}\ ds_1\ \varphi_{12}(\mathbf{x}_1)\ J_z(\mathbf{x}_1)}{\int_{C1}\ ds_1\ J_z(\mathbf{x}_1)} = \langle \varphi_{12}(\mathbf{x}_1)\rangle_{C1} \tag{4.11.10a}$$

and similarly for $A_{z12}(z)$. We do a similar average over the $C_2$ perimeter to obtain $Z_{s2}$, $\varphi_{12}(z)$ and $A_{z12}(z)$,

$$Z_{s2} \equiv \frac{\int_{C2}\ ds_2\ Z_{s2}(\mathbf{x}_2)\ J_z(\mathbf{x}_2)}{\int_{C2}\ ds_2\ J_z(\mathbf{x}_2)} = \langle Z_{s2}(\mathbf{x}_2)\rangle_{C2}$$

$$\varphi_{12}(z) \equiv \frac{\int_{C2}\ ds_2\ \varphi_{12}(\mathbf{x}_2)\ J_z(\mathbf{x}_2)}{\int_{C2}\ ds_2\ J_z(\mathbf{x}_2)} = \langle \varphi_{12}(\mathbf{x}_2)\rangle_{C2} \tag{4.11.10b}$$

We could then define the "active perimeter" distances $p_1$ and $p_2$ for the two conductors as

$$p_1 \equiv \int_{C1}\ ds_1\ Z_{s1}(\mathbf{x}_1)\ J_{z1}(\mathbf{x}_1)$$

The averaged (4.11.9) now becomes,

$$Z_{s1}\ i_1(z) = -\partial_z \varphi_{12}(z) - j\omega A_{z12}(z)\ . \tag{4.11.11}$$

We then slightly redefine V and W as being the difference of the current-averaged potentials,

$$\begin{aligned} V(z) &\equiv \varphi_{12}(z) - \varphi_{12}(z) = \langle \varphi_{12}(\mathbf{x}_1)\rangle_{C1} - \langle \varphi_{12}(\mathbf{x}_2)\rangle_{C2} \equiv \langle V(\mathbf{x}_1,\mathbf{x}_2)\rangle \\ W(z) &\equiv A_{z12}(z) - A_{z12}(z) = \langle A_{z12}(\mathbf{x}_1)\rangle_{C1} - \langle A_{z12}(\mathbf{x}_2)\rangle_{C2} \equiv \langle W(\mathbf{x}_1,\mathbf{x}_2)\rangle\ . \end{aligned} \tag{4.11.12}$$

Performing the same weighted average on (4.11.8b), we arrive at this adjusted version of (4.11.8),

$$[Z_{s1} + Z_{s2}]\, i(z) = -\partial_z V(z) - j\omega W(z)$$
$$\partial_z W(z) = -\, j\,(\beta^2/\omega)\, V(z)\,. \qquad (4.11.13)$$

where now $Z_{s1}$ and $Z_{s2}$ are the constants just defined and V(z) and W(z) are given by (4.11.12).

Comment: For transmission lines made of "thin wires", which means thin relative to their spacing, $J_z$ is close to uniform over each conductor and the above averaging process is unnecessary.

Recall now (4.11.1) which says $W(z) = L_e\, i(z)$. We reinterpret this equation first as $W(\mathbf{x}_1,\mathbf{x}_2) = L_e(\mathbf{x}_1,\mathbf{x}_2)\, i(z)$, then we current-average both sides to get $W(z) = L_e\, i(z)$ where $L_e = \langle L_e(\mathbf{x}_1,\mathbf{x}_2)\rangle$. Then replacing W(z) twice by $L_e\, i(z)$ in (4.11.13) gives,

$$(Z_{s1} + Z_{s2})\, i(z) = -\partial_z V(z) - j\omega L_e\, i(z)$$
$$L_e\, \partial_z\, i(z) = -\, j\,(\beta^2/\omega)\, V(z)$$

which we then rearrange as

$$\partial_z V(z) = -\,[\, Z_{s1} + Z_{s1} + j\omega L_e]\, i(z)$$
$$\partial_z\, i(z) = -\,[\, j\beta^2/(\omega L_e)]\; V(z)\,. \qquad (4.11.14)$$

These are the classical **transmission line equations**. They are usually written in this form:

$$\frac{\partial V(z)}{\partial z} = -\mathbf{z}\, i(z) \qquad \frac{\partial i(z)}{\partial z} = -\mathbf{y}\, V(z)$$

with

$$\mathbf{z} = R + j\omega L \qquad \mathbf{y} = G + j\omega C \,. \tag{4.11.15}$$