The Boltzmann Problem PhL 12.23.14

This is a generalization of the Boltzmann problem but I give that name to the title.

Statement of the General Problem. Consider a real function of m real variables,

$f(x_1, x_2, ...x_m)$. $\quad f : R^m \rightarrow R$

We are interested in finding where this function is a maximum. However, there are also a set of s constraint equations:

$g_1(x_1, x_2, ...x_m) = 0$
$g_2(x_1, x_2, ...x_m) = 0$
....
$g_s(x_1, x_2, ...x_m) = 0$

We want the maximum of f, but subject to all the constraints.

Comments:

1. Each constraint equation is a surface of dimension of dimension m-1 which exists within the larger space R^m. For example, we could rewrite a constraint equations as $x_m = G_i(x_1, x_2, ...x_{m-1})$ so that G_i: $R^{m-1} \rightarrow R$. For each choice of $x_1, x_2, ...x_{m-1}$ values, there is a unique x_m (just assume for now), and that is your surface.

2. Suppose there are only two constraint equations g_1 and g_2 each defining its own m-1 dimensional surface in R^m. In general, the locus of points in R^m on which both these constraints are met is a surface of m-2 dimensions. For example, if g_1 and g_2 are linear constraints, each one defines a hyperplane, and then both constraints are met on the intersection of those two hyperplanes, which is a hyperplane of dimension m-2. If we had m = 3, then g_1 and g_2 would define actual planes in 3D space, and their intersection is a line which is a surface of dimension m-2 = 3-2 = 1.

3. If there are s constraint equations, then it seems that the fully constrained surface would be of dimension m - s. So we certainly need to have $s \leq m$ in order to have a non-null solution surface of constraint.

Question: Where in the mathematical house of many mansions does this problem lie? Perhaps the subject is calculus of variations. I have seen other situations where you might want to maximize things like "the action" in a physics problem, or perhaps you want to minimize some error. In the Boltzmann problem we happen to want to find the maximum number of "states" available to a system of m particles. My PC math world does not have an entry for calculus of variations.

Goldstein Chapter 2. This is on variational principles, but we are there trying to find a set of function solutions $q_i(t)$ [the generalized coordinates] which cause a certain integral to be minimized. That leads to the Lagrange equations of motion. But in our Boltzmann problem, we are not trying to find any functions

like $q_i(t)$. We just have a simple "geometry problem" of finding the max of a function subject to a set of constraints. So I don't think we have a fit here. So "calculus of variations" is not the right topic.

Lagrange Multiplier Method. Here is wiki right off the bat,

> In mathematical optimization, the **method of Lagrange multipliers** (named after Joseph Louis Lagrange[1]) is a strategy for finding the local maxima and minima of a function subject to equality constraints.

So it appears this is the "box" into which this problem falls. Notice that we have equality constraints and not inequality constraints. So this wiki page is going to give a reasonable summary I think. But do I already have this in some book somewhere? Yes, Buck has stuff on this subject, which is referred to as "Extremal properties of functions of several variables" on page 349. Have I ever really read this calculus book? Well, it was the text of Math 105a "advanced calculus" at Harvard Spring 1968 when I was a Sophomore.

Maybe I should do some work in this book just on general principles.