A Model for Air PhL 7.21.11

Summary: We model air as if it were ping pong balls in a 1 m edge box.

Consider a cubic box (edge e) with clear plastic walls containing one ping pong ball in the presence of gravity. A Cartesian coordinate system is set up at the center of the bottom wall of the box with x,y going horizontally and z going up. The bottom plane of the box is z=0, the top is z=e (e = edge dimension of box). We want to study things this ball might do. We set the zero of gravitational potential energy to zero at z = 0 so V = mgz.

When the ball bounces against any wall (and later when there are multiple balls and they bounce off each other as well), we assume elastic collisions, no energy loss. We are now going to do a set of experiments with one ball in the box.

Experiment A. Ball starts in the middle of the box at height h = e/2. It has no initial velocity. At t= 0 we "drop" the ball, meaning we just start the clock for its initial conditions of z = h and v = 0. We can think of this at first as a 1D problem with the ball a point particle and space is just the z vertical direction. The ball bounces forever. We want to know the probability of finding the ball at various heights. We can start with simple energy conservation. v(z) is the velocity of the ping pong ball when it is at height z above the bottom of the box during its bouncing activity. We write Energy(t) = Energy(0) in this way E = KE + PE:

$$1/2\ m\ v(z)^2 + mgz = 1/2\ m\ v_0^2 + mgz_0 = 0 + mgh = mgh$$

From this we find that mass cancels out and the ball's velocity (magnitude) is given by.

$$v(z) = \sqrt{2g(h-z)}$$

The speed is 0 at the top of the bounce when z = h, and maximal at the bottom when z = 0.

We can imagine a narrow vertical slice dz and ask "what is the probability that we will find the ball, which we think of now as a point particle concentrated at the center of the ball, to be in that slice?" The ball moves through this very narrow slice at a constant speed v(z) = dz/dt. Therefore, the time the point ball spends in this slice is dt = dz/v(z). The probability of finding the ball in this slice is proportional to the time it takes for the ball to transit the slice so we write P(z)dz = K dt where P(z)dz is the probability of the ball lying between z and z+dz (which is obviously proportional to dz for very small dz), so P(z) is a probability density and it is the thing we seek. K is a constant to be determined. So we then have

$$P(z)dz = K\ dt = K\ (1/v(z))dz \quad \Rightarrow \quad P(z) = K/v(z).$$

We know that the probability that the ball lies between height z=0 and z=h is 1, so we can evaluate the constant K by requiring that

$$\int_0^h P(z)dz = 1 \quad \Rightarrow \quad K \int_0^h dz/v(z) = 1 \quad \Rightarrow \quad 1/K = \int_0^h dz/v(z)$$

This integral is not hard to do,

$$\int_0^h dz/v(z) = \int_0^h dz/\sqrt{2g(h-z)} = 1/\sqrt{2g} \int_0^h dz\,(h-z)^{-1/2}\,.$$

Change variables to one that measures distance from the top of the bounce, x = h-z, dx = -dz, to get

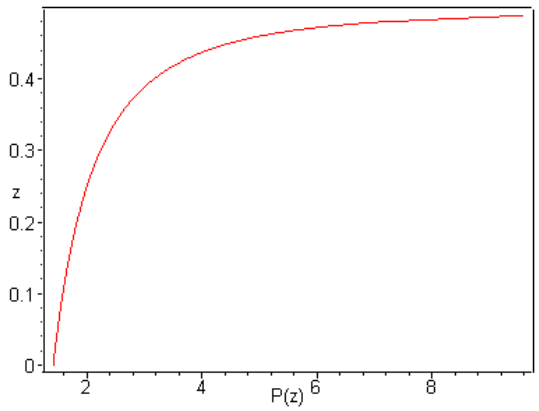
$$= 1/\sqrt{2g} \int_h^0 [-dx]\,(x)^{-1/2} = 1/\sqrt{2g} \int_0^h dx\, x^{-1/2} = (1/\sqrt{2g})\,[2\,x^{1/2}]\,|^h_0 = (1/\sqrt{2g})\,2\sqrt{h}$$

$$= \sqrt{2h/g} \quad = 1/K \qquad\qquad => \qquad K = \sqrt{g/(2h)}$$

Therefore we have found our desired probability of observance function P(z),

$$P(z) = K/v(z) = \sqrt{g/(2h)}\ /\sqrt{2g(h-z)} = \sqrt{2h(h-z)}$$

and here is a plot with z on the vertical axis and P(z) on the horizontal axis, and h = 0.5 .



At the bottom of the bounce, P(0) = $\sqrt{2}$ h = $1/\sqrt{2}$ = 0.7 in our example, small but finite. At the top of the bounce P(z) = ∞ since v(h) = 0. The curve of course integrates to 1 against the z axis. Above z = h, P(z) = 0 since the ball never gets there. So in our first experiment, the probability of finding the ball is 0 in the upper half of our box, and has the above shape in the lower half, and is maximal just below z = h = e/2.

Experiment B. We now extend our 1D vertical box example to include x,y dimensions and we allow that at t = 0, the ball might have some horizontal initial velocity $\mathbf{v_0} = (v_{0x}, v_{0y}, 0)$. In this situation, the ball's horizontal speed components never change (the horizontal velocity changes sign at each wall bounce), and we then have our ball bouncing vertically, and at the same time moving horizontally and bouncing off a wall when it hits a wall. This scene is pretty easy to visualize. The function P(z) is completely unaltered because the z projection of what is going on here is the same as it was in Experiment A. The horizontal component of the ball's kinetic energy never changes.

Experiment C. Now we repeat Experiment A in just the vertical dimension with $\mathbf{v_0} = (0, 0, v_0)$. In this case, our energy equation starts off a little differently, we have an extra term on the RHS

$$1/2\ m\ v(z)^2 + mgz = 1/2\ m\ v_0^2 + mgh$$

Solving for v(z) we find

$$v(z) = \sqrt{2g(h-z) + v_0^2}$$

If we set v(z)=0, we can find that maximum height reached by the bouncing ball, if we assume for the moment that v_0 is small enough so the ball doesn't hit the top of the box,

$$z_{max} = h + v_0^2/2g$$

Notice this is true if v_0 is up or if it is down, in which case z_{max} will be reached after the first bounce. As before, we have P(z) = C/v(z) where C is some constant we could compute, but let's not bother with that detail, and instead discuss qualitatively what we expect to see. The square root appearing in v(z) has two terms, and the maximum size of the first term occurs when z = 0 and there is 2gh. If $v_0^2 << 2gh$, then we can neglect the v_0 term and our P(z) plot looks like our Experiment A result. On the other hand, if $v_0^2 >> 2gh$, then we have $v(z) \approx v_0$ and then $P(z) = C/v_0$ which is a constant. In this scenario, the ball is bouncing up and down between the upper and lower walls of the box, travelling all the time at high speed of v_0 , and the effect of gravity g is negligible. The ball reaches the top of the box if $z_{max} = h + v_0^2/2g > 2h$ which occurs when $v_0^2 > 2gh$. We could compute P(z) precisely for this double-bouncing situation, but we know that as v_0 gets larger and larger compared to 2gh, we approach the limit P(z) = 1/e which says it is equally likely to be at any height in the box and gravity does nothing at all.

Experiment D. We return to the 3D box and assume that $\mathbf{v}_0 = (v_{0x}, v_{0y}, v_{0z})$. The x and y activity is completely decoupled from the vertical activity, and we obtain the results of Experiment C where we replace v_0 by v_{0z}. We have $P(z) \approx 1/e$ as long as $v_{0z}^2 >> 2gh$. In fact, it seems pretty clear in this case that we could define $P(x,y,z)d^3x$ as the probability of finding the ball at some location in our 3D box, and we would have $P(x,y,z) = 1/e^3$. The ball is equally likely to be found anywhere in the box, when observed at many random times, assuming $v_{0z}^2 >> 2gh$.

Experiment E. Now we put some number of balls in the box, perhaps 20, and for each ball i we have some $\mathbf{v}_{0i} = (v_{0xi}, v_{0yi}, v_{0zi})$. We might throw in random starting positions for the balls. It seems pretty clear that as long as $v_{0zi}^2 >> 2gh$, the balls bounce around in random fashion and $P(x,y,z) = 1/e^3$ and gravity has no effect on things. Perhaps on each collision between balls we get some new $\mathbf{v}_{0i} = (v_{0xi}, v_{0yi}, v_{0zi})$ for each ball, randomly determined, but we know that $v_{0xi}^2 + v_{0yi}^2 + v_{0zi}^2 = |\mathbf{v}_{0i}|^2$ never changes because the ball-ball collisions are elastic. If we put a great number of balls in the box, we would expect the ball-ball collisions to dominate over the ball-wall collisions, and we would expect that for the average ball, the three velocity components are about the same, so that $v_{0xi}^2 \approx v_{0yi}^2 \approx v_{0zi}^2 = |\mathbf{v}_{0i}|^2/3$. In this situation, our gravity-has-no-effect condition would be $|\mathbf{v}_{0i}|^2/3 >> 2gh$. Let's let v_0 be the average initial speed of our balls, so then we have $v_0^2/3 >> 2gh$ or $v_0^2 >> 6gh$ as our condition.

It is true that there will be moments when a ball is travelling very slowly horizontally between collisions and that particular ball will then "feel" the effect of gravity and be pulled down. But statistically, such "ball moments" don't contribute much to the average situation. One could do a calculation here to obtain an expression for the effect of such a situation, perhaps based on the assumption that the direction of balls after a collision is random in the solid angle sense. One would find that the

percentage of balls travelling very close to horizontally at slow speed is very small for large v_0. Just thinking of the speed of such balls, we know from statistical mechanics that there will be a certain thermal distribution of ball velocities (now thinking of the balls as air molecules), and not many balls are going slowly enough to qualify for our "ball moment" described above.

If we have a large number of balls in our box, or in fact any number, we could somehow calculate the mean free distance a ball travels between collisions. We expect this to decrease as the number of balls in our box increases. For air in a box, this distance λ is going to be very much smaller than the dimension of any "box" we think about for practical purposes, so the walls have no effect on a cubic box of air "at rest" (except thermally).

A Model for Air. So we replace our ping pong balls with air molecules, mostly N_2 nitrogen molecules each of which has a mass of 28 protons. Statistical mechanics tells us that each degree of freedom will have energy 1/2 kT if the air is at thermal equilibrium (a whole topic in itself, we are not going to attempt to derive this equipartition theorem result here). In particular, the N_2 molecules will have kT/2 energy for each translational degree of freedom, and that tells us that

$$(1/2)mv_{0z}^2 = (1/2)kT \quad \Rightarrow \quad v_{0z}^2 = kT/m$$

where v_{0z}^2 is the average vertical squared speed of an air molecule in our cube of air. We might note that the molecules also have rotational and vibrational degrees of freedom (3 and 1) and each of these also gets 1/2 kT, but that does not affect the translational situation.

So let us now *compute* v_{0z} for an average air molecule at room temperature in SI units. Freezing is 273.15K. If we assume room temperature to be 68K = 20C, we have roughly T = 293K. As noted, we have $m = 28\ m_p$ (assume air = nitrogen) and we collect m_p and k here:

$$m_p = 1.7 \times 10^{-27} \text{ kg} \qquad k = 1.4 \times 10^{-23} \text{ J/K} \quad \text{(joule per kelvin)}$$

then we get ($J = mv^2 = kg\ m^2/sec^2$)

$$v_{0z} = (kT/m)^{1/2} = (1.4 \times 10^{-23} \text{ J/K } 293\text{K} / [28 * 1.7 \times 10^{-27} \text{ kg}])^{1/2}$$

The units do this inside the square root

$$(kg\ m^2/sec^2K)\ K/kg = m^2/sec^2 \qquad \text{// correct}$$

and then the numbers are

$$(1.4 \times 10^{-23} * 293 / [28 * 1.7 \times 10^{-27}])^{1/2}$$

$$= \{1.4 * 293 / [\ 28 * 1.7]\ \}^{1/2} \times 10^2 = \{8.61\}^{1/2} \times 10^2 = 2.9 \times 10^2 = 290$$

so we conclude that

$$\langle v_{0z} \rangle = 290 \text{ m/sec}$$

We know that $\langle v_0^2 \rangle = 3\langle v_{0z} \rangle^2$ since the KE in each direction is the same (and = kT/2), so we find that

$\langle v_0 \rangle = \sqrt{3}\, \langle v_{0z} \rangle = 1.73 * 290 = 502$ m/sec

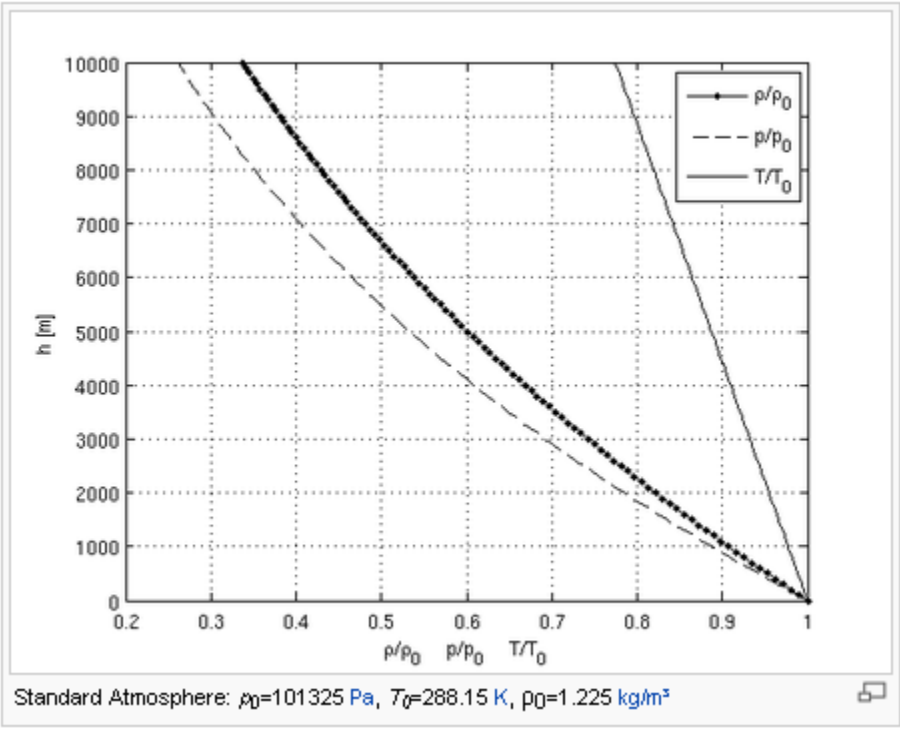
Now we want to test our assumption that $v_0^2/3 >> 2gh$ which results in our having the air molecules evenly distributed through our cubic box. If the height of the box is something like e = 1 meters = 2h, then we have

$v_0^2/3 = 290^2 = 84{,}100$ $\qquad$ $g = 9.8$ m/sec^2

$2gh = 2*9.8*(1/2) = 9.8$

Therefore, since 84,100>> 9.8, we conclude that gravity can be neglected in our consideration of air molecules in a human-sized box at room temperature. If we make the box a mile high = 1609 meters, then h = 800 meters and 2gh is then 1600 times larger or about 16000, and now there will be some small effect of gravity. For a 10 mile high box, a strong effect of gravity -- air molecules significantly more likely to be at the bottom of the box than the top.

Here is a wiki plot where the solid thin line shows air density ρ relative to sea level



It shows that for a box that is 1 mile high (1609 m), ρ drops to perhaps .97 of sea level ρ_0, so we get perhaps a 3% effect. At 5 miles (8045 m) we get maybe .825 so we have an 18% effect. Maybe 10 miles would give a 30% effect. Note that $g \approx 9.8$ at all these relatively small heights since $R_E \sim 4000$ miles.

Answers to some Questions.

Q1. Why don't air molecules in a cubic box of air fall to the bottom of the box and just sit there due to the effect of gravity?

A1: They would do just that at low enough temperature T. They would fall to the bottom and form first a layer of liquid nitrogen and then at lower T that would freeze to nitrogen ice. N_2 becomes a liquid at 77 K and then a solid at 63K, says wiki. But at room temperature, as we have shown above, the thermal motion is such that gravity has no effect whatsoever because the thermal velocity is so large. The molecules on the average are moving so fast between collisions that gravity does not have time to cause any significant deflection. We showed above that this is the situation when $v_0^2/3 >> 2gh$ where $e = 2h$ is the height of our box and v_0^2 is the average squared thermal velocity of the air molecules.

Q2: If you could see the air molecules, how far apart are they compared to their size?

A4: We know roughly that a gas at atmospheric pressure is roughly 1000 times less dense than a liquid. Imagine the liquid consisting of tightly packed cubes where each cube holds one N_2 molecule (even though it doesn't fit well in a cube, this is all just a rough approximation). If we imagine a 3D mathematical cubic lattice with a linear spacing of 10x our N_2 cube edge, and place one of these N_2 cubes at each lattice point, we would then have one small cube per 1000 cubes occupied, and this is about right for air. So roughly speaking, if the gas molecules are 1" diameter ping pong balls, if they were equally distributed in a box they would be 10" from their 6 nearest neighbors. So if we had a cubic box of edge 1 meter = 40 inches, we would have 4 balls across the box, and then 4x4x4 = 64 ping pong balls in the box. So that is our visual "model for air". In this model, if we want the box of balls to be a "movie" of air that is magnified, our magnification factor would be perhaps $1"/ 2A = 2.54cm/2x10^{-8}cm \sim 10^8$. So if we want the actual v_0 to be 500 m/sec, our magnified movie would show $v_0 = 500 \times 10^8$ m/sec for the ping pong balls, 5×10^{10} m/sec. We have $c = 3 \times 10^8$ m/sec, so these balls are truly moving fast. You would just see a blur of balls racing around in your plexiglass box. Web says $\lambda = .1\mu$ so in our room size thing we would have $\lambda = 10$ m which I suppose is ballpark possible (ignoring wall bounces). This paragraph is very crude!

Q3: Derive the ideal gas low PV = nRT. How do we get pressure P into the picture?

Q4: How would you compute the mean free path λ in air?