The Scattering Theory of Waves and Particles PhL 8.27.07
Roger G. Newton 1966

This book was a gift from MRL at my request. I had been looking at a library copy in regard to some scattering problem with Jim Ball relating to CT scanners. Now I am just reading the book a bit. Notice errata on page 744-745. This book also has a 60 page bibliography at the end.

Roger G. Newton

Distinguished Professor Emeritus

Department of Physics, **Indiana University, Bloomington**

Professor Newton's research areas are **field theory, scattering theories, nuclear and high energy physics, elementary particles, quantum mechanics, and mathematical physics.**

Professor Newton received a Ph.D. degree from **Harvard in 1953** and was a member of the Institute for Advanced Study at **Princeton** for the next **two years** [his first and only postdoc] . From **1955**, he has been with the Department of Physics, chairman from 1973-1980, and Director, IU Institute for Advanced Study, 1982-86. He has held visiting appointments at various universities and laboratories including Ohio State, University of Rome, International School of Physics, University of Montpellier, France, and University of Geneva. In 1989 he was the recipient of a Departmental Teaching Award presented by physics graduate students.

Newton has served on various committees in many professional organizations and as associate Editor of the *American Journal of Physics*, and *Inverse Problems*. He is the **Editor** of the *Journal of Mathematical Physics*. His most **recent publications** are *Thinking About Physics*, *What Makes Nature Tick?* (Harvard University Press, 1993) and *The Truth of Science*(Harvard University Press, 1997); *Quantum Physics: A Test for Graduate Students*, (Springer Verlag, 2002); *Galileo's Pendulum: From the Rhythm of Time to the Making of Matter* (Harvard University Press, 2004); and *From Clockwork to Crapshoot: A History of Physics* (Harvard University Press, 2007).

Home: (812) 332-6238; Office: (812) 855-2095; Fax: (812) 855-5533; E-mail: newton@indiana.edu

So this guy is maybe 20 years older than me, maybe age 80 or so. His latter books would probably make very interesting reading. He started at IU in 1955 and I guess wrote this scattering book 1966 after teach classes there for 11 years, a ripe time I would say.

Chapter 1: Formalism and General Results

1.1 The Maxwell Equations.

(1.1) The four equations are stated in the usual time-dependent cgs form, and include the macroscopic aspects as represented by the quantities ε, μ and σ. These quantities are in the three supplementary equations (1.2), and ρ also appears in (1.1).

Before taking even one step forward, we have to admit now to several levels of complexity.

In the time-dependent equations, all quantities including ρ, ε, μ, σ, **E,D,H,B** are real.

First complexity: they may all be dependent on time and space, such as $\rho(\mathbf{r},t)$. The Maxwell equations are stated at a point in 3-space and time.

Second complexity: the three quantities ε, μ, σ can be **tensors**. I have never really dealt with this issue, but here it is. My unread Born and Wolf book shows the idea for $\mathbf{D} = \varepsilon\mathbf{E}$ on page 790 eq (1) where you regard ε as a 3x3 matrix in an x,y,z sense. This matrix can rotate the direction of D away from that of

E, and can scale the components as well. The B&W book is talking about optics of crystals which can of course be anisotropic. The 9 components of the tensor ε could each in theory be functions of space and time. The 3x3 matrix is called a second rank tensor as we well know. For me, it is just a matrix.

Third complexity: What happens to Maxwell's equations and the three supplemental equations if you Fourier transform everything into the frequency domain, replacing ω with t?

Digression on FT: In my own "book" the opening chapter defines the Fourier Transform of a real function x(t) and the conclusion is that $X^*(\omega) = X(-\omega)$ and X(ω) is in general a complex-valued function of ω, even through x(t) is a real-valued function of t. My little book never really talks about transforming differential equations because I was interested in other topics at the time. I don't know if any books I own does a simple presentation of the Fourier Transform conversion of differential equations. Luckily we are in the web era now, so I can refer to http://en.wikipedia.org/wiki/Fourier_transform (saved) which shows the FT with the same phase assumption on the exponent sign that my book uses. This web page then shows that the operator $\partial/\partial t$ is replaced with $+i\omega$ in the Fourier world.

Now, suppose we set $x(t) = h\, e^{+i\omega 1 t} + h^*\, e^{-i\omega 1 t}$ which is a legal example because x(t) is real, so X(ω) must satisfy the rule:

$$X(\omega) = 2\pi\, [\, h\, \delta(\omega-\omega 1) + h^*\delta(\omega+\omega 1)\,] \;= X^*(-\omega)$$

and

$$\partial/\partial t\; x(t) \;\leftrightarrow\; i\omega X(\omega)$$

Note that page numbering is turned on in this template.

However, in the physics world of waves, we like to think of the FT with the "other sign" in the exponent and we prefer to write $x(t) = h\, e^{-i\omega 1 t} + h^*\, e^{+i\omega 1 t}$ so that the non-asterisk thing "h" is associated with the term $e^{-i\omega 1 t}$ and then exp[i(kx - ωt)] is though of as a plane wave traveling "to the right" . The FT world can work this way, but we then have to say $\partial/\partial t\; x(t) \;\leftrightarrow\; -i\omega X(\omega)$.

Using this rule, we quickly obtain the first equation in (1.4) from the first in (1.1). Although Newton uses the same notation for the fields and parameters, we have to now think of them as being in the ω domain, so they are now all complex and just functions of space.

So, looked at from this FT point of view, using $\partial/\partial t\; x(t) \;\leftrightarrow\; -i\omega X(\omega)$, the two equations (1.4) -- with the statement (1.3) and with the negative of my FT convention -- are completely trivial restatements of the first two equations in (1.1).

Newton has made an assumption that he does not clearly state: that ε is not time-dependent. If it were, the first of equations (1.4) [only] would not be true. You would have D(ω) sitting in there, but it would not be true that $D(\omega) = \varepsilon(\omega)E(\omega)$. This equation is diagonal in the t domain, and must therefore be of convolution form in the ω domain. -- end digression

So, we are going to be using the negative-FT sign convention which (1.3) implies.

Next: **why are (1.4) true in the tensor sense?** The $t\leftrightarrow\omega$ aspect we have just done, and we just think of the quantity defined in (1.5) as n'^2 as being a 3x3 matrix. That is, we think of both ε and σ as 3x3 matrices, and therefore the combination of (1.4) and (1.5) are true in the tensor sense. And of course all of equations (1.2) can be tensor equations. The quantity ρ however is always a scalar in the tensor sense.

Derivation of (1.6): To get the RHS of this equation, compute ik∇ x (μH) using the obvious vector identity, but you must be careful in the two identity terms to put the scalar quantity first since both the scalar ϕ and the vector A can be matrices. That is to say, we write:

$$\nabla \text{ x } (\phi \text{ A }) = \nabla(\phi) \text{ x A } + \phi \nabla \text{ x A}$$

with the idea that $\nabla(\phi)$ is a matrix whose elements are $\nabla(\phi_{ij})$ and similarly for ∇ x A_{ij} . I happen to have written this one identity correctly in this matrix sense, but not all are done that way on my little page!

Secondly, in computing the RHS as just stated, you must think of (1/μ) as a matrix inverse to μ, so we are assuming that matrix μ is invertible.

To compute the LHS of (1.6) we again compute ik∇ x (μH) but we replace μH = B (in tensor sense), then replace B with curl of E in (1.4) and we are done.

So, equation (1.6) is therefore true in the tensor sense, and all quantities are in general functions of ω. However, since we made use of the left of (1.4) in computing the RHC of (1.6), we have assumed as noted above that ε(t) is not time-dependent, so $\varepsilon(\omega)$ is independent of ω. This seems to me to be a bad limitation on what we are doing, certainly in general $\varepsilon = \varepsilon(\omega)$ in a dielectric.

So for the moment, I will leave this issue unresolved: I think (1.2) are only true as stated in the time domain, not the frequency domain

Hypothesis: In situations where the "constants" like ε and μ and σ have time-dependence (and in which situations therefore their FT counterparts have ω dependence) the equations of the form (1.2) are true in the frequency domain, not in the time domain. A first example is the expression given for $\sigma(\omega)$ on page 750 of B&W in their chapter on optics of metals. Recall that in general $n^2 = \varepsilon\mu$, and B&W page 97 gives the usual dispersion theory to show $\varepsilon(\omega)$ and this $n(\omega)$. This simple dispersion theory appears in many of my books. The big issue here is your model of the macroscopic media and effective fields, etc etc.
So I think my hypothesis is holding.

Digression: A Comparison of Two FT-like methods of solving a real ODE.

Suppose you have a second order linear differential equation in one variable with everything real because it describes some physical situation:

$$a \ddot{x} + b\dot{x} + cx = d$$

Now imagine that d is complex and we want to solve for complex solution x. Then we write

$$a(\ddot{x}_r + i \ddot{x}_i) + b(\dot{x}_r + i\dot{x}_i) + c(x_r + i x_i) = (d_r + id_i)$$

where we have just expanded the two complex functions into real and imaginary parts. Since the coefficients are real, we can write the above equation as two separate equations:

$$a \ddot{x}_r + b \dot{x}_r + c x_r = d_r \qquad\qquad a \ddot{x}_i + b \dot{x}_i + c x_i = d_i$$

We can think of the left equation here as being our original physical problem with everything real. Suppose we first solve the full complex equation with some complex driving term d. Then we have also solved a physical problem with driving force Re(d). [And we have also solved a physical problem with driving force Im(d) which is the second equation. Usually we ignore this second equation.]

For example, try out $d = d_o e^{-i\omega t}$. We can solve to get a solution of the form $x = x_o(\omega)e^{-i\omega t}$ and we find that $x_o(\omega)$ is a trivial and complex-valued function of the real quantities a,b,c d_o and ω. If we then look at the real equation, we find driving force $d_r = d_o\cos(\omega t)$. This is probably a physical driving force of interest to us and we have now solved our problem by using $x_r(t) = \mathrm{Re}[\, x_o(\omega)\, e^{-i\omega t}\,]$. This will be a mix of $\sin\omega t$ and $\cos\omega t$ terms of course.

So this is just a "method" of solving an ODE with real coefficients. In our full equation x(t) is a complex valued function and $x_o(\omega)$ is in fact the FT of this function.

A full FT approach would be to say:

(1) first, transform the whole equation to the ω plane with FT. We then have (in the opposite of my sign):

$$[\, (-i\omega)^2\, a + (-i\omega)b + c\,]\, X(\omega) = D(\omega).$$

We solve the problem trivially and write out $X(\omega)$ = { function of a,b,c,ω } $D(\omega)$.

(2) We then assume $D(\omega)$ of the form $D(\omega) = 2\pi\, d_o\, \delta(\omega - \omega_1)$. Our answer is then $X(\omega)$ = { function of a,b,c,ω } $2\pi\, d_o\, \delta(\omega - \omega_1)$. We then compute x(t) = { function of a,b,c,ω } $d_o\, e^{-i\omega t}$ and $D(t) = d_o\, e^{-i\omega t}$. We can write the solution as $x(t) = x_o(\omega)\, e^{-i\omega t}$ where $x_o(\omega)$ = { function of a,b,c,ω } d_o. This is exactly the same $x_o(\omega)$ as we found in the "non FT" method above. So the relation between this thing $x_o(\omega)$ and the FT of x(t) is this: $X(\omega) = x_o(\omega)\, 2\pi\, \delta(\omega - \omega_1)$. Since x(t) is complex here, we don't have the fact that $X^*(\omega) = X(-\omega)$. Notice that we never deal with FT[Re(x(t)]

Now what about equations like $D = \varepsilon E$? We cannot write $d(t) = \varepsilon(\omega)d(t)$ in a general sense, because ω has no meaning here. What we really mean is $D(\omega) = \varepsilon(\omega)\, E(\omega)$ and we are completely in the ω plane. Now if we take the example $E(\omega) = E_o(\omega)2\pi\delta(\omega-\omega_1)$, then $D(\omega) = \varepsilon(\omega)\, E_o(\omega)2\pi\delta(\omega-\omega_1)$. Going back to the time domain we then get $e(t,\omega) = E_o(\omega)\, e^{-i\omega t}$ and $d(t,\omega) = \varepsilon(\omega)\ E_o(\omega)\, e^{-i\omega t}$. So we have two ways to write this:

$D(\omega) = \varepsilon(\omega)\, E(\omega)$ // general FT's

$d(t,\omega) = \varepsilon(\omega)\, e(t,\omega)$ // where d and e are plane waves at ω, not general f(t)'s.

So, when we see the equation $D = eE$ sitting in Newton, we have to think of it in either of the two ways just written above. We canNOT think of it as saying $d(t) = \varepsilon(\omega)e(t)$ for a general function e(t), because such a result has no meaning whatsoever. What is ω?

Now let's go back to our general ODE above where our FT plane solution was

$X(\omega)$ = { function of a,b,c,ω } $D(\omega)$

Suppose we now assume $D(\omega) = 2\pi\, d_o(\omega)\delta(\omega - \omega_1) + 2\pi\, d_o^*(\omega)\delta(\omega + \omega_1)$ where $d_o(\omega)$ = real. In this case we really have $D(\omega) = 2\pi\, d_o(\omega)\, [\, \delta(\omega - \omega_1) + \delta(\omega + \omega_1)\,]$. This has the property $D^*(\omega) = D(-\omega)$ since we know that $\delta(-k) = \delta(k)$. Thus, we know that d(t) will be real. The FT of $D(\omega)$ is then

$$d(t) = d_o(\omega)\, e^{-i\omega t} + d_o(\omega)\, e^{+i\omega t} = d_o(\omega)\, 2\cos(\omega t).$$

We can do this exact thing for the solution of our equation, but now we have

$$X(\omega) = 2\pi\, x_o(\omega)\delta(\omega - \omega_1) + 2\pi x_o^*(\omega)\delta(\omega + \omega_1)$$

$$x(t) = x_o(\omega)\, e^{-i\omega t} + x_o^*(\omega)\, e^{+i\omega t}$$

and $x_o(\omega)$ is complex, not real, but x(t) is still real.

In the previous approach where we said $D(\omega) = 2\pi\, d_o(\omega)\, \delta(\omega - \omega_1)$, we were working with the "extended" ODE where we considered x(t) and d(t) to be complex functions, then we just took the real part of x(t) to find the solution to the physical problem driven by Re[d(t)].

In the second approach where $D(\omega) = 2\pi\, d_o(\omega)\delta(\omega - \omega_1) + 2\pi\, d_o^*(\omega)\delta(\omega + \omega_1)$, we are working with the non-extended real equation. The transforms d(t) is x(t) are real, we don't have to take any real parts when we are done.

In either method of solution, the things of interest are $d_o(\omega)$ and $x_o(\omega)$. Many books use the extended method, but Newton when he writes (1.3) is using the second method. -- end digression

Conclusions from the above discussions: The equations (1.4) are in the ω domain. If any of the constants is frequency-dependent, then you have to think of that (1.2) equation as being true only in the ω domain. So really, we should just think generally of (1.2) and (1.4) as being in the ω domain, then we never have any problems or confusions. We first transform (1.1) to the ω domain, we assume (1.2) are already there, and then (1.4) is true in ω space with no assumptions made whatsoever about the "constants". Equation (1.6) we have derived and is true in this general sense. Definitions (1.7) of course need no derivation, but of course we know that n is going to be the index of refraction, not proven at this point.

Derivation of (1.8): We start with (1.1 #3) [which we regard as being in the ω domain]

$$\nabla\bullet(\varepsilon E) = 4\pi\rho$$

It is easy to show that $i\omega\nabla\bullet(\varepsilon E) = 4\pi\, \nabla\bullet(\sigma E)$ because this just says $\nabla\bullet\, (n'^2 E) = 0$ and we know this latter fact taking $\nabla\bullet$ on the left of (1.4). Thus we get

$$4\pi\rho = \nabla\bullet(\varepsilon E) = (4\pi/i\omega)\, \nabla\bullet(\sigma E) \;\Rightarrow\; \rho = (-i/\omega)\, \nabla\bullet(\sigma E) \quad \text{which is (1.8)}$$

The final claims are results of chapter problem 1 which I have shown to be true trivially using the stokes theorems and assuming that fields are never infinite at the boundary.

So, I have now battled my way through the first 1 1/4 pages of Newton's book!!!! Rusty rusty.

1.2 Stokes Parameters and Polarization

1.2.1 Definition of the Stokes parameters

I think in a conducting medium there may be a longitudinal component to E field, but maybe I am wrong and let's ignore that. This whole Stokes subject is new to me. Think of a plane wave where E can have two independent directions which we can choose at right angles to each other. Born & Wolf call these x and y, while Newton calls them || and $\perp$ which is pretty hard to type. The most general plane wave in a given direction k (z) at a given ω (monochromatic) will be an arbitrary linear combination of fields at right angles with arbitrary amplitudes and arbitrary phases (but same ω). You can write down the real fields as in (12) from B&W page 25 using parameters (1 = x, 2 = y): a_1, a_2 are the amplitudes and δ_1 and δ_2 are the phases. If you have arbitrary parameters, you can show (and B&W do show) that the electric vector at any selected point in space traverses an ellipse every period T. The plane of this ellipse is of course perpendicular to the k-vector. B&W show this ellipse on page 27 for a random case. The parameters a and b are the semimajor axes of the ellipse, and angle ψ is the elevation of the ellipse's major axis off the x-axis. Another angle χ is tanχ = b/a and I have drawn this in. The angle $\delta = \delta_2 - \delta_1$ is hard to show, but I have made a little construction to show it (in light pencil page 27 B&W). In general, you need three parameters to describe the ellipse such as a, b, ψ or a_1, a_2, δ.

B&W define the four Stokes Parameters in (43) on page 31, and we see they are a mapping from the set a_1, a_2, δ into $s_{0,1,2,3}$ where the squares of the last three add up to the first, so only 3 independent parameters, as we expect. Clearly s_0 is the power in the wave, the intensity, the sum of the squares of the E field amplitudes, the scale of the ellipse. Large s_0 means large ellipse. Parameter s_1 is a measure of how far from square the inscribing box is. The last two then describe the way the ellipse lies in that box. All four parameters have dimensions of E field squared, something authors regard as useful (rather than some being angles).

The Poincare Sphere is shown B&W p 32 and is very strange and interesting. The vector drawn has nothing to do with the E vector or anything like that. There is no ellipse in this picture anywhere. The radius of the sphere is the intensity $s_o = I = a_1^2 + a_2^2$. Each point on the sphere is defined by a unique set of our angles ψ and χ, and each such pair of angles describes a unique ellipse situation. Thus, all possible ellipses for a given intensity are represented by all the points on this Poincare sphere. The poles are the two circular situations, the equator are the linear cases, and all the other points are ellipses. The upper hemisphere is one rotation direction, the lower the other. The north pole is RHC.

The Stokes parameters are also called I, Q, U, V in the same order, where I of course reminds us of intensity.

As B&W mention on page 630, when you deal with real light sources which have incoherence, you need to replace items in the Stokes expressions with time averages <..>. The net effect is to create a true fourth parameter which is the degree of polarization p: p=0 for unpolarized light, and p=1 for fully polarized light, and in between you have partially polarized light. With this effect in included, the radius of the Poincare sphere is shrunk from I to I*p, as shown on the wiki page.

Now that we have the basic idea, we go back to Newton. Recall that we are in the ω domain all the time here, so E fields are complex. If we write these in terms of our δ_1 and δ_2 of B&W, we see that the U and V parameters duplicate the 2 cosine form of B&W apart from a - sign perhaps. There is ambiguity

somewhat in our definition of phases, perhaps some $\pi/2$ offset for example. Notice that the || and $\perp$ unit vectors are real.

He then restates things using a circular polarized basis (where now the unit vectors are complex) and we see in terms of these amplitudes the Stokes parameters have sort of shuffled around so V is now the difference of the two selected basis intensities. As the wiki page tries to show, the Q parameter has the simple difference form in the x,y basis, while the V has the difference form in the r,ℓ basis, and the U has the simple difference form in the x', y' basis which are just rotated 45 degrees from x,y.

From the closing comments of this section, we see that Newton will use the L and C bases as needed, and has yet another complicated looking capital Greek letter to represent the electric field. The visible arguments are of course the k vector, and the point in space r. I have no problem with this, and end of section!

Doubtless there is some interesting group-theoretic or topological aspect to the Poincare idea and the four parameters, related to changing coordinate systems.

1.2.2 Significance of the Parameters.

We are reminded that Stokes parameter I is related to field average energy density (which unfortunately is called U-bar, confusing this with Stokes U), and also with the magnitude of the Poynting vector. Then Newton starts a discussion much like B&W, first writing the most general lincom of E fields, but in doing so instead of using a_1 and a_2, he uses $E_o\sin\beta$ and $E_o\cos B$. Oddly, he sets both phases the same, α, which seems to be a loss of generality. The basis axes 1 and 2 here are shown in the picture. This restriction allows a simple expo form (1.20). He then writes out the || and $\perp$ E fields in terms of E_0, β and χ, where this last is the ellipse angle between the 1,2 and the ||, $\perp$ coordinates. He then computes the four Stokes parameters, and also the separate || and $\perp$ intensities. Looking at the Q,U,V expressions, we see the 3-vector of Poincare, but this construct is not mentioned. He then inverts to express E_0 χ and β in terms of the four Stokes. The motivation is to what the Stokes parameters do to the two angles. We learn for example that V affects only β, which is the eccentricity of the ellipse, its degree of flatness. Both U and Q affect the angle of the ellipse. Sign of V determines sign of β which it is claimed determines the direction of rotation -- changing β sign reflects the ellipse on its axis, so a rotation would be reversed. Looking at the phasor representation, with its overall minus sign, you see that positive β (and V) implies CW rotation as t increases and this means RH polarized since the e_3 vector goes into the page.

Newton then starts over using the r and ℓ basis instead of the || and $\perp$ basis. He replaces angle B with angle γ, same equal phases α. Again he computes the four Stokes in (1.26). This time, however, he defines the circulars in terms of 1 and 2, whereas 2 pages earlier he defined them in terms of || and $\perp$, so now he needs new notation like χ_{01} . But he draws no conclusions in this basis. $\cos\gamma$ is the relative amount of one circular, and $\sin\gamma$ that of the other, the two of which you are superposing in (1.24). No doubt he has written these here because he will use them at some far distant point in the book.

1.2.3 Partially Polarized Beams.

The claim made in (1.28) is that when you superpose many E fields with *random phases* and average over time, the cross terms all average to zero and you end up with only the diagonal terms where, for example, we know that $\langle\cos^2(\omega t + \alpha)\rangle$ =s ome positive number like maybe 1/2. On page 7 we assumed 2 fields being added with the *same* phase α, and it happened there that we got I = sum of the two separate intensities divided by 2.

Next, Newton finally writes the actual "Stokes vector" in the Poincare sense as (Q,U,V), for each of the superposed fields which he calls wavelets. We are then supposed to add up Stokes vectors for an ensemble of random phase fields, or in fact for any set of superposed fields. If all the fields are the same, then the sums are maximal and the equality holds, but in all other cases we know $I^2 \geq$ sum of others which is the triangle inequality, sounds very reasonable to me. For unpolarized beam meaning all random phases, one would expect time-averaged Q=U=V=0, which is to say the time averaged total Stokes Poincare vector should be 0. This then defines an "unpolarized beam". [he uses the word "beam"] .

But what about $<\sqrt{Q^2 + U^2 + V^2}>$? How could this possibly be zero? This is the average Poincare vector length. Wrong! The above is really $\sqrt{Q^2 + U^2 + V^2}$ because time averaging has already been done to get things like Q. And in a random beam, Q = 0 and in fact Q=U=V = 0. In a beam that is partially random, you can compute $\sqrt{Q^2 + U^2 + V^2}$ and you will get a number between 0 and I, and that number is then defined to be pI, where p is the "degree of polarization". A beam with any amount of randomness $p < 1$ in this sense is called "incoherent". If completely random and p = 0, then it is "unpolarized".

So, for an incoherent beam with some p, we then have Q,U,V and p as four independent parameters, and you can think then of $I = \sqrt{Q^2 + U^2 + V^2}/p$ as itself being the fourth independent parameter. Now, for an incoherent beam, since you still can compute Q,U,V and I, you can still talk about a time averaged ellipse (assuming $p \neq 0$) with angles as in (1.23). And of course we can rescale the vector components down by p and then they will be on a sphere of radius pI. That is, since we know that $IP = \sqrt{Q^2 + U^2 + V^2}$, we can construct a Poincare vector (Q,U,V) as shown in (1.31) of length IP whereas earlier its length was I. So, corresponding to the average ellipse is a point on the shrunk Poincare sphere.

1.2.4 Stokes Vectors.

This term refers to all four components (I,Q,U,V) as a column vector in the usual sense. The claim is made that any optical monochromatic beam you can make can be characterized by this vector, and there are no other hidden variables. I think the claim is that if two beams have some measurably different property by any theoretical optical instrument, then they must have different Stokes vectors. Here is a web page quote:

"The principle of optical equivalence (van de Hulst, 1957) states that the Stokes parameters contain the complete set of quantities needed to characterize the intensity and state of polarization of a beam of light, in practical analysis. It follows that the transformation matrix F between the incident and scattered Stokes vectors fully describe the scattering process to the same level of detail. This principle is based on the fact that optical measurements involve linear transformations only."

I can see now why Stokes has to be mentioned in a book on scattering!

Newton writes the Stokes vector in 3 different "bases" as shown near (1.33) where the first is the official Stokes vector. He then asks what happens if you do a reference frame rotation of angle α about the k vector. This just increases angle χ, and you can mechanically calculate how this alters the Stokes vector. The result is a 4x4 matrix which is no doubt unitary and real orthogonal as a 4-space rotation. He writes the 4x4 matrix for the three different bases.

1.2.5 Relation to the Density Matrix

The density matrix is discussed in Schiff (p 378, 381), it is a QM concept. One writes $P_\alpha = |\alpha><\alpha|$ as a projection operator for state α in the Hilbert space of states for some system. If that system has only 2 states, then you can think of this as a 2x2 matrix. Our light beam has two basis states (in QM, the photon has two transverse polarization states and no longitudinal since goes speed of light, at least normally). Any projection operator is $P^2 = P$ and $tr(P) = 1$. However, when you add the statistical element, then the $P^2 = P$ is violated by amount proportional to (p-1), p = degree of polarization as in (1.45).

Basically, you can construct a little density matrix for a beam having 4 Stokes parameters as shown in (1.43) in the Linear or Circular bases. Notice that it is scaled by I as shown so the matrix itself only has 3 of the parameters. In (1.46) you see these three as p, χ and β and then I is the fourth. So this 2x2 matrix replaces the 3-vector of Poincare.

All fine by me. There is of course a group-theory relation between this stuff. I think the 2x2 matrix is the spinor 2-D representation of the rotation group, while the Poincare vector is in the 3x3 representation. If you do the same plane rotation by α, the 4x4 matrix rotation is replaced with a matrix rotation as in (1.48). This is all very familiar to me from my ancient past lifetime.

Why would you want to rotate the plane? One reason is this: suppose you look at a scattered k' vector. For that direction, you might want to define the plane in which k and k' both lie, and then you might want this to be your polarization plane of interest. As you move k' around in a circle azimuthally by α, you need to think about rotating the beam polarization by this amount if you always want to be measuring things in the k,k' plane. Newton will no doubt be mentioning this, and I remember it generally from particle scattering.

1.3 Scattering

1.3.1 The Scattering Amplitude

Newton writes a superposition of incoming plane wave, outgoing spherical wave, and incoming spherical wave, but he writes it as an integral over ω with a very narrow spectrum $f(\omega)$. He then asks what is the value of this integral as t very large and $t \rightarrow -\infty$. Using the stationary phase method, we set the derivative of each exponent to 0 to find any contributions. The plane wave is non zero at this time basically only when $z = -k|t|$, as you would expect. The incoming spherical wave is large at $r = c\,|t|$ as you would expect, and the outgoing wave has no amplitude at all. An interesting approach! The same argument says what happens at $t \rightarrow +\infty$. All this assumes large r in the usual manner. [I have stationary phase notes in section 4 of my math binder.] So in the far future, you end up with the E field being $\exp(-i\omega t)$ times the E field shown in (1.49) which is incident plus scattered outbound. [There are thousands of words one could write about this development in terms of ODE's, integral equations, boundary conditions, etc. The boy who knew too much.]

Since the outgoing wave has two possible polarization states (you could measure the intensity at each of two polarizations relative to some reference plane, for example). Equation (1.50) then defines what I would call the S matrix, and Newton calls the "scattering amplitude" for k and k'. We certainly expect the scattered wave to be proportional to E_o since all equations are linear (Maxwells), and then (1.50) defines the matrix U.

Now we come to the subject of a "reference plane". Recall the earlier discussion of the two polarization basis directions described as $\parallel$ and $\perp$. The $e_\parallel$ means parallel to a "reference plane". This is the unique plane which contains both **k** and $\mathbf{e}_\parallel$. It is not the plane that contains the ellipse, which is the unique plane perp to **k**. If k = z and $\mathbf{e}_\parallel$ = x, then the reference plane is the xz plane. This is shown just this way on page 12 where I have cross hatched the xz reference plane.

Of course you can select $\mathbf{e}_{\|}$ as any direction you want in the ellipse plane, the k-perp plane. One obvious choice is to select as your reference plane the one containing k and k'. In this case, $\mathbf{e}_{\|}$ lies in this plane. The problem with this choice is that $\mathbf{e}_{\|}$ varies as **k'** varies azimuthally which might be inconvenient experimentally.

1.3.2 The idea of a Fixed Reference Plane

This is a very confusing presentation at first, but then it makes sense. Look first at the figure. The reference plane for the incoming beam is the xz plane, as just discussed above. The reference plane for the outgoing beam is selected as a plane containing k' and the x axis (I have not cross hatched this plane), and you can see how $\mathbf{e}_{\|}'$ lies in this plane. In particular, $\mathbf{e}_{\|}'$ is parallel to the x axis and is parallel to unit vector $\mathbf{e}_{\|}$. No matter how k' is chosen, we can make this choice for $\mathbf{e}_{\|}'$. The reference plane for the outgoing beam is not always the same, but it is always a plane containing the x axis, and this allows $\mathbf{e}_{\|}' = \mathbf{e}_{\|}$ for all k'. (the outgoing $e^{\perp}$' is of course changing as k' changes.).

The "scattering plane" always means the plane containing k and k'.

Now here is an academic question: what is the angle α between the xz plane and the xz' plane? That is to say, what is the angle between the incident beam reference plane and the outgoing beam plane? A nice little geometry problem that I don't think I have ever solved.

Newton says to think of the rotation required as a composite of two separate rotations. First, take the xz plane and rotate it amount ϕ about k to get the scattering plane (seems simple enough). I would tend to think of ϕ as positive when doing this, not negative as he does. Then the second step is to rotate the scattering plane some amount ϕ' about the k' axis to get to the final reference plane. He claims ϕ' is given by (1.51).

Now, our original S matrix called U was defined with these two reference plane selections: for incoming, we use E_0 in terms of say $\mathbf{e}_{\|} = x$. For outgoing, we use $\mathbf{e}_{\|}'$ in the scattering plane. We would then refer to this matrix U as the "scattering-plane S matrix".

Now Newton confuses things a lot. His U_{fix} matrix it seems to me relates E_{scatt} as measured in the above-mentioned outgoing reference plane (which I have not drawn and which contains the x axis and k') to E_o as measured in the scattering plane (the triangle cross hatch region). This latter is called $E_{o,fix}$ for some reason. I am not sure why this matrix basis would be interesting, but I agree is it a possible basis to choose. A matrix basis choice means you need to select a reference plane for both the incoming beam and the outgoing beam. Not surprisingly, the U matrix can be moved from one basis to another by a similarity-like transformation as shown in (1.54).

As an example, suppose you start with a 2x2 matrix in the basis of input = x, output = scattering plane with the circular basis. In this case, your 2x2 matrix has components A_{ij}^{C} . If you convert this matrix to the basis where input = scattering plane output = plane through x and k', then you get (1.55). Newton is very bad here because he has never defined the A_{ij} elements!

What is confusing here is that Newton is changing both the input and output reference planes in a single discussion.

1.3.3 Comparing L and C.

Well, it seems reasonable to claim (1.56) and (1.57). We are rotating a complex 2-vector with a 2x2 matrix B^{CL} (L to C). The claim is that you can change the entire S matrix from L basis to C basis (for both in and out) by (1.58). If you assume that the L matrix form of U has components A_{ij}^{L} , then you can compute the elements of the C-basis version of the U matrix, and you get (1.59).

1.3.4 Do this all for Stokes Vectors.

In (1.60), the symbol F is the S matrix in the Stokes 4-vector basis. But remember that we use 3 different forms of the Stokes basis indicated by letters S, L and C, see (1.33) cf. For each of these three Stokes forms, Newton computes the 4x4 matrix F which describes the scattering. The parameters appearing in the matrix elements are the A_{ij} of the corresponding basis 2x2 U matrix. For the S case, the results are so messy that Newton doesn't have room to write the matrix, so he just writes expressions for all the matrix elements in a list.

The same issue of selecting reference planes of course is present. In the 4x4 world, we change between our two "standard reference plane choices" using (1.65), the analog of (1.54) where now of course we have 4x4 rotation matrices. [Perhaps our method of relating the reference planes has to do with Euler angles, it just occurs to me.]

I will stop here. I just wanted to dive into this book a little and see what it was like. It is good but extremely densepack. Of course we are in a review section here, but I suspect the entire

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