

Physics of sprinting

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Sprinting is described by a simple physical model. The model is used to predict the differences between the recorded times for races on a straight track and on a curve. It is shown that the choice of the running lane makes a non-negligible difference.

I. INTRODUCTION

In the past few years there has been a growing interest in the application of physical models to problems of animal locomotion in general^{1,2} and to human athletic performance in particular.³⁻⁵

A problem of special interest is to devise a model that represents with reasonable accuracy the propulsive and resistive forces acting on a human runner.

One common model assumes that the sum total of resistive forces F_r acting on a runner can be represented by

$$F_r = -\sigma Mv, \quad (1)$$

where M is the mass of the runner, v is the speed, and σ is a parameter of the model, presumed constant or nearly constant for a given runner.⁶ Although such a form is suggested by analogy with viscous forces, it is far from clear that it can adequately represent human runners where the forces are rapidly varying and, in general, complex. Nonetheless such a form was used by Keller³ to model competitive running and draw certain conclusions regarding optimal running strategy and maximal future performances. One significant conclusion reached by Keller is that a well-conditioned athlete is able to sustain a maximum and nearly constant muscular effort for races over distances of 290 m or less. We shall define all such races as sprints.

A basic question is: What are typical values of the parameter σ ? It is to be expected that σ will depend on the runner, his speed, the type race, and possibly on external conditions such as the track surface, altitude, etc. Keller³ uses a value of $\sigma = 0.44 \text{ sec}^{-1}$ while Whitt and Wilson⁵ quote similar values. The latter, however, were measured at walking speeds. To our knowledge there exists no measurement of σ at sprinting speeds.

We propose that for a given sprinter a measurement of σ and the propulsive force parameter f (defined below) can be performed in principle by measuring that sprinter's running times for two straight line races of different distances. Our model will then allow us to use the values of these parameters to predict that runner's performance for a sprint run on the curve.

II. MODEL

We assume as does Keller³ that a sprinter of mass M is subject to two horizontal forces. One is the resistive force expressed by Eq. (1) and the other is a constant propulsive force F_p which we write as

$$F_p = fM, \quad (2)$$

where f is the force per unit mass of the athlete.⁶ The equation of motion for the sprinter is then

$$\frac{d^2x(t)}{dt^2} = \frac{dv(t)}{dt} = f - \sigma v(t), \quad (3)$$

where $x(t)$ is the distance measured from the start of the race and $v(t)$ is the sprinter's instantaneous velocity. The solution to this equation subject to the initial conditions $x(0) = 0$, $v(0) = v_0$ is

$$v(t) = (f/\sigma)(1 - e^{-\sigma t}) + v_0 e^{-\sigma t}, \quad (4)$$

$$x(t) = (f/\sigma)t + (f/\sigma^2 - v_0/\sigma)(e^{-\sigma t} - 1). \quad (5)$$

These well-known solutions describe the motion of a particle sedimenting under the action of a constant force of "gravity" Mf and a viscous force $-\sigma Mv$; the particle (sprinter) approaches the terminal velocity $v_t = f/\sigma$ asymptotically.

Let us suppose that the times t_1 and t_2 for an athlete's performances over two different straight-line sprinting distances x_1 and x_2 are known. Then Eq. (5) can be used (with $v_0 = 0$) to solve for f and σ (see Appendix).

We give in Table I the best running times over two different distances for four world-class sprinters from the 1967-69 period.^{7,8} (We note that two of them are still holders or coholders of three separate world records.) The computed values of σ and f for each runner are shown in columns 6 and 7, respectively.

The last entry in Table I is Tommie Smith who has the distinction of having a recorded time for the straight-track 100-m race and also for the rarely run straight-track 220-yard race for which he holds the world's record. In addition he is the holder of the world's record for the 200 m on the curve which he established in Mexico City in 1968.⁷

The computed values of the parameters f and σ differ widely from runner to runner. We have no way of knowing how reliable the data are but if we assume that the computed values of f and σ for Tommie Smith are accurate, then we can extend our theory to predict his performance in a 200-m race on the curve. The closeness of our prediction to the actual recorded time will be a test of the validity of the basic assumptions of the theory embodied in Eqs. (1) and (2).

III. RUNNING THE CURVE

In a typical 200-m race the starts are staggered so that each runner runs the first 100 m of the race on the curve. However, the radius of curvature of the curved portion of each lane is different. The lanes are 1.22 m wide so that the inner radius of the n th lane is

$$R(n) = 100/\pi + (n-1)(1.22) \text{ m.}$$

Let the runner be running with speed v on a curve of

Table I. Computed values of the parameters f and σ for various runners (*indicates a current world record holder). The times are the runners best times over the given distances.^{7,8}

Runner	x_1	$t_1(\text{sec})$	x_2	$t_2(\text{sec})$	$\sigma(\text{sec}^{-1})$	$f(\text{N/kg})$	$v_t(\text{m/sec})$	$ \delta\sigma/\sigma $	$ \delta f/f $	$ \delta v_t/v_t $
John Carlos	60 yards	6.0	100 yards	9.0	0.667	8.13	12.19	13%	10%	3%
Bill Gaines	60 yards	5.9	100 yards	9.3	1.250	13.45	10.76	25%	23%	3%
Jim Hines*	100 yards	9.2	100 m	9.9	0.581	7.10	12.22	65%	51%	14%
Tommie Smith*	100 m	10.1	220 yards (straight)	19.51	1.252	13.46	10.75	19%	18%	1%

constant radius R and subject to the forces (1) and (2). Now, however, the propulsive force must supply a component that gives rise to the centripetal acceleration of magnitude v^2/R . The athlete's equation of motion then becomes

$$dv/dt + \sigma v = (f^2 - v^4/R^2)^{1/2}. \quad (6)$$

Equation (6) is to be solved with given parameters f , σ , and R subject to the initial conditions $v(0) = 0$, $x(0) = 0$.

An approximate analytical solution can be obtained by noticing that for typical situations the second term under the square root is small, $v^4/R^2 \ll f^2$, and by expanding the square root we obtain

$$dv/dt + \sigma v \cong f - (1/2)v^4/fR^2. \quad (7)$$

As it stands, Eq. (7) is still not subject to a simple analytical solution. We can, however, solve it by treating the term in v^4 as a small perturbation (see Appendix). The solution so obtained subject to the initial conditions $v_c(0) = 0$, $x_c(0) = 0$ gives the speed $v_c(t)$ and the distance travelled along the curve $x_c(t)$ as:

$$v_c(t) = (f/\sigma)(1 - Z^{-1}) - 1/2(f/\sigma)(fR^{-1}\sigma^{-2})^2 F_1(Z), \quad (8)$$

$$x_c(t) = (f/\sigma^2)(\ln Z - 1 + Z^{-1}) - (1/2)(f/\sigma^2)(fR^{-1}\sigma^{-2})^2 F_2(Z), \quad (9)$$

where $Z = e^{\sigma t}$ and

$$F_1(Z) = 1 + (10/3 - 4\ln Z)Z^{-1} - 6Z^{-2} + 2Z^{-3} - 1/3 Z^{-4},$$

$$F_2(Z) = -37/12 + \ln Z + (2/3 + 4\ln Z)Z^{-1} + 3Z^{-2} - 2/3 Z^{-3} + 1/12 Z^{-4}.$$

The accuracy of our perturbation procedure depends on the smallness of the dimensionless parameter $\epsilon = (fR^{-1}\sigma^{-2})^2/2$ which for the case of interest to us has a value of 0.031 39. For running times on the order of 10 sec, terms in Z^{-1} and higher powers of Z^{-1} are negligible and can be dropped from the expressions for F_1 and F_2 .

With these approximations we can write expressions for $x_c(t)$ and $v_c(\infty)$, the latter being the terminal speed of the runner on the curve. We note that with our typical value of σ the sprinter attains 98% of his terminal speed within approximately 3 sec:

$$v_c(\infty) \cong (1 - \epsilon)v_t, \quad (10)$$

$$x_c(t) \cong v_t(1 - \epsilon)t - (v_t/\sigma)(1 - 37/12 \epsilon). \quad (11)$$

Equations (10) and (11) are useful in that they allow us to bring out the effect of curvature of the track represented by the parameter ϵ . Physically, 2ϵ is the square of the ratio of the centripetal acceleration v_t^2/R calculated at terminal speed to the propulsive force per unit mass f . With our pa-

rameters the difference between v_t and $v_c(\infty)$ is of the order of 0.3 m sec^{-1} . Thus the runner accelerates slightly upon entering the straightaway.¹⁰

Equations (10) and (11) allow us to compute the total time t_T for the race in a closed (approximate) form

$$t_T = t_{100} + 100/v_t + \epsilon/\sigma \\ = 2(100)/v_t + 1/\sigma + \epsilon[(100/v_t) - 13/(12\sigma)] \\ + \epsilon^2[(100/v_t) - 25/(12\sigma)] + O(\epsilon^3). \quad (12)$$

Equation (12) then can be used to predict the running time, for example, for Tommie Smith running a 200-m race on the curve. We use the parameters of Table I and assign him to lane 3, the lane in which he ran for his world record. Then $R = 34.27 \text{ m}$, $\epsilon = 0.0314$, $v_t = 10.75$. We obtain

$$t_{100} = 10.35 \text{ sec}, \\ v_{100} = 10.41 \text{ m/sec}, \\ \text{time to run the last } 100 \text{ m} = 9.33 \text{ sec}, \\ \text{total predicted time for the race} = 19.68 \text{ sec}.$$

In the Mexico City Olympics on 16 October 1968 in the men's 200-m final, Tommie Smith running in lane 3 set the still-current world record of 19.83 sec.

In order to test the accuracy of our calculation we numerically integrated the exact Eq. (6) and computed the time necessary to cover the first 100 m of the course on the curve. The comparison of this calculation with the results of our approximate analytical solution is shown in Table II. The two procedures give results that differ at most by 0.3%. This surprisingly close agreement lends support to the validity of the approximations made in the calculations. The close agreement between our "predicted" and the measured value of Tommie Smith's performance in the 200-m race on the curve comes as somewhat of a surprise considering the meagerness of the data and the crudeness of the model.

Finally, we apply our model to the following problem. Since the value of the radius of curvature R enters Eqs. (8)–(12) we would expect that athletes running in the outside lane (lane 8) would enjoy an advantage over those in the inside lane.

In fact, it is well known that sprinters dislike running in

Table II. Comparison of numerical solutions with approximate analytical solutions. Times are calculated for the first 100 m of the race on the curve; $f = 13.46 \text{ N/kg}$, $\sigma = 1.252 \text{ sec}^{-1}$.

Lane	$R(\text{m})$	$t_{100}(\text{sec})$ numerical solution	$t_{100}(\text{sec})$ approximate solution	difference (%)
1	31.83	10.361	10.389	0.27%
3	34.27	10.327	10.348	0.20%
8	40.37	10.266	10.277	0.11%
straight track	∞	10.100	10.100	

the inside lane complaining that the turn is "too tight." Our calculation bears this out. We find that, with the parameters we used for our test case, the total recorded time for a runner in lane 1 would have been 19.72 sec whereas for lane 8 it would have been 19.60 sec. At finishing speeds of about 11 m/sec that difference in time translates into a distance difference of over a meter. Since many sprint races are lost or won by mere centimeters the above effect is quite significant.

Curiously enough sprinters do not consider lane 8 to be the most advantageous one. Because of the staggered start runners in lane 8 run for half the race ahead of the other competitors. This, they claim, puts them at the psychological disadvantage of not being able to see what their competition is doing.

IV. ERRORS

We wish to make a few brief comments about the "experimental" errors. Since neither the records books nor the sport press include error bars with their reported results we will limit ourselves to a few crude estimates. We assume that most of the error is the result of inaccurate time measurements. Many of the times given in Table I were probably the averages of several hand-recorded times. Other times might have been recorded electronically and then rounded off to the nearest tenth of a second.

For the purposes of this estimate, we assume that the accuracy in the recorded times is ± 0.05 sec. We notice that the resulting variation in the calculated values of f and σ are quite large. This is due in a large part to the fact that the calculation involves a denominator which is a difference between two numbers numerically close to one another. One consequence of this effect is that data for races over similar distances such as 50 and 60 yards or 100 yards and 100 m are practically useless.

However, as is apparent from Eq. (12), the total time t_T depends primarily on v , so to some degree the errors in f and σ tend to cancel one another. The uncertainties included in Table I were calculated according to the assumption $\Delta t_1 = -\Delta t_2 = \pm 0.05$ sec. Assuming this error in the measurement of time the resulting error in the predicted t_T for T. Smith is

$$\Delta t_T = \pm 0.054 \text{ sec.}$$

A less conservative error estimate of $\Delta t_1 = -\Delta t_2 = \pm 0.11$ sec brings the predicted and observed t_T within the range of "experimental" error.

V. CONCLUSIONS

In summary we have found that the forces acting on a sprinter at typical sprinting speeds can be reasonably well modeled by Eq. (3). The use of that equation over two different distances can determine the sprinter's parameters f and σ . The values of the parameters that we calculate for a few sample cases are in some disagreement with those commonly found in the literature. Our model, extended to include centripetal effects, then allows us to predict an athlete's time for a race run on the curve. This in turn can be used to compute the difference in the times for two comparable athletes running in different lanes. This difference is solely due to the different radii of curvature of

their respective lanes. We find its magnitude to be non-negligible.

Clearly our results suggest a line of further investigation for those interested in the biomechanics of sports. To the extent that a sprinting human can be characterized by parameters f and σ it might be of interest to measure these parameters under carefully controlled conditions and correlate them with the physical characteristics of the sprinters, such as height, weight, type of build, reaction time, strength of leg muscle, etc. It is conceivable that these parameters f and σ might also depend on external conditions such as track surface, altitude, and other factors.

Finally, we wish to comment that it has been our experience that there is a fairly large audience among today's college students interested in learning about the biomechanics of sports from the point of view of fairly sophisticated physics. Such a trend is certainly indicated by the amount of research done in many countries in the area of sports medicine and technique.

APPENDIX

In order to determine the parameters f and σ for a given sprinter, let us assume that the sprinter has recorded times t_1 and t_2 over two straightaway distances of lengths x_1 and x_2 . Using Eq. (5) twice with $v_0 = 0$ and neglecting the exponential term we solve for f and σ to obtain

$$\sigma = (1/t_1)[(x_2/x_1) - 1]/[(x_2/x_1) - (t_2/t_1)], \quad (13)$$

$$f = \sigma^2 x_2 / (\sigma t_2 - 1). \quad (14)$$

In solving Eq. (7) we proceed as follows. First, we solve it with the v^4 term set equal to zero. This solution which we call w is the same as Eq. (4) with $v_0 = 0$,

$$w(t) = (f/\sigma)(1 - e^{-\sigma t}).$$

Next, we write the solution to Eq. (7) as $v = w + u$ where we treat u as a small perturbation. u then satisfies

$$du/dt + \sigma u = -1/2 v^4 / (fR^2). \quad (15)$$

Replacing v in Eq. (15) by its approximation w , we integrate twice to obtain the solutions (8) and (9).

For our test case (Tommie Smith) the parameters f and σ are directly determined from Eqs. (13) and (14) using the data given in Table I.

Equations (8) and (9) can be further simplified if we notice that for our test case $\sigma \simeq 1.25 \text{ sec}^{-1}$ and for running times of the order of 10 sec, $Z^{-1} \sim 10^{-6}$ so that F_1 and F_2 can be approximated by

$$F_1 \simeq 1, F_2 \simeq t\sigma - 37/12.$$

This gives Eqs. (10)–(12). These equations then predict the time to complete the curve portion of the race t_{100} , the speed at the end of 100 m, and the total time.

For comparison we find that, using the same values of f and σ , a 200-m race run entirely on the straightaway would have taken 19.40 sec as compared with our lane 3 calculated time 19.68 sec. Even for a runner in lane 8 the straightaway time is faster by 0.20 sec.

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⁶We use mks units throughout so that $M = \text{kg}$, $x = \text{m}$, $v = \text{m/sec}$, F

$= \text{N}$, $f = \text{N/kg}$, and $\sigma = \text{sec}^{-1}$.

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⁹Personal observation (IA).

¹⁰We wish to thank an anonymous referee for pointing this out to us.