

Books on GROUP THEORY

Phil Lucht

Phil's Short Bibliography of Group Theory

1. Wybourne (1974): Classical Groups for Physicists.

This 400 page, well-printed, well-written book is a very thorough treatment of Lie Groups. There is much discussion of weights, Dynkin diagrams, Cartan matrices. There is a special section on the 3-parameter groups which show up a lot in physics. The text concludes with three case studies: the isotropic HO , the hydrogen atom, and shell structure.

What is not found in this book are: (1) details of the Poincare and restricted Lorentz groups. (2) details of $SU(3)$.

2. Lyubarskii (1960): Applications of Group Theory in Physics.

This 380 page Russian translation starts out with the finite groups and all the basic group theorems. However, it does have 50 pages specifically on the representations of the Lorentz group and the spinor and tensor algebras. Also, discussion of invariant wave equations. The book is exceedingly dense in print, but good enough that Weinberg references it alone for this Lorentz subject.

3. Ruhl (1970): the Lorentz Group and Harmonic Analysis.

300 pages and much more specialized than the above texts, this book has long sections on the representations of $SL(2, C)$. Somehow, the accent of this book seems to be functions of group spaces, Plancherel measures, and stuff like one would find in a Teller paper. Sort of the generalized Fourier/Laplace business. In fact, the cap of the book is a chapter on Regge and Teller poles.

There is nothing in this book about Lie Algebras or $SU(3)$, although dotted and undotted spinors do get mentioned. Not a particularly good place to start.

4. Naimark (1964): Linear Representations of the Lorentz Group.

A translation from the Russian of Naimark's 1958 text. 450 pages, good printing. Lots of stuff on spinor representations and integral business. Concludes with discussion of Invariant Equations. This book has too many proofs for my liking. Too specialized.

5. Gelfand Minlos Shapiro (1963): Reps of the Rot and Lor groups with Applica.

Another 1958 Russian text translation. Lots and lots of talk about spinors in both the Rot and Lor groups, and invariant equations. This book was recommended by Mahiko. The authors refer to Naimark's book as being more detailed. I agree. Looks good.

6. Lichtenberg (1970): Unitary Sym and Elementary Particles.

230 pages, well organized and printed. Opens with comments on Lie Groups and algebras. A lot on the Young Tableaux methods for describing multiplets. Does the $SU(3)$ Clebsch's, discusses 8-fold way, then chapter on SU_6 . Closes with discussion of quark models. I like. ✓

7. Hammermesh (1962): Group Theory.

A nice chapter on Lie groups, but beyond that it is all basics. There is a small piece about Lorentz way at the end. Do not bother with this book, though it is an excellent group theory text in general.

8. Lipkin (1966): Lie Groups for pedestrians.

Not very organized, but has higher groups mentioned like SU_4 and SU_6 . There is a piece on the Young stuff, and mention of charm. 180 pages. Obviously no mention of the Lorentz group.

Caruthers?

Belinfante + Kolman

Naimark "Linear Reps. of L.G."

Portzugin

Loebel

Helgason (1962)

Caruthers (1966)

Lichtenberg (1970)

Gordin (1967)

Rühl (1970)

phil's biblis

books to remember.

1. "Lie Group Surveys w. Computational Methods"
by Belinfante + Korman 1972 SIAM (QA251 B43)

18,19,20

18,19,20: I am just giving the references for these books here.

18. L.O. Raifeartaigh : Lectures on Local Lie Groups and their Representations
Matscience Report 25, India. QA385 .63, 1964 Inst. of Math. Studies, Madras-20, I.

-- Well written notes. Goes into some of the parameter stuff that Bargmann uses and
-- this was very helpful. He also does root diagrams etc with a special section on the
-- construction of invariants. List of other Matscines given on back cover..

19. "Topological Groups" L. Pontrjagin (Moscow) Princeton U Press, 1939
QA385 P6 1939.

This older book is considered one of the standard references. Contents:

- a) abstract groups
- b) topological spaces
- c) topological groups
- d) representations of compact topological groups
- e) abelian cases
- f) concept of a lie group
- g) structure of compact topological groups
- h) locally isomorphic groups (and universal covering groups)
- i) structure of Lie Groups.

-- 20. Loebl: Group Theory and Its Applications, III - - QA 171 L62 v. 3 PHYS

-- This has an article by Windternitz on two variable expansions and reduction
-- of scattering amplitudes to the ultimately dynamic "Lorentz amplitudes:". Seems
-- rather complicated.

book: { Diff. Geo. and Sym. Spaces.
Sigmundur Helgason (MIT)
Aca. Press 1962 [Pure & App. Math. Series].

contents:	1. elem. diff. geo.	(85)
	2. lie groups & algebras	(40)
	3. SSLA's	(30)
	4. symmetric spaces	(30)
	5. decomposition of $\uparrow$	(33)
	6. compact SS.	
	7. noncompact SS.	
	8. herm. SS.	
	9. classifying SS.	
	10. facts on SS.	

← Hansh Chandra here.

What is the Harish-Chandra formula? Helgason

- ① Def: "symmetric pair" (G, H) . $H = \text{subgroup of } G$. Let $\sigma: G \rightarrow G$. Then some elements of G will stay put under σ ; call these H_σ . One of these is the identity $(H_\sigma)_0$. Then (G, H) is symmetric pair if $(H_\sigma)_0 \subset H \subset H_\sigma$. If $\text{Ad}_G(H)$ is compact, then "Riemannian sym pair". (p174)

- ② Def: $D(G/H)$ = a set of differential operators which are invariant under the action of G , + which act on G/H . So this set includes Casimir, I would say.

Theorem 2.3 (p393) $X \in \mathfrak{g}(L.A.)$, $D \in D(G)$ then

$$[D, X] = 0 \Leftrightarrow D^{R(e^{tX})} = D \quad \forall t \in \mathbb{R}.$$

↑ right shift by e^{tX}

Seems reasonable

Corollary 2.8 (p395) Any $D \in D(G/H)$ can be expressed as

$$D = \sum a_{n_1, n_2, \dots} D_1^{n_1} D_2^{n_2} \dots D_l^{n_l}.$$

if these, ...

Theorem 2.9 (p396) The elements of $D(G/H)$ commute!

Prop 2.10 (397). Only the Laplace-Beltrami Δ is invariant....

- ③ Def: spherical function $K = \text{compact}$



If $f = \text{function on } G/K$.

Then $\tilde{f} = f \circ \pi$ denotes an obvious function on G .

Let $\phi : G/K \rightarrow \mathbb{C}$, $\mathbb{C}^\infty$. Assume $\phi[\pi(e)] = \phi[0] = 1$.

Assume $\phi^{r(k)} = \phi \quad \forall k \in K$
 $\uparrow$ either left or right shifts.

Assume $D\phi = \lambda_0 \phi \quad \forall D \text{ in } D(G/K)$.

Then ϕ is a spherical function.

OR, in terms of $\tilde{\phi}$: $\tilde{\phi} = \text{eigenfunction of all } D_0(G)$
 $\tilde{\phi}(e) = 1$
 $\tilde{\phi}(h g k') = \tilde{\phi}(g)$

prop 3.2 (p399) Let $f: G \rightarrow \mathbb{C}$. f is a spherical if

$$\int_K dk \quad f(xky) = f(x)f(y) \quad \forall x, y \in G.$$

$\uparrow$
measure on K .

Claim: spherical fctns (Thm 6.5) are precisely the fctns of

the form
$$\phi(g) = \int_K dk \quad \chi(\tilde{g}'k)$$

where $\chi = \text{character of finite dim rep } \pi, i.e$

$$\chi(\tilde{g}'k) = \text{tr } \pi(\tilde{g}'k).$$

Thm 6.16 [Harish-Chandra] Let $\rho = \frac{1}{2} \sum_{\alpha \in P_+} \alpha$ (simple roots).

Let $\nu: H_{P_0} \rightarrow \mathbb{C}$, where H_{P_0} has some connection to Cartan s.a. arbitrary. or Iwasawa D .

Then you exhaust the spherical fctns by

$$\phi_\nu(x) = \int_K dk \quad \exp[(i\nu - \rho)H(xk)]$$

This generalizes some result to non-compact groups.

Carruthers: "Introduction to Unitary Symmetry"

ref: QC721 C29 1966 Interscience Tracts.

Contents:

1. $SU(2)$.
2. $SU(3)$. Cartan Weyl standard form. D and F stuff. Weyl group of reflections. Regular rep = generator rep.
3. The IRs. Discusses direct products, Casimirs, and relevance to particles.
4. Properties of the IRs. Clebsch's. Tensor analysis.
5. $SU(3)$ invariant vertices and amplitudes. Ratios.
6. Breaking. Both kinds of mass formulas. Magnetic moments.
7. Crossing symmetry. Isospin crossing matrices. $SU(3)$ crossing matrices!

Appendix has isoscalar factors.

~~Added note on currents.~~

This book is number 27 in a series of monographs in physics. Some of the other ones are of some interest. One on optical models.

book: Unitary Symmetry and Elementary Particles

D.B. Lichtenberg AP (1970) QC721 L486 250 pages

Contents:

(number of pages in chapter)

1. Introduction	10 ✓
2. Properties of Groups	12
3. Symmetry, Reps, Multiplets	14
4. The Symmetric Group and Identical Particles	12 ✓
5. Lie groups and algebras	18
6. Multiplets	23
7. Young Tableaux and Unitary Symmetry	24 ✓ copy
8. Clebsh's	36 ✓ copy
9. The Eightfold way	32
10. The SU(6)	11
11. The Quark Model	32
12. Variants of quark model	10

1. Introduction. An overview with some interesting facts.

a) Jacobi, Schütz, Herglotz in 1842, 1897, 1911 did invariances of class. lagrangia

b) Wigner showed that sym trans are either unitary or anti. Only the unitary ones have conservation laws; thus, no law for T, or TCP.

2. Groups. All old hat. Some facts that are in Tinkham; mention of S_n groups.

3. Grab Bag. Vector space of QM, anilinears, antiunitaries; group representations. Old

4.4. Symmetric Group. Short deal on Young Tables. See copy, see notes.

5. Lie algebras. All this stuff is done in Wybourne. Cartan Weyl standard form. Licht does better by distinguishing root vectors from roots, otherwise same old thing. 30

6. Multiplets. The generators, D and F, d and f, etc. Weight theory. L calls the highest weight "dominant weight." Highest weight p q sum theorem. SU(3) multiplets and tensor operators. Uses m for weight. Again, all known stuff, do not read this.

7. Young Tableaux and SU(n). Very nice stuff. See copy and notes. Class and triality!

8. Clebsch's. Nice, uses Youngs. Lots of tables, Wigner Eckart. See copy and notes.

9. 8Fold Way. The basic multiplets of particles. U-spin section, the u_3 Q axes.

L does not claim EM is a perfect U singlet, just to the extent that U is good for strongs. Does all the standard U-spin predictions: mass splkts, Coleman Glashow 1961 law, reaction-rate ratios, photoproduction ratios, photon decays.

Photon is supposed to be a U singlet, hence ratios easy to get. Then GellMann

Okubos. The invariant D and F couplings written out in full, no formalism chosen.

• Will copy page 178-9.

book: Unitary Symmetries
Gourdin, Wiley, 1967

QC 721 G65 300 pages

Contents:

1. The $SU(3)$ group.	25
2. Physical Interpretations	7
3. Particle Classification	12
4. Strong Interactions	35
5. Electromagnetics	24
6. Weaks	26
7. Nonleptonic weaks	12
8. The $SU(6)$ group.	9
9. The $SU(6)$ model.	15
10. Lie Algebra of Currents	20
11. Physical Conseq's of $SU(6)$	9
12. Lie groups and algebras	7
13. Lie transformation groups	30
14. Topologicals	9
15. Lie algebras of the semisimples	19
16. Representations	21
17. Tensor algebra of the linear group	12
appendix : isometries	5

~~If I have more time to spend, this would be a good place to spend it.~~

1964

Tinkham

1. Rearrangement Theorem: group rearranged by left or right multiply by fixed element. Leads to Haar invariant measure for continuous group.
2. A Subgroup and its cosets (either left or right) partition a group whose order is h . Thus, since order of cosets = order of subgroup = g , you must have $h/g = \text{integer}$.
3. If A is fixed, and X runs over group, then XAX^{-1} generates all elements of the class containing A . Elements of same class are called conjugates. Identity in its own private class. Classes always partition the group.
4. Great Orthogonality Theorem: the matrices for the irreducible reps are orthogonal.
5. Dimensionality Theorem: sum of $l_i^2 = h$, the order, where l_i = dimensionality of representation i .
6. Character is trace of matrix. Characters of two different reps are orthogonal. If two group elements are in the same class, they have the same character. Thus, the rows of a character table are orthogonal when weighted with class multiplicity.
7. Second-Orthogonality Theorem: the different columns of character table are orthogonal.
8. Reduction: to figure out how a rep reduces, realize that the character of the reducible rep must be sum of characters in the direct sum. This is how you do it.
9. Celebrated Theorem: The regular rep, $l_{\text{reg}} = h$, contains each UIR l_i times.
10. Number of UIR's is equal to the number of classes.
11. Abelian groups: obviously each element is in its own class, so h = number of UIR's, so each UIR has dimension 1.

Review of Tinkham's Chapter Two:

"Abstract Group Theory"

2.1 Some definitions

"group" $h \equiv$ order of group

"abelian"

2.2 Examples

integers

covering group of a physical object

$GL(n, C)$

D_6^+ = cover of equilat. triangle. $h=6$

shows a two-dim matrix rep, and the mult. table.

2.3 "Rearrangement Theorem"

If you take your group elements $(E, A_1, A_2, \dots, A_h)$ and right (left) mult. by some R , you simply reorder the group. That is why each el. appears once in a row of the mult. table. Later, this theorem will have a continuum interp:

$$\int_G dg = \int_G d(g'g) \quad g' \in G.$$

2.4 "Cyclic Groups"

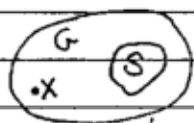
Pick $X \in G$ + form the set $X, X^2, X^3, \dots$. For finite group, this series must eventually repeat. I.e., for some n ,

$$X^n = X \quad X^{n+1} = X^2, \text{ etc.} \Rightarrow X^{n-1} = E$$

This set is a group, clearly. It is a subgroup of G . It is an abelian subgroup.

2.5 Subgroups + Cosets

Let $d \equiv$ a subgroup of G . Let $X \in G$ but $X \notin d$.



Then $L_x = Xd \equiv$ The left coset
 $R_x = dX \equiv$ The right coset.

show cosets. $\rightarrow$



Both cosets lie outside S in G :

proof: if $S_1 = S_2 X$ were in $\mathcal{A}$, then $X = S_2^{-1} S_1$ would have to be in S , contradiction $\Rightarrow S_1$ not in $\mathcal{A}$.

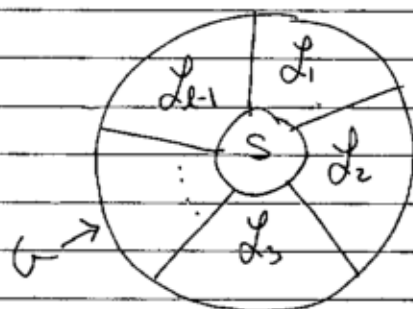
(*) If cosets L_x and L_y have one common el; then $L_x = L_y$ completely.

proof: $XS_1 = YS_2 =$ the common element, assume. Then

$$Y^{-1}X = S_2 S_1^{-1} \in \mathcal{A} \Rightarrow Y^{-1}X \mathcal{A} = \mathcal{A} \text{ by Re. Thm.} \Rightarrow X \mathcal{A} = Y \mathcal{A} \quad \square$$

Since $E \in \mathcal{A}$, $XE = X$ is element of R_x and L_x

Significance of Cosets: we know that every group element lies either in $\mathcal{A}$, the subgroup, or in some left coset. [eg, $X \in X\mathcal{A}$ for sure]. From (*) above, we know that the left cosets partition the space $G - S$.



I.e., cosets cannot overlap, yet they must exhaust $G - S$.

Also, if S is of order g , then each coset has the same order.

Suppose there are $l-1$ cosets. Then $g + (l-1)g = h$

$$\Rightarrow g = h/l \quad \text{or} \quad l = h/g$$

We have proved that: "The order of a subgroup must integrally divide the order of the group!"

2.6 More finite groups

$h=1$: only $\{E\} = C_1$

$h=2$: only $\{E, A\} = C_2$

$h=3$: only $\{E, A, A^2\} = C_3$

The cyclic group is C_n

$\leftarrow$ so all abelian; no matrix reps possible.

$h=4$: two groups are possible: C_4 and D_4
 $D_4 = \text{Viereregruppe} = \text{covering group of a domino brick}$

$h = \text{prime number}$: $h = 1, 2, 3, 5, 7, 11, 13, 17, 19, 23 \dots$

[For $h = \text{prime}$, only C_h is allowed group: proof: if $\exists G \neq C_h$, then G contains a subgroup cyclic $S \neq G$. (The period of any element). But this subgroup's order cannot integrally divide h because $h = \text{prime}$. Thus, subgroup cannot exist $\neq C_h$, $\Rightarrow G = C_h$.

$h = n! = 1, 2, 6, 24, 120, 720$. For such numbers you can always construct a "permutation group". Eg, $h=6 = 3!$

$E = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$ etc. Turns out that $P_{31} = D_6$ of triangle.

2.7 Conjugate Elements and Class Structure

"A conjugate to B" means $A = XBX^{-1}$ for some $X \in G$.

The set of all elements conjugate to B is called ^{the} "class of B"

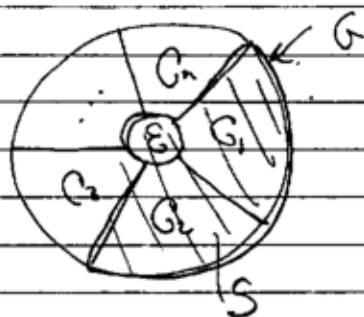
$$B = XBX^{-1} \quad X \in G$$

Note that $E = \text{class of identity} = \{E\} + \text{that's it.}$

Also,

XBX^{-1} cannot = E unless $B = E$.

So, the classes G_i partition a group:



Suppose a subgroup consists of complete classes (red). I.e., no class is chopped up.

Then: $X\mathcal{A}X^{-1} = \mathcal{A}$ as $X \in G$

I.e., the classes within $\mathcal{A}$ simply regenerate themselves, so $\mathcal{A}$ regenerates itself.

Better: if $S_1 \in \mathcal{A}$, then

$XS_1X^{-1} = S_2 \in \mathcal{A}$ for all $X \in G$.

Such a special subgroup is called "invariant subgroup" or "normal divisor".

Think of an operator $f_x: \mathcal{A} \rightarrow \mathcal{A}$ Eq,

$$f_x S_1 = S_2 = X S_1 X^{-1}$$

For any $x \in G$, $f_x: \mathcal{A} \rightarrow \mathcal{A}$ (f_x maps $\mathcal{A}$ into $\mathcal{A}$) if $\mathcal{A}$ is an invariant subgroup.

Suppose we define "commutator" between two group elements.

$$[A, B] \equiv B^{-1} A^{-1} B A$$

Consider: $[X, S_1] = S_1^{-1} (X^{-1} S_1 X) = S_1^{-1} S_2 = S_3$

For $S_1 \in \mathcal{A}$, $X \in \mathcal{A}$, $[X, S_1] \in \mathcal{A}$. I.e., $[X, \mathcal{A}] \in \mathcal{A}$. $[X, S_1] = S_2$.

→ With this def. of "commutator", we take an original group elements and form a special lie algebra. If $\mathcal{A}$ is an invariant subgroup of G , then the elements of $\mathcal{A}$ form an "invariant subalgebra" of the lie algebra $\mathfrak{G}$ of G .

Also known as an ideal.

2.8 Normal Divisors + Factor Groups

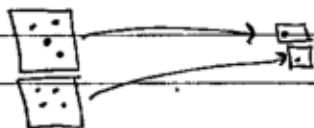
See above discussion. Note:

In abelian group, all elements are in their own classes. A subgroup of an abelian group is always "invariant", therefore, since it cannot help but contain complete classes.

Note: $\mathcal{A} = \text{invar sg} \Rightarrow X \mathcal{A} X^{-1} = \mathcal{A} \Rightarrow X \mathcal{A} = \mathcal{A} X \Rightarrow \mathcal{L}_X = \mathcal{R}_X$ for any $X \in \mathcal{A}$. No distinction between left and right cosets of this kind of subgroup.

The set of objects $\mathcal{A} = \text{invar sg}$ and $\{K_i\}$. The set of all cosets of $\mathcal{A}$, form a group at higher level: The factor group with respect to $\mathcal{A}$. To get groups, we need idea that $K_i = \mathcal{A}_i = \mathcal{R}_i$!

Finally, he mentions homomorphism as many to one. A group $\mathcal{G}$ of order h with I.S. $\mathcal{A}$ is g -to-1 homomorphic to its factor group:



All groups are homomorphic to the identity group $\{E\}$, eg.

2.9 Class Multiplication

Claim: $\bar{X}GX = G$ in the exact sense that every element occurs once on the $\bar{X}$ as the sum over elements of G .

proof: each element of G is mapped into a different element, thus the range is exhausted. Assume not true, then:

$$\begin{aligned} \bar{X}G_1X &= C_a \\ \bar{X}G_2X &= C_a \text{ also} \end{aligned} \Rightarrow G_1 = G_2, \text{ BT contradicts assumption}$$

Now converse theorem: if $\bar{X}RX = A$ for any $X \in \mathcal{G}$, then $A =$ a set of complete classes, not nec. just one class.

Consider: $G_i G_j = \bar{X} (G_i G_j) X$. Thus, the product of two classes is a complete set of classes.

$$G_i G_j = \sum_k c_{ijk} G_k \quad \checkmark$$

Tinkham, Chap. 3 "Theory of Group Representations"

3.1 Definitions

A set of matrices is a "faithful or true" rep of G if homomorphic to G .
Given any matrix rep, the set of determinants of those matrices forms another rep.

All ones is the "identical rep."

If you start with a rep: $\{\Gamma(g)\}$ and define $\Gamma'(g) = \tilde{S}^{-1} \Gamma(g) S$, a "similarity", then Γ' also forms a rep — an "equivalent" rep.

"Block form" is achievable by a similarity if a rep is "reducible."]

3.2 The Orthogonality Theorems

first lemma: given a rep whose matrices are all nonsingular, you can always find an equivalent unitary rep.

sublemma: a hermitean matrix is always diagonalizable, and the diag'ing xformation will be unitary. ✓

Explicitly, if you start with a rep A_i , the equivalent unitary rep is:

$$A_i'' = \Lambda^{-1/2} U^{-1} A_i U \Lambda^{1/2}$$

where $\Lambda^{1/2} = \begin{pmatrix} \lambda_1^{1/2} & & \\ & \ddots & \\ & & \lambda_n^{1/2} \end{pmatrix}$, eigenvalues of $H \propto A_i A_i^\dagger$

Notice the use of rearrangement theorem:

$$\sum_i (R A_i) (R A_i)^\dagger = \sum_i B_i B_i^\dagger = \sum_i A_i A_i^\dagger$$

The only reason this works is because the sum ranges over all elements of a group, each term appears once whether or not you pre-multiply by R .

This is why he chose $H = \sum_{all\ i} A_i A_i^\dagger$.

second lemma (Schur): if M commutes with all matrices of an irred rep, it is const $\times$ identity.

So if $[M, A_i] = 0 \forall A_i$ with M not const $\times I$, then A_i is not an irreducible rep.

third lemma: technical detail.

[great orthogonality theorem:

$$\sum_R \Gamma_{\alpha\beta}^{(i)*}(R) \Gamma_{\gamma\delta}^{(j)}(R) = \frac{h}{l_i} \delta_{ij} \delta_{\alpha\gamma} \delta_{\beta\delta} \leftarrow \text{for irreducible unitary reps.}$$

First of all, if $i \neq j$ you get zero, so must be same rep.

$$\sum_R \Gamma_{\alpha\beta}^{(i)*}(R) \Gamma_{\alpha\beta}^{(i)}(R) = \frac{h}{l_i} \quad \text{no sum on } \alpha, \beta \text{ etc.}$$

$$\Rightarrow \sum_R |\Gamma_{\alpha\beta}^{(i)}(R)|^2 = \frac{h}{l_i} \quad \text{dimensionality}$$

Pick one element, $\alpha\beta$. Sum over all $|\text{entries}|^2$ + you must get this number.

The dimensionality theorem

$$\left[\sum_i l_i^2 = h \right]$$

Think of $\Gamma_{\alpha\beta}^{(i)}$ as $\sum l_i^2$ vectors in a space whose inner product is

$$\langle i\alpha\beta | i'\alpha'\beta' \rangle = \sum_{R=1}^h \Gamma_{\alpha\beta}^{(i)*}(R) \Gamma_{\alpha'\beta'}^{(i')}(R) = \delta \cdot \delta \cdot \delta.$$

These vectors span the space

3.3 The Character of a Representation

"character" of a particular matrix = its trace.

[All elements of a class have the same character]

From the G.O.T. we quickly can show that the characters of the reps. form orthogonal vectors in group space:

$$\sum_R \chi^{(i)}(R)^* \chi^{(j)}(R) = h \delta_{ij}$$

Letting k label the classes: $\sum_k \chi^{(i)}(G_k)^* \chi^{(j)}(G_k) \cdot N_k = h \delta_{ij}$

Again, these K vectors ($K = \#$ classes) span the K -dim vector space [though we have not proved it]. But each vector is an I.R. Thus:

[# of irred. reps = # classes] !
 $\swarrow$ # in class G_3

["character tables" :

	G_1	$3G_2$	$2G_3$
$\chi^{(1)}$	+1	+1	+1
$\chi^{(2)}$	+1	-1	+1
$\chi^{(3)}$	2	0	-1

[We have already proved that rows are orthogonal when weighted with the class multiplicity.

["second orthogonality" shows that columns are also orthogonal, no weighting needed. This is (3-17)

3.4 Construction of Character Tables:

Notice that first entry G_1 is the "identity class". The characters in this column are simply 1's.

Top row is always all 1's.

We simply restate all the rules and show that you can usually do all entries by inspection.

3.5 Decomposition of Reducible Reps.

First rule: if $\Gamma = \Gamma^{(1)} \oplus \Gamma^{(2)} \oplus \Gamma^{(3)} \oplus \Gamma^{(3)}$ then $\chi(\Gamma) = \chi(\Gamma^{(1)}) + \chi(\Gamma^{(2)}) + 2\chi(\Gamma^{(3)})$

$$\chi(R) = \sum_i a_i \chi^{(i)}(R)$$

]

Suppose you have $\Gamma^{\text{red.}}$ and its characters, $\chi(R)$. To compute how many times each I.R. occurs, use:

$$\sum_R \chi(R) \chi^{(i)*}(R) = \text{you can compute this}$$

$$= \sum_i a_i \underbrace{\sum_R \chi^{(i)}(R) \chi^{(i)*}(R)}_{h \delta_{ii}} = h a_i$$

Thus, knowing only the characters, you can trivially compute all a_i !

$$a_i = h^{-1} \sum_k N_k \chi^{(i)}(G_k) \chi(G_k) \quad]$$

↑

example of a reduction

First, form a set of matrices such that

$$\Gamma(R)_{kj} = \begin{cases} 1 & \text{if } R = \bar{A}_k A_j \\ 0 & \text{otherwise} \end{cases}$$

This set does form a rep called "regular".

- a) $\dim = h$
- b) $\chi(E) = h$ $\chi(R \neq E) = 0$
- c) it is reducible

["celebrated theorem" : in fact, the reg. rep contains all your I.R.'s. If $\Gamma^{(i)}$ has $\dim l_i$, then $\Gamma^{(i)}$ occurs l_i times!

$$\Gamma^{(\text{reg})} = l_1 \Gamma^{(1)} \oplus l_2 \Gamma^{(2)} \oplus \dots \oplus l_n \Gamma^{(n)}$$

[bonus: This proves that $h = l_1 \times l_1 + l_2 \times l_2 + \dots = \sum_i l_i^2$ ✓

3.6 Application to Q.M.

Here, we consider R to be a particular 3×3 matrix which rotates, flips, or whatever, the coordinates $\vec{x}$

$$\vec{x}' = R \vec{x}$$

rotating the
↓
contents forward

Now define $P_R \ni$ $P_R f(x) = g(x) = f(R^{-1}x)$

is same as rotating coordinate system backwards

The set of operators $\{P_R\}$ is a realization of the group $\{R\}$, which is the set of all rotations etc which leaves Ham. invariant.

["group of S.E." = set of operators P_R s.t. $[P_R, H] = 0$.

Main idea: if ψ_n is a w.f. of state of energy E_n , then so are $P_R \psi_n$ as R runs over group. There is degeneracy.

Suppose $\text{deg} = l_n$ then (in E_n)

$$P_R \psi_i^{(n)} = \sum_j a_{ij} \psi_j^{(n)}$$

But we can write $= \sum_j \Gamma_{ij}^{(R)} \psi_j^{(n)}$

Each P_R corresponds to a matrix $\Gamma(R)$, but P_R homo. $\{R\} = G$. Thus, Γ must be a l_n dim. rep of the group of the S.E.

Theorem: if you know the group of the S.E., then you can find the reps. All wavefunctions must belong to some rep. Thus, degeneracies can only be 1, a l_2 or ... a l_n = dim. of the reps.

OK: with each eigenvalue of Ham., we may associate a unitary, irred. rep of the G. of the S.E. Unitary if basis fctns in degenerate manifold are orthonormal.

"lifting degeneracy" — add a perturbation to H . This will in general reduce G of SE to a smaller size. If the original G had l dim reps, you might have had l degenerate wavefctns somewhere. If G' does not have an l dim rep, then the degeneracy must be broken.

3.7 Example: Abelian Groups

$$\# \text{ classes} = \# \text{ els} = \# \text{ UIR's} = h$$

$$\text{To get } \sum_{i=1}^h l_i^2 = h, \text{ need all } l_i = 1.$$

Hence: no degeneracies if C_h = group of a S.E.

special case: cyclic abelian groups. Here, if $\Gamma(A) = \omega$,
some # as it must be, then Γ

$$\Gamma(E) = 1 = \Gamma(A^h) = [\Gamma(A)]^h = \omega^h \Rightarrow \omega = (1)^{1/h}$$

There are h such roots of 1, ~~they are~~ they are $e^{2\pi i p/h}$
where $p = 1, 2, \dots, h$. Thus

$$\Gamma^{(p)}(A) = e^{2\pi i p/h}$$

These h numbers form the A-column of the character table!
They are the h one-dim UIR's of A .

Block's theorem: $\Psi_p(x+a) = P_A \Psi_p(a) = \Gamma^{(p)}(A) \Psi_p(x) = e^{2\pi i p/h} \Psi_p(x)$

I.e., any wavefunction a eigenfn of $\hat{H}\Psi = E\Psi$ in a periodic potential
[$\Rightarrow$ G. of SE = cyclic group] must "belong" to some rep.
Assume $\Gamma^{(p)}$ is that rep. Then

$$\Psi_p(x+a) = e^{2\pi i p/h} \Psi_p(x)$$

Therefore $\Psi_p(x)$ may be written as: $\Psi_p(x) = e^{i(2\pi p/h)x} u_p(x)$.
where $u_p(x)$ is periodic.

Q(2) example: if your Ham has rot sym, then any
eigenfn must belong to a rep, $\Gamma^{(m)} = e^{im\phi}$. Then

$$\Psi_m(\phi + \phi_0) = P_{\phi_0} \Psi_m(\phi) = \Gamma^{(m)}(\phi_0) \Psi_m(\phi) = e^{im\phi_0} \Psi_m(\phi)$$

Thus, $\Psi_m(\phi) = e^{im\phi}$ is possibility.

3.8 Basis Functions for UIR's.

"partners" of $\phi_k^{(j)}$

The eigenvalue E_{ij} labels a subspace of dimension l_{ij} .
The "basis fctns" span this subspace.

$$P_R \phi_k^{(j)} = \sum_{\lambda=1}^{l_j} \phi_{\lambda}^{(j)} \Gamma_{\lambda k}^{(j)}(R) \leftarrow \text{note order is reversed from usual sense!}$$

as in $\sum_{m=1}^{l_j} \phi_m^{(j)}$

"projection operator" — if you define

$$P_{\lambda k}^{(j)} = \frac{h_j}{h} \sum_R \Gamma_{\lambda k}^{(j)*}(R) P_R$$

this is an operator acting on the space of all basis fctns.
One can show that:

$$P_{\lambda k}^{(j)} \phi_{k'}^{(i)} = \delta_{kk'} \delta_{ji} \phi_{\lambda}^{(i)}$$

Thus: $\rightarrow P_{\lambda k}^{(i)} \phi_k^{(i)} = \phi_{\lambda}^{(i)}$

[If we know the rep involved, we can in principle construct $P_{\lambda k}^{(i)}$.
Then, given any one fctn. belonging to row k , we can generate all the rest.

But how do you find a $\phi_k^{(i)}$ to start with?

Theorem: any function can be written as a simple sum of functions which belong to reps:

$$F = \sum_{\substack{\text{all} \\ \text{reps } j}} \sum_{k=1}^{h_j} f_k^{(j)}$$

Therefore: $[P_{\lambda k}^{(i)} F = f_k^{(i)}$

$P_{\lambda k}^{(i)}$ will give you a function to prime your Van Vleck machine.
Simply apply $P_{\lambda k}^{(i)}$ to any function you like

Theorem: functions belonging to different reps or to different rows of the same rep are orthogonal. Think of hydrogen atom!

We may repeat all this stuff "at the character level"

$$P^{(j)} = \frac{h_j}{h} \sum_R \chi^{(j)*}(R) P_R$$

$$P^{(j)} f_j = f_j$$

$$P^{(j)} F = f_j$$

Easy to see to which rep. a function belongs.

3.9 Direct Product Groups

Take two groups that are "independent", i.e., $[A_i, B_j] = 0$ for any $A_i \in A$, $B_j \in B$. Then you can show that operators of the form:

$$\{A_i B_j\}$$

also form a group, called $A \otimes B$. Note that $A \otimes B \supset A$ as a subgroup.

A representation of this "direct product group" is

$$\Gamma(A_i B_j)$$

[Define: the "direct product of two matrices":

$$C = A \times B \text{ means } C_{ij,kl} = A_{ik} B_{jl} \equiv (AB)_{ij,kl}$$

$$\text{Theorem: } (A \times B) \cdot (C \times D) = (A \cdot C) \times (B \cdot D)$$

Here, " $\cdot$ " means matrix mult in approp. space.

proof: look at an element on each side:

$$\begin{aligned} \sum_{jj'} (A \times B)_{ii',jj'} (C \times D)_{jj',kk'} &= [(A \times B) \cdot (C \times D)]_{ii',kk'} \\ &= \sum_{jj'} A_{ij} B_{ij'} C_{jk} D_{j'k'} = (A \cdot C)_{ik} (B \cdot D)_{i'k'} \\ &= [(A \cdot C) \times (B \cdot D)]_{ii',kk'} \quad \checkmark \end{aligned}$$

Next, we can show that $\Gamma^{(a)} \times \Gamma^{(b)} \equiv \Gamma^{(a \times b)}$ in fact forms a representation of $A \times B$. In fact, this is an irreducible rep.

Moreover, all the irred reps. of $A \times B$ are of this form.

I.e., $\Gamma^{(a)} \times \Gamma^{(b)} \equiv \Gamma^{(a \times b)}$ exhausts irred reps of $A \times B$
 $\uparrow$
irred reps of A .

The number of irred reps of $A \times B$ is $n_A \times n_B$. Thus, the number of classes is $n_A \times n_B$, recall.

The character of a direct product rep is always equal to the product of the characters of the comprising reps.

$$\text{proof: } \chi^{(A \times B)}(R) = \sum_{ij} [\chi^{(A \times B)}(R)]_{ij,ij} = \text{trace} \\ = \sum_{ij} \chi^{(A)}_{ii} \chi^{(B)}_{jj} = \chi^{(A)}(R) \cdot \chi^{(B)}(R).$$

Thus it is trivial to construct the character table of $A \times B$ if you know the tables of A and B .

example: $D_{3h} = D_3 \times \Sigma$

3.10 Direct products within a group.

Take two reps of G of bases $\{\phi_i\}$ and $\{\psi_i\}$. Then $\{\phi_i \psi_j\}$ is basis for $\Gamma^{(1)} \times \Gamma^{(2)}$. Result is always reducible:

$$\Gamma^{(1)} \otimes \Gamma^{(2)} = \sum_k a_{ijk} \Gamma^{(k)}$$

↑ # of appearances of $\Gamma^{(k)}$

To find a_{ijk} , use former result:

$$a_{ijk} = \frac{1}{h} \left[\sum_R \chi^{(i)}(R) \chi^{(j)}(R) \chi^{(k)*}(R) \right]$$

Eg, $8 \otimes 8$ in $SU(3) = 1 \oplus \underbrace{8 \oplus 8 \oplus 10 \oplus 10^* \oplus 27}_{a_{882} = 2}$

For a continuous group, perhaps

$$\frac{1}{h} \sum_R \rightarrow \int dg, \text{ the Haar measure?}$$

1974
Wybourne

Review of Wybourne.Theory of the Semisimple Lie Algebras.

Chapter Four. connected parameter space; group element as function of parameters $A(x_1, \dots, x_n)$; generators as derivatives w.r.t parameters at identity; generators as diff. ops acting on M -space like $f(x_1, x_2)$ for $SO(2)$; projective group $x' = (ax+b)/(cx+d)$ with $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = 1$ is isomorphic to $SO(3)$; three gens as diff. ops on x .

Chapter Five: a lie algebra is defined as a finite set of vectors in a real or complex vector space, and a linear, distributive, and antisymmetric operation called "commutation", such that Jacobi's Relation is valid for any three vectors; an automorphism is a map of lie algebra onto itself; a subalgebra is closed under commutation. I.e., let $S = \text{s.g. of } G$, then $[s_1, s_2] = s_3$; an ideal is more demanding; $S = \text{ideal}$ if $[s_1, g] = s_3$ for all $s_1 \in S, g \in G$. Thus $S = \text{ideal} \Rightarrow S = \text{subalgebra}$ also; the center of lie algebra is largest set S such that $[s_1, g] = 0$ for all $g \in G, s_1 \in S$; a complex extension means taking the real linear space complex; a lie algebra is simple if it has no ideals at all; if a lie algebra does have ideals, then several cases: if all ideals are non-abelian, then LA is semisimple. Correction: a semisimple LA either has no ideals, or it has some non-abelian ideals, ~~but it never has abelian ideals~~. Thus, the class of semisimple contains the simple. Theorem: any semisimple can be written as direct sum of simple, eg $so(3,1) = A \oplus B$. By direct sum $\oplus$ is meant algebras do not intersect, so $[A_i, B_j] = 0$ eg. The metric tensor $g_{\alpha\beta}$ is constructed from the structure constants $[X_\alpha, X_\beta] = C_{\alpha\beta}^\gamma X_\gamma$ in this way: $g_{\alpha\beta} = g_{\beta\alpha} = C_{\alpha\gamma}^\epsilon C_{\beta\epsilon}^\gamma$. Theorem: $\det(g) \neq 0 \Rightarrow$ lie algebra is semisimple. Converse: not semisimple $\Rightarrow$ abelian ideal $\Rightarrow \det(g) = 0$. Metric tensor also called the Killing form. Above theorem is Cartan's Criterion. Examples: $SO(4) = \text{semisimple}$ since contains $SO(3)$ as nonabelian ideal (twice). $SO(3) = \text{simple}$. $SO(4) = so(3) \oplus so(3)$. Poincaré not a.e. $so(4) = \text{abelian ideal}$



(more chap 5).

- a derived algebra is a bit complicated: The first derived algebra is the subset $S \subset G$ of elements which can be expressed as commutators of other element pairs. Called $A^{(1)} \subset A$. Can show that $A^{(1)}$ is an ideal of A , hence a subalgebra. Similarly, can construct $A^{(2)} \subset A^{(1)}$. If you reach a point where $A^{(k)} = \{\emptyset\}$, then A was "solvable at level k". Theorem: solvable $\Rightarrow$ not simple; Analogous to the derived series where $A^{(k)} = [A^{(k-1)}, A^{(k-1)}]$ is the "descending sequence of ideals" where $A^{(k)} = [A^{(k-1)}, A]$. If this sequence ends in a null set, then A is nilpotent. Claim is that nilpotent $\Rightarrow$ solvable, example: is E_2 in a direct sum, algebras do not intersect and each is an ideal of full A . in a semidirect sum $A = B \oplus C$, both B, C are subalgebras, but only B is an ideal. Theorem: (any) lie algebra may be written as (Solvable) $\oplus$ (semisimple). Of course here (solvable) is your ideal. Example is (Poincaré) = $T_4 \oplus$ (Loeutz) or Poincaré = $\mathfrak{p}^M \oplus \mathfrak{m}^M$. The Casimir Operator is constructed as $g_{\mu\nu} X^\mu X^\nu$. Theorem: a compact lie algebra has a negative definite metric and is therefore semisimple, so compact $\Rightarrow$ semisimple. Example: in $SO(3)$, $C_{ij}^k = i\epsilon_{ijk}$, so $g_{ij} = C_{ik}^l C_{jl}^k = -\epsilon_{ikl}\epsilon_{jkl} = \epsilon_{ikl}\epsilon_{jkl} = 2\delta_{ij}$. This comes out wrong sign because Theorem applies only in a basis s.t. C_{ij}^k are real. Correspondences: subalgebra $\rightarrow$ subgroup; ideal $\rightarrow$ invariant subgroup.

Chapter Six: the order of L.A is called r . When the generators are put into Cartan-Weyl basis, the Lie A. appears as $\{X\} = l \cdot H_i$ plus $(r-l) E_\alpha$. The set $\{H_i\}$ forms an abelian subalgebra known as the Cartan subalgebra whose order is l , known as the rank of the L.A.

This standard basis is obtained by considering the eigenvalue problem $\sigma_A \cdot X = g \cdot X$ where σ_A is the operator which is the adjoint operator to A , i.e., $\sigma_A \cdot X = [A, X]$. By

(more chap 6)

choosing a special $A = \sum_i \lambda_i H_i$, it turns out that $g=0$ is a degenerate eigenvalue of degeneracy $l = \text{rank}$. The corresponding eigenvectors are the H_i $i=1, 2, \dots, l$. All other eigenvalues are nondegenerate + called " α ", so $\alpha = \alpha$ a number, and the corresponding eigenvectors are the E_α .

The explicit Cartan-Weyl basis is:

$$[H_i, H_j] = 0 \quad [H_i, E_\alpha] = \alpha_i E_\alpha \quad [E_\alpha, E_{-\alpha}] = \sum_i \alpha_i H_i$$

$$[E_\alpha, E_\beta] = N_{\alpha\beta} E_{\alpha+\beta} \quad A = \sum_i \lambda_i H_i \quad \alpha = \sum_i \lambda_i \alpha_i$$

Note that with each eigenvalue α is associated a set of numbers α_i where $i=1, \dots, l$. The set $\{\alpha_i\} = \vec{\alpha}$ is a "root vector" in the l -dimensional weight space. In fact, a semi-simple LA is totally characterized by its order n , rank l , and its set of root vectors $\{\vec{\alpha}\}$. Certain properties of roots (length of allowable strings, angles, negatives, etc) allow one to classify all LA's of rank l . The full list of SSA's is this: $A_l = su(l+1)$, $B_l = so(2l+1)$, $C_l = sp(2l)$, $D_l = so(2l)$, plus 5 exceptionals.

Clearly, in Cartan-Weyl, the structure constants have very special form, eg $C_{ij}^x = 0$. The metric tensor looks like

$$\begin{array}{c} 1 \\ \vdots \\ l \\ \hline \alpha_1 \\ \alpha_2 \\ \vdots \end{array} \left[\begin{array}{ccc|ccc} g_{11} & & & & & \\ & \ddots & & & & \\ & & g_{ll} & & & \\ \hline & & & 0 & 1 & 0 & 0 & 0 \\ \alpha_1 & & & 1 & 0 & 1 & 0 & 0 \\ \alpha_2 & & & 0 & 1 & 0 & 1 & 0 \\ \vdots & & & 0 & 0 & 1 & 0 & 1 \\ & & & 0 & 0 & 0 & 1 & 0 \end{array} \right]$$

g_{ij} usually diagonal.

$$g_{ij} = \sum_\alpha \alpha_i \alpha_j$$

Chapter Seven. Of all the root vectors $\vec{\alpha}$, those in a right half space are called positive roots. This half space is actually spanned by a set of l so called simple roots, where $l = \text{rank} = \text{dimension of weight space}$. Suppose you know the simple roots. Then

(One chap?)

There is a tool called the Cartan Matrix which you generate from the simple roots. Its entries give information on string lengths etc and allow construction of the full set of $(r-1)$ root vectors. Thus, a SSLA can be totally defined by its simple roots. These in turn may be coded into a one or two dimensional picture called a Dynkin diagram. Each circle is simple root. Shortest roots have black circles. Connecting lines (0, 1, 2, 3) specify angle between roots. Example: $SU(3) = A_2 = \overset{\circ}{\alpha} - \overset{\circ}{\beta}$. Single line means 120° . The relative lengths of the simple roots are always determined by relative angle.

$\sim \circ \quad \sim \circ$

90°

—

Whenever two or three lines, black dot indicates shorter of two simple roots.

$\circ - \circ$

120°

1:1

$\bullet = \circ$

135°

1: $\sqrt{2}$

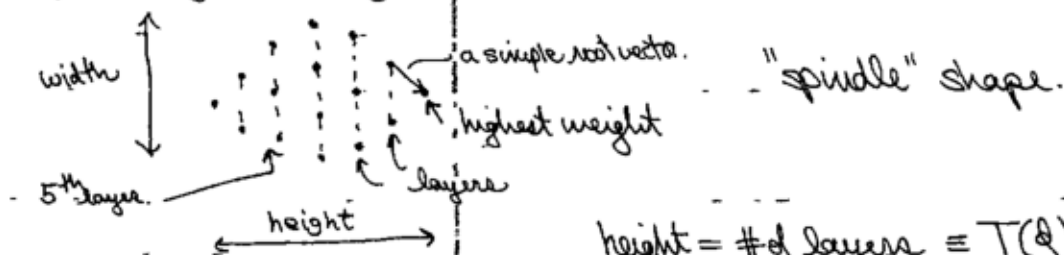
$\bullet = \bullet$

150°

1: $\sqrt{3}$

Chapter Ten: so far, only the root vectors have appeared in the weight space. We now form a set of basis vectors in the space whose operators are the elements of the Lie Algebra. When you act on a basis vector with one of the Lie "generators", something must happen: you get a new vector, or you get 0. The basis vectors are taken to diagonalize all generators of the Cartan subalgebra. Thus, we label basis vectors as $|q, \vec{\lambda}\rangle$, where $\vec{\lambda} = \lambda_1, \lambda_2, \dots, \lambda_N =$ the eigenvalues of the various H_i . (Just like $|T, m\rangle$ or $|N, t_3\rangle$). The set of Cartan Subalgebra eigenvalues $\vec{\lambda}$ which label a basis vector is called a weight. Easy to show that $E_\beta |q, \vec{\lambda}\rangle$ is a new vector whose weight is $(\vec{\lambda} + \vec{\beta})$. Now, a set of basis vectors $|q, \vec{\lambda}\rangle$ forms an irreducible representation of your SSLG if the space spanned by the $|q, \vec{\lambda}_i\rangle$ is invariant under the generators + has no invariant subspaces. Here $\vec{\lambda}_i, i=1, 2, \dots, N$ where $N =$ dimension of rep of q . We know that the weights which characterize a V.I.R. are all connected in weight space by simple root vectors. Some weights may have several basis vectors $|q, \vec{\lambda}, \delta=1, 2, \dots\rangle$.

Then such a weight is called degenerate. It turns out that a irred rep of G is totally defined by the highest weight, which is always simple, i.e., nondegenerate. Moreover, the highest weight is in turn defined by a set of l positive integers (one for each simple root). These integers are called the Λ_i , $i=1, \dots, l$. You can get these integers from the highest weight Λ via this rule: $\Lambda_i = (\Lambda, \alpha_i) / (\alpha_i, \alpha_i)$ where $\alpha_i = i^{\text{th}}$ simple root vector. The scalar product here is via the metric tensor g_{ij} in the $l \times l$ Cartan subspace. Therefore, each IR of your group can be coded as a labelled Dynkin diagram, eg, $\overset{2}{\circ} - \overset{1}{\circ} = \text{the } (1,2) \text{ rep of } SU(3)$, so $\Lambda_1 = 2$ and $\Lambda_2 = 1$. There are various rules for the shape of a rep: layers, height, width.



$\Delta_g \equiv$ the set of weights of the rep of g

height = # of layers $\equiv T(g)$
 $S_k(g) \equiv$ # weights in k^{th} layer

Chapter 1.1: recall that when you take direct product of two VIR's of a group to make the reducible direct or Kronecker product rep, the generators of this new rep are given by star composition of the old generators. Eg, $J_3 = L_3 \otimes S_3 = L_3 \otimes 1 + 1 \otimes S_3$. From this fact, it is easy to show that the weights of a direct product rep are given simply by adding in all ways the weights of the two combined reps. This allows for graphical construction + reduction of a direct product rep, at least in SU_2 and SU_3 .

Chapter 1.2: now here is how you can construct the highest weight as a linear comb of root vectors via the Cartan matrix.

$\vec{\Lambda}_{\text{highest}} = \sum_{i=1}^l a_i \vec{\alpha}_i$ where $a_i \equiv (A^{-1})_{ij} \Lambda_{\alpha_j}$. Here, Λ_{α_j} are those positive integers like (p, q) , and $A_{ij} \equiv \text{cartan}$. Let $\vec{\Lambda}_g^{(h)}$ mean highest weight of g . Then easy to show

that...
$$\vec{\Lambda}_g^{(h)} = \sum_{i=1}^l \Lambda_{\alpha_i} \vec{\Lambda}_{b_i}^{(h)}$$

where $\vec{\Lambda}_{b_i}^{(h)} \equiv$ highest weight of the i^{th} basic representation. (This means a rep like $\circ - \circ - \circ - \circ$). For example,

$$\vec{\Lambda}_{(p,q)}^{(h)} = -p \vec{\Lambda}_{(1,0)}^{(h)} + q \vec{\Lambda}_{(0,1)}^{(h)} = \left(\frac{2p+q}{3}\right) \vec{\alpha}_1 + \left(\frac{p+2q}{3}\right) \vec{\alpha}_2.$$

An elementary rep is when the 1 occurs at a terminal point. In general, you can generate all the basic reps from the few (2 or 3) elem reps.

Remaining chapters. Ch 8 was on the Chevalley Basis, just an alternative to Cartan Weyl. Ch 9 mentions basic group theorems like noncompact $\Rightarrow$ UIRs are inf. dim. Ch 13 reviews some rep "data" for exceptional groups, something Ch 12 did for the classicals like B. Ch 14 gives formulas for the dimensions of all reps of all groups. Ch 15 gives formulas for the Casimir which labels any given IR. Turns out that

$$C = (\Lambda^h, \Lambda^h + \text{sum of positive roots})$$

For semisimple lie groups (which includes simple lie groups) like all the classical lie groups, eg, $SU(3)$, there will be 1 invariants which can be constructed similarly to the Casimir. For SU_3 there are two such invariants:

$$I_2 = g_{\alpha\beta} X^\alpha X^\beta = \text{Casimir} = F^2 = \sum_i F_i^2$$

$$I_3 = C_{\mu\alpha}^\beta C_{\nu\beta}^\gamma C_{\lambda\gamma}^\alpha X^\mu X^\nu X^\lambda = D^2 = \frac{4}{9} \sum_{ijk} d_{ijk} F_j F_k.$$

For each rep of SU_3 , you can compute $I_2(p, q)$, $I_3(p, q)$.

- For a more general Lie group (like Poincaré), the number
- of invariants is set by Beltrami's theorem and apparently
can be anything from 0 to n .

Some elementary linear transformation Groups.

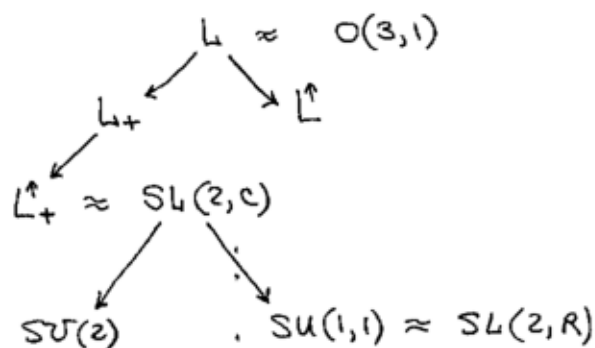
Group	name	parameters	invariants
$GL(n, \mathbb{C})$	general linear, complex	$2n^2$	
$GL(n, \mathbb{R})$	" " , real	n^2	
$SL(n, \mathbb{C})$	special linear, complex	$2(n^2-1)$	
$SL(n, \mathbb{R})$	" " , real	n^2-1	
$U(n)$	unitary group (complex) $a^\dagger a = 1$	n^2	$\sum_{i=1}^n z_i ^2$
$U(p, q)$	no name, but $n = p+q$ and invar is	$-\sum_{i=1}^p z_i ^2 + \sum_{i=p+1}^q z_i ^2$	
$SU(n)$	special unitary	n^2-1	
$O(n, \mathbb{C})$	orthogonal, $a^\dagger a = 1$, complex	$n(n-1)$	$\sum_{i=1}^n z_i^2$
$SO(n, \mathbb{C})$	special orthog, subg. of $O(n, \mathbb{C})$	$n(n-1)$	
$O(n, \mathbb{R})$	real orthogonal	$n(n-1)/2$	$\sum_{i=1}^n x_i^2$
$SO(n, \mathbb{R})$	special real orthog.	$n(n-1)/2$	
$SO(p, q)$	subset of $SL(p+q, \mathbb{R})$ real!		$(-\sum_{i=1}^p x_i^2 + \sum_{i=p+1}^q x_i^2)$
$Sp(2n, \mathbb{C})$	symplectic	$2n(2n+1)$	see p13

"special" always means unimodular

This data was taken from Hiybourne, "classical groups"

QC174.5

Map of Lorentz Group:



/arrow indicates subgroup.

Group Notations:

Wybourne "Classical Groups for Physicists" (1974)

$GL(n, \mathbb{C})$ = "general linear" — set of all $n \times n$ complex, invertible (of course) matrices. There are $2n^2$ real parameters.

$GL(n, \mathbb{R})$ = real values only, so n^2 params only. $O(n, \mathbb{C}) = GL(n, \mathbb{R})$.

$SL(n, \mathbb{C})$ = "special linear" — $\det = \pm 1$. $2(n^2 - 1)$ params.

$SL(n, \mathbb{R})$ has $n^2 - 1$ params.

$U(n)$ = "unitary group" — leaves $\sum z_i z_i^*$ invariant, $A^\dagger A = I$.
Here, $\sum a_{ij}^* a_{jk} = \delta_{ik} \Rightarrow |a_{ij}| \leq 1 \Rightarrow$ bounded parameter
 $\Rightarrow$ a compact group. Still, n^2 parameters, since $A^\dagger A = I$
is n^2 restrictions. why!

$SU(n)$ unitary + unimodular $(n^2 - 1)$ params.

$O(n, \mathbb{C})$ "orthogonal" i.e., $\tilde{A} A^\dagger = I$. $n(n-1)$ params. ✓

$SO(n, \mathbb{C})$ "special complex orth. group" $n(n-1)$ still
 $\det = +1$
[The other half of $O(n, \mathbb{C})$ has $\det = -1$].

$O(n, \mathbb{R})$ real ortho, $\frac{n(n-1)}{2}$ params. [also, # ways to choose pairs of axes = $\binom{n}{2}$. Each choice $\Rightarrow$ a rotation.]

$SO(n, \mathbb{R})$ $\det = +1$.

$U(p, q)$ leaves $\left[\sum_{i=1}^p z_i z_i^* - \sum_{i=p+1}^{p+q} z_i z_i^* \right]$ invariant

Here, $p+q$ = dimension of matrix, so $U(p, q) \subset GL(p+q, \mathbb{C})$.

$SO^*(p, q)$ = $U(p, q) \cap SL(p+q, \mathbb{C})$
 $\uparrow$ unitary in p, q sense. $\nwarrow \det = +1$

$SO(p, q)$ = those $SL(p+q, \mathbb{R})$ such that $-\sum_{i=1}^p x_i^2 + \sum_{i=p+1}^q x_i^2 = \text{const.}$
Thus, Lorentz group must be $SO(3, 1)$.

Scaling of a Cartan Weyl Basis

We can show:

$$g_{\alpha, \alpha} = \sum_{\beta} N_{\alpha, \beta} N_{-\alpha, \beta} + 2(\alpha, \alpha) = \left[\frac{j(k+1)}{2} + 2 \right] (\alpha, \alpha)$$

$$= j(k+1)(\alpha, \alpha)/2$$

$$(\alpha, \alpha) = g^{ij} \alpha_i \alpha_j$$

$$g_{ij} = \sum_{\alpha'} (\alpha')_i (\alpha')_j$$

$$g^{ij} = (g^{-1})_{ij}$$

$$\alpha^i = g^{ij} \alpha_j$$

Scaling rules are

Thus, independent scalings on the E_α and H_i are allowed.

Thus, it is always possible to scale such that g_{ij} and $g_{\alpha, \alpha}$ are whatever you like

α_i	s	1	ie, $\alpha_i \rightarrow s\alpha_i$
g_{ij}	s^2	1	
g^{ij}	s^{-2}	1	
(α, α)	1	f^2	
$g_{\alpha, \alpha}$	1	f^2	
$N_{\alpha, \beta}$	1	f	
α^i	s^{-1}	1	
H_i	s	1	
E_α	1	f	

Example: look at the $SU(3)$ on p 85. Scale s is chosen so that $g_{ij} = \delta_{ij}$. This forces $\alpha + \beta = (1, 0) \times 1/\sqrt{3}$. The f scale is chosen so that $g_{\alpha, \alpha} = 1$.

Metric Tensors for SU(n)

① For SU(2), we have a potential form:

$$[H_1, E_{\pm\alpha}] = \pm E_{\pm\alpha}$$

$$[E_{\alpha}, E_{-\alpha}] = H_1$$

If we choose $\alpha_1 = 1.0$, then we find:

$$g = \begin{matrix} & \begin{matrix} 1 & \alpha & -\alpha \end{matrix} \\ \begin{matrix} 1 \\ \alpha \\ -\alpha \end{matrix} & \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix} \end{matrix}$$

To show this, use:

$$C_{\alpha, -\alpha}^1 = C_{1, \alpha}^{\alpha} = -C_{1, -\alpha}^{-\alpha} = 1 \quad (\text{else} = 0)$$

We can go to another basis via:

$$\begin{aligned} J_x &= \frac{i}{\sqrt{2}} (E_{\alpha} + E_{-\alpha}) \\ iJ_y &= \frac{1}{\sqrt{2}} (E_{\alpha} - E_{-\alpha}) \\ J_z &= H_1 \end{aligned}$$

$$\begin{aligned} E_{\alpha} &= \frac{1}{\sqrt{2}} (J_x + iJ_y) \\ E_{-\alpha} &= \frac{1}{\sqrt{2}} (J_x - iJ_y) \end{aligned}$$

$$C = 2 (E_{\alpha} E_{-\alpha} + E_{-\alpha} E_{\alpha} + H_1^2)$$

In this new basis we have:

$$g = \begin{matrix} & \begin{matrix} x & y & z \end{matrix} \\ \begin{matrix} x \\ y \\ z \end{matrix} & \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \end{matrix}$$

$$C = 2 (J_x^2 + J_y^2 + J_z^2)$$

② For SU(3), if we again choose $(\vec{\alpha} + \vec{\beta}) = (1, 0)$ [see p 85], then I have the feeling that:

$$g = \begin{matrix} & \begin{matrix} F_3 & F_8 & U_{\frac{1}{\sqrt{2}}} & U_{\frac{1}{\sqrt{2}}}^* & E_{\alpha+\beta} & \bar{E}_{\alpha+\beta} & Y & \bar{Y} \end{matrix} \\ \begin{matrix} F_3 \\ F_8 \\ U_{\frac{1}{\sqrt{2}}} \\ U_{\frac{1}{\sqrt{2}}}^* \\ T \\ T^* \\ V \\ V^* \end{matrix} & \begin{bmatrix} 3 & 0 & 0 & 0 & \phi & \phi & 0 & 0 \\ 0 & 3 & 0 & 0 & \phi & \phi & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 & 0 & 3 \\ \phi & \phi & 0 & 0 & 3 & 3 & 0 & 0 \\ \phi & \phi & 0 & 0 & 3 & 3 & 0 & 0 \\ 0 & 0 & \phi & \phi & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 \end{bmatrix} \end{matrix} \quad \text{etc.}$$

Thus, if we take linear combinations like $T_{\pm} = F_1 \pm iF_2$,
The new metric tensor should look like:

$$g = \begin{matrix} & \begin{matrix} F_1 & \dots & F_r \end{matrix} \\ \begin{matrix} F_1 \\ F_2 \\ F_3 \\ \vdots \\ F_8 \end{matrix} & \begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{bmatrix} \end{matrix} \times 3 \uparrow$$

Theorem: The "cartesian" basis is a basis which makes the metric tensor proportional to the unit matrix. In such a basis, the Casimir invariant is always

$$C_1 = g^{ij} F_i F_j = \text{const} \times \sum_i F_i^2$$

③ I conjecture that this same analysis extends to any $SU(n)$. The # of generators is $n^2 - 1 = 15$ for SU_4 , 35 for SU_6 .

Facts to Retain from Wybourne.

- Chapter 3. definitions of basic continuous matrix groups
(like $GL(n, \mathbb{C})$ etc.)
matrix theorems.
- Chapter 4. the parameter space
generators as gradient at identity which is parameter origin.
the commutator of two group elements
structure constants and properties of
the infinitesimal operators
the general exponential form of finite group element.

- Chapter 5. definition of a lie algebra
basis transformations of the X 's
homomorphisms, isomorphisms
automorphisms, inner and outer; the automorphic group
ideals, proper and improper (invariant subalgebras)
subalgebras
adjoint representations
complex extensions
simple and semisimple algebras
the metric tensor or Killing form
the k th derived algebra of an algebra
a solveable lie algebra
direct sums and semidirect sums
the Casimir operator of a semisimple algebra
the corresponding group statements.

theorem: a semisimple lie algebra can always be written as a
direct sum of ideals, each ideal forming a simple algebra.

example: Weinberg's decomposition of the Lorentz group into two
 $SU(2)$ algebras.

theorem: a lie algebra is semisimple iff the metric tensor is invertible.
One can always check this directly from the structure constants.

theorem: any lie algebra can be written as the semidirect sum of a
solveable lie algebra and a semisimple lie algebra.

example: Poincare group, Euclidean groups. Remember that in the def
of semidirect product, the first group (usually translations)
will be an ideal of the complete group.

- Chapter 6. the Cartan Weyl basis, the Cartan subalgebra
roots, their properties and theorems
- Chapter 7. Dynkin diagrams and the Cartan matrix
enumeration of the basic lie algebras.
- Chapter 8. the Chevalley basis

Chapter 9. representations, reducibility of
contragradient reps, adjoint reps, real and complex reps.

theorem: a solveable group has no finite dimensional irreducible
reps except one dimensional reps.

example: translations, abelian groups. There are no matrices used
to describe translations, eg.

theorem: the irreducible unitary reps of a connected, simple,
compact group are all of finite dimension.

example: the rotation group or $SU(n)$.

theorem: the irreducible unitary representations of a noncompact
connected simple lie group are all of infinite dimension,
except for the trivial rep.

example: in $SO(2,1)$ or $SU(1,1)$, the matrices you can construct are
all non-unitary. (eg, the boost matrices)

Chapter 10 weights and how to label irred reps.

Chapter 11 Kronecker products

Chapter 12 more on labelling reps with weights

Chapter 13 the exceptional groups

Chapter 14 dimensions of the irred reps

Chapter 15 the Casimirs
representations of $SO(3)$ and $SO(2,1)$ compared

theorem: a semisimple lie group of rank L has L invariant
Casimirs. All may be constructed from one formula.
Moreover, the values of the Casimirs label completely
any rep. (irred rep)

Chapter 16 global properties of lie groups

Chapter 17 the 3-parameter groups and realizations

Chapter 18 applications of $su(1,1)$

Chapter 19 generalized lie algebra Wigner Eckhart theorem

finally, the case studies.

What is a Lie Group

If we can express the elements of some group as functions of a finite set of parameters $\alpha_1, \alpha_2, \dots = \vec{\alpha}$

$$g = g(\vec{\alpha})$$

And if this function is continuous, differentiable near the identity $\vec{\alpha} = \vec{0}$ $g(\vec{0}) = I$ [ie, we define " $\vec{0}$ " to be the parameters such that $g(\vec{0}) = I$. This is a choice of origin in parameter space], then

$$g(\vec{\alpha}) = g(\vec{0}) + \sum_i \frac{\partial g}{\partial \alpha_i} d\alpha_i + \frac{1}{2} \sum_{i,j} \frac{\partial^2 g}{\partial \alpha_i \partial \alpha_j} d\alpha_i d\alpha_j + \dots$$

is a valid Taylor series expansion.

So, if $g(\vec{\alpha})$ is differentiable in a ball in parameter space about the identity, then the group $\mathcal{G}$ whose elements are these $g(\vec{\alpha})$ is a Lie group. (shown now), This leads to

Now, let $\vec{\alpha}, \vec{\beta}$ be two small vectors, $\vec{\alpha} + \vec{\beta} = \vec{\gamma}$ is also small.
Then:

$$A(\vec{\alpha}) A(\vec{\beta}) = \text{com}(A(\vec{\alpha}), A(\vec{\beta}))$$

$$= [A(\vec{0}) - \sum \alpha_k X_k + \frac{1}{2} \alpha_k \alpha_l X_k X_l] [1 - \beta_k X_k + \frac{1}{2} \beta_k \beta_l X_k X_l]$$

$$\times [1 + \alpha_k X_k + \frac{1}{2} \alpha_k \alpha_l X_k X_l] [1 + \beta_k X_k + \frac{1}{2} \beta_k \beta_l X_k X_l]$$

$$= \alpha_k X_k \beta_l X_l + \frac{1}{2} \alpha_k \alpha_l X_k X_l + \frac{1}{2} \beta_k \beta_l X_k X_l$$

$$= \alpha_k \beta_l X_k X_l + \alpha_k \alpha_l X_k X_l + \beta_k \beta_l X_k X_l$$

$$= X_k X_l [\alpha_k \beta_l + \alpha_k \alpha_l + \beta_k \beta_l]$$

a Lie algebra on the generators

$$X_i = \frac{\partial g}{\partial \alpha_i}$$

$$\begin{aligned} & + (\alpha \cdot X)(\beta \cdot X) - (\alpha \cdot X)(\beta \cdot X) \\ & = (\alpha \cdot X)(\beta \cdot X) - (\beta \cdot X)(\alpha \cdot X) \\ & = (\alpha \cdot X)(\beta \cdot X) - (\beta \cdot X)(\alpha \cdot X) \end{aligned}$$

I left out many terms accidentally:

$$[1 - \alpha \cdot X + \frac{1}{2}(\alpha \cdot X)^2] [1 - \beta \cdot X + \frac{1}{2}(\beta \cdot X)^2] [1 + \alpha \cdot X + \frac{1}{2}(\alpha \cdot X)^2] [1 + \beta \cdot X + \frac{1}{2}(\beta \cdot X)^2]$$

$$= (\alpha \cdot X)^2 + (\beta \cdot X)^2 - (\alpha \cdot X)(\beta \cdot X) - (\beta \cdot X)(\alpha \cdot X) + [(\alpha \cdot X)(\beta \cdot X)] = \alpha_i \beta_j [X_i, X_j]$$

So, through second order, we know:

$$A^{-1}(\alpha) A^{-1}(\beta) A(\alpha) A(\beta) = \alpha_i \beta_j [X_i, X_j] + 1$$

However, another way exists to compute this:

~~$\vec{\alpha} + \vec{\beta} = \vec{\gamma}$, a new point in param space.~~

$$\cancel{A(\alpha) A(\beta) = A(\alpha + \beta) = A(\gamma)}$$

$$\begin{aligned} S_0 &= \cancel{A^{-1}(\gamma) A(\gamma)} = [1 - \gamma \cdot X + \frac{1}{2}(\gamma \cdot X)^2] [1 + \gamma \cdot X + \frac{1}{2}(\gamma \cdot X)^2] \\ &= \cancel{1 + (\gamma \cdot X)^2} \end{aligned}$$

Stop. We know there exists a group element $A(\vec{\gamma})$, such that

$$\begin{aligned} A(\vec{\gamma}) &= A^{-1}(\vec{\alpha}) A^{-1}(\vec{\beta}) A(\vec{\alpha}) A(\vec{\beta}) \\ &= 1 + (\gamma \cdot X) \end{aligned}$$

If all parameters are small, we must have

$$[(\alpha \cdot X), (\beta \cdot X)] = (\gamma \cdot X)$$

$$\text{Or } \alpha_i \beta_j [X_i, X_j] = \gamma_k X_k$$

In particular, choose $\alpha = \hat{i}$ and $\beta = \hat{j}$. Then

$$[X_i, X_j] = \underbrace{(\gamma_k^{ij})}_{\text{constants}} X_k = \underbrace{(C_{ij}^k)}_{\text{constants}} X_k$$

Therefore: a lie group has generators which form
a lie algebra.

Chapter 4 Wybourne: "Local Properties of Lie Groups."

4.1 Parameters

$GL(n, \mathbb{C})$ is a group with $2n^2$ real parameters. It is a huge set of $n \times n$ complex nonsingular matrices. All matrix groups are α subgroups of this monster, such as $SO(3)$, $SO(2,1)$, $SU(3)$, etc.

So consider any one of these subgroups. Say it has r -parameters, where $r \leq 2n^2$. Then

$$A = A(\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_r)$$

4.2 Connectivity

A group is connected if the parameter region is connected. If not, then you have to consider a group as sum of disjoint pieces, like $O(3) = SO(3) \oplus \tilde{SO}(3)$.

4.3 Begin Lie Groups

Basically, all you need is that the group elements must be continuous functions of the parameters. Then you have a Lie group.

4.4 Generators

$$X_i \equiv \left. \frac{\partial A_i(\vec{\alpha})}{\partial \alpha_i} \right|_{\vec{\alpha}=0}$$

ie, at identity

$$[X_i, X_k] = C_{ik}^j X_j \quad \text{etc.}$$

It is shown that $\exists$ structure constants $C_{ij}^k \in \mathbb{R}$. Thus, we have [a Lie algebra on the generators of our Lie group]

4.5 Example $SO(2)$

Shows that this is a one parameter (θ), group (closed). Trivial

4.6 Infinitesimal Rotations

$$x'_i = \sum_j (\delta_{ij} + \epsilon_{ij}) x_j \quad \text{is an infinitesimal rotation}$$

4.7 General Inf.esimal Xforms [Should be "general (nonlinear) xforms"]

Consider $\vec{x}$ to be an n -dim vector in a vector space. The transform operators are then $n \times n$ matrices for linear xforms.
 $\rightarrow$ only.

$$\text{Then: } T_b T_a = T_c \Rightarrow \vec{c} = \phi(\vec{b}, \vec{a})$$

where $\vec{a}, \vec{b}, \vec{c}$ are parameters.

[Perhaps we should not restrict to linear transformations! Let

$$\vec{x}' = \vec{x} T_a$$

$$\vec{x}'' = \vec{x}' T_b$$

$$\Rightarrow$$

$$\vec{x}'' = \vec{x} T_a T_b = \vec{x} T_c$$

$\uparrow$ group.

$$\text{Define: } \vec{x} T_a = f(\vec{x}; \vec{a})$$

$\leftarrow$
vector
in x-space

$\nwarrow$ vector in parameter space.

$$\text{Define: } \vec{c} = \phi(\vec{a}, \vec{b}) \quad \text{as the group map.}$$

So we can demand:

$$\vec{x} T_c = f(\vec{x}; \phi(\vec{a}, \vec{b})) = f(\vec{x} T_a; \vec{b}) = f(f(\vec{x}; \vec{a}); \vec{b})$$

What properties must f and ϕ have in order that set of xforms be a group?

$$\text{Well, identity: } \vec{a} = \phi(\vec{a}, 0) = \phi(0, \vec{a})$$

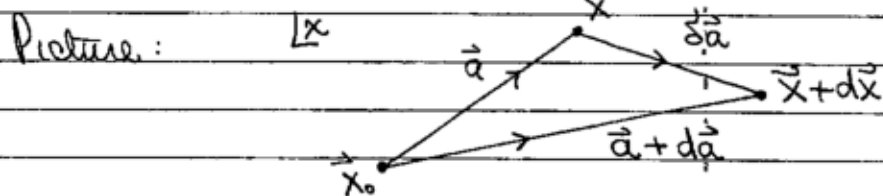
$$\text{inverse: } \vec{0} = \phi(\vec{a}, \vec{a}^{-1})$$

assoc:

You do not have to use addition i.e., $\vec{c} = \vec{a} + \vec{b}$. This is appropriate for linear xforms.

4.8 The infinitesimal operators of a Lie group.

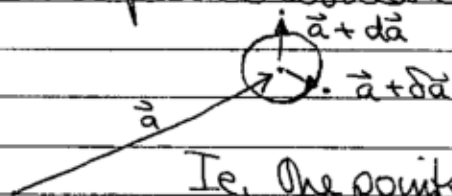
Here it is shown that $X_i = U_i^j \partial/\partial x^j$. I.e., we may represent the generators by certain differential operators acting on functions of the x^i .



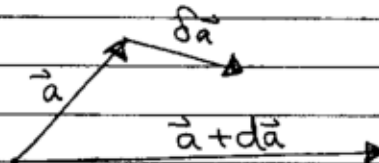
Here are three points, in x -space. The parameter vectors show how you can move among these points with x some.

Notice! If we plot the vectors $\vec{a}$, $\delta\vec{a}$, $\vec{a}+d\vec{a}$ in parameter space, [They do not nec. form a Δ !]

However, as $\delta\vec{a} \rightarrow 0$, $d\vec{a} \rightarrow 0$
So a better picture would be:



I.e., the points $\vec{a}+d\vec{a}$ and $\vec{a}+\delta\vec{a}$ are close together. But $d\vec{a} \neq \delta\vec{a}$.



$$\text{So: } \vec{x}+d\vec{x} = f(\vec{x}; \delta\vec{a}) = f(\vec{x}_0; \vec{a}+d\vec{a})$$

$$\text{where } \vec{x} = f(\vec{x}_0, \vec{a})$$

$$\begin{aligned} \text{Thus: } d\vec{x} &= f(\vec{x}_0; \vec{a}+d\vec{a}) - f(\vec{x}_0; \vec{a}) \\ &= \left. \frac{\partial f(\vec{x}_0, \vec{a})}{\partial a_i} \right|_{\vec{a}=\vec{a}} da_i \end{aligned}$$

Not very interesting. However, we also have: $\vec{a}=0$.

$$d\vec{x} = f(\vec{x}; \delta\vec{a}) - f(\vec{x}; 0) = \left. \frac{\partial f(\vec{x}, \vec{a})}{\partial a_i} \right|_{\vec{a}=0} \delta a_i$$

$$\text{Once more: } x^i = f^i(\vec{x}_0, \vec{a})$$

$$\text{So } \left[dx^i = \left. \frac{\partial f^i(\vec{x}_0, \vec{a})}{\partial a_j} \right|_{\vec{a}=0} \delta a_j \right]$$

These "things" are called $U_j^i(\vec{x})$.

Meanwhile,

$$\vec{a} + d\vec{a} = \phi(\vec{a}, \delta\vec{a})$$

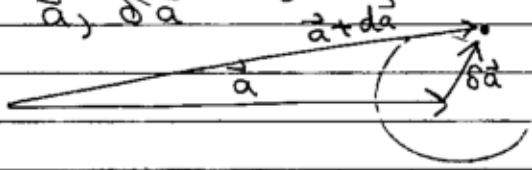
$$\vec{a} = \phi(\vec{a}, 0)$$

$$\Rightarrow da^i = \left(\frac{\partial \phi^i(\vec{a}, \vec{b})}{\partial b^j} \right) \bigg|_{\vec{b}=0} \delta a^j$$

$$\left[\underbrace{\quad}_{\text{this is } V_j^i(\vec{a})} \right]$$

$$\text{So, } da^i = V_j^i(\vec{a}) \delta a^j \quad d\vec{a} = \underline{V} \cdot \delta\vec{a}$$

So, the matrix $\underline{V}$ tells you how to go "direct", given the indirect path $\vec{a}, \delta\vec{a}$

i.e.,  in x-space.

$$\text{Also, } d\vec{x} = \underline{U}(\vec{x}) \cdot \delta\vec{a}$$

A small change in some $\vec{a}$ causes you

If you're at $\vec{x}$ and do a small x from $\delta\vec{a}$, you end up at $\vec{x} + d\vec{x}$. $\underline{U}$ tells you what $d\vec{x}$ is:

So, whereas $\underline{V}$ is a $n \times r$ matrix in parameter space,

$\underline{U}$ is a $r \times n$ matrix with one foot in param space,

+ the other foot in x-space.

$$\text{Define: } [\underline{\lambda} = \underline{V}^{-1}] \quad \delta\vec{a} = \underline{\lambda} \cdot d\vec{a}$$

Then:

$$d\vec{x} = \underline{U} \cdot \underline{\lambda} \cdot d\vec{a}$$

$$\frac{\partial x^i}{\partial a^j} = (\underline{U} \underline{\lambda})^i_j$$

→ Now, consider a function defined on $\vec{x}$. As we apply $\delta\vec{a}$ + go $\vec{x}$ to $\vec{x} + d\vec{x}$, how does $F(\vec{x})$ change?

$$dF(\vec{x}) = \frac{\partial F(\vec{x})}{\partial x^i} U_\alpha^i(\vec{x}) \delta a^\alpha = \delta a^\alpha \underbrace{U_\alpha^i(\vec{x}) \frac{\partial}{\partial x^i}}_{\equiv X_\alpha} F(\vec{x})$$

→ inf x formation on $F(x)$ $\equiv X_\alpha$

Summarize: given an abstract group whose elements are "parametrized", we may speak of "abstract generators X_i " such that

$$X_i = \left. \frac{\partial g(\vec{a})}{\partial a_i} \right|_{\vec{a}=0}.$$

In the case of a group of transformations, we take

$$g(\vec{a}) = F[\vec{X}(\vec{a})]$$

To, to each $\vec{a}$ we assign an $\vec{X}$, then an $F(\vec{X})$.
So:

$$X_i = \frac{\partial F}{\partial \vec{X}} \cdot \frac{\partial \vec{X}}{\partial a_i} = U_i^j$$

Now suddenly we have: $X_i = U_i^j \frac{\partial}{\partial x^j}$.

What is the connection? Later in section 4.10 we shall see that these X_i "realize" the same Lie algebra as the X_i .

$\{X_i\}$ = "generators"

$\{X_i\}$ = "inf'nal operators"

So, in order to form these operators X_i , we need the functional form of

$$\vec{X} = f(\vec{x}_0; \vec{a}) = \vec{x}_0 \cdot T\vec{a}$$

Therefore: given any group of transformations on vectors $\vec{x}$ [real or complex], you can construct differential operators which realize the Lie algebra of the Lie group of transformations. Notably, these transformations need not be linear.

Example: $x' = \frac{ax+b}{cx+d}$ with $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = 1$.

You can show that these things do form a 3 parameter group on $\mathbb{R}$. These transforms are not linear. So

$$x' = f(x; a, b, c) = f(x; \vec{a}) = \frac{a_1 x + a_2}{a_3 x + a_4} \quad \text{where} \quad \begin{vmatrix} a_1 & a_2 \\ a_3 & a_4 \end{vmatrix} = 1.$$

see other sheets for more on this "example"

In this example, the differential generators are:

$$X_a = U_a^i(\vec{x}) \frac{d}{dx^i} \quad \text{where} \quad U_a^i(\vec{x}) = \frac{\partial f^i(\vec{x}, \vec{a})}{\partial a^a}$$

In the real case, there is only one f ,

$$f(x; a_1, a_2, a_3) = \frac{a_1 x + a_2}{a_3 x + a_4}$$

↑ see later sheets.

$$\frac{\partial f}{\partial a_1} = \frac{x}{a_3 x + a_4}$$

$$\frac{\partial f}{\partial a_2} = \frac{1}{a_3 x + a_4}$$

$$\frac{\partial f}{\partial a_3} = (a_1 x + a_2) \frac{(-1) \cdot x}{(a_3 x + a_4)^2}$$

$$\text{So: } X_1 = \frac{x}{a_3 x + a_4} \frac{d}{dx}$$

$$X_2 = \frac{1}{a_3 x + a_4} \frac{d}{dx}$$

$$X_3 = \frac{-x(a_1 x + a_2)}{(a_3 x + a_4)^2} \frac{d}{dx}$$

Now, what Lie algebras do these generators realize? You could compute it by brute force, or you could try to identify this projective group more directly.

[Notice that these generators are always first order differentials. You would never get (18.7) in this way.]

4.9 Examples of Infinitesimal Operators

A. The rotation group. You write out dx^i in terms of small change in parameters:

Then you read off your $U_a^i(\vec{x})$

$$\begin{aligned} dx_1 &= -x_2 \delta\theta \\ dx_2 &= x_1 \delta\theta \end{aligned}$$

$$U_1^1 = -x_2 \quad U_1^2 = x_1$$

$$\text{Then } X = U_1^1 \frac{\partial}{\partial x_1} + U_1^2 \frac{\partial}{\partial x_2} = -x_2 \frac{d}{dx_1} + x_1 \frac{d}{dx_2}$$

Consider S_a to be a transformation acting on $F(\vec{x})$

$$S_a F(x) = G(x)$$

$$\text{For small } x\text{-form: } S_a F(x) \approx (1 + \delta a^a X_a) F(x) = F(x) + dF(x)$$

So, for every group "element" $\vec{a}$, we define a transformation:

$$G(\vec{x}) = S_{\vec{a}} F(\vec{x})$$

If we know the inf. ops X_a , then we know small X forms.
Hopefully, knowing just these X_i , we can generate the finite $S_{\vec{a}}$. (Wait.)

B. the $SO(3)$ group. Again, we construct

$$\begin{aligned} dx &= \delta a_y - \delta b_z && \leftarrow \text{we knew this! because } A \approx I + B \text{ where} \\ dy &= \delta a_x + \delta c_z && B \text{ is antisymmetric.} \\ dz &= \delta b_x - \delta c_y \end{aligned}$$

Read off the $\vec{V}_i$ + construct your three X_i ; then show the algebra.

Both these examples dealt with linear transformations.

4.10 The Structure Constants

Here it is shown that these operators $X_a = \vec{V} \cdot \vec{a}$ satisfy a relation:

$$[X_a, X_b] = C_{ab}^c X_c \quad \text{where } C_{ab}^c \text{ are constants.}$$

Are these the same constants? Same as what?! You start with a group of transformations on a vector $\vec{x}$.

$$\vec{x}' = \vec{F}(\vec{x}; \vec{a})$$

Then you construct the operators X_i . Then you construct the constants C_{ij}^k , being guaranteed their existence by the proof of this section. Then you have a Lie algebra which is related to your transformation group. Of course, your transformations are smooth near $\vec{a} = 0$, so it is a Lie group.

The problem is: what ever happened to those "abstract generators"

$$X_i = \left. \frac{\partial A(\vec{a})}{\partial a_i} \right|_{\vec{a}=0}$$

Well, consider $\vec{x}' = \vec{f}(\vec{x}; \vec{a})$ as $f_{\vec{a}}: \vec{x} \rightarrow \vec{x}'$ so that $f_{\vec{a}}$ is a "transformation operator". For $\vec{a}$ near the identity, we know:

$$f_{\vec{a}} = 1 + \vec{a} \cdot \vec{X} \quad \rightarrow \quad dx = \left(\vec{a} \cdot \vec{X}_i \right) \cdot \vec{x}$$

(The book uses $S_{\vec{a}}$ for $f_{\vec{a}}$). So we have:

$$X_i = \frac{\partial f_{\vec{a}}}{\partial a_i} = \vec{X}_i$$

Summarize: in this application, we have a set of possibly non-linear transformation operators $f_{\vec{a}}: \mathbb{R}^n \rightarrow \mathbb{R}^n$ or $\mathbb{C}^n \rightarrow \mathbb{C}^n$ whatever. This set of operators $\{f_{\vec{a}}\}$ is homomorphic to an abstract group whose elements are $A(\vec{a})$.

Thus, the operators $\vec{X}_i$ are in homomorphism with the abstract group generators X_i . Of course they must satisfy the same Lie algebra ~ that's what homomorphic means!

So far, we have two sets of "generators"

$$1. \quad X_i \text{ of abstract group} = \left. \frac{\partial A(\vec{a})}{\partial a_i} \right|_{\vec{a}=0}$$

$$2. \quad \vec{X}_i \text{ of group of operators } f_{\vec{a}}: \mathbb{R}^n \rightarrow \mathbb{R}^n, \\ \text{such that } \left. \frac{\partial f_{\vec{a}}}{\partial a_i} \right|_{\vec{a}=0} = \vec{X}_i$$

There is now a third set of generators.

$$S_{\vec{a}} F(x) = G(x) \approx (1 + \vec{a} \cdot \vec{X}) F(x)$$

$$\text{Namely, } \chi_i = \left. \frac{\partial S_{\vec{a}}}{\partial a_i} \right|_{\vec{a}=0}.$$

These three sets are in homomorphism. The generators χ_i are differential operators: The generators $\vec{X}_i$ are $n \times n$ matrices which characterize the linearized transformations $\vec{x}' = f(\vec{x}, \vec{a})$. The generators X_i are abstract.

(only for linear & some, however!)

Many levels here!

The Procedure for identifying a Lie group of transformations:

- 1) write your transformation + show that you do have a group. Identify the identity point in parameter space. Define new params $\Rightarrow \vec{a} = (0, 0, \dots)$ is identity.
- 2) write an infinitesimal version of your transformation. Read off the U_a^i , then construct the X_a .
- 3) compute the algebra of the X_a . Done!

4.11 Generation of Finite Group Elements.

In summary, $A(\vec{a}) = e^{\vec{a} \cdot \vec{X}}$ /abstract.

For the set of homomorphic transformations $\vec{R}: \mathbb{R}^n \rightarrow \mathbb{R}^n$, ie, $\vec{x}' = f(\vec{x}; \vec{a})$, I am not sure what the generators are. I do know that

$$dx^i = U_a^i \delta a^a$$

For the set of homomorphic transformations $S_a: \mathcal{L} \rightarrow \mathcal{L}$ [here S_a acts on some $f(\vec{x}) \in \mathcal{L}$, not ~~on~~ only on $\vec{x}$], we know that

$$X_a = U_a^i \partial / \partial x^i$$

Then a finite transformation is:
Now we can compute anything.

$$e^{a \cdot U_a^i \partial / \partial x^i} = e^{\vec{a} \cdot \vec{X}}$$

Example: Let $X_1 = U_1 \partial / \partial x = x \partial / \partial x$
 $X_2 = \partial / \partial x$
 $X_3 = -x^2 \partial / \partial x$

Then: $e^{[(a_1 - a_3)x \partial / \partial x + a_2 \partial / \partial x - a_3 x^2 \partial / \partial x]} \cdot f(x) = S_a f(x)$

= some new function of x

I think $e^{\vec{a} \cdot \vec{X}} x = x' = \frac{(a_1+1)x + a_2}{a_3 x + (1+a_1)}$

I do not know the general effect on some $f(x)$.

4.12 Finite Xforms

[See above. The question which remains unanswered is: given a Lie Algebra, how can you construct all possible transformation groups which are related? We know the linear transformations, these are the matrix groups.

[It is those nonlinear transformations [like the projectives] which seem haphazard.

a) eg, I am still mystified by:

$$S_2 \cdot f(\phi) = g(\vec{\alpha}, \phi) f(\phi') \quad e^{i\phi'} = h(\vec{\alpha}, \phi).$$

Here, S_2 is homomorphic to $SU(1,1)$.

(See p. 2214 of Mukunda) Refer to Bargmann.

Probably this is very closely related to projective

The projective group

① Consider transformations of the form:

$$x' = f(x; \vec{a}) = \frac{a_1 x + a_2}{a_3 x + a_4}$$

$$x'' = f(x'; \vec{b}) = \frac{b_1 x' + b_2}{b_3 x' + b_4}$$

Combining two in a row

$$x'' = f[f(x; \vec{a}); \vec{b}] = \frac{b_1 \left[\frac{a_1 x + a_2}{a_3 x + a_4} \right] + b_2}{b_3 \left[\frac{a_1 x + a_2}{a_3 x + a_4} \right] + b_4}$$

$$= \frac{b_1 (a_1 x + a_2) + b_2 (a_3 x + a_4)}{b_3 (a_1 x + a_2) + b_4 (a_3 x + a_4)} = \frac{(b_1 a_1 + b_2 a_3) x + (b_1 a_2 + b_2 a_4)}{(b_3 a_1 + b_4 a_3) x + (b_3 a_2 + b_4 a_4)}$$

$$= \frac{c_1 x + c_2}{c_3 x + c_4} = f(x; \vec{c})$$

So, this set of transformations is closed.② What about existence of an inverse? Given $\vec{a}$, can we find $\vec{b}$ such that

$$(f_{\vec{b}} \circ f_{\vec{a}})x = x \quad ?$$

$$\text{i.e., } (b_1 a_1 + b_2 a_3)x + (b_1 a_2 + b_2 a_4) = (b_3 a_1 + b_4 a_3)x^2 + (b_3 a_2 + b_4 a_4)x$$

$$\Rightarrow (b_3 a_1 + b_4 a_3)x^2 + (b_3 a_2 + b_4 a_4 - b_1 a_1 - b_2 a_3)x - (b_1 a_2 + b_2 a_4) = 0$$

To work small x , we have three equations:

$$\begin{aligned} a_1 b_3 + a_3 b_4 &= 0 \\ -a_1 b_1 - a_3 b_2 + a_2 b_3 + a_4 b_4 &= 0 \\ -a_2 b_1 - a_4 b_2 &= 0 \end{aligned}$$

Three equations in 4 unknowns, $\vec{b}$. Suppose we choose b_4 out of the blue + then tried to solve for b_1, b_2, b_3

Equations would be:

$$\begin{aligned} a_1 b_3 &= -a_3 b_1 \\ -a_1 b_1 - a_3 b_2 + a_2 b_3 &= -a_4 b_4 \\ -a_2 b_1 - a_4 b_2 & \end{aligned}$$

In order for a solution to exist, we need

$$\begin{vmatrix} 0 & 0 & a_1 \\ -a_1 & -a_3 & a_2 \\ -a_2 & -a_4 & 0 \end{vmatrix} \neq 0 \quad \Rightarrow \quad \begin{vmatrix} a_1 & a_2 \\ a_3 & a_4 \end{vmatrix} \neq 0$$

Also, we need $a_1 \neq 0$. So, we state:

claim: The projective x-form $x' = \frac{a_1 x + a_2}{a_3 x + a_4}$ will have an inverse as long as

$$a_1 \neq 0 \quad \text{and} \quad \begin{vmatrix} a_1 & a_2 \\ a_3 & a_4 \end{vmatrix} \neq 0.$$

③ What is the identity? Well, $a_1, a_3 = 1$, $a_2, a_4 = 0$

Notice that identity has $\det = +1$.

④ Thus, we might reparametrize as:

$$x' = \frac{(a_1 + 1)x + a_2}{a_3 x + (a_4 + 1)} = \frac{(1 + d_1)x + d_2}{d_3 x + (1 + d_4)}$$

Then $\vec{d} = (0, 0, 0, 0)$ does label the identity. Condition is now:

$$d_1 \neq -1 \quad \text{and} \quad \begin{vmatrix} 1 + d_1 & d_2 \\ d_3 & 1 + d_4 \end{vmatrix} \neq 0. \quad (*)$$

⑤ So far, we have a 4-parameter group defining transformation $f_x: \mathbb{R} \rightarrow \mathbb{R}$. We could extend this to an 8 parameter group $f_x: \mathbb{C} \rightarrow \mathbb{C}$. We must avoid certain points in the parameter space (ie, $(*)$)

A subgroup is a 3 param group $\Rightarrow | \cdot | = +1$. [a, 6 param for $\mathbb{C}$].

⑥ Now, examine an inf-mal xform:

$$x' = \frac{(1+d_1)x + d_2}{d_3x + (1+d_4)} \approx (x+d_1x+d_2)(1+d_3x+d_4)^{-1}$$

$$\approx (x+d_1x+d_2)(1-d_3x-d_4) = x - d_3x^2 - d_4x + d_1x - d_1d_3x^2 - d_1d_4x + d_2 - d_2d_3x - d_2d_4$$

$$= x + \left(d_2 + [d_1 - d_4]x - d_3x^2 \right) + \mathcal{O}(d_i d_j)$$

So: $\nwarrow$ ONLY this difference matters, locally

$$dx = d_2 + (d_1 - d_4)x - d_3x^2 = U_i d^i \quad \Rightarrow \quad \vec{a} \cdot \vec{x}$$

So: $U_1 = x \checkmark \Rightarrow X_1 = x \partial/\partial x \checkmark$
 $U_2 = 1 \checkmark \Rightarrow X_2 = \partial/\partial x \checkmark$
 $U_3 = -x^2 \checkmark \Rightarrow X_3 = -x^2 \partial/\partial x \checkmark$
 $U_4 = -x \checkmark \Rightarrow X_4 = -x \partial/\partial x \checkmark$

the same!

These are your inf. operators for this group!

⑦ Now we can define transformations on functions of x :

$$S_{\vec{a}} F(x) \approx (1 + \vec{a} \cdot \vec{X}) F(x) \quad \text{for small ones}$$

The set of objects $\{S_{\vec{a}}\}$ is again homomorphic to the group $A(\vec{a})$. These operators, however, act on a function space:

$$S_{\vec{a}}: \mathcal{L} \rightarrow \mathcal{L} \quad \text{where } \mathcal{L} = \text{space of, say, differentiable fctns of variable } x$$

Whereas:

$$f_{\vec{a}}: \mathbb{R} \rightarrow \mathbb{R}$$

So, now we have three sets of things in homomorphism

$$\{A(\vec{a})\} \cdot \text{h.} \{f_{\vec{a}}\} \cdot \text{h.} \{S_{\vec{a}}\}$$

⑧ What is the group?! What are the X_i type generators? Here they are just numbers

$$X_1 = 1 \quad X_2 = \frac{1}{x} \quad X_3 = -x \quad X_4 = -1$$

The only possible Lie algebra is trivial:

$$[X_i, X_j] = 0.$$

Thus, we seem to have here a 4 parameter abelian group.

Check the operators X_i, X_j ?

$$\begin{aligned} [X_i, X_j] &= ? \quad [X_1, X_2] \neq [x \partial_x, \partial_x] f \\ &= x \partial_x^2 f - \partial_x [x \partial_x (f)] = \cancel{x \partial_x^2 f} - \cancel{x \partial_x^2 f} - \partial_x f \end{aligned}$$

Now it seems that $[X_1, X_2] = -X_2$.

Why do these algebras conflict?

⑨ Well, Hlybourne never claimed that the $X_i \equiv: f_a = 1 + \frac{\partial}{\partial x} X_i$ did satisfy the L.A. Obviously, if you have

$$f_a: \mathbb{R}^1 \rightarrow \mathbb{R}^1$$

Then the X will have no matrix aspect. But can these things X_i be differential ops?

$$\begin{aligned} x' &= f(x; \vec{\alpha}) = \frac{(1+\alpha_1)x + \alpha_2}{\alpha_3 x + (1+\alpha_4)} \\ &\approx x + [\alpha_1 x + \alpha_2 - \alpha_3 x^2 - \alpha_4 x] + O(d_i k_j) \\ &\approx \left[1 + \underbrace{\left(\alpha_1 + \frac{\alpha_2}{x} - \alpha_3 x - \alpha_4 \right)}_{\sum_i \alpha_i X_i} + \dots \right] x \end{aligned}$$

How do you know that X^2 isn't really $\partial/\partial x$!

$$\alpha_2 X^2 \cdot x = \alpha_2 \frac{\partial^2}{\partial x^2} x = \alpha_2$$

The set X does not seem very well defined.

⑩ So, let's go back to the X_i + try to construct our Lie algebra:

$$[X_1, X_2] = [x\partial_x, \partial_x] = x\partial_x - x\partial_x - \partial_x = -X_2$$

$$[X_1, X_3] = [x\partial_x, -x^2\partial_x] = -x(x^2\partial_x + 2x\partial_x) + x^2(x\partial_x + \partial_x) \\ = -2x^2\partial_x + x^3\partial_x = -x^2\partial_x = X_3$$

$$[X_1, X_4] = [X_1, -X_1] = -[X_1, X_1] = 0$$

$$[X_2, X_3] = [\partial_x, -x^2\partial_x] = -\partial_x(x^2\partial_x) + x^2\partial_x \\ = -x^2\partial_x - 2x\partial_x + x^2\partial_x = -2X_1$$

$$[X_2, X_4] = [X_2, -X_1] = [X_1, X_2] = -X_2$$

$$[X_3, X_4] = [X_3, -X_1] = [X_1, X_3] = X_3$$

Now two of our generators are nearly the same! What does that mean?

$$[X_1, X_2] = -X_2$$

$$[X_1, X_3] = X_3$$

$$[X_2, X_3] = -2X_1$$

$$[X_1, X_4] = 0$$

$$[X_2, X_4] = -X_2$$

$$[X_3, X_4] = X_3$$

With 4-generators, we seem to be rank 2 since X_1, X_4 commute.

⑪ Is this algebra simple?

def: abelian ideal: " $\equiv$ is an A.I. if $[X, Y] = 0$ ($\forall X \in \equiv$), ($\forall Y \in A$)"

We do not have any abelian ideals in our set $\{X_1, X_2, X_3, X_4\}$, so this algebra is semisimple. However, it is not simple because $\{X_1, X_2, X_3\}$ is an ideal ~ an invariant subalgebra.

⑫ You cannot have two equal generators! Recall that

$$x' \approx x + [\alpha_2 + (d_1 - d_4)x - \alpha_3 x^2]$$

For small x forms, only $d_1 - d_4 \equiv \alpha_{14}$ is a parameter, one parameter. Why not treat α_4 as a constant, then α_1 is your parameter?

Then we are back to a 3-parameter group with 3 generators:

$$\begin{array}{lll} X_1 = x \partial_x & [X_1, X_2] = -X_2 & X_2 = J_- \\ X_2 = \partial_x & [X_1, X_3] = X_3 & X_3 = J_+ \\ X_3 = -x^2 \partial_x & [X_2, X_3] = -2X_1 \end{array}$$

There are no ideals, so this baby is simple, rank 1, order 3 (degree 3). Suppose we scale X_2 and X_3 down:

$$\begin{array}{lll} X_2 = \frac{1}{\sqrt{2}} \tilde{X}_2 & [\tilde{X}_1, \tilde{X}_2] = -\tilde{X}_2 & \\ X_3 = \frac{1}{\sqrt{2}} \tilde{X}_3 & [\tilde{X}_1, \tilde{X}_3] = \tilde{X}_3 & \\ X_1 = \tilde{X}_1 & [\tilde{X}_2, \tilde{X}_3] = -\tilde{X}_1 & \end{array} \quad \left. \vphantom{\begin{array}{l} \\ \\ \end{array}} \right\} \begin{array}{l} \text{This does not look} \\ \text{like } \mathfrak{su}_2. \text{ Yes it is!} \end{array}$$

Now define: $\begin{array}{l} K_2 = X_2 \\ K_1 = X_1 \end{array}$ $\rightarrow$ Let $\begin{array}{l} J_1 = J_3 \\ J_2 = J_- \\ J_3 = J_+ \end{array}$ $\begin{array}{l} [J_3, J_-] = -J_- \\ [J_3, J_+] = J_+ \\ [J_+, J_-] = J_3 \end{array}$

Define $A_1 = X_2$

Conclusion: The 3 parameter group of transformations

$$x' = \frac{ax+b}{cx+d} \text{ is homomorphic to } \mathfrak{so}(3).$$

Frampson claims $\mathfrak{o}(2,1)$, but I still get $\mathfrak{so}(3)$ after checking.

Nyboerne, Chapter 5: "Lie Groups + Lie Algebras"

5.1 Here, we simply define a lie algebra as a vector space over either $\mathbb{R}$ or $\mathbb{C}$. Any two vectors generate a third, and the operation called "commutator" must be [linear, distributive, antisym, and jacobian] i.e.,

this is called the FIELD "over which" the vector space is defined. "K"

$$\textcircled{1} \quad [(\alpha X + \beta Y), Z] = \alpha [X, Z] + \beta [Y, Z]$$

$$\textcircled{2} \quad [\alpha X, Y] = \alpha [X, Y]$$

$$\textcircled{3} \quad [X, Y] = -[Y, X]$$

$$\textcircled{4} \quad \text{jacobi.}$$

Algebras relating to lie groups are always over $\mathbb{R}$, not $\mathbb{C}$. ✓

5.2 Given one set of basis elements, any invertible transform takes you to an equally valid basis. You can write down what the new structure constants will be. (see 5.8). ✓

5.3 Definition: homomorphism: suppose you have two lie algebras, possibly of different groups. If you can find a mapping $p: A \rightarrow A'$ that preserves certain properties, then A' is a (particular) homomorphism of A . Those conditions are:

$$[X, Y] = Z \Rightarrow [X', Y'] = Z' \quad \text{[commutator preserved]}$$

$$\alpha X + \beta Y = Z \Rightarrow \alpha X' + \beta Y' = Z' \quad \text{[linearity]}$$

The mapping must also be linear, i.e.,

$$p(X+Y) = (X+Y)' = pX + pY = X' + Y'$$

Basically, lie algebras which are homomorphic will have the same structure constants and a map $pX \Rightarrow X'$.

3 definition: isomorphism: a 1-to-1 homomorphism.

Note that [isomorphic lie algebras $\Rightarrow$ only locally isomorphic groups], not globally isomorphic groups. Especially this is true if one group has a piece that is isolated from the identity, whereas the other does not. Or maybe one group is compact, $SO(2)$, while the other is not, $T(1)$. ✓

5.4 def: automorphism — a one-to-one map (isomorphism) of a group onto the same group

def: inner automorphism — This map is $b' = aba^{-1}$ for some fixed $a \in G$. Outer is all the rest.

Theorem: The set of possible automorphisms forms a group, called $\text{aut}(G)$.

def: endomorphism — The relation between a group and one of its subgroups such that G is homo. mapped onto S . [Obviously, this will be a many-1-map].

SUBALGEBRAS.

5.5 def: subalgebra — $S \in G \Rightarrow X \in S \Rightarrow [X_1, X_2] \in S$. ie, just require closure. Here, the commutator is:

$$c: S \otimes S \rightarrow S \quad / S = \text{the subalgebra} /$$

IDEALS

5.6 def: ideal — a subset $I \in G$ is an ideal if

$$[i, g] = \sum c_i \quad \forall i \in I, \forall g \in G.$$

ie, $c: I \otimes G \rightarrow I$. Not quite the same as a subalgebra.
→ An ideal is a special subalgebra.

def: proper ideal: $G \neq I$. ie, G is bigger than I .
Obviously any group is one of its own ideals.
The whole algebra is an "improper" ideal.

def: maximal ideal $\equiv$ center: The largest subset of G such that

$$[i, g] = 0$$

It looks to me like the center of an algebra will be the set of casimirs.

ADJOINT REP.

5.7 def: you start with a representation X . Then

↑ see notes for

Chap 9, p 2,

for more details.

$$X' = [G, X]$$

forms an adjoint representation
[here, G is some fixed $g \in G$].

def: The identity element — suppose an algebra contains an identity element $X_i \ni XX_i = X$. Then $[X, X_i] = 0$. Since $\{0\}$ can be considered part of any set, the identity $\{X_i\}$ forms a trivial and improper ideal.

complex ext.

[5.8] def: complex extension — $Z_i = X + iY$ are the elements of this extension.
see notes on Master Anal Rep Paper

note: several real lies may have the same complex lie for their extensions.

SIMPLE etc.

[5.9] In general, an algebra is not "simple." But, if the only ideals of an algebra are ^{non}abelian, then the algebra is "semisimple." If there are no ideals at all (except the trivial ones), then it is simple.

other than $\{0\}$.
if there are any at all.

If an algebra has abelian ideals, it is neither simple nor semi.
If it has only nonabelian ideals, it is semi, but not simple.

no ideals	simple
only non-abel ideals	semi-simple
other cases	?

Theorem: A is semisimple $\Leftrightarrow A$ can be decomposed into sum of simples.

Metric Tensor.

[5.10] How do you tell if algebra is semi?

Theorem: $[\det |g_{\mu\nu}| = 0] \Leftrightarrow [\exists \text{ an abelian subalgebra}]$
 $\Leftrightarrow [\text{not semisimple}]$

So $[\det |g_{\mu\nu}| \neq 0] \Rightarrow \text{semisimple}$.

def: "metric tensor" : $g_{\mu\nu} = C_{\mu\nu}^{\alpha} \bar{C}_{\alpha}^{\nu}$.
Used above.

Cartan first showed this: $\det g \neq 0 \Rightarrow g^{\mu\nu}$ exists.

example: for $so(3)$, $g_{\mu\nu} = -2\delta_{\mu\nu}$, $\det |g| = 16$, so semisimple!

50(4)

5.11 Very nice example. $so(4)$ has 6 generators (elements), call them M and N . By change of basis, get J and K such that

$$[J, J] = J \quad [J, K] = 0 \quad [K, K] = K.$$

① The algebra $so(4)$ has here two nonabelian ideals. There are no abelian ideals. Thus, $so(4)$ is semisimple.

② These ideals, $\{J\}$ and $\{K\}$, are also subalgebras. Each subalgebra has no proper ideals and is thus simple.

③ This verifies the theorem: $so(4) = so(3) \oplus so(3)$.

④ In groups, $SO(4)$ is locally isomorphic to $SO(3) \otimes SO(3)$.

{Skip: 5.12, 13, 14, 15, 16.} see last sheets.

5.17: You can construct $\epsilon_{\mu\nu}$ out of the $e_i^{\pm}$. Note carefully:

$$C_{\mu\nu} = g_{\alpha\lambda} \epsilon^{\lambda}_{\mu} \epsilon^{\alpha}_{\nu}$$

↑ antisym, full

so suddenly we are thinking of ϵ^{λ}_{μ} as a mixed tensor in the algebra vector space.

Note that $g_{\mu\nu} = \epsilon^{\alpha}_{\mu} \epsilon^{\beta}_{\nu} \epsilon_{\alpha\beta}$

Casimirs

also called the Killing form.

5.18: Theorem: for a semisimple lie algebra [i.e., one for which $g_{\mu\nu}$ exists], the object $C = g^{\mu\nu} X_{\mu} X_{\nu}$ is a Casimir operator, hence by Schur, a multiple of the identity.

Originally, this was the Casimir operator. Only semisimple groups had a C.O. But now the idea is generalized, see p142

def: a negative definite Killing form $\Rightarrow g_{\mu\nu} = a \delta_{\mu\nu} \quad a < 0$

Theorem: $\text{neg def KF} \Rightarrow \text{def } g \neq 0 \Rightarrow \text{semisimple}$

COMPACT

5.20 Theorem: a compact Lie algebra is one whose Killing form is neg def. $\Rightarrow$ [compact $\Rightarrow$ semisimple]

def: a compact Lie group is the usual thing (finite + bounded parameters) but s.s. $\nRightarrow$ compact, since $so(2,1)$ is s.s.

Theorem: compact group $\Leftrightarrow$ compact algebra

5.21 Now we restate things in terms of the group.

def: invariant subgroup: $s \in S \Rightarrow "[s, g]" \in S$. Recall that "[G, G]" means $G_1 G_2 G_1^{-1} G_2^{-1}$ for group elements.

~~ide: subgroup means $s, s_0 \in \mathfrak{g}$~~
~~invar sub means $\text{con}(s, s_0) \in \mathfrak{g}$~~

~~It is the invariant subgroup that connects to a subalgebra, not a subgroup itself.~~

~~And recall: subalgebra is a special case of an ideal. An invariant subgroup $\Rightarrow$ a subalgebra and possibly also an ideal.~~

No invariant subgroups $\Rightarrow$ no ideals, clearly.
 So

def: simple group means no invariant $SG^{\mathfrak{g}}$.

semisimple means all invar subgroups, if any, are non-abelian.

def: direct product: $G = H \otimes K$ means $\exists$ two groups H, K closed with $[H, K] = 0$.

[X direct sum
 $\otimes$ semidirect sum]

Classification:

$$[\text{semisimple}] \stackrel{\text{def}}{\iff} [\exists \text{ no abelian ideals}]$$

$$\begin{array}{c} \nwarrow \quad \nearrow \\ [\det g \neq 0] \implies [\exists \text{ no abelian subalgebras}] \end{array}$$

Thus $[\det g \neq 0] \Rightarrow \text{semi}$.

This is not a $\iff$ as Theorem 5.2 claims, so far as I can see. Well, perhaps by some other route.

So, a semisimple Lie algebra can contain an abelian subalgebra, but no abelian ideals.

5.13 Derivations of Lie Algebras A

Look at your Lie algebra: $\{X_1, X_2, \dots, X_n\}$. Certain of these generators may be written as commutators of other generators. The set of such special generators we can call $A^{(1)} = \{X'_1, X'_2, \dots\}$.

Claim: $A^{(1)}$ is an ideal of A (hence also a subalgebra).

Proof: $[X'_i, X_j] = [X_i, X_j] = X'_k \in A^{(1)}$.

I.e., $A^{(1)} \times A \rightarrow A^{(1)}$. More precisely, I should do all this for linear combinations:

$$\text{eg, } X'_k = \sum_i a_{kij} \dots X_i$$

Now, $A^{(1)} \neq A$ when there is some abelianness floating around.

"first derived algebra" — that's what $A^{(1)}$ is of A .

Now take those elements in $A^{(1)}$ and form an ideal of $A^{(1)}$, namely $A^{(2)}$. Keep going so that:

$$\underbrace{A > A^{(1)} > A^{(2)} > \dots > A^{(k)}}_{\text{"derived series"}} > \dots$$

$\uparrow$ " k^{th} der

5.14 Solvable Lie Algebras

As you go down the derived series, you may come to a point where everything is abelian so that

$$A^{(k+1)} = [A^{(k)}, A^{(k)}] = 0$$

Then A is "solvable at the $(k+1)^{\text{st}}$ level" or just "solvable".

example: For E_2 , we have

$$A = \{X_1, X_2, X_3\}$$

$$A^{(1)} = \{X_2, X_3\}$$

$$A^{(2)} = 0$$

[solvable at the second level.]

example: For $SO(3)$, we have

$$A = \{X_1, X_2, X_3\}$$

$$A^{(1)} = \{X_1, X_2, X_3\}$$

never stops: i.e., $A = A^{(1)} = A^{(2)} = \dots$

Thus, $SO(3)$ is certainly not solvable.]

Similarly, Poincaré is not solvable (see p 346.)

example: an abelian lie algebra is always solvable at the 1st level since $A^{(1)} = 0$.

claim: The only way you stand a chance of being solvable is if A has an abelian subalgebra, like $\{X_2, X_3\} = 0$ of E_2 . So

solvable $\Rightarrow \exists$ an abelian subalgebra.

Note: [solvable $\Rightarrow$ there must be at least one proper ideal [i.e., in order to have $A_1 < A$]]

Thus: [simple $\Rightarrow$ no ideals $\Rightarrow$ not solvable.]

5.15 Nilpotent Lie Algebras

Recall the "derived" series:

$$A^{(1)} = [A, A]$$

$$A^{(2)} = [A^{(1)}, A^{(1)}]$$

$$A^{(3)} = [A^{(2)}, A^{(2)}]$$

⋮

$A^{(1)}$ was an ideal of A ; however, $A^{(2)}$ is not an ideal of A .
 $A^{(2)}$ isn't an ideal of $A^{(1)}$ either. ~~Q, like~~

~~However~~, ~~x~~ However, we do know that $A \supseteq A^{(1)} \supseteq A^{(2)}$ etc. Thus, $A^{(k)}$ is a subalgebra of $A^{(k-1)}$. We have, then, a [nested set of subalgebras] "derived series" $\rightarrow$ solvable.

Consider instead:

$$A^{[2]} = [A, A]$$

$$A^{[1]} = A$$

$$A^{[3]} = [A, A^{[2]}]$$

$$A^{[4]} = [A, A^{[3]}]$$

Now, all the $A^{[k]}$ are ideals of A . Again:

$$A \supseteq A^{[2]} \supseteq A^{[3]} \dots$$

[If this "descending sequence of ideals" stops, your algebra is "nilpotent" [analogous to solvable].

[Nilpotent $\Rightarrow$ Solvable but [Solvable $\nRightarrow$ Nilpotent].

Example E_2 :

$$A^{[1]} = A = \{X_1, X_2, X_3\}$$

$$A^{[2]} = \{X_2, X_3\}$$

$$A^{[3]} = \{X_2, X_3\}$$

$\therefore$ never ends, so not nilpotent.

5.16 Direct + Semidirect Sums (of algebras)

"direct sum" $= A = B \oplus C$ means that B and C are ~~over~~ both subalgebras, and $[B_i, C_j] = 0$.

[If we think of "0" as included in any set,

then $B \oplus C$ means

both B and C are ideals of A . i.e., $[B, A] \rightarrow B$

$$[C, A] \rightarrow C$$

More generally, $A = B \oplus C \oplus D$ means $[B_i, C_j] = 0$
 $[B_i, D_j] = 0$
 $[C_i, D_j] = 0$.

Again, A is partitioned into three sets, and

$[B, A] \rightarrow B, 0$ So B, C, D are each ideals.
 $[C, A] \rightarrow C, 0$
 $[D, A] \rightarrow D, 0$

Comments: This $\oplus$ has no relation to the direct sum of two matrices. Even if we "realize" our Lie generators by matrices, we could have + in fact do have $\{B\}, \{C\}, \{D\}, \{A\}$ all realized by matrices of the same size. This is in sharp contrast to $M_3 = M_1 \oplus M_2$ where the size of matrix M_3 is the sum of the size of $M_1 + M_2$.

Remark: if $A = B \oplus C$ = direct sum of two Lie algebras, then by definition, $A = B \times C$, where R, B, C are the associated Lie groups.

Example: $so(4) = so(3)_1 \oplus so(3)_2$

$$\Rightarrow SO(4) = SO(3)_1 \times SO(3)_2$$

I now hedge on this

Again: This is not a direct product of two matrices. There is the idea that

$$e^{i(\vec{a} \cdot \vec{K} + \vec{b} \cdot \vec{L})} = (e^{i\vec{a} \cdot \vec{K}}) \cdot (e^{i\vec{b} \cdot \vec{L}})$$

But you never think of a direct product of two matrices. In fact, you have here a regular product of two matrices of the same size (well).

Note: when we say $SO(4) = SO(3)_1 \times SO(3)_2$, we mean we can factorize a matrix rep of an $R \in SO(4)$ into two matrices IF we choose our generators in the correct way. The fact that such a choice of generators (such a "basis") exists is enough to say $A = B \times C$.

["semidirect sum" of two Lie algebras:

$A = B \oplus_s C$ means B, C both subalgebras,
but only B is an ideal of A .

i.e., $[B, B] \rightarrow B$

$[B, C] \rightarrow B$

so $[B, A] \rightarrow B \Rightarrow$ ideal

But $[C, C] \rightarrow C$

$[C, B] \rightarrow B$

so C not an ideal

Supertheorem: any Lie algebra A may be written

as:

$A = B \oplus_s C$

$\nwarrow$ a semisimple Lie algebra
 $\swarrow$ a solvable Lie algebra
and ideal of A .

Examples:

$E_3 = T_3 \oplus_s SO(3) \leftarrow$

i.e., $[P, J] \in P$

$[P, P] \in P$

$\nwarrow$ "Euclidean group"

Example:

$E_4 = T_4 \oplus_s SO(4) \leftarrow$

$IG = T_4 \oplus_s HG \leftarrow$

inhomo "galilean"
group

$\nwarrow$ homogal group, 6 parameters

(probably this is related to $SO(4)$,
or $SO(3) \oplus SO(3)$.)

$P = T_4 \oplus_s SO(3,1) \leftarrow$

$\nwarrow$ Poincaré

$\swarrow$ Lorentz

$\rightarrow$ { Wigner has not said much about the reps of
such non-ss. Lie groups. I want to know
more about this: "rays" + "white" + "little groups".

{ Maybe read M+S + go see Wigner's 1939 article.

Nybourne's M. 6: "Root Vectors"

6.1 Intro.

aim: to classify all "complex simple Lie algebras".

method: the "traditional Cartan-Weyl route"

"simple" means no ideals [no invariant subgroups], except of course the two trivial ideals.

"semisimple" means no abelian ideals. This allows for possible non-abelian invar. subgroups. $\sim$ always direct sum of simples

theorem: if a lie algebra is semisimple, then you can decompose it into a "direct sum" of simples.]

$$\text{SSLA} = \sum \text{SLA}$$

["direct sum" $\sim$ If you can partition a L.A. into two separate pieces A_i and B_i such that

$$[A_i, A_j] \in \{A\} \quad [B_i, B_j] \in \{B\} \quad [A_i, B_j] = 0$$

then $LA = A \oplus B$. In some sense, this means they do not "interact".

Chart method:

$\oplus$	A	B
A	A	0
B	0	B

$\oplus_s$	A	B
A	A	A
B	A	B

$\leftarrow$ "semidirect sum"

[In the "semidirect" case, A is an ideal since $A \cdot X \rightarrow A \quad \forall X$.
Convention: write the ideal first:

$$C = A \oplus_s B \leftarrow \begin{array}{l} \text{"residual subalgebra."} \\ \text{"ideal"} \end{array}$$

6.2 Standard Form.

Let $\begin{array}{l} A = a^n X_n \\ X = b^n X_n \end{array} \}$ two linear combs. of the lie generators,

The known algebra is: $[X_a, X_b] = C_{abc} X_c$

Think of commutator as an operator

$$[A, X] = \hat{O}_A X$$

Construct an eigenvalue equation: $[\hat{O}_A X = gX]$

This is:

$$a^\mu b^\nu C_{\mu\nu}^3 X_3 = g b^\nu X_3$$

$$\text{or } (a^\mu b^\nu C_{\mu\nu}^3 - g b^\nu) X_3 = 0 \quad b^\nu = \delta_{\nu}^3 b^\nu$$

$$\Rightarrow (a^\mu C_{\mu\nu}^3 - g \delta_{\nu}^3) b^\nu = 0$$

So think of b^ν as eigenvector in space whose basis vectors are the generators.

$$A_{ij} V_j = \lambda V_i \Rightarrow (A_{ij} - \lambda \delta_{ij}) V_j = 0$$

$\Rightarrow |A - \lambda I| = 0$ determines eigenvalues. So:

$\hookrightarrow |a^\mu C_{\mu\nu}^3 - g \delta_{\nu}^3| = 0$ determines g , given operator $\hat{O}_A$.

Note: in general, there will be n eigenvalues. if n is order of group. In general, there will be degeneracies, BUT:

Cartan's Theorem: if you choose A carefully (ie, the right set of q_μ), then

- 1) $g=0$ will be an eigenvalue, among others.
- 2) of all eigenvalues, only $g=0$ can be degenerate.

"rank" of a lie algebra is, then, the degeneracy of $g=0$.

(Applies to semisimples only)

denote rank by " l "

"semisimple" = direct sum of simples

"simple" = no ideals

["The H_i vectors" — there are l of them, they span the $g=0$ l dim subspace.

Thus: $\mathcal{O}_A \cdot H_i = 0$ since $g=0$, i.e., $[A, H_i] = 0$.

Recall: A is a special, key lincomb of the X_α , not just any old element.

"The E_α " ~ These are the other $n-1$ eigenvectors of A , eigenvalues are α , all non-degenerate:

$$\mathcal{O}_A \cdot E_\alpha = \alpha E_\alpha = [A, E_\alpha].$$

We have now rearranged the generators into "standard form"

$$\{X_1, X_2, \dots, X_n\} \rightarrow \underbrace{\{H_1, H_2, \dots, H_l\}}_{\text{cartan}}; \underbrace{\{E_1, E_2, \dots, E_{n-l}\}}_{\text{roots}}$$

Now A must be some lincomb of these.

$$A = \sum a_i H_i + \sum b_i E_i$$

Thus: $0 = [A, H_j] = \sum a_i [H_i, H_j] + \sum b_i \overbrace{[E_i, H_j]}^{*j E_i}$

We also know:

$$(*) \quad a_i E_i = [A, E_i] = \sum a_i \overbrace{[H_i, E_i]}^{*i E_i} + \sum b_i [E_i, E_j]$$

Cartan shows that, in fact, A is a lincomb only of the H weights, so that A itself lies in the l -dim, $g=0$ subspace:

$$A = \lambda_i H_i$$

6.3 Root Properties.

It will turn out that the various H_i 's all commute with one another. Then we can see why A is a lincomb only of the H_i 's, from (*) above. Somehow, these H_i 's will be the $\mathcal{B}$ basis.

In (6.9) we show that $[H_i, E_\alpha] = \text{const} \times E_\alpha$. Call this constant a_i or just α_i . Then we have some of the standard form structure constants

$$C_{i\alpha} = \alpha_i \delta_\alpha^i$$

Here i and α are restricted to sets, $\{H_i\}$ and $\{E_\alpha\}$.

But, if $[H_i, E_\alpha] = \alpha_i E_\alpha$ and $[A, E_\alpha] = \alpha E_\alpha$ and $A = \lambda^i H_i$
 Then:

$$\alpha E_\alpha = \lambda^i [H_i, E_\alpha] = \lambda^i \alpha_i E_\alpha \Rightarrow \alpha = \lambda^i \alpha_i$$

So $\lambda_i =$ a vector in $g=0$ subspace,
 $\alpha_i =$ same

And $\alpha = \alpha_i \cdot \lambda_i = (\lambda, \alpha)$

Now Jacobi says

$$[A, [E_\alpha, E_\beta]] + [E_\alpha, -\beta E_\beta] + [E_\beta, \alpha E_\alpha] = 0$$

$$\Rightarrow [A, [E_\alpha, E_\beta]] = (\alpha + \beta) [E_\alpha, E_\beta]$$

So $[E_\alpha, E_\beta]$ has eigenvalue $(\alpha + \beta)$ of A .

Case 1: $\alpha = -\beta$, then $[E_\alpha, E_{-\alpha}] = \text{const} \times \text{lincomb of } H_i\text{'s}$

$$[E_\alpha, E_{-\alpha}] = C_{\alpha, -\alpha}^i H_i$$

Case 2: $\alpha \neq -\beta \Rightarrow \alpha + \beta \neq 0 \Rightarrow [E_\alpha, E_\beta] = \text{const} \times E_{\alpha+\beta}$, i.e.,
 must be another one of the $E^{\pm}$. ("Naps")

To summarize so far:

$$[A, H_i] = 0$$

$$[H_i, E_\alpha] = \alpha_i E_\alpha$$

$$[A, E_\alpha] = \alpha E_\alpha$$

$$[E_\alpha, E_{-\alpha}] = C_{\alpha, -\alpha}^i H_i$$

$$A = \lambda^i H_i$$

$$[E_\alpha, E_\beta] = N_{\alpha\beta} E^{\alpha+\beta}$$

$$[H_i, H_j] = 0$$

I think we have the complete set of C_{ij}^k now in standard form.

Recall: "metric tensor" $g_{\mu\nu} = g_{\nu\mu} \equiv C_{\mu\beta}^3 C_{\nu\beta}^3$

LA semisimple $\Leftrightarrow g_{\mu\nu}$ is non-singular, hence invertible

$$C = g_{\mu\nu} X^\mu X^\nu = \text{Casimir}$$

6.4 Symmetry of the Roots

Let us compute the metric tensor:

$$g_{\alpha\beta} = \sum_{\mu, \eta} C_{\mu\alpha}^{\mu} C_{\eta\beta}^{\eta}$$

Need cleaner notation:

$\alpha, \beta, \gamma \dots$ denotes an E element.
 $i, j, k \dots$ denotes an H element.
 $a, b, c \dots$ denotes arbitrary el.

Then:

$$g_{ab} = \sum_{c,d} C_{ac}^d C_{bd}^c$$

Decompose this sum:

$$\textcircled{1} \quad \left[g_{ab} = \sum_{\alpha, \beta} C_{a\alpha}^{\beta} C_{b\beta}^{\alpha} + \sum_{\alpha, i} C_{a\alpha}^i C_{bi}^{\alpha} + \sum_{i, \alpha} C_{ai}^{\alpha} C_{b\alpha}^i + \sum_{i, j} C_{ai}^j C_{bj}^i \right]$$

Now let $a = \gamma$, a particular E subscript. Then

$$g_{\gamma b} = \sum_{\alpha, \beta} C_{\gamma\alpha}^{\beta} C_{b\beta}^{\alpha} + \sum_{\alpha, i} \underbrace{C_{\gamma\alpha}^i}_{=d_i \delta_{\alpha\gamma}} C_{bi}^{\alpha} + \sum_{i, \alpha} \underbrace{C_{\gamma\alpha}^{\alpha}}_{=d_i \delta_{\alpha\gamma}} C_{b\alpha}^i + \sum_{i, j} \underbrace{C_{\gamma\alpha}^j}_{=0} C_{bj}^i$$

Consider:

$$\left[C_{\gamma\alpha}^{\beta} \leftarrow \begin{array}{l} \beta \neq -\alpha, \text{ then this is } \delta_{\beta, \gamma+\alpha} C_{\gamma\alpha}^{\beta+\gamma+\alpha} N_{\gamma+\alpha} \\ \beta = -\alpha, \text{ then this is } 0. \end{array} \right]$$

Also:

$$C_{\gamma\alpha}^i = \delta_{\alpha, -\gamma} C_{\gamma, -\alpha}^i$$

include $i = \alpha$ type too if you like, since $C_{\gamma, -\gamma}^{\beta} = 0$.

$$\text{So far: } g_{\gamma b} = \sum_{\alpha, \beta} C_{\gamma\alpha}^{\beta} C_{b\beta}^{\alpha} + \sum_i C_{\gamma, -\gamma}^i C_{bi}^{-\gamma} + \sum_i d_i C_{b\gamma}^i$$

Redo first term:

$$\sum_{\beta} \left[\sum_{\alpha \neq -\gamma} C_{\gamma\alpha}^{\beta} C_{b\beta}^{\alpha} + 0 \right] = \sum_{\beta, \alpha \neq -\gamma} C_{\gamma\alpha}^{\beta+\gamma+\alpha} \delta_{\beta, \gamma+\alpha} C_{b\beta}^{\alpha}$$

$$\textcircled{2} \quad \left[g_{\gamma b} = \sum_{\alpha \neq -\gamma} C_{\gamma, \alpha}^{\alpha+\gamma} C_{b, \alpha+\gamma}^{\alpha} + \sum_i C_{\gamma, -\gamma}^i C_{bi}^{-\gamma} + \sum_i d_i C_{b\gamma}^i \right]$$

$$\text{So: } g_{\gamma b} = X$$

OK, this section is trying to prove:

"Suppose α is an eigenvalue (a "root"). Then, if $-\alpha$ is not also a root, then $\det(q_{\alpha\beta}) = 0$ and LA is not semi simple"

Thus: "For SLA's $\alpha = \text{root} \Rightarrow -\alpha \text{ also} = \text{root.}$ "]

This is the meaning of "root symmetry". I will skip the proof. [but see below]

6.5 The standard form obtained.

Go back to $q_{\alpha\beta}$ and let $\beta = \gamma$, in the $\alpha, \beta \dots E$ subspace.
Then:

$$q_{\alpha\gamma} = \sum_{\alpha \neq -\gamma} C_{\alpha, \gamma}^{\alpha+\gamma} C_{\gamma, \alpha+\gamma}^{\alpha} + \sum_i C_{\gamma, -\gamma}^i C_{\gamma, i}^{-\gamma} + \sum_i \alpha_i C_{\gamma, \gamma}^i$$

If $\gamma \neq -\gamma$, then $C_{\gamma, \gamma}^i = 0$ so last term vanishes.

Also, $C_{\gamma, i}^{-\gamma}$ vanishes unless $\gamma = -\gamma$. So

$$q_{\alpha\gamma} = \sum_{\alpha \neq -\gamma} C_{\alpha, \gamma}^{\alpha+\gamma} C_{\gamma, \alpha+\gamma}^{\alpha}$$

Now examine $C_{\gamma, \alpha+\gamma}^{\alpha}$

- a) If $\gamma \neq -(\alpha+\gamma)$, this is $\delta_{\alpha, \gamma+(\alpha+\gamma)} \times \text{something}$
- b) If $\gamma = -(\alpha+\gamma)$, this is 0.

In case a), only $\neq 0$ if $\gamma + \alpha + \gamma = \alpha$, i.e., if $\gamma = -\gamma$. If $\gamma \neq -\gamma$, this term vanishes also.

Major conclusion: in the "roots residual space" [i.e., not in the $\rho=0$ subspace], we have proven that:

$$q_{\alpha\beta} = 0 \text{ unless } \alpha + \beta = 0$$

I.e., $q_{\alpha, -\alpha} \neq 0$ but all other $q_{\alpha\beta} = 0$.

Now suppose there is one greek α such that $-\alpha$ is not a root. Then

$$g_{\alpha\beta} = \begin{matrix} & \beta & -\beta & \alpha \\ \begin{matrix} \beta \\ -\beta \\ \alpha \end{matrix} & \begin{pmatrix} 0 & X & 0 \\ X & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{matrix} \leftarrow \text{Then an entire row vanishes.}$$

Now consider the "mixed" elements of g :

$$g_{\alpha\beta} = \sum_{\alpha \neq -\beta} C_{\alpha,\beta}^{\alpha+\beta} \underbrace{C_{j,\alpha+\beta}^{\alpha}}_{=0} + \sum_i C_{\alpha,-\beta}^i \underbrace{C_{ji}^{-\beta}}_{=0} + \sum_i \alpha_i \underbrace{C_{j\beta}^i}_{=0}$$

Remember, $C_{i\beta}^{\alpha} = 0$ unless $\alpha = \beta$

$$C_{ij}^{\alpha} = 0 \quad \forall \alpha$$

$$C_{j\alpha}^i = 0 \text{ because } H \times E \rightarrow E \text{ only.}$$

Thus: $\boxed{g_{\alpha i} = 0} \quad \boxed{g_{\alpha\beta} = \delta_{\alpha,-\beta} \cdot g_{\alpha,-\beta}}$

Now, back to the above: if one greek, say α , is unpartnered with a root $-\alpha$: Then an entire row of the full g_{ab} vanishes $\Rightarrow \det g_{ab} = 0 \Rightarrow$ not SSLA.

This was the proof given so poorly in last section.

By the way,

$$g_{\alpha,-\beta} = \sum_{\alpha \neq -\beta} C_{\alpha,\beta}^{\alpha+\beta} C_{-\beta,\alpha+\beta}^{\alpha} + \sum_i \left[C_{\alpha,-\beta}^i C_{-i,\alpha}^{-\beta} + \alpha_i C_{-\beta,\alpha}^i \right]$$

Suppose we were to rescale separately all the E 's

$$\tilde{E}_{\alpha} = S_{\alpha} E_{\alpha}$$

This would change only the $C_{\alpha,-\alpha}$ and $N_{\alpha\beta} = C_{\alpha\beta}^{\alpha+\beta}$ constants.

$$\tilde{C}_{\alpha,-\alpha} = S_{\alpha} S_{-\alpha} \times C_{\alpha,-\alpha}$$

$$\tilde{N}_{\alpha\beta} = N_{\alpha\beta} \cdot \left(\frac{S_{\alpha} S_{\beta}}{S_{\alpha+\beta}} \right)$$

By so rescaling, we caught $\boxed{g_{\alpha,-\alpha} = 1 \quad \forall \alpha.}$

So, if we order things $\alpha, -\alpha, \beta, -\beta, \gamma, -\gamma \dots$ we have

$$g_{\mu\nu} = \begin{pmatrix} g_{ij} & \phi \\ \phi & \begin{matrix} 0 & 1 \\ 1 & 0 & 1 \text{ etc} \\ & 1 & 0 \end{matrix} \end{pmatrix}$$

$$\text{Now } g_{ij} = \sum_{a,b} C_a^b C_b^i = \sum_{a,b} C_a^b C_{j,b}^a = \sum_a C_a^i C_{j,a}^a$$

$\uparrow$
a must be Greek

$$= \sum_a \alpha_i \alpha_j$$

Compute:

$$\alpha^i \equiv g^{ik} \alpha_k = \sum_k g^{ik} \alpha_k = \left(\sum_b \beta_i \beta_k \right) \alpha_k$$

No:



$$C_{i,-\alpha}^i = g^{ik} C_{k,-\alpha} = g^{ik} C_{i,-\alpha,k} = g^{ik} C_{k,\alpha,-\alpha}$$

Here, we use fact that $C_{\mu\nu}^{\lambda}$ is totally antisym.

$$\text{My own proof: } C_{i,\alpha}^i = g^{ik} C_{k,\alpha}^i = g^{ik} C_{\alpha,k}^i$$

$$\text{But } C_{\alpha,k}^i = \sum_{\beta} g_{\alpha\beta} C_{\beta,k}^i = C_{\alpha,k}^{\alpha}$$

But $g_{\alpha\beta} = \delta_{\alpha\beta}$ from (6.20), so

$$\text{Thus: } C_{i,\alpha}^i = g^{ik} C_{\alpha,k}^i = g^{ik} \alpha_k = \alpha^i$$

Therefore, we have:

A) a metric tensor g_{ab} specified by (6.20) and (6.21).

B) a set of constants α_i

C) another set $N_{\alpha\beta}$

[In fact, the set $\{\alpha_i\}$ determines g_{ik} , + thus $\{\alpha^i\}$.]

So any SSA is specified by:

$2, 1, \dots, \{\alpha_i\}, \{N_{\alpha\beta}\}$

$\uparrow$
rank
 $\uparrow$
order

$\nwarrow$ 2 numbers



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7

+

+

4

+

1

6

"The lowering concept" ~ if you start with some E_γ , you can lower it via:

$$[E_{-\alpha}, E_\gamma] \sim E_{\gamma-\alpha} \quad [\alpha \neq \gamma]$$

$$[E_{-\alpha}, E_{\gamma-\alpha}] \sim E_{\gamma-2\alpha} \text{ etc.}$$

Eventually, you run off the finite set of E 's, so there is some g s.t.

$$[E_{-\alpha}, E_{\gamma-g\alpha}] = 0$$

Crude picture:

$$E_{\gamma-g\alpha} \quad E_{\gamma-2\alpha} \quad E_{\gamma-\alpha} \quad E_\gamma$$

Specifics: $[E_\alpha, E'_{\gamma-j\alpha-\alpha}] = \mu_{j+1} E'_{\gamma-j\alpha}$
 $\uparrow$ raising

Can show:

$$\mu_j = j(\alpha, \gamma) - \frac{j(j-1)}{2}(\alpha, \alpha)$$

? ["strings": if $\begin{cases} \alpha = \text{root} \\ \gamma = \text{root} \\ \alpha + \gamma \neq \text{root} \end{cases}$, then $\gamma, \gamma-\alpha, \gamma-2\alpha, \dots, \gamma-g\alpha$ is a string of roots.

$$g = \frac{2(\alpha, \gamma)}{(\alpha, \alpha)}$$

Repeat: given two roots α, γ , construct g .
 Then you have your self a string of roots:

$$\gamma, \gamma-\alpha, \gamma-2\alpha, \gamma-3\alpha, \dots, \gamma-g\alpha$$

I think that all roots lie on some string.

So theorem says: given α, β roots, g is an integer, and $\beta - g\alpha$ is a root.

Theorem: if α is a root, then of course $(\alpha, 0, -\alpha)$ are roots. But 2α + other multiples of α are not roots.

Theorem: $g = 0, \pm 1, \pm 2, \pm 3$ and no other possibilities.

6.7 Cartan-Weyl Normalization

$$\begin{aligned} N_{\alpha, \beta} &= N_{-\beta, -\alpha} \\ N_{\alpha, \beta} &= -N_{\beta, \alpha} \text{ etc.} \end{aligned} \quad \left. \vphantom{\begin{aligned} N_{\alpha, \beta} &= N_{-\beta, -\alpha} \\ N_{\alpha, \beta} &= -N_{\beta, \alpha} \end{aligned}} \right\} \text{big deal.}$$

6.8 Graphical Root Vectors

α = "root vector" = vector in weight space,

$\mathfrak{g}=0$ subspace = "weight space"]

Think now of $l=2$. Then $\alpha = (\alpha_1, \alpha_2)$ $\beta = (\beta_1, \beta_2)$

Define an angle: $(\alpha, \beta) = \|\alpha\| \|\beta\| \cos \phi$

$$\cos^2 \phi = \frac{(\alpha, \beta)}{(\alpha, \alpha)} \cdot \frac{(\beta, \beta)}{(\beta, \beta)} \in \left[0, \pm \frac{1}{2}, \pm 1, \pm \frac{3}{2}\right] \times \left[0, \pm \frac{1}{2}, \pm 1, \pm \frac{3}{2}\right]$$

Why not $\frac{3}{4}$? Well, > 1 .

I suppose you can do: $(\alpha, \beta) \leq (\alpha, \alpha)(\beta, \beta)$ [Schwarz].

Then: $\cos^2 \phi = 0, \frac{1}{4}, \frac{3}{4}, \frac{3}{4}, 1$

$\Rightarrow \phi = 0^\circ, 30^\circ, 45^\circ, 60^\circ, 90^\circ$

[analyze 30° : we have $\frac{(\alpha, \beta)}{(\alpha, \alpha)} \cdot \frac{(\beta, \beta)}{(\beta, \beta)} = \frac{N}{2} \cdot \frac{M}{2} = \cos^2 \phi$.

since both must be half integers.

For 30° , $\frac{MN}{4} = \frac{3}{4} \Rightarrow MN = 3 \Rightarrow (m, N) = (1, 3) \text{ or } (3, 1)$.

$$\frac{(\beta, \beta)}{(\alpha, \alpha)} = \frac{N/2}{M/2} = \frac{N}{M} = \frac{1}{3} \text{ or } 3 \text{ in this case.}$$

So: $\left[\begin{array}{l} NM = 4 \cos^2 \phi \\ \frac{N}{M} = \frac{(\beta, \beta)}{(\alpha, \alpha)} \end{array} \right] \leftarrow \text{this sets possible values of } MN$

$\leftarrow \text{this sets lengths.}$

Now, the problem is to construct all possible lie algebras.
The general idea is:

- 1) start always with the known root α .
- 2) plot α , and $-\alpha$ of, say, unit length.
- 3) find a second root β by rotating up thru angle apropo. Choose apropo length for chosen angle.
- 4) Having chosen your angle, you now have knowledge

$$\frac{(\alpha, \beta)}{(\alpha, \alpha)} = \frac{N}{2} \quad N = \text{some intgr.}$$

I.e., you have a set of values for N .

- 5) Use these N to construct new roots via:

$$\gamma = \beta - N\alpha.$$

- 6) Note: all pairs of root vectors do not nec. have the same angle.

Question: why does this never occur: 

I.e., why not two roots with $\phi = 0$.

Answer: $\phi = 0 \Rightarrow \cos^2 \phi = 1 \Rightarrow (\alpha, \beta)^2 = (\alpha, \alpha)(\beta, \beta)$.

This can only happen if $\alpha = \beta$. But then, same root.

The rank-1 lie algebras:

One dim. weight space:

$$-\alpha \longleftrightarrow \alpha$$

That's it: no where to add anything. ϕ has to be 0° , so all other vectors are $\pm \alpha$.

This is A_1 [$su(2), so(3) \dots$].

$$-\alpha \quad 0 \quad \alpha$$

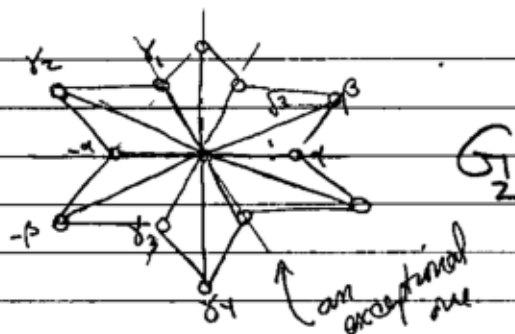
6.9 Lie Algebras of Rank 2

Construct G_2 : $\phi = 30^\circ$.

$$\gamma = \beta - N\alpha \quad N = +1 \text{ or } 3$$

So

$$\gamma_1 = \beta - \alpha \quad \gamma_2 = \beta - 3\alpha$$



Now, just as (α, β) created (γ_1, γ_2) , use (γ_1, γ_2) to create (γ_3, γ_4)

Had we started using $\frac{1}{3}\alpha$, we'd get rotated figure.

Similarly, we can make: B_2, C_2, A_2, D_2
 Why not the same?

6.10 Rank > 2 LA's

First, we can make B_l for any $l = 2, 3, 4, \dots$ as follows:

a) Take l unit vectors along the axes
 $e_i = (1, 0, \dots, 0)$ etc.

b) form all vectors: $\pm e_i$ $\pm e_i \pm e_j$ $= 2l(l-1)$
 How many? $\pm e_i \pm e_j$ $= ?$ $= 2 \times (\text{number of } \pm e_i \pm e_j)$
 $2l$ $l(2 \cdot (l-1)/2)$

So: $2l(l-1) + 2l = 2l^2$ vectors, not null $\Rightarrow l(2l+1)$ in all.
 + l null vectors

Recall: $l = \text{size of the } \mathfrak{g} = \mathfrak{o}$ subspace.
 $2N = \text{number of other roots}$ (must be even)
 $l + 2N = n$, size of algebra (# generators)

So B_l has $l(2l+1)$ generators

$$\begin{array}{ccccccc} \text{SO}(2l+1) & = & \text{SO}(5) & , & \text{SO}(7) & , & \text{SO}(9) & , & \dots \\ & & \uparrow & & \uparrow & & \uparrow & & \\ & & B_3 & & B_5 & & B_7 & & \end{array}$$

$$\boxed{B_l = \text{SO}(2l+1)} \\ l = 2, 3, 4, \dots$$

Table of results:

B_l	$\{\pm e_i, \pm e_i \pm e_j\}$	$l(2l+1)$	$so(2l+1)$
C_l	$\{\pm 2e_i, \pm e_i \pm e_j\}$	$l(2l+1)$	$sp(2l)$
D_l	$\{\pm e_i \pm e_j\}$	$2l(2l-1)$	$so(2l)$
A_l	$?$	$l(l+2)$	$su(l+1)$

6.11 The exceptional algebras

The above list completely exhausts all possible semisimple lie algebras, except for 5. I am amazed. These 5 are the "exceptionals."

	order
G_2 "star of david"	14
F_4	52
E_6	78
E_7	133
E_8	248

Of all these lie algebras, I have only touched $su(2)$ and $su(3)$ [and $su(1,1) \cong su(2)$, $so(1,1) \cong so(2) = su(1)$].

I wonder if any of these higher algebras can be put to work?

- What are these symplectic groups?
- Is $gl(n, \mathbb{C})$ a lie algebra?

Probably yes, but probably not semisimple.

7.1 Simple Roots

Π = set of "simple roots" = subset of Σ = all roots.

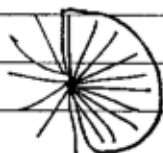
"positive" root means: first non-vanishing component of α_i = root vector is positive. (ie, root is on right half space)

We shall have an interest in: sum of positive roots.

If you write down all + roots of a Lie algebra, some will be expressible as sums of others. Throw out all but those that cannot be so decomposed. This is the set Π of "simple roots".

There are always 2 such roots, + they span the weight space. I do not think the set of simple roots is unique.

Simple roots are selected from the + roots:



Positive roots are on right side of Weyl diagrams.

Data Theorems:

a) if α, β are simple, then $\alpha - \beta$ is not simple.

proof: it is designed that two simple roots must be separated by θ where $\pi/2 \leq \theta < \pi$. The difference would violate this. Of course $\alpha + \beta$ isn't simple either, by def.

b) that quantity we call q , namely $2(\alpha, \beta)/(\alpha, \alpha)$, is equal to a negative integer, $-p$. [if α, β simple]

This is consistent with $(\alpha, \beta) < 0 \Rightarrow \cos \phi$ is in second quadrant $\checkmark$.

c) The angle/length table is:

90°	—	?
120°	—	1
135°	—	$\sqrt{2}$
150°	—	$\sqrt{3}$

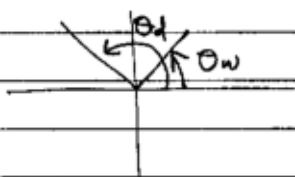
7.2 Examples B_2 and B_3 — listing the simple roots.

Given $\alpha = (0, 1)$ and $\beta = (1, -1)$, how do you compute (α, β) ?
You need to know the metric tensor.

Well, for B_2 we know $\phi = 45^\circ$, so $\cos^2 \phi = \frac{1}{2}$.

$$\cos \phi = \sqrt{\frac{1}{2}} = -\frac{1}{\sqrt{2}} \quad \text{because want } \phi \in \text{sec. quadrant}$$

Hence, $45^\circ \Rightarrow \Theta = 135^\circ$.



Still, I have a feeling that, at least for simple roots,

$$(\alpha, \beta) = \sum \alpha_i \beta_i.$$

Especially since we are thinking of weight space as Euclidean.

7.3 Dynkin Diagrams

Rules: $\bullet$ = short root
 $\circ$ = long root

$$\text{---} \curvearrowright \Rightarrow 120^\circ$$

$$\text{---} \curvearrowright \Rightarrow 135^\circ$$

$$\text{---} \curvearrowright \Rightarrow 150^\circ$$

$$\text{---} \text{---} \Rightarrow 90^\circ$$

Thus,

$$B_2 = \beta \circ \text{---} \bullet \alpha$$

$$\text{And } B_3 = \begin{array}{c} \alpha \text{---} \beta \text{---} \gamma \\ \alpha \quad \beta \quad \gamma \end{array}$$

B_3

What is B_4 ?

$$\alpha \text{---} \beta$$

B_4

$$B_n \text{ has } \begin{array}{c} \alpha \text{---} \beta \text{---} \gamma_1 \text{---} \gamma_2 \text{---} \dots \text{---} \gamma_n \\ \alpha \quad \beta \quad \gamma_1 \quad \gamma_2 \quad \dots \quad \gamma_n \end{array}$$

Claim: no two LA^3 have the same dynkin diagram

7.4 The Cartan Matrix

Now label the simple roots: $\Pi = \{\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_\ell\}$.

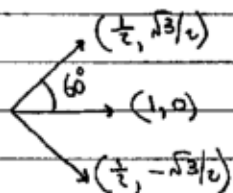
Define:

$$A_{ij} = \frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)} = l \times l \text{ matrix "Cartan Matrix"}$$

Entries are 0, -1, -2, -3 if $\alpha_i \neq \alpha_j$ i.e. $i \neq j$ i.e. off diagonal.
This is Thm 6.4 + Thm 7.1. $\alpha_i = \alpha_j \Rightarrow i = j \Rightarrow A_{ii} = 2$.

7.5 Examples of Cartan Matrices

(a) try $SU(3) = A_2$. List off the roots on right side:



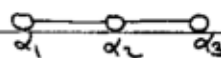
The other 3 roots are negative.
(1, 0) decomposes.

The simple roots are
Neither one is short.



(b) try $SU(4) = A_3$.

Claim:



The picture conveys

- 1) The angle between any pair of simple roots, hence
- 2) Their length ratio via (7.8)
- 3) The short root = length 1.

Thus, we know all (α_i, α_i) .

Hence, can compute all (α_i, α_j) , hence can construct the Cartan matrix. ✓ see rules tucked at end.

We can label each circle [simple root] with its length relative to the short root, i.e. (α_i, α_i) . ✓

In fact, a 3 occurs only in the G_2 exceptional.
Even two's are rare. ✓

To conclude, we can — just by looking at the Dynkin on p 80,
we can at once construct + list the Cartan matrices
for all the classical Lie algebras! ✓

7.6 Enumeration of Roots via Cartan Matrix.

Notation: α_i is not a component of root α
 α_i is now the i^{th} simple root, so $\alpha_i \in \Pi$.

Any root may be written: $\beta = \sum k_i \alpha_i$

Any positive root may be written in same way, but all k_i will be non-neg integers!

"level" of a root β , denoted $|\beta|$ is $\sum |k_i|$.

Let β be a root of level 4; let $l = 3$.

Eg, $\beta = 2\alpha_1 + \alpha_2 + \alpha_3$

Then, to generate a level 5 root, you add an α_i to a level 4 root.

Claim: $\{\alpha^{[n+1]}\} = \{\alpha^{[n]} + \alpha_i\}$.

But: when is $\alpha^{[n]} + \alpha_i$ a root?

If $\alpha^{[n]} + \alpha_i$ is a level $n+1$ root, then ...?

Consider these entities: $\begin{bmatrix} \alpha^{[n]} - \alpha_i \\ \alpha^{[n]} - 2\alpha_i \end{bmatrix}$ etc.

Some of these are roots of lower level. If we know all roots up thru level n , then we may identify the terms in this series which are roots.

But, roots come in "strings", so this string will descend down to

$$\alpha, \alpha - \alpha_i, \alpha - 2\alpha_i, \dots, \alpha - r\alpha_i$$

$\nwarrow$ we know r .

But, string may ascend up to $\alpha + q\alpha_i$, so full string is

$$[\alpha + q\alpha_i], [\alpha + q\alpha_i] - \alpha_i, \dots, \alpha, \alpha - \alpha_i, \alpha - 2\alpha_i, \dots, [\alpha + q\alpha_i] - (r+q)\alpha_i$$

So, $g = \text{"order of string"} - 1 = r + q$.

However, earlier we found that

$$q = \frac{2(\alpha, \delta)}{(\alpha, \alpha)} \quad \text{where } \delta = \text{high end of string} \\ \alpha = \text{step.}$$

In our case, then,

$$q = r + q = \frac{2(\alpha + q\alpha_i, \alpha_i)}{(\alpha_i, \alpha_i)} = \frac{2(\alpha, \alpha_i)}{(\alpha_i, \alpha_i)} + 2q$$

$$\Rightarrow q = r - \frac{2(\alpha, \alpha_i)}{(\alpha_i, \alpha_i)} \quad \text{But } \alpha = \sum k_j \alpha_j$$

$$\text{So } q = r - 2 \sum k_j A_{ji}$$

The point: if we know all roots up to level n , α is one of these, we know where the "lower end" of the string containing α lies, i.e., we know r .

But here we see that, in fact, we also know q from r , α , and the Dynkin diagram (i.e., Cartan matrix).

If $q > 0$, then $\alpha + \alpha_j$ is a root. So, for each α_j , find $r^{(j)}$, compute $q^{(j)}$, then you know whether $\alpha + \alpha_j$ makes it.

This allows you to construct all roots of level $n+1$:

a) for each $\alpha \in \text{level } n$,

b) test each α_j as described above, to see if $\alpha + \alpha_j$ is a $n+1$ root.

7.7. Application to G_2

Execute the above "program" on G_2 . We have $\Pi = \{\alpha_1, \alpha_2\}$ for starters. We are at level 1. To make level 2, we try

$$\begin{array}{l} \alpha_1 + \alpha_1, \alpha_1 + \alpha_2 \leftarrow \text{Since } \alpha_1 - \alpha_2 \text{ not root, } r = 0. \\ \alpha_2 + \alpha_1, \alpha_2 + \alpha_2 \end{array} \quad \text{Then } q = 0 - [1 \cdot (-1)] = 1.$$

This string is, in toto:

The other one here

$r = 0, q = 3$, so

$$\begin{array}{cccc} \alpha_2 & \alpha_2 + \alpha_1 & \alpha_1 & \alpha_1 + \alpha_2 \\ \text{lev 1} & \text{lev 2} & \text{lev 3} & \text{lev 4} \end{array}$$

Now we start with the level 2 roots. Only one: $\alpha_1 + \alpha_2$

$$\Rightarrow \alpha^{[2]} = (\alpha_1 + \alpha_2) = \sum k_i \alpha_i \quad k_1 = k_2 = 1, \text{ else } = 0.$$

Now, consider $\alpha - \alpha_2, \dots \Rightarrow h^{(2)} = 1.$

Look instead at $\alpha^{[4]} = \alpha_1 + 3\alpha_2.$

If we use α_1 as "step", then $n=0$, cannot step down.

But

$q = +1$, thus $2\alpha_1 + 3\alpha_2$ is a root!

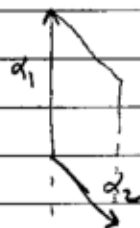
OK, so we have used the procedure to find all + roots of $G_2 = \text{so}(3)$ of David.

Draw this to see where strings are:

$\alpha_2, \alpha_2 + \alpha_1$

$\alpha_1, \alpha_1 + \alpha_2, \alpha_1 + 2\alpha_2, \alpha_1 + 3\alpha_2$

$(\alpha_1 + 3\alpha_2), (\alpha_1 + 3\alpha_2) + \alpha_1$



In the text, I have shown how these three "strings" exhaust the positive roots of G_2 (see p 68).

2.8 Construction of some Simple Lie Algebras.

$\text{so}(3)$: from 6.24 we say:

$$[H_1, H_1] = 0$$

$$[H_1, E_\alpha] = \alpha E_\alpha \quad \text{since } \alpha_i = \alpha, \alpha \text{ has only one component}$$

$$[H_1, E_{-\alpha}] = -\alpha E_{-\alpha}$$

$$[E_\alpha, E_{-\alpha}] = H_1$$

Identify: $H_1 = J_3 \quad E_\alpha = J_+ \quad E_{-\alpha} = J_-$

This is the usual $\text{so}(3)$ algebra.

$SU(3)$: construct your roots $\alpha, \beta, \alpha+\beta$ explicitly.

Then: $\alpha = (\alpha_1, \alpha_2) = (\frac{1}{\sqrt{3}}, \frac{1}{2})$ etc.

$l=2$, so we know all the vectors involved, so $\{d_i\}$ = known.

Thus, we can write out $[H_i, E_\alpha] = d_i E_\alpha$.

Eg, $[H_1, E_{+\alpha}] = (+\alpha)_1 E_{+\alpha} = \frac{1}{\sqrt{3}} E_{+\alpha}$.

Similarly, we know $[E_\alpha, E_{-\alpha}]$ stuff.

Together $N_{\alpha, \beta}$, $\left[\begin{array}{c} \alpha \\ \beta \end{array} \right] ?$

α	$-\alpha$	β	$-\beta$	$\alpha+\beta$	$-(\alpha+\beta)$
0	X	.	.	.	.
		X	.	.	.
			X	.	.
				X	.
					X

10 elements to $N_{\alpha, \beta}$.

Turns out, $N_{\alpha, \beta} = \frac{1}{\sqrt{6}}$... technical detail.

The point: from the dynkin, get α, β your starter roots. (simple roots). From these, construct all (nonzero) roots using the method just illustrated from G_2 . (ie, construct all strings).

Then, once you know all roots explicitly, you know all your d_i . Choose cartesian metric + construct the algebra. The only bitch:

a) pick the string $(\beta+j\alpha, \dots, \beta-k\alpha)$. Then, we know $j, k, (d, d)$, so

$$N_{\alpha, \beta} \cdot N_{-\alpha, (\alpha+\beta)} = \dots \quad (6.36)$$

This relates two elements. Not clear how one sets $N_{\alpha, \beta}$.

ADDED material on chapters 6 + 7.

What is the object $2(\alpha, \beta)/(\alpha, \alpha)$? $\rightarrow$ see last page (-4-)

See p 62. Let δ be on the end of its α -string. This means that $\delta + \alpha$ is not a root, but $\delta - \alpha, \delta - 2\alpha$ etc may be roots.

In fact, we know how to "lower" δ :

$$[E_{-\alpha}, E_{\delta}] = N_{\alpha, \delta} E_{\delta - \alpha}, \text{ as long as } \alpha \neq \delta.$$

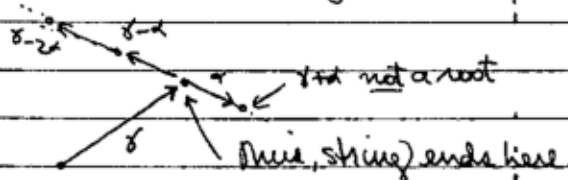
The only way you have a problem here is if you "lower" the same vector α you are lowering by:

$$[E_{-\alpha}, E_{\alpha}] = \alpha \cdot H \quad \text{ie} \quad \longleftrightarrow \alpha$$

You never want to carry a root into the "center". So, assume $\alpha \neq \delta$, then

$$[E_{-\alpha}, E_{\delta}] = N_{\alpha, \delta} E_{\delta - \alpha}.$$

If you think of δ and α as vectors in the weight space [covariant, say], then what we know is: $\vec{\alpha}$ and $\vec{\delta}$ are not colinear. Thus, the sequential lowerings look like:



Some never have to worry about accidentally hitting the center.

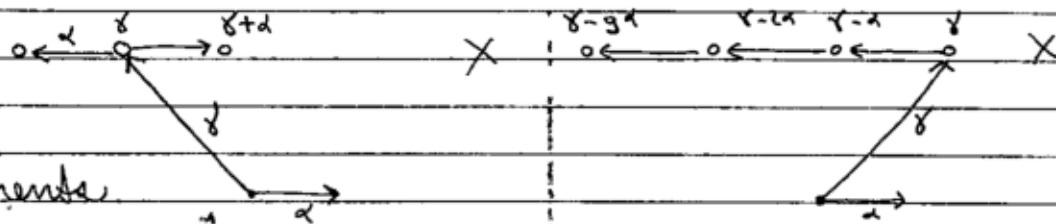
$$\text{Next: } [E_{-\alpha}, E_{\delta - \alpha}] = N_{-\alpha, \delta - \alpha} E_{\delta - 2\alpha}$$

$$[E_{-\alpha}, E_{\delta - 2\alpha}] = N_{-\alpha, \delta - 2\alpha} E_{\delta - 3\alpha}$$

$$\text{In general: } [E_{-\alpha}, E_{\delta - j\alpha}] = N_{-\alpha, \delta - j\alpha} E_{\delta - (j+1)\alpha}. \quad (*) \quad \text{This is 6.25.}$$

Eventually, when $j = g$, there is no more: $[E_{-\alpha}, E_{\delta - g\alpha}] = 0$. We know this because size of the non-center is finite.

Thus:



String has g segments.

Now how do you "raise" the roots?

$$[E_{\alpha}, E_{\gamma-(j+1)\alpha}] = N_{\alpha, \gamma-(j+1)\alpha} E_{\gamma-j\alpha}$$

Now, suppose we lower, then at once raise again:

$$[E_{\alpha}, E_{\gamma-j\alpha}] = (N_{-\alpha, \gamma-j\alpha} E_{\gamma-(j+1)\alpha}) = C$$

$$\text{Then: } [E_{\alpha}, C] = N_{-\alpha, \gamma-j\alpha} [E_{\alpha}, E_{\gamma-(j+1)\alpha}]$$

$$= N_{-\alpha, \gamma-j\alpha} \cdot N_{\alpha, \gamma-(j+1)\alpha} E_{\gamma-j\alpha} \quad \mu_{j+1}$$

$$\text{Thus: } [E_{\alpha}, [E_{-\alpha}, E_{\gamma-j\alpha}]] = N_{-\alpha, \gamma-j\alpha} N_{\alpha, \gamma-(j+1)\alpha} E_{\gamma-j\alpha}$$

$$\text{Shorthand: let } N_{\alpha, \gamma-(j+1)\alpha} = \cancel{d_{j+1}} d_{j+1}$$

$$\text{Then: } [E_{\alpha}, E_{\gamma-(j+1)\alpha}] = d_{j+1} E_{\gamma-j\alpha}$$

$$[E_{\alpha}, [E_{-\alpha}, E_{\gamma-j\alpha}]] = \text{ouch}$$

Let's use Jacobi right here:

$$\text{key!} = \alpha^i H_i$$

$$= -[E_{-\alpha}, [E_{\gamma-j\alpha}, E_{\alpha}]] - [E_{\gamma-j\alpha}, [E_{\alpha}, E_{-\alpha}]]$$

$$= +[E_{-\alpha}, [E_{\alpha}, E_{\gamma-j\alpha}]] - [E_{\gamma-j\alpha}, \alpha^i H_i]$$

$$N_{\alpha, \gamma-j\alpha} E_{\gamma-(j-1)\alpha} *$$

$$= + N_{\alpha, \gamma-j\alpha} \frac{[E_{-\alpha}, E_{\gamma-(j-1)\alpha}]}{N_{-\alpha, \gamma-(j-1)\alpha} E_{\gamma-j\alpha}} + \alpha^i \frac{[H_i, E_{\gamma-j\alpha}]}{(\gamma-j\alpha_i) E_{\gamma-j\alpha}}$$

$$= \underbrace{N_{\alpha, \gamma-j\alpha} N_{-\alpha, \gamma-(j-1)\alpha}}_{\mu_j} E_{\gamma-j\alpha} + (\alpha, \gamma-j\alpha) E_{\gamma-j\alpha}$$

Therefore, we have discovered that:

$$\left[\underbrace{N_{\alpha, \gamma-j\alpha}}_{\mu_j} \underbrace{N_{-\alpha, \gamma-(j-1)\alpha}}_{\mu_{j-1}} = \underbrace{N_{\alpha, \gamma-j\alpha}}_{\mu_j} \underbrace{N_{-\alpha, \gamma-(j-1)\alpha}}_{\mu_{j-1}} + (\alpha, \gamma-j\alpha) \right]$$

where I'm writing these as $N_{\alpha}^{\mu_j}$ & $N_{-\alpha}^{\mu_{j-1}}$

call this $(\alpha \star \star)$ μ_{j+1} , then

this is μ_j ✓

So, our explicit work shows:

$$\mu_{j+1} = \mu_j + (\alpha, \gamma) - j(\alpha, \alpha)$$

where:

$$\mu_j = N_{\gamma} \frac{\gamma - j\alpha}{\gamma - (j-1)\alpha}$$

These eqns don't work for $j=0$ because $E_{\gamma-(j-1)\alpha} = E_{\gamma+\alpha} = 0!$ (see *)
So our first equation is really:

$$\mu_1 E_{\gamma} = 0 + (\alpha, \gamma) E_{\gamma} \Rightarrow \mu_1 = (\alpha, \gamma) \checkmark$$

Then

$$\begin{aligned} \mu_2 &= 2(\alpha, \gamma) - 1(\alpha, \alpha) \cdot 1 \quad \checkmark \\ \mu_3 &= 2(\alpha, \gamma) - 2(\alpha, \alpha) \quad \checkmark \\ \mu_4 &= 2(\alpha, \gamma) - (\alpha, \alpha) [1+2+3] \\ \mu_j &= j(\alpha, \gamma) - (\alpha, \alpha) \left(\sum_{i=1}^{j-1} i \right) = j \frac{j-1}{2} \end{aligned}$$

Thus:

$$\mu_j = j(\alpha, \gamma) - \frac{j(j-1)}{2} (\alpha, \alpha)$$

You could define $\mu_0 = 0$, so what!

We know that $N_{-\alpha} \gamma - g\alpha = 0$, from (**)

This means $\mu_{g+1} = 0$ from (***)

Therefore: $\mu_{g+1} = 0 = (g+1)(\alpha, \gamma) - \frac{(g+1)g}{2} (\alpha, \alpha)$

$$\Rightarrow 2(\alpha, \gamma) = g(\alpha, \alpha) \Rightarrow$$

$$g = \frac{2(\alpha, \gamma)}{(\alpha, \alpha)}$$

So, given two unequal roots (γ, α) , we see that g is the length of their string

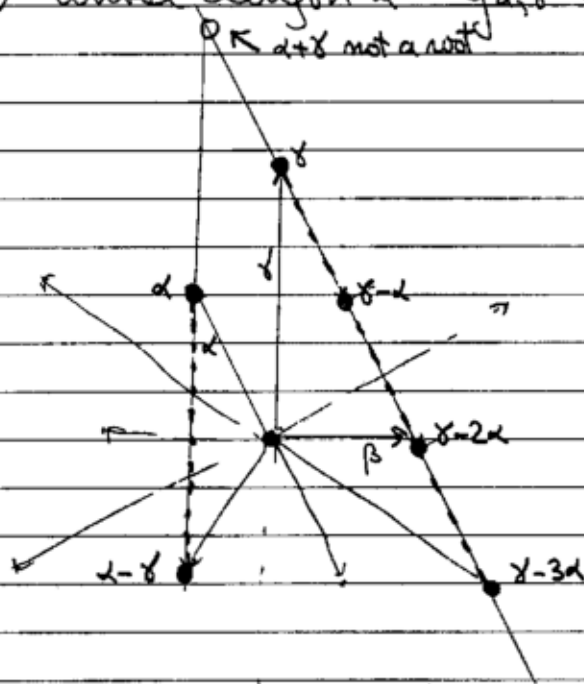
Theorem: given γ and α , $\gamma \neq \alpha$; $\gamma + \alpha$ not a root, then γ will be the "high end" of a string of roots whose length, g , is given by:

$$g_{\gamma, \alpha} = \frac{2(\alpha, \gamma)}{(\alpha, \alpha)}$$

But, if we know that $\alpha + \gamma$ not a root, then there is another string $st(\alpha, \gamma)$ whose length is $g_{\alpha, \gamma} = \frac{2(\gamma, \alpha)}{(\gamma, \gamma)}$

Example:

In G_2 I have chosen two roots α, γ such that $\alpha + \gamma$ not a root.



Then, we can form two strings from these two roots.

Now suppose β is a root on $st(\gamma, \alpha)$. Then $\gamma = \beta + 2\alpha$, in general

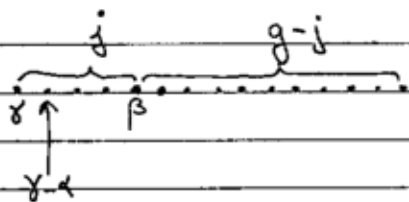
$\gamma = \beta + j\alpha$, so β is j segments down from high end.

Then:

$$(\alpha, \beta) = (\alpha, \gamma - j\alpha) = (\alpha, \gamma) - j(\alpha, \alpha) = \left(\frac{g}{2} - j\right)(\alpha, \alpha).$$

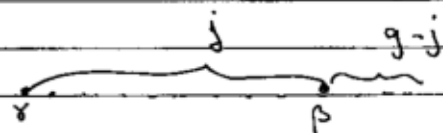
I.e., $\frac{2(\alpha, \beta)}{(\alpha, \alpha)} = g - 2j$

Consider



If j is less than half the string, $(g-j)-j < g-j$, so clearly

$\beta - [g-2j]\alpha$ is a root to the right of β .



If j is a.T. $\frac{1}{2}$, then $(g-j)-j = \oplus$ negative # whose α magn. is less than j . Then

$$\beta - [g-2j]\alpha = \beta + |g-2j|\alpha = \text{root}.$$

So let's summarize.

Pick your stepper root α . Pick any root β . This root β lies on several strings; one of these strings will have a high end γ such that $\gamma + \alpha \neq$ not a root.

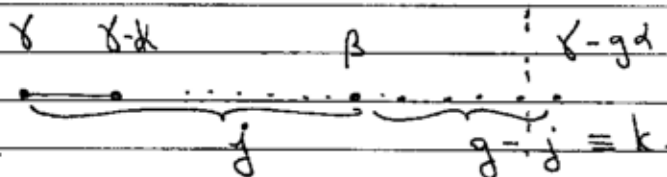
Therefore, $\left[\frac{2(\alpha, \beta)}{(\alpha, \alpha)} = \text{integer and } \beta - \frac{2\alpha(\alpha, \beta)}{(\alpha, \alpha)} \text{ is a root too!} \right]$

Detail on $N_{\alpha\beta}$, p 65

Let $\beta = \gamma - j\alpha$, then we have by our definition

$$N_{\alpha\beta} N_{-\alpha, \alpha+\beta} = \mu_j = \frac{1}{2} j(g-j+1)(\alpha, \alpha)$$

But β lies on a string



So if $\alpha =$ stepper, $\beta =$ point in string, then:

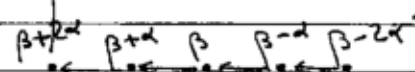
$$N_{\alpha\beta} N_{-\alpha, \alpha+\beta} = \frac{j(k+1)}{2} \cdot (\alpha, \alpha); \quad \begin{array}{l} j = \text{distance to high-end} \\ k = \text{distance to low-end} \end{array}$$

We cannot derive the numbers $N_{\alpha\beta}$ from this statement. For each α , this is g equations.

Why is $|g-2j|$ restricted to be less than or equal to 3?

This seems to say: no string can be longer than three segments. This is proved in p 64 as follows:

Assume 5 elements on a string. Let $\beta =$ center one.



$$\begin{array}{l} (\beta + 2\alpha) - \beta = 2\alpha = \text{not a root} \\ (\beta + 2\alpha) + \beta = (\alpha + \beta)^2 = \text{not a root} \end{array}$$

$[\text{"}\beta\text{-string" means string step-}\beta.]$

Thus, $(\beta+2\alpha)$ is the only element on $sh(\beta+2\alpha, \beta)$. Thus, the g for this string must be 0:

$$g_{\beta+2\alpha, \beta} = 0 = \frac{2(\beta, \beta+2\alpha)}{(\beta, \beta)} \Rightarrow (\beta, \beta+2\alpha) = 0.$$

Repeat at other end, so

$$\left. \begin{array}{l} (\beta, \beta+2\alpha) = 0 \\ (\beta, \beta-2\alpha) = 0 \end{array} \right\} \Rightarrow (\beta, \beta) = 0 \Rightarrow \beta = 0 \Rightarrow \text{contradiction.}$$

This argument does not block length 3 string, eg, because

$(\beta+\alpha)+\beta$ could be a root, not ruled out like $\checkmark$
 $(\beta+2\alpha)+\beta$.

Rules for getting A_{ij}

Let α, β be two simple roots $\Rightarrow \alpha$ is shorter.

$$\begin{aligned} \text{Then: } \cos^2 \phi &= \frac{p}{4} = \frac{(\alpha, \beta)^2}{(\alpha, \alpha)(\beta, \beta)} = \left[\frac{2(\alpha, \beta)}{(\alpha, \alpha)} \right]^2 \cdot \frac{(\alpha, \alpha)}{(\beta, \beta) \cdot 4} \\ &= (A_{\alpha\beta})^2 \cdot \frac{1}{4p} \Rightarrow (A_{\alpha\beta})^2 = p^2 \\ &\Rightarrow A_{\alpha\beta} = -p \end{aligned}$$

$$\begin{aligned} \text{Also: } \cos^2 \phi &= \frac{p}{4} = \left[\frac{2(\alpha, \beta)}{(\beta, \beta)} \right]^2 \cdot \frac{(\beta, \beta)}{4(\alpha, \alpha)} = (A_{\beta\alpha})^2 \cdot \frac{p}{4} \\ &\Rightarrow A_{\beta\alpha} = -1. \end{aligned}$$

Rules for Cartan matrix: $\downarrow$!

Consider $A_{\alpha\beta}$. If α is shorter, then $A_{\alpha\beta} = -p$, where p is # lines joining $\alpha \rightarrow \beta$ on dynkin.

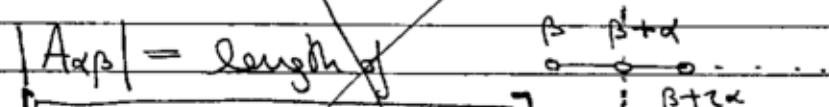
If β is shorter, then $A_{\alpha\beta} = -1$.

If both roots have same length, then $A_{\alpha\beta} = -1$, period.

Extra facts on Cartan

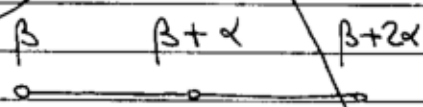
$A_{\alpha\beta} = \frac{2(\alpha, \beta)}{(\alpha, \alpha)}$ is the length of string starting at β and stepping α .

We know it starts at β because $\beta - \alpha = 0$ for two simple roots. Hence



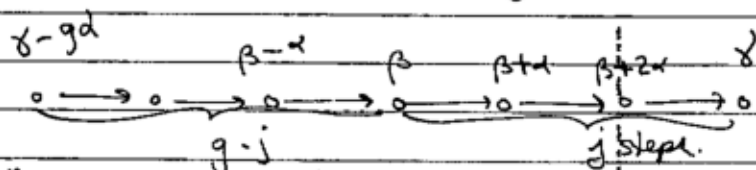
So: $|A_{\alpha\beta}| = g[\beta = \text{low}, \alpha]$ ← interpret cartan elements.

So, if $A_{\alpha\beta} = -2$, we know there is a string of roots:



first index is step, second is low.

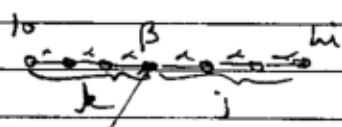
This argument is invalid because $\beta - \alpha$ can be a root. We know only that $\beta - \alpha$ is not a simple root. Thus, the chain does not nec. begin with β :



Nevertheless, β must lie on some real string as shown. Then $\beta = \gamma - j\alpha$, and

$$\begin{aligned} \frac{2(\alpha, \beta)}{(\alpha, \alpha)} &= g - 2j \leftarrow \text{all we really know is that this is an integer.} \\ &= (g - j) - (j) \\ &= \text{difference of distances to string ends} \\ &= k - j \end{aligned}$$

Theorem: if α, β are roots, then β lies somewhere on the α -string containing it $\sim k$ segments up from the low end, j segments down from the high end, so that $k+j = g$. We know that



$$[A_{\alpha\beta} = \frac{2(\alpha, \beta)}{(\alpha, \alpha)} = k - j] \text{ For simple roots, } k > j$$

These notes:

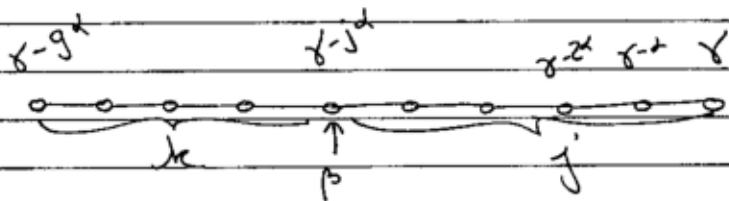
- connect the μ_i to obtain $N_{\alpha\beta}$ & $N_{\beta\alpha}$ functions
- provide an interpretation of $2(\alpha, \beta)/(\alpha, \alpha)$
- prove the basic root theorem on $\uparrow$
- prove that strings have at most 3 segments.
- show how to read the Cartan Matrix elements easily from the Dynkin diagram.

Main result:

$$A_{\alpha\beta} = \frac{2(\alpha, \beta)}{(\alpha, \alpha)} \text{ for any two roots this is}$$

$$= k - j$$

where $k + j = q$



For special case that $\beta = \gamma \Rightarrow \gamma + \alpha$ not a root, we find that $j = 0$ $k = q$ so $A_{\alpha\beta} = q$. The length of $\beta = \text{low}$ end, then $A = -q$.

Wybourne Chapter 8: "the Chevalley Basis"

8.1 Coweights + Chevalley ← 1955.

You start with a set Π of simple roots $\{\alpha_1, \alpha_2, \dots, \alpha_l\}$. As we know, any root may be constructed from this set via:

$$\beta = \sum_{i=1}^l k_i^{(\beta)} \alpha_i$$

Now, for each root β [not just simple roots], define a vector in weight space known as a "coweight"

$$H_\beta = \text{a vector} = \frac{2\beta}{(\beta, \beta)} = \sum_{i=1}^l k_i \frac{(\alpha_i, \alpha_i)}{(\beta, \beta)} \cdot \underbrace{\left[\frac{2\alpha_i}{(\alpha_i, \alpha_i)} \right]}_{H_{\alpha_i}}$$

So, we have $n-l$ such co-weight vectors in the l -dim space. Presumably the "simple coweights" H_{α_i} provide a basis.

$$\beta = \text{lin comb of } \alpha_i$$

$$H_\beta = \text{lin comb of } H_{\alpha_i}$$

Now, recall that $n-l$ is the number of E_α -type generators, and this is also the # of coweights. Perhaps we can assign an E_α to each coweight H_α .

$$[E_\beta, E_{-\beta}] = H_\beta$$

↑ generator ↑ coweight

This equation says: take average

things so that, if you raise/lower + compare to lower/raise, the difference is this number H_β .

Comments: in the Weyl basis, we partitioned the set of n generators into l H 's and $n-l$ E 's. With Chevalley, these are $(n-l)$ "exterior roots", from these we form $(n-l)$ "coweights" H_β and implicitly $(n-l)$ H -type generators. However, for each H generator H_α , he defines two (well one) E type generator E_α .

Consider $SU(3)$. We know there are 6 exterior roots and two interior, so $n-l = 6$.

[Chevalley's 6 E type operators must be what are usually call
 $T^{\pm}, U^{\pm}, V^{\pm}$. The 6 H type operators are $\pm 2T_3, \pm 2U_3, \pm 2V_3$.

Eg, $[E_{\alpha}, E_{-\alpha}] = H_{\alpha}$ becomes $[T_{+}, T_{-}] = 2T_3$ etc

$[E_{\alpha}, E_{\beta}] = E_{\alpha+\beta}$ becomes $[T_{+}, U_{+}] = V_{+}$

So the identification is:

six roots are: $\pm\alpha_u, \pm\alpha_v, \pm\alpha_t$ $\alpha_u + \alpha_t = \alpha_v$
 $\pm\alpha, \pm(\alpha+\beta), \pm\beta$

Then: $T^{\pm} = E_{\pm\beta}$ $U^{\pm} = E_{\pm\alpha}$ $V^{\pm} = E_{\pm(\alpha+\beta)}$

$T_3 = H_{\beta}$ $U_3 = H_{\alpha}$ $V_3 = H_{(\alpha+\beta)}$?

Well, $[T_{+}, U_{+}] = V_{+}$ means $[E_{\beta}, E_{\alpha}] = E_{\alpha+\beta}$
 which is wrong sign.

Improved connection between Chevalley + $SU(3)$:

p91 Wg. p262 Gal. $\left[\begin{array}{lll} T_{\uparrow}^{\pm} = E_{\pm\alpha} & U^{\pm} = E_{\pm\beta} & V^{\pm} = E_{\pm(\alpha+\beta)} \\ \pm T_3 = \frac{1}{2} H_{\pm\alpha} & \pm U_3 = \frac{1}{2} H_{\pm\beta} & \pm V_3 = \frac{1}{2} H_{\pm(\alpha+\beta)} \end{array} \right.$

Example:

$[H_{\alpha}, E_{\pm\beta}] = \mp E_{\pm\beta} \rightarrow [2T_3, U^{\pm}] = \mp U^{\pm}$

$\Rightarrow [T_3, U_{\pm}] = \mp \frac{1}{2} U_{\pm} \checkmark$

In general, E_{α} raise + lower "you", the H_{α} leave you where you "were".

We might associate the H_{α} of Chevalley with the H_{α} of Cartan-Weyl. Eg, T_3 and U_3 would be the generators of the maximal abelian "center", so T_3, U_3 commute, so does V_3 .

[You have more generators than you need in Chevalley.

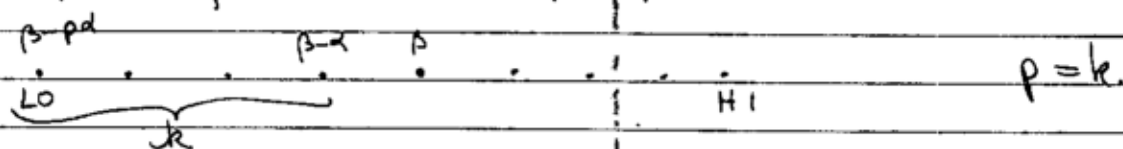
With this redundant set of generators we can write out any Lie algebra.

$$[E_1, E_2, \dots, E_{n-1}; H_1, \dots, H_{n-1}]$$

order rank ↑

$$\left. \begin{aligned} [H_\alpha, H_\beta] &= 0 & \forall H^i \\ [E_\alpha, E_{-\alpha}] &= H_\alpha & \forall E^i \\ [E_\alpha, E_\beta] &= 0 & \text{if } \alpha+\beta \text{ not a root.} \\ [E_\alpha, E_\beta] &= \pm(p+1)E_{\alpha+\beta} \end{aligned} \right\}$$

where p is q. int s.t. $\beta - p\alpha$ is a root.



So " p is distance to $L0$ from β on the α -string of β ."

Finally, $[H_\alpha, E_\beta] = \left[\frac{2(\alpha, \beta)}{(\alpha, \alpha)} \right] E_\beta$

By increasing the number of generators, Chevalley has achieved a simplification of the algebra.

8.2 Phases in the Chevalley Basis

In the Cartan-Weyl basis, we had a problem determining the numbers $N_{\alpha\beta}$. Here, we are given some rules to determine these numbers in the new basis:

$$N_{\alpha\beta} = \pm(p+1) = ?$$

Knowing α, β , you can find p by inspection. It is the sign $\pm$ that needs work.

The theorems relate various N_i/N_j such that you can deduce those signs.

Not interested.

8.3 Example: $SU(3)$ in Chevalley

Here, all $N_{\alpha} = 1$. We construct the Lie algebra of SU_3 in exactly the way all the physics texts do ~ they all use the full set of Chevalley generators.

Procedure:

- a) look at dynkin to get simple roots + data
- b) construct all roots.
- c) identify an H_{α} with each root, you now have your trivial algebra $[H_{\alpha}, H_{\beta}] = 0$.
- d) examine the chain structure of these roots to deduce the $\# p$. For SU_3 , all these p were zero because no chains of the right nature.
- e) fiddle around to find the signs of the N_{α} .
- f) Then you are done.

9.1 Group Representations

Matrices $n \times n$ are operators in an n -dim vector space. Matrices manipulate vectors of (possibly complex) numbers. If you can find a set of r -matrices that satisfy a Lie algebra with r -generators, hence r -parameters, then the set of $n \times n$ matrices "represent" the generators.

But, the set of $n \times n$ matrices tend to form a group: the group of transformations of vectors in the n -dim vector space. If certain transforms "represent" the Lie generators, then the full set of matrices is a "group representation", and the Lie algebra identifies the group.

The n -dim space in which this representation "acts" is $R^{(n)}$ = "representation space"

Theorem: every Lie group has a faithful rep, i.e., a 1:1:1.

Reducibility: if $R^{(n)}$ has an "invariant subspace" $R^{(m)}$. This means no group operation can take a vector in $R^{(m)}$ out of $R^{(m)}$. This of course is clearly seen if $D^{(n)}$ is in "block form".

[There exist representations which are not fully reducible! I do not know how you make one, but I know how not to make one.]

9.2 Real + Complex Representations

If D is a rep [i.e., a set of matrices], then you can trivially construct matrices D^* by just c'ing each element.

The set D^* also forms a rep: $D^*(A)D^*(B) = D^*(AB)$ $\hat{D}^*(A)\hat{D}^*(B) = \hat{D}^*(AB)$

Now D and D^* may not be "equivalent reps".

If D represents some $R \in \mathcal{G}$
Then D^* represents some $S \in \mathcal{G}$. $D^*(R) = D(S)$.

If R and S are not in the same class, then R and S are not conjugate group elements. Then there is no element X such that $R = X^{-1} S X$.

However, there may still be a "constant matrix" X_0 such that

$$D(R) = X_0^{-1} D(S) X_0 = X_0^{-1} D^*(R) X_0$$

In this case, $D^*(R)$ is "equivalent" to $D(R)$, even though R and S are not ~~equivalent~~ "conjugate".

Thus, we are suggesting a theorem:

R, S in same class $\Rightarrow D(R)$ equiv. to $D(S)$.
(ie, R conjugate to S)

But converse may not be true.

If $D(R) \sim D^*(R)$, then $D(R)$ is called a "complex rep.".

If $D(R) \sim D^*(R)$, then $D(R)$ is "real".

In this case, $\exists C$ such that $D = C D^* C^{-1}$

[Now, if D is unitary, we can show that C must also be unitary. In this case [D unitary], it also follows that C is sym or antisym.

$$\text{proof: } D = C D^* C^{-1} \Rightarrow \begin{aligned} D^* &= C^* D C^{-1*} \Rightarrow \pm C^* = C^{-1} \\ D^* &= C^{-1} D C \Rightarrow \pm C = C^{-1*} = \tilde{C} \end{aligned}$$

$$\Rightarrow C = \pm \tilde{C} \Rightarrow C_{ij} = \pm C_{ji}$$

These two choices define: "real of positive sign" "real of neg. sign" for a given unitary rep. }

9.3 The Contragradient Representations

Above, we defined the complex conjugate rep assoc. with D , namely D^* .

Is D^* also a rep? $D(A)D(B) = D(AB) \Rightarrow D^*(B)D^*(A) = D^*(AB)$

No, this is not a rep because $AB = BA \neq BA$.

Nor is D^{-1} a rep. $(D^*(A))^{-1}(D^*(B))^{-1} = (D^*(AB))^{-1}$ ✓

But, $(D^*)^{-1}$ is a rep: This is the "contragradient rep"
"star rep"
"conjugate rep"

9.4 The Adjoint Representation

Define an operator $\sigma_X: \mathbb{R}^2$ by: $\sigma_X Z \equiv [X, Z]$.
It is easy to show that

$$[\sigma_X, \sigma_Y]Z = \sigma_{[X, Y]}Z$$

using the Jacobi. Thus, $[\sigma_X, \sigma_Y] = \sigma_{[X, Y]}$

Or: $[\sigma_{X_i}, \sigma_{X_j}] = \sigma_{[X_i, X_j]}$

Note: $\sigma_{A+B} = \sigma_A + \sigma_B$.

Suppose $[X_i, X_j] = c_{ij}^k X_k$.

Then $[\sigma_{X_i}, \sigma_{X_j}] = \sigma_{c_{ij}^k X_k} = c_{ij}^k \sigma_{X_k}$

Thus, the set of operators $\{\sigma_{X_i}\}$ represents or realizes the generators.

In fact, these operators are as "abstract" as the generators themselves.

If $\{X_i\}$ is a matrix rep of the generators, then $\{\sigma_{X_i}\}$ is another matrix rep of the same algebra.

But, $\{X_i\} \neq \{\sigma_{X_i}\}$ element for element. In general, each σ_{X_i} is a linear comb of the X_i . This looks to me like a basis change only.

We could define a set of matrices:

$$M_{ij} \equiv \sigma_{X_i} X_j = [X_i, X_j] = C_{ij}^k X_k$$

$$\text{Then } [M_{ij}, M_{kl}] = C_{ij}^n C_{kl}^m [X_n, X_m] = C_{ij}^n C_{kl}^m \underbrace{C_{nm}^s}_{M_{nm}} X_s$$

$$\text{Then: } [M_{ij}, M_{kl}] = \left(C_{ij}^n C_{kl}^m \right) M_{nm}$$

Not sure what these are

Remark: if $\{X_i\}$ is a set of $n \times n$ matrices which represent the generators, then $\sigma_{X_i} = \text{adj}(X_i)$ is another set of $n \times n$ matrices which is also a rep.

$$\begin{aligned} \text{Trace Theorem: } \text{tr}(X_i X_j) &= \frac{1}{2} (\text{tr}(X_i X_j + X_j X_i)) = \frac{1}{2} \text{tr}(X_i X_j - X_j X_i + 2X_j X_i) \\ &= \frac{1}{2} (\text{tr}[X_i, X_j]) + \text{tr}(X_j X_i) \end{aligned}$$

$$\Rightarrow \text{tr}(X_i X_j) = \cancel{0} = \text{tr}[C_{ij}^k X_k] \Rightarrow C_{ij}^k \text{tr}(X_k) = 0$$

$$\text{Of course } \text{tr}([X_i, X_j]) = 0.$$

$$\text{Contradiction: if } \sigma_{X_i} = [X_i, \cdot], \text{ then } \sigma_{X_i} Z = [X_i, Z].$$

But, if σ_{X_i} is itself really a matrix, then:

$$\sigma_{X_i} X_i = [X_i, X_i] = 0. \text{ Now } X_i^{-1} \text{ exists? } \Rightarrow \sigma_{X_i} = 0!$$

Or: maybe we don't always have inverses of generators!

If σ_{X_i} is a matrix, then let Z be some invertible matrix

$$\begin{aligned} \text{Then } \text{tr}(\text{ad}(X_i) \text{ad}(X_j)) &= \text{tr}(\text{ad}(X_i) A \bar{A}^{-1} \text{ad}(X_j) B \bar{B}^{-1}) \\ &= \text{tr}([X_i, A] \bar{A}^{-1} [X_j, B] \bar{B}^{-1}) = \text{tr}((X_i A - A X_i) \bar{A}^{-1} (X_j B - B X_j) \bar{B}^{-1}) \\ &= \text{tr}(X_i X_j) - \text{tr}(A X_i \bar{A}^{-1} X_j \bar{B}^{-1}) - \text{tr}(X_i B X_j \bar{B}^{-1}) + \text{tr}(A X_i \bar{A}^{-1} B X_j \bar{B}^{-1}) \end{aligned}$$

If we let $A = B = I$, then $= 0$, but $\text{ad}(X_i)$ is not op on I !

When thinking of $\text{ad}(X_i)$, always treat as an operator. Thus:

$$\begin{aligned} [\text{ad}(X_i) \text{ad}(X_j)] X_k &= \text{ad}(X_i) [X_j, X_k] = \text{ad}(X_i) C_{jk}^l X_l \\ &= C_{jk}^l [\text{ad}(X_i) X_l] = C_{jk}^l [X_i, X_l] = C_{jk}^l C_{il}^m X_m \\ &= \text{ad}(X_i) \text{ad}(X_j) \delta_k^m X_m \end{aligned}$$

Then we make the operator identification:

$$\text{ad}(X_i) \text{ad}(X_j) \delta_k^m = C_{jk}^l C_{il}^m$$

If we choose $m=k$ but do not sum on k ,

$$\text{ad}(X_i) \text{ad}(X_j) = \sum_l C_{jk}^l C_{il}^k$$

Well, considering $\text{ad}(X_i) \text{ad}(X_j)$ is a matrix, its labels are m and k ; it seems to be diagonal in these indices.

$$[\text{ad}(X_i) \text{ad}(X_j)]_{mk} = \text{ad}(X_i) \text{ad}(X_j) \delta_{mk}$$

$$\begin{aligned} \text{So: } k [\text{ad ad}] &= \sum_m \text{ad}(X_i) \text{ad}(X_j) = n \cdot \text{ad}(X_i) \text{ad}(X_j) \\ &= C_{jk}^l C_{il}^k = g_{ji} \end{aligned}$$

Well, hold till we need this.

9.5 The unitary / nonunitary rep. theorem.

Theorem: The VIR^s of a (connected) simple compact group are all of finite dimension!

eg, $\text{SO}(3)$, $\text{SO}(3)$

But, a compact simple group may have other representations which are nasty:

- a) these other reps will all be ∞ dimensional
- b) non-unitary
- c) may be non-decomposable

The VIR^s are always fully reducible.

The section closes with a famous theorem:

"For a simple, connected, noncompact Lie group,
all $\bar{U}IR^1$ are ∞ dim."

Thus, all finite reps are non unitary (eg, boost matrices?).

Wybourne Ch. 10: "Weights and Labeling of IR's"

10.1 Weights and Weight Spaces

" ϕ " = label of some representation] " R_ϕ " = space of " $N(\phi)$ " dimension.

" H_i " = one of the l mutually commuting H -matrices. Here, we label them with $i \in \Pi$, but we can still be using the Cartan basis. Remember that $\Pi \subset \Sigma$ = full set of nonzero roots.

" E_i " = one for each $i \in \Sigma$. Either basis is OK here.

Now take this set of l mutually commuting matrices.

$$H_i |u\rangle = \lambda_i |u\rangle$$

Here, $|u\rangle$ is a particular ket, an eigenket simul. of all l H_i . So we associate with $|u\rangle$ a set of l eigenvalues, λ_i .

$$\Lambda = (\lambda_1, \lambda_2, \dots, \lambda_l) = \text{"weight vector"}$$

This vector is in the same space as the roots [ie, "weight space"]

Claim: if $|u\rangle$ has weight Λ , then $H_i |u\rangle = \lambda_i |u\rangle$
 $\uparrow$ vector of operators $\uparrow$ vector of ev's.

Now:

$$H_i (E_\beta |u\rangle) = (\lambda_i + \beta_i) (E_\beta |u\rangle)$$

So $E_\beta |u\rangle$ is an eigenket of vector H_i with ev. $\lambda_i + \beta_i$.

[Thus, E_β raises the weight of a ket by β .

We decompose R_ϕ :

$$\Lambda_1 | \Lambda_2 | \Lambda_3 | \Lambda_4$$

Δ_ϕ
 space of all weights

$$R_\phi^{\Lambda_1} | R_\phi^{\Lambda_2} | R_\phi^{\Lambda_3} | R_\phi^{\Lambda_4}$$

R_ϕ = space of n dim vectors, a space of the "kets".
 all kets in here have weight Λ_2 . (a subspace)

Every $\lambda_i \in \Delta_\phi$ has an $R_\phi^{\lambda_i} \subset R_\phi$.

"positive" weight is same as a positive root.

" $\lambda' \geq \lambda$ " means $\lambda' - \lambda$ is "positive"

Let the kets $|f_i\rangle$ have weight λ_i . [These kets span R_λ]

You can also think of $|f_i\rangle$ as an n -dim vector not in weight space, but in the n -dim space of the particular rep R_λ . ✓

Let $\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_N$ be decreasing order. There are, by the way, N weight vectors if matrices are $N \times N$.
basis

"series" $\lambda - n\alpha \quad \dots \quad \lambda \quad \dots \quad \lambda + q\alpha$

We know that $r + q = "g" = \frac{2(\lambda + q\alpha, \alpha)}{(\alpha, \alpha)} = \frac{2(\lambda, \alpha)}{(\alpha, \alpha)} + 2q$

$$\Rightarrow r - q = \frac{2(\lambda, \alpha)}{(\alpha, \alpha)}$$

This was our usual interp. of $\frac{2(\alpha, \beta)}{(\alpha, \alpha)} = k - j$. Proof is same

Then, $E_{-\alpha}$ kept lowering E_α to form the string.

Here, $E_{-\alpha}$ keeps lowering the weight of $|u_\lambda\rangle$.

Proof: assume $(\lambda + \alpha)$ is not a weight. Then

$$H_\alpha \left(\underbrace{E_{-\alpha} |u_\lambda\rangle}_{= |u_{\lambda-\alpha}\rangle} \right) = (\lambda - \alpha) \left(\underbrace{E_{-\alpha} |u_\lambda\rangle}_{= |u_{\lambda-\alpha}\rangle} \right)$$

$$H_\alpha |u_{\lambda-j\alpha}\rangle = (\lambda - (j+1)\alpha) |u_{\lambda-j\alpha}\rangle$$

$$E_{-\alpha} |u_{\lambda-j\alpha}\rangle = |u_{\lambda-(j+1)\alpha}\rangle$$

Eventually, $E_{-\alpha} |u_{\lambda-g\alpha}\rangle = 0$

Now: I have ignored normalization; I have defined the "lowering constants" to all be unity

However, $E_{+\alpha} |u_{\lambda-(j+1)\alpha}\rangle = N_{j+1} |u_{\lambda-j\alpha}\rangle$

↑ this we cannot also claim is 1.

We also have:

$$H_\alpha |U_{\Lambda-j+1}\alpha\rangle = (\Lambda - (j+1)\alpha) |U_{\Lambda-j+1}\alpha\rangle$$

We want a recursion relation on the μ_j

How about: $[H_\alpha, E_\alpha] |U_{\Lambda-j+1}\alpha\rangle = \beta E_\alpha |U_{\Lambda-j+1}\alpha\rangle$ / CW base

$$= (\Lambda - (j+1)\alpha) \mu_{j+1} |U_{\Lambda-j}\alpha\rangle - (\Lambda - (j+1)\alpha) \mu_{j+1} |U_{\Lambda-j}\alpha\rangle = \beta \mu_{j+1} |U_{\Lambda-j}\alpha\rangle$$

$$\Rightarrow (\Lambda - j\alpha) - (\Lambda - j\alpha - \alpha) = \beta \Rightarrow \beta = \alpha.$$

This was not helpful because we failed to mix μ_{j+1} and μ_j .

Trial 2: $[E_\alpha, E_{-\alpha}] |U_{\Lambda-j+1}\alpha\rangle = \alpha^\dagger \cdot \vec{H} |U_{\Lambda-j+1}\alpha\rangle$

$$= E_\alpha |U_{\Lambda-j+2}\alpha\rangle - E_{-\alpha} \mu_{j+1} |U_{\Lambda-j}\alpha\rangle = (\alpha, \Lambda - j\alpha - \alpha) |U_{\Lambda-j+1}\alpha\rangle$$

$$= \mu_{j+2} |U_{\Lambda-j+1}\alpha\rangle - \mu_{j+1} |U_{\Lambda-j+1}\alpha\rangle = (\alpha, \Lambda - j\alpha - \alpha) |U_{\Lambda-j+1}\alpha\rangle.$$

Thus: let $j+1=i$ $\mu_{i+1} - \mu_i = (\alpha, \Lambda - i\alpha)$

Or $\mu_{j+1} = \mu_j + (\alpha, \Lambda) - j(\alpha, \alpha)$

Good, this is same as (6.29), with $\gamma \rightarrow \alpha$.

All theorems are now obviously the same:

10.2 Weight Theorems.

First, Theorem 10.1: $\alpha = \text{root}$, $\Lambda = \text{weight}$, then

a) $2(\Lambda, \alpha) / (\alpha, \alpha) = \text{integer}$

b) $\Lambda - 2\alpha(\Lambda, \alpha) / (\alpha, \alpha) = \text{a weight also}$

Second: Every Rep. has at least one weight.

How to prove this? Let Λ_1 (not a vector, just a number) be an eigenvalue of one matrix, H_1 . Perhaps it is degenerate so H_1 has p eigenvectors for Λ_1 . These eigenvectors span a space; call this space R_1 .

Claim: if $|a\rangle \in R_1$, then $H_2|a\rangle \in R_1$ also.

Proof: $|a\rangle \in R_1 \Rightarrow |a\rangle = \sum_{i=1}^p a_i |u_i\rangle$

Then $H_2|a\rangle = \sum_{i=1}^p a_i H_2|u_i\rangle$

Then $H_1[H_2|a\rangle] = \sum a_i H_2 H_1 |u_i\rangle = \sum a_i \Lambda_1 H_2 |u_i\rangle = \Lambda_1 [H_2|a\rangle]$

Thus, $H_2|a\rangle \in R_1$ since $H_2|a\rangle$ is eigenvector of H_1 of e.v. Λ_1 .

I.e., $H_2|a\rangle$ must be a linear comb of the $|u_i\rangle$. $H_2|a\rangle = |b\rangle \in R_1$
 $= \sum b_i |u_i\rangle$.

Symbolically: $H_2 R_1 = R_1$ or $H_2: R_1 \rightarrow R_1$

We say, then: " R_1 is an 'invariant subspace' of H_2 "

Here we need a theorem: "If an $n \times n$ matrix A has an invariant subspace of dimension $r < n$, then A has at least one eigenvector in that subspace."

example: $\begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x' \\ y' \\ z \end{pmatrix}$

Here $(1,0,0)$ and $(0,1,0)$ span an invariant subspace.

i.e., $(a,b,0) \rightarrow (a',b',0)$.

So, we can simply ignore the piece of the matrix, + consider just the $r \times r$ submatrix. This submatrix must have at least one eigenvalue.

Thus, $\exists |u_2\rangle$ such that $H_2|u_2\rangle = \Lambda_2|u_2\rangle$

Next, we show that $H_3: R' \rightarrow R'$ so $\exists |u_3\rangle \Rightarrow H_3|u_3\rangle = \Lambda_3|u_3\rangle$



Because the $\{H_i\}$
all commute.

— 4 —

;

7

7

• • Λ_1 • •	
• •	
• •	

• • Λ_1 • •	
• •	
• •	

1

3

4	4
4	4

7

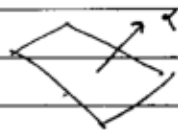
1

If a particular weight is degenerate [eg, Λ_1 in picture on back side has degeneracy 4], it is not "simple".

["simple weight" — nondegenerate, R_ϕ^Λ is a one dimensional subspace of R_ϕ .

["multiplicity" — same as degeneracy.

10.3 The Weyl Reflection Group



Let α be a root. Then α defines a hyperplane. One could do a reflection in weight space, in this plane. Any vector in the plane is of course unchanged.

The form is: $S_\alpha: \Lambda \rightarrow \Lambda' = \Lambda - 2\alpha(\Lambda, \alpha)/(\alpha, \alpha)$.

Here, Λ is an arbitrary vector in weight space. Clearly, Λ on the hyperplane $\Rightarrow (\Lambda, \alpha) = 0 \Rightarrow \Lambda' = \Lambda$, no change. Also, $\Lambda = \alpha \Rightarrow \Lambda' = \alpha - 2\alpha = -\alpha$. ✓

For each root α , you can do such a reflection.

Claim: The set of reflections S_α form a group.

Closure: $S_\beta S_\alpha \Lambda = S_\beta \left(\Lambda - \frac{2\alpha(\Lambda, \alpha)}{(\alpha, \alpha)} \right)$ ✓

$$= \left[\Lambda - \frac{2\beta(\Lambda, \beta)}{(\beta, \beta)} \right] - \frac{2(\Lambda, \alpha)}{(\alpha, \alpha)} \left[\alpha - \frac{2\beta(\alpha, \beta)}{(\beta, \beta)} \right] \quad \text{This is a root, call it } \gamma$$
$$= \Lambda - \left\{ \frac{2\beta(\Lambda, \beta)}{(\beta, \beta)} + \frac{2\gamma(\Lambda, \alpha)}{(\alpha, \alpha)} \right\} = \Lambda - \frac{2\gamma(\Lambda, \gamma)}{(\gamma, \gamma)}$$

where γ is a root. I have not shown this, but believe it.

So, we have a group of order $(n-l)+1$, because $(n-l)$ is # of root vectors, and add an identity

$\{I, S_\alpha\} =$ group of order $n-l+1$. [a finite group].

The point seems to be that the space of root vectors of a Lie group has associated with it certain finite groups. In particular, given an "object" in l -dimensions [eg, the root vector diagram], there are various finite groups which "cover" the object. If you include only reflections, you have the "reflective covering groups". The Weyl group is the reflective covering group for a given root diagram. It also covers the weight diagram.

10.4 Weights, and Classification of IR's.

Some theorems without proof:

#1: if a rep ϕ is irreducible, its highest weight is simple. ie, non degenerate.

#2: if ϕ_1 and ϕ_2 , both IR's, have the same highest weight, then ϕ_1 and ϕ_2 are equivalent IR's. ie, they are connected by a similarity.

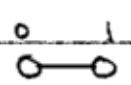
#3: $\{\lambda \text{ is the greatest weight of IR } \phi\} \iff$

$\{\text{the numbers } \frac{2(\lambda, \alpha)}{(\alpha, \alpha)} \text{ for } \alpha \in \Pi \text{ are all non neg integers}\}$

Moreover, given those numbers: [called $\lambda_\alpha, \alpha \in \Pi$], we can construct λ , the largest weight, from our knowledge of the $\alpha \in \Pi$. This knowledge is all contained in the Dynkin!

So: [an IR is completely characterized by its highest weight vector, so you can label your IR's by this maximal weight, $\lambda_{\max}$.

Let $l = \text{the rank} = \# \text{ of simple roots}$. Then for each set of l positive integers $\{\lambda_1, \lambda_2, \dots, \lambda_l\}$, you can construct a max weight, hence an IR. So, each IR can be labelled by a multiplet of $l +$ integers.

(Might as well put these #'s on the Dynkin circles!  eg $\rightarrow$ 1.45

10.5 Computation of the Complete set of Weights

$$|\delta(\Lambda)| = 2 \sum_{\alpha \in \Pi} \Lambda_\alpha = \text{some even integer}, +$$

By the way, for any weight M we can produce a set of numbers

$$M_\alpha = \frac{2(M, \alpha)}{(\alpha, \alpha)} = \text{integer}$$

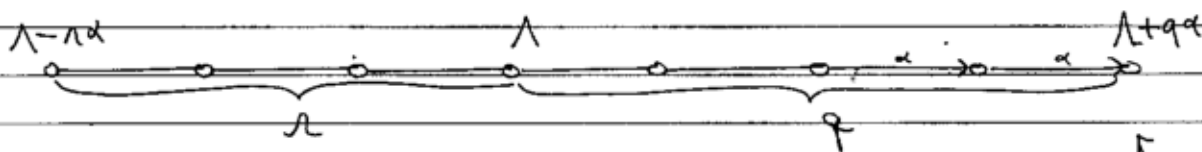
If all such integers were $+$, M would be a "greatest weight" of some rep.

$$\text{So: } |\delta(M)| = 2 \sum_{\alpha \in \Pi} M_\alpha = \text{even integer.}$$

$$|\delta(M)| = \frac{1}{2} [\delta(\Lambda) - \delta(M)] = \text{integer}$$

Consider, eg, $\{\Lambda_\alpha\} = 2, 3, 1, 4 = \text{set for max weight } \Lambda.$

Go back to: Λ arbitrary weight
 α arbitrary root $\Rightarrow \frac{2(\Lambda, \alpha)}{(\alpha, \alpha)} = \Lambda - q$



For each root α , there will exist a "series of weights" as shown. On any series, there is of course a highest weight.

$$\text{Note: } \frac{2(\Lambda_{\max}, \alpha)}{(\alpha, \alpha)} = q = r + q$$

For any other weight on this series, $\frac{2(\Lambda, \alpha)}{(\alpha, \alpha)} \leq \frac{2(\Lambda_{\max}, \alpha)}{(\alpha, \alpha)}$

Now think of $\alpha \in \Pi$, $\Lambda = \text{max weight}$, $M = \text{some weight}$
on this α -series only. Then:

$$\frac{2(M, \alpha)}{(\alpha, \alpha)} \leq \frac{2(\Lambda, \alpha)}{(\alpha, \alpha)} \Rightarrow \boxed{M_\alpha \leq \Lambda_\alpha}$$

Thus, eg, we could have:

$$\begin{aligned} \{\Lambda_\alpha\} &= 2, 3, 1, 4 \\ \{M_\alpha\} &= 2, 2, 1, 3 \end{aligned}$$

Therefore, it follows that

$$\delta(M) = \frac{1}{2} [\delta(1) - \delta(M)] = \frac{1}{2} \cdot 2 \left[\sum_{\alpha} \Lambda_{\alpha} - \sum_{\alpha} M_{\alpha} \right] \\ = \sum_{\alpha} (\Lambda_{\alpha} - M_{\alpha}) = d \text{ positive integer.}$$

I have had to assume here that M is a weight on the α -series topped by Λ for all α ?!

Consider a two dim sample weight space.
If M and Λ are on the same series (α_1),
then clearly



$$\frac{2(\Lambda, \alpha_1)}{(\alpha_1, \alpha_1)} - \frac{2(M, \alpha_1)}{(\alpha_1, \alpha_1)} = (1+q) - (1-q) = 2q$$

where $q = \#$ steps M is to the left of Λ . Specifically,
we are saying:

$$\Lambda = M + q \alpha_1 \quad q > 0$$

In general, you can get from some "lower" weight M up to Λ
via:

$$\Lambda = M + \sum q_i \alpha_i \quad \alpha_i \in \Pi.$$

It is not clear that all integers q_i are positive.

$$M = \Lambda - \sum q_i \alpha_i \quad \checkmark$$

$\nwarrow$ These roots (simple) are being subtracted from Λ
to give M .

So, from this we may compute:

$$= \sum_i q_i \left(\sum_j A_{ji} \right)$$

$$\delta(\Lambda) = 2 \cdot \sum_i \Lambda_i \quad \checkmark = 2 \sum_{i,j} \frac{2(\Lambda, \alpha_j)}{(\alpha_j, \alpha_j)} \quad \checkmark$$

$$\delta(M) = 2 \cdot \sum_j \frac{2(\Lambda - \sum q_i \alpha_i, \alpha_j)}{(\alpha_j, \alpha_j)} \quad \checkmark$$

$$= \delta(\Lambda) - \sum_{i,j} 2 \cdot \frac{1}{2} \cdot q_i \cdot \frac{2(\alpha_i, \alpha_j)}{(\alpha_j, \alpha_j)} \quad \checkmark$$

$$\Rightarrow \delta(M) = \frac{1}{2} [\delta(\Lambda) - \delta(M)] = \frac{1}{2} \left[2 \sum_{i,j} q_i \frac{2(\alpha_i, \alpha_j)}{(\alpha_j, \alpha_j)} \right] = \sum_{i,j} q_i A_{ji} \quad \text{neg int.}$$

I am not convinced that $\chi(M) = k$ is a positive integer. Assume so for now however, then we can group the weights according to their "layer #" $k = \chi(M)$.

$\chi(1) = 0$, so Δ_0 is the layer containing the greatest weight Λ .

["I(q)"] = max value of $\chi(M) \Rightarrow$ least weight, so $\Lambda' \in \Delta^+$

[" $\Sigma_k(q)$ "] = # of weights in a layer = "height"

["III(q)"] = max # weights in a layer = $\max_k \Sigma_k(q) =$ "width"

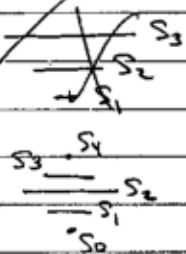
"congruent (mod 2)" = a set of integers is "..." if all elements of the set are even or odd.

"dynkin's 10.7 theorem" = if you know the numbers Λ_k which label a particular rep, then you can compute the height via this formula:

$$[T(q) = \sum_{\Lambda \in \Pi} \Lambda_k \Lambda_k]$$

where Λ_k are given in table 10.1 for all lie algebras

"The spindle theorem" — let $R = \frac{1}{2}T + 1$, or, if $T = 4$, then $R = 3$ + the layer structure looks like



$$\begin{aligned} S_4 &= S_0 \\ S_3 &= S_1 \end{aligned}$$

Another case: $T = 5$ $R = 3\frac{1}{2}$? Nonsense.

$T = 6$

$$S_0 = S_6$$

$$S_1 = S_5$$

$$S_2 = S_4$$

$$S_3 = S_3$$

$R = 4\checkmark$

$$\begin{aligned} S_6 &= S_0 \\ S_5 &= S_1 \\ S_4 &= S_2 \end{aligned}$$

$$\begin{aligned} S_2 &= S_4 \\ S_1 &= S_5 \\ S_0 &= S_6 \end{aligned}$$

Contradictive here!

$$S_4 > S_3 > S_2 > S_1$$

The Spindle Theorem:

Reps. fall into two classes:

- $n = T/2$ a) $T(d) = \text{even} \Rightarrow \text{odd \# layers} \Rightarrow \text{center layer is largest}$
 $n = \frac{T-1}{2}$ b) $T(d) = \text{odd} \Rightarrow \text{even \# layers} \Rightarrow \text{two center layers = size.}$

By "size" we mean # of roots in a layer, i.e., S_k .

example: $T = 6 \Rightarrow 7 \text{ layers} \Rightarrow n = 3 \Rightarrow S_3 \text{ is largest layer}$

$$\begin{array}{c} - S_0 \\ S_1 \\ - S_2 \\ S_3 \\ - S_4 \\ S_5 \\ - S_6 \end{array} \quad \left. \begin{array}{l} S_3 > S_2 > S_1 > S_0 \\ S_5 > S_4 > S_3 > S_2 > S_1 > S_0 \end{array} \right\} \quad S_k = S_{T-k}.$$

Procedure for calculation of weights:

Assume you know weights down thru layer $n-1$.

[Q: why are all weight vectors connected by integral "steps" of simple roots?

Originally, we know that, starting with a weight λ , we could generate a new weight eigenket (+ new weight vector) by applying E_β with $\beta \in \Sigma$.

i.e., $\lambda = \text{weight} \Rightarrow (\lambda + \beta) = \text{candidate for a weight}; \beta \in \mathbb{P}\Sigma$

However, all positive roots (i.e., $\frac{1}{2}$ of Σ) can be constructed from whole multiples of simple roots, by def. of simple root.

i.e., $\beta = \sum_{i \in \Pi} k_i \alpha_i \quad k_i = \text{non-neg. ints.}$

Then you can get all roots as usual: $\beta = \pm \sum k_i \alpha_i$

Thus, given a weight λ , the candidate weights are

$$\lambda \pm \sum k_i \alpha_i$$

Weight vectors — remember, these vectors are "eigenvalue-vectors" of the set of generators which compose the Cartan subalgebra. The number of such weight vectors = The N of $N \times N$ matrices which compose the representation of the generators.

→ So, if you construct the simple roots in the weight space (think $l=2$ for now), then what you know is that any two true weights are connected by a "pathway" of simple roots. A linear pathway is called a "series of weights". This does not require that the weights themselves be integral sums of simple roots [think of the 3 of $SU(3)$].

Now with $l=2$: you know that both simple roots are in the right half plane for $SU(3) = A_2$, so "high" in this case means "to the right".

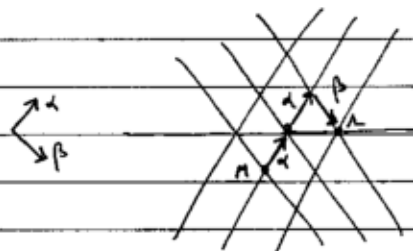
Recall that " Λ_1 higher than Λ_2 " means $\Lambda_1 - \Lambda_2 =$ positive vector. If Λ_1 and Λ_2 are "adjacent", then $\Lambda_1 - \Lambda_2 = \alpha$ positive root.

It now seems clear that, if Λ is the highest weight and M is some general weight in Δ , then

$$M + \sum_{\alpha_i \in \Pi} k_i \alpha_i = \Lambda \quad (k_i = \text{non neg integers}).$$

We are adding "positive" pieces onto M , making Λ more (positive) than M .

Example: the octet in SU_3



$$\Lambda = M + 2\alpha + \beta$$

← more weight

Now we may reinterpret Λ_α : $\Lambda_\alpha = \frac{2(\Lambda, \alpha)}{(\alpha, \alpha)} =$ length of the α -series leading up to Λ .

For the 8, we see that $\Lambda_\alpha = 1$

$$\Lambda_\beta = 1$$

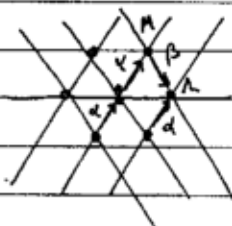
$$\text{Hence } \underline{8} = \overset{1}{\alpha} - \overset{1}{\beta}$$

(like g on α -strings)

Re-examine $\delta(M) \equiv k$:

$$\begin{aligned}\delta(M) &= \frac{1}{2} [\delta(\Lambda) - \delta(M)] = \frac{1}{2} [\delta(\Lambda - M)] = \frac{1}{2} \delta \left(\sum_{\alpha \in \Pi} k_i \alpha_i \right) \\ &= \frac{1}{2} \sum_{\alpha_i \in \Pi} k_i \delta(\alpha_i) = \sum_{\substack{i,j \\ \in \Pi}} k_i \cdot \frac{2(\alpha_i, \alpha_j)}{(\alpha_j, \alpha_j)} = \text{ouch} = \sum_i k_i \left(\sum_j A_{ji} \right)\end{aligned}$$

Let's do an example:



$$\Lambda = M + \beta$$

$$\Lambda_\alpha = 1$$

$$M_\alpha = 2$$

$$\Lambda_\beta = 1$$

$$M_\beta = \cancel{0}$$

Notice that $M_\alpha > \Lambda_\alpha$ here, so that is possible.

$$\delta(\Lambda) = 2 \cdot (1+1) = 4$$

$$\delta(M) = 2 \cdot (0+2) = 4$$

$$\delta(M) = 0 = k. ?$$

Wait: $\frac{2(M, \alpha)}{(\alpha, \alpha)} = M_\alpha = 2 - q$, so $r = \text{distance down to bottom}$
 $q = \text{distance up to top}$.

Then:

$$M_\alpha = 2 - 0 = 2$$

$$M_\beta = 0 - 1 = -1$$

This is why M is not a "highest weight".

So,

$$\begin{matrix} M_\alpha = 2 \\ M_\beta = -1 \end{matrix}$$

$$\Rightarrow \delta(M) = 2(2-1) = 2$$

$$\Rightarrow \delta(M) = 1 = k, \text{ "1st layer" } \checkmark$$

Q: how do we know that $\sum_{\alpha \in \Pi} \Lambda_\alpha > \sum_{\alpha \in \Pi} M_\alpha$?

I would like to see: $\delta(M) = \sum_i k_i \cdot \text{period} = \text{"\# of simple roots sub'd"}$

[The question really hinges on the behavior of $(\sum_j A_{ji}) = \text{sum of the } i^{\text{th}} \text{ column of the Cartan matrix}$.

For A_2 (SU_3), both columns sum to 1, so $\sum_i k_i (\sum_j A_{ji}) = \sum_i k_i$.

In fact, $\sum_j A_{ji} = 1$ for all the A_l algebras [ie, all the $SU(N)$].

For the B_l, C_l , the column sums are +1 for first col, 0 for rest.

For F_4 , there exists a negative sum.

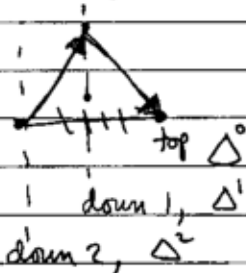
So, if we stick to the A , then I know that

$$\delta(M) = \sum_i k_i = \# \text{ steps away from } \Lambda.$$

More generally, $\delta(M) = \sum_i s_i k_i$ where $s_i = \sum_{j \in \Pi} A_{ji}$.

This may still be a positive integer, however due to the nature of the rep.

Layers: In $SU(n)$, these tell you how far down in "steps" you are from the "highest weight".



These are not the pyramidal layers discussed in some SU_3 text. See exercise 10.5.

$S_k = \# \text{ weights in the } k^{\text{th}} \text{ layer.}$

[highest weight is nondegenerate for an IR.

Q: how do we know that $\Lambda' = -\Lambda$? Or do we know this? NO! Not true for 3, eg. However, due to the spindle theorem, we know that Λ is also a simple weight. We have a reflection symmetry thru the center of the spindle. For SU_3 , this is the particular reflection based on $\alpha + \beta$.

Theorem: (I missed this before): Weights which are "equivalent" via a single reflection on Weyl have the same multiplicity.

ie, Λ_1 and $\Lambda_2 = R_\alpha \Lambda_1 = \Lambda_1 - \frac{2\alpha(\Lambda_1, \alpha)}{(\alpha, \alpha)} \alpha$

Λ_1 and Λ_2 have same mult.

This essentially proves the spindle theorem, in part. It explains why the lowest weight is simple. I think it also explains:

$$\delta(M) = -\delta(R_\alpha M)$$

In particular, $\lambda' \neq -\lambda$, but $\delta(\lambda') = -\delta(\lambda)$.

Height: $T(\lambda) = \frac{1}{2} [\delta(\lambda) - \delta(\lambda')] = \text{number of interlayer segments} = \# \text{ layers minus one}$. But, $\delta(\lambda') = \delta(\lambda)$,
so

$$T(\lambda) = \delta(\lambda) = 2 \cdot \sum \lambda_i$$

Thus, we at once know the height of any rep. Eg $\circ \rightarrow \circ$ has height 4. (caution)

Wybourne corrects his sloppy error in (10.21). Thus, it appears that the analysis of pp 102, 103 applies only to A_2 !

Procedure for computing all weights:

- Choose a rep by putting positive integers on a dynkin.
- compute the highest weight λ from these numbers and from your knowledge of the simple roots.
- compute the height $T(\lambda)$ using (10.21)

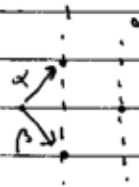
Now you know exactly how many layers you've got, and you know layer $\lambda_0 = \{\lambda\}$.

(d) iterative routine to "build the next layer":

I. Choose a weight M in layer $r-1$.

II. For this one weight, examine each α -series ($\alpha \in \Pi$) which contains this weight. Compute " q_i " for each series [# elements in "high end"]. Also, compute M_i for each series.

III. If $M_i + q_i \geq 0$, then $M - \alpha$ is a weight. [if $r=0$, the M ends the series]

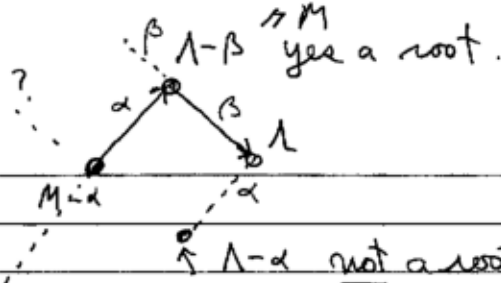


This is the procedure Wybourne espouses, but in his example he does not use it! In fact, in part Three(III), since you are computing R_α , you know how far that series goes to the left, so you can compute all weights in that series.

Example: do $\alpha \rightarrow \beta$ of SL_3 . Start with $\lambda = (\alpha + 2\beta)/3$ from previous

work. Then compute: $q_\alpha = 0$ $q_\beta = 0$. Then $\lambda_\alpha = 0$ $\lambda_\beta = 1$ by definition. Thus $M_i + q_i = 0 \Rightarrow M - \alpha$ not a root. But $\lambda_\beta + q_\beta = +1$, so $\lambda - \beta$ is a root, but is last one.

So far, then, we have:



Now let $M = 1 - \beta$. We already know that the β series has ended, so consider $(M - \alpha)$. Well, $q_\alpha = 0$, and $M_\alpha = \frac{2(M, \alpha)}{(\alpha, \alpha)} = \frac{2(1 - \beta, \alpha)}{(\alpha, \alpha)} = 1 - \frac{2(\beta, \alpha)}{(\alpha, \alpha)} = -\frac{2(\beta, \alpha)}{(\alpha, \alpha)}$

But $(\alpha, \beta) = \sqrt{(\alpha, \alpha)(\beta, \beta)} \cos \theta_{\alpha, \beta} = 1 \cdot 1 \cdot (-\frac{1}{2}) = -\frac{1}{2}$

So $M_\alpha = +1$. Thus, $M_\alpha + q_\alpha = 1 + 0 = 1 > 0$, so $M - \alpha$ is a root, & is last one in series.

Now, let $\tilde{M} = (M - \alpha) - \beta$. No, stop. We know this rep only has 3 layers because $T(d) = 2$. Thus, $\tilde{M}$ cannot be a weight. Thus, we are done.

[Only now, at the end, can we conclude what the dimensionality is. Here, $N = 3$.

[Note: in general, you need only compute half the layers.

.end

11.1 Definition

"Kronecker product of two vector spaces" = "direct product ..."
a very general notion:

$$\mathcal{K}: R_1, R_2 \rightarrow R_3$$

$$\text{Eg. } \mathcal{K}(\mathbb{R}_1, \mathbb{R}_2) = \mathbb{R}_1 \times \mathbb{R}_2$$

some linear operation like multiplication

The main idea is that if $\{e_i\}$ spans R_1 and $\{f_j\}$ spans R_2 , then you must choose a meaning for " $\times$ " such that the set $\{e_i \times f_j\}$ "spans" R . Then " $\times$ " defines, along with R_1 and R_2 , the Kronecker product. Usually you say:

$$R = R_1 \times R_2$$

Example: Let $R_1 = \mathbb{F}^m$ and $R_2 = \mathbb{F}^n$. The basis for R_1 is m vectors like $(1, 0, 0, \dots)$ etc. In R_1 , an operator is a matrix A_{ij} , and so also in R_2 , B_{jk} . From these two operators, we can construct an operator in $R = R_1 \times R_2$. It is:

$$(A \times B)_{i\alpha, j\beta} = A_{ij} B_{\alpha\beta} \quad \text{= product of two numbers.}$$

this index takes on $m \cdot n$ values.

The new basis vectors are: $(\mathbb{F} \times \mathbb{F})_{j\beta} = e_j f_\beta = |e_j\rangle \otimes |f_\beta\rangle$.

Here, $e_j f_\beta$ is a "direct product" of two vectors, but can be thought of as a single vector of $m \cdot n$ dimensions: ✓

If $|e_j\rangle$ is the trivial basis, then the components of vectors $|f_i\rangle$ are $\langle e_j | f_i \rangle$, $j = 1, 2, \dots, m$. Similarly, $|e_i\rangle$ = basis for R_2 .

Then the components of $(\mathbb{F} \times \mathbb{F})_{j\beta}$ are $\langle e_i | f_j \rangle \cdot \langle e_\alpha | f_\beta \rangle$ (i.e. takes on all values). Now we have a product of two numbers.

The standard Kronecker theorems are stated, mainly:

$$(A_1 A_2) \times (B_1 B_2) = (A_1 \times B_1) (A_2 \times B_2)$$

$$\text{i.e. } \mathbb{F} = \mathbb{F} \cdot \mathbb{G}.$$

matrix mult in the $(n \times m) \times (n \times m)$ space.

11.2 Kronecker Product of Representations

$$g = g' \times g'' = \sum_i \Gamma_{g'}^i g'' g''^i, \text{ when "normally" reducible.}$$

Is it possible that a direct product of two reps be "indecomposable"?

W. does not comment on this here. / The direct product is that of two rep spaces, so basically it is "matrix analysis".

We do not yet know how to compute the P_{ij} 's.

11.3 The Weight Space for Kronecker Products.

$$x = \otimes$$

$$R_{g'} = \sum_{\lambda \in \Delta_{g'}} R_{g'}^{\lambda} = \text{sum of the invariant weight subspaces.}$$

$$R_{g''} = \sum_{\lambda' \in \Delta_{g''}} R_{g''}^{\lambda'} = \text{same for rep } g''$$

$$\text{Now, this means: , eg } R_{g'} = R_{g'}^{\lambda_{a_1}} \oplus R_{g'}^{\lambda_{b_1}}$$

$$\text{I.e., } E^4 = E^3 \oplus E^1$$

↑ not and/or, but a sum of two separate spaces.

$$"\oplus" = "+" \text{ . Same as in } D' \otimes D'' = D' \oplus D'' \oplus D''.$$

$$\text{So: Clearly: } (A_1 \oplus A_2) \times (B_1 \oplus B_2) = (A_1 \times B_1) \oplus \text{ etc.}$$

$$\text{The rule is: } (A_1 \oplus A_2) \otimes B = (A_1 \otimes B) \oplus (A_2 \otimes B) \quad \checkmark \text{ OK.}$$

This is "distributive" as in (iv) p 110. ✓

$$\text{Therefore: } (\sum_i A_i) \times (\sum_j B_j) = \sum_{ij} (A_i \times B_j) \quad \checkmark$$

And in particular:

$$R_{g'} \times R_{g''} = \sum_{\lambda \in \Delta_{g'}} \sum_{\lambda' \in \Delta_{g''}} R_{g'}^{\lambda} \times R_{g''}^{\lambda'}$$

Now, how do we apply $\vec{H}_\alpha$ to this?

$$\vec{H}_{g' \times g''} = \vec{H}_{g'} \times \vec{H}_{g''} = (H_{g'} \times 1) \oplus (1 \times H_{g''}).$$

$$\text{So } (\vec{H}_{g' \times g''} | \xi_{g'} \lambda; \xi_{g''} \lambda' \rangle = (\vec{\lambda} + \vec{\lambda}') | \dots \rangle, \text{ it is claimed.}$$

In fact, this really defines our Kronecker product. For any generator H of the Lie algebra:

$$H_{g' \times g''} \equiv (H_{g'} \times 1) \oplus (1 \times H_{g''})$$

In the direct product rep, we know the form of all the $H^i + E^i$, namely:

$$H_{\phi \times \phi'} = H_{\phi} * H_{\phi'} \equiv H_{\phi} \otimes 1 + 1 \otimes H_{\phi'}$$

So, let $|\lambda', \phi'\rangle$ be a vector in the invariant subspace $R_{\phi'}$, and let $|\lambda'', \phi''\rangle$ be in $R_{\phi''}$. Then we can compute the weight of the direct product vector in $R_{\phi' \times \phi''}$:

$$\begin{aligned} H_{\phi \times \phi'} \cdot |\lambda', \phi'; \lambda'', \phi''\rangle &= H_{\phi} * H_{\phi'} \cdot |\lambda', \phi'\rangle \otimes |\lambda'', \phi''\rangle \\ &= (H_{\phi} \otimes 1 + 1 \otimes H_{\phi'}) \cdot |\lambda', \phi'\rangle \otimes |\lambda'', \phi''\rangle \\ &= H_{\phi} |\lambda', \phi'\rangle \otimes |\lambda'', \phi''\rangle + |\lambda', \phi'\rangle \otimes H_{\phi'} |\lambda'', \phi''\rangle \\ &= (\lambda' + \lambda'') |\lambda', \phi'; \lambda'', \phi''\rangle \end{aligned}$$

This is true for each generator H^i , so:

$$\vec{H} |\lambda'; \lambda''\rangle \equiv (\vec{\lambda}' + \vec{\lambda}'') |\lambda'; \lambda''\rangle$$

[Thus, the weight vectors of the new, direct product rep are the vector sums of the weights of the two component reps.]
→ in weight space

This is the fundamental composition theorem for Lie groups.

Reminder: recall that, when you direct sum two reps like $R_1 \oplus R_2$, and if each rep has its set of weights Δ_1 and Δ_2 , then the direct sum (which is a rep, because block form; a reducible rep) has for its set of weights $\Delta_1 \cup \Delta_2$. Also, for the rep $R_1 \oplus R_2$, R_1 is called "an invariant subspace."

Therefore, you can compute the weights of $R_{\phi} \otimes R_{\phi'}$ by using the weight addition rule. Then, if you assume this is equal to a direct sum of irred. reps, then you are assuming that

$$\{\Delta_{\phi} + \Delta_{\phi'}\} = \Delta_1 \cup \Delta_2 \cup \dots \cup \Delta_n$$

Then it is simply a matter of untangling things.

Now, usually we think of direct producting two irred reps. An irred rep has, by definition, no invariant subspaces.

However, we can take the $\otimes$ of two general reps:

$$R_\varphi = R_{\varphi'}^{\Lambda_1} \oplus R_{\varphi'}^{\Lambda_2} \oplus \dots$$

↑ This is a space spanned by the n -degenerate vectors belonging to eigen(value) Λ_i .

For example, the octet 8 of $SU(3)$ has a two-dim subspace for the weight $\Lambda_1=0$; all others are one dim.

So, now we can use the distribution of $\otimes$ over $\oplus$ to say:

$$\begin{aligned} R_{\varphi'} \otimes R_{\varphi''} &= [R_{\varphi'}^{\Lambda_1'} \oplus R_{\varphi'}^{\Lambda_2'} \oplus \dots] \otimes [R_{\varphi''}^{\Lambda_1''} \oplus R_{\varphi''}^{\Lambda_2''} \oplus \dots] \\ &= (R_{\varphi'}^{\Lambda_1'} \otimes R_{\varphi''}^{\Lambda_1''}) \oplus (\dots) \end{aligned}$$

→ Correction: even if a rep. is irreducible, you can still decompose the rep. space into a direct sum of weight subspaces.

E.g., 3 of SU_3 : $R_3 = R_3^{\Lambda_1} \oplus R_3^{\Lambda_2} \oplus R_3^{\Lambda_3}$

Here, each is "singly degenerate"; i.e., each weight is "simple". Thus, an operator in $R_3^{\Lambda_i}$ is a 1×1 matrix, ?

Perhaps, the direct sum in $R_\varphi = \sum_{\Lambda \in \Delta} R_\varphi^{\Lambda_i}$ is not the same as the direct sum of two matrices. I.e., we have not claimed that the $R_\varphi^{\Lambda_i}$ are invariant subspaces [This is only true if all the H_i are symmetric or hermitian].

So, when we say $R_\varphi = \sum R_\varphi^{\Lambda_i}$, we only mean to group the basis vectors of $\mathbb{R}^n$ into certain sets. This idea is completely clear. These sets do form "subspaces". You could, e.g., rotate within a subspace. But in general, a rotation of the full space would mix the subspaces; since not "invariant" (yet).

$$\begin{pmatrix} 1 & & \neq 0 \\ & (3.5) & \\ & & \neq 0 \end{pmatrix} \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix}$$

In (11.9) and (11.10) we have sum rules allowing us to
a) compute the # elements in each layer of $\Delta q \times q'$
in terms of the layer structure of Δq and $\Delta q'$

Well, $\Delta q \times q' = \Delta q + \Delta q'$

Consider a particular layer, say S_k , of $q \times q'$. Weights "arrive"
into this layer in various ways:

$$\begin{array}{l} S_0(q') \cdot S_k(q'') \\ S_1(q') \cdot S_{k-1}(q'') \end{array} \quad \begin{array}{l} \text{is one way} \\ \text{is a second way etc.} \end{array}$$

The idea is that "layer" tells you how many "steps" [ie, simple roots] you are above or below the central layer.

If you add a ^{weight} root that is one layer up from center to a weight root that is 2 layers up, the resultant is a weight that is 3 layers up from center. We are essentially reducing the weight diagram to "one dimension", & that dimension is the layers, or the # of steps away from the center.

Example:

$$\begin{array}{c} 1 & & 1 \\ & 1 & & 1 \\ 1 & \otimes & 1 \\ & 1 & & 1 \\ 1 & & 1 \end{array} = \begin{array}{c} 1 & & 1 \\ & 2 & & 2 \\ 3 & = & 2 \oplus 1 \\ & 2 & & 2 \\ 1 & & 1 \end{array}$$

ie, $3 \otimes 3^* = 8 \oplus 1$

How do we distinguish 3 and 3^* ? In terms of layer structure, you cannot. In order to tear apart your result (12321), you need to know specifically the highest weight.

Thus $3 \otimes 3 = 6 \oplus 3^*$

$$\begin{array}{c} 1 & & 1 \\ & 2 & & 2 \\ 1 & \otimes & 1 \\ & 2 & & 2 \\ 1 & & 1 \end{array} = \begin{array}{c} 1 & & 1 \\ & 2 & & 2 \\ 3 & = & 2 \oplus 1 \\ & 2 & & 2 \\ 1 & & 1 \end{array}$$

to see why this is a 3^* , you need explicitly the highest weight.

ie, these "layer" diagrams do not carry all the info

11.4 Decomposition of the Kronecker Product

See the above discussion. In principle, nothing can stop you from resolving a Kronecker product. It would be nice to have a table of the highest weights [in terms of simple roots] for each rep.

Example given:

$$\overset{1}{\circ} \text{---} \overset{1}{\circ} \otimes \overset{1}{\circ} \text{---} \overset{1}{\circ} = \overset{2}{\circ} \text{---} \overset{2}{\circ} \oplus \overset{1}{\circ} \text{---} \overset{1}{\circ}$$

$$\text{i.e., } 3 \otimes 3 = 8 \oplus 3^*$$

Given $\overset{p}{\circ} \text{---} \overset{q}{\circ}$ as an SU_3 rep, a given $\overset{j}{\circ}$ as an SU_2 rep, or given $\overset{p}{\circ} \text{---} \overset{q}{\circ}$ as a G_2 rep, we do not have general formulas (yet) for the dimensionality of the rep.

For $SU(3)$, the rule is

$$\text{mult}(\overset{p}{\circ} \text{---} \overset{q}{\circ}) = \frac{1}{2} (p+1)(q+1)(p+q+2)$$

For $SU(2)$, the rule is:

$$\text{mult}(\overset{p}{\circ}) = (2p+1)$$

Wybourne Ch. 12 "Rep. Weights + Labels"

12.1 Basic Reps

A rep is "basic" if it has a highest weight which cannot be written as $\Lambda + M$, where Λ, M are highest weights of other reps.
Eg., the 8 of SU_3 is not basic.

depression: since $(\alpha_i, \alpha_j) \neq \delta_{ij}$, i.e., since the simple roots do not form an orthogonal basis for the weight space, we cannot say things like:

$$\Lambda = \sum_i (\alpha_i, \Lambda) \alpha_i$$

Because then $(\alpha_i, \Lambda) = \sum_j (\alpha_i, \Lambda) (\alpha_j, \alpha_i)$ not consistent.

In fact, this non-orthogonality manifests itself in the off-diagonal elements of the Cartan Matrix:

$$A_{ij} = 2(\alpha_i, \alpha_j) / (\alpha_i, \alpha_i) \checkmark$$

Nevertheless, if we know the projections of a vector Λ on the various non-orthogonal basis vectors, we should be able to reconstruct Λ :

$$\Lambda_{\alpha_i} = 2(\Lambda, \alpha_i) / (\alpha_i, \alpha_i)$$

Basically I think we need to be able to invert the Cartan matrix, to wit: let

$$\Lambda = \sum_j a_j \alpha_j \quad ; \quad \Lambda_{\alpha_i} = \sum_j 2(\Lambda, \alpha_i) / (\alpha_i, \alpha_i) \checkmark$$

$$\text{Thus: } \Lambda_{\alpha_i} = \sum_j a_j (\alpha_j, \alpha_i) 2 / (\alpha_i, \alpha_i) = \sum_j A_{ij} a_j \checkmark$$

$$\text{OR: } \Lambda = A \cdot a \checkmark \Rightarrow a = A^{-1} \Lambda \checkmark \quad a_i = (A^{-1} \Lambda)_i = \sum_j A^{-1}_{ij} \Lambda_{\alpha_j}$$

$$\text{Thus: } \Lambda = \sum_i \left[\sum_j A^{-1}_{ij} \Lambda_{\alpha_j} \right] \alpha_i = \sum_i (A^{-1} \Lambda)_i \alpha_i \checkmark$$

$$\text{Example: for } SU_3, \quad A_{ij} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}, \quad |A| = 3 \quad (A^{-1})_{ij} = \frac{1}{3} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \checkmark$$

$$\text{For } \Lambda_{\alpha_i} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \checkmark \text{ we get } (A^{-1} \Lambda) = \begin{pmatrix} 2/3 \\ 1/3 \end{pmatrix} \checkmark \text{ so } \Lambda = \frac{2}{3} \alpha_1 + \frac{1}{3} \alpha_2$$

But p 102 gives $\Lambda = \frac{2}{3} \alpha_1 + \frac{1}{3} \alpha_2$ Why did this fail? It works! I used G_2 by error.

clarify that inner product:

Way back on p61 we used:

$$\alpha^i = g^{ik} \alpha_k$$

The form of the metric tensor was given in (6.20). For generators E_α not in the l -dimensional Cartan subalgebra, we ordered full l -dimensional vectors in this way:

$$\vec{V} = (1, 2, \dots, l; \beta_1, -\beta_1, \beta_2, -\beta_2, \dots, \beta_n, -\beta_n).$$

Then we normalized the E_α freely so that $g_{\beta_i, -\beta_i} = 1$. We already know that $g_{\beta_i, \beta_j} = 0$ unless $\beta_j = -\beta_i$.

Within the l -subspace, we have the tensor g^{ij} . It is given in (6.21) as:

$$g^{ij} = \sum_{\alpha_i \in \Sigma^+} (\alpha_i)_i (\alpha_i)_j$$

For SU_3 , this is a sum over the six nonvanishing roots, not just a sum on the simple roots.

What exactly are the roots of SU_3 ? Look at 6.12 which says:

$$\alpha_k = \sum_i \lambda_i (\alpha_k)_i$$

$$\alpha = \lambda_i \alpha_i$$

Here, " α " was a number in $[A, E_\alpha] = \alpha E_\alpha$. This " α " has no relation to the root vector $\vec{\alpha}$:

$$\vec{\alpha}_k = ((\alpha_k)_1, (\alpha_k)_2, \dots, (\alpha_k)_l).$$

The root vector α_k has l covariant components as shown. Thus:

$$[H, E_\alpha] = \alpha E_\alpha$$

$$[E_\alpha, E_{-\alpha}] = \alpha^i H_i \equiv (\alpha, H)$$

So lets define an inner product:

$$[(a, b) = a \cdot b = \sum_{i=1}^l a^i b_i = g_{ij} a^i b^j = g^{ij} a_i b_j]$$

Clarify the symbol α : originally, we had $\alpha_i E_i = \alpha E_i$ where $\alpha X \equiv [A, X]$. For each of the $l-1$ generators E_i , we had a non-degenerate eigenvalue α of α . Thus, the first meaning of α is

$\{\alpha\}$ = set of $l-1$ complex numbers.

Then, we had a constant set of numbers, $\{\lambda_i\}$. These numbers are fixed for a given Lie algebra, say A_2 . [We found that we could associate with the number α , a set of l numbers α_i .] Knowing the numbers α_i , we could compute the original eigenvalue α via:

$$\alpha = \sum \lambda_i \alpha_i = (\lambda, \alpha)$$

Now, however, we use the symbol α to denote a vector in weight space. However, we still want to use the old α as a label to label the generators E_α .

$$\alpha = (\lambda, \alpha)$$

root $\nearrow$ $\nwarrow$ root vector

Remark: when we say $\beta - 2\alpha$ (α, β) / (α, α) = root, are we thinking of those α, β as roots, or as root vectors? Well, Wybourne ceases to make the distinction between the two, but I had in mind the actual vectors. No doubt, it is also true for roots! :

$$\sum \lambda_i (\beta_i - 2\alpha_i (\alpha, \beta) / (\alpha, \alpha)) = \tilde{\beta} - 2\tilde{\alpha} (\alpha, \beta) / (\alpha, \alpha) = \text{root}.$$

Compute roots for SL_3 : if one followed the exact projection procedure, one would have strange but equal lengths for the 6 root vectors of A_2 . Let's choose them all of unit length:

$$\begin{aligned} \alpha_5 &= (-\frac{1}{2}, \frac{\sqrt{3}}{2}) \\ \alpha_1 &= (\frac{1}{2}, \frac{\sqrt{3}}{2}) \\ \alpha_6 &= (-1, 0) \\ \alpha_3 &= (1, 0) \\ \alpha_4 &= (-\frac{1}{2}, -\frac{\sqrt{3}}{2}) \\ \alpha_2 &= (\frac{1}{2}, -\frac{\sqrt{3}}{2}) \end{aligned}$$

Now we can compute:

$$g_{11} = \sum \alpha_i (\alpha_i)_1 (\alpha_i)_1 = \left[\left(\frac{1}{2}\right)^2 + (1)^2 + \left(\frac{1}{2}\right)^2 \right] \cdot 2 = 3$$

$$g_{22} = \left[\left(\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2 + (0)^2 \right] \cdot 2 = 3$$

$$g_{12} = \sum \alpha_i (\alpha_i)_1 (\alpha_i)_2 = \left[\left(\frac{1}{2} \cdot \frac{\sqrt{3}}{2}\right) + 0 + \left(-\frac{1}{2} \cdot \frac{\sqrt{3}}{2}\right) \right] \cdot 2 = 0$$

$g_{21} = 0$ also.

How to calculate your metric tensor:

$$[g_{ii} = 2 \cdot \sum_{\alpha_k} (\alpha_k)_i^2]$$

↑ sum only over the positive roots.

~~g_{ij}~~ $[g_{ij} = g_{ji} = 2 \cdot \sum_{\alpha_k} (\alpha_k)_i (\alpha_k)_j]$

So, in the case of SU_3 with my unit "length" roots, I find

$$g_{ij} = 3 \cdot \delta_{ij} \quad \text{length} \equiv \sum (\alpha_k)_i^2$$

Had I used $\alpha_3 = (\frac{1}{\sqrt{3}}, 0) = \frac{1}{\sqrt{3}} (1, 0)$

$$\alpha_4 = \frac{1}{\sqrt{3}} (\frac{1}{2}, \frac{\sqrt{3}}{2}) = (\frac{1}{2\sqrt{3}}, \frac{1}{2})$$

I would have found: $g_{ij} = \delta_{ij}$

So, if you use the proper normalization [$1p$, $1/\sqrt{3} \times$ the simple one], then $g_{ij} = \delta_{ij}$ and (α, α) is simple.

But, if you normalize your roots wrong, then $\sum \alpha_i^2 \neq (\alpha, \alpha)$, and things get confused.

Compute g_{ij} for B_2 : here, the + roots are:

$$(0, 1) \quad (1, 1) \quad (1, 0) \quad (1, -1)$$

Thus: $g_{11} = 2 \cdot [1+1+1] = 3/2$

$$g_{22} = 2 \cdot [1] = 3/2$$

$$g_{12} = 2 \cdot [0+1+0+1] = 0 = g_{21}$$

Thus: $[g_{ij} = \frac{3}{2} \delta_{ij}]$

Thus, on p25, I would say: $(\alpha, \beta) = g^{ij} \alpha_i \beta_j = \frac{3}{2} \cdot \sum \alpha_i \beta_i$

$$= \frac{3}{2} \cdot [\alpha_1 \beta_1 + \alpha_2 \beta_2] = \frac{3}{2} \cdot [1 \cdot (-1) + 0] = -3/2.$$

But again, if we shrink all the root vectors by $\sqrt{2/3}$, then $g_{ij} = \delta_{ij}$.
[all ratio calcs are the same]

In our work we encounter things like:

$$A_{ij} = \frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)}$$

$$\Lambda_{\alpha_i} = \frac{2(\Lambda, \alpha_i)}{(\alpha_i, \alpha_i)}$$

Whenever you have a ratio of two inner products, it makes no difference whether $g_{ij} = \delta_{ij}$ or $g_{ij} = \text{const.} \times \delta_{ij}$. [Regardless of the form of g_{ij} , the scale of A_{ij} (or Λ_{α_i}) is immune to changes in the scale of your root vectors. Thus, eg., in SU_3 you might as well use the "simple norm" where $\alpha + \beta = (1, 0)$.]

Also, whether or not $g \propto I$, it is always true that

$$(a, b+c) = (a, b) + (a, c)$$

$$(a, b) = (b, a)$$

So, we can always express the highest weight of a rep. as:

$$\vec{\Lambda} = \sum_{i=1}^l a_i \vec{\alpha}_i$$

$\vec{X}$ = vector in weight space

where $a_i = (\vec{A}^T)_i \Lambda_{\alpha_i}$

Example: for SU_3 , $\vec{A}^T = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix}$. So we can at once show:

$$3 \quad \vec{\Lambda}_{\frac{1}{2}} = (1, 0) \Rightarrow a_i = \frac{1}{3} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{pmatrix} \frac{2}{3} \\ \frac{1}{3} \end{pmatrix}$$

$$\Rightarrow \vec{\Lambda}_{\frac{1}{2}} = \frac{2}{3} \vec{\alpha}_1 + \frac{1}{3} \vec{\alpha}_2$$

$$3^* \quad \Lambda_{\alpha_2} = (0, 1) \quad a_i = \begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \end{pmatrix}$$

$$\Rightarrow \vec{\Lambda}_{(0,1)} = \frac{1}{3} \vec{\alpha}_1 + \frac{2}{3} \vec{\alpha}_2 \quad \checkmark$$

$$8 \quad \Lambda_{\alpha_1} = (1, 1) \quad a_i = \frac{1}{3} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\vec{\Lambda}_{(1,1)} = \vec{\alpha}_1 + \vec{\alpha}_2$$

Thus, in 15 seconds I can tell you the highest weight for any rep. of SU_3 . Once you compute $\vec{A}^T$ for any group, you can at once compute all the highest weights.

[Beware: do not confuse $\Lambda_{\alpha_i} = (1, 2, 0, 0, \dots)$ with $\vec{\Lambda}$ = highest weight. Both are "vectors" in weight space.

Now, suppose you have: $\Lambda_{\alpha_i} = (p, q)$ - example.

This causes:

$$(a_i) = \begin{pmatrix} p \\ q \end{pmatrix} \quad \vec{\Lambda}_{(p,q)} = a_i \vec{\alpha}_i$$

However,

$$\Lambda_{\alpha_i} = p(1,0) + q(0,1)$$

$$(a_i) = p \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + q \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$(a_{(p,q)}) = p \cdot (a_{(1,0)}) + q \cdot (a_{(0,1)})$$

Therefore: $\left[\vec{\Lambda}_{(p,q)} = p \vec{\Lambda}_{(1,0)} + q \vec{\Lambda}_{(0,1)} \right]$

And we know that $\vec{\Lambda}_{(1,0)} = \frac{2}{3} \vec{\alpha}_1 + \frac{1}{3} \vec{\alpha}_2$

$$\vec{\Lambda}_{(0,1)} = \frac{1}{3} \vec{\alpha}_1 + \frac{2}{3} \vec{\alpha}_2$$

Thus: $\vec{\Lambda}_{(p,q)} = \left(\frac{2}{3}p + \frac{1}{3}q \right) \vec{\alpha}_1 + \left(\frac{1}{3}p + \frac{2}{3}q \right) \vec{\alpha}_2$

or: $\left[\vec{\Lambda}_{(p,q)} = \frac{1}{3}(2p+q) \vec{\alpha}_1 + \frac{1}{3}(p+2q) \vec{\alpha}_2 \right]$ $\swarrow$ SU_3

So, here are all the highest weights at once, boom!

Eg, $\vec{\Lambda}_{(1,1)} = \vec{\alpha}_1 + \vec{\alpha}_2$

It is clear that only the reps $(1,0)$ and $(0,1)$ are the only "basic" reps for SU_3 because the highest weight of any higher rep can be written in terms of the highest weights of these reps.

12.2 Kronecker Powers

Start with a rep of having basis vectors [weights] $\{\Lambda_i\}$. For $(1,0)$ of SU_3 , there are 3 weights $\Lambda_1, \Lambda_2, \Lambda_3$. To each weight, we assign the abstract "vector" $f_i = |f_i\rangle$. This vector is not in the weight space, it is in that abstract Hilbert space in which the abstract Lie generators operate.

Now look at $f \otimes f$, " f cross itself". Arrange the

The direct product basis $\{i,j\}$ into sym + antisym sets.

Claim: each such set forms the basis for a rep in the direct product, $\mathfrak{g} \otimes \mathfrak{g}$. Eg,

$$3 \otimes 3 = 6 \otimes 3^* \quad \begin{matrix} \uparrow \text{sym} & \uparrow \text{anti} \end{matrix}$$

Remark: if $\mathfrak{g}$ is some general VTR, after you group your $\{i,j\}$ into sym + antisym sets, there may be pieces left over? Eg, in $8 \otimes 8 = 1 \oplus 8 \oplus 8 \oplus 10 \oplus 10^* \oplus 27$, I think there are 8 sym, and 8 antisym combinations? well: NO!

$$f_{11}^2, f_{22}^2, \dots, f_{nn}^2 = 8 \text{ symmetric terms.}$$

$$f_{12} + f_{21}, f_{13} + f_{31}, \dots =$$

In general, if $S_{ij} = f_{ij} + f_{ji}$, there are $(n^2 - n)/2$ sym. terms to one side of the diag, and n on the diag.

$$\left\{ \begin{array}{l} \text{So } \frac{n(n+1)/2}{n(n-1)/2} \text{ sym. terms} \\ \quad \text{anti-sym. terms.} \end{array} \right\} \Sigma = n^2$$

So, in fact the sym + antisym exhaust all of $\{f_{ij}\}$.

Now, take only the antisym stuff. How many terms? $n(n-1)/2 = \binom{n}{2}$. They claim that these antisym terms form a rep

$$\mathfrak{g}^{\{1,2\}} = \text{the rep of order } \binom{n}{2} \text{ comprised of the antisymmetric combinations}$$

[Q: how do you know that $\mathfrak{g}^{\{1,2\}}$ is in fact a representation?
I am unable to answer this now]

Q: in (12.4), what is λ_1 and λ_2 ? (the case $k=2$)

Perhaps: $\lambda_1 = \text{weight of } f_{12}$
 $\lambda_2 = \text{weight of } -f_{21}$?

→ Time out: The weight associated with $f_{12} + f_{21}$ is the same

as the weight associated with $\xi_1 \xi_2 - \xi_2 \xi_1$. Thus,

$\xi_1 \xi_2 - \xi_2 \xi_1$ does not "mean" $| \xi_1 \rangle \otimes | \xi_2 \rangle - | \xi_2 \rangle \otimes | \xi_1 \rangle$

because $\vec{H}_{\text{diag}} \left[\begin{array}{c} \leftarrow \\ \leftarrow \end{array} \right] = \{ (\vec{\Lambda}_1 + \vec{\Lambda}_2) - (\vec{\Lambda}_2 + \vec{\Lambda}_1) \} \left[\begin{array}{c} \leftarrow \\ \leftarrow \end{array} \right] = 0$.

Rather, $\xi_1 \xi_2 \pm \xi_2 \xi_1$ are polynomials in real variables ξ_1, ξ_2 .

With either polynomial, we associate the state $| \xi_1 \rangle \otimes | \xi_2 \rangle$ whose weight is $(\vec{\Lambda}_1 + \vec{\Lambda}_2)$.

So, the highest weight in the group $\Delta \mathfrak{g}^{\text{sl}}$ is $\vec{\Lambda}_1 + \vec{\Lambda}_2$.

For $\Delta \mathfrak{g}^{\text{su}}$, the symmetric, the highest weight "belongs" to $\mathfrak{g}^{\text{su}}$ and is $2 \vec{\Lambda}_1$.

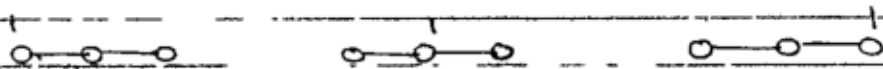
The size of $\mathfrak{g}^{\text{su}}$ is $n(n+1)/2 = \binom{n+1}{2}$.

Now, W has generalized to $\mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g} \otimes \dots \otimes \mathfrak{g}$ k -times.

Let's proceed to find out why these "Kronecker powers" are useful.

Exercise: for A_2 , $\begin{pmatrix} 1 & - \\ 0 & 0 \end{pmatrix}^{\text{sym}} = \text{sym piece} = \begin{array}{c} 2 \\ 0 \end{array} \begin{array}{c} 0 \\ 0 \end{array}$
 $\begin{pmatrix} 1 & - \\ 0 & 0 \end{pmatrix}^{\text{anti}} = \text{anti} = \begin{array}{c} 0 \\ 1 \end{array} \begin{array}{c} 0 \\ 0 \end{array} \leftarrow$

For A_3 , there are supposed to be 4 weights in each basis rep:



Perhaps these form a tetrahedron in 3-space ?? ...
 Call them $\xi_1, \xi_2, \xi_3, \xi_4$.

To find the weights for $\begin{array}{c} 1 \\ 0 \end{array} \begin{array}{c} 0 \\ 0 \end{array} \begin{array}{c} 0 \\ 0 \end{array}$, say, we first find Λ_1 going with ξ_1 , so Λ_1 is the highest weight. We have to invert the Cartan 3×3 to find this, so in principle I could get Λ_1 . Knowing the simple roots, I could then compute all weights for $\begin{array}{c} 1 \\ 0 \end{array} \begin{array}{c} 0 \\ 0 \end{array} \begin{array}{c} 0 \\ 0 \end{array}$. Assume this has been done.

Now, what is $(\overset{1}{\circ}-\overset{2}{\circ}-\overset{3}{\circ})^{\frac{1}{2} \frac{1}{2} \frac{1}{2}}$? Well, $n=4$, so $n(n+1)/2 = 10$, so clearly this is not one of the other reps basic.

$(\overset{1}{\circ}-\overset{2}{\circ}-\overset{3}{\circ})^{\frac{1}{2} \frac{1}{2} \frac{1}{2}}$ has $n(n-1)/2 = 4 \cdot 3/2 = 6$,

Maybe he claims that $(\overset{1}{\circ}-\overset{2}{\circ}-\overset{3}{\circ})^{\frac{1}{2} \frac{1}{2} \frac{1}{2}}$ contains $\overset{1}{\circ}-\overset{2}{\circ}-\overset{3}{\circ}$ } *problem here.*
And maybe $(\overset{1}{\circ}-\overset{2}{\circ}-\overset{3}{\circ})^{\frac{1}{2} \frac{1}{2} \frac{1}{2}}$ contains $\overset{1}{\circ}-\overset{2}{\circ}-\overset{3}{\circ}$?

12.3. Elementary Reps.

OK, correction: $(\overset{1}{\circ}-\overset{2}{\circ}-\overset{3}{\circ})^{\frac{1}{2} \frac{1}{2} \frac{1}{2}}$ means: compute the entire set of antisym vectors + weights. Perhaps this set partitions into several UIR's. Then $\rangle$ means: consider only that UIR with the highest weight?!!

Theorem: from just the terminal reps, the "elementary" ones, one can construct the other "basic" reps via $\mathcal{G}^{\frac{1}{2} \frac{1}{2} \frac{1}{2}}$.

OK, for each terminal point on a Dynkin diagram for a Lie algebra, we define a chain in the diagram called: "the branch for that terminal point". W. list of all branches for all algebras.

The point is this: [The UIR for any "point" on a branch can always be gotten from the "elementary" rep. $\mathcal{G}$ for the terminal point of that branch.] The way you get it is by doing $\mathcal{G}^{\frac{1}{2} \frac{1}{2} \frac{1}{2}}$ on that terminal rep. Here, r is the number of "steps in on the branch" you have come.

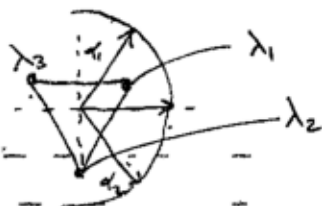
See Exercise 12.1 p 116 for an example of what this all means.

12.4. Weights of Elementary Reps.

The rep $\overset{1}{\circ}-\overset{2}{\circ}$ of SV_3 has 3 weights, call them $\lambda_1, \lambda_2, \lambda_3$ in descending height (ie, right to left). We know these are:

$$\vec{\lambda}_1 = \frac{2}{3} \vec{\alpha}_1 + \frac{1}{3} \vec{\alpha}_2 \quad \vec{\lambda}_2 = \vec{\lambda}_1 - \vec{\alpha}_1 \quad \vec{\lambda}_3 = \vec{\lambda}_2 - \vec{\alpha}_2$$

Specifically:



$$\lambda_2 = \lambda_1 - \alpha_1$$

$$\lambda_3 = \lambda_2 - \alpha_2$$

So: $\alpha_1 = \lambda_1 - \lambda_2$

$\alpha_2 = \lambda_2 - \lambda_3$

the "primary" elem. rep.

Now here is what Dynkin has done: for each lie group, he chooses one particular elementary rep, namely, the one with a 1 on the left end. This elem. rep. of course has some (as yet unknown number of) weights, $\lambda_1, \lambda_2, \dots, \lambda_s$.

In the figures on p. 119, Dynkin shows specifically how you can construct the simple roots from these "primary weights". I infer from these pictures that, for all groups other than A_n , the number of primary weights equals the number of simple roots. For A_n , # pri. wts = # simple roots + 1.

So: A_n : # pri. wts. = $n+1$

all other $\rightarrow G$: # pri. wts. = n

Next, Dynkin shows how to compute all roots from the primary weights. Eg.

$$B_n \quad \Sigma = \{ \lambda_p \}, \{ \pm \lambda_p \pm \lambda_q \} \quad p, q \in [1, n]$$

Next, he shows how you can quickly list only the positive roots.

For the exceptional groups, Dynkin constructs the roots using the primary weights of related groups.

G_2 A_2

F_4 B_4

E_6 A_5

E_8 A_8

Eg, Σ for $G_2 = \{ \pm \lambda_p, \lambda_p - \lambda_q \} \quad p, q \in (1, 3)$

↑ but these are primary weights of A_2 , not G_2

E_6 is a special case

Comment: we now know the size (dimensionality) of certain elementary reps [those I have called "primary" elem. reps, i.e., the left end elem. rep.].

Also, if we know one elem rep for each group (i.e., the primary rep), then we can work backwards to construct all simple, positive, and general roots of an algebra.

I suspect we could reverse this process and, once we know the highest weight, we can construct all weights of the primary rep from a knowledge of the simple roots.

Eg, $\lambda_1 = \frac{2}{3}\alpha_1 + \frac{1}{3}\alpha_2$, we find independently. Then

$$\begin{aligned} \lambda_1 - \alpha_1 &= \alpha_2 \Rightarrow \lambda_1 - \alpha_2 = \alpha_1 - \alpha_2 \Rightarrow \lambda_2 = \lambda_1 + \alpha_1 - \alpha_2 \checkmark \\ \lambda_2 - \alpha_2 &= \alpha_1 \Rightarrow \lambda_2 = \lambda_1 - \alpha_1 \checkmark \end{aligned}$$

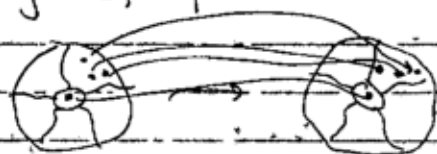
12.5. Spinor Reps and the Groups B_n and D_n

Recall: x. It is possible to map a group onto itself in this way:

aside:

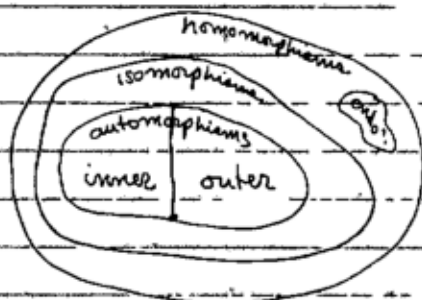
$$f: R \rightarrow ARA^{-1} \quad \text{for some fixed } A.$$

In such a mapping, each element $R \in \mathcal{R}$ is mapped onto one of its conjugates, i.e., each class is separately mapped into itself.

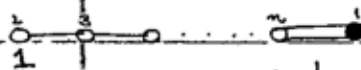


Of course, $E \rightarrow E$.

This connection between $\mathcal{R}$ and $\tilde{\mathcal{R}} = A\mathcal{R}A^{-1}$ is called an "inner automorphism". So, the hierarchy is:



Now, consider a B_n



branch has $n-1$ roots.

Let α_1 be the "primary" elem rep shown.

Let α_2 be the $(0, 1, 0, \dots)$ rep, etc. Then, as suggested earlier:

$$\alpha_k = \alpha_1^{\{1\}^k} \quad k=1, 2, \dots, n-1$$

This repeats what we already know: all the elem reps on a branch can be obtained, via antisym Kronecker power rep, from the primary or terminal rep of that branch.

But, for B_n , the elem rep $\circ - \circ - \dots - \circ - \overset{1}{\circ}$ cannot be obtained in this way; $\overset{1}{\circ}$ is on a different branch. This one left-over elem rep is called

"the spinor rep of B_n " $= \circ - \circ - \dots - \circ - \overset{1}{\circ} = \sigma$

From earlier work, we are supposed to know the weights of the primary rep of B_n , called τ . By doing Kroneckers, we can then compute the weights of any τ_i , but not the weights of σ .

The weights + highest weight of the σ -elem rep is given on p123.

By the way: $SO(3)$ has $l=1$, $= B_1 = ?$ not defined is really $SU(2) = \circ$

$SO(4) = D_2$? But D only starts with $l=4$.

$SO(5) = (\circ - \bullet) = B_2$

$SO(7) = (\circ - \circ - \bullet) = B_3$ not simple

$SO(6) = ?$ [Hey, $SO(4)$ and $SO(6)$ seem a little lost]

Next, for a D_n you can again get all your elem reps by Krn, except now there are two spinor reps.

Theorem: $(\text{spinor rep}) \otimes (\text{spinor rep}) = (\text{a "true rep"})$
 $(\text{spinor rep}) \otimes (\text{true rep}) = (\text{spinor rep})$

A "spinor rep" for $O(3)$ is not here defined. The word "spinor" is unmotivated so far.

Spinor seems to mean that something somewhere is half integral.

12.6 Labelling of IR's.

Dynkin says: compute the primary weights $\{\lambda_i\}$; then the highest weight of any rep is

$$\vec{\lambda} = \sum_i l_i \lambda_i$$

where l_i are certain numbers computable from the λ_i .

Cartan uses $\vec{\lambda} = \sum_{i=1}^l \lambda_i \vec{\Lambda}_i$ which seems to me simpler, certainly for SU_3 .

No matter who you are, you always label your IR's with the highest weights.

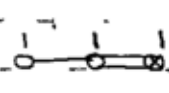
So, Dynkin labels a rep by some fancy numbers $(-l_1, l_2, \dots)$. I find all this very un-useful.

12.7 A matter of Notation.

Look, you can span the weight space with either of these sets:

- $\{\vec{\Lambda}_i\}$, the highest weights of the l basic reps.
- $\{\vec{\lambda}_i\}$, the weights of the primary rep.

Wybourne prefers the latter. Example: $\alpha(7)$

 = a rep. of B_3 with $\Lambda_{di} = (1, 1, 1) = \alpha_i$

Now, from (12.22), $l_k = \frac{a_3}{2} + (a_k + a_{k+1} + \dots + a_{n-1})$

Thus:

$$l_1 = \frac{1}{2} + (a_1 + a_2) = 2\frac{1}{2} = \frac{5}{2}$$

$$l_2 = \frac{1}{2} + 1 = \frac{3}{2}$$

$$l_3 = \frac{1}{2}$$

So $[l_1, l_2, l_3] = [\frac{5}{2}, \frac{3}{2}, \frac{1}{2}]$.

Notice that $\bullet$ means $a_3 \neq 0$, so $\frac{1}{2}$ integers get involved.

Of course, once we know the set $\{l_i\}$ for A_{n-1} , we have:

$$\Lambda = \sum l_i \alpha_i$$

For A_n , the numbers are $\{l_1, l_2, \dots, l_{n+1}\}$ for $SU(n+1)$. These numbers are clearly given in (12.28). There is a different scheme for A_n , namely the $\{l_i'\}$. See p 126 for further details.

Notation:

$$U(n+1) \quad \{ \}$$

$$O(2n+1), O(2n) \quad [\]$$

$$Sp(2n) \quad \langle \rangle$$

$$\text{exceptionals} \quad (\)$$

Q: Note: The Lie groups $X(\dots)$ and $SX(\dots)$ are not the same; not even same # of parameters. The various classical Lie algebras go only with the $SX(\dots)$ groups.

I know nothing at all about the reps of $U(n+1)$, eg, unless there is some kind of isomorphism. I don't think $U(n+1)$ is simple. Of course, $U(n+1)$ contains $SU(n+1)$ as an ideal, a invariant subgroup, so $U(n+1)$ is not "simple".

Is $U(n+1)$ "semisimple"? Its only ideal is $SU(n+1)$. (I think $U(n+1)$ has no abelian ideals, thus it is semisimple).

OK, remark: most of our work with Dynkin diagrams etc. was only for simple Lie algebras. In the "classical Lie algebras" $A_n, B_n, \dots$ exhaust the simple Lie algebras. But, any semisimple can be broken into a direct sum of simples.

~~Answer:~~ Suppose $L_1 = L_2 \oplus L_3$, where L_2 and L_3 are simple Lie algebras. To me, this means that all the generators of L_1 can be brought into block form.

The weights of a rep. of L_1 are the union of weights X_i , no.

→ Q: if you knew all about L_2 and L_3 , how would you construct full data on L_1 ?

- a) does L_1 have a set of roots? What are they?
- b) what are the reps. of L_1 ?

Wybourne has not discussed this question, to my knowledge. He did show, however, back on p 49 that $SO(4)$ was not simple. This is why $SO(4)$ is "left out" of Dynkin's scheme.

$$\begin{array}{ccccc} \text{semisimple} & \uparrow & so(4) = so(3) \oplus so(3) & & / \text{Lie algebras} \\ & & \uparrow \text{simple} \quad \uparrow \text{simple} & & \end{array}$$

$$\begin{array}{ccccc} SO(4) & \sim & SO(3) \otimes SO(3) & & / \text{Lie groups} \\ & \uparrow \text{isomorphic} & & & \end{array}$$

Let's return to these questions later in Ch 16 on Global Properties.

New ideas in this chapter

- 1) "basic" reps
- 2) "elem." reps
- 3) The "primary elem rep"
- 4) Kronecker powers of a rep
- 5) "branch" of points in a Dynkin
- 6) The primary rep weights: λ_i
- 7) construction of various root sets from the λ_i : Π, Σ_+, Σ_-
- 8) spinor reps of B_n, D_n
- 9) Dynkin numbers $\{l_i\}$ to label a rep.

} The construction theorem

Chapter 13:

In general, $\Lambda = \sum_{i=1}^l m_i M^{(i)}$ where $m_i \equiv \lambda_i \equiv a_i$

We have been working in the last chapter on computing the highest weights of the various basic reps of the non-exceptional lie groups. Here is what we found:

a) for the primary basic rep, $M^{(\text{prim})} = \lambda_1$

b) for the basic reps on the chain (branches) we found that:

$$M^{(1)} = \lambda_1$$

$$M^{(2)} = \lambda_1 + \lambda_2$$

$$M^{(3)} = \lambda_1 + \lambda_2 + \lambda_3 \text{ etc. / see (12.4)}$$

Example: $(\overset{1}{0} \rightarrow a)^{F_4}$ had weights:

$$\frac{2\alpha + 2\beta}{3}, \frac{\alpha - \beta}{3}, \frac{-(2\alpha + \beta)}{3}$$

$$\lambda_1 = (2\alpha + \beta)/3$$

$$\lambda_2 = -(\alpha - \beta)/3$$

$$\lambda_3 = -(\alpha + 2\beta)/3$$

But, notice that here,

$$\lambda_1 + \lambda_2 = \frac{2\alpha + \beta - (\alpha - \beta)}{3} = \frac{\alpha + 2\beta}{3} \checkmark$$

Thus, the highest weight in $(\overset{1}{0} \rightarrow a)^{F_4}$ is, in fact, $\lambda_1 + \lambda_2$.

However, there were certain basic reps which we could not reach by the Kronecker method. These were the "spin reps" for B_n and D_n . We had to list off their max weights specially.

[Note: here is a point I missed: The Kronecker theorem on p115 gives you the highest weight of all basic reps on the main branches, plus, it gives you all the other weights of each basic rep.

If, we expect

$\left. \begin{matrix} \lambda_1 + \lambda_2 \\ \lambda_1 + \lambda_3 \\ \lambda_2 + \lambda_3 \end{matrix} \right\}$ to be the weights of $\circ - \circ$,

given that $\lambda_1, \lambda_2, \lambda_3$ were the weights of $\circ - \circ$, and
given that

$$(\circ - \circ)^{\{1\}} = \circ - \circ$$

Let's check:

$$\lambda_1 = (2\alpha + \beta)/3$$

$$\lambda_2 = -(\alpha - \beta)/3$$

$$\lambda_3 = -(\alpha + 2\beta)/3$$

$$\lambda_1 + \lambda_2 = (\alpha + 2\beta)/3$$

$$\lambda_1 + \lambda_3 = (\alpha - \beta)/3$$

$$\lambda_2 + \lambda_3 = -(2\alpha + \beta)/3$$

Yes indeed. Now, how could we have computed the weights of $(\circ - \circ)$ directly? First, we'd have to get $\Delta_{\max}$ from A^T . Then we'd have to compute the height $I(\circ - \circ)$ as on p104. This would tell us: 3 layers. Then we'd use the "weight generation machinery" of ch. 10, p104, to compute the lower two weights of $\circ - \circ$.

It is indeed much faster to use $\circ - \circ = (\circ - \circ)^{\{1\}}$

Now let's try exercise 12.1 once more:

$$(\circ - \circ - \circ)^{\{1\}} = ?$$

Let $\delta_1, \delta_2, \delta_3, \delta_4$ be the basis vecs of $(1, 0, 0)$. These are $n(n-1)/2$ antisym terms, where $n=4$, the "size" of $(1, 0, 0)$. Thus, expect $4 \cdot 3/2 = 6$ antisym terms, of degree 2:

	weight
$\delta_1 \delta_2 - \delta_2 \delta_1$	$\lambda_1 + \lambda_2$
$\delta_1 \delta_3 - \delta_3 \delta_1$	$\lambda_1 + \lambda_3$
$\delta_1 \delta_4 - \delta_4 \delta_1$	$\lambda_1 + \lambda_4$
$\delta_2 \delta_3 - \delta_3 \delta_2$	$\lambda_2 + \lambda_3$
$\delta_2 \delta_4 - \delta_4 \delta_2$	$\lambda_2 + \lambda_4$
$\delta_3 \delta_4 - \delta_4 \delta_3$	$\lambda_3 + \lambda_4$

I guess you are supposed to choose the four highest weights here + ignore the other 2. Those 4 weights are then supposedly the weights of $\circ - \circ - \circ$ X

The basic rep $\circ - \circ - \circ$ is of dimension 6!

What about $(\overset{1}{\circ}-\overset{2}{\circ}-\overset{3}{\circ})^{\{1\}}$? How do you write the antisym poly's? Supposed to be exactly 4 of them.

$\{1\}^2 - \{1\}^2$ - 1/6 of these?

Well, too bad, in principle, one can compute the weights of any main-branch basic rep in this way:

in part Chapter 13 merely states in full the highest weights of the basic reps of the exceptionals. We already have this information for all nonexceptionals:

Eg, highest weight of $\overset{1}{\circ}-\overset{2}{\circ}-\overset{3}{\circ} = \{\overset{1}{\circ}-\overset{2}{\circ}-\overset{3}{\circ}\}^{\{1\}}$

is $\lambda_1 + \lambda_2 + \lambda_3$, where λ_i are weights of $\overset{i}{\circ}-\overset{j}{\circ}$.

So: $\overset{\alpha_1}{\circ}-\overset{\alpha_2}{\circ}-\overset{\alpha_3}{\circ}$

$$\begin{aligned} M^{(1)} &= \lambda_1 \\ M^{(2)} &= \lambda_1 + \lambda_2 \\ M^{(3)} &= \lambda_1 + \lambda_2 + \lambda_3 \end{aligned}$$

$\lambda_1 =$ get from highest wt.
 $\lambda_2, \lambda_3 =$ get from p. 119.

{ Chap 13 merely lists the various $M^{(i)}$ in terms of simple roots for all basic reps of all exceptional Lie algebras

The rest of ch 13 is on label schemes for these exceptionals, skip it!

α_1 and α_2 are given by
: α_1 and α_2 are given by

Pause before Ch 14

There are many unanswered questions still remaining: I will try to list some of those questions:

- ① So far we have really only considered compact lie groups. But where did we inject the idea of "unitary"? Well, recall Tinkham p 20: "any finite rep. is equivalent to a unitary finite rep." Somehow, this must only be true for a compact lie group. Actually, Tinkham only shows it for a discrete group on p 20. I will tentatively assume this to be true:

$$\text{for compact lie group, } \text{F.I.R.} \sim \text{F.V.I.R.}$$

So, with compact, we don't really have to make the distinction. Still, even with a compact lie, there may be some non-finite non-unitaries. I.e.

$$\left\{ \begin{array}{l} \text{Thm: for compact, simple, } \text{V.I.R.} \subset \text{F.} \quad \text{V.I.R.} \Rightarrow \text{F} \\ \text{So, not F} \Rightarrow \text{not V.I.R., but maybe still I.R.} \end{array} \right.$$

Then, as you go $\text{SO}(n)$ to $\text{SO}(n-1,1)$, the V goes from one to the other. So

- ② What are the reps of noncompact lie groups? [The unitary reps of $\text{SO}(2,1)$ are done in ch 15.]

- ③ What are the reps of non-simple lie groups, like $\text{SO}(4)$ and $\text{SO}(3,1)$? Note that $\text{SO}(3,1)$ is non-simple and noncompact.

- ④ What are Young Tableaux, and how do you distinguish the two 8^3 's in $8 \otimes 8$ of SU_3 ? What are the Clebsch's, what are the D+F in var. couplings. All this in general group form?

- ⑤ Why must the $\text{SO}(3,1)$ spectrum of J_3 be int/half int?

- ⑥ What is the deSitter Group? What are all the "significant" lie groups?

Wybourne Chapter 14: "Dimensions of IR's"

14.1 Scalar Products of Basic Weights

preliminary

Recall: $g_{ik} = \sum_{\alpha \in \Sigma} (\alpha_n)_i (\alpha_n)_k$

(i.e., primary)

Suppose g_{ik} were δ_{ik} . Then we could say:

$$(\alpha_n)_i (\alpha_n)_i = g^{ii} (\alpha_n)_i (\alpha_n)_i = \sum_i (\alpha_n)_i^2$$

Relevant: ? Suppose roots were orthonormal [under some strange metric]. This means:

$$(d_i, d_j) = \delta_{ij}$$

But we know that the Cartan matrix has off diag elements, so this is impossible to have. It won't even be true for the simple roots.

Examine (14.13): $\sum_{\alpha} (d_i, d_j) = \delta_{ij}$

In order to be inside a S.P., α_i must be a root vector, it cannot be a component of a root vector. If α_i is a root vector [the ith one], then $\sum_{\alpha}$ has no meaning. Fishy indeed.

Perhaps (14.13) should read: $\sum_{\alpha} (\alpha)_i (\alpha)_j = \delta_{ij}$

This would be saying: $g_{ij} = \delta_{ij}$. Assume this for now.

Then: $(\alpha, \alpha) = \sum_{ij} g^{ij} \alpha_i \alpha_j = \sum_i (\alpha_i)^2$

And $\sum_{\alpha} (\alpha, \alpha) = \sum_i \left[\sum_{\alpha} (\alpha_i) (\alpha_i) \right] = \sum_i \delta_{ii} = \sum_i 1 = n$

where n = total number of components in α [i.e., $n=1$].

So, for each group, we can compute: (λ_i, λ_j) . Example:

$(\lambda_i, \lambda_j) = K \cdot \delta_{ij}$

for the $B_n, C_n, D_n + F$ we find:

However, for A_n + certain exceptions, we get:

$$(x_i, x_j) = \begin{cases} nK & i=j \\ -K & i \neq j \end{cases}$$

But what is K ? Well, recall that the scale of your weights depends on the scale of your simple roots. There is usually one special scale such that:

$$\sum_i (\alpha)_i (\alpha)_j = \delta_{ij}$$

For example, for B_2 you need $\sqrt{\frac{2}{3}}(0,1)$, $\sqrt{\frac{2}{3}}(1,1)$, etc. in order to have $g_{ij} = \delta_{ij}$.

So, K reflects this scale freedom. Now if we choose the one scale on our α_i so that $g_{ij} = 1 \cdot \delta_{ij}$, then we of course set the scale as our λ_i weights.

Example: B_n , $\alpha = \lambda_i \pm \lambda_j \neq \lambda_i$ $(i,j) \in 1, n$.

Well, ok. The results are:

A_2 : $K = \frac{1}{2 \cdot 3^2}$ etc. All scales are listed in (14-15).

Procedure for getting K :

1) compute all roots in the $g_{ij} = \delta_{ij}$ scale.

2) compute from these the (x_i, x_j) .

Then of course you will obtain K .

14.2 Dimensions of $\mathbb{R}$'s

Theorem: $(\sum_{\alpha \in \Sigma^+} \alpha, \beta) = \sum_{\alpha \in \Sigma^+} (\alpha, \beta) = 1$ if $\beta \in \Pi$.

This has not been proved by me.

Now $["g"] \equiv \frac{1}{2} \sum \alpha^2 = \frac{1}{2}$ sum of $\pm$ roots.

So g is a vector in Δ . Expand: $\vec{g} = \sum g_i \vec{\alpha}_i$

We may compute the g_i mechanically for each group.

Eg, A_n : $g_i = (2-i)$

The reason we are playing with these g_i 's + g_i 's is Weyl's Formula for multiplicity:

$$N(\Phi) = \prod_{\alpha \in \Sigma^+} \frac{(A + g, \alpha)}{(g, \alpha)} = \frac{\prod (1 + g, \alpha)}{\prod (g, \alpha)}$$

Now, for A_n : $\prod (g, \alpha) = \prod \sum_i g_i (\lambda_i, \alpha) = \prod_{p > q} \sum_i g_i (\lambda_i, \lambda_p - \lambda_q)$

Now we make use of all that (λ_i, λ_j) stuff. Namely:

$$\sum_i g_i (\lambda_i, \lambda_p) = -K(n) \sum_{i \neq p} g_i + nK \cdot g_p$$

$$-\sum_i g_i (\lambda_i, \lambda_q) = +Kn \sum_{i \neq q} g_i - nK \cdot g_q$$

But: $\sum_{i \neq p} g_i = (\sum_{\text{all}} g_i - g_p)$ so:

$$\sum_i g_i (\lambda_i, \lambda_p - \lambda_q) = -Kn \left(\sum_{\text{all}} g_i - g_p \right) + nK g_p + Kn \left(\sum_{\text{all}} g_i - g_q \right) - nK g_q$$

$$= 2nK (g_p - g_q)$$

↑ They get it without a 2 here.

The end result, for A_n is: eg.

$$N(\Phi) = \prod_{p > q} \left(\frac{m_p - m_q}{g_p - g_q} \right) \quad \text{where } \begin{cases} \{g_i\} \text{ given on p 136} \\ \{\lambda_i\} \text{ given on p 125} \\ m_i = \lambda_i + g_i \end{cases}$$

So, the data as to which rep you're dealing with is contained in these m_i 's thru the λ_i

Note: for A_n : $\lambda_k = \lambda_{n+1} + \sum_{i=k}^n a_i$ $\lambda_{n+1} = \frac{-1}{n+1} \sum_{i=1}^n (i a_i)$

$$a_i = \lambda_i = \text{labels of rep}$$

$$g_i = \frac{n}{2} - i + 1 \quad p, q \in 1, n+1 \text{ I guess.}$$

This stuff is getting rather cookbook

By the way, Wybourne has a whole other book called

"Sym. Principles in Atomic Spectroscopy"

It is big on: symmetric groups (i.e., permutation groups) & their connection to $GL(n)$. Also, Young Tableaux, and all that stuff about Schur & S-functions, etc, etc. This stuff apparently is important for many-electron symmetry ideas. $SU(3)$ gets no mention. See QCD21 W9XX (1970)

The Butler appendix does not contain SU_3 multiplets, by the way, just U_3 . I don't even understand the notation for the reps. of U_3 . Do they look like SU_3 reps.

Wybourne Ch 15: "the Casimir Invariants"

Review 5.18. Recall that $C_{\alpha\beta\gamma} = g_{\alpha\delta} C^{\delta}_{\beta\gamma}$

But, $g_{\alpha\beta} = C^{\lambda}_{\alpha\lambda} C^{\sigma}_{\beta\sigma}$

Thus: $C_{\alpha\beta\gamma} = \cancel{C^{\lambda}_{\alpha\lambda}} \cancel{C^{\sigma}_{\beta\sigma}} C^{\delta}_{\gamma\delta}$

Compose: $C_{\beta\alpha\gamma} = C^{\lambda}_{\beta\lambda} C^{\sigma}_{\alpha\sigma} C^{\delta}_{\gamma\delta} = C_{\alpha\gamma}$

By playing with Jacobi's deal: $C^c_{ab} C^a_{cd} = 0$

$$\cancel{C^m_{kl} C^p_{mn}} + \cancel{C^n_{lm} C^p_{kn}} + \cancel{C^n_{mk} C^p_{ln}} = 0$$

You can show that: $\{C_{\alpha\beta\gamma} = \text{totally antisym.}\}$

For a semisimple algebra, we know that $g^{\mu\nu}$ exists, and I think you have:

$$[g^{\mu\nu} g_{\nu\lambda} = \delta^{\mu}_{\lambda}] \quad \text{as in relativity.}$$

I.e., $g^{\mu\nu} = (g^{-1})^{\mu\nu}$, so $(g^{-1})^{\mu\nu} (g)_{\nu\lambda} = \delta^{\mu}_{\lambda}$.

[In order for g^{-1} to exist, you need $\det g \neq 0$, i.e., you need semisimple. [This is Cartan's Criterion].

So, it is this fact that $C_{\mu\nu\lambda}$ is antisym that gives the proof that:

$$C = g^{\mu\nu} X_{\mu} X_{\nu} \quad [C, X_r] = 0$$

See Eq. 5.50, p 54 in particular.

[Generalized Casimirs? Reluctant this one:

$$C = g_{\mu\nu} X^{\mu} X^{\nu} = \cancel{C^{\alpha}_{\mu\alpha}} \cancel{C^{\beta}_{\nu\beta}} X^{\mu} X^{\nu}$$

How's about $\tilde{C} = \cancel{C^{\alpha}_{\mu\alpha}} \cancel{C^{\beta}_{\nu\beta}} \cancel{C^{\gamma}_{\lambda\gamma}} X^{\mu} X^{\nu} X^{\lambda}$

You can keep on going in this way.

15.1 Eigenvalues of The Quadratic Casimir

$$C = g^{\alpha\beta} X_\alpha X_\beta = g^i H_i H_i + \sum_\alpha E_\alpha E_{-\alpha}]$$

in (Cartan-Weyl basis). Now:

$$\sum_{\alpha^+} E_\alpha E_{-\alpha} |u_\Lambda\rangle = \sum_{\alpha^+} [E_\alpha, E_{-\alpha}] |u_\Lambda\rangle \quad \Lambda = \text{highest}$$

$$= \sum_{\alpha^+} (\alpha, H) |u_\Lambda\rangle = \sum_{\alpha^+} (\alpha, \Lambda) |u_\Lambda\rangle$$

$$= \left(\sum_{\alpha^+} \alpha, \Lambda \right) |u_\Lambda\rangle = (2g, \Lambda) |u_\Lambda\rangle$$

by def. of "g" (14.17). So:

$$[C |u_\Lambda\rangle = (\Lambda, \Lambda + 2g) |u_\Lambda\rangle] \quad \text{see table.}$$

Now, use: $\Lambda = \sum_i h_i \alpha_i$

$$g = \sum_i g_i h_i$$

$$\begin{aligned} \text{Thus: } (\Lambda, \Lambda + 2g) &= \sum_i h_i = \sum_i h_i \left(\sum_j h_j \alpha_j, \sum_j (h_j \alpha_j + 2g_j \alpha_j) \right) \\ &= \sum_{i,j} h_i (h_j + 2g_j) (\alpha_i, \alpha_j) \end{aligned}$$

Now, read $h_i, g_i, (\alpha_i, \alpha_j)$ off our tables + compute $(\Lambda, \Lambda + 2g)$ for all groups:

Example: A_n : $C = \frac{1}{2(n+1)} \sum_{i=1}^{n+1} h_i (h_i - 2i)$

note that the numbers $\{h_i\}$, as shown on p 125, carry the info spec as to which rep you are in.

So, boom: in one swoop, we now know the value of C in all reps of all simple (classical) lie algebras.

15.2 Generalized Casimirs

For $SO(3)$, $I_3 \propto I_2$, gives nothing new. For a rank l classical group, p 141 shows you exactly which I_n are independent. For $SU_3 = A_2$, I_2, I_3 as you'd guess.

15.3 Invariants for non semisimple Lie groups

Eg, Poincaré is: $P = T_4 \oplus SO(3,1)$

Others:

$$E_2 = T_2 \oplus R_2$$

$$E_3 = T_3 \oplus SO(3)$$

$$E_n = T_n \oplus SO(n)$$

solvable: semisimple

} all "inhomogeneous"
Lie groups.

[Apparently, compact or noncompact of $SO(n+1)$, $SO(n,1)$, does not affect semisimple property. (Yes, see p. 48)]

For the semisimples we know that:

$$\begin{aligned} \text{Number of invariants} &= \text{rank of Lie group} = l \\ &= \text{size of Cartan subalgebra.} \end{aligned}$$

i.e. for any Lie group

For nonsemisimples, we can say: (Beltrametti & Blasi, 1966 Phys. Lett.!)

$$\begin{aligned} \text{Number of invariants} &= \text{order of Lie group} \text{ minus } \text{Rank} \| C_{\alpha}^{\alpha} \alpha \| \\ &= n - R \| C_{\alpha}^{\alpha} \alpha \|. \end{aligned}$$

i.e. $C_{\alpha}^{\alpha} \alpha$ is an $n \times n$ matrix, α are the n -parameters, as in

$$G = \bigoplus_{\alpha} i \vec{\alpha} \cdot \vec{X}_{\alpha} \quad \begin{matrix} \text{generators} \\ \text{parameters} \end{matrix}$$

Thus, there is somewhere between 0 and n invariants.
and 4 invariants

→ Eg, T_4 has 4 parameters, because $C_{\alpha}^{\alpha} \equiv 0$.

→ For E_3 , there are 6 parameters, $n=6$, but 2 invariants. These are P.P. and P.T. Note that T^2 is not translation invariant!

15.4 Casimirs for $SO(3)$ and $SO(3,1)$

Assume T_i are all hermitian, so we are only considering unitary reps. like $O(2,1)$. Then

$$\begin{aligned} \text{a) } T_+ T_-, T_- T_+ &\text{ are positive definite, because} \\ T_+ T_- &= T_- T_+ \end{aligned}$$

b) Therefore, $X - a(a \pm 1) \geq 0$ [i.e., the eigenvalues of both $J_+ J_-$ and $J_- J_+$ must be positive]

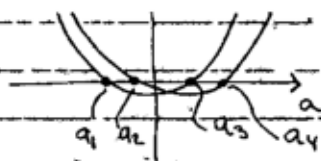
c) X is real + positive because J^2 is pos. def., herm.
 d) a is real since J_3 herm.

e) J_+ raises eigenvalue of J_3 by one unit, see 15.26.

Frankel Now, make a little plot:

$$f_{\pm}(a) \equiv a^2 \pm a - X. \text{ We require } f_{\pm}(a) \leq 0.$$

But:



This implies there must be upper + lower bounds on a !
 Of course $a_+ - a_- = \text{integer}$, since J_+ connects them.

$$\text{Well, } f_{\pm}(a) = a^2 \pm a + \frac{1}{4} - X - \frac{1}{4} \\ = (a \pm \frac{1}{2})^2 - (X + \frac{1}{4})$$

$$\text{So } f_{\pm}(a) = 0 \Rightarrow (a \pm \frac{1}{2})^2 = X + \frac{1}{4}.$$

$$\Rightarrow a \pm \frac{1}{2} = \pm \sqrt{X + \frac{1}{4}}.$$

The smallest a on the right is a_3 :

$$(a \pm \frac{1}{2}) = \begin{cases} \sqrt{X + \frac{1}{4}} \\ -\sqrt{X + \frac{1}{4}} \end{cases} \Rightarrow a = \begin{cases} \sqrt{X + \frac{1}{4}} + \frac{1}{2} = X + \frac{3}{4} = a_4 \\ -\sqrt{X + \frac{1}{4}} - \frac{1}{2} = -X - \frac{3}{4} = a_1 \\ \sqrt{X + \frac{1}{4}} - \frac{1}{2} = X - \frac{1}{4} = a_3 \\ -\sqrt{X + \frac{1}{4}} + \frac{1}{2} = -X + \frac{1}{4} = a_2 \end{cases}$$

It would seem that:

$$a_- = a_2 = -\sqrt{X + \frac{1}{4}} + \frac{1}{2}$$

$$a_+ = a_3 = +\sqrt{X + \frac{1}{4}} - \frac{1}{2}$$

$$a_{\pm} = \mp \frac{1}{2} \pm \frac{1}{2} \sqrt{1 + 4X}$$

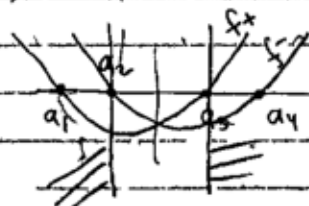
$$a_+ - a_- = \left(-\frac{1}{2} + \frac{1}{2} \sqrt{1 + 4X} \right) - \left(+\frac{1}{2} - \frac{1}{2} \sqrt{1 + 4X} \right)$$

$$= -1 + \sqrt{1 + 4X}$$

$$(a_+ + \frac{1}{2})^2 = \frac{1}{4}(1 + 4X)$$

$$a_+ = -\frac{1}{2} + \frac{1}{2} \sqrt{1 + 4X}$$

$$a_+ + \frac{1}{2} = \frac{1}{2} \sqrt{1 + 4X}$$



Question: given some + values of X , I see that

a) J_+ raises a by integers, J_- lowers.

b) a cannot exceed the range $(-a^+, a^+)$ where $a^+ = -\frac{1}{2} + \frac{1}{2}\sqrt{1+4X}$

But these facts alone do not explain why a takes on all values from a^+ to $-a^+$. Is, it certainly could stop short:

I conclude that certain facts have been left out. Well, one idea is that you want $J_+ |X, a^+\rangle = 0$. This happens if $J_- J_+ |X, a^+\rangle = 0$, and this in turn means $X - a^+(a^++1) = 0$ which in turn means $X = a^+(a^++1)$.

Similarly, $J_- |X, a^-\rangle = 0 \Rightarrow J_+ J_- |X, a^-\rangle = 0 \Rightarrow X - a^-(a^--1) = 0 \Rightarrow a^-(a^--1) = X$

Well $a^-(a^--1) = a^+(a^++1) \Rightarrow a^- = -a^+$

Finally, $a^- = -a^+ \Rightarrow a^+ + a^+ = 0 \Rightarrow 2a^+ = n$
 $a^+ - a^- = n$

So, the usual $|jm\rangle$ results come out.

Finally,

$$\langle jm | J_- J_+ | jm \rangle = \sum_{m'} \langle jm | J_- | jm' \rangle \langle jm' | J_+ | jm \rangle = | \langle jm+1 | J_+ | jm \rangle |^2$$

$$= j(j+1) - m(m+1) \quad \text{etc.}$$

Now let's do $SO(2,1)$. Change one sign:

$$\begin{aligned} [J'_1, J'_2] &= -i J'_3 \\ [J'_2, J'_3] &= i J'_1 \\ [J'_3, J'_1] &= i J'_2 \end{aligned}$$

$$\text{Define } J_{\pm} = \frac{1}{\sqrt{2}} [i J'_1 \mp J'_2]$$

$$R = e^{i \theta \cdot J'}$$

So, if J' is herm, R is unitary, but $J_{\pm}$ are:

$$\begin{aligned} (J_+)' &= -J_- \\ (J_-)' &= -J_+ \end{aligned}$$

We have defined $J_{\pm}$ in this strange way to recover the $SO(3)$ Lie algebra:

$$[J_+, J_-] = J_3$$

$$[J_3, J_{\pm}] = \pm J_{\pm}$$

but, $J_+ \neq J_-$

as in $SO(3)$ in unitaries.

However, as shown on p. 48, the Cartan metric tensor for $SO(2,1)$ in terms of J_1, J_2, J_3 is:

$$g_{\mu\nu} = 2 \begin{pmatrix} -1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$$

So: $C \propto J_1'^2 + J_2'^2 - J_3'^2$

→ So, X can now be + or minus since C is no longer + def.

Meanwhile, $J_+ J_- = -J_+ J_+ = \text{neg. def.}$

So now we require:

$$X - a(a \pm 1) \leq 0. ? X$$

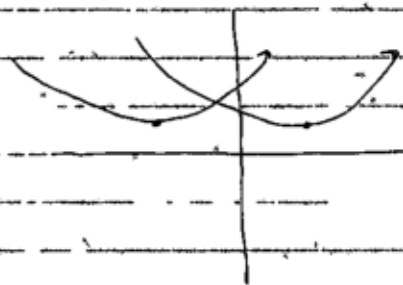
Be careful!

$$J'^2 = -[J_+ J_- + J_- J_+] - J_3'^2 = -[J_+, J_-] \text{ etc.}$$

The sign of X is different, so $[X + a(a \pm 1)] \geq 0 \checkmark$

So, how exactly do you find the seps? You must guarantee that $X + a(a \pm 1) > 0$. Let's plot these:

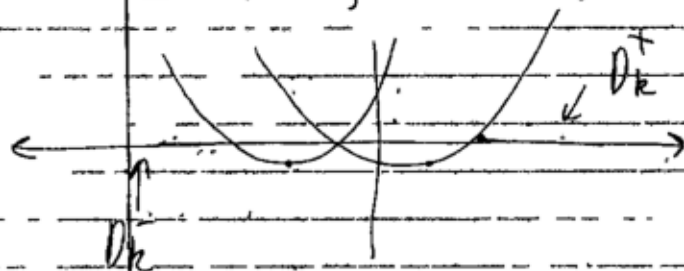
$$\begin{aligned} g_{\pm}(a) &= X + a^2 \pm a = a^2 \pm a + \frac{1}{4} + X - \frac{1}{4} \\ &= (a \pm \frac{1}{2}) + (X - \frac{1}{4}) \end{aligned}$$



If $X > \frac{1}{4}$, any value of a is OK. Still, you want them with one unit spacing.

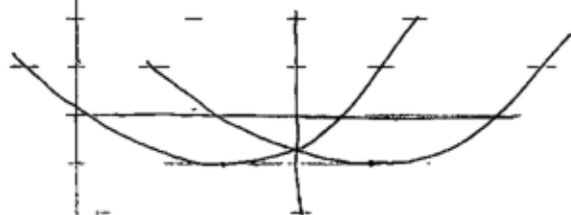
So, this gives you the continuous series.

However, if $X = \text{negative or below } \frac{1}{4}$, then you have:



Now, you have to make sure $J_- |X, a_{min}\rangle = 0$, so this will make:

The discrete series: suppose $X < \frac{1}{4}$, so we have:



The $\pm a$ series will need: $\sum_{n=1}^{\infty} |X, a_{\min}| = 0$

$$\Rightarrow \sum_{n=1}^{\infty} |..| = 0 \Rightarrow (-X - a_{\min}(a_{\min} - 1)) = 0$$

$$\Rightarrow -X = a_{\min}(a_{\min} - 1) \Rightarrow X = a_{\min}(1 - a_{\min})$$

Choose $a_{\min} = k$, then $X = k(1-k)$ and $a = k, k+1, \dots$

Not clear why we should restrict k to $\frac{1}{2}$ a full integers.

For the $\bar{I}_k$ series, want $\sum_{n=1}^{\infty} |X, a_{\max}| = 0$

$$\Rightarrow -X - a_{\max}(a_{\max} + 1) = 0 \Rightarrow -X = a_{\max}(a_{\max} + 1) \Rightarrow a_{\max} = -a_{\min} = -a_{\min}(a_{\min} - 1)$$

So, again, choose $a_{\min} = k$,

but this time:

$$X = k(1-k) \quad a = -k, -k-1, \dots$$

Q: OK, all this makes sense except the idea that $\bar{I}_3$ has to take on integral / half integral values. Why must this be? Has something to do with "singlevaluedness", I suspect.

Wyllbourne Chapter 16: global properties of Lie groups

16.1 Topological Neighborhoods The concept of open ball as a neighborhood in $\mathbb{R}^n$ euclidean space. Very basic stuff.

16.2 Topological Spaces First you need some kind of set M , to know what you are dealing with. In M there are, presumably, tons of subsets V_i . Let $p \in M$. Associate with this p , a certain set of subsets of M , call these $V_i(p)$. These subsets are supposed to have certain niceness properties, such as: V_1, V_2 neighborhoods of $p \Rightarrow V_1 \cap V_2$ also a neigh.

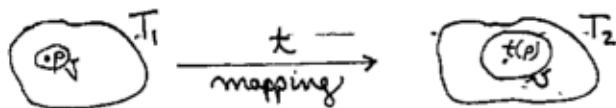
Such a set of subsets $V_i(p)$ can be defined for each $p \in M$. The full set of subsets V_i is called a "topology for M ." So, the set M together with its topology V is called a "topological space." $T = \{M, V\}$.

If $T_1 \subseteq T$ somehow, then T_1 = "topological subspace" and the T_1 subsets V_i are called "the topology induced on T_1 by T ."

The most inferior topology is called "coarsest". This topology is just $V =$ two sets: M and $\emptyset$. As V gets more complicated for a given M , your topology is said to get "finer".

16.3 Examples of Topological Spaces First he mentions some geometric topologies. $S =$ sphere surface has a topology induced from $\mathbb{R}^3$. Second, he does an abstract one with integers. $M =$ integers. If $n \in M$, then $V^q(n, r)$ is a set of neighborhoods of n defined in a strange way, so r, q are parameters which label the subsets $V(n)$. So, $T = \{M, V\} =$ a topological space. Finally, the "natural topology of the real line" is done: there, the subsets V are defined as intervals.

16.4 Homeomorphisms Consider two top. spaces T_1, T_2



If any open ball in T_2 maps back into an open ball in T_1 , map is "continuous at p " which has $t(p)$ as center of open ball in T_2 . If continuous $\forall p \in T_1$ then just plain "continuous."

Now, a mapping $t: T_1 \rightarrow T_2$ which is 1-to-1 and such that open sets are also 1-to-1 (i.e., all open balls in T_2 map uniquely into an open ball in T_1 ; I think this means "bi-continuous"), then this map is a homeomorphism. (Not homomorphism).

Notice that all these words are in the Topologic language: continuous mapping, homeomorphism, topological equivalence,...

16.5 Products of Topological Spaces. If you know what "neighborhood" means in T_1 and T_2 on M_1 and M_2 (i.e., if you have two topologies T_1 and T_2 on sets M_1 and M_2), then automatically there is a concept of neighborhood induced onto the direct product set $M_1 \times M_2$ whose elements are (p, q) , $p \in M_1$, $q \in M_2$. This new sense of neighborhood is the "direct product topology". So, this new topology on the new set is a new topological space, $T = T_1 \times T_2$, the "topological product".

Examples: torus = circle $\times$ circle, all with natural interval topologies. Also, $E_2 = E_1 \times E_1$, etc.

16.6 Hausdorff Spaces. In general, top. spaces are too general to deal with, so we restrict to "Hausdorff Spaces" which are defined by the "Hausdorff Separation Axiom": If $p, q \in M$ and $p \neq q$, then $\exists$ open balls $V(p)$ and $V(q)$ such that $V(p) \cap V(q) = \emptyset$. I.e., you can separate any two points by going to sufficiently small neighborhoods about each point.

Consequences: a convergent sequence in a Hausdorff space always has a unique limit. Not true in general!

16.7 Metric Spaces. Take a set M , add a metric $d(p, q)$ with the usual 3 properties: born, $(M, d) =$ a metric space. Note: $\nRightarrow$ existence of a norm. Note: metric spaces are always Hausdorff, because the existence of "distance" $d(p, q) > 0$ $p \neq q \Rightarrow$ separation is possible.

16.8 Connected Spaces. "Connected" means you cannot write as union of "disjoint" sets, obvious notion. Results: T_1, T_2 connected $\Rightarrow T_1 \times T_2$ also connected. Obvious homeomorphism result. Example: The rationals are a "totally disconnected subspace" of the reals. $\mathbb{R}$ is connected. If T_1 is disconnected, its pieces may still be connected: "locally connected".

16.9 Compact Spaces. First, recall the Heine-Borel Theorem + significance. In Buck p.31 we have this theorem: if $C =$ closed + bounded set in $\mathbb{E}^n$, then any open covering contains a finite subcover. So theorem says: closed + bounded $\Rightarrow$ Heine Borel Property, at least in $\mathbb{E}^n$. Another word for Heine Borel Property is "compact". So closed, bounded $\Rightarrow$ compact. Now Wyb. makes this point: we do not want to define "topological compactness" in terms of "boundedness" because bound $\Rightarrow$ metric, and in general top. spaces are "pre-metric". So, define compactness via the, or as, the Heine-Borel Property. In fact, Buck does this.

16.10 Homotopic Paths. A new idea for me. First, define a "path" as a continuous map from $I = (0,1)$ to your top space T . $t(0)$ and $t(1)$ are the "endpoints of the path". If $\forall p, q \in T \exists$ a path, then T is "arcwise connected". A path is really the same as a "curve".

Now, suppose you have two curves in T with the same endpoints. $t_1(s), t_2(s) \quad s \in I. \quad t_1(0) = t_2(0)$
 $t_1(1) = t_2(1).$

Suppose there exists a function continuous on $I \times I = I^2$, call it $t(r,s)$, such that $t(0,s)$ gives you one path $t_1(s)$, and $t(1,s)$ gives $t_2(s)$. Endpoint equality means $t(0,0) = t(1,0); \quad t(0,1) = t(1,1)$. If such a $t(r,s)$ exists, then the two paths are homotopic. The idea of course is that r is a continuous variable which allows us to continuously deform one path to the other. The set of curves homotopic to a curve is the homotopy class of that curve.

Consider the curve $\gamma_p(s) = p$, a constant, as s ranges in $(0,1)$. On T , this path appears as a point, the "null path", denoted by $1_p = \gamma_p(s)$. The homotopy class of 1_p is denoted by $[1_p]$.

Let $\gamma_p(s)$ be some closed path through p . Is $\gamma_p(s) \in [1_p]$? Geometrically, yes, as long as $\gamma_p(s)$ doesn't catch up on a "hole" in space T . But mathwise, all we need to show is existence of $\lambda(t,s)$ continuous on $(0,1)^2$ with

$$\begin{aligned} \lambda(0,s) &= \gamma_p(s) \\ \lambda(1,s) &= p, \forall s \end{aligned} \quad \left(\begin{aligned} \lambda(0,0) &= p \\ \lambda(1,0) &= p \end{aligned} \right) ?$$

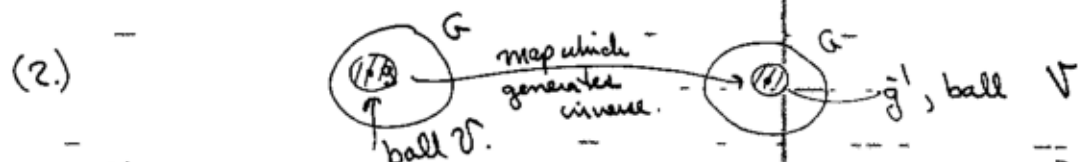
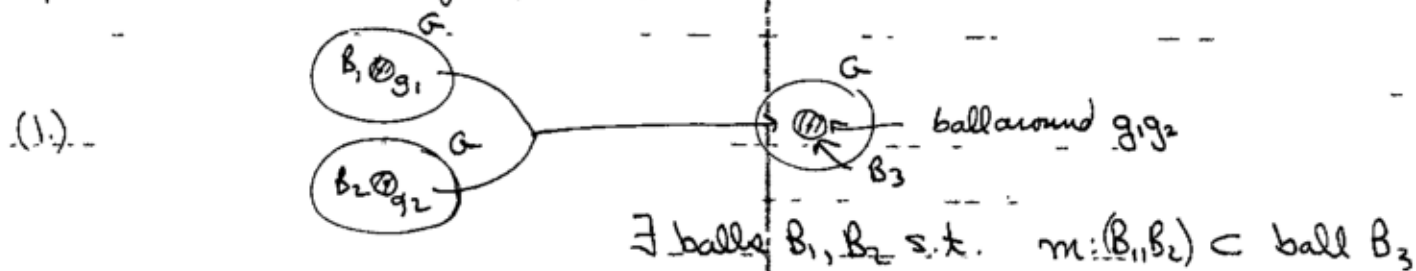
16.11 Multiple + Simple Connectivity: An annulus is "infinitely connected" because the number of homotopy classes of a point is ∞ . Could label these classes by "homotopy number" — # times curve wraps the hole. Disc is simple. Sphere is simple.

16.12 The Fundamental Group. Start with a top space T . Pick a point $p \in T$. Let $[\gamma]$ be a particular homotopy class at this point, i.e., $[\gamma] =$ set of all curves deformable to some standard curve? — closed curves, let's say. Then $\{[\gamma]\} =$ the set of such classes. We claim that the set of homotopy classes of a point $p \in T$ defines a group on T . Multiply two classes $[\gamma_1], [\gamma_2]$, you get a new class $[\gamma_3] \in G$. Inverse class is $[\gamma^{-1}]$. Identity class is $[1]$, etc. This group is called $\pi_1(T, p)$, the "fundamental group of T ".

16.13 Universal Covering Groups. Spaces. First, define a "covering space": $p: W \rightarrow T$, W "covers" T if the map p hits all points in T at least once. Eg., $p: \mathbb{R} \rightarrow S^1$ with $p(s) = e^{i2\pi s}$. So $\mathbb{R}$ covers S^1 , the circle. When the covering space is simply connected, it is called "universal". Theorem: if T is a top space which is (a) locally connected (b) locally compact, then $\exists$ a unique universal covering space. (1925)

16.14 Topological Groups Recall that a group is defined by $\{G, m\}$ where $m: G \times G \rightarrow G$. I.e., m is the multiplication. Also, $\{G, \tau\}$ defines a topological space on the same set G ; here τ defines the open set structure, a topology.

If G is a group and a top space, it will be a "topological group" provided that the group map is "continuous" in this sense:



Here, we want there always to exist a ball around g which maps into a ball around (g^{-1}) , called V^{-1} . Really the same idea as naive continuity.

As an example, consider $GL(n, \mathbb{C})$ as group $g \in G =$ a matrix of n^2 complex #s. But, also, consider g as a point in space $\mathbb{C}^{n^2}$, + "induce" the topology of $\mathbb{C}^{n^2}$. Now we have $\{G, m, \tau\}$. This is a topological (natural) group.

16.15 Products of Topological Groups. We know already how to make a direct product of two groups (= sum of the Lie algebras). And, we know topological product. So, can do a direct product of two Topo. Groups. Example: $GL(n_1, \mathbb{R}) \times GL(n_2, \mathbb{R})$ is set of $(n_1 + n_2) \times (n_1 + n_2)$ matrices, but only $n_1^2 + n_2^2$ generators, so in the "parameter space" we have $\mathbb{R}^{n_1^2 + n_2^2} = \mathbb{R}^{n_1^2} \times \mathbb{R}^{n_2^2}$, a topological product.

16.16. Isomorphisms of Topo. Groups In order for two Top. Groups to be "topologically isomorphic" you need both:

- groups must be isomorphic
- topological spaces must be homeomorphic.

16.17 Topological Subgroups. You need subgroup in the usual sense, plus there must be a closed topological subspace. For example, the "integers" is not a top. subspace of the "reals" with the natural topology of the reals, although integers is subset of reals. However, additive integer group is a subgroup of additive reals group. But no topo. subgroup here.

16.18 Invariant Topologic Subgroups. A carry over of the idea of a usual invariant subgroup, i.e., $gSg^{-1} = S \quad g \in G$, i.e., contains only complete classes. I learn here the definition of the "center" of a group, as opposed to the center of a Lie algebra: - the subset of G which commutes with all of G . Continuous groups often have discrete centers ~ semisimples always do. Eg, the center of $SU(2)$ is E and $-E$ (also called the "kernel"?). For $GL(2, \mathbb{R})$, $\lambda E \quad \forall \text{ real } \lambda$ form a continuous center, isomorphic to add group of reals with usual mult. The center is always an abelian subgroup of G . It is also always an invariant subgroup.

16.19 Coset Spaces + Factor Groups. If H is a subgroup of G , you can talk cosets gH . If H is invariant sg , then $gH = Hg$. The invariant subgroup H , together with its cosets, comprise the "factor group", called G/H . The elements of G/H are cosets gH , with g taking value e to include H itself in G/H .

Why is this a group? $(g_1H)(g_2H) = \text{product of two cosets}$
 $= (g_1Hg_2)H = \text{some } (g_3H) = \text{another coset.}$

OK. Now, define a map $\phi: G \rightarrow G/H$ which takes you from group to factorgroup, obviously a homomorphism, many-to-one in general. This map is called the "canonical" or "natural" homomorphism of G wrt H . For example, $SU(2) \xrightarrow{\phi} SU(2)/Z_2$ where $H = Z_2 =$ the two element center. The # elements in G/H is $|G| \div |H|$, hence name "quotient group."

Finally, we can talk about a topology for the quotient group generated somehow by the topology of G , thought of as a topological group. Namely, let ϕ be the natural map $G \rightarrow G/H$. Then we require that open ball in G/H , call it V , must map from an open ball in G , $\phi^{-1}(V)$, and vice versa. This makes G/H a topological space, call it the quotient space.

16.20 Homogeneous Spaces Let $\{G, \tau\}$ be some topo. space. Let g_1, g_2 be points in G . If there is always a g_3 such that $g_3 \cdot g_1 = g_2$, then you have a homogeneous space. If G is also a group [ie, a topological group], then such a g_3 always exists by def. Thus, a topo. group is always homogeneous.

16.21 Manifolds and Lie Groups Consider a group G . Endow it with a "nice" topology $\Rightarrow G =$ topological group. If the topological space is locally Euclidean + connected, then the topo. space is called a "topo. manifold", and G is then a Lie group. All topo. manifolds [hence, all Lie groups] are locally compact, locally connected. If locally T_n looks like $\mathbb{R}^n$, you have a "real manifold". If looks like $\mathbb{C}^n$, then a "complex manifold". If coordinates on T_n have m_x ^{partial} derivatives relative to another set of coords, then you have a "differentiable manifold of class C^m ". Type of manifold determines type of Lie group ($\mathbb{R}$ or $\mathbb{C}$).

16.22 Real Simple Lie Groups + Algebras In the classification we did earlier in this book, all the complex semisimple Lie algebras were worked out via Dynkin's methods. We have yet to deal with the enumeration of the real forms of these complex algebras, like $so(3)$ and $so(2,1)$ both being real forms of A_1 . ~~$so(3)$~~ We do know that of all the possible real forms (whose complex extensions are equivalent), only one is compact, ie, only one has a negative definite metric g .

We now attack this problem. Let $A_{\mathbb{C}}^{\mathbb{C}} =$ the compact form (real) of a complex simple Lie algebra $A_{\mathbb{C}}$. Now, $A_{\mathbb{C}}^{\mathbb{C}}$ has some standard "basis" - [generators]. You can often transform your basis to get a new, nonisomorphic Lie algebra. Eg, if (X_1, X_2, X_3) is $\mathfrak{so}(3)$, then (Y_1, Y_2, Y_3) related as shown p168 is $\mathfrak{so}(2,1)$.

So, how do you write down all possible basis x forms P which will give you new real algebras?

(a) first, construct all matrices S where $S^2 = I$, and S diagonal (hence ± 1 on diag).

(b) then let $P = \sqrt{S}$, so P has either $+1$ or $+i$ on diagonals.

So, for each P so formed, you get a new real form. (Note: a basis change S s.t. $S^2 = I$ is called an "involution automorphism" of the group in question).

Finally, table 16.3 lists off all real forms of the nonexceptionals; however, for low n , there are isomorphisms.

16.23 Isomorphisms of the Lie Groups + Algebras. The Table 16.5 shows isomorphic Lie algebras. Note, eg, that $\mathfrak{su}(1,1)$ which Vilenkin calls $\mathfrak{ou}(2)$ relates to $\mathfrak{sl}(2, \mathbb{R})$, whereas $\mathfrak{su}(2)$ does not so relate. Also, note that for Lie groups, these are only local isomorphisms.

16.24 Universal Covering Groups. Let G be a multiply connected group. It turns out that G has certain partner groups, and all have the same "covering group" $\tilde{G}$ which is simply connected. The set of partner groups to G , called set Γ , are all locally isomorphic. In fact, you can generate the partners of G in Γ by writing G as $\tilde{G}/K$ where K is a (discrete) inv. sg. of the center $\mathbb{Z}$. Example: ~~$\mathfrak{so}(2)$~~ $\mathfrak{su}(2) = \mathfrak{so}(3)/\mathbb{Z}_2$, where here the full center is used of $\mathfrak{su}(2)$. $\uparrow$ $\mathfrak{so}(3) = \mathfrak{su}(2)/\mathbb{Z}_2$.
Not too clear on this.

Chapter 18: $su(1,1)$ spectrum-generating algebras

18.1 Introduction

The game here is to "realize" an $su(1,1)$ algebra by using as generators certain differential operators. Our knowledge of the Lie algebra should help us understand a differential equation!

18.2 A realization

The generators are $\Gamma_1, \Gamma_2, \Gamma_3$, with Γ_3 being the compact one. The suggested forms have " α " as a parameter. Somehow, α is a label of our realization + sets the value of the only Casimir:

$$\Gamma^2 = -\frac{1}{4}\alpha - \frac{3}{16}$$

$$\Gamma_3 = \frac{\partial^2}{2y^2} + \frac{\alpha}{y^2} - \frac{y^2}{16} \quad \text{etc.}$$

A typical equation of interest is:

$$\left\{ \frac{d^2}{dy^2} + \left[\frac{a}{y^2} + by^2 + c \right] \right\} Y(y) = 0$$

This can be trivially rewritten as:

$$\left\{ \left(\frac{1}{2} + 8b\right)\Gamma_1 + \left(\frac{1}{2} - 8b\right)\Gamma_3 + c \right\} Y = 0$$

Here, $a = \alpha$.

18.3 Discrete Eigenvalue Spectra

① Spectral Theory: we could rewrite our differential equation in operator form as:

$$D(a, b)Y = -cY$$

This is an eigenvalue equation. Presumably, not all values $c \in \mathbb{R}$ will be eigenvalues.

② example: $DY = cY$ with $D = \partial/\partial x$. Here, $Y = e^{cy}$ is always a solution, so the "spectrum" of operator $\partial/\partial x$ is continuous.

③ example: $D = \partial^2/\partial x^2$. Then $DY = dY \Rightarrow Y = e^{\pm \sqrt{d}y}$, again you have a continuous spectrum. However, boundary conditions can force the spectrum to be discrete.

④ I am not familiar with discrete spectrums except those caused by boundary conditions.

⑤ However, in this section it is shown that

$$c = \sqrt{-b} [\sqrt{1-4a} + 2 + 2n] \quad n = 0, 1, 2$$

are special values of c which are eigenvalues of D . Clearly this spectrum is discrete.

In fact, such values of c correspond to the various values of $E+x$ is $|\Phi, E+x\rangle$ which label the elements of the representation $D^+(\Phi)$.

⑥ Are our generators hermitean? If not, then we have no right to claim, eg, that the values of $c/\sqrt{-b}$ are "quantized" in (18.16).

Well, in fact, as long as we restrict to L_2 , these generators are hermitean!

$$\begin{aligned} \text{Eg, } & \int_{-\infty}^{\infty} f(y) \left[-i/2 \left(y \frac{\partial}{\partial y} + \frac{1}{y} \right) \right] g(y) dy \\ &= \cancel{\text{part}} + \int_{-\infty}^{\infty} \left[-i/2 \left(y \frac{\partial}{\partial y} + \frac{1}{y} \right) \right]^* f(y) \cdot g(y) dy \end{aligned}$$

Notice that $(iy \partial_y)$ by itself is not hermitean, but

$$\begin{aligned} \int f^*(y) \left[i \left(y \frac{\partial}{\partial y} + \frac{1}{y} \right) \right] g dy &= i \left[\int f^*(y) y \left(\frac{\partial}{\partial y} g \right) dy + \frac{1}{2} \int f^* g \right] \\ &= i \left[- \int \left(y \frac{\partial f^*}{\partial y} + f^* \right) g + \frac{1}{2} \int f^* g \right] \\ &= i \left[+ \int \left(y \frac{\partial}{\partial y} + \frac{1}{y} \right) f^* \cdot g \right] = \int (-i) \left[y \frac{\partial}{\partial y} + \frac{1}{y} \right] f^* \cdot g \\ &= \int \left[i \left(y \frac{\partial}{\partial y} + \frac{1}{y} \right) \right]^* f^* \cdot g \end{aligned}$$

So, Weybourne has in fact been very careful to set up hermitean differential operators. The space of acceptable functions is something like L_2 .

So: $[L_2 + \text{def of } P_i \Rightarrow P_i \text{ herm} \Rightarrow \text{eigenvalues labeled int'g reps} \Rightarrow \text{possible spectral discreteness}]$

1965

VILENKIN

July 15, 1977

Vilenkin Chapter 7

Very unfortunately I overlooked this chapter. It contains much information I thought was being originally presented in LBL-5527. 50 pages long, -I have no Xerox of it.

1. Hypergeometric Function.

1. Definition
2. Relations
3. Integrals involving
4. Jacobie (first kind only) polynomials expressed in terms of HGF.

2. The Group SL(2,R).

1. Remarks: accent here will be on diagonal matrices
2. Parametrization. Theorem: you can write $g = d_1 p d_2$ or similar, where d_i are diagonal (unimodular) and p is either a y -boost or y -rotation form matrix. Close to the Euler angles but not quite due to details I have left out.
3. Lie algebra stated

3. Irreducible Representations.

1. Description: converts the QU(2) shift rep to SL(2,R). Converts to usual multiplier rep in one real variable x .
2. Another realization: Mellin's of $f(x)$ and $f(-x)$ are called $F(\lambda)$ and $F(\lambda)$. Group action on $f(x)$ is then converted to action on $F(\lambda)$. This group action is then written as an integral, there are always two terms because 2x2 sense. Kernels of integrals are the famous $K_{++}(\lambda, \mu; \chi; g)$. You have:

$$R_x(g).F(\lambda) = \sum_{\mu} \int_{a-i\infty}^{a+i\infty} d\mu K_{++}(\lambda, \mu; \chi; g) F(\mu)$$

The reps are labelled by $\chi = (l, a)$ so we are doing Bargmann's C_q series. The kernels are written as integral representations only here.

4. Infinitesimal operators. I.e., generators are written in terms of variable x

4. Calculation of the Kernels.

1. Calculation of the K's for case of y -boost (called h) or y -rotation (called u). Also does for trihedral matrices. For the y -rotation matrix he gets:

$$h = \begin{pmatrix} \cosh \theta & \sinh \theta \\ \sinh \theta & \cosh \theta \end{pmatrix}$$

$$K_{++}(\lambda, \mu; \chi; h) = \frac{1}{2\pi i} \frac{\Gamma(\lambda) \Gamma(-\lambda - 2\ell)}{\Gamma(-2\ell)} \frac{(\sinh \theta)^{4\ell + \lambda + \mu}}{(\cosh \theta)^{2\ell + \lambda + \mu}} F(-2\ell - \lambda, -2\ell - \mu; -2\ell; -\frac{1}{\sinh^2 \theta})$$

By doing a $c-a, c-b$ thing I could write this in terms of my function $Q_{\lambda}^{-j-1}(\cosh \theta)$.
By the way, $K_{+-} = 0$.

2. The case of triangular matrices: K's recomputed
3. The general case. Multiply earlier results together.
4. Some hypergeometric integral transforms. Nothing exciting here.

5. Recurrence Formulas. Derives the usual recurrence formulas as in Bateman by first writing the operation of differentiation in terms of generators, then converting the generators to the Mellin space. The K 's are used here.

6. Integral reps, and composition formula.

1. Intro: from the group property, the kernels must compose with no constants at all, just Mellin-Barnes contour with adjacent "helicity" indices touching.

Rest: by choosing simple K 's inside (ie, simple g 's) you can make lots of integral representations for the outside K function. Also, various special cases of the composition formula (addition theorem) are done.

7. Triangular and Hankel: a different representation of same group. Same representations, but new way to do it. The kernel now is some kind of Hankel function. Very short sec nothing super.

/end

Comments: Nowhere in this chapter do we find any reference to Azimov's work. Nor is anything said about "Legendre". Nothing about "second kind Legendre". Why bother if you are interested in the HGF per se. No need to convert to Legendre. No one would know that these kernels are close to associated Legendre functions. Vilenkin simply does not associate the word "Legendre" with the second kind functions.

Also, there is no comment about the D_k^+ representations.

Of course nothing on diagonalization.

This work was referenced by one of the Abarbanel and Saunderson's papers. They converted to e -functions.

I would criticize Vilenkin for omitting to mention Legendre, and also for making the artificial separation between $QU(2)$ and $SL(2, R)$. The real difference in these two chapters is not the group but rather the basis. The chapter 6 is in the discrete basis and so involved first kind Legendre functions; chapter 7 has chosen the continuous basis and involves the second-kind Legendre functions.

Vilenkin

Check on α, β conventions to get correct $SU(2)$ elements!

① Lock in on Vilenkin's conventions system. He uses ω^a, A^a, H^a etc. etc.

$$\omega_1(t) = \begin{pmatrix} \cosh \frac{t}{2} & \sinh \frac{t}{2} \\ \sinh \frac{t}{2} & \cosh \frac{t}{2} \end{pmatrix} = e^{-i t K_1} = e^{-i t K_1} \text{ in my book.}$$

So: $\frac{d}{dt} e^{-i t K_1} \Big|_{t=0} = -i K_1$

$$\omega_2(t) = \begin{pmatrix} \cosh \frac{t}{2} & i \sinh \frac{t}{2} \\ -i \sinh \frac{t}{2} & \cosh \frac{t}{2} \end{pmatrix} = e^{+i t K_2}$$

$$\omega_3(t) = e^{+i t J_3}$$

So we have:

$$\begin{cases} \omega_1(t) = e^{-i t K_1} \\ \omega_2(t) = e^{+i t K_2} \\ \omega_3(t) = e^{+i t J_3} \end{cases} \leftarrow \text{where } K_{..} = 2 \times 2 \text{ d. m.}$$

$$\omega_i(t) = e^{t a_i}$$

② Next: $a_1 = \frac{1}{2} \sigma_1 = J_1 = -i(i \frac{1}{2} \sigma_1) = -i K_1$
 $a_2 = -\frac{1}{2} \sigma_2 = i K_2$
 $a_3 = \frac{1}{2} \sigma_3 = i J_3$

$$\begin{cases} a_1 = -i K_1 \\ a_2 = i K_2 \\ a_3 = i J_3 \end{cases}$$

③ Next, I read

$$T_x[\omega_i(t)] = T_x[e^{t a_i}] = e^{t A_i}$$

so $\frac{d}{dt} (e^{t A_i}) \Big|_{t=0} = A_i$

Therefore:

$$A_i \leftrightarrow a_i$$

Thus:

$$\begin{cases} A_1 = -i K_1 \\ A_2 = i K_2 \\ A_3 = i J_3 \end{cases}$$

④ Next: -----

$$H_{\pm} = -A_1 \mp iA_2 = -(-iK_1) \mp i(iK_2) = iK_1 \pm K_2$$

$$= i(K_1 \mp iK_2) = iK_{\mp}$$

$$H_3 = iA_3 = i(iJ_3) = -J_3$$

$\Rightarrow$

$$\boxed{\begin{aligned} H_{\pm} &= iK_{\mp} \\ H_3 &= -J_3 \end{aligned}}$$

⑤ Now (5) reads:

$$\boxed{T_X(g) f(z) = (\beta z + \alpha)^{l+\epsilon} (\bar{\beta} \bar{z} + \bar{\alpha})^{l-\epsilon} f\left(\frac{\alpha z + \bar{\beta}}{\beta z + \alpha}\right), z = e^{i\theta}$$

Corresponding generators are:

$$K_+ = -iH_- = ie^{i\theta} \left[(l+\epsilon) - \frac{1}{i} \frac{\partial}{\partial \theta} \right]$$

$$K_- = -iH_+ = ie^{-i\theta} \left[(l-\epsilon) + \frac{1}{i} \frac{\partial}{\partial \theta} \right]$$

$$J_3 = -H_3 = -\epsilon + \frac{1}{i} \frac{\partial}{\partial \theta}$$

So:

$$\boxed{K_{\pm} = ie^{\pm i\theta} \left[(l \pm \epsilon) \mp \frac{1}{i} \frac{\partial}{\partial \theta} \right]}$$

$$\boxed{J_3 = -\epsilon + \frac{1}{i} \frac{\partial}{\partial \theta}}$$

⑥ The subspaces:

$$\bar{f}(\bar{z}) = (\bar{z})^{-l+\epsilon} \sum_{n=0}^{\infty} a_{-n} \bar{z}^n$$

" D_l^+ "

$$f(z) = (z)^{l+\epsilon} \sum_{n=-\infty}^0 a_n z^n$$

" D_l^- "

⑦ Scalar product:

$$\boxed{H_l^+(f, g) = \cancel{(f, g)} = \sum_{n=-l+\epsilon}^{\infty} \frac{\Gamma(n+l+\epsilon)}{\Gamma(n-l-\epsilon)} f_{-n} \bar{g}_n}$$

$$H_l(f, g) = \sum_{n=-\infty}^{l+\epsilon} \frac{\Gamma(-n+1+l+\epsilon)}{\Gamma(-n-l+\epsilon)} \tilde{f}_n \bar{g}_n \quad \text{for } D_k$$

⑧ Claim on disc form:

$$H_l^+(f, g) = \frac{1}{\pi \Gamma(-2l-1)} \int d^2z (1-|z|^2)^{-2l-2} |z|^{2(l+\epsilon)} \tilde{f}(z) \bar{g}(\bar{z})$$

⑨ For C_g he says:

$$(f, g) = \sum_{n=-\infty}^{\infty} \tilde{f}_n \bar{g}_{-n} = \frac{1}{2\pi} \int_0^{2\pi} \tilde{f}(z) \bar{g}(\bar{z}) d\theta$$

⑩ Basis for C_g : $\tilde{f}_m(z) = \bar{z}^{-m}$ $z = e^{i\theta}$

Matrix elements:

$$t_{mn}(g) \equiv (\tilde{f}_m, T_X(g) \tilde{f}_n)$$

$$\Rightarrow t_{mn}(g) = \frac{1}{2\pi} \int_0^{2\pi} d\theta e^{+im\theta} \cdot T_X(g) e^{-in\theta}$$

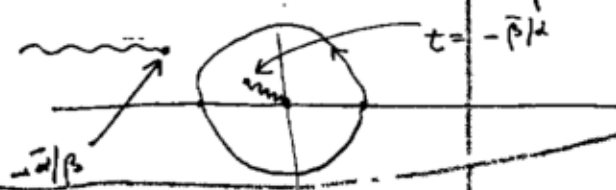
$$= \frac{1}{2\pi} \int_0^{2\pi} d\theta e^{+im\theta} (\beta z + \bar{\alpha})^{l+n+\epsilon} (\bar{\beta} \bar{z} + \alpha)^{l-n-\epsilon} e^{-in\theta}$$

$$= ? \quad \text{Let } z = e^{i\theta} \quad dz = i d\theta z$$

$$\begin{aligned} t_{mn}(g) &= \frac{1}{2\pi i} \oint dz \cdot \bar{z}^{-l+m-n} (\beta z + \bar{\alpha})^{l+n+\epsilon} (\bar{\beta} \bar{z} + \alpha)^{l-n-\epsilon} \\ &= [\bar{z} (\bar{\beta} + \alpha z)]^{l-n-\epsilon} \\ &= \bar{z}^{n+\epsilon-l} (\bar{\beta} + \alpha z)^{l-n-\epsilon} \end{aligned}$$

agree w. (5), p 307

$$t_{mn}^X(g) = \frac{1}{2\pi i} \oint dt \cdot t^{[\epsilon+m-l]-1} (\beta t + \bar{\alpha})^{l+n+\epsilon} (\bar{\beta} + \alpha t)^{l-n-\epsilon}$$



Change variables to $s = -\frac{\bar{\beta}}{\alpha} t$ Then: $t = \left(-\frac{\alpha}{\bar{\beta}}\right) s \checkmark$

$$(\bar{\beta} + \alpha t) = \bar{\beta}(1-s) \quad dt = \left(-\frac{\alpha}{\bar{\beta}}\right) ds \checkmark$$

$$(\beta t + \bar{\alpha}) = \bar{\alpha}(1-s|\frac{\beta}{\alpha}|^2)$$

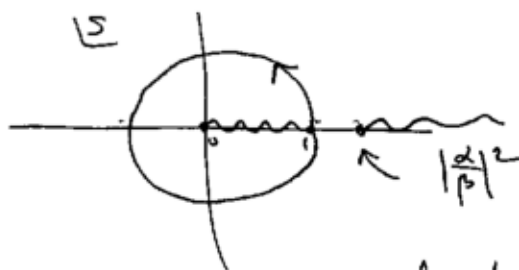
Then:

$$\frac{1}{2\pi i} \oint_{\Gamma} \frac{1}{z} (\bar{\beta})^{l-n-\epsilon} (\alpha)^{l+n+\epsilon} \left(-\frac{\alpha}{\bar{\beta}}\right)' \left(-\frac{\alpha}{\bar{\beta}}\right)^{\epsilon+m-l-1}$$

$$\times \oint ds \, \bar{z}(s)^{[\epsilon+m-l]-1} (1-s|\frac{\beta}{\alpha}|^2)^{l+n+\epsilon} (1-s)^{l-n-\epsilon}$$

where contour is:

(Contour direction needs cal.)
Let $\bar{\beta}/\alpha = \frac{1}{2}$, eg.)



$$\frac{1}{2\pi i} \oint_{\Gamma} \frac{1}{z} (\bar{\beta})^{m'-n'} (\alpha)^{l+n+\epsilon} (-\alpha)^{l-m'}$$

$$\times \frac{1}{2\pi i} \oint_{\Gamma} t^{[\epsilon+m-l]-1} (1-t)^{l-n-\epsilon} (1-t|\frac{\beta}{\alpha}|^2)^{l+n+\epsilon}$$

Identify:

$$\begin{aligned} b &= \epsilon - l + m & c - b - 1 &= l - n - \epsilon \\ a &= -l - \epsilon - m & c - b &= l - n - \epsilon + 1 \\ c &= 1 - n + m \end{aligned}$$

$$c = l - n - \epsilon + 1 + \epsilon - l - m = 1 - n + m$$

So integral is

$$\frac{2 \Gamma(\epsilon - l + m) \Gamma(l - n - \epsilon + 1) \sin \pi(\epsilon - l + m)}{-i \Gamma(1 - n + m) \exp(-i\pi[\epsilon - l + m])}$$

$$\times F(-l - \epsilon - m, \epsilon - l + m; 1 - n + m; |\frac{\beta}{\alpha}|^2)$$

And so one gets:

$$t_{m,n}^x(q) = (\bar{\beta})^{\cancel{m'-n'}} (\bar{\alpha})^{(l+\epsilon)+n} (-\alpha)^{\cancel{l-m'}} \frac{\Gamma(\epsilon-l+m) \Gamma(l-\epsilon-n+1)}{\Gamma(1-n+m)} \frac{\sin \pi(\epsilon-l+m)}{\pi} \frac{1}{\exp(-i\pi[\epsilon-l+m])} \\ \times F(-l-\epsilon-n, \epsilon-l+m; 1-n+m; |\frac{\beta}{\alpha}|^2)$$

⑪ Now define:

$$m' = m + \epsilon$$

$$n' = n + \epsilon$$

Then:

$$t_{m,n}^x(q) = (\bar{\beta})^{\cancel{m'-n'}} (\bar{\alpha})^{l+n'} (-\alpha)^{+l-m'} \frac{\Gamma(-l+m') \Gamma(l+1-n')}{\Gamma(1-n'+m')} \frac{\sin \pi(-l+m')}{\pi} e^{i\pi[-l+m']} \\ \times F(-l-n', -l+m'; 1-n'+m'; |\frac{\beta}{\alpha}|^2)$$

⑫ Now use Azizov (2a) to make sense of this:

$$\frac{z-1}{z+1} = |\frac{\beta}{\alpha}|^2$$

$$m' = \nu$$

$$j = l$$

$$n' = \mu$$

$$\Rightarrow z = |\alpha|^2 + |\beta|^2. \text{ So Azizov says:}$$

$$P_{n'm'}^l(|\alpha|^2 + |\beta|^2) = \frac{1}{\Gamma(m'-n'+1)} \left(\frac{|\alpha|^2 + |\beta|^2 + 1}{2} \right)^l \left| \frac{\beta}{\alpha} \right|^{m'-n'} \\ \times F(-l-n', -l+m'; 1-n'+m'; |\frac{\beta}{\alpha}|^2)$$

$$\Rightarrow t_{m,n}^x(q) = (\bar{\beta})^{\cancel{m'-n'}} (\bar{\alpha})^{l+n'} (-\alpha)^{+l-m'} \left| \frac{\beta}{\alpha} \right|^{m'-n'} e^{i\pi[-l+m']} \\ \times \frac{\Gamma(l+1-n')}{\Gamma(l+1-m')} \cdot \left(\frac{|\alpha|^2 + |\beta|^2 + 1}{2} \right)^l \cdot P_{n'm'}^l(|\alpha|^2 + |\beta|^2)$$

As I see it, this result must be true for all "el" and for (m'n') on either lattice.
After all, we obtained this result from Vilenkins general representation.

③ Now I will choose a parametrization. Or should I use this:

$$\alpha = \text{ch} \frac{\tau}{2} e^{i(\frac{\phi+\psi}{2})} \quad \beta = \text{sh} \frac{\tau}{2} e^{i(\frac{\phi-\psi}{2})}$$

Then $|\alpha|^2 + |\beta|^2 = \text{ch} \tau \equiv \mathbb{Z}$.

$$\frac{|\beta|^2}{|\alpha|^2} = \frac{\text{ch} \tau - 1}{\text{ch} \tau + 1}$$

Then I get:

$$t_{m,n}^l(g) = (\text{sh} \frac{\tau}{2})^{l+m'-n'} e^{-i(\frac{\phi-\psi}{2})(l+m'-n')} \\ (\text{ch} \frac{\tau}{2})^{2l+n'-m'} e^{i(\frac{\phi+\psi}{2})(-n'-n')}$$

$$\frac{\Gamma(l+1-n')}{\Gamma(l+1-m')} \left(\frac{\text{ch} \tau + 1}{2} \right)^l P_{n',m'}^l(\text{ch} \tau) = \left(\text{ch} \frac{\tau}{2} \right)^{-2l}$$

where I have cancelled the shown phases. Now group terms:

$$t_{mn}^l = \left\{ \begin{aligned} & (\text{sh} \frac{\tau}{2})^{2l} (\text{ch} \frac{\tau}{2})^{2l} e^{-i\phi m'} \\ & e^{-i\psi n'} \frac{\Gamma(l+1-n')}{\Gamma(l+1-m')} P_{n',m'}^l(\text{ch} \tau) \end{aligned} \right.$$

So:

$$t_{mn}^l(g) = \frac{\Gamma(l+1-n')}{\Gamma(l+1-m')} \cdot e^{-i(\phi m' + \psi n')} P_{n',m'}^l(\text{ch} \tau)$$

↑ but add $(-1)^{n'-m'}$ if $\tau < 0$.

⑭ Redo back of page 2: want $s = -\frac{\alpha}{\beta}t$ so that

$$s_1 = \left(-\frac{\alpha}{\beta}\right)(-\bar{\beta}) = +1 \text{ as end of cut.}$$

$$\text{So } t = -\frac{\bar{\beta}}{\alpha}s \quad dt = -\frac{\bar{\beta}}{\alpha}ds$$

$$(\beta t + \bar{\alpha}) = \left(-\frac{\beta\bar{\beta}}{\alpha}s + \bar{\alpha}\right) = \bar{\alpha}\left(1 - \frac{\beta\bar{\beta}}{\alpha^2}s\right) = \bar{\alpha}\left(1 - \left|\frac{\beta}{\alpha}\right|^2 s\right) \checkmark$$

$$(\bar{\beta} + \alpha t) = \bar{\beta} - \bar{\beta}s = \bar{\beta}(1-s) \checkmark$$

$$\begin{aligned} \text{So: } dt \cdot t^{(m'-l-1)} (\beta t + \bar{\alpha})^{l+n'} (\bar{\beta} + \alpha t)^{l-n'} \\ = ds \cdot \left(-\frac{\bar{\beta}}{\alpha}\right)^{m'-l-1} s^{m'-l-1} \cdot (\bar{\alpha})^{l+n'} \left(1 - s\left|\frac{\beta}{\alpha}\right|^2\right)^{l+n'} (\bar{\beta})^{l-n'} (1-s)^{l-n'} \\ = \underbrace{(\bar{\alpha})^{l+n'}}_{\checkmark} \underbrace{(\bar{\beta})^{m'-n'}}_{\substack{\uparrow \\ \text{these differ and}}} \underbrace{(-\alpha)^{l-m'}}_{\substack{\uparrow \\ \text{these differ and}}} \underbrace{ds s^{m'-l-1} \left(1 - s\left|\frac{\beta}{\alpha}\right|^2\right)^{l+n'} (1-s)^{l-n'}}_{\checkmark} \end{aligned}$$

Go back + correct these exponents.

⑮ Now, looking at (9), p. 310, we conclude:

$$\frac{\Gamma(l+1-n')}{\Gamma(l+1-m')} P_{n',m'}^l(\text{chr}) = \beta_{m',n'}^l(\text{chr})$$

At last I know what β^l really is!

⑯ Check consistency of $V_l(s)$, (9) p 316 at

$$\frac{\Gamma(l+1-n)}{\Gamma(l+1-m)} P_{m,n}^l(\text{chr}) \neq \beta_{m,n}^l(\text{chr})$$

Wrong way:

$$\frac{1}{\Gamma(l+1-m)} P_{0,m}^l(\text{chr}) = \beta_{0,m}^l(\text{chr}) = \frac{\Gamma(l+1)}{\Gamma(l+1+m)} P_l^m(\text{chr})$$

he claims, IF " $l = \text{neg. int and pos. } |m| \leq l$ "

But I get: $P_{lm}^{(l)}(\text{chr}) = \left(\prod_{m=0}^l P_{ma}^{(l)} = \frac{\Gamma(l+1-m)}{\Gamma(l+1+m)} \cdot P_e^m(\text{chr}) \right)$

So we get: $\frac{P_e^m(\text{chr})}{\Gamma(l+1+m)} = \frac{\Gamma(l+1)}{\Gamma(l+1+m)} P_l^m$

So something is sick here. But how does he define his P_l^m ? God only knows:

⑩ What are VL's $P_{lm}^{(l)}$?

$$P_{lmn}^{(l)}(z) = (i)^{m-n} \sqrt{\Gamma(l+1-m)\Gamma(l+1+n)\Gamma(l+1-n)\Gamma(l+1+n)}$$

$$\left(\frac{1-z}{1+z}\right)^{\frac{m-n}{2}} \left(\frac{1+z}{z}\right)^l \frac{1}{\Gamma(j+1)\Gamma(l+1-m-j)} \left(\frac{z-1}{z+1}\right)^j \Gamma(l+1+n-j) \times \Gamma(m-n+j+1)$$

ugly.

Compare to Jacob's

~~$$P_{l-m}^{(l)}(z) = \frac{\Gamma(l+1-m)}{\Gamma(l+1+m)} \left(\frac{1-z}{1+z}\right)^{\frac{m-n}{2}} \left(\frac{1+z}{z}\right)^l$$~~

~~$$\frac{\Gamma(l+1+n)\Gamma(l+1-n)}{\Gamma(l+1+m)\Gamma(l+1-m)}$$~~

~~$$\left(\frac{1-z}{1+z}\right)^{\frac{m-n}{2}} \left(\frac{1+z}{z}\right)^l = P_{lmn}^{(l)}(z) |^{VL}$$~~

✓

our guy says:

~~$$P_{j-\nu}^{\nu-\mu, \nu+\mu}(z) = \frac{\Gamma(j+1-\mu)}{\Gamma(j+1-\nu)} \left(\frac{z-1}{z}\right)^{-\nu+\mu} \left(\frac{z+1}{z}\right)^{-\nu-\mu} \times P_{\mu\nu}^j(z)^{AZ}$$~~

So: set $\nu=m$
 $\mu=n$. So

Thus: $\left[\frac{\Gamma(l+1+\mu)\Gamma(l+1-\mu)}{\Gamma(l+1+\nu)\Gamma(l+1-\nu)} \right] = \frac{P_{\mu\nu}^l(z)}{P_{\mu\nu}^j(z)} |^{VL}$

$$= \frac{\Gamma(l+1-\mu)}{\Gamma(l+1-\nu)} P_{\mu\nu}^j(z) |^{AZ}$$

$$\Rightarrow P_{\mu\nu}^j(z) |^{AZ} = \sigma_{\mu\nu}^l \left(\frac{P_{\mu\nu}^l(z)}{P_{\mu\nu}^j(z)} \right) |^{VL}$$

But l, m, n are sense sensal, so

$$P_{\nu\mu}^l(z)|^{VL} = G_{\nu\mu}^l P_{\mu\nu}^l(z)|^{Az} = P_{\nu\mu}^l(z)|^{Az}.$$

So

$$P_{mm}^l(z)|^{VL} = P_{mm}^l(z)|^{Az} \quad \text{in } SU(2) \text{ region.}$$

⑬ Next, what is VL 's "associate":

$$P_l^m(z)|^{VL} = i^m \sqrt{\frac{\Gamma(l+1+m)}{\Gamma(l+1-m)}} P_{l, m, 0}^l(z) \quad m \geq 0.$$

But for Az we have:

$$P_l^m(z)|^{Az} = P_{l, m, 0}^l(z)$$

But this cannot be true because look at Vilenkin's symmetry rule top of page 124. This shows clearly that his P 's are not the same as Azimov's. Therefore, he must have faked us out by using different Jacobis. His P functions must be more like my d functions. So this throws lots of things off.

Comment: I simply do not trust Vilenkin's calculations because he refuses to standardize things using hypergeometric functions. I hate that, no way to tell where you are. So I should really only use him for ideas, not explicit results.

Comment on a Common Error in Group Rep Theory:

1. Usually, the standard group rep formula reads:

$$T(g)f(x) = f(\bar{g}^{-1}x)$$

One gets the erroneous impression that g_2^{-1} will be the first group element to "touch" x :

$$\begin{aligned} T(g_1)T(g_2)f(x) &= T(g_1)f(\bar{g}_2^{-1}x) \\ &= f(\bar{g}_1^{-1}\bar{g}_2^{-1}x) \end{aligned}$$

These equations are wrong. In fact, g_1^{-1} is the first group element to touch x . The above equations are simply a misunderstanding of the notation. The bra-ket notation makes this very clear:

$$\langle x | T(g)f \rangle = \langle \bar{g}^{-1}x | f \rangle \quad \text{is the meaning of } T(g)f(x) = f(\bar{g}^{-1}x).$$

$$\text{So: } T(g_1)T(g_2)f(x) \text{ means } \langle x | T(g_1)T(g_2)f \rangle$$

$$\begin{aligned} \text{which} &= \langle x | T(g_1)f' \rangle = \langle \bar{g}_1^{-1}x | f' \rangle = \langle x' | f' \rangle \\ &= \langle x' | T(g_2)f \rangle = \langle \bar{g}_2^{-1}x' | f \rangle = \langle \bar{g}_2^{-1}\bar{g}_1^{-1}x | f \rangle \\ &= \langle (g_1g_2)^{-1}x | f \rangle = \langle \bar{g}_3^{-1}x | f \rangle \\ &= \langle x | T(g_3)f \rangle = \langle x | T(g_1g_2)f \rangle = T(g_1g_2)f(x). \end{aligned}$$

$$\begin{aligned} \text{In short: } \langle x | T(g_1)T(g_2)f \rangle &= \langle \bar{g}_1^{-1}x | T(g_2)f \rangle \\ &\quad \leftarrow \text{notice: } \bar{g}_1^{-1} \text{ touches } x \text{ first} \\ &= \langle \bar{g}_2^{-1}\bar{g}_1^{-1}x | f \rangle = \langle (g_1g_2)^{-1}x | f \rangle \\ &= \langle x | T(g_1g_2)f \rangle. \end{aligned}$$

Comment on Normalizers of Vilenkin.

1. So far I have been thinking in terms of having complete freedom in choosing a normalizer A_m for a representation. However, I now think certain choices are more natural than others. For example, look at page 307 of Vilenkin where he deals with $SU(1,1)$ matrix elements. There, equations (1) and (2) are undeniably true: these equations give you the representations irreducibles. Now, if you choose basis functions in this space which are normalized and orthogonal, ie,

$$f_m(\theta) = \frac{1}{\sqrt{(j+m)!(j-m)!}} e^{-im\theta}$$

... well, this choice is orthonormal with respect to the simple scalar product given in (4) page 304. When you do this, you get a very explicit matrix element without any "normalizer freedom". See (4).

2. How then can we understand the idea of changing normalizers?

$$f_m(\theta) = A_m^j e^{-im\theta}$$

If you choose $A_m^j = 1$, you get (4) as noted. But you could also choose:

$$A_m^j = \frac{1}{\sqrt{(j+m)!(j-m)!}} \Rightarrow \langle f_m | f_m \rangle = \frac{1}{(j+m)!(j-m)!}$$

OK, then all you have to do is redefine the scalar product to normalize your basis states. In fact, now look at Vilenkin page 118 to see that this is exactly what he does for $SU(2)$ matrix elements. !

3. Thus, here is what I have learned. First, for $SU(2)$, Vilenkin used the standard normalizer that everybody uses. The only other thing you have to know is that he chooses $\exp(+i\theta J_0)$ for his central group element thus obtaining the rotation matrix functions which he calls $P_{mn}^j(z)$.

Second, for $SU(1,1)$ in the continuous series Vilenkin chooses the unit normalizer $A_m = 1$. Then, as shown on page 309, he chooses for his central element $\exp(-i\tau K_1)$. The resulting matrix elements are then called $\beta_{mn}^j(\text{ch } \tau)$.

$$\tau = i\theta$$

4. Keep in mind that Andrews and Gunson use J_2 as their central generator.

Vilenkin: conversion of his $SL(2, \mathbb{C})$ version of t_{mn}^l to HGF.

① P112 (4) says:

$$t_{mn}^l(q) = \left[\frac{(l-m)!(l-n)!}{(l+m)!(l+n)!} \right]^{\frac{1}{2}} \cdot \frac{\sum_{k=-\infty}^{\infty} \frac{(\beta x)^k}{(l-k)!(k-m)!(k-n)!}}{\beta^m \gamma^n} \times \text{SUM}$$

where

$$\text{sum} = \sum_{k=-\infty}^{\infty} \frac{(l+k)!}{(l-k)!(k-m)!(k-n)!} (\beta x)^k$$

$\max(m, n)$, but 0 is same thing; better assume $m = \text{integer}$ for say this.

Assume $m, n = \text{integers}$ + write

$$\text{sum} = \sum_{k=-\infty}^{\infty} \frac{(l+k)!}{(l-k)!(k-m)!(k-n)!} (\beta x)^k$$

$$k = s + m$$

$$l+k = l+s+m$$

$$l-k = l-s-m$$

$$k-m = s$$

$$k-n = s+m-n$$

Now shift sum to $s = k-m \Rightarrow k = s+m$

Then

$$\text{sum} = \sum_{s=-\infty}^{\infty} \frac{(l+s+m)!}{(l-s-m)!(s)!(s+m-n)!} (\beta x)^{s+m}$$

$$= (\beta x)^m \cdot \sum_{\text{all } s} \frac{(\beta x)^s}{s!} \times \frac{(l+s+m)!}{(l-s-m)!(s+m-n)!}$$

convert to top.

Now must use: $\sin \pi p P(p) P(1-p) = \pi$

$$\Rightarrow (l-s-m)! = P(l-s-m+1) = P(1-[m+s-l]) = P(1-p)$$

so

$$\frac{1}{(l-s-m)!} = \frac{1}{P(1-p)} = \frac{\sin \pi p \cdot P(p)}{\pi \cdot P(p)} = \frac{\sin \pi [m+s-l] \cdot P(m+s-l)}{\pi \cdot P(m+s-l)}$$

$$= (-1)^{m+s} \sin \pi [-l] \cdot P(m+s-l) = \text{stop trouble.}$$

② I will need another tack. Try:

$$\left[\frac{(l-m)!}{(l+m)!} \frac{(l-n)!}{(l+n)!} \right]^{\frac{1}{2}} = \left\{ \left[\frac{(l-m)!}{(l+m)!} \frac{(l-n)!}{(l+n)!} \right]^{\frac{1}{2}} \times \frac{(l-m)!}{(l-m)!} \right\} \times \frac{(l-m)!}{(l+m)!}$$

$$= \left[\frac{(l+m)!}{(l-m)!} \times \frac{(l-n)!}{(l+n)!} \right]^{\frac{1}{2}} \times \frac{(l-m)!}{(l+m)!}$$

Then I can use "rule"

$$\frac{(l-m)!}{(l-m-s)!} = \frac{(l-m)!}{(l-m')!} = (-1)^{m-m'} \frac{\Gamma(m'-j)}{\Gamma(m-j)} = (-1)^{m+s} \frac{\Gamma(m+s-j)}{\Gamma(m-j)}$$

$$m' = m+s$$

③ Combining all these steps I get

$$t_{mn}^l = \left[\frac{(l+m)! (l-n)!}{(l-m)! (l+n)!} \right]^{\frac{1}{2}} \times \frac{z^{m+n} \bar{z}^{m-n}}{\Gamma(l+1+m) \Gamma(-l+m)} \\ \times \sum_{s=0}^{\infty} \frac{(-\beta \gamma)^s}{s!} \frac{\Gamma(l+1+m+s) \Gamma(-l+m+s)}{\Gamma(m-n+1+s)}$$

$$= \left[\frac{1}{2} \right]^{\frac{1}{2}} z^{m+n} \bar{z}^{m-n} \frac{1}{(m-n)!} F(-l+m, l+1+m, m-n+1, -\beta \gamma)$$

Therefore:

$$t_{mn}^l \equiv H_{mn}^j$$

$$t_{m'm}^j = \left[\frac{(j+m')! (j-m)!}{(j-m')! (j+m)!} \right]^{\frac{1}{2}} \cdot \frac{1}{(m'-m)!} \cdot z^{m'+m} \bar{z}^{m'-m} \cdot F(-j+m', j+1+m', m'-m+1, -\beta \gamma)$$

④ This is Valentia $SL(2, \mathbb{C})$ matrix element for $m' \geq m$.
Two special cases:

$$SU(1,1) \Rightarrow \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} \Rightarrow \gamma = \bar{\beta}, \delta = \bar{\alpha} \text{ so}$$

$$m' \geq m \quad t_{m'm}^j = H_{m'm}^j \cdot (\alpha)^{m'+m} \cdot (\bar{\beta})^{m'-m} \cdot F(-j+m', j+1+m', m'-m+1, -\beta \bar{\beta})$$

If α, β are real, this agrees with my $D_{m'm}^j$

⑤ Thus, when α and β are real,

$$(m' \geq m: \quad t_{m'm}^j = D_{m'm}^j) \text{ precisely for } SU(1,1).$$

Vilenkin: HGF form for p308 (6) (for $SU(1,1)$).

① This equation reads

$$T_{mn}^x(g) = (\alpha)^{l-n-\epsilon} (\bar{\alpha})^{l+m+\epsilon} (\beta)^{n-m} \times \left\{ \sum_{s=0}^{-\infty} \left(|\beta|^2 / |\alpha|^2 \right)^s \cdot \binom{l+n+\epsilon}{n-m+s} \binom{l-n-\epsilon}{s} \right\}$$

From my HGF notes, let $M = n + \epsilon$, $M' = m + \epsilon$ $\Rightarrow n - m = M - M'$; $\phi = l$

Then $\{ \} = \frac{(l+n+\epsilon)!}{(l+m+\epsilon)!} \frac{1}{(n-m)!} F(-l-m-\epsilon, -l+n+\epsilon; n-m+1; \frac{|\beta|^2}{|\alpha|^2})$

This is for $n \geq m$, that's all I care about now. Thus: $n \geq m$

$$T_{mn}^x(g) = (\alpha)^{l-n-\epsilon} (\bar{\alpha})^{l+m+\epsilon} (\beta)^{n-m} \times \frac{(l+n+\epsilon)!}{(l+m+\epsilon)!} \frac{1}{(n-m)!} F(-l-m-\epsilon, -l+n+\epsilon; n-m+1; |\beta|^2 / |\alpha|^2)$$

② Convert to $mn \rightarrow m'm$ where now $m \geq m'$. Then

$$T_{m'm}^x(g) = (\alpha)^{l-m'-\epsilon} (\bar{\alpha})^{l+m'+\epsilon} (\beta)^{m-m'} \cdot \frac{(l+m'+\epsilon)!}{(l+m'+\epsilon)!} \frac{1}{(m-m')!} \times F(-l-m'-\epsilon, -l+m'+\epsilon; m-m'+1; |\beta|^2 / |\alpha|^2)$$

This disagrees with Bant+Ph by $(-1)^{m-m'}$ which we know is not of any significance. (I.e., use $A_m = 1$ in BP).

Vilenkin, Chapter 1, section 2: "transformation groups." (12 pages)

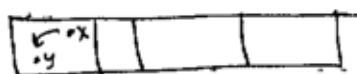
1 Transformation Groups.

- ① "transformation of a set M " — a 1-to-1 map of $M \rightarrow M$.
- ② " G is a transformation group of the set M " — start with set M ; want to test some group G to see if it is a t.g. of M : Let $x \in M$. Then for each $g \in G$, we want some transformation $x \rightarrow gx$ where $gx \in M, \forall x \in M$. Not necessarily linear! Also, this xform must be such that if $e \in G = \text{identity}$, then $ex = x$. Also, need $(g_1 g_2)x = g_1(g_2 x)$. If your xforms g do all this, then $G = \text{xform group of } M$.
- ③ Example: Let $M = \text{set of } n \text{ objects}$. Let $G = \text{group of permutations of these } n \text{ objects}$. Then say $x = [123]$. $gx = [213]$ for some g . Clearly $gx \in M$. So $G = \text{example of a xform group}$.
- ④ Example: $G = GL(n, \mathbb{C})$. Set of invertible $n \times n$ complex matrices. Each represents a linear xform on n -space. Let $M = \text{set of complex } n\text{-vectors}$. Then $gx \in M$, obviously. $e = (1, \dots, 1)$. So, $GL(n, \mathbb{C})$ is a linear example of a xform group.
- ⑤ " G of M is effective" — let $g \in G$. As long as there is some $x \in M$ s.t. $gx \neq x \forall g$, then G is effective.
- ⑥ Example: let $M = G$ itself. Let $g_0 \in G, g \in M$. Then clearly $g_0 g \in M$. This set $g_0 g \forall g \in G$ is called the left shift of G by g_0 .

2 Transitive Transformation Groups.

- ① " G is transitive on M " — This means that any two elements of M , say x and y , must be connected by an xform $g \in G$. I.e., $\exists g \in G \mid gx = y$. If M is such a set for G , then $M = \text{"homogeneous space with xform group } G"$.
- ② Example: $G = E^3 \text{ rotations}$ $M = \text{set of all real 3-vectors}$. Not transitive! But $M = \text{set of 3-vectors on some sphere}$, then yes.

- ③ "transitivity classes" — if G is not transitive on M , still you can partition M into subsets:



such that each subset is transitive. These subsets are the t -classes. Of course if G, M is bad case, each t -class could have only one element.

- ④ "fixed point of M " — perhaps $\forall h \in H \subseteq G, ha = a, a \in M$. Then a is a fixed point. Also, easy to show that H is a group, hence a subgroup of G . So H is the "stationary subgroup of the point A ".

- ⑤ Relation to cosets: let H be the s.s. of a point a . Let $x = ga$, where $g \notin H$. Then:

$$\begin{array}{c} \bigcirc \\ G \end{array} a \xrightarrow{g} x$$

Clearly: $g_1 = gh$ also takes $a \rightarrow x$, if $h \in H$. So $gH: a \rightarrow x$. But gH is a left coset of the subgroup H . So, each $x \in M$ is associated with a left coset of H , if H is the s.s. of a , and if $ga = x$. But: so what?!
(see below.)

- ⑥ Consider:
- $$\begin{array}{c} \bigcirc \\ \in H \end{array} G a \xrightarrow{g} \begin{array}{c} \bigcirc \\ b \end{array} \tilde{g} h g.$$
- $\tilde{g} H g = \text{stat. sb. of } b!$

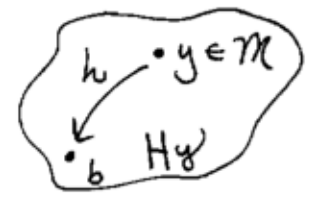
- ⑦ "conjugate subgroup" — H_1 and H_2 — both s.g. of G , are conjugate if $H_2 = \tilde{g} H_1 g$ for some $g \in G$. Of course, if $g \in H_1$ or H_2 , then $H_1 = H_2$ and subgroups are the same, so usually let $g \in G - H_1 - H_2$.

- ⑧ So, we know that given M as a homo. space for G (which of course $\Rightarrow G$ is transitive), we have ~~partitioned~~ M into a ~~bunch~~ of concocted a set of subgroups of G , $H(a)$. Perhaps given these H 's, we could construct M ?

⑨ Let's try constructing M . Take a particular H . For each $g \in G$, we have a coset gH . Let $M = \text{set of these left cosets}$, $M = \{gH\}$. So $gH \in M$. Then $g(g'H) = (gg')H = g'H \in M$ also! Thus, G is a trans. group on M , & we have discovered a particular M that works.

⑩ This particular ... what is the " a " which, $\in M$, is the fixed point which defines H ? I.e., $ha = a$. Try $a = eH$, the particular coset corresponding to e . Then $h(eH) = h(H) = H$. Notice that the elements of M are cosets, not group elements.

⑪ A "sphere in a homogeneous space" — The set H_y is a piece of M called the "sphere centered at a passing through y ".



Obviously, if H were $O(3)$, a would be the origin because $Ha = a$. Then H_y would really be a sphere.

⑫ In the special case that $M = \text{set of left cosets } gH$, we'd say $y = g_1H$, some element of M . Then $H_y = \text{sphere} = Hg_1H$. As H varies, you come out the "sphere" centered at ($a = H$) passing thru $y = g_1H$. Very abstract.

⑬ Example: $G = SO(3)$. $S^2 = \text{unit sphere}$. Let $N = \text{pole}$. Then let this unit sphere be M , so $N \in M$. Then $H(N) = \text{stat. sq} = O(2)$. So " $M = G/H$ " becomes $SO(2) \subseteq SO(3)$. $S^2 = SO(3)/SO(2)$. Just a notation!! For any group & subgroup, there is a homogeneous space M . All elements of M are connected by some $g \in G$.

⑭ Comment:

3 Invariant Measures

① Let A be a "patch" on M , with some measure (area): $\mu(A)$.
 Then suppose $gA = A' =$ some other patch. (Must be, since $gx \in M$ by def. of x from group on M). If $\mu(A) = \mu(gA)$, then μ is called a "left-invariant measure".
 Similarly $\nu(A) = \nu(Ag) =$ a R.I.M.

② We are not told how to find such a measure.

③ Invariant measures have powerful implications for doing integrals.

4 Group Representations by Shift Operators

① " $\mathcal{L}$ " — space of functions $f: M \rightarrow \mathbb{C}$. Let G be group of x 's on M , so $x \in M \Rightarrow gx \in M$ also. Then $f(x) \in \mathcal{L}$, say. We may define another function: $f(gx)$. If $f(gx) \in \mathcal{L}$ also, then $\mathcal{L}$ is "invariant".

② Claim: if $T(g)f(x) = f(g^{-1}x)$, then $T(g)$ is a representation of group G . (Trivial proof).

③ Special case: let $\mathcal{L}$ be space of scalar functions s.t. w/ an (assumed) invariant measure $d\mu$. Then we can define a scalar product over $\mathcal{L}$:

$$(f_1, f_2) = \int_M f_1(x) \overline{f_2(x)} d\mu(x)$$

↖ notice antilinearity on f_2 , not f_1 .

④ Claim: on such an $\mathcal{L}$, $T(g)$ is a unitary rep! (relative to said scalar product).

proof: $(T(g)f_1, T(g)f_2) = \int d\mu(x) f_1(g^{-1}x) \overline{f_2(g^{-1}x)}$ $y = g^{-1}x$
 $x = gy$

$$= \int d\mu(gy) f_1(y) \overline{f_2(y)}$$

$$= \int d\mu(y) \dots = (f_1, f_2), \text{ hence } \underline{\text{unitary}}.$$

- ⑤ "Regular Reps." — go again to the special case where $M = G$ itself. Then instead of $x \in M$, we can use $g \in M$. Then consider:

$$L(g_0) f(g) = f(g_0^{-1} g)$$

Somewhat, this rep. $L(g_0)$ depends only on G , since $M = G$, and $f(g)$ are functions sq. int over $dg(g)$. L is called the "left-regular rep." Similarly, one can have $R(g_0)$.

- ⑥ Now comes a theorem which is going to relate each I.R. of G to a shift rep. Later, I suspect this will be used to find the I.R.'s of $O(2,1)$.

Let $\underline{a} \in \mathcal{L}$; $\mathcal{L}$ is "the rep. space of $T(g)$ " Also, $\underline{f} \in \mathcal{L}$.
Now define a scalar F by:

$$f(g) = \langle \underline{a} | T(g) \underline{f} \rangle \quad \leftarrow \text{my notation} \quad = (T(g) \underline{f}, \underline{a}) \quad \leftarrow \text{Vilenkin's.}$$

Claim: $\underline{f} \rightarrow T(g_0) \underline{f} \Rightarrow f(g) \rightarrow f(gg_0)$.

But there should be a rep $Q(g_0)$ s.t. $\underline{f} = Q(g_0) f(g)$. Also, we shall show that $Q(g_0)$ is equivalent to $T(g)$.

Notice that we have reduced the idea of $T(g) \underline{f}(x) = \underline{f}(g^{-1}x)$ to the scalar problem $Q(g_0) f(g) = f(gg_0)$.

- ⑦ We have a mapping here of $\underline{f} \rightarrow f(g)$, where $\underline{f} \in \mathcal{L}$, $f(g) \in \mathcal{L}^2(G)$. If $\underline{e}_i$ are basis vectors in $\mathcal{L}$, then they map into $e_i \in \mathcal{L}^2(G)$. Choose $\underline{e}_i$ such that $\underline{e}_1 = \underline{a}$, which defines the map. Then

$$e_j(g) = (T(g) \underline{e}_j, \underline{e}_1) = t_{1j}(g)$$

So, basis elements of $\mathcal{L}^2(G)$ are the first row of $t_{ij}(g)$.

- ⑧ Now use t_{1j} as $f(g)$, so $Q(g_0) \underline{t_{1j}(g)} = \underline{t_{1j}(gg_0)}$

$$= \sum_k t_{1k}(g) t_{kj}(g_0) = \sum_k t_{kj}(g_0) \underline{t_{1k}(g)}$$

Therefore, the matrix elements of $Q(g)$, in the basis $e_j = T_{ij}$, are the same as the elements of $T(g)$ in basis e_j . Thus, of course T and Q are equivalent. One of course also needs to know that only Q maps into 0 under $f \rightarrow f(g)$, so both spaces have same size ["kernel is null"]. This is only true if $T(g)$ is ir-reducible.

[5] Class 1 reps, and Spherical Functions

- ① Suppose $H = \text{s.g. of } G$ and suppose $\exists \underline{a} \in \mathcal{L}$ such that $T(h)\underline{a} = \underline{a}$ for all $h \in H$. Then " $\underline{a}$ (of $\mathcal{L}$) is invariant relative to H ."
- ② If there is such a fixed point $\underline{a} \mid T(h)\underline{a} = \underline{a} \quad h \in H = \text{s.g. of } G$ (where $\underline{a} \neq \underline{0}$), and if $T(g)$ is unitary on H (ie, when restricted to $H \subset G$), then $T(g)$ is called "class 1".
- ③ Repeat: if $\exists H \text{ s.g. of } G$ with a nontrivial fixed point in $\mathcal{L}$ under $T(h)$, and if T is unitary on H , then T is a "class 1 rep relative to H ."
- ④ If there is only one such vector $\underline{a}$ in $\mathcal{L}$ (modulo length) that is a fixed point of $T(H)$, then $H =$ a "massive" subgroup of G .
- ⑤ Suppose H is massive s.g. of G with $\underline{a}$ as invariant vectn $T(h)\underline{a} = \underline{a}$. Then, use this particular $\underline{a}$ in:

$$f(g) = (T(g)f, \underline{a})$$

Then $f(g)$ are called "spherical functions of the rep $T(g)$ relative to H ".

- ⑥ Because spherical functions are associated with class 1 reps $T(g)$, and because class 1 reps are unitary on H , we can show that $f(g) = f(hg) \quad \forall h \in H$. Thus, on entire right-coset Hg , f takes value $f(g)$

This means that we can think of f as being defined on the set of right cosets of H . But this set of right cosets is a possible way to have M on which one defines the group of transformations G . Therefore, $f(g) = \text{defined on } M = G/H$.

⑦ In particular, consider $a(g) = (T(g)a, a)$. These are the "zonal spherical functions". You can show that

$$a(hg) = a(g), \text{ as before.}$$

$$a(gh) = a(g) \text{ in this case only.}$$

Thus, a is constant on both cosets of H . I.e., $a(h_1 g h_2) = a(g)$. But HgH ~~is a coset~~, the two-sided coset, is the sphere centered at $eH = a$, passing through $y = gH$. So:

"zonal spherical functions are constants on 'spheres'."

⑧ Comment: $G = O(3)$ $H = O(2) \Rightarrow$

sphericals = Yew.

zonals = k .

More generally, Gegenbauers!

6. The induced representations

① Here is how you construct an "induced rep."

(a) let G be group with subgroup H .

(b) let $Q(h)$ be a rep. of H "on a Hilbert Space $\mathcal{H}$ ". ~ i.e., Q is going to operate on elements of $\mathcal{H}$ to give new elements of $\mathcal{H}$, like $Q(h)f = f'$, say.

(c) let $\mathcal{L}$ be the function space, $\mathcal{L} = \{f\}$ where $f: G \rightarrow \mathcal{H}$.

(d) if $f \in \mathcal{L}$ and $a \in \mathcal{H}$, then $f(g) \in \mathcal{H}$ too, and

(e) $\phi(g) = (f(g), a) \leftarrow \text{Hilbert space scalar product}$

(f) Assume $\mathcal{L} = \{f\}$ is such that

$$\int (f(g), f(g)) dg < +\infty$$

$\nwarrow$ invariant G -measure.

(g) Also, $Q(h)f(g) = f(gh)$

Note: basically, we are defining the set of functions $\mathcal{L}$

② So far, unclear. Assume a set of functions $\{f\}$ have been found $\Rightarrow$ (1) and (2) are true. Then claim is that $f(g)$ OK $\Rightarrow f(g_0^{-1}g)$ OK. Proof:

Let $f(g_0^{-1}g) = f'(g)$. Then:

$$\text{RHS} = Q(h)f'(g) = Q(h)f(g_0^{-1}g) = Q(h)T(g_0)f(g)$$

$$= f(g_0^{-1}gh) \text{ by (2) for } f = f'(gh) = \text{LHS} \checkmark$$

③ Therefore: the space so constructed, which we called $\mathcal{L}$, is invariant under $\times$ forms G because all the shifted vector functions are also in $\mathcal{L} = \{f\}$.

④ Because $\mathcal{L}$ is invariant, $T: \mathcal{L} \rightarrow \mathcal{L}$ so closed. Then we can say that T is a rep.

⑤ So $T(g_0)$, which does $T(g_0)f(g) = f(g_0^{-1}g)$, is the representation induced by the rep $Q(h)$ of subgroup H .

Vilenkin Ch. III (SU(2)), Section 2.1 : Reps in the homopoly subspace.

① Let: $\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$. Define $T(g)$ by:

$$T(g) f(z_1, z_2) = f(w_1, w_2) \text{ where } w_1, w_2 \text{ as shown above.}$$

② Why is $T(g)$ a rep? $T(g_1) T(g_2) f(z_1, z_2) = T(g_1) f(g_2 \begin{pmatrix} z_1 \\ z_2 \end{pmatrix})$
 $= f(g_1 g_2 \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}) = f((g_1 g_2) \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}) = T(g_1 g_2) f \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \checkmark$

③ Consider: $f(z_1, z_2) = a_{2l} z_1^0 z_2^{2l} + a_{2l-1} z_1^1 z_2^{2l-1} + \dots + a_0 z_1^{2l} z_2^0$

This is a "homogeneous" polynomial of degree $2l$ in z_1, z_2 .

The space of all such degree $2l$ poly's is called: $\mathcal{H}_l$.

Initially, $T(g)$ is defined on the full space $\mathcal{H}$ of functions of z_1, z_2 . If we consider only $\mathcal{H}_l$, then we are dealing with a piece of $T(g)$ called $T_l(g)$.

④ Define: $\phi(z) = f(z, 1)$.

⑤ Claim: $f(z_1, z_2) = (z_2)^{2l} \phi(\frac{z_1}{z_2})$, trivial, but only true in subspace $\mathcal{H}_l$ where f is a homo-polynomial.

⑥ $T(g) f(z_1, z_2) \equiv f_g(z_1, z_2) = f(w_1, w_2)$ by (2).
 $= (w_2)^{2l} \phi(\frac{w_1}{w_2}) = (\beta z_1 + \delta z_2)^{2l} \phi(\frac{\alpha z_1 + \gamma z_2}{\beta z_1 + \delta z_2})$

⑦ $T(g) \phi(z) \equiv \phi_g(z) = T(g) f(z, 1) = f_g(z, 1)$
 $= (\beta z + \delta)^{2l} \phi(\frac{\alpha z + \gamma}{\beta z + \delta})$

Thus: $T_l(g) \phi(z) = (\beta z + \delta)^{2l} \phi(\frac{\alpha z + \gamma}{\beta z + \delta})$

⑧ So what?! I don't see any significance to this.

Section III.2.2. : Infinitesimal Operators on $T_\ell(g)$.

- ① We know there is a subgroup of $SU(2)$ called Ω_1 , of one parameter, wherein

$$\omega(t) \in \Omega_1 = \begin{pmatrix} \cos \frac{t}{2} & i \sin \frac{t}{2} \\ i \sin \frac{t}{2} & \cos \frac{t}{2} \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}$$

I.e., $\omega \in SU(2)$ but is not the general element. Well, using our previous result:

$$T_\ell(\omega(t)) \phi(x) = \left(i \sin \frac{t}{2} x + \cos \frac{t}{2} \right)^{2\ell} \phi \left(\frac{\cos \frac{t}{2} x + i \sin \frac{t}{2}}{i \sin \frac{t}{2} x + \cos \frac{t}{2}} \right).$$

- ② The infinitesimal operator for a O.P.S.O. is:

$$A = \left. \frac{d}{dt} T(g(t)) \right|_{t=0}.$$

So here $g(t) = \omega(t)$ and LHS gives $A_1 \cdot \phi(x)$. But RHS gives:

$$(1)^{2\ell} \frac{d\phi(q)}{dq} \cdot \left[\frac{1 \cdot (i \cdot \frac{1}{2}) - x (i x \cdot \frac{1}{2})}{1} \right]$$

$$+ 2\ell (1)^{2\ell} \cdot (i x / 2) \phi(x)$$

$$= \left\{ \frac{i}{2} [1 - x^2] \frac{d}{dx} + i \ell x \right\} \phi(x) = A_1 \cdot \phi(x).$$

Thus: $A_1 = i \ell x + \frac{i}{2} (1 - x^2) \frac{d}{dx}$

- ③ Comment: we have here a realization of the generators of $SU(2)$ as differential operators, providing we are restricted to a space $\mathcal{H}$ whose functions $\phi(x)$ are polynomials of degree 2ℓ in x .

- ④ As a basis in $\mathcal{H}$, clearly we take $\{x^{2\ell}, x^{2\ell-1}, \dots, x^2, x^1, 1\}$.
Recombine A_i to get H_+, H_- and H_3 . Then:

$$H_+ = - \frac{d}{dx}$$

$$H_3 = \ell - x \frac{d}{dx}$$

$$H_- = -2\ell x + x^2 \frac{d}{dx}$$

Then x^{l-n} becomes eigenfunction of d.o. H_3 , etc., see (11), (12) and (13).

Section III. 2.3. Irreducibility

Let's review this magic show starting with section 2.1. First, we show that you can set up a group representation $T(g)$ which is an operator acting on a function of 2 complex variables (for $SL(2, C)$). This operator takes your function of two variables and generates a new function by replacing those variables with new complex variables obtained simply by multiplying the old variables backwards by a 2×2 matrix rep for g . Oddly enough, we start with a 2-dimensional matrix representation of group G and from that we construct a more general representation as an operator on a function space. Since this function space has invariant subspaces under $T(g)$ consisting of homogeneous polynomials of degree $2L$, we conclude that $T(g)$ is a highly reducible representation.

In fact, we are easily able to deduce the effect of $T(g)$ on a function $\phi(z)$ which is restricted to the Hilbert subspace H_L of degree $2L$ polynomials. This is shown in equation (8).

In this section 2.3, we show that the representation T_L does not itself have invariant subspaces and is therefore irreducible. Later, we will show how equation (8) can be used to get the matrix elements of the irreducible reps in some basis.

Section III. 2.4. Invariant Scalar Product

① Compute: $T_L(h) x^{l-k} = (e^{-it/2})^{2L} \left(\frac{e^{it/2} x + 0}{0 + e^{-it/2}} \right)^{l-k}$
 $= e^{-itL} (e^{it} x)^{l-k} = e^{-itk} x^{l-k}$

② Scalar product must have this property:

$$(\phi_1(x), \phi_2(x)) = (T(g)\phi_1(x), T(g)\phi_2(x))$$

This is so because Theorem 1, page 44 shows that such a s.p. always exists for a compact group.

③ Thus, letting $g=h$, we conclude: $(x^{l-k}, x^{l-m}) = 0$, in this special but not yet explicitized scalar product.

④ By choosing $(x^0, x^0) = (1, 1) = 2L!$, one gets:

$$(x^{l-k}, x^{l+k}) = (l-k)!(l+k)!$$

So define $\psi_k(x) = \frac{x^{l-k}}{\sqrt{(l-k)!(l+k)!}}$, then $(\psi_k(x), \psi_k(x)) = 1$.

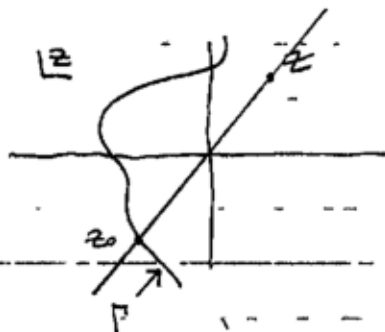
Vilenkin Chapter 6 on $QU(2)$, Section 2: The IR's of $QU(2)$.

I The Space D_x

① For some reason, the reps are going to be labelled by $\chi = (l, \epsilon)$, where $l \in \text{complex}$, and $\epsilon = 0$ or $1/2$. In $SU(2)$ we dealt with $\mathcal{H}_l$, the space of homo-polynomials of degree $2l$. Here, we shall instead deal with space D_x of functions of one complex variable such that:

- 1) $\phi(z)$ are infinitely differentiable except at $z=0$
- 2) $\phi(az) = a^{2l} \phi(z)$, i.e., degree of homo = $2l$.
- 3) $\phi(-z) = (-1)^{2\epsilon} \phi(z)$, i.e., definite parity

② Significance of these conditions? Suppose we know $\phi(z)$ everywhere on some curve Γ . We want to find $\phi(z)$ at some z off the



curve, say z . To find $\phi(z)$, draw a straight line through the origin. If Γ is chosen right, this line intersects Γ . Call that intersection z_0 . Notice that $z/z_0 = \text{real}$, either $+$ or $-$. [- in my picture].

If $z/z_0 = +$, then:
$$\phi(z) = \phi\left(z_0 \cdot \frac{z}{z_0}\right) = \phi(z_0 a) = a^{2l} \phi(z_0) = \left(\frac{z}{z_0}\right)^{2l} \phi(z_0).$$

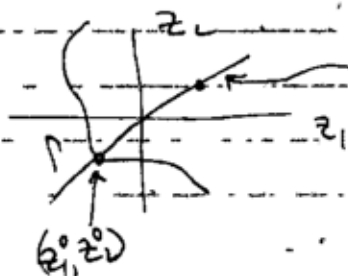
If $z/z_0 = -$, then:
$$\begin{aligned} \phi(z) &= \phi\left(z_0 \cdot \frac{z}{z_0}\right) = (-1)^{2\epsilon} \phi\left(z_0 \cdot \left[-\frac{z}{z_0}\right]\right) \\ &= (-1)^{2\epsilon} \phi(z_0 a) = (-1)^{2\epsilon} a^{2l} \phi(z_0) \\ &= (-1)^{2\epsilon} \left[-\frac{z}{z_0}\right]^{2l} \phi(z_0). \end{aligned}$$

We can handle both cases with one notation

$$\left. \begin{aligned} \phi(z) &= \left|\frac{z}{z_0}\right|^{2l} \left(\frac{z}{z_0}\right)^{2\epsilon} \phi(z_0) \\ \left|\frac{z}{z_0}\right|^{2l} (\text{sign } \frac{z}{z_0})^{2\epsilon} \phi(z_0) \end{aligned} \right\} \rightarrow \begin{aligned} \left(\frac{z}{z_0}\right)^{2l} \phi(z_0) & \quad \frac{z}{z_0} > 0. \\ \left(-\frac{z}{z_0}\right)^{2l} (-1)^{2\epsilon} \phi(z_0) & \quad \frac{z}{z_0} < 0. \end{aligned}$$

③ The point is this: if we know $\phi(z)$ on a nice contour Γ , then we know it everywhere in $\mathbb{C}$.

④ We can go back now to ^{top} page 109 & pick up comments there. In complex z_1, z_2 space, imagine a hypersurface.



complex line

If we want $f(z_1, z_2)$, we can get it from $f(z_1^0, z_2^0)$ by the homogeneity of f .

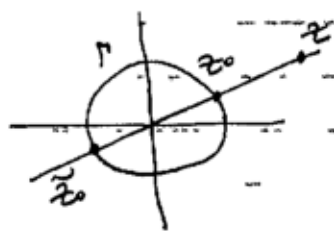
$$f(az_1, bz_2) = \sum_n a_n (a)^{l+n} (b)^{l+n} z_1^{l+n} z_2^{l+n} \\ = a \dots \text{ugh}$$

$$f(az_1, az_2) = a^{2l} f(z_1, z_2)$$

$$\text{So } f(a\vec{z}) = a^{2l} f(\vec{z}).$$

Thus, if $\vec{z} = a\vec{z}_0$, then $f(\vec{z}) = f(a\vec{z}_0) = a^{2l} f(\vec{z}_0)$.
But this idea was never used here.

⑤ Example of a Γ : here: $\phi(z) = \phi(az_0) = a^{2l} \phi(z_0)$



$$= \phi(-az_0) \\ = (-1)^{2l} \phi(az_0) \\ = (-1)^{2l} a^{2l} \phi(z_0)$$

$$\Rightarrow \phi(z_0) = (-1)^{2l} \phi(-z_0)$$

So, $\phi(z)$ is even (odd) on the circle Γ when $l = 0$ ($\frac{1}{2}$).

⑥ So, functions on a circle in effect ~~realize~~ realize space D_x !

① The forms $f(e^{i\theta})$. Suppose we define, for $\theta=0$,

$$f(e^{i\theta}) \equiv \phi(e^{i\theta/2})$$

Then consider first $\theta=\pi$. Then $f(e^{i\pi}) = \phi(e^{i\pi/2})$. Fine.

Now consider $\theta=-\pi$. Then $f(e^{-i\pi}) = \phi(e^{-i\pi/2})$. But we want these to be the same. Since ϕ is even, they are the same. So, no double valuedness problem. So: the evenness of ϕ becomes the singlevaluedness of f .

$$\theta=0 \Rightarrow f(1) = \phi(e^{i0}) = \phi(1)$$

$$\theta=2\pi \Rightarrow f(1) = \phi(e^{i2\pi}) = \phi(-1) = \phi(+1) \checkmark$$

② For $\theta=\frac{1}{2}$, use $f(e^{i\theta}) = e^{i\theta/2} \phi(e^{i\theta/2})$. Then

$$f(1) = 1 \cdot \phi(1)$$

$$f(1) = (-1) \phi(-1) = \phi(1) \checkmark$$

← now odd

2 The representation $T_X(g)$

① Try this: $T_X(g) \phi(z) = \phi(\alpha z + \bar{\beta} z)$.

First, is this consistent. Is $\phi \in D_X$? $z \rightarrow \alpha z$ pulls a $\alpha^{2\bar{q}}$ on both sides $\checkmark$. $z \rightarrow -z$ pulls $(-1)^{2\bar{q}}$ on both sides $\checkmark$.

The real test is to show that this is a rep!

$$T_X(g_1) T_X(g_2) \phi(z) = T_X(g_1) [\phi(\alpha_2 z + \bar{\beta}_2 z)] = T_X(g_1) \xi(z)$$

$$= \xi(\alpha_1 z + \bar{\beta}_1 z) \quad \text{where} \quad \xi(z) = \phi(\alpha_2 z + \bar{\beta}_2 z) \checkmark$$

$$= \phi(\alpha_2 [\alpha_1 z + \bar{\beta}_1 z] + \bar{\beta}_2 [\alpha_1 z + \bar{\beta}_1 z])$$

$$= \phi(\{\alpha_1 \alpha_2 + \bar{\beta}_1 \bar{\beta}_2\} z + \{\alpha_2 \bar{\beta}_1 + \bar{\beta}_2 \alpha_1\} \bar{z})$$

$$\text{But } g_1 g_2 = \begin{pmatrix} \alpha_1 & \beta_1 \\ \bar{\beta}_1 & \alpha_1 \end{pmatrix} \begin{pmatrix} \alpha_2 & \beta_2 \\ \bar{\beta}_2 & \alpha_2 \end{pmatrix} = \begin{pmatrix} \alpha_1 \alpha_2 + \bar{\beta}_1 \bar{\beta}_2 & \alpha_1 \beta_2 + \bar{\beta}_1 \alpha_2 \\ \bar{\beta}_1 \alpha_2 + \alpha_1 \bar{\beta}_2 & \bar{\beta}_1 \beta_2 + \alpha_1 \alpha_2 \end{pmatrix}$$

$$= \phi(\alpha_{12} z + \bar{\beta}_{12} \bar{z}) = T(g_1 g_2) \phi(z)$$

Hence yes: $T_X(g)$ is a representation.

② Confirm $e^{i\psi/2}$...

$$e^{i\psi/2} = \frac{(\alpha e^{i\psi/2} + \bar{\beta} e^{-i\psi/2})}{|- \text{ same} |} \quad \text{has unit magnitude, } \psi \text{ real } \checkmark$$

$$\begin{aligned} \text{Try } e^{i\psi/2} &= \frac{\alpha e^{i\psi} + \bar{\beta}}{\beta e^{i\psi} + \bar{\alpha}} \cdot \frac{(\bar{\beta} e^{-i\psi} + \alpha)}{(\bar{\beta} e^{-i\psi} + \alpha)} \\ &= \frac{(\alpha e^{i\psi} + \bar{\beta})(\alpha + \bar{\beta} e^{-i\psi})}{|\beta e^{i\psi} + \bar{\alpha}|^2} = \frac{(\alpha e^{i\psi/2} + \bar{\beta} e^{-i\psi/2})^2}{1 \cdot 1} \\ &= (e^{i\psi/2})^2 = e^{i\psi}. \text{ Thus } \psi' = \psi \checkmark \end{aligned}$$

$$③ |\alpha e^{i\psi/2} + \bar{\beta} e^{-i\psi/2}| = |\bar{\alpha} e^{-i\psi/2} + \beta e^{i\psi/2}| = |\bar{\alpha} + \beta e^{i\psi}| \checkmark$$

④ I have at last derived this result:

$$T_x(g) f(e^{i\psi}) = |\beta e^{i\psi} + \bar{\alpha}|^{2l} f\left(\underbrace{\frac{\alpha e^{i\psi} + \bar{\beta}}{\beta e^{i\psi} + \bar{\alpha}}}_{= e^{i\psi}}\right)$$

⑤ Let's try the $l = 1/2$ case. Then:

$$\begin{aligned} T_x(g) f(e^{i\psi}) &= T_x(g) [e^{i\psi/2} \phi(e^{i\psi/2})] \checkmark \\ &= (\bar{\alpha} e^{i\psi/2} + \bar{\beta} e^{-i\psi/2}) \phi(\alpha e^{i\psi/2} + \bar{\beta} e^{-i\psi/2}) \checkmark \\ &= (\dots)^{2l} \phi(e^{i\psi/2}) \underbrace{(e^{i\psi/2} e^{-i\psi/2})}_{\text{add on}} \checkmark \\ &= (\alpha e^{i\psi/2} + \bar{\beta} e^{-i\psi/2}) |\beta e^{i\psi} + \bar{\alpha}|^{2l} \underbrace{(\bar{\alpha} e^{-i\psi/2} + \bar{\beta} e^{i\psi/2})}_{|\beta e^{i\psi} + \bar{\alpha}|} \checkmark \\ &= (\alpha^2 + |\beta|^2 + \alpha \bar{\beta} e^{i\psi} + \bar{\beta} \bar{\alpha} e^{-i\psi}) = |\alpha|^2 + |\beta|^2 + 2 \operatorname{Re}(\alpha \bar{\beta} e^{i\psi}) \\ &= |\beta e^{i\psi} + \bar{\alpha}|^2 \end{aligned}$$

$$\text{So } = |\beta e^{i\psi} + \bar{\alpha}|^{2l+1} f(e^{i\psi})$$

disagrees with (4).

Valentin III.3: Matrix Elements of the Reps $T_l(g)$. $G = SU(2)$.

[1] Calculation of the Matrix Elements

① We have our orthonormal basis ψ_n , so all you have to do is turn the crank to get the reps of $SL(2, \mathbb{C})$:

$$T_{mn}^l(g) = (T_l(g)\psi_n, \psi_m) = \frac{(T_l(g)X^{l-n}, X^{l-m})}{\sqrt{(l-m)!(l+m)!(l-n)!(l+n)!}}$$

So:

$$\begin{aligned} T_l(g)X^{l-n} &= (\beta x + \delta)^{2l} (\alpha x + \gamma)^{l-n} (\beta x + \delta)^{n-l} \\ &= (\beta x + \delta)^{l+n} (\alpha x + \gamma)^{l-n} \end{aligned}$$

Now we need: $([\beta x + \delta]^{l+n} [\alpha x + \gamma]^{l-n}, X^{l-m})$

Well, use binomial theorem:

$$[\beta x + \delta]^{l+n} = \sum_{j=0}^{l+n} (\beta x)^{l+n-j} (\delta)^j \binom{l+n}{j}$$

$$[\alpha x + \gamma]^{l-n} = \sum_{j'=0}^{l-n} (\alpha x)^{l-n-j'} (\gamma)^{j'} \binom{l-n}{j'}$$

So $\Rightarrow = \sum_{j,j'} \binom{l+n}{j} \binom{l-n}{j'} \delta^j \gamma^{j'} \beta^{l+n-j} \alpha^{l-n-j'} (X^{2l-j-j'}, X^{l-m})$

But $(X^{2l-j-j'}, X^{l-m}) = \delta_{l-m, 2l-j-j'} \frac{1}{\sqrt{(l-m)!(l+m)!}}$, so

$$l-m = 2l-j-j' \Rightarrow j+j' = l+m \Rightarrow j' = l+m-j, \text{ so}$$

$$T_{mn}^l(g) = \frac{\sqrt{(l-m)!(l+m)!}}{\sqrt{(l-n)!(l+n)!}} \sum_j \binom{l+n}{j} \binom{l-n}{l+m-j} (\delta)^j (\gamma)^{l+m-j} (\beta)^{l+n-j} (\alpha)^{j+m-n}$$

$$l-n-j' = l-n-l+m+j = j+m-n$$

② OK, get (1), I've done this before But here is for $SL(2, \mathbb{C})$!

[2] Various expressions for these elements.

① When $\delta \neq 0$, you can decompose g into product of triangulars, apply rule (1) twice:

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \tilde{\alpha} & \beta \\ 0 & \delta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \gamma/\delta & 1 \end{pmatrix}$$

② Or, from obvious observation,

$$T_{lm}^l(g) = \sqrt{\dots} \frac{\beta^{n-m}}{\alpha^{n+m}} \left\{ \left(\frac{d}{dz} \right)^{l-m} [z^{l-n} (z+1)^{l+n}] \right\} \Big|_{z=\beta\delta}$$

③ Now we need that "trigonometric" thing. Here is how that goes:

$$\Phi(z) = \sum_{n=-l}^l a_n \bar{z}^n \quad \text{where } z = e^{i\phi}$$

Thus, Φ is a polynomial of order l in $\cos\phi, \sin\phi$,

$$\begin{aligned} \text{Ex: } e^{2i\phi} &= (\cos\phi - i\sin\phi)^2 \\ &= \cos^2\phi - \sin^2\phi - 2i\cos\phi\sin\phi \end{aligned}$$

So, claim: $f(z_1, z_2) = (z_1 z_2)^{-l} \Phi(z_1/z_2)$ replaces (5).

$$\text{Then: } f_g(z_1, z_2) = f(w_1, w_2) = (w_1 w_2)^{-l} \Phi\left(\frac{w_1}{w_2}\right)$$

$$\text{Now } f_g(z_1, 1) = (z_1)^{-l} \Phi(z_1)$$

$$\text{So } f_g(z_1, 1) = \dots$$

④ Another idea: $T_l(g)(\phi_1(z)\phi_2(z)) = (\beta z + \delta)^{2l} \phi_1\left(\frac{\alpha z + \gamma}{\beta z + \delta}\right) \phi_2\left(\frac{\alpha z + \gamma}{\beta z + \delta}\right)$ provided

that $\phi_1(z)\phi_2(z)$ is polynomial of order $2l$. ^{not true.} Let $\phi_1(z) = z^{-l}$ and $\phi_2(z) = \phi(z)$. Then $\phi_1\phi_2 = \Phi(z)$. Then:

$$T_l(g) \Phi(z) = (\beta z + \delta)^{2l} \cdot \left(\frac{\alpha z + \gamma}{\beta z + \delta}\right)^{-l} \phi\left(\frac{\alpha z + \gamma}{\beta z + \delta}\right) \quad ?$$

Not logical at all!

⑤ Another idea: $f(z_1, z_2) = \sum_n a_n \bar{z}_1^{-n} z_2^{l+n} = \text{hmo.}$
poly of degree l .

~~Let $\phi = (4)$.~~

$$\text{Then Let } \Phi(z_1) = f(z_1, 1) = \sum a_n \bar{z}_1^{-n} \quad (\text{replaces 4})$$

$$\begin{aligned} \text{Then } f(z_1, z_2) &= z_2^l \Phi\left(\frac{z_1}{z_2}\right) \\ f(z_1, 1) &= \Phi(z_1). \end{aligned} \quad \text{I give up. (replaces 5)}$$

⑥ Because I do not understand §2.1 (10), I cannot accept the integral rep. (15), yet.

[3] Euler Angles. Decompose $g(\phi, \theta, \psi) = g(\phi, 0, 0) g(0, \theta, 0) g(0, 0, \psi)$.

① We have:

$$\text{Te}[g(\phi, 0, 0)] X^{l-n} = e^{-in\phi} X^{l-n}$$

$$\text{So } (\text{Te}(h) \psi_n, \psi_m) = e^{-in\phi} (\psi_n, \psi_m) = \delta_{n,m} e^{-in\phi}$$

② So all we need is $\text{Te}(g(0, \theta, 0))$. Use version (4), p.11? with

$$\begin{aligned} \alpha = \delta = \cos \frac{\theta}{2} \quad \text{so} \quad & (\cos \frac{\theta}{2})^{m+n} (\sin \frac{\theta}{2} i)^{-m-n+2j} \\ \beta = \gamma = i \sin \frac{\theta}{2} & = (\cot \frac{\theta}{2})^{m+n} \times \sin^2 j \times (i)^{2j-m-n} \end{aligned}$$

Thus, we have

used the triangle decomposition result! Note

$$t_{mn}^l(\theta) = d_{mn}^l(\theta)? \equiv P_{mn}^l(\cos \theta). \quad (\text{not quite Jacobi's})$$

But beware: θ is about x-axis, not y-axis! Easy to relate to other form, however.

[4] Playing with these $P_{mn}^l(\cos \theta)$

Here, I go into all the properties of these functions and related functions. All the real work has already been done.

§1 (1.) The group $QU(2)$ is defined to be set of complex 2×2 matrices g such that $\det g = +1$ and $gsg^t = s$ where $s = \sigma_3$ of Pauli. Since $gg^t = 1$ is called unitary, this $gsg^t = s$ is called "quasi-unitary". So: "quasiunitary, unimodular matrices of the second order. This group is isomorphic to $SL(2, R)$, also to $SH(3)$, just as $SU(2)$ is to $SO(3)$. Linear fractionals are mentioned.

(2.) Three 1-param subgroups of $SL(2, R)$ are mentioned: $SO(2)$, $SH(2)$, and Z . But, the $SO(2)$ in $SL(2, R)$ is not diagonal; in $QU(2)$, it is diagonal; we will use this $SO(2)$ as basis. Other bases $\rightarrow$ other functions as reps.

(3.) $QU(2)$ is a s.g. of $SL(2, C)$, so we can still use the complex Euler angles. He uses $\tau = i\theta$, so (ϕ, τ, ψ) replaces Baymann's (μ, ξ, ν) .

(4.) states the invariant measure $dg = \sinh \tau d\tau d\phi d\psi$.

(5.) produces three generator matrices a_1, a_2, a_3 from the selected one parameter subgroups $\Omega_1, \Omega_2, \Omega_3$. Ω_1 and Ω_2 are "boosts", i.e., $SH(2)$, whereas $\Omega_3 = SO(2)$ = diagonal. Lie algebra stated.

§2 (1.) Here we set up a certain space of functions $\phi(z)$ called D_X , where $X = (l, \epsilon)$. l measures the homogeneity (scaling), ϵ measures the parity under negation of z . This space D_X is "realized" in terms of functions defined on a circle, because the D_X properties of $\phi(z)$ tell you $\phi(z)$ anywhere of the circle. The cases $\epsilon = 0, \pm$ are always separated.

(2.) On this special function space — cleverly chosen — we can define a $QU(2)$ representation $T_X(g) \phi(z) = \phi(\alpha z + \beta \bar{z})$. The beauty of this rep is that $\phi(z)$ is an invariant subspace [characterized by $(l, \epsilon) = X$] of the space of all $f(z)$, so $T_X(g)$ will be an irreducible rep! We want these I. reps only. By inserting (4) into (1), we get the famous result

(3) [for $\epsilon = 0$]:

$$T_X(g) f(e^{i\theta}) = |\beta e^{i\theta} + \bar{\alpha}|^{2l} f(e^{i\psi}) \quad \psi = \dots$$

Everything follows from this equation! (Another one for $\epsilon = 1/2$)

(3.) By playing with this master equation for the reps as operators on functions on a circle, one can obtain the $SO(2)$ generators as differential operators. We then go from the A_i basis to the $H_{\pm}, H_3$ basis where $H_3 = E + i \frac{d}{d\theta}$. Then at once we recognize $\{e^{-ik\theta}\}$ as our basis in the space $\mathcal{D}_X$ realized as functions on the circle. I.e., we shall work in a basis $\Rightarrow H_3$ is diagonal. Had we chosen alternatives to R_1, R_2, L_3 , we could have had a noncompact diagonal generator. The action of $H_{\pm}$ on basis $e^{-ik\theta}$ is found.

(4.) When $l \pm \epsilon$ and $l - \epsilon$ are integers, you say that X is "integral", a special case. As long as X is not integral, it's easy to show that $T_X(g)$ is an irreducible rep. The proof is just that if $e^{-ik\theta}$ is in a subset rep, then $H_{\pm}$ force all $\{e^{-ik\theta}\}$ to be, + then your sub. rep is all of T_X , so T_X could not be reduced.

(5.) When $X = \text{integral}$, the above arg fails because certain coefficients vanish. In this case, T_X is reducible to $T_l^+ \oplus T_l^-$, or $T_l^+ \oplus T_l^- \oplus T_l^0$.

(6.) When are two $SO(2)$ reps equivalent?

(7.) When is $T_X(g)$ a unitary rep? I do not understand how he arrives at his conclusions, but the conclusions are: reps are unitary in nonintegral X case if $l = -\frac{1}{2} + i\epsilon$, or if $l = \text{real}$, and $-1 < l < 0$, and $\epsilon = 0$. For integral X , T_l and T_{-l} are unitary (discrete series).

(8.) skip, "unitary adjoint reps"

§3. (1.) Here, the matrix elements T_X are done in 1 minute's work, again using the master equation and the $e^{ik\theta}$ basis. Hence, $T_{mn}^X(g) = (5)$.

(2.) Now (5) here shows $T_{mn}^X(g)$ in terms of τ , for the subgroup $g \in$. So:

$$T_{mn}^X(\phi, \tau, \psi) = e^{-i(m'\phi + n'\psi)} \beta_{m'n'}^l(\text{ch } \tau)$$

where $m' = m + \epsilon$. Recall that $H_3 \cdot e^{-ik\theta} = (k + \epsilon)e^{-ik\theta}$, not $ke^{-ik\theta}$.

Lyubanski

Lyubarski: (referenced from chapters 15)

Chapter 10: "Reps of the rotation groups"

Comment: 35 pages. Since the L.G. contains the rotation group, one should really understand all about rotations first. Eg, it too has a spinor algebra. Outline follows:

Chapter 10: "Reps of rotation groups"

45. the irred reps of $O(2) \cong \mathbb{Z}$
46. classific. of irred reps of $O(3)$
47. matrix elements of the irred. reps
48. properties of the irred reps
49. product of reps
50. spinor algebra
51. tensor algebra
52. reps of full orthog. group
53. double valued reps of point groups.

45. The irred reps of $\mathbb{Z}$. Since abelian, reps are one dim, just a phase. Write as general rep:

$$\chi_m(\phi) = e^{-im\phi}$$

If $m = \text{real}$, rep is unitary; otherwise not.

If $m = \text{integer}$, $\chi_m(\phi) = \chi_m(\phi + 2\pi)$, called "singlevalued".

The generator here is: $I_\phi = -im$, a number.

46. Classifying the irred reps of $O(3)$. First, a list here of 6 reminders. Remember that "characters" of any matrix rep is the trace, so in 3 dimensions,

$$\chi = 1 + 2 \cos \phi$$

Secondly, since $SO(3)$ is compact, all cont. SV reps are "equivalent" to unitary reps. In fact, even the non SV reps are unitary in this sense.

Let $D(g)$ be a rep, $g \in R$, of dimension s . But choose g like $(0, 0, \alpha_3)$ so $D(g)$ is also a (reducible) rep of Σ . Thus, "block form" is:

$$D(0, 0, \alpha_3) = \begin{bmatrix} e^{-im_1 \alpha_3} & & \\ & e^{-im_2 \alpha_3} & \\ & & \ddots \end{bmatrix} = \sum_{m_\nu} e^{-im_\nu \alpha_3} \cdot e_{m_\nu} \quad \uparrow \text{"unit vectors"}$$

This sum is in the sense of $1 \otimes 2 = 1 \oplus 2 \oplus 3$.

Construct operators:

$$A_{+1} = iI_1 - I_2$$

$$A_{-1} = iI_1 + I_2$$

$$A_0 = I_3$$

How do these "transform under Σ "? (wait)

Algebra:

$$[A_0, A_k] = -ik A_k \quad k=0, \pm 1$$

$$[A_1, A_{-1}] = 2i A_0$$

Now $A_0 e_{m_\nu} = (A_0)_{ij} (e_{m_\nu})_j = -im_\nu (e_{m_\nu})_j$. This means that, as a matrix:

$$A_0 = \begin{bmatrix} -im_1 & & \\ & -im_2 & \\ & & -im_3 \end{bmatrix} = \text{diagonal}$$

Clearly $A_1 e_{m_\nu}$ will not be diagonal. Next, he shows that:

$$I_3 (A_1 e_{m_\nu}) = -i(m_\nu + 1) (A_1 e_{m_\nu}) \quad \text{from commutators}$$

$$I_3 (e_{m_\nu}) = -i(m_\nu) (e_{m_\nu})$$

In English, " e_{m_ν} transforms under Σ according to the rep of index m_ν " or " $A_1 e_{m_\nu}$ transf. under Σ ac. to rep whose index is $m_\nu + 1$!"

Now, let m_j be the value of the largest index. Lower that index using A_{-1} :

$$\begin{aligned} e_{j-1} &= (A_{-1} e_j) & \text{has } j-1 &= m \\ e_{j-2} &= (A_{-1}^2 e_j) & \text{has } j-2 &= m \end{aligned}$$

You cannot get more than s vectors here.

Suppose there are r nonzero vectors. They must come in order!

$$e_j, e_{j-1}, e_{j-2}, \dots, e_{j-(r-1)}$$

$$A_{-1} e_{j-(r-1)} = -e_{j-r} = 0.$$

These are lin indep because they "belong" to different "indices" under $\mathbb{Z}$. (orthonormal too). Also, $A_{\pm} e_j = 0$ for obvious reasons ($j \equiv \max(m_j)$). Proceeding,

$$\begin{aligned} A_{+1} e_{j-1} &= 2j e_j & / \text{raising} \\ A_{+1} e_{j-2} &= 2(j-1) e_{j-1} \end{aligned}$$

We have shown that the subspace of size r is "invariant" under all 3 generators! Thus, our supspace is the whole space, $r=5$.

$$A_1 e_m = \lambda_m e_{m+1}$$

$$A_{-1} e_m = e_{m-1}$$

$$A_0 e_m = -im e_m$$

$$-r+1 \leq m \leq j$$

$$\exists r \text{ values } m = j, j-1, j-2, \dots$$

$$\left. \begin{aligned} m_j &= j \\ m_{j-1} &= j-1 \\ m_{j-2} &= j-2 \end{aligned} \right\} \text{The value of } m \text{ is } m, \text{ but so far } j \text{ is arbitrary}$$

In order to satisfy the comrels, you need:

$$\lambda_m = j(j+1) - m(m+1)$$

$$\lambda_m = 0$$

$$\lambda_{m-1} = \lambda_{j+1} = 2j \text{ etc.}$$

$$\lambda_{j-2} = 2(2j-1) \dots$$

Also, you want $\lambda_{j-s} = 0$. To get this, need:

$$\begin{aligned} j &= \frac{1}{2}(S+1) \\ S &= 2j+1 \end{aligned}$$

$$\begin{aligned} \lambda_{j-s} &= j(j+1) - (j-s)(j-s+1) \\ \Rightarrow j-s &= j+1 \\ \Rightarrow s &= 2j+1 \end{aligned}$$

So $j = 0, \frac{1}{2}, \dots$. For each j , you get one rep.

j = the weight. Now we can go back + say $m_0 = j$ $m_{s-1} = j-1 \dots m_1 = -j$

$$D(a_0, a_1, a_2) = \begin{bmatrix} e^{i j a_0} & & \\ e^{i(j-1)a_1} & & \\ & \ddots & \\ & & e^{-i j a_2} \end{bmatrix}$$

Finally, he goes to a new normalized "canonical" basis called e'_j , such that

$$(e'_j, e'_k) = 1 \cdot \delta_{jk}$$

Then the standard relations obtain. Note that the $\vec{I}$ generators are anti hermitean, so if $\rightarrow$ to get unitary reps

$$L_i = i I_i$$

L_i are hermitean.

$$A_+ = L_1 + i L_2$$

$$A_- = L_1 - i L_2$$

$$A_0 = -i L_3$$

$$A_0 E_m = -i m E_m$$

$$\Rightarrow L_3 E_m = m E_m$$

Theorem: summarizes this long, 8 page section

Lyubarski

Comments: in a concise 20 pages, he deals with the reps of the Lorentz group. Outline follows.

Chapter 15: "Reps of the Lorentz Group"

p 286

- ✓ 65. The Lorentz Group
- ✓ 66. The inf. operators of the L.G.
- ✓ 67. Classification of the irred. reps of the L.G.
- ✓ 68. Product of irred. reps.
- ✓ 69. Complex - conj. reps.
- ✓ 70. Spinor algebra.
- ✓ 71. Tensor algebra.
- 72. Reps of the Full Lorentz G.

65. The Lorentz Group. He considers all inertial frames whose spatial origins coincide, time $t = 0$. Restrict to this set of frames. Thus, $(0, 0, 0, 0) = x^\mu$ has same coords in all these frames. A Lorentz transform is:

$$x'_i = \alpha_{ij}(K) x_j$$

going from frame K to K' . (Note use of Latin indices.) We have a group because the matrices α satisfy:

$$\alpha(K''K) = \alpha(K''K') \alpha(K'K)$$

Any two LT's generate another one. Names $LG \equiv \mathcal{L}$.

If you factor off the parity operator I , you are left with the proper L.G. I think this is correct:

$$\mathcal{L} = \mathcal{L}_+ \otimes \mathbb{Z}_2$$

$\swarrow$ L.G. $\nwarrow$ proper L.G.

Lyub's metric is $\text{diag}(c, -1, -1, -1)$. No i 's etc.
These are 6 generators.

66. The infinitesimal operators of L.G. As the 6 parameters, why not use the cosine direction. For rotations, use a little vector $\hat{u}$ whose mag is θ , $\hat{u}$ = direction. Then use.

$$\alpha_x \equiv \vec{u} \cdot \hat{x}$$

$$\alpha_y \equiv \vec{u} \cdot \hat{y}$$

$$\alpha_z \equiv \vec{u} \cdot \hat{z}$$

$$\beta_x = \text{obv.}$$

So the $\vec{\alpha}$ = rotation params, $\vec{\beta}$ = boost params. Generators will be:

$$\vec{I} \equiv \text{rotation generators}$$

$$\vec{J} \equiv \text{boost generators}$$

We know that $d_{ik}(\text{x-boost}) = \begin{bmatrix} x & -\gamma\beta \\ -\gamma\beta & x \\ & & 1 & \\ & & & 1 \end{bmatrix} = [e^{\beta_i J_i}]_{ik}$

Thus, $\left. \frac{d d_{ik}}{d\beta_i} \right|_{\beta_i=0} = (J_i)_{ik}$. So we can find the generators in this way:

Let E_{ij} be a 4×4 matrix with a 1 in the (ji) position & all else 0. Then:

$$J_i = -E_{0i} - E_{i0}$$

In general

$$J_i = -E_{0i} - E_{i0} \leftarrow \text{boost generators}$$

$$d_{ik} = [e^{\vec{\beta} \cdot \vec{J}}]_{ik}$$

Here, system K' is boosted $\vec{\beta}$ relative to K . This means a passive boost in my book.

Similarly, by writing the passive rotations,

$$d_{ik}(\text{x rot}) = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & c & s \\ & & -s & c \end{bmatrix} \quad \left. \frac{\partial d_{ik}}{\partial c} \right|_{s=0} = \begin{bmatrix} & & & \\ & & & \\ 0 & 1 & & \\ & & -1 & 0 \end{bmatrix} = E_{23} - E_{32}$$

$$\text{So } I_1 = E_{23} - E_{32}$$

$$I_i = \epsilon_{ijk} E_{jk}$$

$$\text{Note that } I_2 = E_{31} - E_{13} = \begin{bmatrix} & & & \\ & & & \\ & & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$\text{By the way, } [\epsilon_{ij}]_{\alpha\beta} = \delta_{i\alpha} \delta_{j\beta}$$

We can write:

$$[J_i]_{\alpha\beta} = -(\delta_{\alpha\beta} \delta_{i\alpha} + \delta_{\alpha\beta} \delta_{i\alpha})$$

$$[I_i]_{\alpha\beta} = (\delta_{j\alpha} \delta_{k\beta} - \delta_{j\beta} \delta_{k\alpha})$$

The commutators are:

$$\begin{aligned} [I_i, J_j] &= -\epsilon_{ijk} J_k \\ [I_i, I_j] &= \epsilon_{ijk} I_k \\ [J_i, J_j] &= +\epsilon_{ijk} I_k \end{aligned}$$

Now define

$$A = -\frac{1}{2}(I + iJ)$$

$$B = -\frac{1}{2}(I - iJ)$$

Then:

$$\begin{aligned} [A_i, A_j] &= \epsilon_{ijk} A_k \\ [B_i, B_j] &= \epsilon_{ijk} B_k \\ [A_i, B_j] &= 0 \end{aligned}$$

67. Classify the fixed reps. Find all "forms" of A, B that satisfy the commutators. These will be the reps. So define:

$$\begin{aligned} A_+ &\equiv iA_1 - A_2 \\ A_- &\equiv iA_1 + A_2 \\ A_0 &\equiv A_3 \end{aligned}$$

Same for B .

▷ reference to section 46 ✓

Construct vectors e_{pq} which span the whole shebang. Then,

$$\begin{aligned} B_+ e_{pq} &\equiv \alpha_{p,q+1} e_{p,q+1} \\ A_+ e_{pq} &\equiv \alpha_{p+1,q} e_{p+1,q} \end{aligned}$$

We know what all operators do to these e_{pq} 's.

For each choice of P, Q (like j_1, j_2), $\exists$ one rep, call it

$$\mathcal{G}_{PQ} \quad (P, Q) = \text{the weight.}$$

If $P+Q = \text{half}$, double valued
If $P+Q = \text{int}$, single valued

Comments: so far we have restricted ourselves to matrix reps of finite size. Recall we said " s cannot be more than s ". If $s = \infty$, there is no truncation, and the results are different.

He says that Wigner + Gelfand discovered these reps in maybe 1947. What about Bargmann?

In general, the ∞ reps are: $\mathcal{G}(k_0, k_1)$

$$k_0 = 0, \pm \frac{1}{2}, \pm 1 \text{ etc.}$$

$$k_1 = \text{any complex number.}$$

If $k_0 - k_1 = \text{integer}$, you recover the finite dim reps.

Basis $\mathcal{E}_p^k$ $k = |k_0|, |k_0|+1, \dots$ of rep $\mathcal{G}(k_0, k_1)$
 $p = (k_1 - k) \text{ step } 1.$

A multiple infinity of basis vectors.

Define: $H^0 = I_1$, $H^\pm = I_3 \pm i I_2$ weird!
 $F^0 = -J_1$, $-F^\pm = J_3 \pm i J_2$

Now: $H^0 \mathcal{E}_p^k = i p \mathcal{E}_p^k$

$$2 \cdot F^0 \mathcal{E}_p^k = -\sqrt{k} B_k \mathcal{E}_p^{k-1} + p A_k \mathcal{E}_p^k + \sqrt{k+1} B_{k+1} \mathcal{E}_p^{k+1}$$

Very strange relations. The number k_1 enters via

$$A_k = f(k, k_0, k_1)$$

$$(k_0, k_1) \text{ equiv. } (-k_0, -k_1)$$

$$B_k = g(k, k_0, k_1)$$

$$\mathcal{G}(k_0, k_1) = \text{unitary when}$$

$$a) k_1 = \text{imag}$$

$$b) k_0 = 0 \text{ and } k_1 \text{ real}$$

} i.e., I, J are antiherm.

So, this ∞ rep stuff is really just slightly more complicated than the finites.

68. Product of irreps of L.G. First, he shows that

$$\tilde{C}_{p0} \times \tilde{C}_{0q} = \tilde{C}_{pq} \quad \text{ie} \quad u_{p0}^{p0} \cdot v_{0q}^{0q} = e_{pq}^{pq}$$

This in a sense is like $D^{(1)} \times D^{(1)} = D^{(2)}$, in that two irrep's can yield a third irrep, even tho in general, $2 \times 2 = \text{reducible}$. The reason for this is that

$$A \begin{bmatrix} u_{p0}^{p0} & v_{0q}^{0q} \end{bmatrix} = (A u_{p0}^{p0}) \cdot (A v_{0q}^{0q}) = (A u_{p0}^{p0}) v_{0q}^{0q}$$

↑ trivial

Or, in operator sense:

$$\tilde{C}_{pq} = \tilde{C}_p^{(A)} \otimes \tilde{C}_q^{(B)} = P \otimes Q$$

$$\tilde{C}_{p0} = P \otimes 1$$

$$\tilde{C}_{0q} = 1 \otimes Q$$

$$\tilde{C}_{p0} \times \tilde{C}_{0q} = \begin{bmatrix} A & B \\ P \otimes 1 & 1 \otimes Q \end{bmatrix} \times \begin{bmatrix} A & B \\ 1 \otimes Q & 1 \otimes Q \end{bmatrix} = \begin{bmatrix} A \\ P \end{bmatrix} \times \begin{bmatrix} B \\ Q \end{bmatrix} = \tilde{C}_{pq}$$

$$= (P \times 1) \otimes (1 \times Q)$$

Next, $\tilde{C}_{p_1 0} \times \tilde{C}_{0 p_2} = \sum_P \tilde{C}_{P0}$ exactly as

$$D^{(1)} \times D^{(1)} = \sum_i D^{(i)}$$

same Clebsch's too. The end result is:

$$\tilde{C}_{p_1 q_1} \times \tilde{C}_{p_2 q_2} = \sum_{PQ} \tilde{C}_{PQ} \quad \text{where } P \text{ in } (|P_1 - P_2|, P_1 + P_2)$$

$$Q \text{ in } (|Q_1 - Q_2|, Q_1 + Q_2)$$

The full Clebsch's are:

$$e_{pq}^{pq} = (P_1 P_2 p_1 p_2 | P) (Q_1 Q_2 q_1 q_2 | Q) u_{p_1 p_2}^{PQ} v_{p_2 q_2}^{PQ}$$

sums over all $\Rightarrow p_1 + p_2 = P, q_1 + q_2 = Q$, as obvious.

In other words, the whole thing is just what you'd expect!

In closing out this section, Lin. shows a few ways to construct scalars using either u, v , or the vector components ξ, η . Eg,

$$\xi_{00}^{\infty} = \text{scalar} = \sum_{pq} (-1)^{(p+q)-(p+q)} \xi_{pq} \eta_{-p-q}$$

Eg, if $Q=0$, then $= \sum_p (-1)^{p-p} \xi_{p0} \eta_{-p0} = \sum \xi \cdot \eta$ some way.

Finally, he remarks that $\mathcal{R}_{pq} \equiv D_p \times D_q$, which now seems very obvious.

69. Complex Conjugate Reps. Just start all the I, J generators, but only these [In the canonical reps, these are real anyway.] Nevertheless,

$$A_k^* = -\frac{1}{2}(\bar{I}_k + i\bar{J}_k) = -\frac{1}{2}(\overline{I_k - iJ_k})^* = \bar{B}_k$$

Note that "*" now has a special meaning. $A_k^* \neq \bar{A}_k$. Also,

$$(A_+)^* = -(B_+)$$

$$(B_+)^* = -(A_+)$$

Let E_{pq} span L of the $A, B = \mathcal{R}_{pq}$
 $\bar{E}_{pq}$ span $\bar{L}$ of the $A^* B^* = \mathcal{R}_{pq}^*$

Turns out that, $E_{pq} = U_{pq}$, the canonical basis

$$\text{then } \bar{E}_{pq} = (-1)^{(p+q)(p+q)} U_{q-p}$$

That scalar can now be written:

$$\sum_{p,q} \bar{E}_{pq} \eta_{qp} = \text{scalar}$$

$\nwarrow$ vector components in L

$\nearrow$ components of other vectors, but in $\bar{L}$

Maybe this is "the dual space"?

Turns out: $\bar{G}_{pq}^* = \mathcal{R}_{qp}$

The * reverses the roles of A, B ; hence switches these labels p and q . Not too deep.

70. Spinor Algebra

first kind spinors: $\mathcal{R}_{\pm 0}$ rep,

$$C_{\pm \frac{1}{2}, 0} \equiv C_{\pm 1}$$

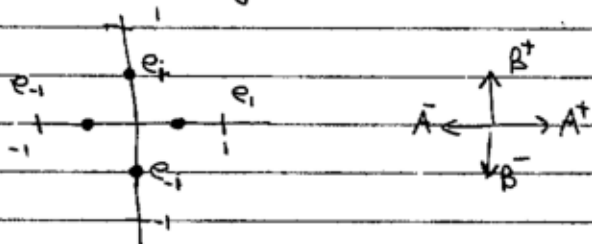
second kind spinors: $\mathcal{R}_{0 \pm}$ rep,

$$C_{0, \pm \frac{1}{2}} \equiv C_{\pm i}$$

$$\text{Eg: } A_+ C_{-1} = C_1 \quad \text{etc.}$$

$$A_+ C_{\pm} = 0$$

I can imagine some kind of weight diagram for this



A spinor of first kind is: $\sum_i e_i + \sum_{-i} e_{-i}$ "undotted spinor"

" " " second " " : $\sum_i e_i + \sum_{-i} e_{-i}$ "dotted spinor"

The scalars are now:

$$S_{\text{undotted}} = \sum_{pq} (-1)^{p+q-p-q} \bar{e}_{q-p} e_{p-q}$$

$$S_{\text{dotted}} = \sum_{pq} (-1)^{p+q-p-q} \bar{e}_{q-p} e_{p-q}$$

In our simple case, the undotted scalars $(\frac{1}{2}, 0) = (P, Q)$ are:

$$S = \sum_p (-1)^{p-p} \bar{e}_{0-p} e_{p-0} = (-1)^{\frac{1}{2}-\frac{1}{2}} \bar{e}_{0-\frac{1}{2}} e_{\frac{1}{2}-0} + (-1)^{\frac{1}{2}+\frac{1}{2}} \bar{e}_{0+\frac{1}{2}} e_{-\frac{1}{2}-0}$$

$$= \bar{e}_{-\frac{1}{2}} e_{\frac{1}{2}} - \bar{e}_{\frac{1}{2}} e_{-\frac{1}{2}}$$

Remember that $\bar{e}$ = basis in the dual space where vectors are $v = \sum \bar{e}_{pq} e_{pq}$. $\bar{e}$ relates to some u_{pq} , but do not say $u_{pq} = e_{pq}$ because u are still used for the dual space.

Let $g = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Then these scalars are:

$$g_{kl} e_k e_l$$

$$g_{kl} \bar{e}_k \bar{e}_l$$

and so on.

So certain combinations of spinors form scalars.

Example: let $\bar{e}$ and $\bar{e}$ be two undotted spinors. Then

$$g_{kl} \bar{e}_k \bar{e}_l = \text{scalar}.$$

So g_{kl} is kind of a little metric tensor, this is just like combining two vectors to get a scalar.

Direct products of spinors will give "higher spinors" — never a vector!

Notation: $C_{\alpha_1 \alpha_2 \dots \alpha_m; \beta_1 \beta_2 \dots \beta_n} =$ components of a fancy spinor.

It is a combination of
 m ~~undotted~~ undotted
 n dotted

element of $(C_{\pm 0})^m (\bar{C}_{\pm 0})^n$.

Of course it can be reduced.

$m+n =$ total rank.

One can "expose" some of the subscripts.

$$\underbrace{C_{\alpha_1 \alpha_2 \alpha_3; \beta_1}}_4 = \underbrace{\sum_{\alpha_1}}_1 \underbrace{C_{\alpha_2 \alpha_3; \beta_1}}_3$$

$$= \sum_{\alpha_1} \eta_{\alpha_1} C_{\alpha_3; \beta_1} \quad \text{~~g}_{\alpha_1 \alpha_2} \eta_{\alpha_2} C_{\alpha_3; \beta_1}~~$$

Lots of ways to break off indices, one at a time, or two.

If we close with $g_{\alpha_1 \alpha_2}$, we get

$$g_{\alpha_1 \alpha_2} C_{\alpha_1 \alpha_2 \alpha_3; \beta_1} = \underbrace{(g_{\alpha_1 \alpha_2} \sum_{\alpha_1} \eta_{\alpha_1})}_{\text{scalar}} \cdot C_{\alpha_3; \beta_1}$$

In fancier notation, we might say

$$C^{\alpha_1}_{\alpha_2 \alpha_3; \beta_1} = \text{scalar} \cdot C_{\alpha_3; \beta_1}$$

"Simplification" is like tensor contraction. The result must be proportional to a "rank (1,1) spinor."

Closing theorem is: "Spinors that are symmetric on each side transform like tensors of T_{mn} " A little vague unless I read earlier section 50.

71. The tensor algebra. A "vector" has a D^3 piece + D^1 piece, but the "space" subspace ~~is~~ does not coincide with the $A \otimes B$ algebra; not an invariant subspace.

So, vectors $\sim C_{\pm \frac{1}{2}}$. Basis will be e_0, e_1, e_2, e_3 .

These relate to the $C_{\pm \frac{1}{2}}$ as shown, if we use the standard 6 generators.

For example, $e_{\frac{1}{2}\frac{1}{2}} = (0, -\frac{i}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0) = \frac{1}{\sqrt{2}}(e_2 - ie_1)$.
 $(1, 2), (1, 3)$ states all relations.

$e_1, e_2, e_3, e_4 \equiv$ "natural basis"
 $e_{\frac{1}{2}\frac{1}{2}}, \dots \equiv$ "canon. basis"

A rank- n tensor in natural basis is

basis $e_{p_1 p_2 \dots p_n}$

$\uparrow$
 tensor. $\uparrow$ components of them.

How do tensors "transform"? $(\mathcal{C}_{\frac{1}{2}\frac{1}{2}})_{ik}$ = the "vector" matrix representation of the L.G. This is just the usual Lorentz transformation. So:

$$(\mathcal{C}_{\frac{1}{2}\frac{1}{2}})e = (\mathcal{C}_{\frac{1}{2}\frac{1}{2}})_{ik} e_k = \delta_{ik} e_k \quad \text{generalize to rank } n.$$

Scalars? $\delta_{ik} e_i e_k = \text{scalar}$. [$= \delta_{ii} e_i e_i = e_1 e_1 - \dots$]

"Simplification" is again demonstrated, always reduces rank by 2.

72. Reps of the full L.G. The new element here is the spatial inversion operator (We do not mean Poincaré here). Call $\mathcal{Z}(I)$ the operator. Then he shows that, eg,

$$\mathcal{Z}(I) \mathcal{Z}(000 \times 00) \mathcal{Z}(I)^{-1} = \mathcal{Z}(000 - \times 00).$$

$$\Rightarrow \{\mathcal{Z}(I), J_k\} = 0 \quad \text{[3, boost]} \neq 0!$$

$$[\mathcal{Z}(I), I_k] = 0 \quad \text{[parity trans. com with rotations]}$$

From this we can deduce action of $\mathcal{Z} \cdot \mathcal{Z}^{-1}$ on all A, B type generators.

Now, let e_{pq} span $\mathcal{R}^6$. Define $e'_{pq} = \mathcal{Z}^{-1}(I) e_{pq}$.
 Then $e'_{pq} \equiv u_{qp}$ form a $\mathcal{R}^6$ themselves [the other "half" of the space.]

$$u_{qp} = \mathcal{Z}(I) e_{pq}.$$

So, the $C_{pq} + U_{qp}$ span all of $\mathcal{L}$.
 Next, if $P \neq Q$, then define $\tilde{Z}_{PQ} = \tilde{Z}_{QP} =$ the $2(2P+1)(2Q+1)$ full space. Under $\mathcal{L}$,

$$\tilde{Z}_{PQ} = \tilde{Z}_{PQ} + \tilde{Z}_{QP}$$

But, if $P=Q$, the sets U_{qp} , C_{pq} are related by

$$U_{pq} = \pm C_{qp}$$

Here, if $\pm$ they $\tilde{C}_P^{\pm}$ where $\tilde{C}_P^{\pm}(I) C_{pq} = \pm C_{qp}$. So there are two reps of $\mathcal{L}$; both are $\tilde{C}_P^{\pm}$ under $\mathcal{L}$.

In case $P \neq Q$, there was only one rep, " $\tilde{Z}_{PQ}$ ".
 In case $P=Q$, there are two reps.

The $+$ is regular
 The $-$ is pseudo ("axial", etc.)

$$\begin{array}{lll} \text{So } \tilde{C}_0^+ = S & \tilde{C}_{\frac{1}{2}}^+ = V & \tilde{S}^+ = ? \\ \tilde{C}_0^- = P & \tilde{C}_{\frac{1}{2}}^- = A & \tilde{S}^- = ? \end{array}$$

What about "tensor"? Combine two vectors + get

$$\tilde{C}_{\frac{1}{2}}^+ \otimes \tilde{C}_{\frac{1}{2}}^+ = \tilde{S}_{11} \oplus \tilde{S}_{10} \oplus \tilde{S}_{01} \oplus \tilde{S}_{00}$$

What does this mean? $\tilde{S}_{10}$ = some 3 component object
 $\tilde{S}_{11}$ = some 9 component thing

Why have I never
 seen a $\tilde{S}_{11}$ type object?

Syukarski

Ch 16

Ann: am looking for the group nature of the Dirac spinors. Not at all clear to me right now. Outline of this chapter is:

Chapter 16: "Rela. Invar. Equas."

pp 307 → 334

- 73. the wave function
- 74. rel. invar. equa's
- 75. the lagrangian
- 76. conservation laws
- 77. spin
- 78. time inversion + Pauli Theorem
- 79. the Dirac equation

I should read at least 73, 74, 79.

73. the wave function Every free elem particle has a "wave function" called ψ_k $k=1,2,\dots$ of course, or goes to ∞ for some cases. There is a $L(\psi)$ "functional" and an action $\int d^4x L(\psi)$ and the "theory" is $\delta S = 0$, or Euler Lagrange these latter always boil down to,

$$A_{jk} \partial_j \psi_k + X_{jk} \psi_k = 0 \quad j=0,1,2,3$$

Let's see why this is:

Usually,

$$L = A_{jk} \partial^\mu \psi_j \partial_\mu \psi_k + B_{jk} \psi_j \partial_\mu \psi_k + C_{jk} \psi_j \psi_k$$

This is the

most general bilinear form with no "hanging" Lorentz indices + no $D > 2$ terms. So

$$\partial^\mu \frac{\partial L}{\partial (\partial_\mu \psi_j)} = \frac{\partial L}{\partial \psi_j} \Rightarrow \partial^\mu [2A_{jk} \partial_\mu \psi_k + B_{jk} \psi_k] = B_{jk} \partial_\mu \psi_k + 2C_{jk} \psi_k$$

Well, this does have a $\partial^\mu \partial_\mu \psi$ term, but usually this is replaced with mass for plane waves! So general form is correct

Now, in general these wave functions transform as:

$$\psi'_k(x') = T_{ik}(g) \psi_k(x)$$

$$\Rightarrow \text{oooo} \rightarrow \text{oooo.}$$

How do these connect to reps of LG ? Easy to show that $T(g_1 g_2) = T(g_1) T(g_2)$, so by definition, $T(g_1)$ is a rep of the L.G. (the full L.G.)

Now lets say T is space E which is reducible, $\Rightarrow$ can go to block form W like,

$$E = \sum_{\sigma} \sum_{\lambda} E_{\sigma\lambda}$$

In $E_{\sigma\lambda}$, use basis

$$e_{pq}^{(\sigma)} = e_{\sigma, l} = e_{\sigma, l} \quad l = pq \text{ index.}$$

So ψ can be broken down into $\psi_{\sigma, l}$.

74. Rela. Invar. Equas. Start with (73,1). We want this equation to have same form in all L.F.'s. Assume X_{ik} = invertible + red basis $\Rightarrow X$ is diagonal. Then:

$$A_{ik} \partial_{\lambda} \psi_k + X \psi_i = 0$$

Now look at $\psi_{\sigma', l}$ in a particular σ' . We can say,

$$\frac{\partial \psi_{\sigma', l}}{\partial x_j'} = [A]_{ll'} \cdot \frac{\partial x_j'}{\partial x_j} \cdot \frac{\partial \psi_{\sigma', l}}{\partial x_j'}$$

$$\text{Now, } x_j' = \alpha_{jj'} x_j \quad \frac{\partial x_j'}{\partial x_j} = \alpha_{jj'}(g)$$

$$\text{Also, } x_j = \alpha_{jj'}^{-1}(g') x_j' \quad \text{so} \quad \frac{\partial x_j}{\partial x_j'} = \alpha_{jj'}^{-1}(g')$$

$$\text{So, } \frac{\partial \psi_{\sigma', l}}{\partial x_j} = 4\sigma' \# \text{'s as } \sigma' \times 3_{\frac{1}{2}}$$

$$\text{Rewrite the equation: } A_{\sigma\sigma', 33'} \partial_{\lambda} \psi_{\sigma', l} = -X \psi_{\sigma, l}$$

"If σ' does not "link" to σ , then $A_{\sigma\sigma', 33'} = 0$!" Here, link means:

$$\sigma' \text{ links } \sigma \Leftrightarrow \sigma' \times 3_{\frac{1}{2}} \supset \sigma$$

The reason is that both sides must x form together.

Note: $\sigma' \times 3_{\frac{1}{2}} = \sum \sigma'$, does not include any rep twice, just as in the rotation group.

Question: for a fixed $\sigma' = (\sigma')$ [ie, we look at only a piece of the full ψ_k], and for fixed σ , they link once or none. If they link once, then Clebsch set the A 's.

Now use

- (a) (14,5) to recover the "natural" derivatives ∂_i
- (b) (14,7) for the P^{mn}
- (c) (14,4) for general relativity (under L_1) equation

Use $\mathcal{Z}_0 = PQs$ etc. Result:

$$\sum_{(P'Q'S'), P'Q'} \sum_{PQ, P'} \{ \text{clebsch} + \delta_i^{i'} \{ \Psi(P'Q'S'), P'Q' \} = - \chi \Psi(PQ, P) \quad (14,8)$$

Then (14,9) sees explicit Clebsch inserted.

This is the most general L_1 invariant equation.

Symmetry of $P_{\alpha\beta}^{mn} = P_{\beta\alpha}^{mn}$ etc. Lots of fancy properties here.

Next, he notes that the constants A_{ik}^{α} have "certain constraints", namely:

$$T(g)_{ij} A_{jk}^{\alpha} T(g)_{kl} = A_{il}^{\beta} \quad \alpha \leftrightarrow \beta \quad \Lambda^{\alpha\beta}$$

Now, $T_{ij} =$ a matrix rep of $\mathcal{G}$; $A_{ij} =$ matrix of same size.
We next show that,

$$[T_k, A^0] = A^k \quad k=1,2,3$$

So what...?

↑ boost generators

Inversion considerations: need $[A^0, I] = 0$

Ends up with $[A^k, I^0] = 0$

a $\mathcal{Z}_0$ condition for inversion.

79. The Dirac Equation. Let's specialize. The supergeneral (14,9) results for the special case

$$T = 1 \cdot \mathcal{Z}_{0\frac{1}{2}} + 1 \cdot \mathcal{Z}_{\frac{1}{2}0}$$

↑ once!

↑ once!

Think about (14,9). The RHS is $-\chi \Psi(PQ, P)$ where $PQ = 0\frac{1}{2}$, then $\frac{1}{2}0$. For $PQ = 0\frac{1}{2}$, we consider

$$\mathcal{Z}' \otimes \mathcal{Z}_{\frac{1}{2}\frac{1}{2}} = \mathcal{Z}_{0\frac{1}{2}} \Rightarrow \mathcal{Z}' = \mathcal{Z}_{\frac{1}{2}0} \text{ or } \cancel{\mathcal{Z}_{\frac{1}{2}\frac{1}{2}}}$$

But T does not include $\mathcal{Z}_{\frac{1}{2}\frac{1}{2}}$, so

Remember that T must include $\mathcal{I}$ and $\mathcal{I}'$. That was a theorem:

Theorem: if $\mathcal{G} \subset T$, then T must include at least one $\mathcal{G}'$ which links $\mathcal{G}$, otherwise $\Psi_{\mathcal{G},T} = 0$.

apply: Thus, you cannot have an equation with $T = \mathcal{G}_{0\frac{1}{2}}$ alone. Very revealing! There is no $(\frac{1}{2})$ rel ~~was~~ equation!

Now getting back to $(14,9)$ or $(14,4)$, if $\mathcal{G} = 0\frac{1}{2}$, then $\mathcal{G}'$ must be ± 0 . Thus, there is no (P'Q's) sum.

$$\left. \begin{array}{l} \Psi_{\pm 0,0} = \Psi_{\pm} \\ PQ \end{array} \right\} \text{ belong to first, or } \pm 0' \quad \text{"undotted"}$$

$$\Psi_{\pm 0,-0} = \Psi_{\mp}$$

$$\left. \begin{array}{l} \Psi_{0\frac{1}{2},0\frac{1}{2}} = \Psi_{\pm} \\ PQ \\ \Psi_{0\frac{1}{2},0-\frac{1}{2}} = \Psi_{\mp} \end{array} \right\} \text{ dotted.} \quad \text{Now } \begin{array}{l} PQ = \frac{1}{2} 0 = \mathcal{G} \\ P'Q' = 0 \frac{1}{2} = \mathcal{G}' \end{array} \text{, say.}$$

Then we write out all Clebsch's in $(14,9)$, get first equation p 331. Other choices in $PQ P'Q'$ yield 3 more equations. (1989)

Dimension invariance says: $\sum_{\pm 0,0\frac{1}{2}} = -\sum_{0\frac{1}{2},\pm 0}$
Equations boil down to (19,1).

$$\text{i.e., } A_{ik}^x \partial_k \Psi_i + x \Psi_i = 0.$$

$$\text{Then } \mu = 0, 1, 2, 3. \quad A^0 = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \\ -i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix} \begin{matrix} -1 \\ 1 \\ -i \\ i \end{matrix}$$

In this basis, using standard Pauli's, we have:

$$A^0 = \begin{bmatrix} 0 & -iI \\ -iI & 0 \end{bmatrix} \quad A^x = \begin{bmatrix} 0 & \\ & 0 \end{bmatrix}$$

$$A^1 = \begin{bmatrix} 0 & i\sigma_x \\ i\sigma_x & 0 \end{bmatrix} \quad A^2 = \begin{bmatrix} 0 & i\sigma_y \\ -i\sigma_y & 0 \end{bmatrix} \quad A^3 = \begin{bmatrix} 0 & -i\sigma_z \\ i\sigma_z & 0 \end{bmatrix}$$

Nonstandard δ matrices.

Now do a basis change:

$$\phi_1 = \psi_1 + \psi_i$$

$$\phi_2 = -\psi_1 - \psi_i$$

$$\phi_3 = -\psi_1 + \psi_i$$

$$\phi_4 = \psi_1 - \psi_i$$

Then $\gamma_0 = \begin{pmatrix} \mathbb{I} & \\ & -\mathbb{I} \end{pmatrix}$, $\gamma_k = \begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{pmatrix}$.

This is the BjD choice. Now those components are all mixed up.

$$\psi_1, \psi_{-1} \in \mathcal{I}_{\pm 0}$$

$$\psi_i, \psi_{-i} \in \mathcal{I}_{0 \pm}$$

$$\phi_n \in ? \quad \phi_n \notin \mathcal{I}_{\pm 0} \oplus \mathcal{I}_{0 \pm} = \mathcal{I}_{\pm \pm}$$

Here is the point: The "usual" Dirac spinor is most definitely not a vector of $\mathcal{I}_{\pm \pm}$. Rather, its components are linear combs of $\mathcal{I}_{0 \pm}$ spinors + $\mathcal{I}_{\pm 0}$ spinors. If you undo the shuffle, you get

$$\psi_n = \begin{bmatrix} \psi_{-1} \\ \psi_i \\ \psi_{-i} \\ \psi_1 \end{bmatrix} \quad \text{This is the meaning of}$$

$$\Gamma = \mathcal{I}_{0 \pm} \oplus \mathcal{I}_{\pm 0}$$

Also, there is no covariant linear equation that involves only $\mathcal{I}_{0 \pm}$ or $\mathcal{I}_{\pm 0}$. Any invar. equation that has one must have the other.

STOP. Return to P19 after reading 75 and 76 and 77 and 78.

75. The Lagrangian. Define an inner product on these ψ .

$$(\phi, \psi) \equiv B_{ij} \phi_i \bar{\psi}_j \quad (\text{note where bar goes!})$$

$$L = -(\Gamma^\mu \partial_\mu \psi, \psi) + x(\psi, \psi)$$

But this does not work for scalars...?

Now using this version of L , least action $\delta S = 0$ gives us our claimed "general" equations only if

$$\Gamma^k = -(\Gamma^k)^{\dagger}, \text{ anti-hermitean. [he uses } * \text{]}.$$

Recall def of herms: $(A\phi, \psi) \equiv (\phi, A^{\dagger}\psi)$
 $= A^k$

From the "wave" equation, $\Gamma^k \partial_k \psi$ transforms as ψ , so

$$L = \text{scalar if } (\phi, \psi) = \text{scalar } \forall \phi, \psi.$$

Now go examine (ϕ, ψ) "in detail".

$$\psi_k = \psi_{\alpha\beta, l} \text{ recall.}$$

$$(\phi, \psi) = B_{ij} \psi_i \bar{\psi}_j = B_{\alpha\beta, \gamma\delta} \phi_{\alpha\beta} \cdot \bar{\psi}_{\gamma\delta}$$

So these $\bar{\psi}_{\gamma\delta}$ span $\tilde{3}'$, not $\tilde{3}$ itself, since bar.

Theorem: $\tilde{3}_1 \times \tilde{3}_2$ contains $\tilde{3}_0$ only when $\tilde{3}_1 = \tilde{3}_2$, so if we want a "scalar", require:

$$\tilde{3}' = \tilde{3}$$

Debreak to P69. There, we

constructed scalars for profit. We found there that

$$\text{scalar} = \sum_i \bar{F}_i \eta_i \quad l = pq.$$

where $\eta = \eta_i e_i^{(3)} \equiv$ a "vector" in $\tilde{3}$

$\bar{F} = \bar{F}_i e_i^{(3')} \equiv$ another vector in $\tilde{3}$.

Actually, I was confused during 69.

$e_i =$ basis in L_1 $\cdot \mathbb{C}P^1$

$\bar{e}_i =$ basis in $\bar{L}_1$ $\cdot \mathbb{C}P^1$ $\tilde{3}_{pq}^* = \tilde{3}_{pq} = \tilde{3}_{pq}$ so $* = \sim$ here.

So, if $\bar{F} = \bar{F}_i e_i$, then $\bar{F} = \bar{F}_i \bar{e}_i =$ vector in $\bar{L}_1$.

Now $\bar{F}, \eta$ belong to the same $\tilde{3}_j$ if η is in $\tilde{3}_j$
 $\bar{F}$ is in $\tilde{3}_j$

So $\overline{\Psi}_{3s'qp}$ means what? Error, go back:

The original invariant was:

$$\text{scalar} = \sum_{p,q} (-1)^x \xi_{pq} \eta_{p-q}$$

where $\xi = \xi_{pq} C_{pq}$
 $\eta = \eta_{pq} C_{pq}$ both in $\mathcal{Z}$.

Then we said: $\eta = \eta_{p-q} C_{p-q}$ also, rename

but, $C_{p-q} = (-1)^x \bar{C}_{qp}$, so we can say:

$$\eta = (-1)^x \eta_{p-q} \bar{C}_{qp}$$

$$\eta = \eta_{qp} \bar{C}_{qp} \quad \text{directly}$$

Therefore, $\bar{\eta}_{qp} = (-1)^x \eta_{p-q}$

Thus, you can write that scalar as: $\sum_{pq} \xi_{pq} \bar{\eta}_{qp}$

Now $\xi =$ in $\mathcal{Z}$ with index l
 $\bar{\eta} =$ in $\tilde{\mathcal{Z}}$ with " $\hat{l}$

So, $\text{scalar} = \sum_l \xi_{l\mathcal{Z}} \bar{\eta}_{\tilde{\mathcal{Z}}\hat{l}}$ arb.

This then gives: $\text{scalar} = \sum_{p,q} \phi_{\mathcal{Z}\mathcal{S},pq} \bar{\psi}_{\tilde{\mathcal{Z}}\mathcal{S}',qp}$

Keep in mind: $\bar{\psi} = \bar{\psi}_{pq} \bar{C}_{pq} \in \tilde{\mathcal{Z}}$ $\uparrow$ reversed

General scalar is any sum over $\mathcal{Z}, \mathcal{S}, \mathcal{S}'$ (75,4) ✓

Now, to get a real action, need

$$(\phi, \psi) = (\psi, \phi)$$

Note: (ϕ, ψ) is not nec. a $\mathbb{C}$ -number; it is an operator, +
 real eigens. $\Rightarrow$ hermitian. Think of (ϕ, ψ) as a
 $\mathbb{C}$ number; then

$$(\phi, \psi) = B_{ij} \phi_i \psi_j = (\psi, \phi) \text{ if } B \text{ real.}$$

76. Conservation laws. How does one write the conservation in terms of these ψ_i ? Translation says:

$$\psi'(x') = \psi(x' + a)$$

For rotations we would have said:

$$f(R\vec{x}) = f(\vec{x})$$

Ide: I have just completed my massive review + study of the $\mathcal{L}$ generators + fields etc. So now I want to translate everything I've seen into this new language!

$$T(a) \psi(x') = \psi(x)$$

This is a type ③: I would have said: $\phi_i(x') = e^{i\partial_\mu \tilde{P}_\mu} \phi_i(x)$ with $x' = x + a$. But Liu uses: $x' = x - a$, so

$$\phi_i(x') = e^{+i\partial_\mu \tilde{P}_\mu} \phi_i(x) \Rightarrow \phi_i(x) = e^{-i\partial_\mu \tilde{P}_\mu(x)} \phi_i(x')$$

$$\text{Thus, I say: } \tilde{P}_\mu \psi(x') = i\partial_\mu \psi(x)$$

Thus: $-i\tilde{P}_\mu = \partial_\mu = Y$. This Y is a differential operator!

$$\text{Now we arrive at: } \psi'_i(x') = T_{ij}(h) \psi_j(x) \quad (76,10)$$

This is type ①, so $T_{ij}(h) = [e^{\delta M^{\mu\nu}}]_{ij}$, eg

$$\text{Next: } \psi'_i(x') = T_i(h) \psi(x')$$

This is type ②, so $T_i = [e^{\delta M^{\mu\nu}}]_{ij}$.

Now return to the rel. inv. equations:

$$(A^k)_{ij} \partial_k \psi_j = -x \psi_i$$

$$(A^k)_{ij} \partial_k [M_{jk} \psi_k] = -x [M_{jk} \psi_k]$$

a type ①?

Both must be true since A^k indep of frame,

$$\partial_k [M_{jk} \psi_k] = M_{jk} \partial_k \psi_k + (\partial_k M_{jk}) \psi_k$$

Go derive (76,12) elsewhere, then return.

We had better pay attention to the inner product here:

$$(q, \psi) \equiv \sum_{\alpha\beta\gamma, \gamma\beta\alpha'} B_{\alpha\beta\gamma, \gamma\beta\alpha'} q_{\alpha\beta\gamma} \overline{\psi_{\gamma\beta\alpha'}}.$$

Recall that $\psi_{\alpha\beta\gamma}$ is the γ component of a vector ψ .

Repeat Sidebar: For a "complex-conjugate" rep. to $\mathbb{Z}_{pq}$, called $\mathbb{Z}_{pq}^*$. The generators are $\overline{I}$ and $\overline{J}$.

$$I^* = \overline{I} \quad J^* = \overline{J}$$

This $\Rightarrow A^* = \overline{B}$ and $B^* = \overline{A}$, so there is a mixup.

$\mathbb{C}_{pq}$ = basis elements $\{ \text{like } |e_i\rangle \}$ for $\mathbb{Z}_{pq}$

What is a basis? $|e_j\rangle = \mathcal{O}^{\otimes M_{pq}} |e_j\rangle = [\mathcal{O}^{\otimes M_{pq}}]_{jk} |e_k\rangle$. i.e., a basis transforms according to some rep.

Then: $|e_j\rangle = [\mathcal{O}^{(aA+bB)}]_{jk} |e_k\rangle.$

Now conjugate everything in sight

$$\begin{aligned} \overline{|e_j\rangle} &= [\mathcal{O}^{(a\overline{A}+b\overline{B})}]_{jk} \overline{|e_k\rangle} \\ &= [\mathcal{O}^{(aB^*+bA^*)}]_{jk} \overline{|e_k\rangle}. \end{aligned}$$

Then, the $\overline{|e_j\rangle}$ form a basis for $\mathbb{Z}_{pq}^*$ negate and switch:

Actually, turns out that $\mathbb{C}_{pq} = (-1)^{Q+P-Q-P} \mathbb{C}_{-q-p}$ $\downarrow$
 $\mathbb{C}_{-q-p}$

This allows the scalars to be written:

$$\text{inv.} = \sum_{P, Q} \mathbb{E}_{pq} \mathbb{N}_{qp} \quad \left[\text{see inserted sheets} \right]$$

Liubanski + his Complex Conjugate Representation

- ① The meaning of the basis $\bar{C}_{pq}^{PQ}$ is clear. This basis is "canonical" in the sense of the following "canonical algebra":

$$A_- \bar{C}_{pq}^{PQ} = \alpha_p^P \bar{C}_{p-1,q}^{PQ}$$

$$A_+ \bar{C}_{pq}^{PQ} = \alpha_{p+1}^P \bar{C}_{p+1,q}^{PQ}$$

The first subscript on $\bar{C}$ is changed by actions of the A operators.

- ② Liu defines some new operators A_-^*, A_+^* , etc. These operators also have a canonical basis, called u ; namely,

$$A_-^* u_{qp}^{QP} = \alpha_q^Q u_{q-1,p}^{QP}$$

$$A_-^* = -B_+$$

The A^* still act on the first subscript, but since A^* is a form of B , that subscript ranges over $(2Q+1)$, not $(2P+1)$ as in ①.

- ③ One can easily show that:

$$A_-^* \bar{C}_{pq}^{PQ} = -\alpha_{q+1}^Q \bar{C}_{p,q+1}^{PQ}$$

Notice that A_-^* raises the q index. Obviously we can relate the basis $\bar{C}_{pq}^{PQ}$ to the basis u_{qp}^{QP} . The relation is:

$$\bar{C}_{pq}^{PQ} = \lambda_{pq} u_{-q,-p}^{QP} \quad \text{where} \quad \lambda_{pq} = (-1)^{Q+P-q-p}$$

- ④ What is the point of doing all this A^* stuff ??? Wait:

- ⑤ I find it useful to use a little $*$ to notate a basis for A^* and B^* . Then:

$$\bar{C}_{pq}^{PQ} = (-1)^{Q+P-q-p} (C^*)_{-q,-p}^{QP}$$

- ⑥ One use of this formalism is a simplification in the writing of scalars [E.g. to write a Lagrangian you always want scalars]. Watch:

$$\text{invar.} = \sum_{pq} u_{pq}^{PQ} v_{-p,-q}^{PQ} (-1)^{P+Q-p-q} = \sum_{pq} u_{-p,-q}^{PQ} (-1)^{P+Q+p+q} v_{pq}^{PQ}$$

$$= \sum_{pq} (u^*)_{qp}^{QP} v_{pq}^{PQ} \Rightarrow \text{scalar} = \sum_{pq} \bar{v}_{pq} u_{qp}^*$$

Some Liebariski Dabbolings

- ① Back on page 294, we learned how to construct the $D^{(00)}$ unit a basis vector from the $D^{(PQ)}$:

Theorem: $D^{(PQ)} \otimes D^{(P'Q')}$ contains $D^{(00)}$ iff $P'=P$ and $Q'=Q$.

When $P'=P$ and $Q'=Q$, then:

$$|00; 00\rangle = \sum_{P, Q} (-1)^{P+Q-P-Q} |PQ; P, Q\rangle \otimes |PQ; -P, -Q\rangle$$

- ② How does this work for linear combos of "vector" components? Let's reduce back to $SU(2)$ for a while. Here we have:

$$|0; 0\rangle = \sum_P (-1)^{P-P} |P; P\rangle \otimes |P; -P\rangle$$

This is like the combining of 2 angular momenta to get a $J=0$ state. In fact, maybe clearer if we say:

$$|0; 0\rangle = \sum_m (-1)^{j-m} |j; m\rangle \otimes |j; -m\rangle$$

Think maybe of two spins. Then:

$$\langle n | j; m \rangle = \mathcal{N}_{jm}^{(1)} \quad |n\rangle = |j; m\rangle \langle j; m | n \rangle = \mathcal{N}_{jm}^{(1)} |j; m; 1\rangle$$

The numbers $\mathcal{N}_{jm}^{(1)}$ characterize the spin state of # (1).

Then $\langle n | = \langle n | \otimes \langle n | =$ the spin state of the total system.

$$|n\rangle = |j; m\rangle \langle j; m | n \rangle = \mathcal{N}_{jm}^{(T)} |j; m\rangle.$$

In particular, $\mathcal{N}_{00}^{(T)}$ tells you how much of $|n\rangle$ is in the spin-0 ["scalar"] state. In fact, this number $\mathcal{N}_{00}^{(T)}$ is a scalar, whereas the numbers $\mathcal{N}_{jm}^{(T)}$ [m ranging] form a vector object belonging not to $D^{(0)}$ but to some $D^{(j)}$.

So we can construct $\mathcal{N}_{00}^{(T)}$ as follows:

$$\mathcal{N}_{00}^{(T)} = \sum_m (-1)^{j-m} \mathcal{N}_{jm}^{(1)} \mathcal{N}_{j, -m}^{(1)}$$

- ③ Similarly, we can go back to $\mathcal{L}_1$ and construct a scalar:

$$\mathcal{N}_{00; 00}^{(T)} = \sum_{P, Q} (-1)^{P+Q-P-Q} \mathcal{N}_{PQ; P, Q}^{(1)} \mathcal{N}_{PQ; -P, -Q}^{(1)}$$

- ④ So far, we have considered coupling.

$$\mathcal{Z}_{PQ} \otimes \mathcal{Z}_{PQ} = \mathcal{Z}_{00} \oplus \dots$$

What about

$$\mathcal{Z}_{PQ} \otimes \mathcal{Z}_{QP}$$

?!

I had hoped that this might generate the "trivial scalar" with no $(-1)^{...}$ factors.

Again, we can read off: (using (68, 6):

$$\begin{aligned} \mathcal{Z}_{00} &= \sum_{P, Q} (P, Q | P, P | 00) (Q, P | Q, P | 00) \mathcal{N}_{P, Q}^{PQ} \mathcal{N}_{P, P}^{QP} \\ &= \sum_{P, Q} (-1)^{P-P} (-1)^{Q-Q} \sum_{P, Q} \mathcal{N}_{P, Q}^{PQ} \mathcal{N}_{P, P}^{QP} \end{aligned}$$

It did not work. Signs reversed on the second factor, but negatives still survive!

⑤ Claim: although $(\mathcal{Z}^{PQ})^*$ and $(\mathcal{Z}^{QP})$ may be the same, (2)
 The basis $(\mathcal{C}^*)_{qp}^{QP} = (-1)^{Q+P+q+p} \overline{\mathcal{C}_{-p-q}^{PQ}}$ of $(\mathcal{Z}^{PQ})^*$

is not the same as the canonical basis of $\mathcal{Z}^{QP}$ which we might call $\mathcal{C}_{qp}^{QP}$.

This is why the $(\mathcal{Z}^{PQ})^*$ basis makes "simple scalars", whereas $\mathcal{Z}^{QP}$ makes "messy ones".

⑥ We should be able to express a "vector" in any basis

$$\mathcal{N} = \sum_{p,q} \mathcal{N}_{pq}^{PQ} \mathcal{C}_{pq}^{PQ}$$

$$\Rightarrow \overline{\mathcal{N}} = \sum_{p,q} (\mathcal{N}^*)_{qp}^{QP} (\mathcal{C}^*)_{qp}^{QP} = \sum_{p,q} \overline{\mathcal{N}_{pq}^{PQ}} \cdot \overline{\mathcal{C}_{pq}^{PQ}}$$

where

$$(\mathcal{N}^*)_{qp}^{QP} = (-1)^{Q+P+q+p} \overline{\mathcal{N}_{-p-q}^{PQ}}$$

Thus, if we know the components of $\mathcal{N}$ in terms of the basis vectors $\mathcal{C}_{pq}^{PQ}$, then we also know the components of $\overline{\mathcal{N}}$ in terms of $(\mathcal{C}^*)_{qp}^{QP}$ basis vectors.

⑦ Now construct some scalars:

Let:

$$\mathcal{N} = \sum_{p,q} \mathcal{N}_{pq}^{PQ} \mathcal{C}_{pq}^{PQ}$$

$$\mathcal{S} = \sum_{p,q} \mathcal{S}_{pq}^{PQ} \mathcal{C}_{pq}^{PQ}$$

From (68,8) we know that:

$$\text{scalar} = \sum_{p,q} (-1)^{P+Q-p-q} \mathcal{S}_{pq}^{PQ} \mathcal{N}_{-p-q}^{PQ} = \sum_{p,q} (-1)^{P+Q+p+q} \overline{\mathcal{S}_{-p-q}^{PQ}} \mathcal{N}_{pq}^{PQ}$$

But in ⑥ we showed that

$$\mathcal{S}_{-p-q}^{PQ} (-1)^{P+Q+p+q} = \overline{(\mathcal{S}^*)_{qp}^{QP}}$$

Thus we know that:

$$\text{scalar} = \sum_{pq} \overline{(\mathcal{S}^*)_{qp}^{QP}} \mathcal{N}_{pq}^{PQ}$$

Of course we could reverse the roles to get

$$\overline{\text{scalar}} = \sum_{pq} \overline{\mathcal{S}_{pq}^{PQ}} (\mathcal{N}^*)_{qp}^{QP}$$

This is my interpretation of (69,5).

Note that $\mathcal{S}_{pq}^{PQ}$ are components of a vector in $\mathcal{Z}^{PQ}$,

whereas $(\mathcal{N}^*)_{qp}^{QP}$ are components of a vector in $\mathcal{Z}^* = \tilde{\mathcal{Z}} = QP$.

⑧ Rewrite this last:

$$\text{scalar} = \sum_{pq} f_{pq}^{pq} \cdot (\overline{\eta})_{qp}^{qp}$$

Now, with the last statement of ⑦ in mind, we could write this as:

$$\text{scalar} = \sum_{\tilde{z}} f_{\tilde{z}, \tilde{z}} \overline{\eta}_{\tilde{z}, \tilde{z}}^* \quad \tilde{z} = pq, \quad \tilde{z} = qp.$$

⑨ Of course this is a scalar formed from two "wavefunctions" related to a particular $\tilde{z}$. If you are in some huge reducible situation, there may be several $\tilde{z}$ reps labeled by s . [but all the same]. Thus:

$$\text{scalar} = \sum_{s, s', \tilde{z}} f_{s, s', \tilde{z}} \overline{\eta}_{\tilde{z}, s', \tilde{z}}^* \cdot f_{ss'}$$

⑩ Finally, you can have a scalar contribution from different reps. So:

$$\text{general scalar} = \sum_{\tilde{z}} \cdot \sum_{s, s' \in \tilde{z}} B_{ss'} \cdot \sum_{\tilde{z}} f_{ss\tilde{z}} \overline{\eta}_{\tilde{z}, s', \tilde{z}}^*$$

This is how I read (75, 4). Now, this is a general scalar formed from the components of two (reducible) vectors:

$$f = \sum_{\tilde{z}, s} f_{\tilde{z}, s} \cdot \sum_{\tilde{z}} f_{\tilde{z}, s\tilde{z}} \cdot e_{\tilde{z}, s}$$

$$\eta = \dots$$

As such, every (arbitrary) choice for $B_{ss'}$ defines an inner product of vectors in this huge, reducible space.

$$(f, \eta)$$

It is in terms of such an inner product that the Lagrangian is defined!!

⑪ Remarks: The $\tilde{z}$ -sum is the thing I have been using all the time, except I have been calling it a "spin-sum". It should really be called the "component sum".

Thus, the inner product that Liu is using is exactly the "spin space" inner product I was using before, except your choice of $B_{ss'}$ [think of Dirac case] prescribes a special basis.

This Liu "inner product" is not inner in the coordinate sense.

Just imagine

$$(u, v) = \sum_{i=0}^{\infty} u_i v_i^*$$

with some appropriate definition of components u_i, v_i .

Now, back to those conservation laws. The objects A^k are large matrices in the secible space. They define the invariant equation. (Think χ matrices).

Of course, invariant equation $\Leftrightarrow$ scalar lagrangian
 $\Leftrightarrow$ conserved currents under Poincaré
 in fact:

$$L = -(A^k \partial_k \psi, \psi) + x(\psi, \psi)$$

Somehow I should be able to pull the conserved currents right off this lagrangian:

$$\frac{\partial^k L}{\partial(\partial_k \psi)} = \frac{\partial L}{\partial \psi} ? \rightarrow \text{Wave equations only.}$$

Well, we first need a "mover". Then, OK, I have now reviewed Noether.

So the conserved current from such an L should be:

$$j^k = \cancel{M^k} \left\{ -(A^k \tilde{\partial}_i \psi, \psi) \right\} + \text{another piece.}$$

Lie does not use that "extra piece", so his energy momentum tensor is a little different.

① At last Lie defines his version of $M^{\mu\nu}$:

$$T_{12}|^{LH} = -T_{21}|^{LH} = -R_3|^{me} = M^{12}|^{me} = M_{12}|^{me}.$$

$$T_{01}|^{LH} = -T_{10}|^{LH} = -B_1|^{me} = -M^{10}|^{me} = +M_{10}|^{me}.$$

Well, you can never tell.
 It seems that

$$T_{\mu\nu}|^{LH} = M^{\mu\nu}|^{me}.$$

OK, fine. On to the next + Sects.

77. Spin OK, just a fast reading here.

78. Dirac OK. quick read. I have analyzed this (mostly) before.

O'RAIPH.

Very nice proof that any one parameter Lie Group must, due to the assoc. condition, be basically $\phi(x, y) = x + y$. This is the "translation group", locally speaking. [compare $SO(1,1)$ and $SO(2)$]. Obviously abelian.

Lecture 3. Here again is what identity said:

$$\phi_\alpha(x_\beta, x'_\gamma) = x_\alpha + x'_\alpha + C^\alpha_{\beta\gamma} x_\beta x'_\gamma + g^\alpha_{\beta\gamma\delta} x_\beta x_\gamma x'_\delta + f^\alpha_{\beta\gamma\delta} x_\beta x'_\gamma x'_\delta + \dots \quad (3.4)$$

We are always interested in ϕ 's near $X = 0$, i.e., near the identity.

What does the associativity condition tell us?

$\bar{x}'$ means $\phi(x, \bar{x}') = 0$.

First, show that $\phi[\phi(x+dx, y), \phi(x, y)] = \phi(x+dx, \bar{x}')$

Take coeff of dx on both sides:

$$\frac{\partial \phi^\alpha[\phi(x, y), \bar{x}']}{\partial \phi^\beta(x, y)} \frac{\partial \phi^\beta(x, y)}{\partial x^\gamma} = \frac{\partial \phi^\alpha}{\partial x^\gamma}(x, \bar{x}') \quad (3.12)$$

Define:

$$V^\alpha_\beta(x) \equiv \frac{\partial \phi^\alpha(x, \bar{x}')}{\partial x^\beta} = \frac{\partial \phi^\alpha(x, y)}{\partial x^\beta} \Big|_{y=\bar{x}'}$$

Then $V(\phi) \frac{\partial \phi(x, y)}{\partial x} = V(x) \quad (3.14)$

Define: $U^\alpha_\beta(x) \equiv \left[\frac{\partial \phi^\alpha(y, x)}{\partial y^\beta} \right] \Big|_{y=0}$

Look again at (3.12). Set $x=0 \Rightarrow \bar{x}'=0$ since $\phi(0,0)=0$.

RHS:

$$\left. \frac{\partial \phi^\alpha(t, \bar{x}')}{\partial t^\beta} \right|_{x=0} = \frac{\partial \phi^\alpha(t, 0)}{\partial t^\beta} = \frac{\partial t^\alpha}{\partial t^\beta} = \delta^\alpha_\beta.$$

Now do LHS: $\frac{\partial \phi^\alpha(\phi(x, y), \phi'(x, y))}{\partial \phi^\beta(x, y)} = V^\alpha_\beta(\phi(x, y))$ by def.
 $\rightarrow V^\alpha_\beta(y)$ when $x \rightarrow 0$.

Also:

$$\left. \frac{\partial \phi^\beta(x, y)}{\partial x^\gamma} \right|_{x=0} = U^\beta_\gamma(y).$$

$$\Rightarrow V^\alpha_\beta(y) U^\beta_\gamma(y) = \delta^\alpha_\gamma.$$

So $V \cdot U = 1 = U \cdot V$ as matrices. Now rewrite (3.14):

$$V^\alpha_\beta(\phi(x, y)) \frac{\partial \phi^\beta(x, y)}{\partial x^\gamma} = V^\alpha_\gamma(x)$$

This is Bargmann's (1.1)

Apply V which inverts to get:

$$\frac{\partial \phi^\alpha(x, y)}{\partial x^\beta} = U^\alpha_\gamma(\phi(x, y)) V^\gamma_\beta(x)$$

(3.18)

Now, compute V and U from expansion of ϕ :

$$U^\alpha_\beta(x) = \left[\frac{\partial}{\partial y^\beta} \{ y^\alpha + x^\alpha + C^\alpha_{\beta\gamma} y^\beta x^\gamma + \dots \} \right] \Big|_{y=0} = \delta^\alpha_\beta + C^\alpha_{\beta\gamma} x^\gamma + \frac{1}{2} f^\alpha_{\beta\gamma\delta} x^\gamma x^\delta + \dots \quad (3.20)$$

$$V^\alpha_\beta(x) = \left[\frac{\partial}{\partial x^\beta} \{ x^\alpha + y^\alpha + C^\alpha_{\beta\gamma} y^\beta x^\gamma + \dots \} \right] \Big|_{y=\bar{x}'} = \delta^\alpha_\beta + C^\alpha_{\beta\gamma} (\bar{x}')^\gamma$$

But when x is very small, $\bar{x}' = -x$ because $\phi(x, y) \approx x + y$.

Thus: $V^\alpha_\beta(x) = \delta^\alpha_\beta - C^\alpha_{\beta\gamma} x^\gamma + \dots \quad (3.21)$

Lecture 4: A little theorem is proved: if $U_\beta^\alpha(x)$ are given, and if (3.18) can be integrated, then you will thereby produce a viable $\phi(x, y)$ which specifies a local Lie Group! So we need only find the integrability conditions.

Lecture 5: He accidentally switches $U \leftrightarrow V$ symbols, so correct (5.9) is as I have on data sheet. But not important, except remember diff:

consistent: 1) $V_\beta^\alpha(x) = \frac{\partial \phi^\alpha(x, y)}{\partial x_\beta} \Big|_{y=\bar{x}^{-1}}$

2) $U_\beta^\alpha(x) = \frac{\partial \phi^\alpha(y, x)}{\partial y_\beta} \Big|_{y=0}$

3) $\frac{\partial \phi_\alpha(x, y)}{\partial x_\beta} = U_\gamma^\alpha(\phi) V_\beta^\gamma(\bar{x})$

(5.10) 4) $\left[\frac{\partial U_\mu^\alpha(x)}{\partial x_\lambda} U_\tau^\lambda(x) - \frac{\partial U_\tau^\alpha(x)}{\partial x_\lambda} U_\mu^\lambda(x) \right] V_\alpha^\sigma(x) = \hat{C}_{\tau\mu}^\sigma$

This (5.10) follows from 3), nothing added. Correspondence with Bargmann:

$$x \leftrightarrow U$$

$$y \leftrightarrow V$$

(5.9) $\left[\frac{\partial V_\beta^\sigma(x)}{\partial x_\gamma} - \frac{\partial V_\gamma^\sigma(x)}{\partial x_\beta} \right] = \hat{C}_{\mu\tau}^\sigma V_\beta^\mu(x) V_\gamma^\tau(x)$

The problem has now been reduced to finding solutions of (5.9) or (5.10). Once you know V , you can construct U , and you can then integrate 3) to get ϕ .

Analogy with

$$\partial_\mu A_\nu - \partial_\nu A_\mu = F_{\mu\nu} \quad 1) F_{\mu\nu} = -F_{\nu\mu}$$

You can solve for A only if: $2) F_{\mu\nu, \lambda} + F_{\nu\lambda, \mu} + F_{\lambda\mu, \nu} = 0$

Then solution has gauge freedom: $A_\mu \rightarrow A_\mu + d\mu \Lambda$

So let $F_{\mu\nu}^\alpha = \hat{C}_{\sigma\tau}^\alpha V_\mu^\sigma V_\nu^\tau$

Then $V_{\mu, \nu}^\alpha - V_{\nu, \mu}^\alpha = F_{\mu\nu}^\alpha \quad (5.9)$

Conditions for solution are same: $\hat{C}_{\sigma\tau}^\alpha = -\hat{C}_{\tau\sigma}^\alpha \quad (5.16)$

and $\hat{C}_{\mu\tau}^\alpha \hat{C}_{\nu\lambda}^\alpha + \dots + \dots = 0 \quad (5.22) \text{ Jacobi}$

Conclusion: if you choose constants $\hat{C}_{\sigma\tau}^\alpha$ which satisfy (5.16) and (5.22), then (5.9) can be solved + you get a Lie algebra! Very nice!

Can show: $\hat{C}_{\mu\nu}^\alpha = \hat{C}_{\mu\nu}^\alpha - \hat{C}_{\nu\mu}^\alpha$
 s.c.'s $\nearrow$ coefficient in ϕ

Lecture 36 Suppose you change parameters. To preserve location of identity, must be:

$$x'_\alpha = f(x) = a_{\beta}^\alpha x_\beta + a_{\beta\gamma}^\alpha x_\beta x_\gamma + \dots$$

Write as two kinds of x forms:

first kind: $x'_\alpha = x_\alpha + a_{\beta\gamma}^\alpha x_\beta x_\gamma + \dots$

second kind: $x'_\alpha = a_\alpha^\beta x_\beta$

You find that under first kind, $c \rightarrow c + 2a_{\beta\gamma}^\alpha$, so

One ~~set~~ ^c constants change, so will the V 's. But $\hat{C}$ do not change under first kind.

The new V 's solve same equation as old V 's, but not the same. By doing parameter changes of 1st kind, you generate all possible V 's. (not proven)

It is possible to go to "normal parameter" where sym. part of $C_{\alpha\beta} = 0$. These parameters will then solve:

$$U_{\beta}^{\alpha}(x) X_{\beta} = X_{\alpha} \quad \text{same with } V.$$

Now second kind. $x'^{\alpha} = A^{\alpha}_{\beta} x^{\beta}$. Assume $|A| \neq 0$.

Then $x' = Ax$ - $\phi' = A\phi$. This time the structure constants do change.

$$\hat{C}' = (A \hat{C} A^{-1}) A^{-1}.$$

But same Lie Group. Normal parameters not unique either.

Lecture 7. Now he starts transformations on an M -space,

$$y' = f(y, x) \quad y \in M.$$

↑
parameters.

f = group of continuous transformations

(you need, of course:
(homogeneous space).

$$f[f(y, x), x'] = f[y, \phi(x, x')]$$

↙ connection to a
Lie Group.

Among the transformations you can do, two are special:

$$f(\xi, y) = \phi(\xi, y)$$

"second parameter group"

$$f(y, x) = \phi(x, y)$$

"first " "

ξ = point in M space.

In terms of transformations, the former purely group results reappear.

$$\frac{\partial f_i(y, x)}{\partial x_\alpha} = V_\alpha^i(f) U_\alpha^i(x)$$

$$f_i(y, x) \quad \begin{matrix} \uparrow \\ y \in \mathcal{M}. \end{matrix} \quad \begin{matrix} \uparrow \\ \text{parameter} \end{matrix}$$

$$f[f(y, x), x'] = f[y, \phi(x, x')]$$

Lecture 8.

$$V_\alpha^i(\xi) = \left[\frac{\partial f_i(\xi, x)}{\partial x_\alpha} \right]_{x=0}$$

$$V_\alpha^i(\xi) \frac{\partial V_\alpha^i(\xi)}{\partial \xi_j} \dots \quad \text{generators:}$$

$$X_\alpha = V_\alpha^i(\xi) \frac{\partial}{\partial \xi_i}$$

The transformations so far need not be linear.

Now assume linear. Then

$$\xi'_i = f_i(\xi, x) = A_{ij}(x) \xi_j$$

$$U_\alpha^i(\xi) = \left[\frac{\partial f_i(\xi, x)}{\partial x_\alpha} \right]_{x=0} = \underbrace{\left[\frac{\partial A_{ij}(x)}{\partial x_\alpha} \right]_{x=0}}_{= I_{ij}^\alpha} \xi_j$$

$$\text{So } U_\alpha^i(\xi) = I_{ij}^\alpha \xi_j.$$

You can then show that these matrices satisfy LA.

$$[I^\alpha, I^\beta] = C_{\alpha\beta}^\gamma I^\gamma, \text{ so } \quad \uparrow \quad \uparrow \quad (8.18)$$

$$X^\alpha = I_{ij}^\alpha \xi_j \frac{\partial}{\partial \xi_j} = \text{generators too, also satisfy (8.18).}$$

So why not start with matrices I^α . Compute V, U , then solve for $f(\xi, x)$.

$$A(x) = e^{x_\alpha I^\alpha} = \text{resultant x-form!}$$

$\uparrow (8.29)$, valid for normal coordinates only!

← good discussion here

The adjoint rep: (sometimes called "regul" rep).

$$I_0 = C_{0\mu}^\lambda \quad \text{works!}$$

Conditions on C 's for existence of

- 1) subgroup
- 2) invariant subgroup.

Table of results:

$$i, j = 1, \dots, m$$

$$\alpha = \dots, m+1, \dots, \nu$$

$$\mu, \lambda = 1, \dots, \nu$$

$$\text{Subgroup} \quad \hat{C}_{ij}^\alpha = 0$$

$$\text{invar. sg.} \quad \hat{C}_{i\lambda}^\alpha = 0$$

$$\text{abel sg.} \quad \hat{C}_{ij}^\mu = 0$$

$$\text{invar. ab. sg.} \quad \hat{C}_{i\lambda}^\mu = 0 \quad \text{and} \quad \hat{C}_{i\lambda}^\alpha = 0$$

Now lectures go into Cartan-Weyl analysis of SSLA, starting lecture 10.

not vectors. OK, enough. 240 pages.

General refs:

Hermann Boerner "Reps of Groups" NH 1963
Nathan Jacobson "L.A.'s" ISP 1962
Dynkin AMST I, 9
Racah Lecture Notes, CERN 61-8 (1961)
Weyl — two books
Pontrjagin — continuous groups 1938
topological groups 1946 (Pineau).

vectors: X^μ Y^μ $\phi^\mu(x,y)$, but set all indices to lower.

matrices: $V_{\alpha\beta}$ $U_{\alpha\beta}$ $\delta_{\alpha\beta}$

equation as derived was: $\partial_\beta \phi_\alpha(x,y) = [U(\phi)V(x)]_{\alpha\beta}$

int conditions: $\partial_i \partial_j \phi_\alpha(x,y) = \partial_j \partial_i \phi_\alpha(x,y)$

∂_i always means $\frac{\partial}{\partial x_i}$. But $\partial_j \phi_\alpha = [U(\phi)V(x)]_{\alpha j}$
 $\partial_i \phi_\alpha = [U(\phi)V(x)]_{\alpha i}$

Thus:

$$\partial_i [U(\phi)V(x)]_{\alpha j} = \partial_j [U(\phi)V(x)]_{\alpha i}$$

work out: $\frac{\partial U(\phi)_{\alpha\sigma}}{\partial \phi^\lambda} [\partial_i \phi^\lambda(x,y)] V_{\sigma j}(x) + U(\phi)_{\alpha\sigma} \partial_i V_{\sigma j}(x) = (i \leftrightarrow j)$

replace:

$$\frac{\partial U_{\alpha\sigma}(\phi)}{\partial \phi^\lambda} [U(\phi)V(x)]_{\lambda i} V_{\sigma j}(x) + U_{\alpha\sigma}(\phi) \partial_i V_{\sigma j}(x) = (i \leftrightarrow j)$$

apply: $\sum_k V_{k\alpha}(\phi)$ on both sides, $\sum_k V_{k\alpha}(\phi) U_{\alpha\sigma}(\phi) = \delta_{k\sigma}$

but this kills sum on σ + freezes $\sigma = k$ in second term.

$$V_{k\alpha}(\phi) \frac{\partial U_{\alpha\sigma}(\phi)}{\partial \phi^\lambda} [U(\phi)V(x)]_{\lambda i} V_{\sigma j}(x) + \partial_i V_{kj}(x) = (i \leftrightarrow j)$$

So write as:

$$A_{k\alpha} V_{\alpha i}(x) V_{\sigma j}(x) + \partial_i V_{kj}(x) = A_{k\alpha} V_{\alpha j}(x) V_{\sigma i}(x) + \partial_j V_{ki}(x)$$

Apply to both sides: $\sum_{ij} U_{ip}(x) U_{jq}(x)$ $\partial_j V_{kj}(x) \frac{U_{ip}(x)}{U_{jq}(x)}$

Get: $A_{k\alpha} \delta_{\alpha p} \delta_{\sigma q} + \partial_i V_{kj}(x) U_{ip}(x) U_{jq}(x) = A_{k\alpha} \delta_{\alpha q} \delta_{\sigma p} + \partial_j V_{ki}(x)$

$$\Rightarrow V_{k\alpha}(\phi) \frac{\partial U_{\alpha\beta}(\phi)}{\partial \phi^\lambda} U_{\lambda\gamma}(\phi) - V_{k\alpha}(\phi) \frac{\partial U_{\alpha\beta}(\phi)}{\partial \phi^\lambda} U_{\lambda\gamma}(\phi) = \partial_i V_{kj}(x) U_{ip}(x) U_{jq}(x) - \partial_j V_{ki}(x) U_{ip}(x) U_{jq}(x)$$

Ans:

$$V_{ka}(\phi) \left[\frac{\partial U_{ar}(\phi)}{\partial \phi^\lambda} U_{rp}(\phi) - \frac{\partial U_{ap}(\phi)}{\partial \phi^\lambda} U_{lr}(\phi) \right] \quad (5.7) = (5.8)$$

$$= [\partial_i V_{kj}(x) - \partial_j V_{ki}(x)] U_{ip}(x) U_{jr}(x) = \hat{C}_{p\lambda}^k$$

So, I have derived (5.9) with $U \leftrightarrow V$.

Next, use $U_\beta^\alpha(x) = \delta_{\alpha\beta} + C_{\beta\sigma}^\alpha X^\sigma + \dots$

$$V_\beta^\alpha(x) = \delta_{\alpha\beta} - C_{\beta\sigma}^\alpha X^\sigma + \dots$$

$$\begin{aligned} \text{So } \hat{C}_{\mu\tau}^{\lambda\sigma} &= \left[\left(\frac{\partial U_\mu^\alpha}{\partial \phi^\lambda} \right)^{\phi=0} U_\tau^\lambda(0) - \frac{\partial U_\tau^\alpha(\phi)}{\partial \phi^\lambda} U_\mu^\lambda(0) \right] V_\alpha^\sigma(0) \\ &= ? \quad \frac{\partial}{\partial \phi^\lambda} [U_\mu^\alpha(\phi)] \approx \frac{\partial}{\partial \phi^\lambda} [\delta_{\alpha\mu} + C_{\mu\nu}^\alpha \phi^\nu + \dots] \\ &= C_{\mu\nu}^\alpha \delta_{\nu\lambda} = C_{\mu\lambda}^\alpha \end{aligned}$$

$$\text{So } \hat{C}_{\mu\tau}^{\lambda\sigma} = C_{\mu\tau}^\sigma - C_{\tau\mu}^\sigma$$

Hermann

1. Introduction: the purpose of this chapter is to learn about the relationship between the first and second kind functions. As his starting point, H reminds us of certain simple formulas which relate the P_j and Q_j functions. He shall generalize these Neuman formulae. The trick for doing so is outlined: you want your e -functions to be matrix elements of an operator $D(g)$ in some appropriate space. For example:

$$E_{jk}(g) = \langle \psi_j | D(g) \psi_k \rangle \quad (1)$$

When label j takes special values (integers, eg), the vectors shown are supposed to span a certain space H^j . This is the usual space connected with D^j , ie,

$$D^j(g) \psi_k = \sum_{k'} D_{kk'}^j(g) \psi_{k'} \quad (2)$$

Now suppose it happens that these basis functions are not really the usual things, and that (j/k') is not just a Kronecker delta, but rather, suppose this quantity is divergent, or otherwise poorly defined. Ie, we are not using a real scalar product here, just a bilinear form. Clearly, then, we do not have an orthonormal basis in our space H^j of dimension $2j+1$. Therefore, you cannot simply substitute equation (2) into (1) and conclude that the E functions are the same as the D functions. This would be the conclusion if we did have an orthonormal basis, so we had better not have that.

Although you cannot put (2) into (1) as shown, you can do what follows:

$$\begin{aligned} E_{jk}(g_0 g) &= \langle \psi_j | D^j(g_0 g) \psi_k \rangle = \langle \psi_j | D^j(g_0) D^j(g) \psi_k \rangle \\ &= \langle D^j(g_0)^{\dagger} \psi_j | D^j(g) \psi_k \rangle \quad (3) \end{aligned}$$

Pause before finishing. The first step follows because $D(g)$ is supposed to be a group representation by operators in our space. The second step involves some kind of "adjoint" operator with respect to this crazy scalar-like product. Now from (2), taking the adjoint we get: wrong! Instead, the next step is this: you can easily prove that the adjoint object is also a representation, the one that H calls $D'^j(g)$. It is this representation which is supposed to do (2) above. If it does, then you can insert (2) (with a prime on D) into (3) to get:

$$\begin{aligned} &= \sum_{k'} D_{k'k}^{j*}(g_0) \langle \psi_{k'} | D^j(g) \psi_k \rangle \\ &= \sum_{k'} D_{k'k}^{j*}(g_0) E_{k'k}(g) \end{aligned}$$

$$E_{jk}(g_0 g) = \sum_{k'} D_{jk'}^j(g_0) E_{k'k}(g)$$

So, what H is showing us is how that "group-like property" might come about. we repeat that there is something rather tricky about the scalar product like form.

2. General Remarks about Cauchy Kernels for Complex Lie Groups.

Here we obtain a method of sort of "extending" a function on G to a new function on G_c (minus G , however), where G_c is the complexification of G . The new function $F(g_c)$ is generated by integration of original $f(g)$ over G , weighted by a special so-called "Cauchy kernel".

By requiring that this kernel have the special form $C(g_c, g) = C(g^{-1}g_c)$, the new function $F(g_c)$ bears a special relation to $f(g)$, namely: if $f(g)$ maps into $F(g_c)$, then $f(g_0g)$ maps into $F(g_0g_c)$. Somehow, left-translation "shines through" this "transformation". Right translation does not, however. (Here, g_0 must of course be in G).

-- Next, for reason unknown, H points out that you can, if you like, also insist that the your Cauchy kernel be "invariant under $\text{Ad } K$ " where K is some subgroup of G . The notation $\text{Ad } K$ means the set of inner automorphisms of G which are generated by the elements of K . I.e, g maps into $k^{-1}gk$ is an automorphism. "invariance under $\text{Ad } K$ " is a fancy way of saying that $C(g_c) = C(kg_ck^{-1})$.

If this condition is also met, then not only will your Cauchy transformation be left-translation "invariant" for all G , it will also be right-translation invariant for K . Of course this implies that for K , the map is right- and left-translation invariant. The significance of all this is: in this case, if $f(g)$ is a spherical function, then so is $F(g_c)$. We have in mind, of course, that $f(g)$ is like $P_j(z)$ and $F(g_c)$ is like $Q_j(z_c)$.

For the record, the phrase "spherical function" means two things

- 1) the function is eigenfunction of differential casimir
- 2) the function is left-and-right translation invariant. Clearly, P_j and Q_j are both spherical.

3. Hua's construction of Cauchy Kernels.

This is a strange digression which is rather vague. Apparently, Hua has a method which gives you a formula for the Cauchy kernel for $G = \text{GL}(n, \mathbb{R})$. H claims, without explaining why, that this nice general method simply does not apply to the Lorentz group cases. Hua was 1963, by the way.

4. Wigner d-functions of the second kind.

I will simply state the gist of this section which is very clear. Suppose you actually have some D and E functions which are related as shown in (4.6) and (4.7) (the E and D functions are so-related!). Then, you can invent operators $E_j(g)$ and $D_j(g)$ in obvious way. It follows, all in reverse, that these operators are related by the Cauchy kernel transformation shown in (4.3), and the actual kernel itself is given in (4.1). However, it is given as a sum over the d and e functions. The point is not that there is some fancy compact form for this Cauchy kernel. The point is that there exists such a kernel, and the D and E operators are so related.

5. Group theoretic meaning of the integral representations.

Here at last we get specific. First H quotes the standard integral reps for the associated P and Q functions and notes that they look similar. Then he sets up a standard (to me, by now) realization of the $SL(2, R)$ Lie algebra as differential operators in a real variable which he calls THETA, but which is very much like the q variable of Mukunda. Notice that you can talk about these generators and their eigenfunctions without using any scalar product. Next, he sets up what he hopes are going to be the e-functions by choosing a particular scalar product, suggested by the integral representation. He says:

$$E_{mm'}^{j-}(\theta) = \int_{-\infty}^{\infty} d\phi e^{m\phi} \exp(i D_j(\phi)) e^{m\phi}$$
$$= \langle m' | e^{i D_j(\phi)} | m \rangle$$

The most important thing to realize at the start is that these are not the states $|m\rangle$ we are used to. These are actually states which diagonalize one of the hyperbolic generators. But, he is taking this generator to be anti-hermitian (formally, and also wrt to his scalar product). That is why he gets m, where I would get ip. He never says his m is real, but I suspect it is supposed to be real. Therefore:

$$\langle m' | m \rangle = \int_{-\infty}^{\infty} d\phi e^{i(m+m')\phi} = \text{undefined!} \neq \delta_{m,m'}$$

After this, H comes up with a very nice trick for exponentiating the differential generator. This was a problem I could never seem to solve. I knew that first order diff gens always corresponded to multiplier reps, but could never do it so explicitly!!!!

So, having shown how to find the multiplier rep, H proceeds to show that the integral in question does in fact lead to the function Q_j^m in its usual integral rep. So huray, he has explicitly written the Q functions as matrix elements of certain operator in some space. Back in section 4 we had done this formally, but we never stated anything explicit for the operators D and E , and nothing explicit for the scalar product. Here, things are made explicit.

Next, we change variables to $w = e^q$ or as H says, $u = e^\theta$. He rewrites the generators in the new variable and notes that these are the ones having to do with the linear fractional transformations of the u -plane. For $SL(2, \mathbb{R})$, these map the real line into the real line, and the circle into the unit circle. Thus, either of these sets of the u -plane is a good candidate for a scalar product (because, functions-on-the-real-line are mapped into functions-on-the-real-line, etc. We have already used the real axis to get the Q rep. It turns out, as one might expect, that the unit circle gives the integral rep for the P functions, and H shows it.

This section is then closed out by H computing the explicit "adjoint" rep and showing the $D = DE$ addition theorem.

6. The technique in general terms.

I am not quite up to this, but will try anyway. You have a group G and a subgroup L , and somehow this gives a space G/L whose points are denoted simply by p . This must be that transitive space M that Vilenkin mentions, the coset space. Recall that all homogeneous spaces must be homomorphic to G/L for some subgroup L .

So, now make up some functions on this M -space, $f(p)$. Then, make a multiplier representation as shown in ~~ex 2~~ (6.1).

Then, take everything complex, so that M goes to space $M_c = G_c/L_c$. Then in this space M_c , take some other manifold, call it N , a certain loci of points p_c . Use this manifold to define a scalar product as shown in p 113.

I am not sure I am getting the point. It seems that you generate "second kind" functions by somehow changing the domain of your scalar product. The reason I am unclear is that I do not see exactly how you make $G/L =$ the unit circle. I will go look at H 's other book to see this. I bet fiber bundles will get in there.

--- Hermann's trick for expo of differential generators.

① We are given $D_\lambda(x) = x \frac{d}{dx} + \lambda x$.

We want $e^{\pm D_\lambda(x)} [e^{m_0}]$. How would you naively compute this?

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} (x \frac{d}{dx} + \lambda x)^n e^{m_0}$$

Rather messy, since $\frac{d}{dx}$ keeps hitting x and x .

② His method is this. Rewrite

$$\begin{aligned} D_\lambda(x) &= (x \frac{d}{dx})^{-\lambda} : x \frac{d}{dx} (x \frac{d}{dx})^\lambda \\ &= \cancel{x} \cdot \cancel{x} \cdot \lambda (\cancel{x})^{-1} x + x \frac{d}{dx} \checkmark \end{aligned}$$

$$\text{Then: } e^{\pm D_\lambda(x)} = e^{\pm (x \frac{d}{dx})^{-\lambda} x \frac{d}{dx} (x \frac{d}{dx})^\lambda}$$

$$= \sum_n \frac{t^n}{n!} [(x \frac{d}{dx})^{-\lambda} x \frac{d}{dx} (x \frac{d}{dx})^\lambda]^n$$

$$= 1 + t (x \frac{d}{dx})^{-\lambda} x \frac{d}{dx} (x \frac{d}{dx})^\lambda$$

$$+ \frac{1}{2!} (t (x \frac{d}{dx})^{-\lambda} x \frac{d}{dx} (x \frac{d}{dx})^\lambda) (t (x \frac{d}{dx})^{-\lambda} x \frac{d}{dx} (x \frac{d}{dx})^\lambda) + \dots$$

$$+ \frac{1}{3!} (t (x \frac{d}{dx})^{-\lambda} x \frac{d}{dx} (x \frac{d}{dx})^\lambda) (t (x \frac{d}{dx})^{-\lambda} x \frac{d}{dx} (x \frac{d}{dx})^\lambda) (t (x \frac{d}{dx})^{-\lambda} x \frac{d}{dx} (x \frac{d}{dx})^\lambda) + \dots$$

= Notice that $(x \frac{d}{dx})^{-\lambda}$ always sits on the left, $(x \frac{d}{dx})^\lambda$ on the right

$$= (x \frac{d}{dx})^{-\lambda} \left[1 + t x \frac{d}{dx} + \frac{1}{2!} (t x \frac{d}{dx})^2 + \dots \right] (x \frac{d}{dx})^\lambda$$

$$= (x \frac{d}{dx})^{-\lambda} \cdot e^{t x \frac{d}{dx}} \left[(x \frac{d}{dx})^\lambda \right] \leftarrow$$

--- The result is that a simpler object is now expo'd!

③ The problem is now reduced to analyzing:

$$e^{t_1 s} \frac{d}{ds} \tilde{f}(s).$$

Now define a new variable $t(s)$ such that

$$s \frac{d}{ds} = \frac{d}{dt}$$

This means $s \frac{ds}{dt} = 1$ which has the solution

$$e^s = \frac{1 + e^{t+c}}{1 - e^{t+c}} \quad \text{OR} \quad \frac{e^s - 1}{e^s + 1} = e^{t+c}$$

Here, $c = \text{integration constant}$. Now, let

$$f(s) = f(s(t)) = \tilde{f}(t).$$

$$\text{Then, } e^{t_1 s} \frac{d}{ds} f(s) = e^{t_1 \frac{d}{dt}} \tilde{f}(t).$$

This is the one expo generator we know how to deal with

$$e^{t_1 \frac{d}{dt}} \tilde{f}(t) = \tilde{f}(t + t_1)$$

$$\text{So, } e^{t_1 s} \frac{d}{ds} f(s) = f(s[t + t_1]).$$

④ Example: if $f(s) = e^{s(t)}$, then

$$f(s[t + t_1]) = e^{s[t + t_1]} = \frac{1 + e^{(t+t_1)+c}}{1 - e^{(t+t_1)+c}}$$

$$= \frac{1 + e^{t_1} \cdot \left(\frac{e^s - 1}{e^s + 1} \right)}{1 - e^{t_1} \cdot \left(\frac{e^s - 1}{e^s + 1} \right)} = \frac{e^s + 1 + e^{t_1} (e^s - 1)}{e^s + 1 - e^{t_1} (e^s - 1)}$$

This proves (5.7).

⑤. Why not do this at the start?

$$sho \frac{d}{ds} + hcho = \frac{d}{ds} ?$$

Not so simple. You really would need $= \frac{d}{ds} [f(s)]$.

$$\text{Then } sho \frac{d}{ds} + hcho = f(s) \frac{d}{ds} + \frac{df}{ds}$$

$$\Rightarrow \frac{sho}{f(s)} = \frac{ds}{ds} \quad \text{and} \quad hcho = \frac{df}{ds}$$

$$\Rightarrow \left. \begin{aligned} f(s) &= sho \cdot \frac{ds}{ds} \\ \frac{df}{ds} &= cho \cdot h \end{aligned} \right\} \Rightarrow \frac{d(\ln f)}{ds} = \frac{cho \cdot h}{sho \cdot (ds/ds)}$$

$$\frac{df}{ds} = sho \frac{d}{ds} \left(\frac{ds}{ds} \right) + \frac{d}{ds} (sho) \frac{ds}{ds} = sho \frac{ds}{ds} \frac{d}{ds} \left(\frac{ds}{ds} \right) + \underbrace{\frac{ds}{ds} \cdot \frac{d}{ds} (sho)}_{cho}$$

$$\text{So } \frac{df}{ds} = sho \frac{ds}{ds} \cdot \frac{d}{ds} \left(\frac{ds}{ds} \right) + cho$$

$$\Rightarrow cho(h-1) = sho \cdot \frac{\frac{d}{ds} \left(\frac{ds}{ds} \right)}{\frac{ds}{ds}} = sho \frac{d}{ds} \left[\ln \left(\frac{ds}{ds} \right) \right]$$

$$\Rightarrow cho(h-1) ds = d \left[\ln \left(\frac{ds}{ds} \right) \right] \quad \text{etc etc.}$$

You could go this way, but much simpler if only one term in your expo.

⑥ Next: $e^{tcho ds} (e^0) = \frac{e^0 + 1 + e^t (e^0 - 1)}{(e^0 - 1)}$

$$\text{So } e^{tcho ds} (sho) = \frac{1}{2} \left[\frac{(e^0 + 1) + e^t (e^0 - 1)}{(e^0 + 1) - e^t (e^0 - 1)} - \frac{(e^0 + 1) + e^t (e^0 - 1)}{(e^0 + 1) + e^t (e^0 - 1)} \right]$$

$$= \frac{1}{2} \left[\frac{\{(e^0 + 1) + e^t (e^0 - 1)\}^2 - \{(e^0 + 1) - e^t (e^0 - 1)\}^2}{(e^0 + 1)^2 - [e^t (e^0 - 1)]^2} \right]$$

$$= \frac{1}{2} \cdot 4 e^t (e^0 + 1) (e^0 - 1)$$

this gives (5.8) with $\rightarrow$

⑦ Assume 5.8-last is OK. Then

$$(sho)^{-\lambda} e^{t sho \frac{d}{ds}} [(sho)^{\lambda} \cdot f(o)].$$

$$= (sho)^{-\lambda} e^{t, d/dt} [sho(t)^{\lambda} f(o(t))].$$

$$= (sho)^{-\lambda} \cdot [sho(t+t_1)]^{\lambda} f(o(t+t_1)).$$

$$= (sho)^{-\lambda} \cdot \left[e^{t_1, d/dt} (sho)^{\lambda} \right] \cdot \left[e^{t_1, d/dt} f(o) \right].$$

$$= \left[e^{t_1, d/dt} (sho)^{\lambda} \right]$$

$$(sho)^{-\lambda} = \left[\frac{sho}{sho + dsho} \right]^{\lambda} \cdot e^{t_1, d/dt} f(o).$$

Therefore:

$$e^{t D_{\lambda}(x)} f(o) = [sho + dsho]^{-\lambda} \cdot e^{t sho \frac{d}{ds}} f(o).$$

And $e^{t D_{\lambda}(x)} e^{mo} = [sho + dsho]^{\lambda} \cdot [e^o \dots]^m.$

⑧ Do not overlook our powerful result:

$$e^{t sho \frac{d}{ds}} [f_1(o) f_2(o)] = \left(e^{t sho \frac{d}{ds}} f_1(o) \right) \times \left(e^{t sho \frac{d}{ds}} f_2(o) \right).$$

This is the great simplifying feature!

① Therefore,

$$\int_{-\infty}^{\infty} dx e^{m'x} \exp(i\lambda(x)) e^{mx}$$

$$= \int_{-\infty}^{\infty} dx e^{m'x} \left[\frac{e^{\theta+1}}{e^{\theta+1}} \right]^m (st\ ch\theta + ch\theta)^{-1}$$

Is this an integral rep for Q_j^{mn} ?

② How did he know this was going to work?? How did he get his scalar product?

addendum.

① We showed that $e^{t_1 \frac{d}{ds}} f(\theta(t)) = f(\theta(t+t_1))$. Now:

$$\theta(t) = \ln \left[\frac{1 + e^{t+c}}{1 - e^{t+c}} \right] \quad \text{so}$$

$$\theta(t+t_1) = \ln \left[\frac{1 + e^{t+t_1+c}}{1 - e^{t+t_1+c}} \right] = \ln \left[\frac{1 + e^{t_1} \left(\frac{e^c - 1}{e^{c+1}} \right)}{1 - e^{t_1} \left(\frac{e^c - 1}{e^{c+1}} \right)} \right]$$

$$= \ln \left[\frac{(e^{c+1}) + e^{t_1}(e^c - 1)}{(e^{c+1}) - e^{t_1}(e^c - 1)} \right]$$

Therefore we have:

$$e^{t \frac{d}{ds}} f(\theta) = f \left(\ln \left[\frac{(e^{c+1}) + e^t(e^c - 1)}{(e^{c+1}) - e^t(e^c - 1)} \right] \right).$$

② Now, using the "product rule" we find:

$$\begin{aligned} e^{t \frac{d}{ds}} f(\theta) &= (shc)^{-\lambda} e^{t \frac{d}{ds}} \left[(shc)^{\lambda} f(\theta) \right] \\ &= (shc)^{-\lambda} \cdot \left[\frac{shc}{shc \, ds + dht} \right]^{\lambda} \cdot f(\ln \dots \end{aligned}$$

Thus:

$$e^{t \frac{d}{ds}} f(\theta) = (shc \, ds + dht)^{-\lambda} f \left(\ln \left[\frac{(e^{c+1}) + e^t(e^c - 1)}{(e^{c+1}) - e^t(e^c - 1)} \right] \right)$$

③ We have produced, from the diff gen, the multiplier rep for this one parameter subgroup!

check. Hermann's (5.8)

① We shall assume (5.7) :

$$e^{+t \operatorname{tanh} \frac{d}{2}}(e^{\theta}) = \frac{(e^{\theta}+1) + e^t(e^{\theta}-1)}{(e^{\theta}+1) - e^t(e^{\theta}-1)}$$

② Note that $\operatorname{tanh} \frac{d}{2}$ is a linear operator; if $A = \text{linear operator}$,
 then A^n also linear; e^A also linear. Therefore:

$$e^{+t \operatorname{tanh} \frac{d}{2}}(\sinh \theta) = \frac{1}{2} \left[e^{+t \operatorname{tanh} \frac{d}{2}}(e^{\theta}) - e^{+t \operatorname{tanh} \frac{d}{2}}(e^{-\theta}) \right].$$

So we need this second term.

③ We showed that $e^{t \operatorname{tanh} \frac{d}{2}} f(\theta) = f(\theta')$ where $\theta' = \theta(t+t_1)$.
 Also, we showed:

$$\theta(t+t_1) = \ln \left[\frac{e^{\theta}+1 + e^{t_1}(e^{\theta}-1)}{e^{\theta}+1 - e^{t_1}(e^{\theta}-1)} \right].$$

Thus: $e^{t \operatorname{tanh} \frac{d}{2}}(e^{\theta}) = e^{\theta'} = \frac{e^{\theta}+1 + e^{t_1}(e^{\theta}-1)}{e^{\theta}+1 - e^{t_1}(e^{\theta}-1)}$ ✓

So: $e^{t \operatorname{tanh} \frac{d}{2}}(e^{-\theta}) = e^{-\theta'} = \frac{e^{\theta}+1 - e^{t_1}(e^{\theta}-1)}{e^{\theta}+1 + e^{t_1}(e^{\theta}-1)}$

④ Shorthand notation:

$$\begin{aligned} e^{\theta'} &= \frac{e^{\theta}(e^{t_1}+1) - (e^{t_1}-1)}{(e^{t_1}+1) - e^{\theta}(e^{t_1}-1)} \\ &= \frac{e^{\theta} \operatorname{sh} \frac{t_1}{2} - \operatorname{sh} \frac{t_1}{2}}{\operatorname{sh} \frac{t_1}{2} - e^{\theta} \operatorname{sh} \frac{t_1}{2}} = \frac{(e^{\theta} - \operatorname{sh} \frac{t_1}{2})}{(1 - e^{\theta} \operatorname{sh} \frac{t_1}{2})} \end{aligned}$$

Thus, $e^{-\theta'} = \frac{1 - e^{\theta} \operatorname{sh} \frac{t_1}{2}}{e^{\theta} - \operatorname{sh} \frac{t_1}{2}} = \frac{e^{-\theta} - \operatorname{sh} \frac{t_1}{2}}{1 - e^{-\theta} \operatorname{sh} \frac{t_1}{2}}$

$$\Rightarrow e^{+t \operatorname{tanh} \frac{d}{2}}(\sinh \theta) = \frac{1}{2} \left[\left(\frac{e^{\theta} - \operatorname{sh} \frac{t_1}{2}}{1 - e^{\theta} \operatorname{sh} \frac{t_1}{2}} \right) - \left(\frac{e^{-\theta} - \operatorname{sh} \frac{t_1}{2}}{1 - e^{-\theta} \operatorname{sh} \frac{t_1}{2}} \right) \right]$$

Now need only simplify:

$$= \frac{1}{2} \left[\frac{(e^0 - \eta \frac{t}{2})(1 - \bar{e}^0 \eta \frac{t}{2}) - (\bar{e}^0 - \eta \frac{t}{2})(1 - e^0 \eta \frac{t}{2})}{(1 - e^0 \eta \frac{t}{2})(1 - \bar{e}^0 \eta \frac{t}{2})} \right]$$

Mult top+bot by $\eta \frac{t}{2}$

$$= \frac{1}{2} \left[\frac{(\eta \frac{t}{2} e^0 - 1)(1 - \bar{e}^0 \eta \frac{t}{2}) - (\eta \frac{t}{2} \bar{e}^0 - 1)(1 - e^0 \eta \frac{t}{2})}{(1 - e^0 \eta \frac{t}{2})(1 - \bar{e}^0 \eta \frac{t}{2})} \right] \quad \text{not useful.}$$

$$= \frac{1}{2} \left[\frac{e^0 + \bar{e}^0 \eta^2 \frac{t^2}{2} - 2\eta \frac{t}{2} e^0 \bar{e}^0 - e^0 - \bar{e}^0 \eta^2 \frac{t^2}{2} + 2\eta \frac{t}{2} \bar{e}^0 e^0}{1 + \eta^2 \frac{t^2}{2} - \eta \frac{t}{2} (e^0 + \bar{e}^0)} \right]$$

$$= \frac{1}{2} \left[\frac{(e^0 - \bar{e}^0) - \eta^2 \frac{t^2}{2} (e^0 - \bar{e}^0)}{1 + \eta^2 \frac{t^2}{2} - \eta \frac{t}{2} (e^0 + \bar{e}^0)} \right] = \frac{1}{2} \left[\frac{\text{sh} \theta - \eta^2 \frac{t^2}{2} \text{sh} \theta}{(1 + \frac{\eta^2 t^2}{2}) - \eta \frac{t}{2} \cosh \theta} \right]$$

$$= \frac{\text{sh} \theta (1 - \eta^2 \frac{t^2}{2})}{1 + \eta^2 \frac{t^2}{2} - 2\eta \frac{t}{2} \cosh \theta} = \frac{\text{sh} \theta (\eta \frac{t}{2} - \eta \frac{t}{2})}{\eta \frac{t}{2} + \eta \frac{t}{2} - 2 \cosh \theta}$$

$$= \frac{\text{sh} \theta \cdot 2 \cosh \theta}{2 \cosh \theta - 2 \cosh \theta} = \frac{\text{sh} \theta}{\sinh (\frac{\cosh \theta}{\sinh \theta} - \cosh \theta)}$$

$$= \frac{\text{sh} \theta}{(\cosh \theta - \sinh \theta)}$$

Therefore:

$$e^{\frac{t \cosh \theta}{\sinh \theta}} (\text{sh} \theta) = \text{sh} \theta (\cosh \theta - \sinh \theta)^{-1}$$

⑤ Now repeat using Hermite's method to locate error:

$$\frac{1}{2} \left[\frac{(e^0 + 1) + e^t (e^0 - 1)}{(e^0 + 1) - e^t (e^0 - 1)} \right] - \left[\frac{(e^0 + 1) - e^t (e^0 - 1)}{(e^0 + 1) + e^t (e^0 - 1)} \right]$$

$$= \frac{1}{2} \left[\frac{[(e^0 + 1) + e^t (e^0 - 1)]^2 - [(e^0 + 1) - e^t (e^0 - 1)]^2}{(e^0 + 1)^2 - e^{2t} (e^0 - 1)^2} \right]$$

$$= \frac{1}{2} \left[\frac{4e^t (e^0 + 1)(e^0 - 1)}{(e^0 + 1)^2 - e^{2t} (e^0 - 1)^2} \right] = \frac{2e^t (e^{2\theta} - 1)}{1 - e^{2t}}$$

Now proceed:

$$\begin{aligned} D &= (e^{\theta} + 1)^2 - e^{2t} (e^{\theta} - 1)^2 \\ &= e^{2\theta} + 2e^{\theta} + 1 - e^{2t} (e^{2\theta} - 2e^{\theta} + 1) \\ &= e^{2\theta} (1 - e^{2t}) + 2e^{\theta} (1 + e^{2t}) + (1 - e^{2t}) \end{aligned}$$

Now mult top by $e^{-t} e^{-\theta}$

$$= \frac{2(e^{\theta} - e^{-\theta})}{e^{\theta} (e^{-t} - e^t) + 2(e^{-t} + e^t) + e^{-\theta} (e^{-t} - e^t)}$$

$$= \frac{2 \sinh \theta}{-e^{\theta} \sinh t + 2 \cosh t - e^{-\theta} \sinh t}$$

$$= \frac{2 \sinh \theta}{2 \cosh t - \sinh t \cdot (e^{\theta} + e^{-\theta}) 2}$$

$$= \frac{\sinh \theta}{\cosh t - \sinh t \cosh \theta}$$

he made a sign err.

Can we reduce general formula down to Newman?

① We are told that $E^j(g_c) = \int_0 d\bar{g} \ C(\bar{g}^j g_c) D^j(\bar{g})$ as operators. Therefore, seems that

$$Q_j(z_c) = \int_0 d\bar{g} \ C(\bar{g}^j g_c) P_j(\bar{z})$$

since $\langle 0 | D^j(\bar{g}) | 0 \rangle = \bar{e}^0 \cdot e^0 \cdot d_{\infty}^j(\bar{z})$. Not at all clear how this will work. Let

$$d\bar{g} = (8\pi^2)^{-1} d\alpha d\beta d\gamma. \text{ Then}$$

$$Q_j(z_c) = (8\pi^2)^{-1} \int_{-1}^1 d\alpha \ P_j(\alpha) \cdot \int_0^{2\pi} d\alpha \int_0^{2\pi} d\gamma \cdot \sum_{j'} (2j'+1) \sum_{m'=-j'}^{j'} E_{m'm'}^j(\bar{g}^j g_c)$$

Now,

$$\int_0^{2\pi} d\alpha \int_0^{2\pi} d\gamma \cdot D_{mm'}^j(\bar{g}^j)$$

$$\sum_m D_{mm'}^j(\bar{g}^j) E_{mm'}^j(g_c)$$

$$= (2\pi)^2 \delta_{m',0} \delta_{m,0} d_{\infty}^j(\bar{z}) \text{ so RHS is:}$$

$$Q_j(z_c) = \frac{1}{2} \int_{-1}^1 d\alpha \ P_j(\alpha) \cdot \left\{ \sum_{j'} (2j'+1) Q_{j'}(z_c) \right\} = Q_j(z_c) \quad \checkmark$$

Yes, it works.

② Is there some better way to compute C to see the Newman $\frac{1}{z-z'}$?

$$\begin{aligned} Q_j(z_c) &= \frac{1}{2} \int_{-1}^1 d\alpha \ P_j(\alpha) \cdot \left[\frac{1}{(2\pi)^2} \int_0^{2\pi} d\alpha \int_0^{2\pi} d\gamma \cdot \sum_{j'} (2j'+1) \sum_m \sum_{m'} D_{mm'}^j(\bar{g}^j) E_{mm'}^j(g_c) \right] \\ &= \sum_{j'} (2j'+1) P_j(\alpha) Q_{j'}(z_c) \\ &= \frac{1}{z_c - z} \end{aligned}$$

It is so simple because m -stuff drops out.

Hermann and $E = ED$ etc.

- ① First of all, what sum rules do we really know of? In AG, I see this rule:

$$E_j^{mm'}(R_3) = \sum_{m''=m, m \pm 1, \dots} D_j^{mm''}(R_2) E_j^{m''m'}(R_1)$$

where $R_3 = R_2 R_1$. Not really clear where the various R 's lie w.r.t. $SO(3, \mathbb{C})$, say, but sum is clear.

- ② Herman says:

$$E_j^{mm'}(g g_c) = \sum_{m''} d_j^{mm''}(g) E_j^{m''m'}(g_c)$$

so I presume that g itself must lie in $SU(2)$, or at least stands out there.

- ③ What about the $E = ED$ rule? If $K = SO(2, \mathbb{R})$, then

$$d_j^{m''m'}(k) = e^{-im''\theta(k)}$$

where I take $k = (0, 0, \theta(k))$ in Euler angles.

Similarly, if we put $g_c = (\alpha, \beta, \gamma)$ in complex Euler angles, then $g_c k = (\alpha, \beta, \gamma + \theta k)$

$$E_j^{mm'}(g_c k) = E_j^{mm'}(g_c) \cdot e^{-im'\theta k}$$

$$D_j^{m''m'}(0, 0, \theta k) = \delta_{m''m'} e^{-im''\theta k}$$

Therefore, the $E = ED$ rule for subgroup $K = SO(2, \mathbb{R})$ is rather trivial.

④ Since we have demonstrated (4.6) and (4.7) for each $j = 0, \frac{1}{2}, 1$, etc. (true j), we should be able to get the fancy Cauchy kernel from (4.1)

First, make sure norms are OK. We say

$$\begin{cases} \hat{f}(j) = \int_G dg f(g) D_j(\bar{g}') \\ f(g) = \sum_j c(j) \text{trace} [\hat{f}(j) D_j(g)] \end{cases}$$

This means, of course, in each j space

$$\hat{f}_{mm'}^j = \int_G dg f(g) D_{mm'}^j(\bar{g}')$$

$$f(g) = \sum_j c(j) \sum_{m=-j}^j \sum_{m'=-j}^j \hat{f}_{mm'}^j D_{m'm}^j(g).$$

where we show the trace. Then

$$f(g) = \sum_j c(j) \sum_{m, m'} [\int_G dg' f(g') D_{mm'}^j(\bar{g}')] D_{m'm}^j(g)$$

But

$$\begin{aligned} \sum_{m, m'} D_{mm'}^j(\bar{g}') D_{m'm}^j(g) \\ = \text{trace} [D^j(\bar{g}'g)] \end{aligned}$$

Well I know that

$$\sum_{j=0}^{\infty} \text{trace} [D^j(\bar{g}'g)] \cdot \frac{(2j+1)}{8\pi^2} = \delta(g'-g) \quad \uparrow \text{defined by me.}$$

If you want $\int_G dg = 1$, then you need

$$dg = d(\phi) d(\phi') d(\cos\theta) \times \frac{1}{8\pi^2}$$

$$\Rightarrow \delta(g-g') = \delta(\phi-\phi') \delta(\phi'-\phi') \delta(\cos\theta-\cos\theta') \cdot 8\pi^2$$

Then $\int dg f(g) \delta(g-g_1) = f(g_1)$

So, with this $\delta(g-g_1)$, I know:

$$\sum_{j=0}^{\infty} (2j+1) \text{trace} [D_j(\bar{g}'g)] = \delta(g'-g).$$

Thus, it must be that $c_j = 2j+1$ ✓

⑤ So, our Cauchy kernel is

$$\begin{aligned} C(g'g) &= C(\bar{g}'g_c) = \sum_{j=0,1,\dots}^{\infty} (2j+1) \text{trace} [E_j(g_c) D_j(\bar{g}')] \\ &= \sum_{j=0,1,\dots}^{\infty} (2j+1) \sum_{m=-j}^j \sum_{m'=-j}^j \underbrace{E_{mm'}^j(g_c)}_{\substack{\text{problem direction} \\ \uparrow}} \underbrace{D_{m'm}^j(\bar{g}')}_{\substack{\text{but} \\ = D_{m'm}^j(\bar{g}') E_{mm'}^j(g_c)}} \end{aligned}$$

Notice that this is the "problem direction" but

$$= D_{m'm}^j(\bar{g}') E_{mm'}^j(g_c)$$

Use up the $\sum_m$ to tie together, so

$$C(\bar{g}'g_c) = \sum_{j=0,1,\dots}^{\infty} (2j+1) \sum_{m=-j}^j E_{mm'}^j(\bar{g}-g_c)$$

So, in terms of E-functions, here is your Cauchy kernel.
Then

$$E^j(g_c) = \int_G dg C(\bar{g}'g_c) D^j(g)$$

The notion of Cauchy kernels.

① I am reading randomly in Hermann's book: "Fourier Analysis on Groups and PNA", Benjamin 1969, QC174.5 H46

② Suppose you have a function $f(g)$ defined for $g \in G = \text{some group}$. Let $G_c = \text{complexification of } G$. Consider

$$f(g_c) = \int_G dg \, C(g_c, g) f(g)$$



where $g_c \in G_c - G$. This extends, via G , the definition of f to the area shown, $G_c - G$. This looks like a Cauchy formula where $C(g_c, g)$ is like the $\frac{1}{z_c - z}$, so $C(g_c, g)$ is called a Cauchy Kernel.

③ Now consider the standard Fourier pair

$$\begin{cases} \hat{f}(\lambda) = \int_G dg \, f(g) D_\lambda(g) & \lambda \in \Lambda. \\ \text{where } D \text{ and } \hat{f} \text{ are operators in some space } H_\lambda. \end{cases}$$

pair

$$\begin{cases} f(g) = \int_{\Lambda_0} d\lambda \, \text{trace} [\hat{f}(\lambda) D_\lambda(g)] c(\lambda) \\ \uparrow \text{some portion of full } \Lambda \quad \uparrow \text{measure.} \end{cases}$$

④ Now, the idea is to combine this standard pair with the Cauchy-extension idea. You get:

$$f(g_c) = \int_{\Lambda_0} d\lambda \, c(\lambda) \cdot \text{trace} [\hat{f}(\lambda) D_\lambda(g_c)]$$

where of course

$$D_\lambda(g_c) = \int_G dg \, C(g_c, g) D_\lambda(g)$$

extends $D_\lambda(g)$.

⑤ Now, suppose you put $g_c = g_u \in G_u$ where G_u is some compact subgroup in G_c . Then somehow you know that

$$f(g_c) = \sum_{\lambda} d_{\lambda} c(\lambda) + [\hat{f}(\lambda) D_{\lambda}(g_c)]$$

has to reduce to a sum, because G_u has discretely labelled reps. This means $c(\lambda)$ has poles, and the residues $\rightarrow$ reps of G_u , of course.

I see a glimmer of something useful here!

⑥ Think of $G = SO(2,1)$, $G_u = SO(3)$. $G_c = SL(2, \mathbb{C})$.
a $SO(3, \mathbb{C})$. We have discussed the idea

$$D_1(g_c) = \int_G dg C(g_c, g) D(g)$$

Can we arrange so

$$D_1(g_c g'_c) = D_1(g_c) D(g'_c) \quad ?$$



This seems to require that $C(g_c, g)$ have a special property:

$$C(g_c g'_c, g) = \int_G dg_1 C(g_c, g_1) C(g'_c, g_1^{-1} g)$$

I.e., "convolution form" in second variable.

Let $C(g_c, g) = C_{g_c}(g)$ to show second variable.

$$C_{g_c g'_c}(g) = \int_G dg_1 C_{g_c}(g_1) C_{g'_c}(g_1^{-1} g)$$

$$\text{So } C_{g_c g'_c}(g) = [C_{g_c} * C_{g'_c}](g)$$

Very fancy object $\uparrow$.

⑦ The Basic Example:

$$G_c = \mathbb{C}$$

$$G = \mathbb{R} \quad \text{under addition}$$

$$G_m = \mathbb{I}.$$

Consider $C(z, t) = \frac{1}{2\pi i} \frac{1}{(t-z)}$

$$f(z) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} dt \frac{1}{(t-z)} f(t)$$

extends $f(t)$ to complex $z \in \mathbb{C} - \mathbb{R}$. $dt = dg$. $G = \mathbb{R}$.

Does this have the fancy property in ⑥?

$$C(z, t) = \int_{-\infty}^{\infty} dt_1 C(z, t_1) C(z', -t_1 + t)$$

$$\frac{1}{t - z z'} = \frac{1}{2\pi i} \int_{-\infty}^{\infty} dt_1 \frac{1}{(t_1 - z)} \frac{1}{(-t_1 + t - z')}$$

Is it true? Only if z, z' both are in UHP. Example of a "sub-semi-group".

Next, how is group rep "extended" $\swarrow$ a rep of G .

$$D_{\lambda}(z) = \int_{-\infty}^{\infty} dt \frac{1}{2\pi i} \frac{1}{(t-z)} e^{\lambda t} = e^{\lambda z} \quad \uparrow$$

Very nice. This was all in chapter two.

⑧ Now into ch3: "Cauchy Integrals on Lie Groups".

$G = \text{group}$

$D(g) = \text{operator in space } H.$

$\langle \psi | \psi' \rangle = \text{herm. bilinear } \psi, \psi' \in H.$

$\swarrow$ symbol for herm. adjoint

Assume adjoint exists for operator O : $\langle O^* \psi | \psi' \rangle = \langle \psi | O \psi' \rangle$

Let $\sigma = D(g)$ eq. Then $\langle D(g)^* \psi | \psi' \rangle = \langle \psi | D(g) \psi' \rangle$.
 Then $\bar{D}(g) = D(g^{-1})^*$ is a new rep.

Let $H' \equiv$ finite subspace of H , basis $\psi_1, \dots, \psi_n$.

Define: $e_{jk}(g) \equiv \langle \psi_j | D(g) \psi_k \rangle$.

We do not assume that $D(g): H' \rightarrow H'$. I.e., H' is not an invariant subspace of rep space G , so $e_{jk}(g)$ is not a matrix representation.

But we are to assume that $\bar{D}(g): H' \rightarrow H'$, so H' is invariant under the adjoint rep, though not D itself. Thus.

$$\bar{D}(g) |\psi_j\rangle = \sum_k d_{kj}(g) |\psi_k\rangle$$

where $d_{kj}(g)$ is a matrix rep of $\bar{D}$. Now watch:

$$e_{jk}(\bar{g}_0 g) = \langle \psi_j | D(\bar{g}_0 g) \psi_k \rangle$$

$$= \langle \psi_j | D(\bar{g}_0) D(g) \psi_k \rangle = \dots \quad \text{why?}$$

This seems to assume that D has the group property. Oh yes, D is a rep on H still.

$$= \langle D(\bar{g}_0)^* \psi_j | D(g) \psi_k \rangle$$

$$= \langle \bar{D}(\bar{g}_0) \psi_j | D(g) \psi_k \rangle$$

$$= \sum_{k'} d_{k'j}(\bar{g}_0) \langle \psi_{k'} | D(g) \psi_k \rangle$$

$$= \sum_{k'} d_{k'j}(\bar{g}_0)^* e_{k'k}(g)$$

S

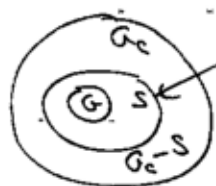
$$e_{jk}(g_0^{-1}g) = \sum_k d_{kj}(g_0) e_{kk}(g)$$

Under "left translation", e transforms as does d .

⑨ So H was invariant under $D(g) \Rightarrow$

This e_{jk} is called, in general, "second kind".
end section 1.

⑩ Now let



$S =$ only a "closed subset" of G_c .

Invent $C(g_c, g)$, analytic in g_c .

Define with this Cauchy kernel: as usual,

$$F(g_c) = \int_G dg \cdot C(g_c, g) f(g)$$

Now, suppose you demand that

$$F(\underline{g_0 g_c}) = \int_G dg \cdot C(g_c, g) f(\underline{g_0 g})$$

left-translation invariance
for $g_0 \in G$.

where $g_0 \in G$ so $g_0 g \in G$ + this is sensible. (He calls this "intertwine"). Then you need:

$$C(\underline{g_0 g_c}, g) = C(g_c, \underline{g_0^{-1} g})$$

Some sort of invariance condition on C . Try.

$$C(g_c, g) = C(\bar{g}^{-1} g_c).$$

Then $C(g_0 g_c, g) = C(g^{-1} g_0 g_c)$
 $C(g_c, g_0^{-1} g) = C((g_0^{-1} g)^{-1} g_c)$ ✓ it works.

- ⑪ Now, in addition to left invariance on G , he demands right invariance. & $k \in K$, where $K \equiv \text{subgroup of } G$.

$$F(g \underset{\sim}{k}) = \int dg \underbrace{C(g, g)}_{C(\tilde{g}^T g_c)} f(g \underset{\sim}{k})$$

To get this also, you need

$$C(k g c k^{-1}) = C(g_c) \quad \text{"equivariance condition."}$$

- ⑫ So what? Take example:

$$\begin{aligned} G_c &= \text{SO}(3, \mathbb{C}) \\ G &= \text{SO}(3, \mathbb{R}) \\ K &= \text{SO}(2, \mathbb{R}). \end{aligned}$$

If $f(g)$ is invariant left and right under k , and is also a Casimir eigenfunction, then $f(g)$ is a spherical function.

Assume that $f(g)$ is such a function on G with respect to K . What, then, is

$$F(g_c) \text{ as defined above}$$

- a) right invariant under K .
- b) left invariant under all G . (hence also K)

Well you can show that $F(g_c)$ is also a Casimir eigenfunction, right + left invar under K

$$\begin{aligned} \text{Claims: } F(e) &= \int_G \underbrace{C(e, g)}_{C(\tilde{g}^T e)} f(g) dg \\ &= \int_G C(\tilde{g}) f(g) dg. \end{aligned}$$

claims this is singular, like $\Delta(\mathbb{R}^2)$.

⑬ In fact, we shall get this correspondence:

$$F(g) = \int dg \ C(g, g) f(g)$$

$$\updownarrow = \int dg \ C(\bar{g}^1 g) f(g)$$

$$Q_L(z) = \int_{-1}^{+1} dz' \ \frac{1}{z^* - z'} \ P_L(z)$$

where dg = usual Euler. So Neumann is connected to a certain special "Cauchy extension."

end section 2

⑭ Construction of Hua:

V = finite dim space.

E = operators on V .

Imagine this weird object

$$\text{map: } V \otimes V \rightarrow \mathbb{C}$$

$$(v_1/v_2) = \text{some complex \#.}$$

$$= \text{a bilinear form.}$$

Wants it to be anti linear
(looks linear to me.)

$$(cv_1/v_2) = c^* (v_1/v_2)$$

$$(v_1/cv_2) = c (v_1/v_2)$$

Define an adjoint-like object

$$A \in E, A^* \in E$$

$$(A^* v_1/v_2) = (v_1/A v_2)$$

$$\forall v_1, v_2 \in V$$

Then certain rules are true like

$$(AB)^* = B^* A^*$$

"Positive" means

$$(v|Av) > 0 \quad \forall v \in V$$

I.e., " $A > 0$ "

$A = \text{"unitary"} \quad \tilde{A} = A^*$

Now decompose E , the space of operators, into pieces



here, if $A \in E_+$, then the operator A^*A^{-1} is "negative"

Claim: E_+, E_- are semigroups! /// C

Proof: Let $A, B \in E_+$. Then $\begin{matrix} A^*A^{-1} < 0 \\ B^*B^{-1} < 0. \end{matrix}$ D

Then $(AB)^*(AB)^{-1} = \underbrace{B^*CB}_{< 0} + \underbrace{D}_{< 0} \quad \sim < 0.$

Q.E.D.

⑮ So what? Well, let $D(g_c)$ be an extended rep of G_c , where $D(g)$ is unitary rep $g \in G$. Let $D(g_c)$ be operators in the space E .

Let $g(g_c) =$ an operator in E , say.

Consider $C(g_c, g) = [\det(1 - g(g_c)g(g^{-1}))]^{-n}$

where, I presume, $n = \dim V$.

$f(g) \rightarrow F(g_c)$ under this ↑

Oh! $g(g_c)$ is a representation of G_c as operators in the finite dim space V , so $g(g_c)$ is a matrix rep of $G_c = GL(n, \mathbb{C})$ eg.

The kernel so obtained. I presume has certain desired properties. like

$$C(g_0, g) = C(\tilde{g}^{-1} g_0)$$

We are told that this all works great when

$$G_0 = GL(n, \mathbb{C})$$

$$G = U(p, q) \quad p+q=n. \quad \text{end section 3}$$

But it works poorly in $SL(2, \mathbb{C})$, which means after.

④.1 Section 4: G, K, G_0 . Let $C(g_0) =$ function on G_0 (an open subset).

want. We want $C(g)$ to have certain conditions, namely

1) analytic in variables g .

2) invariant under K in this sense $C(k g_0 k^{-1}) = C(g_0)$

3) differentiable ∞ .

Pause a moment. This was the condition for a cauchy kernel to preserve right translation under extension, as on p 93.

$G =$ compact, reps. are $D_j \quad j=0, 1, \dots$. Treat $C(\tilde{g}^{-1} g_0)$ as function on G , then expand it as usual:

$$f(g) = \int_0^\infty d\lambda \alpha(\lambda) \tau[\hat{f}(\lambda) D_\lambda(g)]$$

But what is (4.1)? There is no $\hat{f}(\lambda)$. He requires

$$(*) \quad C(\tilde{g}^{-1} g_0) = \sum_j c(j) \text{ trace } \left[\int dg D_j(\tilde{g}) C(\tilde{g}^{-1} g_0) D_j(\tilde{g}^{-1}) \right] \\ \equiv E_j(g_0).$$

OK. He wants now to define the kernel in terms of E_j .

④.2 Notice: $E_j(g_0) = \int_G dg \quad C(\tilde{g}^{-1} g_0) D_j(g)$

$$\Rightarrow E_j(g_0 g_0) = D_j(g_0) E_j(g_0) \leftarrow \text{trivial to show, don't}$$

need any properties.

E on the right.

But you can also show, using the k-condition

$$E_j(g_k) = E_j(g_c) D_j(k)$$

$\uparrow$ $\uparrow$
 $\in K$ E on the left
only

Now, let $|\psi_1\rangle \dots |\psi_n\rangle =$ basis in the rep. space H_j , so $n = \dim H_j$. Then we get

$$d_j^{mm'}(g) = \langle m | D_j(g) | m' \rangle$$

$$e_j^{mm'}(g) = \langle m | E_j(g) | m' \rangle$$

now let's see

$$e_j^{mm'}(g g_c) = \sum_{m''} d_j^{mm''}(g) \cdot e_j^{m''m'}(g_c)$$

$$e_j^{mm'}(g_k) = \sum_{m''} e_j^{mm''}(g_c) d_j^{m''m'}(k)$$

① So here is the idea. Find some functions which work here. Then go backwards to define operators $D_j(g)$, $E_j(g)$ via matrix els. This, you also must have your Cauchy kernel via $(*)$ (see last page)! Very nice!

But what is K ?

. end section 4

Section 5: Let $H =$ functions $f(\phi)$. $G = SL(2, \mathbb{R})$.
 algebra X, Y, Z . Some generators given. $\uparrow$ $\text{twiddle} \Rightarrow \text{Lie Algebra}$
 real:

$$\begin{aligned} \vec{Z} &= \partial/\partial\phi \\ \vec{X} &= \sinh \partial/\partial\phi + \lambda \partial/\partial\psi \\ \vec{Y} &= \cosh \partial/\partial\phi + \lambda \sinh \end{aligned}$$

$\psi_m = e^{m\phi}$ Now look at matrix elements:

$$e(m, m', t, \lambda) \equiv \int_{-\infty}^{\infty} d\phi \cdot e^{m\phi} \exp(t \vec{X}) e^{m'\phi}$$

Note: the differential generators should really carry a label, he uses

$D_\lambda(X)$ eg.

$\nwarrow$ W in G .
 $D_\lambda(W)$ ok.

(5.2) Now all of a sudden we have $D_\lambda(g) = e$.

$$e(m', m, g) = \int_{-\infty}^{\infty} d\phi \cdot e^{m'\phi} D_\lambda(g) e^{m\phi} = \langle e^{m'\phi} | D_\lambda(g) | e^{m\phi} \rangle.$$

Now claim:

$$e(m', m; \vec{g}_0, \vec{g}) = \sum_{m''=-l}^l d(l, m'', m', g_0) e(m'', m, g)$$

ie, the finite dim reps appear (of $SL(2, \mathbb{R})$). The e -functions transform under left translation via a finite dim rep, so λ must be integer.

You do not have full right translation except under K which is

$$e^{t\vec{Z}} \quad (\text{ie, } e^{t\vec{D}_3}).$$

(5.3) Summarize: We have been looking for an operator such that

$$e_s^{mm'}(g) = \langle m | E_j(g) | m' \rangle.$$

His candidate is -

$$\int_{-\infty}^{\infty} d\theta \, e^{-im\theta} e^{iD_X(X)} e^{im\theta}$$

After lots of algebra, he shows that

$$e^{iD_X(X)}(e^{im\theta}) = \frac{[e^{\theta} + 1] e^{i\theta} (e^{\theta} - 1)^m}{[(e^{\theta+1}) - e^{\theta}(e^{\theta}-1)]} (sh + ch + ct)^{-\lambda}$$

↑ a hard result.

Then $\int_{-\infty}^{\infty} d\theta \, e^{-im\theta} [e^{iD_X(X)}(e^{im\theta})] \Big|_{m \rightarrow 0}$

$$= 2 \int_0^{\infty} d\theta \, ch(im\theta) (sh + ch + ct)^{-\lambda}$$

$-j-1 \rightarrow$

$$\sim Q_{j+1}^{m'}(ct)$$

So he really has obtained Q as a "matrix element".

Meanwhile

$$\int_0^{2\pi} d\theta \, e^{-im\theta} [e^{iD_X(X)} e^{-im\theta}] \Big|_{m \rightarrow 0} = P \text{ function.}$$

So she gets both "kinds" of functions from the same type of integral. Different scalar product.

(5.1) The more general idea:

G = real, noncompact, SSLG

L = subgroup

G/L = "boundary homo space"

$$\begin{aligned} &\rightarrow G^c \\ &\rightarrow L^c \\ &\rightarrow G^c/L^c \end{aligned}$$

Now, consider functions on G/L . I.e., let $p \in G/L$, and set up

$$[D_X(g) \psi](p) = [m(g,p)]^{\lambda} \psi(g^{-1}p)$$

ie, set up standard multiplier rep on $M \equiv G/L$.
 The space of fctns on M is called H . Now try to
 continue

$$g \rightarrow g_c$$

Let $N =$ piece of G/L_c . Define

$$\langle \chi | \chi' \rangle = \sum_N dp \overline{\chi(p)} \chi'(p). \quad dp = \text{volume in } G/L.$$

Now try to get $D_\lambda(\tilde{g})^{* \text{ (adj) }} = D_{\lambda'}(g)$

where $\lambda' = \lambda'(\lambda)$. Why want this $\uparrow$? Chapter ends abruptly.

OK, was interesting, needs "restudy".

Chapter IV: "deformation of the Fourier integral from compact to non-compact".

2. Consider at once $SO(2,1)$. X_1, X_2, X_3 .

$$g(t) = e^{tX_3} = \text{usual } O(2) \text{ subgroup } [SO(2,1)].$$

Define: $K_t = g(t) K g(-t) = e^{tX_3} K e^{-tX_3}$

where $K =$ some operatn.

Wait: $\{e^{\theta X_1}\} =$ subgroup called K (compact).

$$\{e^{a(X_1+X_2)}\} = \text{subgroup } L.$$

Define $K_t = ~~g(t)~~ e^{tX_3} [K] e^{-tX_3}$

He says as $t \rightarrow \infty$, $K_t \rightarrow L$ as subgroups.

Let $f(g)$ be defined on G , hence on $K_t = \text{sg of } G$.

Claim:

$$f(k_t) = \sum_{j=-\infty}^{\infty} (a_j^t) \cdot e^{ij\theta(k_t)}$$

ie, θ is the label of our element k_t . This is just a fourier series expansion on this subgroup.

The idea is to let $t \rightarrow \infty$ and watch what happens.

$$\lim_{t \rightarrow \infty} f(k_t) = f(l)$$

where $l = \lim_{t \rightarrow \infty} k_t$. Then you will end up with

$f(l) =$ an integral, not a sum.

OK, this chapter is about a method. This application is $SO(3) \rightarrow E(2)$, as notation suggests. See this $\rightarrow$ for later ref.

Chapter 5: "pwa of scatt amp."

① a note on Dual Space. Let ψ = elements of a Hilbert Space H .

Let $\mathcal{D}$ be the space of all linear functionals defined on H . Then $\mathcal{D}$ is a "Dual Space" for H .

Let $\alpha \in \mathcal{D}$. Then $\langle \alpha | \psi \rangle = \alpha(\psi)$ = "the value of functional α on ψ "

Insist that $\langle \alpha | \psi \rangle = \langle \psi | \alpha \rangle^*$.

Define new functional $(c\alpha)$: $c\alpha(\psi) = c^* \alpha(\psi)$

Then $\mathcal{D}$ = a complex vector space.

② Let $A: H \rightarrow H$. Suppose $A^*: H \rightarrow H$ also. Then

$$\langle A\alpha | \psi \rangle = \langle \alpha | A^* \psi \rangle$$

$$A\alpha(\psi) = \alpha(A^*\psi)$$

So A generates from $\alpha \in \mathcal{D}$ a new $\alpha' = A\alpha \in \mathcal{D}$.
So $A: \mathcal{D} \rightarrow \mathcal{D}$, if you like.

This process is referred to as "the extension of operator A from the space H to the space $\mathcal{D}$."

③ Now, let $A = D(g)$, a rep of some group G . I.e., $D(g)$ is a rep. on H . Claim: The extension is also a rep on $\mathcal{D}$. Seems reasonable.

Let $D(x)$ = differential generates on H space. Extnd to space $\mathcal{D}$ as usual.

① Now, G = group

A = abelian subgroup.

$\mathcal{C}$ = center of the "universal enveloping algebra $U(\mathcal{C})$ of G "

Huh? See page 72 of LGFP. There, $D(x)$ = differential generates on space $\mathcal{V}$ (of functions).

If $X, Y \in \mathcal{C}$ (a L.A.), what is (XY) ? No such object, formally speaking. The U.E.A. is a gimmick to give meaning to XY .

Let $U(\mathcal{C})$ = U. Assoc. E.A. of $\mathcal{C}$ be defined by

A) elements of $U(\mathcal{C})$ are elements of $\mathcal{C}$

B) rules:

$$X(YZ) = (XY)(Z)$$

$$[X, Y] = XY - YX$$

recall that this already was defined in G new objects here.

So, whenever you actually write out $XY - YX$, you are really in the U.E.A.

"Casimir Operators" $\equiv$ the center of $U(\mathfrak{g})$, i.e.

$$\uparrow D \in \text{set of Casimirs} \Rightarrow XD = DX \quad \forall X \in \mathfrak{g}.$$

So OK, $\mathcal{L} = \text{casimir of } \mathfrak{g}$.

So far we had $\mathcal{L}$ as generators representing $\mathfrak{g}$.
Extend these to $U(\mathfrak{g})$, no big deal.

$$\underline{\text{So:}} \quad \begin{aligned} D(A) &= \text{abelians} \\ \mathcal{L}(\mathfrak{g}) &= \text{casimirs} \end{aligned}$$

All mutually commute.

Note: since Casimirs are "quadratic", you have to go to the $U(\mathfrak{g})$ to talk about them.

⑤ Where are we? Let $p \in \mathcal{M} \quad \psi_p \in \text{some } H$.

$$\text{Then we want } \left. \begin{aligned} D(a) \psi_p &= \lambda(a, p) \psi_p \\ \mathcal{L}(\Delta) \psi_p &= \lambda(\Delta, p) \psi_p \end{aligned} \right\}$$

where $\Delta = \text{a casimir} \quad \Delta \in \mathcal{L}$
 $a = \text{an abelian}$

I.e., ψ_p diagonalize both. Here $\lambda = \lambda(A, \mathcal{M})$

$$\lambda: A \otimes \mathcal{M} \rightarrow \mathbb{C}.$$

Hamermesh

3-1. Linear vector spaces.

3-2. Linear dependence.

3-3. Notion of basis vectors in a space.

3-4. Linear operators, equivalence of two matrices (similarity).

3-5. Representation of a group by matrices. Multiplication formula.

3-6. Character.

Notion of two representations being equivalent by similarity. Character is easily seen to be invariant under similarity, so a useful quantity. Called CHI. Notation is:

$\chi(R)$ = the character in some rep for group element R

$K_1, K_2, \dots, K_v$ = the classes.

Elements of same class have same character.

3-7. Notion of direct sum representation. Block form.

3-8. Idea of a function being invariant under some group (belongs to identity rep). QM.

3-9. Scalar products, unitary matrices.

3-10. Adjoint and Unitary operators.

3-11. Theorem: for finite groups, you can always arrange all representations to be unitary. I.e., every representation is equivalent to a unitary rep.

3-12. Definition of a Hilbert Space.

3-13. Reducibility.

Fully reducible means block form. This is a direct sum. You keep looking for invariant subspaces until there are no more, then you have fully reduced to irreducibles.

a_i = multiplicity of some rep in a reduction.

3-14. Shur Lemmas.

1. Two UIR's of different dimensions cannot be intertwined by any matrix.

2. The only matrix which intertwines a UIR with itself is the identity.

3-15. Orthogonality Relations.

Here based on the Shur lemmas is an excellent derivation of the Great Orthogonality Theorem: $\sum_R \chi^u(R) \chi^v(R)^* = g \delta_{uv}$

$$\sum_R D^u(R)_{il} D^v(R)_{jm}^* = (g/n_u) \delta_{uv} \delta_{ij} \delta_{lm} \quad (3-143)$$

From this, character orthogonality is derived.

UIR's:

$$\sum_R \chi^u(R) \chi^v(R)^* = g \delta_{uv}$$

, can also write as class sum.

Theorem: number of UIR's equals number of classes

Theorem: sum of squared of dim's of UIR's = g, the order

$$\sum_i \chi_i^{(\mu)} \chi_i^{(\nu)*} g_i \quad \# \text{ in class } i$$

$$\chi^{(\nu)}(R)^* = \chi^{(\nu)}(R^{-1}) \text{ since unitary}$$

$$= g \delta_{\mu\nu}$$

13-16. If a rep is a sum of reps, the characters add in the obvious way. Thus:

$$D(R) = \sum_{\nu} a_{\nu} D^{(\nu)}(R)$$

a_{ν} = multiplicity

$$\Rightarrow \chi(R) = \sum_{\nu} a_{\nu} \chi^{(\nu)}(R) \quad \text{or} \quad \chi_i = \sum_{\nu} a_{\nu} \chi_i^{(\nu)}$$

Use character orthog to learn:

$$a_{\mu} = \frac{1}{g} \sum_i g_i \chi_i^{(\mu)*} \chi_i$$

/class sum.

This tells you how many times a UIR occurs in the reduction of an arbitrary rep!!!

While were here, lets write the other main theorems out:

$$\sum_R D_{ij}^{(\mu)}(R) D_{jm}^{(\nu)*}(R) = \frac{g}{n_{\mu}} \delta_{\mu\nu} \delta_{ij} \delta_{lm}$$

$$\sum_i \chi_i^{(\mu)} \chi_i^{(\nu)*} g_i = g \delta_{\mu\nu}$$

Next comes a potenti little theorem. Take some arbitrary rep, get its character χ_i . Then compute the LHS of this equation:

$$\frac{1}{g} \sum_i g_i |\chi_i|^2 = \sum_{\mu} a_{\mu}^2 \quad (= 1 \text{ if a UIR!})$$

Thus, you have computed the sum of the squares of the multiplicities of reps present in your original rep. If that sum comes out 1, your rep was an UIR! If the sum was 2, you know your thing reduces into a sum of two different UIR's, each occurs once, and so on. This is a criterion for irreducibility. The idea is extended, I skip for now.

3-17. Group Algebra used to prove main theorems.

Here group algebra is defined, fine. Celebrated Theorem is proved: regular rep contains each UIR n_{μ} times. This is then used to confirm that $g = \sum n_{\mu}^2$ of the UIR's, an equality. Next, the group algebra is used to show that:

$$\sum_{\nu=1}^r \chi_i^{(\nu)} \chi_j^{(\nu)*} = \left(\frac{g}{g_i} \right) \delta_{ij}$$

18. Sum
overall
UIR's

This is "second orthogonality" ie the columns are orthogonl. This also confirms that the number of UIR's equals the number of classes.

3-18. Projective A-OK
3-19. Direct Prod ✓

Comment: I just want to track down a few new ideas I see sitting in this chapter.

5-1. Gives the concept of a direct product representation. This is very similar to my lectures on this subject. In particular:

$$D_{ik,jl}^{(uxv)}(R) = D_{ij}^{(u)}(R) D_{kl}^{(v)}(R) = \text{the direct product matrix}$$

You pair the first index in each pair, and the second indices.

Theorem: the character of the direct product rep is simply the product of the two factor characters. Could not be simpler. To see this, just set $jl = ik$ in the above formula and then sum.

5-2. The notion of even and odd reps. Well, not yet.

Theorem: if you take $D^u \times D^u$, ie the direct product of two representations which are the same, you can do a preliminary reduction of the result into two pieces, each of which may then be further reducible. The two basis functions are:

$$D^u \otimes D^u = [D^u \times D^u]_{\text{sym}}^S \oplus \{D^u \times D^u\}_{\text{antisym}}^A$$

$$[D^u \times D^u]_{\text{sym}}^S{}_{kl,ij} = S_{kl} [D_{ki}^u D_{lj}^u] = S_{kl} [D_{kl,ij}^{u \times u}(R)]$$

$$[D^u \times D^u]^A_{kl,ij} = A_{kl} [D_{ki}^u D_{lj}^u] = A_{kl} [D_{kl,ij}^{u \times u}].$$

I will use S and A superscripts instead of H's bracket notation. The idea is to symmetrize and antisymmetrize on the first-indices. Your direct product space then breaks into two invariant subspaces. The basis functions are:

$$S_{kl} \psi_k^{(\mu)} \phi_l^{(\mu)} = \text{basis for } [D^u \times D^u]^S$$

$$\dim = \frac{1}{2} n(n+1)$$

$$A_{kl} \psi_k^{(\mu)} \phi_l^{(\mu)} = \text{basis for } [D^u \times D^u]^A$$

$$\dim = \frac{1}{2} n(n-1)$$

It is trivial to compute the character of the S and A representations, just from their definitions. You get:

$$\chi^{(u \times u)^S}(R) = \frac{1}{2} [\chi^u(R)^2 + \chi^u(R^2)]$$

$$\chi^{(u \times u)^A}(R) = \frac{1}{2} [\chi^u(R)^2 - \chi^u(R^2)]$$

5-3. Definition: the adjoint representation of $D(R)$ is $D(R)^T^{-1}$, called $\bar{D}(R)$ with bar.
 def: the complex conjugate rep of $D(R)$ is obviously $D(R)^*$.

fact: for unitary reps, ~~xxxxx~~ $\bar{D}(R) = D^*(R)$ so no need to distinguish them.

fact: the various theorems involving $X(R)^*$ began with that being $X(R^{-1}) = \bar{X}(R)$.
 Thus, the character of the adjoint rep appears naturally in various statements.

5-4. Existence of hermitian invariants. Skip.

5-5. Classification of Representations.

UIR

This section is new to me and interesting. If you pick a particular IR of some group, it must fall into one of three classes: (we are only talking unitary reps here)

(2) D equivalent to D^* but D cannot be made real such that $D=D^*$

(1) D equivalent to D^* and can be brought to real form $D=D^*$

/this type of IR is called a "real representation"

In both the above cases, the character is real for all group elements.

(3) D and D^* are not equivalent (and here character must be complex).

There is a tricky way to tell which of these categories a rep falls. You have this:

$$\sum_R \chi^{(\mu)}(R^2) = c^{(\mu)} \cdot g \quad c^{(\mu)} = \begin{cases} +1 & (1) \\ -1 & (2) \\ 0 & (3) \end{cases}$$

Wigner calls: type (1) = "integer"

type (2) = half integer.

Example from self: I wanted to say that the spin- $\frac{1}{2}$ matrices involve complex numbers which you can never get rid of by equivalencing, even thout the trace (characters) are real. This is a "half-integral rep". On the other hand, the $J=1$ matrices can be made real. This is an integral rep. On,ly problem is that $g = \text{infinity}$ for rotation group, but probably this classification can be carried over somehow.

Next comes a bunch of strange theorems about things like $\zeta(S)$ which is the number of square roots of group element S . I will skip these for now. There is something here about ambivalent classes which I skip.

5-6. Clebsch -Gordon Series for product reduction.

Here is the basic stuff, the character multiplication idea. Here is the multiplicity:

$$a_r = (\mu \nu \sigma) = \frac{1}{g} \sum_R \chi^{(\mu)}(R) \chi^{(\nu)}(R) \bar{\chi}^{(\sigma)}(R)$$

$\uparrow$
 $\chi^* \text{ for } \nu \in R.$

Theorem: (proved easily from the preceding equation). Consider $D^u \times D^{\bar{v}}$.

- a) if $u=v$, the identity rep occurs exactly once in the reduction
- b) if $u \neq v$, the identity does not occur at all.

Example: In $SU(3)$, the 3 and $\bar{3}$ reps are not equivalent, but $3 \times \bar{3}$ contains the identity exactly once. I think this is an example.

5-7. The Clebsch Gordon Coefficients. This is the section I have been after all this time. The Clebsch symbols are introduced and of course you have to include a multiplicity index, so :

$$\psi_s(\lambda \tau_\lambda) = \sum_{j, l} \psi_j^{(\mu)} \phi_l^{(\nu)} (\mu_j, \nu_l | \lambda \tau_\lambda s)$$

← copy index

If the representations are unitary, then the Clebsch's are unitary and have various standard properties that I recall from $SU(2)$ analysis, like the $DD = D$ theorem etc.

5-8. Simply Reducible Groups

Here we restrict interest in two ways: first, assume that all characters are real which excludes groups with type-3 representations. second, restrict to groups without multiplicity. Such groups are then called SR. For such groups we can prove several interesting theorems. By the way, these restrictions apply to the rotation group, S_3 and S_4 .

Lemma: in direct products, integrality behaves reasonably. Eg, direct product of two half-integral reps gives only integral reps, etc.

Sublemma: $D^u \times D^u$ can contain only integral (type-1) reps. This notion is of course familiar from the rotation group.

Theorem: if a UIR occurs in $D^u \times D^u$, then that UIR must belong either to $D \times D)^S$ or $D \times D)^{\bar{S}}$, one or the other. A rep must be either "even" or "odd". For reps which don't appear in some $D^u \times D^u$ no statement may be made.

Example: Consider 1×1 in rotation group. You get $0+1+2$. Thus 0,1,2 are either even or odd. since they occur in a $D^u \times D^u$. Since 1 in the 1×1 is integral, the symmetric ones are the "evens", so 0 and 2 are even, and 1 is odd. Consider $\frac{1}{2} \times \frac{1}{2} = 0+1$. Here, since $\frac{1}{2}$ is half-integral, the symmetric which is 1 is called "odd". Consistent! . The half-integral reps do not occur in $D^u \times D^u$ and are thus neither even or odd in this sense.

5-9. The 3-j symbols. Skip.

June 15, 79.

Hammermesh on Symmetric Group S_n Chap 7.

Comment: This is an excellent reference. All those tricky rules just stated by Licht. are derived in detail here, though I have not studied it. Chapter is 100 pages or so.

7-1. This section discusses a method of constructing character tables for the S_n groups by a sort of building-up method. I.e., you use information on the S_{n-1} characters to get the S_n characters. I have not studied the section, but its goal is clear. Here for example are some character tables:

S_2	1	1
	(1)	(2)
$\chi^{(1)}$	1	1
$\chi^{(2)}$	1	-1

S_3	1	3	2	1
	(1)	(2,1)	(3)	
$\chi^{(1)}$	1	1	1	
$\chi^{(2)}$	1	-1	1	
$\chi^{(3)}$	2	0	-1	

Lets get clear what these tables are doing. Since group elements in same class always have same character, you need only list the classes across the top. For the symmetric groups classes are characterized by their cyclic structure. Thus, in the S_3 table above you have (1,1,1) referring to the identity, whereas (2,1) refers to transpositions and (3) refers to full cycles made by combining two transpositions. All the single transpositions are in the same class, etc. The other number tells how many elements are in the class shown.

7-2. This section derives Frobenius' general formula for characters of S_n . By the way, the phrase "simple character" refers to the character of an irreducible rep, whereas compound character refers the character of a reducible rep. Here we get into all kinds of polynomials and things. Looks horrible.

7-3. But here are the mnemonic results of all the above work. First, you can deduce the dimensionality of each UIR of S_n simply by counting the "standard arrangements" of the tableau. In a standard arrangement, all the n numbers must appear, numbers must increase both across and down.

7-4. A graphical method for getting the characters.

7-5. Characters by recursion formulas (based on Frobenius formula)

7-6. How to use Frobenius formula to compute characters.

Comment: All of the preceding 6 sections, 33 thick pages, are devoted to construction of character tables for symmetric groups!

7-7. Here is a method for constructing not the characters but the actual detailed representation matrices. Follows paper of Yamanouchi 1937. The section ends with a large table which gives the matrices for all representations of S_1, S_2 , through S_5 . Here is the S_3 table, for example:

$$S_3: \quad \begin{array}{c} \boxed{} \\ \boxed{} \\ \boxed{} \end{array} \quad [1] \quad \text{Elements.}$$

$$\begin{array}{c} \boxed{} \\ \boxed{} \end{array} \quad e \quad (ij) \\ [1] \quad [-1]$$

$$\begin{array}{c} \boxed{} \\ \boxed{} \end{array} \quad e \quad (12) \quad (23) \quad (13) \\ [21] \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \quad \begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$$

Details: the matrices ^{for transposition} are always symmetric so lower off diagonal not stated, you can fill it in obviously. Also, matrices are only given for transpositions. You can get the matrices for the rest of the group elements by multiplying transposition matrices.

By the way, these are the same as the matrices of Tinckham for E rep of D_3 . Identify (12)=A; (23)=B; (13) = C etc. See page 8 of Tinckham.

7-8. This section discusses "Hind's Method" of constructing wavefunctions $f(1,2,\dots,n)$ of maximal symmetry or antisymmetry. There is a direct connection between such functions and the Young business. I have not studied this and will not because I think later sections are more relevant.

7-9. Group Algebra Approach. This seems to be the best way of doing things.

First, if you take any linear combination of group operators, you have a linear combination of $h = n!$ (for S_n) operators. The coefficients may be regarded as components of an h -fold vector. Two such linear combinations may be multiplied together in an obvious way to get a new vector. Each linear combination is regarded as a vector in group algebra space. Pozzi had much to say about this stuff. Given an irreducible representation of the group, you can generate an irreducible representation of the group algebra in obvious way.

The idea is to decompose the group algebra into a direct sum of minimal ideals, and these guys in turn correspond to the UIR's of the group. It is possible to construct projection operators called "idempotents" which take any element x of the group algebra and project out that portion which belongs to a particular ideal minimal. These are called $e_1, e_2, \dots$ by Hammermesh. If you can find these projectors, then the problem of finding basis functions belonging to UIR's is mostly solved.

7-10. Young Operators. Here is the fundamental stuff: consider a given Young Tableau, ie, a young "pattern" or "diagram" which is filled in with numbers in some standard arrangement, all numbers appearing once. The obvious one is all numbers in increasing order, called normal tableau by Messiah. For this particular tableau, you can construct an operator S and another one called A . These are best defined by Messiah page 1114 and are called P and Q by Hammermesh. The P or S operator is a special symmetrizer: it is a sum of all the permutations (group elements) which do not mix rows. The symmetrizer is supposed to not mix rows, the anti is supposed to not mix columns.

Example 1: S_3

1	2	3
---	---	---

$$P = e + (12) + (13) + (23) + (123) + (132)$$

Here there is nothing to antisymmetrize. Note how the 6 elements of S_3 are labelled here. The last two are made of two transpositions and thus have even "parity".

Example 2: S_3

1
2
3

$$Q = e - (12) - (13) - (23) + (123) + (132)$$

And here there is nothing to symmetrize.

Example 3: S_5

1	2	3
4	5	

$$P = [e + (12) + (13) + (23) + (123) + (132)] [e + (45)]$$

$$Q = [e - (14)] [e - (25)] = e - (14) - (25) + (14)(25).$$

In the first line I have added all possible permutation operators which do not mix the two rows. There are 12 such operators. Q is the sum, with parities, of all operators which don't mix columns.

wrong Theorem: I think this is the main idea: in order to generate the "basis functions" which belong to a UIR of S_n , you first list off the standard arrangements. Recall that there are exactly as many of these as there are dimensions of the UIR. Then, for each standard arrangement, construct P , then Q , and then construct $Y = QP$ (this corresponds to the j objects of Messiah). This operator Y will be a projector which will give you the basis function corresponding to your chosen tableau.

Now, here is the main poop: You know that a given UIR occurs in the regular rep (ie, the group algebra) N times, where N is the dimensionality. These N occurrences are, I think, correspondent to the N possible standard arrangements you can make in the pattern or diagram. I am now going to show how you find a set of basis functions for each of these N uir's. (well ...)

First, each ~~UIR occurrence~~ ^{and of S_n} gets its own projection operator called $Y = QP$, the j of Messiah. You take each projection operator and apply it to all possible group algebra elements until you obtain a basis for each copy ✓.

Example: Consider S_3 . The two-dim rep occurs twice. For each copy we can construct the projector:

$$\begin{array}{l} \boxed{\begin{smallmatrix} 1 & 2 \\ 3 \end{smallmatrix}} \quad P = e + (12) \\ Q = e - (13) \end{array} \quad Y_1 = QP = A_{13} S_n = [e - (13)][e + (12)] \\ = e - (13) + (12) - (312) = "Y_1" = Y \text{ of } H.$$

$$\boxed{\begin{smallmatrix} 1 & 3 \\ 2 \end{smallmatrix}} \quad Y_2 = A_{12} S_{13} = [e + (12)][e + (13)] = e - (12) + (13) - (213) = Y'_1 \text{ of } H.$$

Lets work first with the top copy. Take the projector Y_1 and apply it from the right (as Messiah shows) onto various elements of the group algebra:

$$(12)Y_1 = (12)[e - (13) + (12) - (123)] = (12) - (213) + e - (23) = Y'_1 = "Y'_1" \text{ of } L.$$

$$(13)Y_1 = (13)[e - (13) + (12) - (312)] = (13) - e + (312) - (12) = -Y_1$$

$$(23)Y_1 = (23)[e - (13) + (12) - (312)] = (23) - (231) + (321) - (31) = Y_1 - Y'_1$$

I am convinced that if you go on and apply Y_1 to all other group elements (thereby spanning the group algebra), you will never get anything other than a linear combination of Y_1 and Y'_1 . Thus, these may be taken as "spanners" of the first copy of the two-dim rep of S_3 . In fact they correspond exactly to Lickenberg's PSI1 and PSI2.

If you want an orthonormal basis, you just do it in obvious way.

In similar manner, I could take Y_2 and generate a Y'_2 such that Y_2 and Y'_2 are operators spanning the other uir's "ideal":

Then finally to get functions which correspond to representations, just apply Y_1 and Y'_1 to $f(x_1, x_2, x_3)$ and you have it!

For example, we have:

$$f_1(1,2,3) \equiv Y_1 f(123) = f(123) + f(213) - f(321) - f(231)$$

$$f_1'(123) \equiv Y_1' f(123) = f(123) + f(213) - f(132) - f(312).$$

Now any permutation acting on one of these functions can only linear-combine them because we have already shown that Y_1 is a right-acting projector [this proves that $(12) f_1(123) = f_1'(123)$, eg]. But, $Y_1' = (12) Y_1$. Thus $G Y_1' = G' Y_1 =$ in the manifold. In other words, Y_1' is also a right-acting projector into this little UIR manifold.

Conclusion: Both of the above linear combinations "belong" to the first copy of the 2-dim UIR occuring in the regular rep decomposition.

Normalization: Since I failed to normalize P and Q above, $Y_1^2 =$ constant times Y_1 so not quite normalized right. No big deal.

Conclusion: I now know exactly how to construct a set of basis functions for each copy of each UIR appearing in a regular-rep decomposition of S_n . For each pattern (ie, each UIR) draw the various standard arrangements. Each arrangement corresponds to one copy of the UIR whose pattern the arrangement is in. Then, for each arrangement construct the projector. Apply that projector from-the-right-only elements of the perm group until a basis has been generated. Then apply the basis elements to an arbitrary function to get the basis functions. Ie, a function $f(123....n)$.

By the way, I think the basis functions for different copies of the same UIR or different UIR's are automatically orthogonal because the projectors really are projectors in this sense.

Hammermesh then wraps up this section 7-10 by finding the symmetrizers and implicitly the projectors for the various representations of S_4 . Fine, now lets continue to the next section.

7-11. Fock, and construction of definite symmetry wavefunctions.

Up to now we have been considering very general stuff, functions $f(1,2,3)$. Suppose now that this function $f(1,2,3,\dots,n)$ is really a product of n different, orthonormal functions. IN quantum mechanics, in the independent particle approximation the energy of such a state is the sum of the single state energies, and therefore the energy is invariant under permutations. You can then apply permutations to get linear combinations which belong to the UIR's of group S_n .

Example: $n=2$, you can have $u(1)v(2)+u(2)v(1)$, and the other one. You start here with two independent functions, and you regroup them so that they classify into the UIR's.

For $n=3$ there are 6 degenerate starting functions. He classifies them into the UIR's. He only gives two functions for the 2-dim UIR, so he only has a total of 6 functions. I know how to make the other two.

Next, he says: suppose instead of all the little functions being different, only two different functions were available. Think what happens: you go ahead and construct your projectors $Y = QP$. When you apply this Y to any starting function, you will always get zero if your tableau had more than two rows: reason is that you are trying to antisymmetrize. I.e., suppose you start with a tableau having three rows. You simply cannot make an antisymmetric rank-3 function if only two functions are available.

In general, if you have k different states only (and not the full n), you can never have any functions belongs to UIR's with more than k rows!

The rest of this section is some Fock thing I skip for now. I think the above thing is what is important here.

7-12. Outer products. I have to do a simpler example than he gives to see what is going on here. Consider this: you have a function which is symmetric, $f(1,2)$ and is a function for particles 1,2. Then imagine you have another "system" consisting of particles 3 and 4, also described by a symmetric function $g(3,4)$. The function $f(1,2)$ belongs to a definite representation of S_2 , and so does $g(3,4)$.

Now suppose you multiply these functions together: $F_1(1,2,3,4) = f(1,2) g(3,4)$. If you only apply (12) or (34) type operations, this function always goes into itself and is in that sense irreducible. But, what happens if you apply arbitrary group permutations of the group S_4 ??? Eg, (23) on this function gives you a new function.

So here is what you might do: Apply elements of S_4 to this function to generate all possible new functions. I will do this on the next page:

$$(23) F_1(1234) = f(1,3) g(2,4) \equiv F_2(1234)$$

$$(24) F_1(1234) = f(1,4) g(3,2) \equiv F_3(1234)$$

$$(12) F_2(1234) = f(2,3) g(1,4) \equiv F_4(1234)$$

$$(34) F_4 = f(2,4) g(1,3) \equiv F_5(1234)$$

$$(23) F_5 = f(3,4) g(1,2) \equiv F_6(1234)$$

Further applications of S_4 operators will never yield any new functions.

So we now have a set of 6 functions which transform among themselves under S_4 , ie, we have a 6-dimensional representation of S_4 . The claim is that this thing is in fact reducible, reducible into UIR's of S_4 . For example, if you took the sum of these 6 functions, you would get a function belonging to four boxes horizontale.

$$\begin{array}{|c|c|} \hline & \\ \hline \end{array} \otimes \begin{array}{|c|c|} \hline & \\ \hline \end{array} = \begin{array}{|c|c|c|c|} \hline & & & \\ \hline \end{array} + \begin{array}{|c|c|} \hline & \\ \hline \end{array} + \begin{array}{|c|c|} \hline & \\ \hline \end{array}$$

So this is an example of an "outer product" representation.

In general, here is what you do to make an outer product rep: take a set of $n+m$ particles. Assume that you have a function $f(1,2,\dots,n)$ describing the first set of n particles and this belongs to one of the UIR's of S_n . This function has lets say $N-1$ partners, because this rep of S_n has dimension N . This set of N functions spans the UIR of S_n .

Similarly the second set of particles is described by a set of M functions which span the M dimensional rep of S_m . Now multiply these two sets of functions together in all possible ways: you now have a set of MN product functions. Next, apply all elements of the larger group S_{m+n} to all of these product functions. In this way you will generate a huge set of functions which transform among themselves under group action of elements of S_{m+n} . Since there are $(m+n)/n$ ways to choose n particles to belong to the first set, I would claim this:

$$\binom{m+n}{n} MN = \text{dimensionality of outer product rep}$$

Hammermesh then does an example which agrees with my analysis above. He then gives the rules for decomposing the outer product rep into irreducibles. Looks familiar to me. Many examples are given.

This is clear, but I don't yet know the usefulness (applicability) of an outer product. Notice that in the outer product you can take the "product" of representations of two different groups (possibly the same) and you then reduce this product relative to the larger group S_{n+m} . Very different from inner product.

7-13. Inner products. Now we take two UIR's of S_n and reduce the result according to S_n . What does this mean? Take an N dimensional rep of S_n and take its N basis functions. Also take an M dimensional rep of S_n and its M basis functions. Multiply these functions together in all possible ways. You now have NM functions. Now application of group elements of S_n to this set of functions transforms them among themselves. Eg,

$$N=4 \quad M=5 \quad n=3:$$

$$\begin{aligned} & (12) \overbrace{f_1(1,2,3) g_4(1,2,3)} \\ &= f_1(2,1,3) g_4(2,1,3) \\ &= \left(\sum_{i=1}^4 f_i(1,2,3) a_i \right) \left(\sum_{j=1}^5 b_j g_j(1,2,3) \right) \\ &= \sum_{ij} a_i b_j \underbrace{f_i(1,2,3) g_j(1,2,3)} \end{aligned}$$

In this example, the inner product rep has dimension 20 and it can be somehow reduced.
So what are the rules for reducing inner products?

Special Case: Suppose you want to compute $(y) \times (n-1,1)$. This is to say, you want $(y_1, y_2, \dots) \times (n-1,1)$. Recall that the numbers in this notation are just telling you about the Young pattern: y_1 = boxes in first row, and so on. These are partitions of n , ie, $y_1 + y_2 + \dots = n$ because there must be n boxes if you are talking about a rep of S_n .

The answer is this: pull a box off (y) and stick it back on in all possible reasonable places. Each rep you thereby get is contained in the inner product. On top of that, (y) itself is contained in the inner product a number of times equal to the number of different values of the y_i less one.

Example: S_2

$$(y) \quad (n-1,1)$$

$$a) \begin{array}{c} \boxed{1} \\ (2) \end{array} \times \begin{array}{c} \boxed{1} \\ (1,1) \end{array} = \begin{array}{c} \boxed{1} \\ (2) \end{array} + (\text{zero times}) = \begin{array}{c} \boxed{1} \\ (2) \end{array}$$

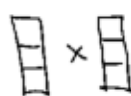
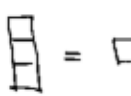
$$b) \begin{array}{c} \boxed{1} \\ (1,1) \end{array} \times \begin{array}{c} \boxed{1} \\ (1,1) \end{array} = \begin{array}{c} \boxed{1} \boxed{1} \\ (2,1) \end{array} + (\text{zero times}) = \begin{array}{c} \boxed{1} \boxed{1} \\ (2,1) \end{array}$$

c) of course the fact that $\boxed{1} \times \boxed{1} = \boxed{1}$ is "obvious".

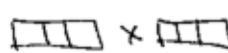
This last result follows since $f(1,2) g(1,2)$ is obviously 12 symmetric if both functions are separately. Thus, the totally symmetric representation of S_n is like the singlet of rotation group or any $SU(n)$ group.

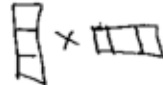
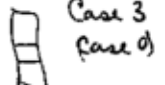
Notice that these results tell you: "the product of a symmetric function with an antisymmetric function must be antisymmetric" and so on.

Example S_3 :

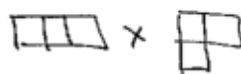
Case 3
 $\times$  = 

/ obvious. $f_{TA}(123) f_{TA}(123)$ must = $F_S(123)$ because any transposition does nothing to product.

Case 1, (Case 0)
 $\times$  =  / see above.

Case 3 (Case 0)
 $\times$  =  / since $f_{TA}(12) f_S(123) = TA$.

For any S_n group these three products are always obvious: total symmetry and total antisymmetry. The difficult stuff is what happens with the other reps:

(Case 0)
 $\times$  =  + (zero) = 

Case 1 Case 2 not covered.
 $\times$  =  +  + (one) =  +  + 

Case 3
 $\times$  =  + (zero) = 

And there you have it: I finally see where Feynman got those facts (eg, mixed times mixed = S + A + M etc.

Hammermesh gives some useful general formulas and has a table for S_4 and S_5 .

7-14. Now comes the Clebsch problem: for example, we see above how the inner product of two (2,1) reps of S_3 can be reduced into a (3) + (2,1) + (1³). What we do not yet know is which linear combination of the 4 functions you get by multiplying the two starting rep functions belong to which reps in the reduction. Ie, we don't yet know the Clebsch Gordon coefficients!

Aside on Yamanouchi Symbols. Recall that a UIR of S_n of dimension N has N so-called standard arrangements. We sort of identify each standard arrangement with a row of the N dimensional representation matrix. The Yama symbol can be regarded as a linear way to describe the standard arrangement. Easier to write the Y symbol than to draw the picture everytime. Example:

$$\begin{array}{|c|c|c|} \hline 1 & 2 & 6 \\ \hline 3 & 5 & \\ \hline 4 & & \\ \hline \end{array} \leftrightarrow [123211]$$

How did I do this? Look for the highest number, the 6. It is in row 1 of the tableau, so put a 1 as first symbol in $[\quad]$. Where is the 5? Row 2, so next symbol is a 2. And so on.

Fact: if you read the symbol in reverse, it is always a "lattice permutation", ie, there are always more 1's than 2's, more 2's than 3's and so on. I guess this can be shown to be true, given the rules for making the standard arrangement: increase to the right and down!

Now back to computing the Clebsch's. Hammermesh gets very general and complicated recursion relations and so on, but he also does some special cases.

Case 1: When you combine a rep with itself, the identity (totally symmetric thing) occurs once in the result. The Clebsch for that combination is very simple. Example:

$$\begin{array}{|c|c|} \hline (2,1) & (2,1) \\ \hline \end{array} \times \begin{array}{|c|c|} \hline (2,1) & (2,1) \\ \hline \end{array} = \begin{array}{|c|c|c|} \hline (3) & (2,1) & (1^3) \\ \hline \end{array} + \begin{array}{|c|c|} \hline (2,1) & (2,1) \\ \hline \end{array} + \begin{array}{|c|} \hline (1^3) \\ \hline \end{array}$$

(we are now looking here. $\Rightarrow$ like $|j, m_j\rangle$)

$$S_{1 \begin{array}{|c|c|} \hline (3) & (2,1) & (2,1) \\ \hline \end{array}}^i j = \frac{1}{\sqrt{2}} \delta_{ij}$$

$$\Rightarrow [(3), 1] = \frac{1}{\sqrt{2}} \{ \overbrace{[(2,1), [211]] \cdot [(2,1), [211]]}^{\text{like } |j, m_j\rangle} + [121] \cdot [121] \}$$

a Yama symbol to label the rows of $\begin{array}{|c|c|} \hline (3) & (2,1) & (2,1) \\ \hline \end{array}$

$$\text{Eg } \begin{array}{|c|c|} \hline (1,2) & \\ \hline (3) & \\ \hline \end{array} \Rightarrow i = [211]$$

$$\begin{array}{|c|c|} \hline (1,3) & \\ \hline (2) & \\ \hline \end{array} \Rightarrow i = [121]$$

$$\begin{array}{|c|c|c|} \hline (1) & (1) & (1) \\ \hline \end{array} = [111]$$

Notice that the Y-symbol actually includes information about which rep you have: ie, all by itself it tells you which UIR and which standard arrangement (row). My results here agree with the table they give later.

$$\text{ie, } [(n), 1] = \frac{1}{\sqrt{n}} \sum_i [\lambda_i] [\lambda_i]'$$

Let me repeat this last example:

$$[111] = \frac{1}{\sqrt{2}} \left\{ [211] \cdot [211]' + [121] \cdot [121]' \right\}$$

OR

$$[123] = \frac{1}{\sqrt{2}} \left\{ \begin{matrix} 1 & 2 \\ 3 & \end{matrix} \cdot \begin{matrix} 1 & 2 \\ 3 & \end{matrix}' + \begin{matrix} 1 & 3 \\ 2 & \end{matrix} \cdot \begin{matrix} 1 & 3 \\ 2 & \end{matrix}' \right\}$$

This tells you exactly what linear combination of "basis objects" goes with the totally symmetric rep. I think it would be useful to see exactly what this means. Each of these M (mixed) representations has two basis functions. Then here is the claim:

$$F(1,2,3) = \frac{1}{\sqrt{2}} \left\{ f_1(1,2,3) g_1(1,2,3) + f_2(1,2,3) g_2(1,2,3) \right\}$$

$$(12) F(123) = F(123), \text{ etc.}$$

It certainly is not obvious to me that the function so constructed is totally symmetric.

However, we can show that it really works as follows: consider combining an arbitrary rep with itself, and you are looking at the identity in the product. You have:

$$[(n), 1] = \sum_{ij} S_{1 \ i \ j}^{(n) \ \lambda \ \lambda} [\lambda_i] [\lambda_j]' = \frac{1}{\sqrt{n_\lambda}} \sum_i [\lambda_i] [\lambda_i]'$$

$$\begin{aligned} \text{Consider: } R[(n), 1] &= \frac{1}{\sqrt{n_\lambda}} \sum_i \underbrace{R[\lambda_i]} \otimes \underbrace{R[\lambda_i]}' \\ &= \sum_j D_{ji}^\lambda(R) [\lambda_j] \sum_{j'} D_{j'i}^\lambda(R) [\lambda_{j'}]' \\ &= \frac{1}{\sqrt{n_\lambda}} \sum_{jj'} D_{ji}^\lambda(R) D_{j'i}^\lambda(R) [\lambda_j] [\lambda_{j'}]' \end{aligned}$$

$$\text{But } \sum_i D_{ji}^\lambda(R) D_{j'i}^\lambda(R') = D_{jj'}^\lambda(e) = \delta_{jj'} \Rightarrow \frac{1}{\sqrt{n_\lambda}} \sum_j [\lambda_j] [\lambda_j]' = [(n), 1].$$

So now I believe it. This is a good Clebsh set to know because you are often combining two like objects to see what combination is totally symmetric or whatever.

Case 2: When you combine a UIR with the transposed UIR, among other things you get the totally anti sym rep once, the (1^n) rep. Here are the Clebsh's for this case:

$$[(1^n), 1] = \frac{1}{\sqrt{n}} \sum_i (\overset{\leftarrow \text{parity}}{\lambda_i}) [\lambda_i] [\tilde{\lambda}_i']$$

Here the parity is just the number of transpositions away from the standard normal arrangement. Here is a simple example:

$$\begin{array}{|c|c|c|} \hline & & \\ \hline \end{array} \times \begin{array}{|c|} \hline \\ \hline \\ \hline \end{array} = \begin{array}{|c|} \hline \\ \hline \\ \hline \end{array}$$

$$\begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline 3 \\ \hline \end{array} = + \frac{1}{\sqrt{1}} \cdot (+1) \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline \end{array} \cdot \begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline 3 \\ \hline \end{array}$$

OR.

$$[321] = [111] \cdot [3\bar{1}].$$

Here you are combining two one-dim reps, so not much to do. Since only one possible arrangement in each, parity is +1.

Another example:

$$\begin{array}{|c|c|} \hline & \\ \hline & \\ \hline \end{array} \times \begin{array}{|c|c|} \hline & \\ \hline & \\ \hline \end{array} = \begin{array}{|c|} \hline \\ \hline \\ \hline \end{array} + \text{other.} \quad \text{note that } [\tilde{\lambda}_1'] = [1\bar{2}] \text{ in this case.}$$

$$\begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline 3 \\ \hline \end{array} = \frac{1}{\sqrt{2}} \left\{ \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & \end{array} \cdot \begin{array}{|c|} \hline 1 \\ \hline 3 \\ \hline \end{array} + (-1) \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & \end{array} \cdot \begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline \end{array} \right\}$$

$$a \quad [321] = \frac{1}{\sqrt{2}} \left\{ [211] \cdot [12\bar{1}]' - [121] \cdot [211]' \right\} \quad / \text{ agrees with hel}$$

Case 3: General Rule: when you combine any rep with the TA rep you get the transpose of the original rep and nothing else! In this case we also know the clebsches.

$$[\tilde{\lambda}\tilde{\mu}] = \Lambda_i [\lambda_i] [(1^n), 1].$$

Example:  $\times$  = 

$$\Rightarrow \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & \\ \hline \end{array} = \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & \\ \hline \end{array} \cdot \begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline 3 \\ \hline \end{array}'$$

$$\begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & \\ \hline \end{array} = (-1) \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & \\ \hline \end{array} \cdot \begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline 3 \\ \hline \end{array}'$$

Example:

$$\begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline 3 \\ \hline \end{array} \times \begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline 3 \\ \hline \end{array} = \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline \end{array}$$

$$\Rightarrow \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline \end{array} = \begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline 3 \\ \hline \end{array} \cdot \begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline 3 \\ \hline \end{array}'$$

Example:

$$\begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline \end{array} \times \begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline 3 \\ \hline \end{array} = \begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline 3 \\ \hline \end{array} \Rightarrow \begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline 3 \\ \hline \end{array} = \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline \end{array} \cdot \begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline 3 \\ \hline \end{array}'$$

Conclusion: for S_3 , almost all the Clebsches are covered by the above special cases, see labelling in red. The case missed is also given in a table which I will quote below.

Note, by the way, that the Clebsch's are based on a definite convention for the representation matrices. In other words, you really have to coordinate your representation tables with your Clebsch tables.

On the next page I give a complete S_3 Clebsch table:

(see next page) By the way, I think there is a symmetry property which says it doesn't matter which order you put your two objects.

End Of Hammermesh Chapter 7 on Symmetric Groups

Complete Clebsch Table for S_3

(see earlier for 2-dim rep matrices)

$\begin{array}{ c c c } \hline & & \\ \hline \end{array} \times \begin{array}{ c c c } \hline & & \\ \hline \end{array}$	$\begin{array}{ c c c } \hline & & \\ \hline \end{array} \cdot \begin{array}{ c c c } \hline & & \\ \hline \end{array}$
$\begin{array}{ c c c } \hline 1 & 2 & 3 \\ \hline \end{array}$	1

$\begin{array}{ c } \hline \\ \hline \end{array} \times \begin{array}{ c } \hline \\ \hline \end{array}$	$\begin{array}{ c } \hline \\ \hline \end{array} \cdot \begin{array}{ c } \hline \\ \hline \end{array}$
$\begin{array}{ c c c } \hline 1 & 2 & 3 \\ \hline \end{array}$	1

$\begin{array}{ c c c } \hline & & \\ \hline \end{array} \times \begin{array}{ c } \hline \\ \hline \end{array}$	$\begin{array}{ c c c } \hline & & \\ \hline \end{array} \cdot \begin{array}{ c } \hline \\ \hline \end{array}$
$\begin{array}{ c } \hline \\ \hline \end{array}$	1

$\begin{array}{ c c c } \hline & & \\ \hline \end{array} \times \begin{array}{ c c } \hline & \\ \hline \end{array}$	$\begin{array}{ c c c } \hline & & \\ \hline \end{array} \cdot \begin{array}{ c c } \hline & \\ \hline \end{array}$	$\begin{array}{ c c c } \hline & & \\ \hline \end{array} \cdot \begin{array}{ c c } \hline & \\ \hline \end{array}$
$\begin{array}{ c c } \hline 1 & 2 \\ \hline \end{array}$	1	0
$\begin{array}{ c c } \hline 1 & 3 \\ \hline \end{array}$	0	1

$\begin{array}{ c } \hline \\ \hline \end{array} \times \begin{array}{ c c } \hline & \\ \hline \end{array}$	$\begin{array}{ c } \hline \\ \hline \end{array} \cdot \begin{array}{ c c } \hline & \\ \hline \end{array}$	$\begin{array}{ c } \hline \\ \hline \end{array} \cdot \begin{array}{ c c } \hline & \\ \hline \end{array}$
$\begin{array}{ c c } \hline 1 & 2 \\ \hline \end{array}$	0	-1
$\begin{array}{ c c } \hline 1 & 3 \\ \hline \end{array}$	1	0

$\begin{array}{ c c } \hline & \\ \hline \end{array} \times \begin{array}{ c c } \hline & \\ \hline \end{array}$	$\begin{array}{ c c } \hline 1 & 2 \\ \hline \end{array} \cdot \begin{array}{ c c } \hline 1 & 2 \\ \hline \end{array}$	$\begin{array}{ c c } \hline 1 & 2 \\ \hline \end{array} \cdot \begin{array}{ c c } \hline 1 & 3 \\ \hline \end{array}$	$\begin{array}{ c c } \hline 1 & 3 \\ \hline \end{array} \cdot \begin{array}{ c c } \hline 1 & 2 \\ \hline \end{array}$	$\begin{array}{ c c } \hline 1 & 3 \\ \hline \end{array} \cdot \begin{array}{ c c } \hline 1 & 3 \\ \hline \end{array}$
$[111] = \begin{array}{ c c c } \hline 1 & 2 & 3 \\ \hline \end{array}$	$\frac{1}{\sqrt{2}}$	0	0	$\frac{1}{\sqrt{2}}$
$[321] = \begin{array}{ c } \hline 1 \\ \hline \end{array}$	0	$\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	0
$[211] = \begin{array}{ c c } \hline 1 & 2 \\ \hline \end{array}$	$\frac{1}{\sqrt{2}}$	0	0	$-\frac{1}{\sqrt{2}}$
$[12] = \begin{array}{ c c } \hline 1 & 3 \\ \hline \end{array}$	0	$-\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	0

S₂ Problem: How could one compute the Clebsch's from scratch for this group?

1. First lets find out how to make a totally symmetric combination since this is the easiest to do. We know there is some linear combination that works, call it V_s :

$$V_s = \sum_{ij} C_{ij}^s F_i G_j \quad (1) \quad \text{where } F_i \in \mathbb{F} \quad G_j \in \mathbb{F}.$$

$$R V_s = V_s \quad \forall R \quad (2)$$

If we insert equation (1) into equation (2) and apply group operation R to the M basis functions we get this:

$$\sum_{ijkk'} C_{ij}^s D_{ki}(R) D_{kj}(R) F_k G_{k'} = \sum_{ij} C_{ij}^s F_i G_j$$

$$\Rightarrow C_{kk'}^s = \sum_{ij} C_{ij}^s D_{ki}(R) D_{kj}(R) \quad \forall R.$$

$$\Rightarrow \underline{\underline{C}}^s = \underline{\underline{D}} \cdot \underline{\underline{C}}^s \cdot \underline{\underline{D}}^T \quad \text{but } \underline{\underline{D}}^T = \underline{\underline{D}}^\dagger = \underline{\underline{D}}^{-1}$$

$$\Rightarrow \underline{\underline{C}}^s \underline{\underline{D}}(R) = \underline{\underline{D}}(R) \underline{\underline{C}}^s.$$

I examined this in some detail, but now I see a fast way: C_s is a matrix which commutes with all the matrices of a UIR. Shurt tells us that C^s is proportional to the identity. Thus we may conclude that:

$$C_{ij}^s = K^s \cdot \delta_{ij} \quad K^s = \text{some constant.} \quad C_{ij}^s$$

To figure out the constant, we can apply the completely general formula (5-111) which reads:

$$\sum_{ij} (\lambda' \mu' s' | \mu_i, \nu_j) (\mu_i, \nu_j | \lambda \mu s) = \delta_{\lambda\lambda'} \delta_{\mu\mu'} \delta_{ss'}$$

For our application here $u=v=$ the mixed rep, there are no multiplicity labels needed so we have a shorter version:

$$\sum_{ij} C_{ij}^{\lambda s'} [C_{ij}^{\lambda s}]^* = \delta_{\lambda\lambda'} \delta_{ss'} \quad (3)$$

where here s is the row index for the rep you are "combining into". For our present

case we insert our result in the diagonal version of (3) to get:

$$\sum_j |C_{ij}^{\lambda_s}|^2 = 1 \Rightarrow |K^S|^2 \sum_j (\delta_{ij})^2 = 1$$

But $\sum_j \delta_{ij} \delta_{ij} = \sum_i \delta_{ii} = \sum_i 1 = 2 \Rightarrow |K^S|^2 = \frac{1}{2} \Rightarrow K^S = \frac{1}{\sqrt{2}} \cdot e^{i\phi}$

This phase is chosen zero and we have duplicated the results of Ham's table.

2. Comment: Notice that this family of Clebsch coefficients into the S rep does not change sign under interchange of i and j. I.e, we have:

$$(S|M_i, M_j) = (S|M_j, M_i).$$

According to later discussion in Hamermesh, I think this means that the rep S is contained in the symmetrized subspace of the product $D^M \times D^M$ and its delta=+1. For the group S_3 we know that all reps are type (1), real representations, i.e, they are all integral representations in Wigner sense. Thus, rep S is an "even" rep.

3. Now lets look for Clebschs into the A rep:

Let λm = any final rep & its row index. Then

$$\begin{aligned} R V_m^\lambda &= \sum_{m'} D_{m'm}^{(\lambda)}(R) V_{m'}^\lambda = \sum_{m' k k'} D_{m'm}^{(\lambda)}(R) C_{kk'}^{\lambda m'} F_k G_{k'} \\ &= \sum_{j k k'} C_{ij}^{\lambda m} D_{ki}(R) D_{k'j}(R) F_k G_{k'} \end{aligned}$$

$$\Rightarrow \boxed{D_{m'm}^\lambda(R) C_{kk'}^{\lambda m'} = \sum_j C_{ij}^{\lambda m} D_{ki}(R) D_{k'j}(R)} \quad (4)$$

This appears as (5-114) Hamermesh. This is the type of relation that allows you to compute any Clebsch's. Along with the normalizing consitions which I'll record here:

$$\boxed{\begin{aligned} \sum_j C_{ij}^{\lambda' m'} C_{ij}^{\lambda m *} &= \delta_{\lambda' \lambda} \delta_{m' m} \quad (5) \\ \sum_\lambda C_{ij}^{\lambda m} C_{i'j'}^{\lambda m *} &= \delta_{ii'} \delta_{jj'} \quad (6) \end{aligned}}$$

Now rewrite in matrix notation, then specialize to the A-rep:

$$\sum_{m'} D_{m'm}^{\lambda}(R) C_{kk'}^{\lambda m'} = \sum_{ij} D_{ki}(R) C_{ij}^{\lambda m} D_{jk'}^T(R)$$

$$\Rightarrow \sum_{m'} D_{m'm}^{\lambda}(R) [\underline{C}]^{\lambda m'} = [\underline{D}(R)] [\underline{C}]^{\lambda m} [\underline{D}(R)]^{-1}$$

If $\lambda = A$, then $D_{m'm}^A(R) = \delta_R \delta_{m'm}$ so

$$\delta_R [\underline{C}^A] = [\underline{D}(R)] [\underline{C}^A] [\underline{D}(R)]^{-1}$$

$$\Rightarrow \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} \begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix} \delta = \begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix} \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}$$

Because of that delta, I no longer have a "Shur" bet, so have to look at various R's and see what we get.

$$R = C(2) \Rightarrow \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} (-1) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} \Rightarrow \begin{aligned} C_{11} &= -C_{11} \\ C_{21} &= -C_{21} \end{aligned}$$

Thus, $C_{11}^A = C_{22}^A = 0$.

Then:
$$\begin{bmatrix} C_{12} D_{21} & C_{12} D_{22} \\ C_{21} D_{11} & C_{21} D_{12} \end{bmatrix} \delta = \begin{bmatrix} D_{12} C_{21} & D_{11} C_{12} \\ D_{22} C_{21} & D_{21} C_{12} \end{bmatrix}$$

$$\Rightarrow \frac{C_{12}}{C_{21}} = \frac{D_{12}}{\delta \cdot D_{21}}$$

Let $R = (13)$, $\delta = (-)$, $D = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \Rightarrow \frac{C_{12}}{C_{21}} = \frac{(-\sqrt{3}/2)}{(-1)(-\sqrt{3}/2)} = -1$.

So: $\underline{C}^A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} K^A$

$$\underline{C}^{A\dagger} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} K^{A\dagger}$$

Norm says: $\text{tr}(\underline{C}^A \underline{C}^{A\dagger}) = 1 \Rightarrow |K^A|^2 \text{tr} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 2 |K^A|^2$

3. Now let's get these phases right. Suppose I choose the S-rep phase to be zero. Then I am probably not free to choose this phase also because we have a condition:

$$\left(\text{tr} [\underline{C}^{\lambda'm'} \underline{C}^{\lambda m \dagger}] = \delta_{\lambda\lambda'} \delta_{m'm} \right) \quad (?)$$

$$\Rightarrow \text{tr} [\underline{C}^S \underline{C}^{A\dagger}] = 0$$

$$\text{But } \underline{C}^S = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \underline{C}^A = \frac{1}{\sqrt{2}} e^{i\delta_A} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \Rightarrow \underline{C}^{A\dagger} = \frac{1}{\sqrt{2}} e^{-i\delta_A} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

But then $\underline{C}^S \underline{C}^{A\dagger}$ is traceless for any δ_A . And δ_S .

I think that each family of C coefficients can have its own arbitrary overall phase (the same for all m if there is an m row label). You can see by inspection that equation (4), the primal one, is invariant under an overall phase change of the C families. The same applies to (5) and (6) since the * appears there.

Since the D's are all real, you know you can make the C's all real. But there is still an arbitrary overall ± 1 for each C family.

So I will choose to agree with Hamermesh:

$$\underline{C}^A = + \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \Rightarrow C_{12}^A = \frac{1}{\sqrt{2}} \quad C_{21}^A = -\frac{1}{\sqrt{2}}$$

$$\Rightarrow (A | M_i, M_j) = - (A | M_j, M_i)$$

Therefore we find that the A-rep is an "odd" representation.

4. Now comes the toughie. Let λ be the mixed rep. Then you have:

$$\sum_{m'} D_{mm'}(R) [\underline{C}^m] [D(R)] = [D(R)] [\underline{C}^m]$$

Let $R = (12)$. Then

$$[\underline{C}^1] \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} [\underline{C}^1] \Rightarrow$$

$$- [\underline{C}^2] \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} [\underline{C}^2] \Rightarrow$$

$$\left(\begin{array}{c|c} C_{11}^1 & -C_{12}^1 \\ \hline C_{21}^1 & -C_{22}^1 \end{array} \right) = \left(\begin{array}{c|c} C_{11}^1 & C_{12}^1 \\ \hline -C_{21}^1 & -C_{22}^1 \end{array} \right)$$

$$\text{same but } - \Rightarrow \left(\begin{array}{c|c} C_{11}^2 & C_{12}^2 \\ \hline C_{21}^2 & -C_{22}^2 \end{array} \right)$$

$$C_{12}^1 = C_{22}^1 = 0$$

$$C_{11}^2 = C_{22}^2 = 0$$

So that was rather helpful. Now let's go to $R=(13)$:

$$\sum_{m'} D_{m1}(13) [C^m] = D_{11}(13) [C^1] + D_{21}(13) [C^2]$$

$$= -\frac{1}{2} [C^1] - \frac{\sqrt{3}}{2} [C^2] = \left[\begin{array}{c|c} -\frac{1}{2} C_{11}^1 & -\frac{\sqrt{3}}{2} C_{12}^2 \\ \hline -\frac{\sqrt{3}}{2} C_{21}^2 & -\frac{1}{2} C_{22}^1 \end{array} \right]$$

So factor out $(-\frac{1}{2})$ + we get:

$$(-\frac{1}{2})^2 \left(\frac{C_{11}^1}{\sqrt{3} C_{21}^2} \mid \frac{\sqrt{3} C_{12}^2}{C_{22}^1} \right) \begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix} = (-\frac{1}{2}) \begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix} \begin{pmatrix} C_{11}^1 & 0 \\ 0 & C_{22}^1 \end{pmatrix}$$

$$\Rightarrow \frac{1}{4} [C_{11}^1 + 3 C_{12}^2] = -\frac{1}{2} C_{11}^1$$

$$\frac{1}{4} \sqrt{3} [C_{11}^1 - C_{12}^2] = -\frac{1}{2} \sqrt{3} C_{22}^1$$

$$\frac{1}{4} \sqrt{3} [C_{21}^2 + C_{22}^1] = -\frac{1}{2} \sqrt{3} C_{11}^1$$

$$\frac{1}{4} \sqrt{3} [C_{21}^2 - C_{22}^1] = (-\frac{1}{2}) (-C_{22}^1)$$

$$\Rightarrow \checkmark C_{11}^1 + 3 C_{12}^2 = -2 C_{11}^1$$

$$C_{11}^1 - C_{12}^2 = -2 C_{22}^1$$

$$\checkmark C_{21}^2 + C_{22}^1 = -2 C_{11}^1$$

$$3 C_{21}^2 - C_{22}^1 = +2 C_{22}^1$$

$$\Rightarrow C_{11}^1 + C_{12}^2 = 0 \Rightarrow$$

$$\boxed{C_{11}^1 = -C_{12}^2}$$

$$\boxed{C_{21}^2 = C_{22}^1}$$

$$\Rightarrow C_{22}^1 + C_{21}^2 = -2 C_{11}^1 \Rightarrow$$

$$\boxed{C_{22}^1 = -C_{11}^1}$$

$$\Rightarrow C_{11}^1 + C_{11}^1 = -2 C_{22}^1, \text{ consistent.}$$

Let $C_{11}^1 = a$. Then

$$[C^1] = \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}$$

$$\text{and } [C^2] = \begin{pmatrix} 0 & -a \\ -a & 0 \end{pmatrix}$$

So we have reduced it to a single parameter, a .

Now apply equation 7 in all ways to see what gives:

$$h[C^1 C^{1\dagger}] = 1 \Rightarrow a^2 h\left[\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right] = 1 = 2a^2 \Rightarrow \boxed{a = \pm \frac{1}{\sqrt{2}}}$$

$$h[C^2 C^{2\dagger}] = 1 \Rightarrow (-a)^2 h\left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right] = 1 = 2a^2 \Rightarrow \text{same.}$$

$$h[C^1 C^{2\dagger}] = 0 \Rightarrow -a^2 h\left[\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right] = 0 = -\frac{1}{2} h\left[\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\right] = 0 \checkmark$$

So let's convention that $a = +1/\text{SQRT}2$. Then we have:

$$C_{11}^1 = \frac{1}{\sqrt{2}}$$

$$C_{12}^2 = -\frac{1}{\sqrt{2}}$$

$$C_{22}^1 = -\frac{1}{\sqrt{2}}$$

$$C_{21}^2 = -\frac{1}{\sqrt{2}}$$

This agrees exactly with the table.

The M rep must be even because switching the labels 12 does not change sign.
I.e.,

$$(M_m | M_i, M_j) = + (M_m | M_j, M_i) \quad m = 1 \text{ or } 2.$$

Conclusion:

- 1) I have derived all Hamermesh's coefficients for S_3
- 2) In each family, an overall phase can be added. Don't do it.
- 3) The S and M reps are "even" having $\delta = +$, whereas the A rep is "odd", $\delta = -$.
The significance is this: suppose you are doing $M \times M$ into M, S, or A. If you switch your initial labels, the coefficient will change sign if it is going into an odd rep. You can check that this rule works for the whole subtable.
- 4) If you switch in this sense in any of the other subtables, you get a plus sign. This is a general symmetry rule derived by H in later chapter, something I could probably show but am too lazy now.

Hammermesh, Chapter 10: Irreducible Tensors and all that.

10-1. The most general linear group of transformations is of course $GL(n)$, . All groups of interest are subgroups of $GL(n)$, so anything you can say in general about $GL(n)$ is very nice.

The first thing you can do is define a "vector" with respect to $GL(n)$, x_i , having of course n components. In $SU(3)$, eg, a "vector" therefore has 3-components. Of course there are two inequivalent 3-dim reps of $su(3)$ so there are really two kinds of vectors, but consider either kind.

Next, a tensor of rank r with respect to $GL(n)$, or with respect to some n -dim matrix group, is defined. Just a transformation law. One obvious way to make such a tensor is $F_{i_1 i_2 \dots i_r} = x_{i_1}^{(1)} x_{i_2}^{(2)} \dots$. Ie, you can combine r different vectors just multiplied together. Notice that there are subindices on the indices. This tensor has n times r independent components if all vectors are different.

10-2. Decomposition of space of rank- r tensors into invariant subspaces.

Consider some rank-4 tensor, say $G_{i_1 i_2 i_3 i_4}$. Pick one of the Young symmetrizers for the group S_4 and apply it to the subindices of this tensor. In that way you ~~produce~~ produce a tensor having a definite symmetry wrt group S_4 .

Now take this new tensor and think about it. Eg, you might have this:

 This tensor is associated with a particular row of a particular representation of group S_4 . What happens if you apply a general $GL(n)$ transformation to this tensor?

$$\begin{aligned}
 \left[\begin{array}{|c|c|c|c|} \hline i_1 & i_2 & i_3 & i_4 \\ \hline \end{array} \right] &\equiv a_{i_1 j_1} a_{i_2 j_2} a_{i_3 j_3} a_{i_4 j_4} \left[\begin{array}{|c|c|c|c|} \hline j_1 & j_2 & j_3 & j_4 \\ \hline \end{array} \right] \\
 &= a_{i_1 j_1} a_{i_2 j_2} a_{i_3 j_3} a_{i_4 j_4} \left[Y \left[\begin{array}{|c|c|c|c|} \hline 1 & 2 & 4 \\ \hline 3 \\ \hline \end{array} \right] G_{j_1 j_2 j_3 j_4} \right] \\
 &= Y \left[\begin{array}{|c|c|c|c|} \hline 1 & 2 & 4 \\ \hline 3 \\ \hline \end{array} \right] G'_{i_1 i_2 i_3 i_4}
 \end{aligned}$$

You see that the "rotated" tensor has the same symmetry classification of the original tensor. Any tensor can be projected into irreducible components which belong to UIR rows of UIR's of S_r where r is the tensor rank. Of course the reason this is true is that the $GL(n)$ or whatever group "transformations" commute with the permutation group action of the S group.

This is important and is worth some time. A rank- r tensor can be acted upon by group operators in two different senses: elements of S_r can permute the subindices, and elements of say $GL(n)$ can act on the tensor in the usual sense. The reason these actions commute is rather obvious, eg,

$$\begin{aligned}
 (12) F'_{i_1 i_2} &= (12) T F_{i_1 i_2} = (12) \sum_{j_1 j_2} a_{i_1 j_1} a_{i_2 j_2} F_{j_1 j_2} \\
 &= \sum_{j_1 j_2} a_{i_2 j_1} a_{i_1 j_2} F_{j_1 j_2} = \sum_{j_1 j_2} a_{i_1 j_2} a_{i_2 j_1} F_{j_2 j_1} \quad \swarrow \text{rename.} \\
 &= \sum_{j_1 j_2} a_{i_1 j_1} a_{i_2 j_2} [(12) F_{j_1 j_2}] = T \cdot (12) F_{i_1 i_2}
 \end{aligned}$$

Thus, $[T, (12)] = 0$, commute

Now, is there any connection between the rank r and the vector size n ? Not really, except this: in any S_r tableau there are going to be a certain number of rows. This number is of course some number less than or equal to r . For that tableau with R rows, there will exist no tensors at all if n is larger than R . Reason: in first column you must put R numbers, but at least one number must be repeated and this kills you under antisymmetrization: Remember that $Y=QP$ so the antisym is done last.

In particular, if n is greater than r you get no tensors. Another way to say it: if you are given a value of n , then the only tensors you can have correspond to Young diagrams with less than or equal to n rows.

Once more: suppose someone asks you: What irreducible tensors can you make wrt group $SU(3)$? Answer: you can make tensors of any rank r you want. For given rank r , the tensor is classified by Young diagram with r boxes but cannot have more than three ~~boxes~~ rows in Young pattern.

By the way, at this point, "irreducible tensor" means a tensor of some rank r which belongs to a particular UIR of S_r . So far we have said nothing about representations of the group $GL(n)$.

Next, H shows how to make TS or TA tensors out of vectors. It is very easy to make a TS tensor of rank- r . All you need is one vector x , then multiply r of them together and you are done. To make an TA rank- r tensor, you need r different vectors. The TA tensor is then just the Slater determinant. Of course if you are making TA tensors, you cannot make one with rank r greater than n , as discussed above. If $r=n$ the tensor you get is maximal in some sense and is proportional to the epsilon tensor.!!

10-3. UIR's of $GL(n)$, their dimensionality.

WE know that when the $GL(n)$ group acts on a rank- r tensor, the components of this tensor get mixed around. In particular, suppose we have an irreducible rank- r tensor. We know that after action by $GL(n)$ operator, tensor keeps its symmetry relative to S_r .

Lets get to the point. Suppose you have a rank- r tensor and you have $GL(n)$. This tensor then has $r \times n$ components which are mixed among themselves by $GL(n)$. Thus, the components of this tensor form a $m \times r$ dimensional representation of $GL(n)$. But such a representation in general is reducible.

Here is the deal: now take a rank- r tensor which belongs to some UIR of S_r . Because this tensor has certain symmetry properties (eg, might be totally symmetric), not all $n \times r$ elements are "independent" (ie, they are not linearly independent; eg, perhaps $F_{123} = F_{213}$). Suppose there are really only N independent elements. Then clearly these N elements are getting mixed among themselves by $GL(n)$. Thus, this irreducible tensor is giving you a representation of $GL(n)$ of dimensionality N ; of course it is also in a representation of S_r of some other dimension.

We need an example: Consider a rank-3 tensor belonging to (3) rep (ie, the TS). Under the action of elements of S_3 , this tensor transforms as a one dimensional rep of S_3 — regardless of the value of n . But, it turns out that the number of independent elements of this tensor is $(n+2)/3$ binomial coefficient. If $n=2$, this tensor transforms as a four-dimensional rep of $GL(n)$.

Recall that the elements of S_3 here act only on the subindices; that is to say, elements of S_3 act on the positions of indices. When you are counting the dimensionality of the S_3 representation, you are counting linear combinations of the three objects (three positions, or three subindices) which transform among themselves. On the other hand, when you are counting the dimensionality of the $GL(n)$ UIR, you are interested in how many values an index can take.

So here is the big question: for $GL(n)$ for some fixed n , and for a tensor of fixed rank r , and for a particular Young diagram with r boxes, exactly how many components of this tensor are independent? How do you count the independent components of a tensor of rank r which is irreducible with respect to $GL(n)$ according to some Young diagram?? Answer: you count the "standard arrangements". When we compute the dimensionality of a UIR of S_r when also count standard arrangements, but in that case every integer from 1 to r must appear once. In this case, only numbers in the range 1 to n can appear (n may be smaller or larger than r), and most importantly, numbers can appear more than once, as long as they give a "standard arrangement" with numbers the same or increasing across, increasing down.

I had better do an example to understand the origin of this counting rule, otherwise I'll forget it quickly.

Example: Consider a rank-3 irreducible tensor belonging to the M rep of S_3 . Lets count the components of this tensor

$$\begin{array}{c|c|c} F_{\begin{array}{|c|c|} \hline 1 & 1 \\ \hline 2 \\ \hline \end{array}} \equiv a & F_{\begin{array}{|c|c|} \hline 2 & 1 \\ \hline 1 \\ \hline \end{array}} = -a & F_{\begin{array}{|c|c|} \hline 1 & 2 \\ \hline 1 \\ \hline \end{array}} = F_{\begin{array}{|c|c|} \hline 1 & 1 \\ \hline 2 \\ \hline \end{array}} = F_{\begin{array}{|c|c|} \hline 2 & 1 \\ \hline 1 \\ \hline \end{array}} = F_{\begin{array}{|c|c|} \hline 1 & 2 \\ \hline 2 \\ \hline \end{array}} = 0. \\ F_{\begin{array}{|c|c|} \hline 1 & 2 \\ \hline 1 \\ \hline \end{array}} \equiv b. & F_{\begin{array}{|c|c|} \hline 2 & 2 \\ \hline 1 \\ \hline \end{array}} = -b & \end{array}$$

So, in general a tensor of this form would have $n^3 = 2^3 = 8$ components. But for this particular case you see that 4 elements vanish, and the other four are pairwise related, so really there are only two independent components.

I don't really understand this counting stuff in more gmplicated cases, but I accept the result. I could figure it out if I wanted to invest the time, but right now I do not.

But I know the rule: just count the standard arrangements.

I also know a shortcut to the rule which I call the a/b method. This is a fast method of counting the standard arrangements. Here it is:

$$F_{\begin{array}{|c|c|} \hline 1 & 2 \\ \hline 1 & 2 \\ \hline \end{array}} = \frac{n(n+1)(n-1)}{3 \cdot 1} = \frac{(n^2-1)n}{3}, \text{ agree.}$$

I learned this shortcut in Lichtenberg.

So, by examining irreducible tensors and counting their independent elements, we are really counting the dimensionality of UIR's of $GL(n)$.

Outer Product Analysis. Now, observe that a "vector" is a rank-1 tensor and it belongs to the $\square$ representation of group S_1 , rather trivial.

Now suppose you want to combine two different vectors (two systems) to make a rank-2 tensor. This is precisely what that outer product deal did for you!!! Recall that you combined an S_n with an S_m and classified results into S_{n+m} . So consider:

$$\begin{array}{c} \square \otimes \square = \square + \square \\ \begin{array}{cc} x_i & y_j \\ \uparrow & \uparrow \\ n \text{ comp} & n \end{array} & \begin{array}{c} x_i y_j + y_j x_i \\ + (x_i y_j - x_j y_i) \end{array} \end{array}$$

Here we are combining two rank-1 tensors to get some rank-2 tensors. BUT, each irreducible tensor is associated with a UIR of $GL(n)$, so this picture also does reduction of products of $GL(n)$ UIR's. (Of course in that sense it is an inner product since each diagram refers to a rep of $GL(n)$.)

Example 2: Here is something interesting. What rank-3 tensors can you make by combining a symmetric rank-2 tensor with a vector??

$$\square \otimes \square = \square\square + \square$$

Now I will be very explicit with an example: $g_{\mu\nu}$ plus a momentum, $n=4$ Lorentz.

$$Y_{\square\square\square} = \frac{1}{6} \sum P \quad \text{so}$$

$$Y_{\square\square\square} \sim g_{\mu_1\mu_2} p_{\mu_3} \rightarrow \frac{1}{3} [g_{\mu_1\mu_2} p_{\mu_3} + g_{\mu_2\mu_3} p_{\mu_1} + g_{\mu_3\mu_1} p_{\mu_2}]$$

This is the only completely symmetric tensor you can make from these objects. Next:

$$\begin{aligned} Y_{\square\square\square} &= \frac{1}{3} A_{15} S_{12} \rightarrow \frac{1}{3} [1 - (15)] [1 + (12)] g_{\mu_1\mu_2} p_{\mu_3} \\ &\quad \underbrace{\hspace{10em}}_{2 g_{\mu_1\mu_2} p_{\mu_3}} \\ &= \frac{2}{3} [g_{\mu_1\mu_2} p_{\mu_3} - g_{\mu_3\mu_2} p_{\mu_1}] \end{aligned}$$

$$\begin{aligned} Y_{\square\square\square} &\rightarrow \frac{1}{3} [1 - (12)] [1 + (15)] g_{\mu_1\mu_2} p_{\mu_3} = \frac{1}{3} [1 - (12)] (g_{\mu_1\mu_2} p_{\mu_3} + g_{\mu_2\mu_1} p_{\mu_3}) \\ &= \frac{1}{3} [g_{\mu_3\mu_2} p_{\mu_1} - g_{\mu_3\mu_1} p_{\mu_2}] \end{aligned}$$

Recompose:

$$T_{\square\square\square} = \frac{1}{3} [g_{\mu_1\mu_2} p_{\mu_3} + g_{\mu_2\mu_3} p_{\mu_1} + g_{\mu_3\mu_1} p_{\mu_2}]$$

$$T_{\square\square} = \frac{2}{3} [g_{\mu_1\mu_2} p_{\mu_3} - g_{\mu_3\mu_2} p_{\mu_1} -$$

$$T_{\square} = \frac{1}{3} [\quad + g_{\mu_3\mu_1} p_{\mu_2} - g_{\mu_3\mu_2} p_{\mu_1}]$$

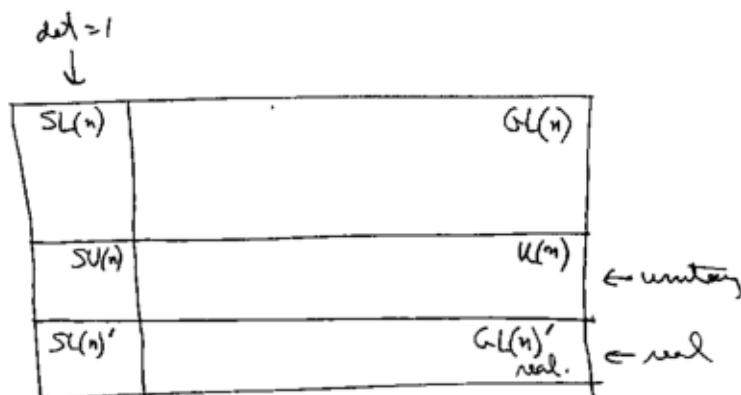
$$T = g_{\mu_1\mu_2} p_{\mu_3}$$

Notice that the "other" combination $g_{\mu_2\mu_3} p_{\mu_1} - g_{\mu_3\mu_2} p_{\mu_1}$ is a linear combination of the two M tensors shown, it is not linearly independent. So, we have constructed all the tensors having any S_3 symmetry that it is possible to construct from these objects!! And they add up right. Obviously there is no possible TA contribution here.

10-4. About the subgroups of $GL(n)$.

The point here is this: if you restrict yourself to certain subgroups of $GL(n)$, the UIR's you found (and counted) in the preceding sections still apply. I.e, if you have a UIR of $GL(n)$ and you restrict it to $GL(n)'$ (real only, $GL(n, R)$ if you like), the rep is still irreducible. That might not have been the case.

Here is a picture showing the subgroups where irreducibility is maintained:



Thus, the above analysis gives the dimensionalities of the reps of $SU(n)$, for example.

In passing, H has a few comments about Lie Algebras for some of these groups

$GL(n)$ Lie algebra: $X_{ij} = \delta_{ij} \Rightarrow n^2$ basis elements. ($2n^2$ real param)

$[X_{ij}, X_{lm}] = \delta_{il} X_{jm} - \delta_{im} X_{jl}$ Lie Alg. $\hat{X}_{ij} = X_{ij}$

$U(n)$ Lie Algebra:

$$\left. \begin{aligned} X_{ij}^{(lm)} &= \delta_{il} \delta_{jm} + \delta_{im} \delta_{jl} \\ X_{ij}^{(lm)'} &= i \delta_{il} \delta_{jm} - i \delta_{im} \delta_{jl} \\ X_{ij}^{(rl)} &= \delta_{il} \delta_{jl} \end{aligned} \right\} n^2 \text{ basic matrices}$$

Fact: if you take any Young tableau which stands for a rep of one of these groups, and if you add a column of n boxes, the claim is that you make no significant change to the representation: you causes it to have extra factor $(\det \lambda)^s$ carried along if you add s columns of n boxes. Thus, essentially you are free to always cross out "full columns". Thus, the UIR's of $GL(n)$ or one of the above subgroups are really enumerated by Young diagrams having less than n rows.

10-5. The case of $O(n)$. Now something slightly different happens: here, when you contract a tensor on two indices (take its "trace" on that pair of indices) the object you get transforms as a tensor of rank 2 lower than the original tensor. In the $GL(n)$ case you could still do this contraction, but the object you get would not transform as a tensor: in $O(n)$ you have that extra property of the a_{ij} which makes this work: namely, $a_{ij}a_{ik} = \delta_{jk}$.

So what is the significance of this fact??? Suppose you take a tensor $F_{i_1 i_2 i_3}$ which belongs to some symmetry of S_3 . You might think that this tensor has independent components and these are shuffled around in an invariant subspace by the $O(n)$ operations and this subspace is irreducible: not so! The reason is that this tensor, or this invariant subspace, actually is reducible further into two pieces. Consider:

$$F_{i_1 i_2 i_3} = (F_{i_1 i_2 i_3}^{\circ}) + (\delta_{i_1 i_2} H_{i_3} + \delta_{i_1 i_3} L_{i_2} + \delta_{i_2 i_3} K_{i_1})$$

The claim is that any tensor can be broken into two pieces as suggested here: the first piece is completely traceless (ie, contract on any pair of indices and you get zero). The second piece is the remainder.

You can show that each of these pieces transforms only into itself under $O(n)$ transformations, the reason being that "contraction" operation, like the symmetries, commutes with the $O(n)$ transformations. You can compute H, L, K the three vectors appearing in the above just by taking traces.

The usual simplest case is this:

$$F_{i_1 i_2} = F_{i_1 i_2}^{\circ} + \delta_{i_1 i_2} J \quad \Rightarrow \quad F_{ii} = nJ \quad \Rightarrow \quad F_{i_1 i_2} = F_{i_1 i_2}^{\circ} + \delta_{i_1 i_2} \frac{F_{ii}}{n}$$

$$\Rightarrow F_{i_1 i_2} = [F_{i_1 i_2} - \delta_{i_1 i_2} \frac{F_{ii}}{n}] + \delta_{i_1 i_2} \frac{F_{ii}}{n}$$

Here you see the general idea: first you compute the piece with the trace, then subtract to find the traceless tensor.

Any tensor can be decomposed in this way? Into traceless plus δ_{ij} times lower rank tensors.

10-6. Now consider the traceless tensors. The space of traceless tensors is invariant under S_n permutations! So if you apply a Young symmetrizer to traceless tensors, you generate irreducible tensors wrt $O(n)$.

The problem now for $O(n)$ is to count the dimensionality of tensors of various S_r symmetry which are traceless. Obviously this extra tracelessness condition is going to lower the number of independent components.

Example:

$$F_{\begin{array}{|c|c|c|} \hline i & i & i \\ \hline \end{array}}$$

$$F_{111}, F_{112}, F_{122}, F_{222}$$

$$\cancel{5n} \quad GL(2)$$

$$F_{111}, F_{112} \text{ only}$$

$$1n \quad O(2).$$

Here the trace conditions easily give you the second pair of elements in terms of the first, hence only 2 independent elements.

Before we learn the general rule for counting in $O(n)$, here is a theorem:

Theorem: if your Young diagram of S_r for a rank r tensor has more than n boxes in the sum of the first two columns, then the corresponding traceless tensor vanishes identically, ie, has zero independent elements.

example:

$$F_{\begin{array}{|c|c|c|} \hline i & i & i \\ \hline \end{array}}$$

$$\equiv 0 \quad \text{since } \Sigma = 3 > 2 \quad \text{for } \cancel{GL(2)}. \quad O(2),$$

Thus, you simply cannot make a traceless rank-3 tensor relative to $O(2)$ which has this symmetry!

Now H goes on to get counting rules for special cases like $O(3)$ and $O(5)$. Basically you just count how many conditions tracelessness implies, then subtract those off the count for $GL(n)$.

Next H shows how $O(n)^+$ is special case of $O(n)$

10-7. Now reconsider the decomposition of tensor = traceless tensor + extra piece. The traceless tensor with S_r rep choice forms an irreducible $\mathfrak{sp}$ of $O(n)$. But the extra piece can be further reduced. Write second piece as (double traceless tensor + residual lower piece having two d_{ij} functions and so on.

The point of doing this is simply that you are showing how to "reduce" a $GL(n)$ representation into $O(n)$ representations: I don't care now about this.

10-8, 10-9. Symplectic groups. Skipped.