

CRITICAL PHENOM.

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. Reif

Chapter 1: An introduction to statistics. Considers drunk in two-dimensions. You compute his mean number of steps-to-the-right in terms of total N .

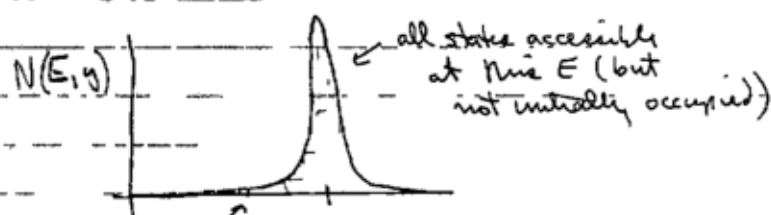
Solution shows answer given by binomial distribution. For given N , you can compute the mean number of right-steps $\bar{n}_1$ for ensemble of drunks. Can also compute the rms dispersion. For large N the dispersion gets small relative to mean, and binomial becomes a gaussian distribution. Chapter illustrates how a statistical peak gets sharp for large N ; also shows how to compute means and dispersions and average values of things.

Chapter 2: Consider one system A made up of many particles. Has energy E . Typically the number of states "accessible" to one particle (or degree of freedom) is proportional to total system E , so total states accessible to entire system is order of E^f , a rapidly increasing function of energy if $f = N_A$. This is the key fact! System A also described by some parameters y_i like volume. System A spontaneously spreads itself equally into all accessible states, ie, an ensemble of system- A 's would do this.

To talk about a "process" or "interaction" you need two systems A and A' . If work-parameters like volume are held fixed during an interaction, only "thermal interaction" can occur. Heat will flow to maximize the accessible states of entire system, ie, to maximize entropy of $A+A'$. If adiabatic shielding, you can have a purely mechanical interaction, maybe work $p dv$ done. General interaction allows both kinds of interaction. In order to have well-defined "generalized forces" like $\bar{p}$ conjugate to V , interaction must always be quasi-static. It is noted that dW and dQ are "inexact" differentials.

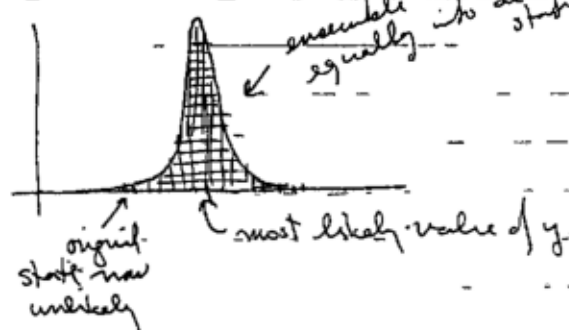
Once again on that thermal interaction. In ensemble of $A+A'$ systems, you will find that mean temperatures equalize. But in a particular ensemble this might not quite be the case, actual temps may differ. Its just that the probability that a particular ensemble has $T = T'$ is huge because this configuration has by far the greatest number of states. When $A+A'$ comes to "equilibrium", its "existence probabilities" are all described by the statistics of accessible states. That is why $T = T'$. If you initially put a system like $A+A'$ in an accessible state that has relatively low probability, (like different temperatures), the thing just spreads out into all its accessible states [process of achieving equilibrium] and peak happens to be at $T=T'$. The only postulate we make is this: in equilibrium, ensemble spreads out equally into its accessible states.

Chapter 3: As another example of the above discussion, consider two-compartment box with movable partition at position y . How can you explain the fact that the partition adjusts in such a way that the pressures are equal on the two sides? Answer: system was initially set in an accessible state of low probability. Ensemble then spreads out. Most likely accessible states happen to have equal pressures. So same analysis as before. Another way to say it is this (but no new postulate is implied): parameters vary in such a way as to maximize number of accessible states. Statement is a little unclear because all states for all y are accessible. You can make another cut and speak of $N(E; y)$ as number of states accessible for E and particular y . Then y will "spontaneously" adjust so that $N(E, y_1)$ is the max. I think a picture will help here:



all systems in ensemble are initially set here.

(INITIALLY)



(AT EQUILIBRIUM)

Because numbers are so large, dispersions are usually extremely small, super narrow peaks. Thus we say $p_1 = p_2$ at equilibrium, there is no real need to specify that actually p_1 and p_2 are likely to differ by one part in 10^{12} . Certainly $\bar{p}_1 = \bar{p}_2$, exactly.

In pictures and example above, $N(y_{\text{initial}})$ was less than $N(y_{\text{final}})$. This process (the partition sliding to center of box) is called "irreversible". Once the system has spread out and the most probable values set, it is not going to ever again squeeze back into the initial configuration, except maybe one in 10^{23} cases or something. For a reversible process, like charging a battery, the accessible-state density must stay the same as parameter (voltage) varies.

In the above box example, as y changed, the number of accessible states $N(E, y)$ increased, so the entropy increased. A process in which entropy increases is therefore irreversible. The fact that entropy wants to increase is nothing new to be added, just a restatement of the above spreading discussion.

When A and A' interact thermally, you get $N_{\text{tot}}(E) = N(E)N'(E_{\text{tot}} - E)$, the product of two steep functions. System goes to the peak. At this peak both systems have the same value of a certain parameter $d \ln N / dE$. I.e., further transfer of E does not increase N . This parameter is called Temperature. Entropy $\bar{S} = k \ln(N)$.

Zeroth Law states that A.te.B and B.te.C implies A.te.C and B is called

a "thermometer".

If you dump heat Q into a reservoir, you increase E to $E+Q$ and thus you increase entropy of reservoir. Trivially, $dS = dQ/T$. This assumes no work done on the reservoir. This result must be true no matter how non-quasi-statically Q is dumped.

Now consider A and A' interacting with some work done. In order even to talk about the work done like pdV interaction has to be quasi-static so that p is defined. In a general interaction, the entropy of system A' (now not necessarily a reservoir) changes for two reasons: 1) by doing work on A' you change the total energy of A' ; 2) by changing the work-parameters you change the states-structure of system A' ; e.g., you might drastically increase the density of states at all energies. If you compute dS taking both these things into account, it turns out by magic that you get precisely the result $dS = dQ/T$, but now the process must be quasistatic or the derivation of this result is no good. This is the Second Law. The Third Law says that as T goes to zero, S goes to some constant, the minimal disorder. Often nuclear spins are still random at very low temp where everything else has stopped. The First Law is just that $dE = dQ - dW$ for system A ; change in its internal energy E is given by heat added to system A minus work done by system A on some other system.

Chapter 4: We have at this point derived from microphysics the Laws. These laws govern the macrophysics and there is much you can do without even knowing about the microphysics, if you accept these laws. This is the subject of thermodynamics. This chapter tells you how to specify a macrostate of a system (give its energy, pressure, volume, temperature, etc) Also tells you how to measure various things like specific heat. You can even measure S by integrating dQ/T from a point of reference.

Chapter 5: Thermodynamic Applications in the Real World. The equation of state for and "ideal gas" (one with negligible particle interactions; even a fermi degenerate gas can be ideal) is taken as empirical in thermo, though we know also how to derive it from the micro. Using this equation of state and the first-second laws $TdS = dE + pdV$ you can show that $E = E(T)$ only. Thus in a free Joule expansion in isolated box the T does not change, no cooling. You can also show that $c_p = c_v + R$ for any ideal gas and $\gamma = c_p/c_v$. For monatoms $c_v = 3R/2$ and $\gamma = 5/3$. Adiabatic gas law $pv^\gamma = \text{const}$ is derived.

A "pure" or "homogeneous" substance is something whose only descriptive external parameter is volume, like a solid or a gas. For such substances $dW = pdV$ only, and you can state various "Maxwell's relations: between partials.

Very mysteriously the functions H, F and G are defined:

$$H = E + pV$$

$$F = E - TS$$

$$G = E - TS + pV$$

and the point of doing so is not made clear. Recall that $dE = TdS - pdV$. Thus, E is associated automatically with variables S, V . In a quasi-static process of constant entropy and volume, E cannot change. The reason for defining the other "energies" is that each of them gets connected with a special pair of variables:

$$dH = TdS + Vdp$$

$$dF = -SdT - pdV$$

$$dG = -SdT + Vdp$$

Thus, eg, in a quasi-static process where T and p are held fixed, G cannot change.

These functions are "enthalpy" H , Helmholtz free energy F , and Gibbs free energy G .

So what else can be said about a pure substance? Two easily measureable parameters are $(dV/dT)_p$ and $(dV/dp)_T$ called expansion coefficient and compressibility. These are called α and κ . Using them you can relate C_V to C_P in a simple way. You find that for a solid which is highly non-compressible (small κ), C_P and C_V are about the same.

Van der Waals gas: (not ideal) Equation given, conclude that $C_V = C_V(T)$ only like ideal gas, but $E = E(T, V)$, unlike ideal gas. Reif computes temperature change when van der Waals gas is Joule-free expanded. In this "process", note that $dE = 0$. Also, $dQ=0$ and $dW=0$.

Joule-Thomson Throttling, very interesting and basic. When steady state reached, pipe walls thermalize so $dQ = 0$ for mass dM going thru the throttle valve. But pressure and volume of dM changes, work is done by gas ~~against~~ on hi-pressure side against gas on low pressure side, so dE does change. Turns out that enthalpy H is constant, so this process described by $dH=0$. I presume this is why H was invented. For ideal gas $H_1 = H_2$ implies $T_1 = T_2$ so not useful. A real gas typically has a throttling curve with a certain "inversion" curve. This is in the T - P plane. Throttling may either heat or cool gas depending on where you are in the plane (sign of μ). Usually you have to pre-cool a gas before throttling will cool it more. The effect is physically explained as low-temperature attractions versus high temperature repulsions in collision.

Chapter ends with discussion of heat engines (to get out work) and refrigerators (to cool things off). Condition that dS positive gives theoretical max efficiencies. A "heat pump" heats your house by refrigerating the outdoors. In effect, it sucks heat out of the low temperature outdoors and adds it to its dW to give you more heat than you would get with resistance heating. Kelvin and Clausius "statements of the second law" say: you cannot make a perfect heat engine or refrigerator.

Chapter 6: Theory of Statistical Mechanics. In last two chapters we did the macro theory and applications, now comes the micro theory. First comes the famous "Boltzmann factor": Consider a small system A in contact with a large system A' [eg, A might be one gas molecule, A' the rest of the gas, and "contact" is made by collisional thermalization]. Since E' swamps E, easy to show that $N'(E_{\text{tot}} - E) = e^{-\beta E}$. Thus, number of accessible states is given by $N(E) \exp(-\beta E)$. The idea is simply that if you suck energy dE out of the large system A' and give it to A, you have done a severe restriction on the number of states available to A'; this is what the exponential factor says. Of course if A is also large, N(E) rapidly increases and both combine to make the gaussian. But now we are more interested in A being rather small system.

Immediate examples of the Boltzmann Factor are: 1) paramagnetism; system A is one spin, Boltzmann Factor causes it to want to align with external H. At low temp all tend to align, at high temp you can show that $\chi = \text{const} \times T^{-1}$, known as Curie's Law. Thus in a huge ensemble of systems (pick one in each, one spin), the distribution of alignments μ , or Prob(μ), is really controlled entirely by the Boltzmann Factor. Such a distribution is called a "cannonical distribution". 2) Next example is Maxwell Velocity distribution and Speed distribution, also with gravity. Gives usual atmosphere decay with height.

Now finally comes the idea of Partition Function. $Z = \sum e^{-\beta E_r}$. This thing is a tool. Its various derivatives tell you things you want to know, like mean energy, mean pressure, dispersion. This object Z is closely connected to entropy: $S = k(\ln Z + \beta \bar{E})$ where $\bar{E}$ is mean energy of an ensemble of systems. Also, $F = -kT \ln Z$ to Z is very close to the Helmholtz free energy. Essentially Z contains the same energy as S and therefore anything can be calculated from it. Remember that it is the structure of accessible states that tells all.

For weakly coupled systems (like two molecules in a gas) entropies are additive so Z's are multiplicative.

Calculation Trick: in computing Z you want to sum over all accessible states. If system A under inspection has definite energy E to E+dE, this sum can be hard to do. Put A in contact with reservoir whose T is same as T of A. Then energy can transfer and restriction removed from your sum. "Grand Canonical Ensemble" is the same idea except you also allow free flow of number of particles as well as energy. You make sure that $\bar{E}_{\text{gce}} = \bar{E}_{\text{isolated}}$ and similarly for N.

Chapter 7: Applications of Statistical Mechanics, the Micro-theory.

1. Ideal Gas. For monatoms, you can compute $Z' = \xi^N$ where ξ = partition function for one molecule (atom). This in turn is computed using the phase space available to the atom, and of course the Boltzmann factor. In the end you have to correct your Z' by saying $Z = Z'/N!$ because particles are identical and we multiply counted them in our partition sum. If you fail to correct like this, you find your entropy S is not "extensive", a problem known as Gibbs Paradox. This ideal gas, by the way, is only useful for real gases when T is high or n is low enough so that quantum makes no difference. A condition is given for when quantum is significant. Condition involves h , n and T and also m . He at room T is OK, fermi electron gas in metal at room T is degenerate.

2. Equipartition Theorem. For non-quantum situations only (!), each quadratic term in Hamiltonian yields contribution $\frac{1}{2}kT$ to the mean energy of a subsystem in equilibrium. Thus, a non-quantum harmonic oscillator (mass on spring) has $\bar{E} = kT$. Good application of this is solids: for the crystal lattice, you get $3kT$ since each is a H.O., so you expect that $C_V = 3R$, which is really true! The temperature barrier below which quantum must be considered for the lattice ions is called $\propto \theta_D$, the "Einstein Temperature". Below this temperature you can make simplest model that ions are on single springs of some Ω that is fixed; this model then gives C_V goes to 0 as T does, which must always be true.

3. More paramagnetism. This time general spin J considered, Z is computed, and then $\bar{M}$ given by a functional form known as the "brillouin function".

4. More on Maxwell Velocity Distribution. This time shown for general ideal gas. The various peak, mean, and rms values are given. This chapter ends with kinetic theory of gases; exact flux against a surface is $\frac{1}{4}n\bar{v}$, from which you can find the balance condition for effusion-connected cavities filled with gas. Exact pressure on wall is $(1/3)n\bar{v}^2_{rms}$.

Chapter 8: Phase Transitions. This is why I did this review in the first place, to learn about second-order phase transitions. Turns out that Reif says nothing about them, but background will now help me find out from some other source.

Free Energy: suppose you put water, ice and water vapor in a constant volume container and call the whole thing system A. A' is the outside, a reservoir at temperature T which is fixed. A is prevented from doing any work on A'. A process of some sort goes on inside system A (some ice melts), and some heat is transmitted to A'. You can easily show that: since dS_{tot} for A+A' must be positive, dF for A must be negative. Thus any process inside A can only lower F, the Helmholtz free energy. Equilibrium is then determined by F = minimum. The system A internally adjusts in such a way that F is at a minimum. Now suppose an equilibrium is reached inside A. Thing just sits there forever. Notice that free energy F is useful when T, V held constant. The condition that F of A = minimum can then tell you how much ice, etc.

Similarly, suppose you put ice, water, and vapor in a balloon, and immerse this in air bath of constant T and p. As a process occurs inside A, the volume V may change and so A can now do work on A'. If no other work is done other than this pdV work, you can show that the requirement that dS_{tot} be positive requires that any process in A will lower G; and A will equilibrate when G = minimum.

Example of latter case: consider mathematical chunk of matter in bulk matter. This chunk is at constant T and P. Condition that G = minimum yields these two results: 1) $T = T_{\text{bulk}}$; 2) C_V positive.

Now consider a 2-phase system in a P, T reservoir. If both phases are to be present in equilibrium, the condition that G = min tells you that $g_1 = g_2$. This in turn leads at once to the "Clapeyron Equation" for the 2-phase line in the pT-plane. As an application of the Clapeyron Equation, Reif computes vapor pressure over a liquid, shows exponential dependence on temperature.

Chemistry: suppose you have some reaction going on. Now you can (partially) change, say, G by changing the number of particles of some species, since $(dG/dN_i)_{T,p,N}$ does not vanish. So you simply add a term to your dG equation: watch:

$$dG = -SdT + Vdp + \sum_i \left(\frac{\partial G}{\partial N_i} \right)_{T,p,N} dN_i = \sum_i \mu_i dN_i \quad \text{But } dN_i = \lambda b_i \Rightarrow \sum_i \mu_i b_i = 0$$

Thus you get an equilibrium condition in terms of the potentials. But you can go on to compute the potential for each species: how? Potential u is (dF/dN) but F related to Z, and you can always compute Z in terms of the single-molecule partition function z, this all leads to the famous law of mass action with its temperature-dependent equilibrium constant. Of course we are talking about gases, not chemicals in a solvent.

The lines in the P-T plane show where phase transitions occur. When you cross a line, the volume and entropy etc undergo finite changes. Such a transition is called first order. I don't know anything yet about the more subtle second-order transition. The "critical point" here is a location (P_c, T_c) where volume of liquid and gas becomes the same, so through this point you might be talking about a second-order phase transition.

Chapter 9: Quantum Statistics. The point here is simple: when you compute the total partition function Z for a gas, say an ideal gas, how should you count? I.e., what are the constraints on your sum which defines the partition function Z ? Classically, you consider all particles different and you just do the sum freely, no constraints. This leads to the "Maxwell-Boltzmann" predictions for the occupation $\bar{n}_s$ of level of energy E_s , and also leads to a certain Z .

What about the Photon Gas Case? This is exactly like the classical summation, except you leave out all the factorials because indistinguishable. ~~Leads~~ As in the classical case you sum freely with no constraints. You get a certain Z and a certain $\bar{n}_s$ known as the Planck distribution.

For Bose-Einstein you do the same sum as for photons, but you have a number constraint. You remove this number constraint by doing a trick, then you sum freely. Result is a Z and an n_s . In the free sum you allow any # particles in a state. For Fermi Dirac you repeat the trick but sum only $n=0,1$, get answer that is similar but signs changed. A certain constant called α appears in the latter answers. It happens to be the chemical potential, but can be calculated by insisting that the $\bar{n}_i$ add up to N . Of course this requires knowledge of all energies E_s of your quantum system; since this information is already contained in Z , you might just as well compute α from Z in usual way. Its different for different quantum gases of course, for photons it is zero.

In the so-called "classical limit" of low density n or high T , the occupation of any level is so small that both BE and FD statistics predictions for $\bar{n}_s$ reduce to the MB predictions. Their Z 's reduce to the MB Z but corrected for Gibbs paradox.

I choose not to summarize the rest of my review. Deals with applications of computing equilibrium constant for H dissociation, polyatomic molecules Z calcs, application of Planck distribution to get Blackbody formula, the radiation spectrum from a hot object computed by the BB trick, and the idea of a Fermi degenerate gas.

Chapter 1: (46 pages) Statistical Methods

1.2 Random Walk in 1-D: What is prob., after N steps, to be m steps to the right of starting point? if $p = \text{prob to right}$, $q = \text{prob to left}$, for each step? Let $n_1 = \text{total steps taken to right}$, so $m = n_1 - n_2$. Then answer is given by binomial distribution: $P_N(m) = \binom{N}{n_1} p^{n_1} q^{n_2}$ where $q = 1-p$.

1.3 Basic Statistics. $\bar{u} = \sum P(u) u / \sum P(u) = \text{mean value}$; also for $f(u)$. The dispersion is $(\Delta u)^2 = (u - \bar{u})^2 = \bar{u}^2 - \bar{u}^2$, tells you amount of spread. $\Delta u = \text{SQR}(\overline{u^2}) = \text{The rms dispersion}$.

1.4 Statistics applied to Random Walk. Easy computations show: $\bar{n}_1 = Np = \text{av. \# steps to right}$ for ensemble of drinkers. $\bar{n}_1 = N(p-q)$; $[\Delta n_1 / \bar{n}_1] = 1/\sqrt{N}$, says as $N \uparrow$, the thing peaks up near $n_1 = \bar{n}_1$ with ever-narrowing peak. $\Delta n_1 = \sqrt{N}$ shows how guy moves away from pole. $(\Delta n_1) = \sqrt{Npq}$.

1.5 Limit for large N . As $N \rightarrow \infty$, the binomial distribution $W(n_1)$ peaks very strongly near its mean $\bar{n}_1 = Np$. For practical purposes, when $Npq \gg 1$, you can approximate $W(n_1)$ by a gaussian distribution: $W(n_1) = \exp[-(n_1 - \bar{n}_1)^2 / (2(\Delta n_1)^2)] / (\sqrt{2\pi} \Delta n_1)$. You see the peak dropping as $N \uparrow$ since $\Delta n_1 \propto \sqrt{N}$, and you see width $\sim (\Delta n_1)$.

1.6 Gaussian Distrib. Now let $x = \text{dist. variable}$. Then $P(x) = \exp[-(x - x_0)^2 / (2\sigma^2)] / (\sqrt{2\pi} \sigma)$ where $\sigma = \text{rms dispersion away from mean } x_0$. So $\sigma = \sqrt{2.1 \sqrt{Npq}}$.

rest of chapter: on further statistics; several variables; skip in this review

Chapter 2: (35 pages) Statistics and Systems of Particles

2.1 Specify each state. I.e., the problem of labelling states of a system. The "microstate" of N particles.

2.2 Ensemble. Large # of systems, each is in one of the "accessible states".

2.3 Postulates. If the probability that a system is in some state A is time independent, you have equilibrium. For isolated, equilibrium system, there is equal prob. to be in any accessible state. No state is favored.

2.4 Notation: $\Omega(E) \equiv$ # states near energy E ; $\Omega(E; y_k) =$ subset of these states wherein parameter y has value y_k . Then $P(y_k) = \sum \Omega(E, y_k) / \Omega(E)$.

2.5 Density of States. Consider system of f degrees of freedom. Let each degree have Φ_i accessible states. Then $\Phi =$ total # states $\approx \phi_1^f$, $\ln \Omega \approx \sigma(f)$, Ω grows very rapidly with f ! Also, $\Omega \propto E^f$ roughly, ∞ states proliferate madly with increased energy. The form $\Omega(E) = \text{const} \times V^N \times E^{3N/2}$ is derived for monatomic ideal gas. Notice N in exponents!

2.6 Thermal Interaction. If two systems exchange energy $\Delta E \propto Q$, but their separate "external parameters" stay fixed, then "thermal-only" interaction. An ext. param. is a macrostate param. like volume or applied B field. When these ext params stay fixed, the microstate energy levels do not shift at all.

2.7 Mechanical Interaction. Two systems can also exchange energy with no thermal interaction at all, i.e., one can do physical work on the other by changing its external parameters, eg, weight compresses gas. Here you require thermal insulation. (this is adiabatic interaction).

2.8 General Interaction. Let $dQ =$ heat absorbed by system; $dW =$ work done by the system. Let $dE =$ change in energy of system. Then $dE = dQ - dW$. These are all of course mean energy changes for ensemble of systems.

2.9 Quasi-static process. If you change external param x_i slowly/relative to relaxation time, system $\in$ ensemble is in some well-defined state with energy $E_i(x_i)$. You then interpret the ensemble-average of $\partial E_i / \partial x_i \equiv \bar{X}_i$ as a generalized force. Eg, a pressure when no turbulence.

2.10 Pressure. $\bar{P} = \partial \bar{E} / \partial V =$ example of a gen. force. So $dW = \bar{P} dV$. No big deal.

2.11 Inexact Differentials. Consider $dW = \vec{F} \cdot d\vec{r}$. If $\vec{F}$ is a non-conservative force, we know that there exists no work-function (potential) $W(x, y, z)$ such that $dW = W(\text{here}) - W(\text{there})$. In this case you write dW and you have an "inexact differential". dE is an exact diff because it is always $E_f - E_i$. First law: In adiabatic system ($dQ=0$), $dW \equiv dE$ + therefore dW is "path independent" a "state function". In general, first law is $dE = dQ - dW$. Same comment holds when $dW=0$.

Chapter 3: Statistical Thermodynamics (40 pages)

3.1 Removal of Constraints Let y_i = parameters of macrosystem (eg, pressure of air in box). Claim: if you allow y_i to vary, the set $\{y_i\}$ will end up taking values such that $\Omega(y_i)$ is maximized. System adjusts so as to maximize the # of states available to itself. Examples: y = coordinate of movable partition. Where does partition come to rest? When $\Omega = \max$!

3.2 Reversibility If $\Omega_f > \Omega_i$, process is irreversible (entropy increased); if $\Omega_f = \Omega_i$, process is reversible.

3.3 Entropy & Temperature Let two systems interact. Let A have energy E , A' have E' and $A^\circ = A + A'$ have $E^\circ = E + E'$. The prob. that A ends up with some energy E is $P(E) \propto \Omega(E) \propto \Omega(E) \Omega(E') = \Omega(E) \Omega(E^\circ - E)$. Product of two steep fctns has a narrow max. So $\ln P = \ln \Omega(E) + \ln \Omega(E^\circ - E)$. Set $d \ln P / dE = 0$ to get $d \ln \Omega(E) / dE = d \ln \Omega(E') / dE'$ or $\beta = \beta'$. Now $\beta = 1/kT$, so condition of max probability is: final temperatures are the same. Also, $S \equiv \ln \Omega(E) \equiv S(E) \equiv$ entropy. So $P(E) \propto \Omega(E) \Omega(E')$
 $\Rightarrow S^\circ = S + S'$: "Equal temperatures" is condition that maximizes final $\Omega(E)$ or entropy S .

$$S(E) = k \ln \Omega(E)$$

$$\beta(E) = \frac{\partial \ln \Omega}{\partial E} = \frac{1}{k} \frac{\partial S}{\partial E} = \frac{1}{kT}$$

$$\text{so } \frac{1}{T} = \left(\frac{\partial S}{\partial E} \right)$$

3.4 Approaching Thermal Equilibrium. As $A^\circ = A + A'$ does so, $\Delta S^\circ = \Delta S + \Delta S' \geq 0$! Obviously entropy can only increase.

3.5 Temperature The "zeroth law" says: A.T.E.B, B.T.E.C $\Rightarrow$ A.T.E.C. Then system B can be used as a "thermometer" to see whether A + C would interact thermally or not. Θ = thermo parameter. Recall $\Omega(E) \propto E^f$. Then $\beta = \frac{\partial \ln \Omega}{\partial E} = \frac{\partial (\ln E^f)}{\partial E} = \frac{f}{E} \Rightarrow kT \equiv E/f =$ mean energy per degree of freedom! A way to interpret T , given that $\Omega(E) \propto E^f$.

3.6 Heat Reservoirs What happens if you dump heat Q into a large reservoir? Well, $\ln \Omega_f = \ln \Omega(E + Q)$, $\ln \Omega_i = \ln \Omega(E)$, so $\Delta S = k \frac{\partial \ln \Omega}{\partial E} \cdot Q = Q/T$. Rule then is $\Delta S = Q/T =$ entropy increase of the reservoir when Q is dumped into it at temp T . Also, $dS = dQ/T$.

3.6? How sharp is the Gaussian max? Let $\beta = \partial \ln \Omega / \partial E$ and $\lambda = -\partial^2 \ln \Omega / \partial E^2 = -\partial \beta / \partial E$. When two systems interact, you get $P(E) \propto \exp[-\frac{1}{2}(\lambda + \lambda')(E - \bar{E})^2]$ as you saw. Gaussian. Now use rule of thumb: $\Omega(E) \propto E^f \Rightarrow S = k \ln \Omega = k f \ln E$. $\beta = f/E$ (as before), $\lambda = f/E^2$. Then dispersion $\Delta E^* = 1/\sqrt{\lambda + \lambda'} \propto 1/\sqrt{\delta}$ $\approx \bar{E}/\sqrt{f}$. So main result is: $(\Delta E^*)/\bar{E} = 1/\sqrt{f} \approx 10^{-12}$ for $f = N_A$, i.e., a very narrow gaussian peak.

3.8 How does Ω depend on external parameter X ? As you vary some parameter X (like the volume), the energy of each microstate changes by $(\partial E / \partial X) dX = X_k$ where $X =$ gen force (pressure). Since all states move a little, the # states at energy E changes. The answer turns out to be: $\partial \ln \Omega(E) / \partial X_i = -\beta X_i$ where X_i is "force" conjugate to x_i . Eg, increase volume by dV , you increase $\ln \Omega$ by $d[\ln \Omega] = \beta \bar{p} dV$, for $dS(E) = dW / T(E)$.

3.9 General slow interaction. Let A. int. A'. During a quasi-static process, A undergoes $d\bar{E}$ and receives dX_i . Then $d(\ln \Omega) = (\frac{\partial \ln \Omega}{\partial \bar{E}}) d\bar{E} + (\frac{\partial \ln \Omega}{\partial X_i}) dX_i = \beta d\bar{E} + \beta X_i dX_i$ or $dS = \frac{1}{T} (d\bar{E} + dW) = dQ/T$. Conclusion: in any quasi-static process, $dS = dQ/T$, even if external params. are changing, i.e., even if there is a combined thermal and mechanical interaction. Corollary: for a quasi-static process which is adiabatic, entropy cannot change. I.e., $QS: dQ = 0 \Rightarrow dS = 0$.

3.10 About Entropy. Since $S = k \ln \Omega$, S is a state function and $dS = dQ/T$ is exact differential even though dQ is not. So idea that it is hard to cool off nuclear spins, so as $T \rightarrow T_0$ (cool enough to stop thermal atomic motions), $S \rightarrow S_{\text{spin}}$. So also, $S =$ unique function since $h_0 = h$ in QM, - phase space cell size cannot be made smaller.

3.11 Summary of the Laws of Thermo.

Zero: if A. te. B and B. te. C, then A. te. C, $\Rightarrow$ thermometers.

First: $d\bar{E} = dQ - dW$; $\bar{E} =$ state fcn; isolation $\Rightarrow d\bar{E} = 0$.

Second: a) for ~~reversible~~ thermally isolated system, $dS \geq 0$
b) for general quasi-static change, $dS = dQ/T$.

Third: as $T \rightarrow 0^+$, $S \rightarrow S_0$ (S_{spins} , eg.).

Note: These laws are all macroscopic + form basis for "Thermo." even with no microscopic knowledge of what is going on.

3.12 Equation of State appears microscopically as $\bar{X}_i = \frac{1}{\beta} \partial \ln \Omega / \partial x_i$. Eg, for ideal gas we showed that $\Omega \propto V^N X(E)$. Thus, you get at once $\bar{p} = nkT$ or $\bar{p}V = \nu RT$ where $R = kN_A$. I.e., $\bar{p} = \bar{p}(V, T)$. As a bonus, since $\beta = \partial \ln \Omega / \partial E$, you learn that $\bar{E} = \bar{E}(T)$ only, not $\bar{E}(T, V)$.

Chapter 4: How to Measure Macroscopic Parameters. (22 pages)

4.1 Macrostate. To specify a macrostate, you give values all all state fctrs such as $\bar{E}$, external params like V , gen forces like $\bar{p}$. A macrostate is a point on diagrams.

4.2 Measure W & Q. Insulate system, do wkt W_{ext} , cause change $\Delta \bar{E}$. Or, if not insulated, Q is the excess.

4.3 Measure T. Use ideal gas thermometers. Triple point of water is arbitrarily set at $T_E = 273.16$; $T_{\text{boiling}} = 0$. This then sets the size of a Kelvin degree.

4.4 Specific Heat. Let $y =$ things held constant. Then $C_y \equiv (\partial Q / \partial T)_y =$ the specific heat generally. Usually, one says $c_y = (\partial Q / \partial T) \frac{1}{\nu} =$ specific heat per mole, or per gram. For water, $C_p = (\partial Q / \partial T)_p = 1 \text{ calorie/gm} = 4.18 \text{ joules/gm}$. Methods to measure are given. Since in C_p case V can change, injected ∂Q goes into wkt and $\bar{E}$, so dT is less with C_p than C_v . Thus, $C_p > C_v$ usually.

4.5 Measure S: do this by $\Delta S = \int \partial Q / T$, so $S = \text{joules/degree}$.

4.6 Absoluteness of S. Let's you measure S exactly...? ship

4.7 Intensive/Extensive parameters. If param doubles when system doubles, then it is extensive (like V etc); $\bar{p}$ is intensive, as is β, T, S .

Chapter 5: Applications of Macro-Therm. (40 pages)

5.1 Ideal Gases. Variables may be taken as T, V . Gas law here taken as empirical fact, $pV = \nu RT$. For quasistatic processes, we derive the famous result $TdS = dE + p dV$. By doing partials, this plus gas law shows that $(\partial E / \partial V)_T = 0$, i.e., $E = E(T)$ only (as we found earlier by micro-analysis). "Joule Expansion" of ideal gas shows no change ΔT in water tank $\Rightarrow Q = 0 \Rightarrow \Delta E = 0$ since $\partial W = 0$ also. This proves explicitly that $E = E(T)$ and not $E(T, V)$.

5.2 Specific Heat Consider $dQ = dE + p dV$. For c_v , $c_v = \frac{1}{\nu} \left(\frac{\partial E}{\partial T} \right)_V$ and thus $c_v = c_v(T)$ only, ideal gas. Gas law itself does not tell us $E(T)$ function, though. By fiddle, $c_p = c_v + R$, specific heats per mole. $\gamma = c_p/c_v = 1 + R/c_v$. For air, $\gamma = 1.4$ or so. This enters into speed of sound. As special case, consider monatomic ideal gas. From microphysics we learn that $E = \frac{3}{2} \nu R T$, explicit! Then $c_v = \frac{3}{2} R$ and $c_p = \frac{5}{2} R$ and $\gamma = 5/3$.

5.3 Adiabatic Gas Law. Use $pV = \nu R T$ and $dQ = 0$ to show that $\gamma dV/V = -dp/p$. But $\gamma = 1 + R/c_v$ and $c_v = c_v(T)$ for ideal gas. Actually, $c_v = \text{constant}$, at least over reasonable range of T , so get $pV^\gamma = \text{const}$. Isothermal gas law of course is $pV = \text{const}$.

5.4 Entropy of Ideal Gas = get by integrating $T dS = \dots$ skip.

5.5,6 Maxwell Relations. Of the variables S, V, T, p, E which describe a system "of the $p dV$ type", not necessarily an ideal gas, only two are independent. One relation among them is $dE = T dS - p dV$, the first law. This single relation gives a set of relations:

$$\left(\frac{\partial T}{\partial V} \right)_S = - \left(\frac{\partial p}{\partial S} \right)_V \text{ and 3 more.}$$

But in auxiliary fashion introduces several mysterious new state fctns:

$$dE = T dS - p dV$$

$$dH = T dS + V dp$$

$$dF = -S dT - p dV$$

$$dG = -S dT + V dp$$

$$E(S, V)$$

internal energy

$$H(S, p) = E + pV$$

"enthalpy"

$$F(T, V) = E - TS$$

"Helmholtz free energy"

$$G(T, p) = E - TS + pV$$

"Gibbs free energy".

Each "energy" seems connected with a certain pair-choice of variables. Not clear why all these guys introduced.

5.2 Specific Heats. A note on "the minus sign". Let $dA = \left(\frac{\partial A}{\partial B} \right)_C dB + \left(\frac{\partial A}{\partial C} \right)_B dC$. Then we want to divide by, say, dC , holding A constant. Then: $0 = \left(\frac{\partial A}{\partial B} \right)_C \left(\frac{\partial B}{\partial C} \right)_A + \left(\frac{\partial A}{\partial C} \right)_B \Rightarrow \left(\frac{\partial A}{\partial C} \right)_B = - \left(\frac{\partial A}{\partial B} \right)_C \left(\frac{\partial B}{\partial C} \right)_A$. The chain rule is a little weird here.

Now we seek a relation (for $p dV$ material, solid, whatever) which relates c_v and c_p generally. Good illustration of thermo partial derivative manipulation. You want to avoid appearance of $\left(-\frac{\partial A}{\partial B} \right)_V$ since constant volume expts hard to do for a solid; also avoid any $(\partial S/\partial T)$ except of course $(\partial S/\partial E) = 1/T$.

If you define: $\alpha \equiv \frac{1}{V} \left(\frac{\partial V}{\partial T} \right)_P$ = volume coeff. of expansion.

$$K \equiv -\frac{1}{V} \left(\frac{\partial V}{\partial P} \right)_T = \text{isothermal compressibility.}$$

Now you find that $C_P = C_V + VT^2/K$. For solid, $\alpha V/AT$ is small, α small, α^2 smaller, so $C_P \approx C_V$ for a solid. So $\alpha \approx 1$. Note that as $T \rightarrow 0$, all C_P, C_V and $\alpha \rightarrow 0$. Also, as $T \rightarrow 0$, ideal gas approx fails!

5.8 Entropy & Internal Energy. Everything may be computed from knowledge of S. So what do you need in order to know S (macroscopic & statistical). Answer: a) equation of state! b) $C_V(T)$ at some particular V. Proof given (AT No).

Aside: Van der Waals gas equation of state is $\left(P + \frac{a}{V^2} \right) (V - b) = RT$. Here, b = volume occupied by molecules [clear range for effect], and a = long range attractive effect which seems to increase pressure. V = molar volume. Conclusions: $C_V = C_V(T)$ only, as with ideal gas [$a=b=0$]. But molar energy $E = E(T, V)$, unlike ideal gas.

5.9 Free Expansion of Gas. Assume two chambers and adiabatic. Since volume, $dV = 0$ and $dQ = 0$, so $dE = 0$ even though not quasi-static process. [Quasi-static thing is needed in $dS = dQ/T$, not just law]. So, you want to know final temperature of gas. Set $E(T, V) = E(T_2, V_2)$. For ideal gas, conclude $T_2 = T_1$. For Van der Waals, $T_2 = T_1 - \frac{a}{V_2} (V_1^{-1} - V_2^{-1})$.

5.10 Joule-Thomson Throttling. This is continuous flow thru constriction. In steady state, for SM of gas, $dQ = 0$ since walls have same T as gas. Gas data unit, though, unlike in free expansion. If, $dU = P_2 V_2 - P_1 V_1$, then, $dE = E_2 - E_1 = -P_2 V_2 + P_1 V_1$, a $E_2 + P_2 V_2 = E_1 + P_1 V_1$. On, $H_1 = H_2$. In throttling, entropy H is conserved - a justification for defining H ! E not conserved since work being done. For ideal gas, $H = E(T) + PV = E(T) + VRT = H(T)$, so $\Delta T = 0$. In throttling of ideal gas.

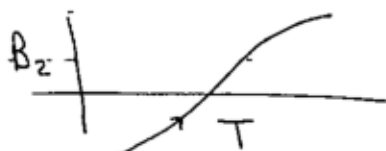
The "throttling curve" for a typical real gas (N_2) are shown in Fig. 5.10.3. Curves of const. H . In order to use throttling to cool a gas, you must be inside the inversion curve, otherwise you will actually heat the gas. Curve has a T_{max} . Thus, to cool the this way, you must pre-cool to $37^\circ K$ with liquid N_2 prior to applying Joule-Thomson effect. How to explain existence of this inversion curve made of which $\mu > 0$ + gas cools? Consider this so-called.

"virial expansion" for the gas law when $n = N/V$ is small:

$$p = kT [n + B_2(T)n^2 + B_3(T)n^3 + \dots]$$

↑ virial coeffs.

At low temp, long range attractive tend to reduce the pressure, i.e., gas sucks itself together. Then $B_2 < 0$. (Same in vdW eqn). At high T , collisions dominate, atomic size increases p , $B_2 > 0$. In general, you have:



Now $\mu = \frac{N}{C} (T \frac{\partial B_2}{\partial T} - B_2)$, so this explains boundary at $\mu = 0$ as battle between these opposing effects.

5.11 Heat Engines. Perfect heat engine $q \rightarrow w$ cannot work because you are reducing the entropy of bath. But note $dS = dQ/T$. If you dump off some heat to a low- T output reservoir, small dQ gains you lots of $+dS$. This leads to quasi-static efficiency $\eta = (T_{in} - T_{out})/T_{in}$. Recall the initial heat engine: Carnot Engine uses two reservoirs. All engines which run on heat must reject heat to the environment.

5.12 Refrigerators. Again, in order to pull q_2 out of ice box, you must dump q_1 at T_1 . So, $w = q_1 - q_2 > 0$, so $q_1 = q_2 + w$, you dump heat = work + q_2 . Efficiency is smaller. Note: statement that "you cannot make a perfect heat engine (refrigerator)" is Kelvin (Clausius) form of Second Law.

Chapter 6: Basics of Statistical Mechanics (i.e., microscopic theory). (32p).

6.1 The "equally likely" postulate. If ensemble is in E to $E + \delta E$, equally likely to be in any accessible state.

6.2 The Boltzmann Factor. Put small A in contact with large reservoir A' . We already showed that $P_A(E) = C \Omega(E) \Omega'(E^0 - E)$. We also showed that if both Ω^s are rapidly increasing, this thing looks like a gaussian. [page 109]. Both A and A' had to be large systems. Now we consider A large but A' could be a small system. We leave $\Omega(E)$ as is, + write

$$\ln \Omega'(E^0 - E) = \ln \Omega'(E^0) - E \left(\frac{\partial \ln \Omega'(E^0)}{\partial E^0} \right) + \sigma(E^2)$$

$$\Rightarrow \Omega'(E^0 - E) = \Omega'(E^0) e^{-\beta E}$$

= β (reservoir) (drop if $E \ll E^0$)

Note that you need $E \ll E_0$ to be able to say this. So result is

$$P(E) = c \Omega(E) e^{-\beta E}, \text{ valid for } A \ll A', A \text{ arbitrarily small.}$$

Interpret: if $\Omega(E) = \text{const}$, as for a single system $A = \text{atom}$, then as you increase energy to A , that energy is subtracted from reservoir, severely reducing # states accessible to reservoir. Thus, unlikely!

6.3 Application of $e^{-\beta E}$ Paramagnetism. Consider a solid with N_0 spins in external H field. Let $A = \text{one spin}$, $A' = \text{the phonon bath somehow}$. If a spin goes against H , it takes up energy, making less likely. So

$$P_{\pm} = C e^{\pm \frac{\mu H}{kT}} \Rightarrow \bar{\mu} = \mu \tanh\left(\frac{\mu H}{kT}\right).$$

At high T , $\bar{\mu} = \mu \cdot \frac{\mu H}{kT}$ and $\bar{\mu} \rightarrow 0$ as $T \rightarrow \infty$ as $\bar{\mu} \propto T^{-1}$, this is Curie's Law.

At low T , $\bar{\mu} \approx \mu$, complete alignment. At high T , $M = N_0 \bar{\mu} = \left(\frac{N_0 \mu^2}{kT}\right) H$ so $\chi = N_0 \mu^2 / kT$, or $\chi \propto T^{-1}$.

By the way, an ensemble of such spins each obeying $e^{-\beta E}$ is called a canonical ensemble.

The Maxwell Velocity distribution is same idea: $A = \text{free molecule}$ $A' = \text{the rest of the gas} = \text{collision bath}$. So $P(v) = \exp(-\frac{1}{2}mv^2/kT)$. If there is gravity, $E = \frac{1}{2}mv^2 + mgz$. Thus you have a $P(v, z)$. Integrate over v to get $P(z) = \exp(-mgz/kT)$ as usual.

6.4 Mean Energy If A can be in any of states r with E_r , then clearly $\bar{E} = \frac{\sum_r E_r e^{-\beta E_r}}{\sum_r e^{-\beta E_r}}$. Just a mean.

6.5 Partition Function. Define $Z = \sum_r e^{-\beta E_r}$ for a system A . Then

$$\bar{E} = -\frac{\partial \ln Z}{\partial \beta} \quad [\Delta^2 E] = \frac{\partial^2 \ln Z}{\partial \beta^2} = \text{square of dispersion away from } \bar{E}.$$

Now suppose you change volume V . Each E_r shifts a little, so you can compute the change in $\bar{E}$, which is then work done. But $dW = pdV$, so conclude $\bar{p} = \frac{1}{\beta} \frac{\partial \ln Z}{\partial V}$. Generally:

$$X_\alpha = \frac{1}{\beta} \frac{\partial \ln Z}{\partial x_\alpha} \quad \text{General formula for equation of state for a canonical ensemble.}$$

6.6 How is Z related to S ? By direct calc,

$$S = k(\ln Z + \beta E) = \text{entropy of system } A = k \ln Z.$$

So compute $Z = \sum_r \tilde{c}^{\beta E_r}$ to get S ; easier than computing R . Also,

$$F \equiv E - TS = \text{Helmholtz f.e.} = -kT \ln Z = F \quad F \approx Z$$

Also, if A, A' are two weakly coupled systems, $Z^0 = Z Z' \Rightarrow S^0 = S + S'$, obvious.

6.7 Approx. Trick Suppose you want to compute $\bar{y}$ (some mean) for an isolated system of width δE . Hard to do because of energy constraint. As a good approx, let your system A contact a bath A' at same T . Then you have created a canonical ensemble and $\bar{y} = \frac{\sum_r \tilde{c}^{\beta E_r} y_r}{\sum_r \tilde{c}^{\beta E_r}}$ without any restriction on sum. Answers nearly the same! Do not use this trick to compute a dispersion, however!

6.8 skip

6.9 Grand Canonical Ensemble Here you allow your small system A to exchange energy and particles with reservoir A' . Now, if you take either energy or particles from A' , you reduce R' . Thus, you get $P = \exp(-\beta E_r - \alpha N_r)$ where $\beta = \partial \ln R' / \partial E$ and $\alpha = \partial \ln R' / \partial N'$. This type ensemble is useful when it is helpful to remove restriction to fixed # particles when you are doing a calc. I.e., take an isolated system and "grand ensemble" it. Of course $E_{\text{isolated}} \approx E_{\text{gce}}$ and $N_{\text{iso}} \propto N_{\text{gce}}$. There are also other types of ensembles.

6.10 Another Derivation of $\tilde{c}^{\beta E_r}$. On what way should particles distribute themselves in the E_r levels so that $P(a_1, a_2 \dots a_r \dots)$ is a max? 5 in state r , but $r+1$?? I.e., want to max out $\ln P = \alpha \ln a - \sum_r \ln a_r$ subject to $E_{\text{tot}} = a$ and $\sum_r a_r E_r = a E$. Answers: each a_r takes value $\tilde{a}_r = e^{-\beta E_r}$. Hence, you get the canonical ensemble. Note that here we consider all particles at once, no separation into $A+A'$. Deviations from $\tilde{c}^{\beta E_r}$ simply reduce $R(E)$, no more no less. Note: no restriction as to how many particles in level E_r !

Chapter 7: Applications of Stat Mech (46 pages)

7.1 Partition Fcn If you can compute $Z = \sum e^{-\beta E}$, then the entire problem is solved. You can then compute anything by taking derivatives!
 • Zero of energy scale has no effect, really.

7.2 Ideal Monatomic Gas For any gas of N particles, energy is $E = \sum \frac{p_i^2}{2m_i} + V(r_1, r_2, \dots)$. Treat this entire gas as system A. Then sum over states, i.e. usual phase space. For ideal gas $V(r_1, \dots) = 0$, no attractive. Monatomic $\rightarrow$ no internal energies to deal with.
 For ideal gas $V=0 \Rightarrow V^N$. Direct calculation yields:

$$\ln Z' = N \ln \xi \quad \text{where } \xi = V \left(\frac{2\pi m}{h^2 \beta} \right)^{\frac{3}{2}}$$

From this Z' you compute: $\bar{p} = \frac{1}{\beta} \frac{\partial \ln Z'}{\partial V} = nkT$ (state!). Also $\bar{E} = \frac{3}{2} kT$; also $C_V = \dots$. But

$$S' = k(\ln Z' + \beta \bar{E}) = Nk \left[\ln V + \frac{3}{2} \ln T + \text{const } \sigma \right] \quad \text{is wrong!}$$

7.3 Gibbs Paradox. First paradox is $S' \rightarrow -\infty$ as $T \rightarrow 0$. This is because we have assumed continuum of states for molecules which is not true at low temp where discreteness more apparent. Gibbs Paradox is this: S' as given above is not extensive! A box of volume $2V$ has more than twice the entropy of two boxes of volume V ! This is true if two separate gases. But if molecules are identical, then we have overcounted and

$$Z = Z' / N! \Rightarrow \ln Z = \dots \Rightarrow S = kN \left[\ln \frac{V}{N} + \frac{3}{2} \ln T + \sigma \right]$$

and now S is extensive.

7.4 When can you ignore quantum? For an ideal gas, apart from the identity problem, you can use "classical stat mech" as long as $R\bar{p} \gg \hbar$, where R is mean separation. Criterion is then:

$$\text{classical OK if } \left(\frac{1}{n} \right)^{\frac{1}{3}} \gg \frac{h}{\sqrt{3mkT}}$$

Say you are OK at low n , high T , high m . He at room temp is OK (STP) but electrons in metal no go (electrons are light!!).

7.5 Equipartition Theorem. For each quadratic p^2 term in H , $\bar{\epsilon}_i = \frac{1}{2} kT$. Theorem is stated very generally. Only valid when $\Delta E_{\text{levels}} \ll kT$. I.e., spacings narrow compared to kT .

7.6 Examples of Equipartition. For any gas, $\bar{K} = \frac{3}{2} kT$. For Brownian motion, $\bar{x} = 0$ but $\bar{x}^2 = kT/m$, observable for small m . For harmonic oscillators, each of two quadratic terms gets $\frac{1}{2} kT$, so $\bar{E} = kT$. Reif shows how this is limit of quantum result at high T .

7.7 Solids' Specific Heats. For ideal monatomic gas, $C_V = \frac{3}{2} R$ because $\bar{E} = \frac{3}{2} NkT$. For a "simple" solid, you add $\frac{3}{2} kT$ for the motion of each atom, plus $\frac{3}{2} kT$ stored in "the springs". I.e., each atom in lattice behaves as 3 HO's. Thus expect $C_V = 3R$.

Above the Einstein temperature $\Theta_E \equiv \hbar\omega/k$ [where ω = frequency of each HO], all simple solids should have $C_V \approx 3R$. Note that C_p and $C_V \approx$ same for incompressible solids. This result is true! $C_p = 3R \approx 25$ joules/mole/degree, see Table p 255.

Below Θ_E , you can use the quantum HO to learn that $C_V \rightarrow 3R(\Theta_E/T)^2 \exp(-\Theta_E/T) \rightarrow 0$ as we know on general grounds. This too is only an approx.; springs do not have a single $\omega = \omega_0$, and collective motions also neglected.

7.8 More Paramagnetism. The earlier discussion is here generalized to case of an atom (or nucleus) of arbitrary spin $\mu = (gJ)\mu_B$ Bohr magneton. g allows for combination of L and S in atom. You compute Z in obvious way, a $(2J+1)$ -fold sum. Results:

$$Z = \frac{\sinh((J+\frac{1}{2})\eta)}{\sinh \frac{1}{2}\eta} \quad \text{where } \eta = gJ\mu_B/kT$$

$$\bar{\mu} = \frac{1}{\beta} \frac{\partial \ln Z}{\partial H} = (gJ\mu_B) \times B_J(\eta), \quad \text{The "Brillouin function"}$$

Again, at hi- T you get a Curie Law. Note: Since $\mu_N \approx \frac{1}{1836} \mu_B$, you need 1836x lower T to line up nuclear spins !!

7.9 More Maxwell Velocity Distrib. Using basic canonical ensemble, we find as before that $f(\vec{r}) d^3\vec{r} d^3\vec{v} = n (m/2\pi kT)^{3/2} \exp[-\frac{1}{2} m\vec{v}^2/kT] d^3\vec{r} d^3\vec{v} =$ velocity distribution. This holds for a dilute (but possibly polyatomic) gas where $\epsilon^{\text{internal}}$ = independent of all separations $\vec{r}_i$. In computing $P(\vec{r}, \vec{p})$, you sum over internal states, they have no effect on the result. Although polyatomic, this gas is still "ideal", not eg. Van der Waals. Note: ideal gas has $pV = NkT$.

2.10 More velocity stuff. $g(v_x)dv_x = \dots$, centered at $v_x = 0$, so $\bar{v}_x = 0$. But $\bar{v}_x^2 = kT/m$ of course by E.Th. The speed distribution is $F(v) \propto v^2 \exp(-\frac{1}{2} \frac{mv^2}{kT})$ where v^2 arises from d^3v . $\bar{v} = \sqrt{8kT}$ will all be close.

$$\bar{v} = \sqrt{\frac{8}{\pi}} \sqrt{\frac{kT}{m}} \quad v_{rms} = \sqrt{3} \sqrt{\frac{kT}{m}} \quad v_{peak} = \sqrt{2} \sqrt{\frac{kT}{m}}$$

2.11 Flux Against a Surface. This is "kinetic theory". Let $\Phi_0 = \text{flux}$. By crude avg. $\Phi_0 = \frac{1}{6} n \bar{v}$; exact calc using $F(v)$ gives $\Phi_0 = \frac{1}{4} n \bar{v}$. With gas laws you can learn that $\Phi_0 = \bar{p} / \sqrt{2\pi m kT}$, so flux increases with $\bar{p}$. At constant $\bar{p}$ (balloon), raising T decreases Φ_0 .

2.12 Effusion. If, flow thru a small hole based on $\Phi_0 = \frac{1}{4} n \bar{v}$. If hole connects two bottles, you learn that

$$\frac{\bar{p}_1}{\sqrt{T_1}} = \frac{\bar{p}_2}{\sqrt{T_2}}$$

at balance point $\Rightarrow$ Higher T will be in higher p compartment. If, pressure does not equalize!! Pressure does not "feed" through a small hole!

Note: due to mass in Φ , effusion can be used to separate gaseous isotopes.

2.13 Pressure. Crude arg. says $\bar{p} = \frac{1}{3} n m \langle \bar{v}^2 \rangle$. Exact gives same but with $\bar{v}^2$ this plus equipartition $\langle \frac{1}{2} m \bar{v}^2 \rangle = \frac{3}{2} kT$ gives ideal gas law!

Chapter 8: Equilibrium Between Phases (38 pages)

8.1. Isolated System. Let y be a parameter describing isolated system, eg, fraction of ice in water-ice mix. We expect y to assume a value which maximizes the states available to system, i.e., maximizes $S(y)$ or $S(y)$. If you pull y away from $\bar{y}$, it will try to return.

8.2 System touching T-reservoir. Some process occurs within system A. Suppose A does some work W , and absorbs some heat Q from A' . By simple calculation, $\Delta S_{tot} = \Delta S = (-\Delta E_0 - W)/T_0$ where $F = E - TS$. But this change ΔS must be ≥ 0 . Thus, $W \leq -\Delta E_0$. This is a bound on how much work A can do. If you clamp all work-parameters $\Rightarrow dW = 0$ in process, then we find that $\Delta E_0 \leq 0$. The Helmholtz free energy can only decrease! Equilibrium attained when $F_0 = \min$; no further spontaneous process can occur.

8.3 System with T & P reservoir. Breaking $W = W^* + SpdV$. Then obviously can conclude $W^* \leq -\Delta G_0$ where $G = E - TS + pV$. This puts a bound on the max non-pdV work you can get out of all params (except V). If we clamp, any process must have $\Delta G_0 \leq 0$. In equilibrium

you will then have $G = \min$.

Moreover: in the Helmholtz case, you can deduce likelihood of being away from the min: if $W=0$, then $\Delta S = -\Delta F/T$ so $S \equiv S_{\min} - \Delta F/T$
 $\Rightarrow P(y) \sim \exp[-F(y)/kT]$ i.e., as you vary y either way, $F(y)$ increases, $P(y)$ decreases.

8.4 Stability of Matter Consider piece A of a substance. A is embedded

in a T, P reservoir. At equilibrium we expect to have $G(A) = \min$.

This occurs when $T_A = \bar{T}$, same μ value. Condition that $G = \min$ is

shown same as $\bar{T} = T_0$ (reservoir), as expected. The fact that G curves up

$\Rightarrow V > 0$, as earlier.

So line $C > 0$ guarantees stability against temperature fluctuations.

Repeat the discussion for volume fluctuations: the zero-slope of G

gives $\bar{P} = P_0$, the curve-up gives $K = -1/V(\partial V/\partial P)_T > 0$. Obviously it would collapse if negative compressibility!

We can actually compute the dispersion for volume (density) fluctuations.

One gets $\langle \delta n^2 \rangle = -\partial^2 / (\partial \mu^2) (kT_0 K / \bar{V})$. Large volume has smaller dispersion.

If material "goes soft" and $K \rightarrow \infty$, you get huge fluctuations, a "critical point". (Clean fluid) becomes white.

8.5 Phase Equilibrium. Let A = system with one "component" (H_2O)

and two "phases" 1 and 2. A is in contact with a T, P reservoir.

A adjusts so as to minimize its G . But $G = \sum_i \mu_i + V_0 g_2$ where

$\mu_2 = \mu$ (chemical potential of phase 2). But $\mu_1 + \mu_2 = \mu = \text{fixed}$. Condition for a

minimum is $g_1 = g_2$. If $g_1 \neq g_2$, system will go entirely over to one or the other phase. (The one with lower g).

Each phase is described by a function $g_i(T, P)$. The 2-phase

line is given by $g_1(T, P) = g_2(T, P)$. This leads at once to the

Clausius-Clapeyron equation: gives slope of equilibrium line in

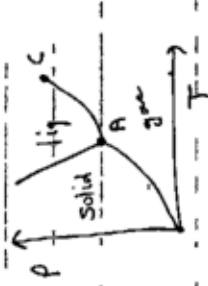
the $P-T$ plane in terms of entropy change ΔS and

$\frac{dp}{dT} = \frac{\Delta S}{\Delta V}$ or volume change ΔV for a sample crossing the

line at point of interest. Since $\Delta S = \Delta Q/T$,

and $\Delta Q = q = \text{moles heat change}$ given off,

you get $dp/dT = \Delta H/T \Delta V$. This is the "latent heat". Typical phase diagram is shown:



A = the triple point $\equiv$ temperature standard 273.16000°

C = critical point beyond which no gas-liquid difference.

For liquid-gas line, if you assume $n_{\text{gas}} \gg n_{\text{liq}}$

and $V = \text{volume of } T$,

$P_{\text{gas}} = P_0 \exp(-\Delta H/RT)$ has slope $\frac{dP}{dT}$ independent.

8.6 Phase Changes. For a typical non-ideal gas, Fig 8.6.1 shows iso thermas in p - V plane. For ideal gas, curves are all like T_c on the curve, give in graphical form the complete liquid/gas equation of state. If p is above the critical pressure, only one volume V is possible, given p , as in ideal gas. For example, liquid/gas CO_2 above critical pressure. Below point, for a given temp (say room T) there are two potentially stable V 's on same p ; system chooses phase of least g . Fig 8.6.3 shows how, at some intermediate pressure the system kicks over from gas to liquid as you reduce the system volume isothermally (Fig 8.6.2).

At critical point, $(\partial p / \partial V) = 0$. Also $\partial^2 p / \partial V^2 = 0$. The "both phases" have same volume at same p . You then have just a "fluid".

8.7 Many Components. Suppose you have many species $N_1 + N_2 + \dots + N_m$. If N_i can change in a process, it becomes a thermodynamic variable. What you learn is that:

$$dE = TdS - pdV + \sum_i \mu_i dN_i$$

where $\mu_i = -T \left(\frac{\partial S}{\partial N_i} \right)_{E, V, N_{j \neq i}} = \left(\frac{\partial F}{\partial N_i} \right)_{T, V, N} = \left(\frac{\partial G}{\partial N_i} \right)_{T, p, N} =$ "chem potential" per molecule.

Assume to be energy, studying a molecule, adds to dE . For one component present $\mu = g$, the Gibbs free energy (per molecule).

8.8 Another view of 1-component, 2-phase equilibrium. Conclude that $\mu_1 = \mu_2$, nothing more.

8.9 Chemical Equilibrium. Let's review our various external principles:

- for fixed E and V (no work), S is max.
- for fixed T and V , F is min.
- for fixed T and p , G is min.

All three cases lead to same equilibrium condition: $\sum_i \mu_i = 0$.

Since the μ_i are fctns of all chosen variables like

$\mu_i = \mu_i(T, V, N_i, \dots, N_r)$, one condition will determine the N_i for given T, V .

8.10 Chemical Potentials. Let's choose a fixed V, T situation. A chemical reaction goes on described by chem equation $\sum_i \nu_i B_i = 0$. The condition of equilibrium is to minimize the Helmholtz

free energy F . This leads at once to $\sum b_i \mu_i = 0$, so all we need are the μ_i . But $\mu_i = (\partial F / \partial N_i)_{T, V, N}$ so let's compute F , then differentiate. But recall $F = -kT \ln Z$ so we need only Z . But $Z = \tilde{Z}_1^{N_1} \tilde{Z}_2^{N_2} \dots \tilde{Z}_m^{N_m}$ where $\tilde{Z}_i =$ partition fun for one molecule of type i . But $\tilde{Z}_i \equiv \mathcal{E}_i =$ internal state sum $\sum e^{-\beta \epsilon_i}$, which you determine from energy levels of that molecule (I guess $\epsilon_0 = 0$ for ground state; higher states less important. Now throw in Gibbs paradox to get

$$Z = \prod_{i=1}^m \tilde{Z}_i^{N_i} \quad \text{where} \quad \tilde{Z}_i = (\mathcal{E}_i)^{N_i} / N_i!$$

This product rule leads to various obvious results ($\sum p_i = 1$). Note that we are always in a weak-coupling limit! So $\ln Z_i = N_i \ln \mathcal{E}_i - \ln N_i!$ Use Stirling, get $F = -kT \sum N_i (\ln \mathcal{E}_i - \ln N_i + 1)$.

$$\mu_i = -kT \ln \left(\frac{\mathcal{E}_i}{N_i} \right) \quad \mu_i = -kT \ln \left(\frac{\mathcal{E}_i(T)}{n_i} \right)$$

where $\mathcal{E}_i = \sum e^{-\beta \epsilon_i}$ for one molecule, including regular phase space! Thus $\mathcal{E}_i \propto V \mathcal{E}_i'$ and then
Now group things

$$\sum \mu_i b_i = 0 \Rightarrow \sum b_i [-kT \ln \mathcal{E}_i + kT \ln N_i] = 0$$

$$= -\Delta F_0(T, V)$$

$$\Rightarrow \frac{\Delta F_0}{kT} = \ln(N_1^{b_1} N_2^{b_2} \dots) \Rightarrow N_1^{b_1} \dots N_m^{b_m} = e^{-\Delta F_0(T, V)/kT}$$

But better way is this: $\sum b_i [-kT \ln \mathcal{E}_i'(T) + kT \ln n_i] = 0$.
 $\Rightarrow n_1^{b_1} \dots n_m^{b_m} = (\mathcal{E}_1')^{b_1} (\mathcal{E}_2')^{b_2} \dots (\mathcal{E}_m')^{b_m} = K(T)$ ← equilibrium constant.

So this is the old high-school chem rate equation. In principle we can calculate $K(T)$ by getting the $\mathcal{E}_i'$.

Note: for a many-gases experiment, const. V makes sense. I also easy to do. Constant p requires a float of some sort. A "piston".

Chapter 9: Quantum Statistics for Ideal Gas.

9.1 Definitions. Maxwell-Boltzmann statistics treats all particles as distinguishable, so $\exists$ many states. If 2 particles + three levels, then $3 \times 3 = 9$ possibilities. Are MB stats ever correct??

In BE statistics, indistinguishable + sym Ψ . Thus, only 6 states. In FD, indist. + antisym $\Psi \Rightarrow$ only 3 states.

9.2 Formulation of the problem. A gas has N objects; individual object energies are the ϵ_i (level energies); occupation is n_i . Whole-gas states are labelled by R . For any statistics you get:

$$Z = \sum_R e^{-\beta E_R} = \sum_R e^{-(\sum_i n_i \epsilon_i) \beta} \quad \text{subject to } \sum_i n_i = N \text{ and} \\ \text{subject to stat requirements.}$$

Once you know Z , then

$$\bar{n}_s = -\frac{1}{\beta} \frac{\partial \ln Z}{\partial \epsilon_s} \quad (\Delta n_s)^2 = -\frac{1}{\beta} \frac{\partial \bar{n}_s}{\partial \epsilon_s}$$

It is Z that differs in various cases.

9.3 Compute the $\bar{n}_s$. Here, $\bar{n}_s = \sum n_s e^{-\beta \epsilon_s} / \sum e^{-\beta \epsilon_s}$ is calculated directly for each statistics class, i.e., we do not compute Z first. The main problem here is remembering that if you use up $n_s = 1$ in first sum, you are left with $N-1$ in second sum. I just quote the results: well, actually, I prefer the Z -derivations in next sections.

9.4 MB statistics. For each set of #'s $\{n_1, n_2, \dots\}$ there are many different states you can make, since distinguishable. In fact $N!$ ways, corrected by $n_1! n_2! \dots$. So

$$Z = \sum_{n_1, n_2, \dots} \left(\frac{N!}{n_1! n_2! \dots} \right) e^{-\beta(n_1 \epsilon_1 + n_2 \epsilon_2 + \dots)} \quad \text{subject to } \sum n_i = N.$$

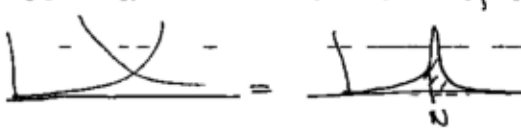
But $(a_1 + a_2 + \dots + a_m)^N = \sum_{n_1, n_2, \dots} \frac{N!}{n_1! n_2! \dots n_m!} (a_1)^{n_1} (a_2)^{n_2} \dots (a_m)^{n_m}$ / multinomial theorem.

$$Z = [e^{-\beta \epsilon_1} + e^{-\beta \epsilon_2} + \dots]^N = Z^N \quad \text{where } Z = \sum_i e^{-\beta \epsilon_i}$$

$$\text{Then } \bar{n}_s = -\frac{1}{\beta} \frac{\partial}{\partial \epsilon_s} [N \ln(e^{-\beta \epsilon_1} + e^{-\beta \epsilon_2} + \dots)] = N \left[\frac{e^{-\beta \epsilon_s}}{Z} \right]$$

9.5 Photon Statistics. At once, $\ln Z = -\sum_r \ln[1 - e^{-\beta \epsilon_r}] \Rightarrow \bar{n}_s = \frac{1}{e^{\beta \epsilon_s} - 1}$
 $\uparrow$ "Planck"

9.6 Bose-Einstein Case. A trick is used; i.e., we do a trick which allows a full sum over N , because then you can do the sum. Similar to putting gas in contact with a reservoir with which it can exchange particles (grand canonical idea). Now $N \rightarrow \bar{N}$ which has small fluctuations. Here, trick is to realize that $Z(N)$ rapidly increases w. N , so much by a dying expo such that peak is at N .

I.e., define $\sum_{N'} Z(N') e^{-\alpha N'} = \mathbb{Z}$. Look at various lens: i.e., consider as an integral $\frac{\sum_{N'} Z(N') e^{-\alpha N'}}{\int dN Z(N) e^{-\alpha N}}$.  $\approx Z(N) e^{-\alpha N} \cdot \Delta^*$ (width) $\rightarrow \ln \mathbb{Z} \approx \ln Z - \alpha N + \ln \Delta^*$ neglect.

Recall that $Z \sim \Omega$ and $\Omega \sim (E)^f$ for system energy $E \Rightarrow \ln Z \sim N \ln E \sim N$. But now $\mathbb{Z}$ is easy to compute since no restriction ~ same as photon Z except α in there. Result:

$$\ln Z|^{BE} = \alpha N - \sum_r \ln [1 - e^{-\alpha - \beta \epsilon_r}] \quad \text{where } \alpha = -\beta \mu.$$

$$\Rightarrow \bar{n}_s = \left[\frac{1}{e^{\alpha + \beta \epsilon_s} - 1} \right]. \quad \text{Note that } \alpha = \frac{\partial \ln Z}{\partial N}, \quad \mu = \left(\frac{\partial F}{\partial N} \right). \quad \text{I don't}$$

yet know how to compute μ . (oh so I!!) For single component gas, $\mu = -kT \ln(\xi'/n)$ so $\alpha = \ln(\xi'/n)$ where $\xi' =$ p.f. of one gas molecule, excluding the volume.

9.7 Fermi-Dirac Statistics. Similar to BE. Result uses same $\mathbb{Z}$ trick. We get

$$\ln Z|^{FD} = \alpha N + \sum_r \ln [1 + e^{-\alpha - \beta \epsilon_r}]$$

$$\Rightarrow \bar{n}_s = \left[\frac{1}{e^{\alpha + \beta \epsilon_s} + 1} \right]. \quad \left\{ \begin{array}{l} \frac{1}{\bar{n}_s} \text{ MB. Typical, } \rightarrow 0 \text{ for large } \bar{n}. \\ \frac{1}{\bar{n}_s} + 1 \text{ photon. Even as } \bar{n} \rightarrow \infty, \text{ always } \frac{\Delta n}{\bar{n}} \approx 1! \end{array} \right.$$

Dispersions: $\frac{\Delta n^2_{\text{rms}}}{\bar{n}^2} = \left\{ \begin{array}{l} \text{mess BE} \\ \frac{1}{\bar{n}_s} - 1 \text{ FD if } \bar{n}_s \rightarrow 1, \text{ state full, cannot have any fluct.} \end{array} \right.$

9.8 The Classical Limit. Let us consider a huge quantum gas of definite particle count N . In this section it is shown that: if the gas is "sufficiently" dilute (small N), or if T is sufficiently high ($\beta \rightarrow 0$), then the occupation number $\bar{n}_r$ for any state is very small $\bar{n}_r \ll 1$ and is in fact correctly given by the classical Maxwell-Boltzmann occupation:

$$\text{This is true for both } \bar{n}_r = N e^{-\beta \epsilon_r} / \sum_r e^{-\beta \epsilon_r} \ll 1.$$

The BE & FD gases!!

The occupation is so low that Pauli effect makes no difference. Fermions + Bosons act the same! In this "classical limit", both Z_{BE} and $Z_{FD} \rightarrow Z_{MB}/N!$; hence the earlier Gibbs Paradox business is justified. He at room T is a nondegenerate gas, hence MB OK.

9.9. Density of Quantum States for Gas. Good 8 pages. Using either periodic BC's or walls for box of volume V , it is shown that

$$\Delta n_x \Delta n_y \Delta n_z = dN = \# \text{ of states} = V \cdot d^3 p / h^3.$$

It is shown that h is the correct cell size in phase space. Uses κ as wave # so $\vec{p} = \hbar \vec{\kappa}$, relates momentum to deBroglie λ .
 $\lambda = 2\pi / |\vec{\kappa}|$.

9.10. Repeat 2-calc for ideal gas. Same as before except discrete values of $\vec{p}$ in box, and $h_0 \rightarrow h$, Planck. Results are same as before (monatomic), eg, $\xi = V (2\pi m kT)^{3/2} / h^3$ with h , not arbitrary h_0 . (ξ = single molecule partition fn).

9.11. Applications. Since we now know how to compute single-particle ξ 's without arbitrariness ($h_0 \rightarrow h$), let's do it. First example is $H_2 \rightleftharpoons H + H$. We know that law of mass action constant is $\xi_+ \xi_- / \xi_H$, and each of these is easily computed. For monatoms, it is just the phase space integral $\times (2s+1)$. For H , approximate integral sum or $e^{-\beta \epsilon_0}$. Overall result is (9.11.9), computed from first principles! I wonder if this stuff applies in a solvent? Not an ideal gas, so cannot use simple phase-space integral.

Next application is vapor pressure over a solid. Here you want to set $\mu_v = \mu_s$. For the vapor, $\mu = -kT \ln(\xi/N)$ recall, + we now know ξ (assume vapor = monatoms). Unfortunately, to get μ_s you need a messy double integral of the solid's specific heat $c(T)$. I.e., $\mu_s \sim 2P/2N_s \sim 2 \ln k/2N_s$... very messy. Result is similar to that obtained by integrating the Clapeyron equation, but more specific.

9.12. Calculate ξ for a polyatomic. Since we're complacent, $\xi = \xi_t \xi_e \xi_r \xi_v$ (translational, electronic, rotational, vibrational). We have already done ξ_t (add $(2s+1)$ if "spin"). $\xi_e = \sum e^{-\beta \epsilon_i}$ [degen of ground state] $\times e^{-\beta \epsilon_0}$, even at room T, essentially in ground state; ξ_r is done for a diatomic, you sum over all levels, include identicals - $\frac{1}{2}$ if needed. So $\xi_r = 2 \pi^2 I kT / h^2 \sigma$ where $\sigma = 1, 2$ and $I = \text{moment of inertia}$. Vibrations treated like a H.O. Again, essentially in vib ground state [$\hbar \omega \gg kT$] so $\xi_v = e^{-\frac{1}{2} \beta \hbar \omega}$ where $\omega = \text{H.O. param}$.

So in principle you can compute ξ for any molecule. Then you know all about an ideal gas of such molecules. $\xi, \mu = -kT \ln(\xi/N)$, mass action constants, etc etc.

9.13 Blackbody Rad. We already know the "Planck distribution" $\bar{n}_\omega = (e^{\beta\epsilon_\omega} - 1)^{-1}$. We also know the phase space d^3p/h^3 and two pol states for each ω , hence we conclude this for the energy density in photon gas held at T : $\bar{u}_T(\omega) \equiv \omega^3 d\omega / [e^{\beta\hbar\omega} - 1]$, the usual blackbody curve. Thing peaks at $\hbar\omega/kT \approx 3$, so double T and you will double $\bar{\omega}$, "Wien's Displacement Law". The total energy density is $\bar{u}_T \propto T^4$, the "Stefan-Boltzmann Law". Radiation pressure is $\frac{1}{3} \bar{u}_T$.

9.14 The Enclosure. In a "large enclosure" held at T , a photon gas in equilibrium must also "be at T ". The photon distribution is shown to be the same at any two points: isotropic, homogeneous, indep of box shape, equal amounts of both pol.

9.15 Radiated Power from Hot Object. Trick: if object were inside a BB at its T , then $P_{rad} = P_{abs}$. But, $P_{abs} = a \cdot P_{inc}$. We know P_{inc} because we know about the photon gas. Thus $(P_{rad}/a) = P_{inc} = f(T)$ only. $a = \text{absorptivity}$. This shows: good absorber (black) surface is also a good radiator, "Kirchhoff's Law". a is usually only $a(\omega)$ + determined by surface. You find that radiated power is $\propto (\cos\theta)$ going into $d\Omega d\omega$, "Lambert's Law". Integrate over $d\Omega$ to find that $P_{rad}(\omega) = c a(\omega) \frac{1}{4} \bar{u}_T(\omega)$. Then integrate over $d\omega$ to get $P_{rad} = c \int a(\omega) \frac{1}{4} \bar{u}_T(\omega) d\omega$. But, $\bar{u}_T = \text{energy density} = \text{const} \times T^4$, so $P_{rad} = a \cdot (\sigma T^4)$ where a is a mean value of $a(\omega)$. Here, $\sigma = \text{"S.B. constant"}$, and "SB Law". This is the rate at which energy comes off unit area of hot object!! If burner is in kitchen, this does not mean that entire kitchen is at T . My own example: a 1600 watt burner of radius $r = 10 \text{ cm}$ radiating off the top, no convection or conduction, must be at $\approx 1000^\circ \text{K}$. It radiates a spectrum as if it were in an enclosure (detailed balance principle). Wien says: $\tilde{\lambda}_1 T_1 = \tilde{\lambda}_2 T_2$. Sun has $\tilde{\lambda}_e = 5500$ and $T_2 = 6000$ so $\tilde{\lambda}_1 \approx 6 \times 5500 \approx 33,000 \text{ \AA}$. Maybe dull red due to tail at $10,000 \text{ \AA}$...

9.16 Fermi Gas. Electron cloud in box volume V . $\bar{n}_\epsilon = [e^{\beta(\epsilon - \mu)} + 1]^{-1}$ is now called Fermi fct. Below $\epsilon = \mu$, occupation is 1, above ≈ 0 , so $\mu = \text{chem potential} = \epsilon_f = \text{Fermi energy}$. At $T \ll T_f$ (Fermi temp $kT_f = \epsilon_f$), electrons fill a ball in k -space up to $k_f = \hbar (3\pi^2 n)^{1/3}$, where $\frac{1}{2} \hbar k_f = \epsilon_f$. Thus, $\mu = \epsilon_f = \hbar^2 / 2m_e (3\pi^2 n)^{2/3}$ depends on electron density! Only those electrons within kT of ϵ_f contribute to specific heat, (Equipartition fails) so $C_V(\text{electrons})$ tends to be very small.

Sources of Information on Critical Phenomena + Renormalization group

<u>author(s)</u>	<u>ref</u>	<u>year</u>	<u>title</u>
Barber	PR29,1	1977	"An Intro. to the Fund's of the Renorm. Grp. in Crit. Phenom."
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Chapter 1: averaging ~ a "simple system" = large, homogenous, etc. — internal energy
 "Equilibrium state" is defined by $U=E, V, N$. Fixed wall is "restrictive". "Diathermal wall" passes heat ~ quasi-static ~ imperfect diathermal. The basic problem: find the final equilibrium state. Postulates given.

Chapter 2: T, P , potentials $\mu_i = (\partial U / \partial N_i)_{S, V}$. $dU = TdS$ quasi-static. Equilibrium state. Temperature "Extensive parameter" adjust to extremize S . (2nd Law $\approx$).
 When state is in equilibrium, $S = \max$, so $dS = 0$ for "virtual process" from the equilibrium state. $T \equiv (\partial U / \partial S)_{V, N}$. Equilibrium $dS = 0$ with U, V as variables.
 $\Rightarrow T_1 = T_2$ (thermal eqn) and $P_1 = P_2$ (mechanical eqn.)

Chapter 3: "Formal Relations"

Aside: Entropy $S^{(x)}$ of system (x) is a fn of that subsystem's extensive parameters $U^{(x)}, V^{(x)}, N^{(x)}$. Entropy is itself extensive. Therefore we have this "first-order homogeneous" scaling law:

$$S(\lambda U, \lambda V, \lambda N_i) = \lambda S(U, V, N_i) \quad \text{same goes for } U(S, V, N_i)$$

Generally, $\{X_i\}_{S,0}$ = set of all extensive parameters like V, μ_i, N_i .

$$U(\lambda S, \lambda X_i) = \lambda U(S, X_i)$$

Intensive parameter is "zero-order homogeneous": $T(\lambda S, \lambda X_i) = T(S, X_i)$
 i.e., λ^0 .

Either $U = X_0$ is an independent variable [entropic variables] $S(U, X_i)$
 or S is indep. [energetic rep $U = U(S, X_i)$] Take your choice

Theorem: due to the first-order homogeneity of U , easy to show by taking $\partial / \partial \lambda$ that

$$U = TS + \sum P_i X_i = U(S, X_i)$$

Also, $dU = TdS + \sum P_i dX_i$

"Equation of state" is one for intensive parameter: like $P = P(S, V, N_i)$. But all intensives are derivatives of U . Claim:

[know all "equations of state"] = [know the fundamental relation $U = U(S, X_i)$.] very basic

The Gibbs-Duhem Relation. Consider system with r "components", hence $V, N_1, N_2, \dots, N_r, S$ are your extensive variables. There are then $r+2$ equations of state for the intensive params ($r+2$ intensives). These are $P(f, n, v)$, $T(f, n, S)$, and $\mu_i(f, n, v)$. So you think there are $r+2$ independent intensives, but not so. Compare dU to dH and conclude that

$$SdT - VdP + \sum_i N_i d\mu_i = 0 \quad \text{Gibbs Duhem}$$

This is a differential relation betwixt the $r+2$ intensives. Can integrate to eliminate one of the intensives:

r components $\Rightarrow r+1$ indep intensives $\Rightarrow r+1$ degrees of freedom.

Note: $X_i = V, N_i$ here.

Thus: one component has only 2 variables. Note:

$$U = TS + \sum_i P_i X_i = \text{"Euler Form"}$$

Proof tried: all state equations for $r+2$ intensives $\Rightarrow$ fund relation.

Ideal Gas as Example: $P = NRT/V = P(?)$ $U = \frac{3}{2} NRT$.

$$T = \frac{2}{3} NR U = T(V, N) = \text{"state"}$$

$$P/T = NR/V = P/T(N, V) = \text{"state"}$$

Can use Gibbs-Duhem to get 3rd equation of state.

End of chapter 3: α, K , specific heats, Maxwell Relations

Chapter 4: Engines, Fudges, Etc.

① Thermodynamic Configuration Hyperspace. Consider space whose axes are S, U, X_i . In this space, $S = S(U, X_i) =$ The surface of equilibrium states. A "quasi-static process" means a curve along this surface!



qs process = succession of equilibrium states.

Reversible $\Rightarrow$ curve along $S = \text{const}$
(is quasi-static one).
isobaric

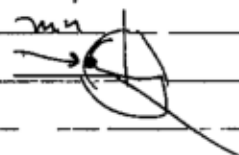
isochoric, etc!

Chapter 5: "Alternative Formulations"

"Nernst Postulate" = 3rd law.

A final equilibrium state is unique a) $\max S$ for given V

b) $\min U$ for given S



I.e., the surface must be convex in a sense. ($T > 0$)

Legendre Transform: In our original and complete formulation, we use the extensive $X_0, X_1, \dots$ as independent variables + the intensive as desired. Eg., $T(S, V, N_i)$

However, it is easier practically to have the intensive as controllable, independent variables. This leads to reformulation + complication!

Atank Problem: Curve is $Y=Y(X)$, describes curve fully. $P=Y'(X)=P(X)$ is slope at X . You could write $Y(P)=\int P(X)dX+C$, but there is that unknown constant! $Y(P)$ not unique.

Instead, regard the curve $Y(X)$ as set of lines [Bluecher]. $\Psi=y$ -intercept and $P=\text{slope}$, so $\Psi=\Psi(P)$ = set of curves which describe $Y(X)$ fully. I.e., P now indep variable, Ψ = intercept.

How do you compute $\Psi=\Psi(P)$, given $Y=Y(X)$?

$$\Psi = Y - PX \quad P = dy/dx = P(X)$$

$$X = -d\Psi/dP = X(P) \leftarrow \text{we have successfully inverted}$$

usual Legendre Transform!

General Problem: $Y=Y(X_0, X_1, \dots, X_r)$ ^{new removed} $P_k = \partial Y / \partial X_k$

$$\Psi \equiv Y - \sum P_k X_k \Rightarrow d\Psi = - \sum X_k dP_k$$

$$X_k = -\partial \Psi / \partial P_k$$

compute the extensive

↑ now these are the indep's. (intensive)

Partial Leg. Transform also Possible. Eg., $L(q_i, v_i)$ you have $P_i = \frac{\partial L}{\partial v_i}$. You replace v_i with p_i to get

$$H(q_i, p_i) = L - P_i v_i$$

$$dU = TdS - PdV + \sum \mu_i dN_i$$

Helmholtz: Replace S with T

$$F = U - TS$$

$$F(T, V, N_i)$$

$$T = \frac{\partial U}{\partial S}$$

$\Rightarrow$

$$-S = \frac{\partial F}{\partial T}$$

$$dF = -SdT - PdV + \sum \mu_i dN_i$$

Enthalpy: Replace V with P .

$$H = U + PV$$

$$H(S, P, N_i)$$

$$P = -\frac{\partial U}{\partial V}$$

$\Rightarrow$

$$+V = \frac{\partial H}{\partial P}$$

$$dH = TdS + VdP + \sum \mu_i dN_i$$

Gibbs: Replace S and V by T and P .

$$G = U + PV - TS$$

$$G(T, P, N_i)$$

$$T = \partial U / \partial S$$

$\Rightarrow$

$$-S = \partial G / \partial T$$

$$P = -\partial U / \partial V$$

$$+V = \partial G / \partial P$$

$$dG = -SdT + VdP + \sum \mu_i dN_i$$

Grand Canonical: Replace S and N_i by T and μ_i .

$$\Omega = U - TS - \sum \mu_i N_i$$

$$T = \partial \Omega / \partial S$$

$$-S = \partial \Omega / \partial T$$

$$\mu_i = \partial \Omega / \partial N_i$$

$$-N_i = \partial \Omega / \partial \mu_i$$

$$d\Omega = -SdT - PdV - \sum N_i d\mu_i$$

All these new functions are called "thermo potentials".

Note: if all this stuff is done starting with $S(U, X_i)$ in place of $U = U(S, X_i)$, you get "Mossion functions" in place of the "thermo potentials".

Chapter 6: External Principles with these new Potentials

Good discussion: When a system is in contact with a reservoir at constant $\overset{A}{T}$, the system adjusts so that the therm potential $\overset{B}{F}$ is minimized:

$\overset{A}{T}$
P
 T, P
 T, μ_i

$\overset{B}{F}$
H
G
 μ_i

Ex: in first case,
 $dF = d(U - TS)$
 $= d(U - T \Delta S)$
 $= dU + dU_c$
 $= dU_{tot}$
 $= 0$ by original principle

Nothing to it! Much better than Reif confused discussion

(F)

Why "free energy"? Consider process, reversible and $dF =$ change in F, also so therm. Then $dF =$ the work you get out!! If not quasi-static, $w \leq dF$.

but adiabatic so T changes.

(H) Open vessels are at constant p_i

Heat often transferred at constant pressure.

"potential for work"

at constant $\rightarrow$ $\frac{S}{dU} \quad \frac{T}{dF} \quad \frac{P}{dT} \quad \frac{T+P}{dG}$ is $\downarrow$

Nothing new at all!

Good Joule-Thompson throttling discussion.

(G) Chemistry done at open vessel, const. T, P more a loss.

$G = \sum \mu_i n_i$ so $dG = 0 \Rightarrow \sum \mu_i dn_i = 0$ and $\sum \mu_i b_i = 0$ recall

Chapter 7: Maxwell Relations

Every order-equivalent second derivative $\Rightarrow$ Max Rel. Just a matter of a table.

How many first-derivs are independent?? Consider $U = U(S, V, N_i)$
 $\wedge U = U(S, X_1, \dots, X_r)$. There are $t+1$ derivatives, $t+1$ equations of state, only t are degrees of freedom since Gibbs-Duhem couples them

So $t+1$ derivatives, $t+1$ vars. There are then $\frac{(t+1)(t)}{2}$ Maxwell relations!

Ex: $V = V(S, V, N)$ single comp $t \leq 3$. $t=2$. $\text{Max}=3$.

Is 3 for each potential, here V . But there are many potentials!!

For single-component, $t+1=3 \Rightarrow$ There are three derivatives, namely T, P, μ . eg, $T = (\partial V / \partial S)$.

There are $t+1$ variables, choose them as you will. ~~(T, V, μ)~~
Why are only $t+1$ indep derivs?

$$\frac{\partial I_i}{\partial I_j}, \text{ two intensives} = \frac{\partial I_i(E_n)}{\partial I_j} = \frac{\partial}{\partial I_j} \left[\frac{\partial E_n}{\partial E_j} \right]$$

intensive is always a basic deriv

$$\text{So } \frac{\partial^2 V}{\partial X_1} \frac{\partial^2 V}{\partial X_2} \frac{\partial^2 V}{\partial X_1 \partial X_2} = 3 \text{ indep.}$$

Conclusion: for single component system, only 3 indep deriv
Usually they are

$$\alpha = \left(\frac{\partial V}{\partial T} \right)_P \frac{1}{V} \quad -K_T = \left(\frac{\partial V}{\partial P} \right)_T \frac{1}{V} \quad C_P = \left(\frac{\partial S}{\partial T} \right)_P \frac{1}{N} = \left(\frac{dQ}{dT} \right)$$

$\frac{\text{change } T}{\text{look at } V}$ $\frac{\text{change } P}{\text{look at } V}$ $\frac{\text{change } T}{\text{look at } dQ}$

All easily measured.

Callen gives a systematic method for reducing any derivative to the basic chosen one!

Also he comments on the Jacobian Technique.
Examples are given.

and 2.

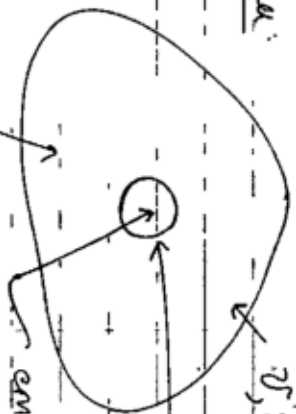
Chapter 8: Stability Now book is more relevant to my problem: 2nd order phase here.

$dS=0 \Rightarrow$ equilibrium. $d^2S < 0 \Rightarrow$ stable.

Theory will be that of L. Tissera, 1950.

Consider:

$U', S', V', N' =$ 10th order single-component



surface deformed, non rigid by holds N .

$u, s, v =$ molar parameters here.

u, s, v $\sim$ $\downarrow$
equilibrium by
subsystem

$$\Delta U' = N dU + N [du + d\epsilon + d\epsilon' + \dots]$$

$\leftarrow$ all higher order terms are neglected.

$$du = u_s ds + u_v dv = \text{first order}$$

$$d^2u = \frac{1}{2} u_{ss} (ds)^2 + u_{sv} ds dv + \dots = \text{second order}$$

$$d^3u = \frac{1}{6} u_{sss} (ds)^3 + \dots$$

Since $\Delta U' = 0$ by extremal, we get $N d\epsilon + N du = 0 \Rightarrow T = T', P = P'$.
I.e., we separate first order pieces + get obvious results.

But stability requires that universal part of $\Delta U'$ be positive. (I.e., $U' = m \dots$).
So, as,

$$d^2u > 0. \quad \text{But } d^2u = \frac{1}{2} [u_{ss} (ds)^2 + 2u_{sv} ds dv + u_{vv} (dv)^2]$$

$$\text{Change never do get } d^2u > 0 = \left[\frac{1}{2} \right] \left[\frac{1}{u_{ss}} (dT)^2 + f_{uv} (du)^2 \right]$$

Conclusions: $C_p > 0$ $K_T > 0$ Basically there are Le Chatelier's

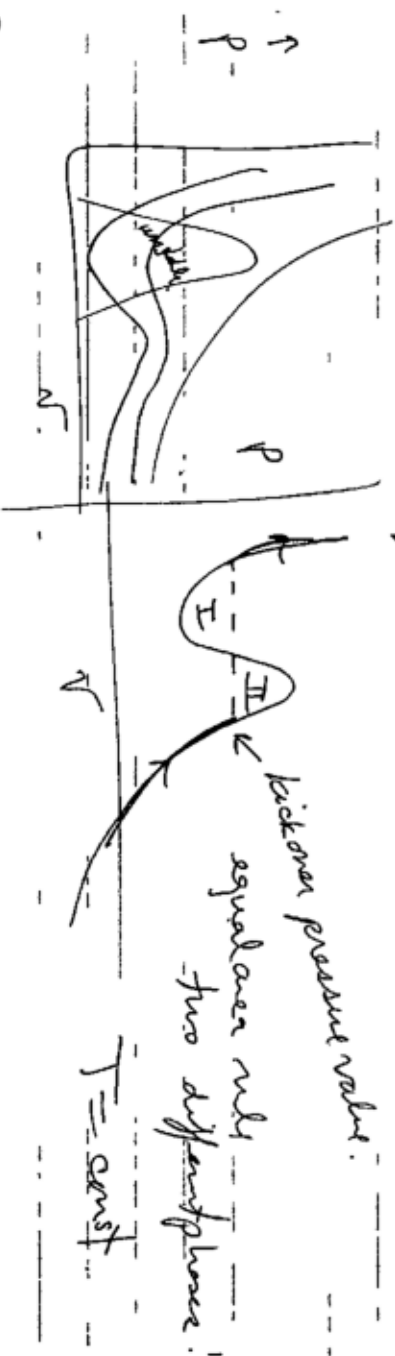
Principle, single-component system is "intrinsically stable".

Results on stability are generally do multi-conf. systems. Big deal.

Chapter 9 "First and Second Order Phase Transitions"

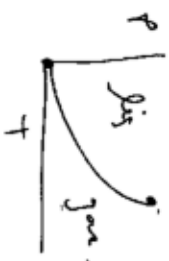
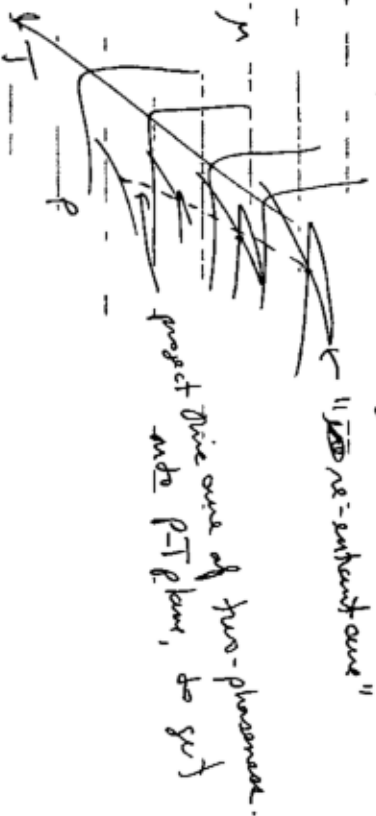
An excellent discussion!

positive.



Key to the discussion + worked it. Note that temperature changes as you see varies: high and low phase.

Moreover you have a subgrade in your P - V isotherm, you get this two-phase effect. Applies to all cases really, just just gas-liquid. Should buy book.



At this point, however, diagram is derived.

totally unstable region - eg, supercooled water. Kicks me either when you stop or spot. Start on micro scale.

$$f = \frac{5 - M + 2}{\# \text{ phases for each component}}$$

phases which can break set.

Gibbs Phase Rule.

$$F = 1 \quad M = 3 \Rightarrow f = 0$$

triple point of water.

We have several "polymorphic" phases.

Proof of Gibbs Rule: r components $\Rightarrow$ $r+2$ ^{intensive} params $T, P, \mu_1, \dots, \mu_r$

But in each (M) phase $\exists$ one Gibbs-Duhem relating these, so

$r+2-M = f$. You can arb. choose f of these T, P, X_j

Note: $\mu_i^I = \mu_i^{II} = \mu_i^{III} \dots$

binary $r=2$
ternary $r=3$

Discussion of binary systems of metals. "Eutectic" solution freezes but both solid phases freeze together, so material stays mixed. This is how you make a strong alloy !!

Higher order: what is latent heat? $dU = TdS \Rightarrow Q = T\Delta S$, so

$L = T\Delta S$, arises because phases have different entropy!

Recall that

$$-s = \left(\frac{\partial g}{\partial T}\right)_P \quad v = \left(\frac{\partial g}{\partial P}\right)_T$$

Both these derivatives are discontinuous in a 1st order phase transition: (g is continuous $\Rightarrow g_1 = g_2$)

In second-order, 1st derivs are continuous $\Rightarrow \Delta S = 0, \Delta V = 0$, but 2nd derivs are disc.

n^{th} order $\Rightarrow$ all derivs continuous up to n^{th} , $2^{\text{nd}} = \text{disc. int.}$

Ehrenfest proposed this. He thought changes in derivs could be finite, but this turns out very rare.

Ehrenst. theory only good for finite jumps

Tisza's theory good for all jumps

book is Wiley 1951 (solids)

also Rice 1955

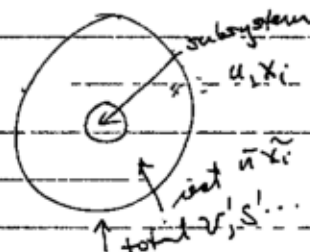
Must go back + work on 8.4 on stability. System described by ~~$u = u(S, X_1, \dots, X_r)$~~ $u = u(S, X_1, \dots, X_r)$ as usual. Now rescale

$$u \equiv U/X_t \quad x_i \equiv X_i/X_t \quad x_0 \equiv S/X_t$$

$$\text{Then } u \equiv u(x_0, x_1, \dots, x_{r-1})$$

$$U' = X_t u(x_0, \dots) + \tilde{X}_t \tilde{u}(\tilde{x}_0, \dots)$$

$$\text{Let } X_t = \text{total mole number, say } \Delta U' = X_t [du + d^2u + \dots] + \tilde{X}_t d\tilde{u}$$



Again, stability if $d^2u > 0$, but $u = u(x_0, \dots, x_{t-1})$, more variables!

So
$$d^2u = \frac{1}{2} \sum_{j,k=0}^{t-1} u_{jk} dx_j dx_k \quad \text{where } u_{jk} = \frac{\partial^2 u}{\partial x_j \partial x_k}$$

We want to diagonalize u_{jk} by change of var. Game: eliminate cross terms one at a time.

$$d^2u = \frac{1}{2} \left[u_{00} (dx_0)^2 + \underbrace{2 \sum_{i=1}^{t-1} u_{0i} dx_0 dx_i}_{\text{kill this}} + \sum_{j,i=1}^{t-1} u_{ij} dx_i dx_j \right]$$

$$dP_0 \equiv u_{00} (dx_0) + \sum_{k=1}^{t-1} u_{0k} (dx_k)$$

$$(dP_0)^2 = u_{00}^2 dx_0^2 + \left(\sum_{k=1}^{t-1} u_{0k} dx_k \right)^2 + 2 u_{00} dx_0 \sum_{k=1}^{t-1} u_{0k} dx_k$$

$$\Rightarrow \frac{(dP_0)^2}{u_{00}} = u_{00} (dx_0)^2 + 2 \sum_{i=1}^{t-1} u_{0i} dx_0 dx_i + \frac{1}{u_{00}} \left(\sum_{k=1}^{t-1} u_{0k} dx_k \right)^2$$

$$= \frac{1}{u_{00}} \sum_{i,j=1}^{t-1} u_{0i} dx_i u_{0j} dx_j$$

$$\text{So } d^2u = \frac{1}{2} \left[\frac{(dP_0)^2}{u_{00}} + \sum_{i,j=1}^{t-1} \left(u_{ij} - \frac{u_{0i} u_{0j}}{u_{00}} \right) dx_i dx_j \right]$$

This is $\psi_{ij}^{(0)}$ where $\psi^{(0)} = u - P_0 x_0$.

So we have "completed the square" in $i=0$ variable $\bar{S}$.

$$d^2u = \frac{1}{2} \left[\frac{(dP_0)^2}{u_{00}} + \sum_{i,j=1}^{t-1} \psi_{ij}^{(0)} dx_i dx_j \right]$$

start all over:

$$\psi_{11}^{(0)} (dx_1)^2 + \underbrace{2 \sum_{j=1}^{t-1} \psi_{0j} dx_0 dx_j}_{\text{kill}} + \sum_{i,j=2}^{t-1} \psi_{ij} dx_i dx_j$$

$$dP_1 \equiv \sum_{j=1}^{t-1} \psi_{0j}^{(0)} dx_j \Rightarrow$$

$$d^2u = \frac{1}{2} \left[\frac{(dP_0)^2}{u_{00}} + \frac{1}{\psi_{11}^{(0)}} (dP_1)^2 + \sum_{j,k=2}^{t-1} \left(\psi_{jk}^{(0)} - \frac{\psi_{0j}^{(0)} \psi_{0k}^{(0)}}{\psi_{11}^{(0)}} \right) dx_j dx_k \right]$$

$$\equiv \psi_{jk}^{(1)}$$

where $\psi^{(1)} = u - P_0 x_0 - P_1 x_1$

Result of all this:

$$d^2u = \frac{1}{2} \sum_{j=0}^{t-1} \left(\frac{1}{\psi_{jj}^{(j-1)}} \right) [dP_j^{(j-1)}]^2 = \text{diagonal form.}$$

Where $\psi_{jj}^{(j-1)} = u - \sum_{i=0}^{j-1} P_i x_i \equiv u[P_0, P_1, \dots, P_{j-1}]$.

We have replaced the variables x_i with Legendre ones P_i .

Now:

$$\psi_{jj}^{(j-1)} = \frac{\partial^2}{\partial x_j \partial x_j} \left[u - \sum_{i=0}^{j-1} P_i x_i \right] = \text{some mess.}$$

Now back to Tisza theory: You want $d^2u = 0$ $d^4u > 0$. Then need some $\psi_{ss}^{s+1} = 0$, ie, matrix needs less than full rank.

So critical point $\Rightarrow d^2u = 0$.

If $s \in$ relevant variables which "set" the critical point, then "order-disorder transition".

There must be a better matrix method to do this!!

Typically $C_p \rightarrow \infty$ at a crit point. This is obvious when you go from liquid to gas or in gas diagrams.

Chapter 10 Nernst (3rd Law).

$$C_p \rightarrow 0 \text{ as } T \rightarrow 0.$$

$$\alpha \rightarrow 0$$

Chapter 11: Summary of Thermo.







MISC

Ising Models

[Taken from Arnold Munster, VR 2]

1974

One-Dim model

$\uparrow \downarrow \uparrow \uparrow \downarrow \downarrow \uparrow \uparrow = \text{state}$

Assume:

"parallel" adjacent pairs $\Rightarrow -J$ to energy (favourable) $\uparrow \uparrow$
antiparallel pairs $\Rightarrow +J$ $\uparrow \downarrow = +1$

Let $K = +J/kT$

$C = -[J]H/kT$ $[J] = \text{mag. moment}$

$$Z_1 = \sum \exp \left[C \sum_{i=1}^N \sigma_i + K \sum_{NN} \sigma_i \sigma_j \right]$$

" Q_C " = part-fcn. η

overall alignment $\nwarrow$ interaction energy
 $NN = \text{sum over nearest neighbor pairs}$

By many manipulations we can eventually show that

$$Z_1 = \text{above} = \left[e^{KJ} + \sqrt{e^{2KJ} + e^{-2KJ}} \right]^N$$

where we assume they lie in a ring, N large.

$$M = \frac{[J]}{N} \frac{\partial \ln Z}{\partial C} = \frac{[J]}{N} \frac{\partial \ln C}{\partial C} = M(C)$$

As $C \rightarrow 0$ ($H \rightarrow 0$), $M \rightarrow 0$: The 1-D Ising has no spontaneous mag.
The interaction do not all want to line up by themselves.

Two-Dim model - solved by Onsager in 1944. Has a "transition".

(Kramers/Wannier showed in 1941 the power of the matrix method.
Result is a formula for D as a limit of an integral. You see an infiniteness in specific heat at some value of $kT/5.18$, some special T .

Earlier treatments of 2-D Ising failed to show this infinite specific heat. Bethe's approx not very good.

Model 2-D also shows ferromagnetism!

Properties in general are singular at critical temperatures.

3-D Ising - has not been exactly solved!!

refs: Newell, Marshall RMP 25,353 (1953) — 9 years after Onsager.



Onsager's Famous 1944 paper (PR 65, 117; 1944)

INTRO. Methods of successive approx are OK for first-order phase transition. You expect things to be analytic in T . But second-order more subtle. Latent heat vanishes, asymmetry vanishes, but $C_p \rightarrow \infty$ as if there still were 2 phases. In α - β Quartz, $C_p \sim (T_c - T)^{-\frac{1}{2}}$ blow up. At T_c , things have weird discs.

When non-analytic, you cannot solve by finite series. 1-D Ising has no transition.

History: 1925 German E. Ising proposes 2-D model for ferromagnet.

1936 R. Peierls shows that $\exists$ a transition.

1941 HA Kramers, GH Wannier locate T_c .

METHOD. Set up eigenvalue problem, put lattice around a cylinder. Use clever algebraic method. Will get a log sing in C_p .

LINEAR ISING. Let $u(\mu^{(k)}, \mu^{(k+1)}) = -J \mu^{(k)} \mu^{(k+1)}$ where $\mu^{(k)} = \pm 1$, obvious. Note that $H=0$. So

$$Z = \sum_{\mu^1} \exp \left(- \sum_k u[\mu^k, \mu^{k+1}] / kT \right)$$

$\uparrow$ re, pair sum

$$= \sum_{\mu^1, \mu^2, \dots, \mu^N} e^{-u_{21}} e^{-u_{32}} e^{-u_{43}} \dots e^{-u_{N,N-1}} = \sum_{\mu^1, \mu^N} \sum_{\mu^2, \dots, \mu^{N-1}} \frac{V_{12} V_{23} V_{34} \dots V_{N-1,N}}{V^{N-1}}$$

Define $V_{21} = e^{-u_{21}} = \text{matrix}$. then

$$= \sum_{\mu^1, \mu^N} [V^{N-1}]_{\mu^1, \mu^N}, \text{ not a trace. ETC.}$$

Follows Kramers/Wannier closely.

RECTANGULAR ISING (asym)

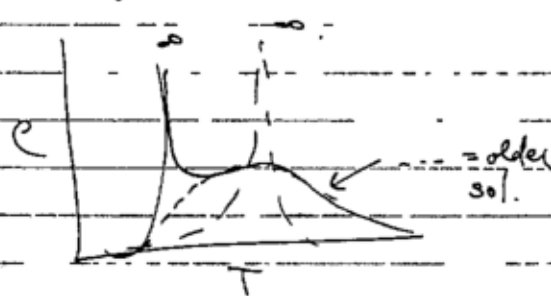
quaternion algebra

commutators

irreducible reps

solution of problem here

famous graph



end



✓ Superfluids

Lemansky Ch 15

(15.1) Liquid He II. Take some He in atmosphere dewar. Cool to 4.2° , it becomes two-phase, liquid-vapor. Cool to 2.172°K (the λ point) it enters He II also. Volume Exp goes negative below T_λ , bubbles stop, due to anomalously high thermal conduct. of He II. Viscosity goes very low! He⁴ does this, He³ does not do it.

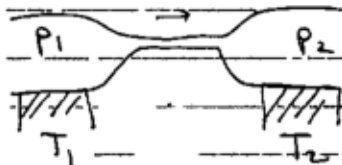
Theory? Landau had one in 1941, then London said it was Bose Condensation in momentum space. Feynman cleaned things up. Tisza made up a simple, phenomen. theory called "two-fluid model".

$$\rho = \rho_n + \rho_s \quad \text{density} \quad \text{At } T_\lambda, \rho_n/\rho = 1. \quad \text{At } T=0, \rho_s/\rho = 1.$$

Expansion that $\rho_n/\rho \sim T^4$ up to $.6^\circ\text{K}$.

(15.2) Fountain Effect. Only the superfluid atoms go thru small tube, & they carry no entropy since "frozen". Usually slow thru a tube does carry entropy, so you need to redo your therm. result is a little equation.

$$\left(\frac{\partial P}{\partial T}\right)_\rho = \frac{s_2}{v_2}$$



For this throttling process.

it turns out that $g_1 = g_2$ (not enthalpy),

this pressure difference causes

the "fountain effect".

$$s_2 = \text{entropy/gm} \cdot \text{m}^3 \text{K}$$

$$v_2 = \text{volume/gm} \cdot \text{m}^3 \text{K}$$

(15.3) Second Sound. Long waves of sound which causes dg called "first sound". But you can also have $dg=0$ but $\delta(\rho_s/\rho_n) \neq 0$, a sound due to relative n.v.s.s. motion: "second sound". w_2

First sound is 237 m/sec at low T, but has a cusp at $T_\lambda = 2.2^\circ$, then drops off as T raised more. Second-sound waves are produced by certain temperature-varying discs. w_2 -sound carries ΔT in pulse, w_1 does not.

You can show that

$$w_1 = \text{usual classical}$$

$$\text{sound speed but}$$

$$\gamma = 1, \Delta T = 0$$

$$w_2 = \sqrt{\frac{\rho_s}{\rho_n} \frac{s^2 T}{c}} \quad \text{entropy disc. significant}$$

$$\neq 0 \text{ only when } \rho_s \neq 0.$$

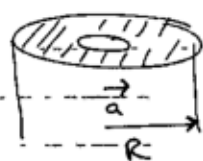
If you plot $\omega_2(T)$ versus T you see it goes away at T_λ .

15-4 Fourth Sound. If you put porous packing in your He-II, it stops motion of $\rho_{n,r}$ but is still moves since not so little viscosity. This is "fourth sound". Predicted 1948, found 1960? Why is between ω_1 and ω_2 .

15-5 Third Sound. He-II "creeps" in a thin film ≈ 100 atoms thick over top of a vessel. By 1968 even experimenters could not agree on how creep rate depends on parameters. The creep does stop at the λ point.

Waves through such a film predicted by Atkins 1959, found by same in 1962. Like a shallow-water wave.

15-6 Other Effects (1) Persistent flow - The He-II maintains angular momentum given it when you stop container, in a strange reversible way. Somehow T is stored internally. More than just a low viscosity effect. (2) Vortexes -



$$\begin{aligned} \omega &= K/r & \text{for } a < r < R \\ \omega &= 0 & \text{for } 0 < a. \end{aligned}$$

This is called a "vortex line"

Can bend it into a ring. T quantized in units of h , causes the $\omega \propto 1/r$ rule.

Fred Reif did experiments with vortex rings! Some real new work.

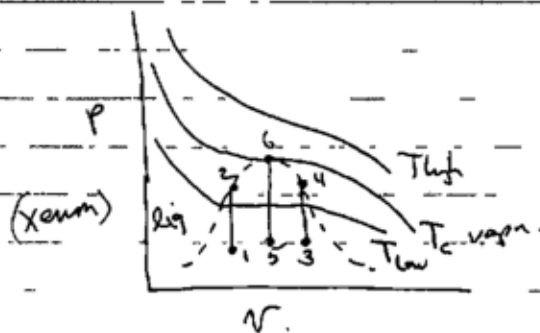
15-7 Mixture of He^3 + He^4 . He^3 has no superphase itself (as of 1968). When you mix these two, (ie, a solution), below 1.9° it separates into two phases:

phase 1: concentrated He^3 in He^4
phase 2: dilute He^3 in He^4 .

He^3 diffuses toward dilute phase. As He^3 crosses into dilute, it absorbs heat. This is a "He dilution refrigerator", can hold exper space at $\leq 0.1^\circ\text{K}$.

Criticality: Zemansky Ch 12, pieces of.

12-8 Critical State Look again at a $P-V$ diagram:

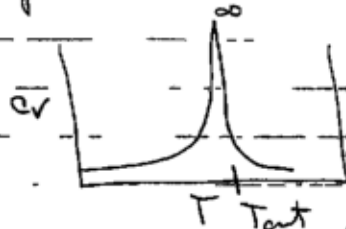


Explicitly, how do you see crit point? Fill a constant volume cylinder with some xenon. If you vary T , you go up down on picture. If too little xenon, you can go $3 \rightarrow 4$. If, you see two phases (Meniscus). As $T \rightarrow T_c$, meniscus falls to bottom + goes away.

If too much xenon, you can do $1 \rightarrow 2$ + see meniscus rise, because you end up on the liquid side.

If volume/mass correct, you can do $5 \rightarrow 6$. As $T \uparrow$, meniscus stays put in middle of tube. As $T \rightarrow T_{crit}$, you get all the weird effects.

If you do $5 \rightarrow 6$ and plot C_p vs. T , you get the characteristic logarithmic sing !!



This is then how the exper is true.

12-10 Higher-Order Transitions Some terminology here:

"second-order phase trans." means no change in first derivs of G (ie. S, V, μ, p, T all const; $q = \text{const too}$) and finite change in second-derivs. This definition refers to Ehrenfest's idea. Examples of second-derivs of G are: $C_p, K, \alpha (= \beta \text{ in Zemansky})$. This is observed only in superconductivity.

"lambda transition" means same but second-derivs go to ∞ . (Looks like a λ . Maybe it's just disc (sec. der.) $= \infty$).

Many Examples:
 1. order/disorder in alloys
 2. ferromagnetism
 3. He-II appearance
 all at T_c "lambda point"

Only the 2-D Ising Model allows calculation of a λ -transition. Graphs show C_p diverges at T_c of, say, Helium.

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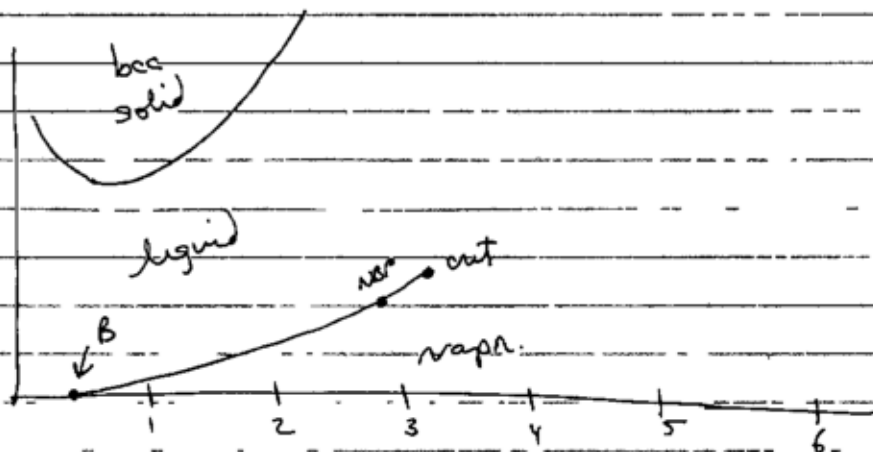
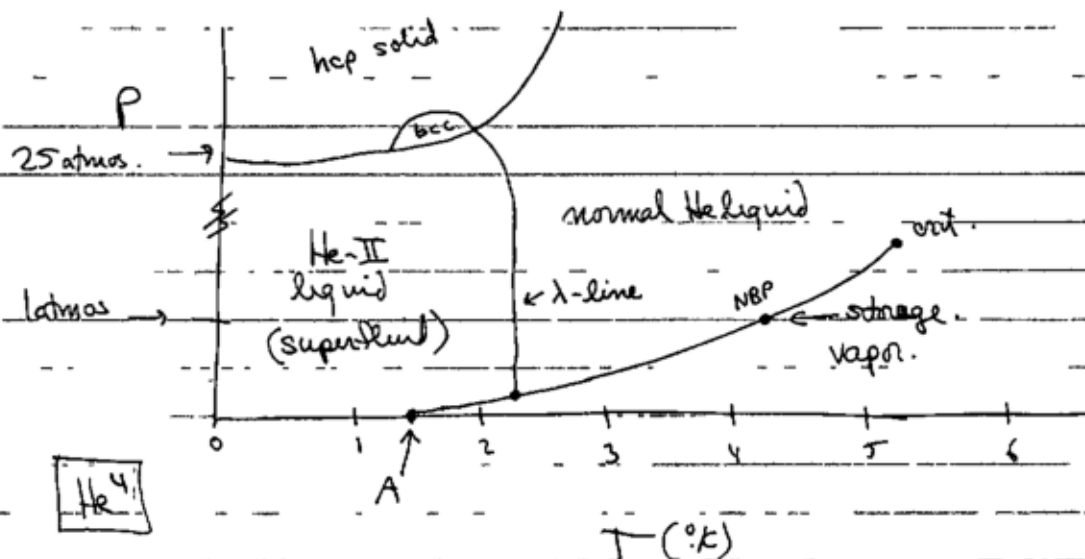
Helium Phases

A = 1.3°K = lowest temp you can get by ~~putting~~ pumping vapor off liquid

B = .4°K - .6

He^4

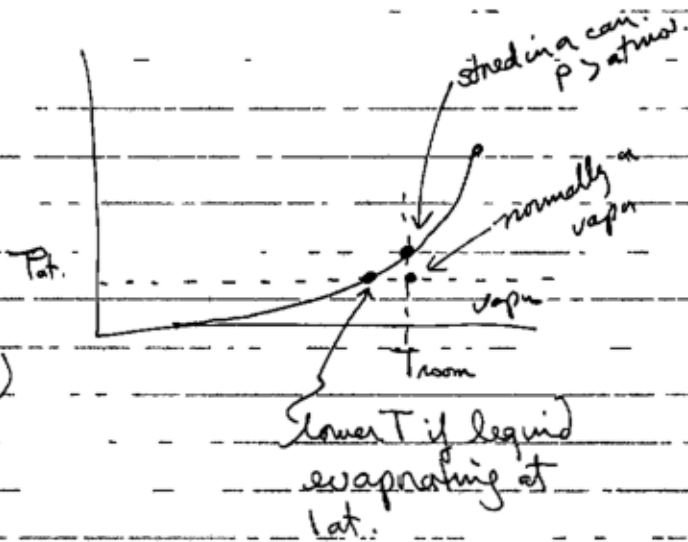
But $He^3/He^4 = 10^{-6}$ in abundance.



He^3

(no superfluid)
(no triple point!)

A diffusion pump sucks He^3 off surface of He^3 - He^4 mixture, thus lowering temp to $T_B \approx .6°K$. Called an "evaporator". Simply lowers P. You stay on the two-phase line.



(acetone?)



Newell and Montroll
RMP 25, 2 (1953)
"The Ising Model"

36 RMP pages.
(54 std. pages)

6/14/98.

1. Introduction The usual setup.

$$Z = \sum_{\sigma_1 = \pm 1} \sum_{\sigma_2 = \pm 1} \cdots \sum_{\sigma_N = \pm 1} \exp \left\{ \frac{1}{2} K \sum_{j,k=1}^N a_{jk} \sigma_j \sigma_k + \mu H \sum_{j=1}^N \sigma_j \right\}$$

where

$$K = J/kT$$

$$J = \text{interaction energy}$$

$$a_{ij} = \begin{cases} 1 & \text{NN} \\ 0 & \text{not NN} \end{cases}$$

$$H = H/kT$$

If you can compute Z , you can get all the thermodynamic info.

$$\bar{E} = \text{mean energy per particle} = - \frac{1}{N} \frac{\partial \ln Z}{\partial \beta} \quad \text{Ref (6.54)} \quad \beta = 1/kT$$

$$= - \frac{1}{N} \frac{\partial \ln Z}{\partial T} \frac{\partial T}{\partial (kT)} = + \frac{kT^2}{N} \frac{\partial \ln Z}{\partial T} \quad / (1.6)$$

$$\bar{m} = \text{mean mag. per particle} = \frac{1}{N} \frac{\partial \ln Z}{\partial H} \quad \text{since} \quad \mu \sum \sigma_j = \mu_{\text{tot.}}$$

By the way, notice that $Z \neq \prod Z_i$ because of interaction which is the whole point! Note that spin is treated as a scalar $\sigma = \pm 1$, not as a vector. Model is just a Testing Ground for techniques.

Last Review was Nix + Shockley in 1938

Ising not really good as a model of ferro, but

- good model for alloys - binary, $\sigma = +1, -1$ for type of atom at site
- lattice gas model $\sigma = \pm 1 \Rightarrow$ yes/no occupied.

Unsolved problems in 1953:

- 1) 3-D Ising
- 2) 2-D Ising with Next.
- 3) 2-D Ising with NN and NNN interaction

2. The Two Basic Math Formulations (Setups) [no H]

A. The Matrix Method Connect n^{th} layer to 1^{st} (no effect for large system).

$\mathbf{r}_j$ = coordinates of the j^{th} layer.

System energy is
$$\sum_{j=1}^{m-1} \underbrace{V(v_j, v_{j+1})}_{NN \text{ layer interaction}} + \sum_{j=1}^m \underbrace{V(v_j)}_{\text{internal layer energy}} = V.$$

Then write Z as:

$$Z = \sum_{v_1, v_2, \dots, v_m} e^{-V/KT} = \sum_{v_1, v_2, \dots, v_m} e^{-\sum_j V_j/KT}$$

$$= \sum_{v_1, v_2, \dots, v_m} \prod_{j=1}^m e^{-V_j/KT} = e^{-\left[\sum_{j=1}^m \left[V(v_j, v_{j+1}) + \frac{1}{2} \left[\frac{V(v_j) + V(v_{j+1}))}{2} \right] \right] / KT}$$

average the
to get symmetric

$$Z = P(v_1, v_{j+1}).$$

$$= \sum_{v_1} \sum_{v_2} \dots \sum_{v_m} P(v_1, v_2) P(v_2, v_3) \dots P(v_{m-1}, v_m) P(v_m, v_1)$$

hooked around

$[Z = \text{trace}(P^m)]$!! A simple trace a symmetric

$$= \sum \text{eigenvalues} = \sum_{i=1}^m \lambda_i = \sum_{i=1}^m (\lambda_i)^m$$

But, if λ_i are the eigenvalues of P , then $(\lambda_i)^m$ are those of P^m .

$$\log Z = \log \sum (\lambda_i)^m = \ln \left[(\lambda_1)^m + (\lambda_2)^m + \dots \right]$$

$$= \ln \left[(\lambda_1)^m \left\{ 1 + \left(\frac{\lambda_2}{\lambda_1} \right)^m + \left(\frac{\lambda_3}{\lambda_1} \right)^m + \dots \right\} \right]$$

$$= m \ln \lambda_1 + \ln \left[1 + (\lambda_2/\lambda_1)^m + \dots \right]$$

But we are really only interested in very large m .

$\lim_{m \rightarrow \infty} \frac{1}{m} \ln Z = \ln \lambda_1$! So all you need know is largest eigenvalue.

medium model: $V(v, v') = V(\sigma, \sigma') = -J \sigma \sigma'$

$$P(\sigma, \sigma') = e^{+J \sigma \sigma' / kT} = e^{K \sigma \sigma'} \quad K = J/kT.$$

$$= \begin{pmatrix} e^K & e^{-K} \\ e^{-K} & e^K \end{pmatrix} = e^{-K} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \quad \text{where } Q = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}.$$

As is trivial to show, eigenvalues of P are $2 \cosh K, 2 \sinh K$

$\uparrow \lambda_1 \quad \uparrow \lambda_2$

Thus, for one-dim Ising we get $\underline{Z} \approx (\lambda_1)^m \approx (2 \cosh K)^m$

Appendix 2: Since 1-D so easy, add in $\mathcal{K} = H/kT$. Then

$$P(\sigma, \sigma') = \begin{bmatrix} e^{K + \mu\mathcal{K}} & e^K \\ e^{-K} & e^{K - \mu\mathcal{K}} \end{bmatrix} = \begin{pmatrix} ab & a^{-1} \\ a & ab^{-1} \end{pmatrix} \quad \begin{matrix} a = e^K \\ b = e^{\mu\mathcal{K}} \end{matrix}$$

So $\lambda_{\pm} = e^{K \cosh(\mu\mathcal{K})} \pm [e^{2K \sinh^2(\mu\mathcal{K})} + e^{-2K}]^{\frac{1}{2}}$ ✓

Thus, $\underline{Z} = \{ e^{K \cosh(\mu\mathcal{K})} + \sqrt{e^{2K \sinh^2(\mu\mathcal{K})} + e^{-2K}} \}^m$

Now $K = J/kT$ $\mathcal{K} = H/kT$. $\underline{Z}$ is analytic in T , so cannot show a "transition" wherein $\bar{\epsilon} \rightarrow \infty$.

$$M = \tilde{N} \frac{\partial \ln \underline{Z}}{\partial \mathcal{K}} = \tilde{N} m \left[e^K \mu \sinh(\mu\mathcal{K}) + \frac{1}{\sqrt{\dots}} 2 \sinh(\mu\mathcal{K}) e^{2K} \cosh(\mu\mathcal{K}) \right]$$

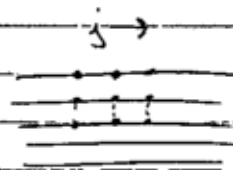
$\propto \sinh(\mu\mathcal{K})$

As $\mathcal{K} \rightarrow 0$, $M \rightarrow 0$: no spontaneous mag! Specific heat is $\partial \bar{\epsilon} / \partial T$ again analytic, so no lambda pt.

Note: when $\mathcal{K} = 0$, easy to compute $\underline{Z}$ directly:

$$\sum_{\sigma_i} e^{K \sigma_i \sigma_j} = \sum_{\sigma_N} \sum_{\sigma_{N-1}} e^{K \sigma_{N-1} \sigma_N} = \sum_{\sigma_{N-1}} 2 \cosh(K \sigma_{N-1})$$

$$\Rightarrow \underline{Z}(N) = 2 \cosh(K) \underline{Z}(N-1) \Rightarrow \underline{Z}(N) = 2^N \cosh^{N-1}(K)$$



— So much for the 1-D problem.

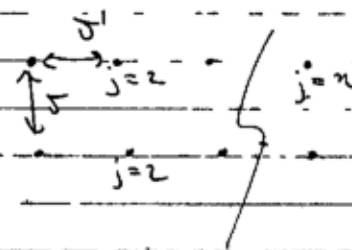
Two dim model. Now "layer" = row, and $V(i) = \text{total energy}$ of row i . Entirely due to NN:

$$V(i) = -J \sum_{\substack{NN \\ \text{row } i}} \sigma_i \sigma_{i+1} \quad \text{Wrap around!}$$

Between two rows, get

$$V(i, i') = -J \sum_j \sigma_j \sigma'_j$$

← same!

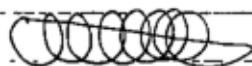


$$S_0 \quad P(\mathbf{v}, \mathbf{v}') = \exp \left(K' \underbrace{\sum_{j=1}^n \sigma_j \sigma_{j+1}}_{\text{within row}} + K' \underbrace{\sum_{j=1}^n \sigma_j \sigma'_j}_{\text{between rows}} \right)$$

This ends up as a huge direct-product of n 2×2 matrices. (i.e., $\mathbf{v} \times \mathbf{v}'$)

$$\Phi \quad \underline{P} = (2 \sinh 2K)^{\frac{n}{2}} \underline{V}_3 \underline{V}_2 \underline{V}_1 \quad \text{and}$$

Kramers-Wannier used a "screen" condition, so all particles end-to-end.



Matrix for 3-D can also be "setup", not done here.

B. The Combinatoric Method. (van der Waerden)

This is a small- K series (high T), not exact. You write:

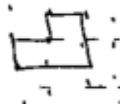
$$e^{K \sigma_i \sigma_j} = (chK) (1 + u \sigma_i \sigma_j) \quad \text{where} \quad u = thK \ll 1.$$

Then

$$\prod_{i,j} e^{K \sigma_i \sigma_j} = (chK)^{\frac{N}{2}} \prod_{i,j} (1 + u \sigma_i \sigma_j)$$

To a given order in $u = thK$, you include certain # of "closed graphs".

This is done graphically.



But I thought series solution could not show the non-analyticity in T !!

3. Phase Transitions

For self-dual lattices (like square), KW showed that if $\exists$ a sing, it should be at $|\sinh 2K_c| = 1 \Rightarrow K_c = \pm .44$.

Long-range order in $n \times \infty$ lattice requires degenerate max eigenvalue. For $\infty \times \infty$, not true.

4. Two-Dim Square Lattice Burst of activity in 1944-48 after Onsager's first solution. Better methods were found to do it. Then other lattices were solved. Kaufman (she) used spin- $SO(3)$ rep!

square.

4.1 combinatoric method: Long hairy mess involving counting closed graphs, was done by Kac + Ward in 1952. Here is the answer:

$$\left[\lim_{n, m \rightarrow \infty} (nm)^{-1} \ln Z = \ln 2 + \frac{1}{2\pi^2} \int_0^\pi \int_0^\pi d\omega \int_0^\pi d\omega' [\operatorname{ch} 2K \operatorname{ch} 2K' - \operatorname{sh} 2K \operatorname{ch} \omega - \operatorname{sh} 2K' \operatorname{ch} \omega'] \right]$$

where recall $K = J/kT$ and $K' = J'/kT$, directions different.

It is clear that loop counting is even harder in 3-D, no one had done it in 1953.

4.2 algebraic method: Another long barange, similar to Onsager approach. Certain matrices generate a Lie Algebra $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$. End result is the same, a double-integral. If $K = K'$, reduces to

$$(nm)^{-1} \ln Z = \frac{1}{2} \ln [2 \operatorname{sh} 2K] + \frac{1}{2\pi} \int_0^\pi d\omega \operatorname{arccosh} [\operatorname{ch} 2K \operatorname{coth} 2K - \cos \omega]$$

Then $E = -J \operatorname{arccosh} [1 + \frac{1}{2} \{2 \operatorname{sh}^2 2K - 1\} K_1(k_1)]$

where $k_1 = \frac{2 \operatorname{sh} 2K}{\operatorname{ch}^2 2K}$ and $K_1 =$ complete elliptical.

Again:

E_{ex}

at $k_1 = 1$, i.e., $|\operatorname{sh} 2K| = 1$, i.e., $K = .44$ as noted by KW.

But $K_1(z)$ has a logsing at $z=1$, i.e.,

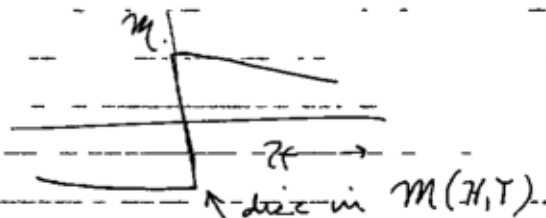
However, $E(T)$ is continuous at $T=T_c$ because K_1 coeff. vanishing.
But $\partial E / \partial T$ shows blow up.

So there you are! Problem is solved, you get a "lambda transition".

5. Other lattices — skip

6.1 Ferromag. For fixed $T \leq T_c$, expect

You have to resolve the Ising with a small H field. Then let $H \rightarrow 0^+$. Done by Yang.



6.2 Correlations — long range influence between spins.

7. 3-D Lattices — various power series.

8. Spherical Model — replace lattice sum with int'l.

and.





1953 Review

The Ising Model, Comments:

In 1925 student Ising proposed this thing as a real model for a ferromagnet. Turned out not to be useful for such, but is now used as model for binary metal alloys. The interest in the model is now mainly theoretical: in 2-D you can solve the model and you get a second-order phase transition where various thermo quantities, notably the specific heat, are singular at the critical temperature.

In 1941 Kramers and Wannier located the critical temperature, but could not solve the model. Earlier in 1936 Peierls argued that there should be a transition. Then the ice broke in 1944 when Onsager actually solved the 2-D model completely. The solution (no magnetic field) shows a logarithmic divergence of the specific heat at a certain critical temperature.

This is therefore in the class of transitions called lambda: there is no latent heat, so not first order. Really a second-order with infinite c .

In 1953 the 2-D model had been solved by two methods: Onsager's original matrix method with subsequent Lie Algebraic commutator stuff. Then others invented a combinatoric method wherein you could do-called closed loops in the lattice. By the way this all refers to a simple square lattice. Both methods are very messy. Kaufmann found a simple method in 1952, not duplicated in the 1953 Review

No one in 1953 had a complete solution to the 2-D Ising with a H field, but you could do it approx for small H and indeed you did find that there was "ferromagnetism". The spins want to spontaneously line up. Also no one had solved the 3-D even without an H field.

The 1-D model with H can be solved completely but is not interesting: there is no transition, and no magnetization spontaneously.

The methods involved are messy but the idea is simple: all you have to do is calculate the partition function for the lattice. This is some function of T . If analytic in T for all positive real T , then there is no transition. To get a critical temperature you need your $Z(T)$ to be singular somewhere. Specifically, when you compute $d \ln Z / dT$ you construct the mean energy E , and then dE/dT is the specific heat: here is where you are looking for a singularity.

I guess before this Onsager work it was not known that you could actually calculate anything near a critical point. Various earlier approximation methods simply failed to produce the log singularity because something essential was left out of the analysis.

I will now have to read what has happened since 1953. Ken Wilson in 1971 did some big thing involving the renormalization group. I don't think I will pursue the solid state stuff much further. But I had to review Reif to recall what a partition function was and its relation to specific heat and the meaning of a second-order phase transition. I feel I now have the basics and can resume the particle physics course.

The Reciprocal Lattice and Fourier Analysis [Kittel]

1. Diffraction used to study crystals:

- a) photons $\lambda = 12/\text{E (keV)}$
- b) neutrons $\lambda = 0.3/\text{E (eV)}$
- c) electrons $\lambda = 12/\text{E (eV)}$

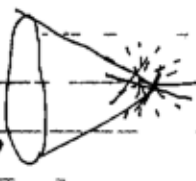
scattering of atomic electrons
scatter from "magnetic electrons" or nuclei
used only for thin films + surface study

Bragg's law says: shoot in beam, add all layers - reflective. They will fire only when $d \sin \theta = n\lambda$ where d = layer sep. θ = grazing angle. You need $\lambda \leq 2d$ to get Bragg diffraction scattering.

2. Explanation of the powder method: say $\theta_{\text{peak}} = 45^\circ$. If surfaces are randomly oriented 1, 2, 3... in the plane, only those at 45° make a peak. Now suppose $\theta_{\text{peak}} = 10^\circ$. Only those at 10° make a "diffracted beam" (really a reflected beam). But due to 3-D and you see why or come!



Thus, diffraction pattern is set of rings concentric. Reduce the θ_n , you can then figure out what it's structure. I.e. you get one set of θ_n for each crystal plane family



3. Laue Method of Diffraction: just add up all the scattered waves. You assume elastic scatter at any point. Easy to show that

$$E_{\text{sc}}(\vec{r}) = \sum_{\vec{r}'} e^{i\vec{k} \cdot \vec{r}'} e^{-i\vec{k}' \cdot \vec{r}'} e^{-i\vec{r}' \cdot \vec{r}} = \sum_{\vec{r}'} e^{i(\vec{k} - \vec{k}') \cdot \vec{r}'} e^{-i\vec{r}' \cdot \vec{r}}$$

where $\Delta \vec{E} = E - E' = \text{doppler}$, $\vec{r}_{\text{scat}} = \text{location of scatterer}$ at lattice site m, n, p .
 Consider: $e^{-i\vec{r}_{\text{scat}} \cdot \Delta \vec{k}} = \exp(-i[m\vec{a} + n\vec{b} + p\vec{c}] \cdot \Delta \vec{k})$

We require for $\Delta \vec{k}$ such that $\Delta \vec{k} = 1/V m, n, p$. Then a large part! This requires:

$$\begin{aligned} a \cdot \Delta \vec{k} &= 2\pi h \\ b \cdot \Delta \vec{k} &= 2\pi k \\ c \cdot \Delta \vec{k} &= 2\pi l \end{aligned} \quad \text{where } h, k, l = \text{integers} \quad (\text{Laue Equations})$$

To solve these in general non-orthogonal case, we need S. Wilson & Gibbs invention of: The Reciprocal Lattice.

Recip. lattice: We want to solve lattice. Try: $\vec{G} = h\vec{A} + k\vec{B} + l\vec{C}$.

This says:

$$\vec{a} \cdot \vec{A} = 2\pi$$

$$\vec{a} \cdot \vec{B} = 0$$

$$\vec{a} \cdot \vec{C} = 0$$

$$\vec{A} = 2\pi \frac{\vec{b} \times \vec{c}}{\vec{a} \cdot \vec{b} \times \vec{c}}$$

So $\vec{A} \perp \vec{b}$ and $\vec{A} \perp \vec{c}$. Result.
 $\vec{B} \perp \vec{a}$ and $\vec{C} \perp \vec{a}$ etc.

Better notation: $\sum m_i \vec{a}_i = m_1 \vec{a}_1 + m_2 \vec{a}_2 + m_3 \vec{a}_3 = \sum m_i \vec{a}_i$

Want: $\vec{G} \cdot \sum m_i \vec{a}_i = 2\pi \times \text{integer} \quad \forall \text{ integer } m_i$.

So let $\vec{a}_i \cdot \vec{G} = 2\pi n_i \quad i=1,2,3$.

Try $\vec{G} = \sum n_i \vec{A}_i$. Then: $\vec{a}_i \cdot \sum n_j \vec{A}_j = 2\pi n_i$

$\Rightarrow \vec{a}_i \cdot \vec{A}_j = 2\pi \delta_{ij}$. Try:

$$\vec{A}_i = \epsilon_{ijk} \vec{a}_j \vec{a}_k \cdot 2\pi C \quad C = \text{some const.}$$

Then: each!

$\vec{a}_i$

Go back.

$$\vec{a}^{(i)} \cdot \vec{A}^{(j)} = 2\pi \delta_{ij}$$

Try $\vec{A}_i = 2\pi C \epsilon_{ijk} \vec{a}_j \vec{a}_k$

$$\vec{A}^{(i)} = 2\pi C \epsilon^{jkl} \vec{a}^{(j)} \times \vec{a}^{(l)}$$

Then $\vec{a}^{(i)} \cdot \vec{A}^{(j)} = \vec{a}^{(i)} \cdot [2\pi C \epsilon^{jkl} \vec{a}^{(j)} \times \vec{a}^{(l)}]$ etc etc.

OK I accept (28) as solution.

$$\vec{A} = 2\pi C \vec{b} \times \vec{c} \quad \text{and cyclic,} \quad C = (\vec{a} \cdot \vec{b} \times \vec{c})^{-1}$$

So you get a diffraction peak at any $\vec{G} = \sum n_i \vec{A}_i$. I.e., at every "point in the reciprocal lattice".

So $\vec{G} = h\vec{A} + k\vec{B} + l\vec{C}$ = points of reciprocal lattice.

$\vec{G} = \vec{G} \Rightarrow \text{diff max.}$

① Comment: Try to Fourier transform a function $f(\vec{r})$ where $\vec{r}$ = position in crystal:

$$f(\vec{r}) = \sum_{\vec{k}} F(\vec{k}) e^{i\vec{k} \cdot \vec{r}}$$

If $f(\vec{r})$ is periodic in the lattice, then the only $\vec{k}$ which contribute in this sum are $\vec{k} = \vec{G}$, the reciprocal lattice vectors. Eg, electron density is

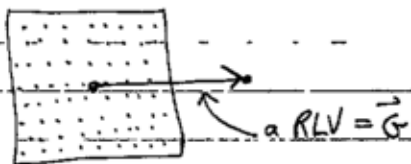
$$n(\vec{r}) = \sum_{\vec{G}} N_{\vec{G}} e^{i\vec{G} \cdot \vec{r}}$$

② Note, then, that in scattering we have peaks when $\vec{k}' = \vec{k} + \vec{G}$. But

$$\vec{k}'^2 = k^2 = k^2 + 2\vec{k} \cdot \vec{G} + \vec{G}^2 \Rightarrow 2\vec{k} \cdot \vec{G} + \vec{G} \cdot \vec{G} = 0, \text{ another way}$$

to write the Bragg condition in 3-D.

③ Now try to explain (2.3) of Barker. In reciprocal lattice space there are Brillouin Zones. Look at 1st BZ. If you are talking about, say, phonon modes, then for a finite crystal $\vec{k}$ can take only discrete values



Consider Fourier analysis of fcn of period L. Discrete k 's! So, consider

$$\tilde{f}(\vec{k}) \equiv \sum_{m,n,p} e^{-i\vec{k} \cdot \vec{r}_{mnp}} f(\vec{r}_{mnp})$$

where $f(\vec{r}_{mnp})$ is a function defined only at lattice sites. And $\vec{k} \in$ the " $\vec{k}$ lattice" $\neq$ "reciprocal lattice". Suppose $\vec{k} \in 1^{st}$ BZ, $\vec{k}' \in$ some other BZ. Then $\vec{k}' = \vec{k} + \vec{G}$ and

$$\tilde{f}(\vec{k}') = \tilde{f}(\vec{k}) \text{ since } e^{-i\vec{G} \cdot \vec{r}_{mnp}} = 1$$

Thus, the only unique Fourier components are those where $\vec{k} \in 1^{st}$ BZ. Now consider the inverse:

$$\begin{aligned} f(\vec{r}_{mnp}) &= \sum_{\text{all } \vec{k}} \tilde{f}(\vec{k}) e^{i\vec{k} \cdot \vec{r}_{mnp}} = \text{special case of fourier.} \\ &= N \cdot \left(\sum_{\vec{k} \in \text{F0B}} \tilde{f}(\vec{k}) e^{i\vec{k} \cdot \vec{r}_{mnp}} \right) \text{ where } N = \# \text{ of Brillouin zones.} \end{aligned}$$

[So, more or less, to Fourier expand a "lattice function" $f(\vec{r}_{\text{lattice}})$, you need only sum over $\vec{k} \in \text{FBZ}$.]

Another way to say this: for a "lattice function", the values k takes in the first BZ are a complete set of values!

$$\text{I.e., } \delta_{\vec{r}_{\text{lattice}}, \vec{r}'_{\text{lattice}}} = \sum_{\vec{k} \in \text{FBZ}} e^{i\vec{k} \cdot (\vec{r}_{\text{lattice}} - \vec{r}'_{\text{lattice}})}$$

$$\text{or } \sum_{\vec{k} \in \text{FBZ}} e^{i\vec{k} \cdot [\vec{r}_{\text{lattice}}]} = \delta_{m,0} \delta_{n,0} \delta_{p,0}.$$

Wilson RMP

The renormalization group: Critical phenomena and the Kondo problem*†

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This review covers several topics involving renormalization group ideas. The solution of the s -wave Kondo Hamiltonian, describing a single magnetic impurity in a nonmagnetic metal, is explained in detail. See Secs. VII-IX. "Block spin" methods, applied to the two dimensional Ising model, are explained in Sec. VI. The first three sections give a relatively short review of basic renormalization group ideas, mainly in the context of critical phenomena. The relationship of the modern renormalization group to the older problems of divergences in statistical mechanics and field theory and field theoretic renormalization is discussed in Sec. IV. In Sec. V the special case of "marginal variables" is discussed in detail, along with the relationship of the modern renormalization group to its original formulation by Gell-Mann and Low and others.

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INTRODUCTION

One of the most basic themes in theoretical physics is the idea that nature is described locally. The basic equations of all physics are local. For example, Maxwell's equations specify the behavior of electric and magnetic fields in an infinitesimal neighborhood of a point x . In order to be able to specify local equations it is necessary to define continuum limits, namely the limits which define derivatives. The idea of the derivative and the idea of a continuum limit that underlies the derivative is therefore of great importance in all of physics.

It is now becoming clear that there is a second form of continuum limit, called the statistical continuum limit, which also has a very broad range of applicability throughout physics. In the statistical continuum limit functions of a continuous variable are themselves independent variables. For example, the electric and magnetic fields throughout space can be the independent variables in a statistical continuum limit. This happens in statistical or quantum mechanical problems where there are field fluctuations, so that one has to compute averages over an ensemble of fields. In statistical calculations one does not compute the field at a

point x . Instead one computes correlation functions; that is, expectation values of products of fields such as $\langle E(x,t)E(y,t') \rangle$. In quantum mechanical problems the correlation functions are sometimes replaced by vacuum expectation values of products of fields. In the simplest cases a field average, determining a correlation function can be written formally as a functional integral. In the functional integral the fields are the independent variables of integration.

There are two ways in which a statistical continuum limit can arise. The obvious way is when the independent field variables are defined on a continuous space; the case of statistical or quantum fluctuations of the electromagnetic field is an example. If one were to replace the continuum by a discrete lattice of points, the field averages would consist of integrals over the value of the field E at each lattice site n . Thus for the discrete lattice case one has a multiple integration, $\prod_n \int dE_n$, the variables of integration being the fields E_n . In the continuum limit one has infinitely many integration variables E_n . Problems with infinitely many variables can be very difficult to solve.

The second source of statistical continuum limits is the situation where one has a lattice with a fixed lattice spacing, usually an atomic lattice. The number of independent variables (i.e., independent degrees of freedom) at each lattice site is fixed and finite. The continuum limit arises when one considers large size regions containing very many lattice sites. When the lattice is viewed on a macroscopic scale one normally expects the lattice structure to be invisible. That is, large scale effects should be describable by a continuum picture making no reference to the lattice spacing.

Consider, for example, critical phenomena in a magnet. A magnet is built of atoms and the atomic spacing provides a fixed shortest length which does not go to zero. At the critical point (the critical point occurs at the Curie temperature) there are long wavelength fluctuations of the magnetization signalling the onset of spontaneous magnetization. The maximum wavelength of the fluctuations is the corre-

* Supported in part by the National Science Foundation.

† This paper is a compilation of material presented as a series of nine lectures at Cargèse in Summer 1973.



Stanley

Notes on Stanley Book

Introduction to Phase Transitions and Critical Phen.

H.E. Stanley (Oxford Univ Press, 1971). Based on

some 1969 notes. Pre-Wilson, pre-renorm group.

Copy in Physics Utah library. Read parts, 6/78

Thermodynamics of magnetic systems.

Comparison with gases:

Preparation for critical theories.

$\left\{ \begin{array}{l} T = \text{its usual self} \\ H \text{ replaces } P, \text{ the intensive} \\ -M \text{ replaces } V, \text{ the extensive} \end{array} \right.$

$$PdV \rightarrow -HdM$$

$$C_P = -T \left(\frac{\partial^2 G}{\partial T^2} \right)_P \rightarrow C_H = -T \left(\frac{\partial^2 G}{\partial T^2} \right)_H, \text{ still heat capacity.}$$

$$K_T = \text{isothermal compressibility} = -\frac{1}{V} \left(\frac{\partial V}{\partial P} \right)_T = -\frac{1}{V} \left(\frac{\partial^2 G}{\partial P^2} \right)_T$$

$$\rightarrow \chi_T = \left(\frac{\partial M}{\partial H} \right)_T = - \left(\frac{\partial^2 G}{\partial H^2} \right)_T = \text{isothermal susceptibility.}$$

$$\chi_S = \left(\frac{\partial M}{\partial H} \right)_S = - \left(\frac{\partial^2 E_{\text{th}}}{\partial H^2} \right)_S = \text{adiabatic suscept} \quad (dQ = Tds)$$

note absence of norm factor.

$$C_H = \left(\frac{\partial E}{\partial T} \right)_H = -T \left(\frac{\partial^2 F}{\partial T^2} \right)_H$$

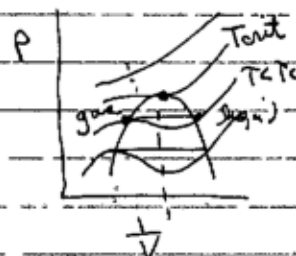
$$\alpha_H = \left(\frac{\partial M}{\partial T} \right)_H$$

$$\text{gas: } \left\{ \begin{array}{l} C_P = T \left(\frac{\partial S}{\partial T} \right)_P \\ K_T = -\frac{1}{V} \left(\frac{\partial V}{\partial P} \right)_T \\ \alpha = \frac{1}{V} \left(\frac{\partial V}{\partial T} \right)_P \end{array} \right.$$

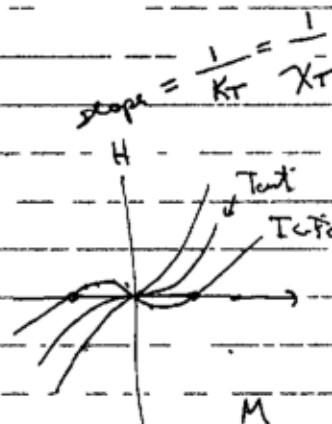
Recall the 3 "basic deriv" of gas therm:

$$C_P \rightarrow T \left(\frac{\partial S}{\partial T} \right)_H = C_H$$

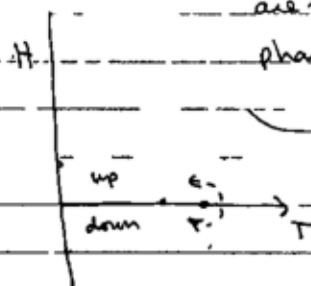
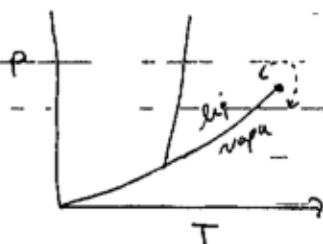
$$\left. \begin{array}{l} K_T \rightarrow + \left(\frac{\partial M}{\partial H} \right)_T = \chi_T \\ \alpha \rightarrow \left(\frac{\partial M}{\partial T} \right)_H = \alpha_H \end{array} \right\} \text{both change in norm.}$$



for $T < T_c$, there are two possible phases.



for $T < T_c$, there are two possible phases: $\uparrow$ and $\downarrow$



compare.

The separation of the two phases is: $(g_L - g_G)_{P=P'}^{H=0}$ for gas

Over are the "order parameters": $(\partial M)_{H=0}$ for magnet.

Normalized Exponents: Let $\epsilon \equiv (T - T_c)/T_c$.

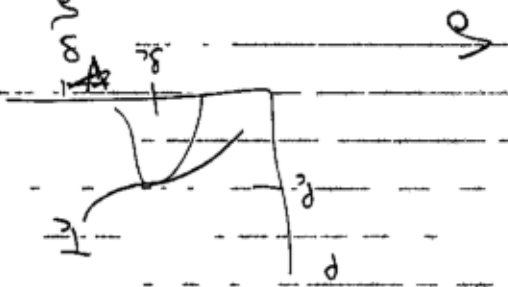
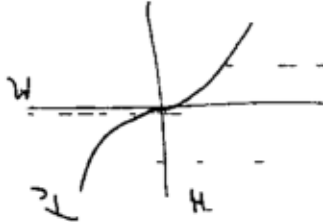
$$\beta \begin{cases} \text{gas: } (g_L - g_G)_{P=P'}^{H=0} \sim (-\epsilon)^\beta \\ \text{magnet: } (M)_{H=0} \sim (-\epsilon)^\beta \end{cases} \quad \alpha \rightarrow 0. \quad (\beta > 0) \quad [\beta = .35 \text{ He}] \quad [\beta \approx .3]$$

$\alpha \rightarrow 0$ these being: finite means approach from low T side.

$$\alpha \begin{cases} \text{gas: } C_V = C \sim (-\epsilon)^{-\alpha'} \\ \text{magnet: } C_H = 0 \sim (-\epsilon)^{-\alpha'} \end{cases}$$

$$\gamma \begin{cases} \text{gas: } K_T \sim (-\epsilon)^{-\gamma'} \\ \text{magnet: } (\chi_T)_{H=0} \sim (-\epsilon)^{-\gamma'} \end{cases}$$

Other: small critical iso lines



$$\beta \begin{cases} \text{gas: } (P - P_c) \sim |(S - S_c)|^{\beta-1} \\ \text{magnet: } H \sim |(M)|^{\beta} \end{cases} \quad \delta \approx 4.3 \approx \alpha$$

Correlation length: $\xi \sim (-\epsilon)^{-\nu}$, blows up $\nu \rightarrow \infty$

$$\text{Pair Correlation Fcn: } G(r) \sim \frac{1}{r^{d-2+2\eta}}$$

$\eta \rightarrow \infty$

$$\text{One gap exponents: } G^{(1)} \equiv (a/aH)^{1/2} \quad G^{(2)} \equiv (a/aH)^{1/2} \quad G^{(1)} \equiv G$$

then define Δ' by $G^{(1)} = (\epsilon)^{\Delta'}$

Δ^s Now, maybe $G^{(s)}$ is ... well, basically:

$$(\epsilon)^{-\Delta_2'} = \frac{G^{(s)}}{G^{(1)}} = \frac{(\partial^2 G / \partial H^2)_T}{(\partial G / \partial H)_T} = \frac{\chi_T}{M} \sim \frac{(\epsilon)^{-\gamma'}}{(-\epsilon)^\beta}$$

$$\Rightarrow \Delta_2' = \gamma' + \beta \quad \checkmark$$

$$\text{I will take } \Delta_1' = 2 - \alpha' + \beta$$

Inequality assuming exponents:

① The magnetic $C_V - C_P$ equation leads to: $\alpha' + 2\beta + \gamma' \geq 2$ (Rushbrooke)

Book Contents:

Ch 1. "Classical + Modern Eras of Critical Phenomena."

a) classical = 1869 Andrews on CO_2

1872 van der Waals theory.

b) modern = 1944 Onsager.

recognition of exponents as interesting / order parameters.

Guggenheim plot shows many gases have same gas/liquid coexistence curve in the $T-V$ plane, same β .

universality.

Ch 2. "Thermos for fluid + magnetic systems" - all basic formulas (see above).

Ch 3. "The Critical-Point Exponents."

definitions of $\alpha, \beta, \gamma, \gamma', \Delta_1, \Delta_2, \nu$ + some numerical values (see above)

Ch 4. "Exponent Inequalities."

Ch 5. "Van der Waals Theory of Liquid-Gas Transition". (1st classical theory)

This theory involves only a, b and generates those S-curves in the $P-V$ plane.

You can compute that $P_c = a/27b^2$, $V_c = 3b$, and $RT_c = 8a/27b$ (one mole).

$Z_c \equiv P_c V_c / RT_c = .375$ in VDW theory = $3/8$, not so good empirically. You can

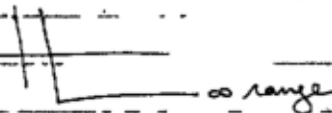
calculate the various critical exponents: $\delta = 3$, $\gamma = 1 = \gamma'$, $\beta = \frac{1}{2}$. Also,

$\alpha = \alpha' = 0$ so specific heat does not blow. Van der Waals is a good theory, but not

very good in terms of critical behavior. The "Maxwell" equal-area modification

is also needed, by the way, to remove those unstable regions. In 1965 Fred

Rief showed how you could derive VDW equation by assuming a trivial single particle hard-core potential.



2nd Classical Theory.

Ch 6: "One Magnetic Mean Field Theory"

- a) Very simple idea. First let $H = -g\mu_B \sum_i \vec{S}_i \cdot \vec{H}_{ext}$, i.e., no interaction between spins. Naturally this theory yields no spontaneous mag. Next, replace $H \rightarrow H_{ext} + \lambda M(T, H_{ext})$, sort of a feedback. The spins make a field which adds to H . $\lambda =$ "molecular field parameter." In this case you do get $M|_{H=0}$ below a T_{crit} . You then calculate your exponents: $\beta = \frac{1}{2}$, $\delta = 3$, $\alpha = \alpha' = 0$, finite disc in C_H . $\gamma = \gamma' = 1$. $\Delta_1' = \frac{3}{2}$, $\Delta_2' = \frac{3}{2} = \Delta_2$.

"Law of Correspondence" = equation of state in terms of $\tilde{P}, \tilde{V}, \tilde{T}$, $\tilde{P} = P/P_c$.

Onsager model invented 1907 by Weiss: Weiss molecular field. Exponents are same as VDW fluid model, since both models have ∞ range interaction.

- b) Heisenberg Theory: $\mathcal{H} = -\mu \sum_i \vec{S}_i \cdot \vec{H} - \sum_{ij} J_{ij} \vec{S}_i \cdot \vec{S}_j$ interactions.

Now approximate $\sum_{ij} J_{ij} S_i^z S_j^z \approx \sum_{ij} J_{ij} \langle S^z \rangle S_j^z = \hat{J}_0 \langle S^z \rangle S_i^z$.

This means $H_{eff} = \dots$ & model reduces to Weiss model with $\Delta = 2\hat{J}_0/N\mu^2$.

I.e., this model at least explains a model for Weiss speculation.

Chapter closes by showing that [any theory with spins interacting equally with all other spins leads to a mean field theory like Weiss].

I.e., a long range interaction yields same set of (wrong) exponents.

Model is not very good, but has correct qualitative features, i.e., a T_c and M when $H=0$.

Ch 7: "Pair Correlation Function and Ornstein-Zernike Theory" (see below)

Ch 8: "Results of exactly soluble models". Let $d =$ lattice dimensionality, $D =$ spin dimens. Then here are exactly solvable cases.

$D \backslash d$	1	2	3	
1	Ising ($\frac{n \cdot n}{H=0}$)	Ising ($\frac{n \cdot n}{H=0}$)	(unsolved)	
2	XY ($\frac{n \cdot n}{H=0}$)	—	—	XY = plane rotator
3	Heis. ($\frac{n \cdot n}{H=0}$)	—	—	
∞	Spherical ($\frac{n \cdot n}{H=0}$)	→ same	→ same	

these words refer to D , not d .

The "spherical" models: In 1947 Mark Kac said: use Ising but don't restrict spins to be unit length. Just insist that $\sum_i S_i^2 = N$. I.e., in N -dim space, instead of summing over corners

of a cube, integrate over surface of sphere. Easier to integrate than to sum! In 1952 Berlin showed Kac how to do the integrations via steepest descent. Hence: "Berlin-Kac spherical model" 1952. This was in 3-dimensions, $d=3$. Later (1968), Stanley showed this model equivalent to a $D=\infty$ model. Amazing model, but not practical.

"transfer matrix" method - Onsager's matrix method, recall $2 = \text{trace}(P)$. See notes elsewhere.

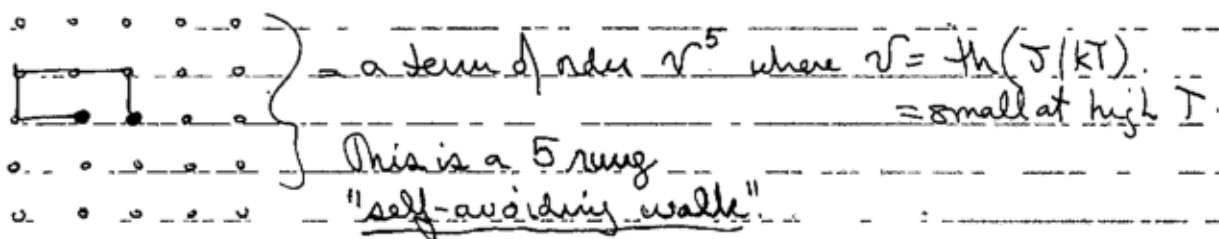
Ch 9: "Results from approximately solved models"

Basically, this means hi-T and lo-T series expansions which allow you to approach T_c from either side. In general, to get 1% accuracy you have to add more than two terms. Hence, various Feynman-Diagram-like methods have appeared. Of course you can test your methods against the exactly soluble models!

Basic idea is this

$$Z = \sum e^{-\beta \mathcal{H}} = \sum [1 - \beta \mathcal{H} - \frac{1}{2} \beta^2 \mathcal{H}^2 \dots] = \text{high temperature expansion}$$

Example: suppose you try to compute $\langle S_n S_m \rangle = \sum S_n S_m e^{-\beta \mathcal{H}} / Z$, the two-spin correlation fun. Suppose moreover that m, n are nearest neighbors. Then the series has this graphical interpretation (note $\langle S \rangle = 0$ $T > T_c$).



Results are better (closer to exact) than in soluble models.

Ch 10: "Landau's Classic Theory of Exponents" (see below)

Ch 10: "Scaling ideas"

Ch 12: "Kadanoff scaling"

Ch 13: "Dynamic Critical Phenomena" [i.e., non-equilibrium: transport, light scatter, hydrodynamics]

Ch 14: "Dynamic Exponents"

Ch 15: "Mode-mode coupling."

Back now to

Ch 1: "Pair correlation" "O2 Theory"

$$\langle n(\vec{r}) n(\vec{r}') \rangle = \sum \int d\vec{r}_1 d\vec{r}_2 \dots \int d\vec{r}_N \dots \int d\vec{r}_N n(\vec{r}_1) n(\vec{r}_2) \dots n(\vec{r}_N) \propto \text{prob of finding}$$

a particle at $\vec{r}'$ if you know there is one at $\vec{r}$. For large $|\vec{r} - \vec{r}'|$ you would guess that the "correlation" makes no diff. so $\langle n(\vec{r}) n(\vec{r}') \rangle \rightarrow \langle n(\vec{r}) \rangle \langle n(\vec{r}') \rangle = n^2$.

$$\text{Let } G(\vec{r}, \vec{r}') = \langle [n(\vec{r}) - \langle n(\vec{r}) \rangle] [n(\vec{r}') - \langle n(\vec{r}') \rangle] \rangle$$

which measures "correlation between density fluctuations" here and here. Obviously,

$$G(\vec{r}, \vec{r}') = \langle n(\vec{r}) n(\vec{r}') \rangle - n^2 = G(\vec{r} - \vec{r}') \text{ by translation invariance} \\ \equiv \text{"density-density correlation function"}$$

So let's use the grand ensemble including $\mu N \beta$ in $\mathcal{H}$. We compute the fluctuation in N . Result

$$\left[\frac{\langle (N - \langle N \rangle)^2 \rangle}{\langle N \rangle} \right] = \text{dispersion as function} = -k_B T (\chi_T) \rightarrow \infty$$

Thus as $T \rightarrow T_c$, you know $k_T \rightarrow \infty$, so you see big fluctuations in total N . But

$$(N - \langle N \rangle)^2 = \int d\vec{r} G(\vec{r}) \rightarrow \infty$$

Thus, $G(\vec{r})$ must be increasing its range to ∞ as $T \rightarrow T_c$.

Next, he shows that scattered light intensity is given by

$$I(\vec{q}) \sim \int d\vec{r} G(\vec{r}) e^{-i\vec{q} \cdot \vec{r}} = S(\vec{q}) \text{ "structure factor"}$$

So as $T \rightarrow T_c$, $I(\vec{q})$ very large at $\vec{q} = 0$ (forward). This is critical opalescence.

Now the Ornstein-Zernike theory in a way to calculate $S(\vec{q})$. They find that

$$G(\vec{r}) \sim \frac{e^{-k_F r}}{r} \quad \text{and } k_F \rightarrow 0 \text{ as } T \rightarrow T_c$$

$$k_F = 0. \text{ Their theory shows that } k_F^2 \sim k_F^{-1} \sim (-\epsilon)^{-1/2}$$

and therefore that $(\nu' = \frac{1}{2}\delta')$ and $(\nu = \frac{1}{2}\delta)$. Note also that $\xi \sim (-\epsilon)^{-\frac{2\nu'}{d-2}}$ But $\xi \sim (-\epsilon)^{\nu'}$ by def, so their theory predicts

$$\chi(r) \sim \frac{e^{-r/\xi}}{r} = \frac{1}{r} \text{ at } T \rightarrow T_c$$

But $\chi(r) \sim \frac{1}{r^{(d-2+\eta)}}$. So if $d=3$, $\eta=0$. Etc.

I don't know what set of assumptions = OZ theory, has to do with power series + calc. assumptions.

3rd Classical Theory.

Q10: "Landau Theory". For magnet, take say $F(T, M)$ and expand in a series around $M=0$ and $T=T_c$, since you are interested only in the critical region. Theory predicts same set of classical exponent values as Weiss field and Heisenberg models. The Landau Theory is not great because coefficients [second-derivs] are just those objects which diverge at $T=T_c$, $M=0$. Maybe OK for Ehrenfest-type critical points where C_p has a disc but no divergence.

copy pages (44-45 / 46-47) shows exp values

more Stanley book notes.

Stanley again

Ch 11: "scaling law hypothesis" [for thermo funcs]

From thermodynamics and elsewhere we know there are certain exponent inequalities. But the data suggest these are actually equalities. An idea has arisen to suggest (though not prove) why this is so: scaling. Certain forms are assumed for thermo functions.

Who? Widom 1965, Domb + Hunter, then Kadanoff 1966. (next chap)

def: a "homogeneous function" is $f(\lambda r) = g(\lambda) f(r)$, i.e., one r determines all. Consistency requires that $g(\lambda) = \lambda^p$.

"generalized homo Schr"

$$f(\lambda^a x, \lambda^b y) = \lambda^p f(x, y)$$

Theremlet: $f(x, y) = y^p F(x/y)$ if $f(\lambda x, \lambda y) = \lambda^p f(x, y)$
 $= x^p G(y/x)$

Basic Assumption: $G(\epsilon, H)$ where $\epsilon = T - T_c/T_c$ is homogeneous, after you subtract non-singular terms. [Static Scaling Hypothesis]. $\uparrow$ Most generally this means:

$$G(\lambda^{-a_\epsilon} \epsilon, \lambda^{a_H} H) = \lambda^p G(\epsilon, H) \quad \left(\begin{array}{l} \text{two-exponent scaling} \\ \text{"hyper-scaling"} \end{array} \right)$$

(Could do this with any

of the other potentials: if G homo, then so do V, F, H etc.)

Take derivative wrt H :

$$\lambda^{a_H} \frac{\partial G(\epsilon, H)}{\partial H} = \lambda \frac{\partial G(\epsilon, H)}{\partial H} \quad \text{where } \epsilon' = \lambda^{-a_\epsilon} \epsilon \quad H' = \lambda^{a_H} H$$

n $M(\epsilon', H') = \lambda^{1-a_H} M(\epsilon, H)$. Now first, to get β , set $H=0$.

$$M(\lambda^{-a_\epsilon} \epsilon, 0) = \lambda^{1-a_H} M(\epsilon, 0)$$

Now set $\lambda^{-a_\epsilon} \epsilon = -1 \Rightarrow \lambda^{-a_\epsilon} = -1/\epsilon \Rightarrow \lambda = (\epsilon)^{-1/a_\epsilon}$. Then

$$M(-1, 0) = (-\epsilon)^{-1/a_\epsilon} M(\epsilon, 0) \Rightarrow M(\epsilon, 0) = (-\epsilon)^{1/a_\epsilon} M(-1, 0) \quad H=|M| \text{ recall.}$$

$\Rightarrow \frac{1-a_H}{a_\epsilon} = \beta$! A similar trick shows: $\frac{a_H}{1-a_H} = \delta$ $\left[\begin{array}{l} \epsilon=0 \\ \text{then } H \rightarrow 0 \end{array} \right]$

Next, do the second derivative to get rule on χ_T :

Result

$$\chi' = \frac{2a_H - 1}{a_\epsilon} \Rightarrow [\chi' = \beta(\beta - 1)]^{\odot} \quad \leftarrow \text{"Widom"}$$

Next, easy to show $\gamma = \gamma'$.

Next, the "gap" exponents: Recall: $\frac{G^{(l)}}{G^{(l-1)}} = e^{-\Delta_l}$

where $G^{(l)} \equiv (\partial^2 G / \partial H^2)_T$.

What does "static scaling" have to say here? Do I derivative to conclude:

$$[\Delta_l = \text{same for all } l! = \frac{a_H}{a_\epsilon} = \beta\delta = \gamma' + \beta]^{(2)}$$

Note: $a_\epsilon = \frac{1}{\beta} \frac{1}{\delta+1}$ and $a_H = \delta \frac{1}{\delta+1}$ But we just showed

$$\text{that } \gamma' = \beta(\delta-1) = \beta\delta - \beta \quad \text{so}$$

Next: Π -derivative: $\alpha' = 2 - \frac{1}{a_\epsilon} \Rightarrow \alpha' + \beta(\delta+1) = 2$ (was Griffiths inequality)

This plus $\gamma' = \beta(\delta-1) \Rightarrow [\alpha' + 2\beta + \gamma' = 2]^{(3)}$ (was Rushbrooke inequality)

The point here is very clear: [if you simply make the ansatz that any of the thermodynamic potentials (like Gibbs) ~~is~~ is homogeneous in the most general sense that it can be with its two variables, then you can compute all critical exponents in terms of the two numbers appearing in your homogeneity relation. Thus you end up with many relations between exponents. Many of these were earlier proved to be inequalities from basic thermo and other basic assumptions.]

Scaling and Equation of State: We already showed

$$M(\epsilon, H) = \lambda^{a_H - 1} M(\lambda^{a_\epsilon} \epsilon, \lambda^{a_H} H)$$

$$\text{Let } \lambda = |\epsilon|^{-\frac{1}{a_\epsilon}} \text{ to get } M(\epsilon, H) = |\epsilon|^{(1-a_H)/a_\epsilon} M\left(\frac{\epsilon}{|\epsilon|}, \frac{H}{|\epsilon|^{a_H/a_\epsilon}}\right)$$

$$\text{or } \underbrace{\frac{M(\epsilon, H)}{|\epsilon|^\beta}}_{\equiv m} = M\left(\underbrace{\frac{\epsilon}{|\epsilon|}}_{\equiv \pm 1}, \underbrace{\frac{H}{|\epsilon|^{a_H/a_\epsilon}}}_{\equiv h}\right) \Rightarrow m = M(\pm 1, h) \equiv F_{\pm}(h)$$

"scaled mag." "scaled H"

Thus, $m = F_{\pm}(h)$ or $h = F_{\pm}(m)$: all M, H data when scaled to m, h should fit on same curve. It works!!! Data fits. This gives you a way to measure a_ϵ, a_H for a magnetic substance. like CNO scaling.

Ch 12 "Scaling of the Correlation Functions"

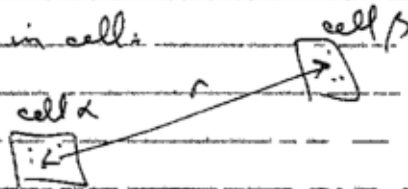
Kadanoff's Idea: Very clearly explained by Stanley: Consider say a $d=2$ Ising model for spins. We know the Hamiltonian. Now group spins into little blocks which are large but which are still small relative to the correlation length. Thus, you can do this if you are close to the critical temperature. Since a block is now smaller than a cluster of magnetization (which is the order of correlation length), these blocks themselves form an Ising model. Thus, if you consider these blocks as your basic Ising elements, you expect the Hamiltonian to have the same form as with the original, except the parameters are different, like J' in place of J , and maybe a differently normalized critical temperature. If Hams have the same form (since same physics just on a different scale) the Gibbs also has the same form. Finally, you might assume that $J' = L^a J$ and $m' = L^b m$ etc, ie, the parameters are proportional with some factor of L , the block size. Also free for temperature t : both systems should go critical at same point. Of course $G(\text{blocks})$ is larger than $G(\text{spins})$ by L^d which is the number of spins in a block. Put all these things together and what do you get: you have shown that G is most-general homogeneous, and thus you get all the relations of last chapter.

[Kadanoff has therefore motivated why the thermodynamic potentials should be homogeneous in the critical region!!!!] He is the father of the renormalization group idea, I would say.

Connection to spin-correlation fcts.

$$P(r, \epsilon) = \langle (s_i - \langle s \rangle)(s_j - \langle s \rangle) \rangle$$

$L^d = \# \text{ spins in cell}$



$$\frac{1}{(L^d)^2} \sum_{\substack{i,j \\ \in \alpha \\ j \in \beta}} P(r, \epsilon) = \left\langle \left(\frac{1}{L^d} \sum_{i \in \alpha} s_i \right) - \langle s_{\alpha} \rangle \right\rangle \left(\frac{1}{L^d} \sum_{j \in \beta} s_j - \langle s_{\beta} \rangle \right)$$

$\underbrace{\quad}_{\text{avg of effective } s_{\alpha}}$
 $\underbrace{\quad}_{\text{effective } s_{\beta}}$

$$= \langle (\tilde{s}_{\alpha} - \langle s \rangle)(\tilde{s}_{\beta} - \langle s \rangle) \rangle = P(\tilde{r}, \tilde{\epsilon})$$

But $\tilde{s}_{\alpha} \neq \pm 1$, who knows. So write $\tilde{s}_{\alpha} \rightarrow L \tilde{s}_{\alpha} \leftarrow \pm 1$.

But $\frac{1}{(L^d)^2} \sum_{\substack{i \in \alpha \\ j \in \beta}} P(r, \epsilon) \approx P(r, \epsilon)$ since cells small w.r.t. ξ .

Thus $P(r, \epsilon) \approx L^2 P(\tilde{r}, \tilde{\epsilon})$ where $\tilde{r} = r/L = \text{rescaled } \tilde{r}$.

ie, in order to make "cell picture" look like "site picture", we have to shrink the cell picture by L . (This point is not clear). (Seems to me that $\tilde{r} = r$, maybe.)

Next:

$$\beta \mathcal{H}_{\text{site}} = -h \sum_{\alpha=1}^m \left(\sum_{i \in \alpha} s_i \right) = L^d L \tilde{S}_2$$

$$\beta \mathcal{H}_{\text{ell}} = -\tilde{h} \sum_{\alpha=1}^m \tilde{S}_2$$

To make $\mathcal{H}_{\text{site}} = \mathcal{H}_{\text{ell}}$,

$$\ln L^d L = \tilde{h}$$

[but why $\mathcal{H}$'s equal, why not a factor?]

Now let $L = L(h) = L^z$.

$$\Rightarrow \ln L^d L = L^x h \Rightarrow d+z=x \Rightarrow z=x-d.$$

Then: $\left[P(r, \epsilon) = L^{2(x-d)} \tilde{P}(r/L, \tilde{\epsilon}) \right] \leftarrow \text{note: } x, y \text{ already known in terms of } d, \delta, \nu$

Thus our scaling argument has shown that the spin correlation function is also a most general two-variable homogeneous function. Obviously we shall now be able to bring in the two remaining critical exponents, η and ν :

Choose $L = (\epsilon)^{1/y} \Rightarrow P(r, \epsilon) = (\epsilon)^{2(x-d)/y} f(r \epsilon^{1/y}) = P(r, 1)$

Choose $L = r \Rightarrow P(r, \epsilon) = (r)^{2(x-d)} g(\epsilon r^y) = P(1, t)$

From this second we get:

$$\Rightarrow \left[-(d-2-\eta) = 2(x-d) \right] \quad P(r, 0) = (r)^{2(x-d)} g(0) \sim \frac{1}{r^{d-2-\eta}} \quad \text{recall } \eta.$$

But, look also at $P(r, \epsilon) = r^{2(x-d)} g(\epsilon r^y)$ when $\epsilon \neq 0$.

$$\text{So } g(\epsilon r^y) \div e^{-r \epsilon^{\nu}} = e^{-[\epsilon r^y]^{\nu}} \Rightarrow e^{-r/\delta} = e^{-r \epsilon^{\nu}} \Rightarrow e^{-r/\delta} = e^{-r \epsilon^{\nu}} \Rightarrow e^{-r/\delta} = e^{-r \epsilon^{\nu}}$$

$$\Rightarrow \left[y = \frac{1}{\nu} \right]$$

But earlier we showed and had just shown $(\alpha = \frac{y}{d})$

$$\alpha' = 2 - \frac{1}{\alpha \epsilon}$$

$$\Rightarrow y = \alpha \epsilon d = d \left(\frac{-1}{2-\alpha} \right)$$

So $[d\nu = 2-\alpha]$. Then solve for ν

$$x = d\alpha_H = d \cdot \frac{\delta}{\delta+1} \Rightarrow \eta = \eta(d, \delta). \text{ Instead, use } [(2-\eta)\nu = \delta]$$

Tricky stuff!

$$2-\eta = d \left(\frac{\delta+1}{\delta+1} \right) \text{ alternate.}$$

End of Ch 12: use of the Fourier Transform of $G = \Gamma$, called S .

① Instead of using $P(\epsilon, r)$, by $\tilde{P}(\xi, r)$, i.e., replace ϵ with ξ as your indep variable. Then you might say

suppose: $\tilde{P}(\xi, r) = \xi^\alpha f(r/\xi)$ ratio

i.e. $\tilde{P}(\lambda \xi, \lambda r) = \lambda^\alpha \tilde{P}(\xi, r)$.

This is equivalent to our earlier assumption on $P(\epsilon, r)$.

② Let $K = \xi^{-1}$. You equivalently can use the Fourier transform:

$$S(K, q) = q^\nu \tilde{P}(q/K).$$

But, for $\xi q \gg 1$, maybe $\tilde{P}(K/q) \approx 1 + \frac{\sigma(K/q)}{\sigma(1/2)} + \dots$

Then $S(K, q) \sim q^\nu = q^{-2+\alpha}$

$$\Rightarrow [\nu = -2 + \alpha]$$

But, we also know that

$$\chi_T \approx \int d\mathbf{r} G(\mathbf{r}) = S(q=0)$$

$$\xi^{-\gamma} = [\xi^{-1}]^{-\gamma} = K^{-\gamma/2}$$

But for $q=0$, $S(K, q) = K^\nu \tilde{P}'(1/K)$

since know

$$\sim K^\nu$$

so $[\nu = -\frac{\gamma}{2}]$

⑤ again We got this earlier by using $P(r, \epsilon)$ know.

Compare: $-\frac{\gamma}{2} = -2 + \alpha \Rightarrow [\alpha = (2 - \frac{\gamma}{2})]$

Models generally tend to consistently respect all exponent scaling relations!!



Note on d-dim Fourier Transform. (see p. 10.6 Stanley).

① The Ornstein-Zernicke theory says:

$$\frac{S(q)}{n} = \frac{R^2}{K_1^2 + q^2} \doteq \frac{1}{q^2} \text{ at } T_c \text{ since } K_1 \rightarrow 0$$

where $R^2 K_1^2 = K_T^0 / K_T$ so $K_1^2 \sim \epsilon^2$

Now, $S(q) = \int d\vec{r} e^{-i\vec{q} \cdot \vec{r}} G(\vec{r})$ is simply the F.T. of the correlation fun. So work it either way.

② But how do you do a $d\vec{r}$ FT?

$$[d\vec{q} = (q^{d-1} dq) [(\sin \theta)^{d-2} d\theta] \cdot d\Omega_{d-1}]$$

useful to know.

Thus:

$$G(r) = \int d\vec{q} e^{i\vec{q} \cdot \vec{r}} S(q) \quad \text{assume isotropic} \quad e^{iqr \cos \theta}$$

$$= \int q^{d-1} S(q) dq \times \int_0^\pi d\theta (\sin \theta)^{d-2} \times \int d\Omega_{d-1} \quad \text{const}$$

$$= \left(\frac{J_{\frac{d}{2}-1}(qr)}{(qr)^{\frac{d}{2}-1}} \right)$$

$$\text{So } G(r) = \text{const} \times \int_0^\infty q^{d-1} dq \cdot \frac{J_{\frac{d}{2}-1}(qr)}{(qr)^{\frac{d}{2}-1}} \times S(q) \quad \left(= \frac{\sin qr}{qr} \text{ } d=3 \right)$$

Fisher '62 did these. You get:

$$G(r) = \frac{e^{-kr}}{r^{(d-1)/2}} \text{ as } r \rightarrow \infty.$$

Theorem: if $S(q) = q^{-2+\eta}$ as $T \rightarrow T_c$, then $G(r) = r^{-(d-2+\eta)}$ at $T = T_c$. Phenomen.

Not obvious you have to look at that Bessel fun!

The diagram illustrates the experimental design as a sequence of four stages: Stimulus presentation, Response, Feedback, and Inter-trial interval. These stages are connected by arrows in a linear fashion. A 'Start' box points to the first 'Stimulus presentation' box, and an 'End' box points to the last 'Inter-trial interval' box. Each stage is represented by a box containing a question mark, suggesting a choice or decision point.

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[illegible]

doi:10.1371/journal.pone.0142101.g001

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	1987	1988	1989	1990	1991	1992	1993	1994	1995	1996	1997	1998	1999	2000	2001	2002	2003	2004	2005	2006	2007	2008	2009	2010	2011	2012	2013	2014	2015	2016	2017	2018	2019	2020	2021	2022	2023	2024	2025	2026	2027	2028	2029	2030	2031	2032	2033	2034	2035	2036	2037	2038	2039	2040	2041	2042	2043	2044	2045	2046	2047	2048	2049	2050	2051	2052	2053	2054	2055	2056	2057	2058	2059	2060	2061	2062	2063	2064	2065	2066	2067	2068	2069	2070	2071	2072	2073	2074	2075	2076	2077	2078	2079	2080	2081	2082	2083	2084	2085	2086	2087	2088	2089	2090	2091	2092	2093	2094	2095	2096	2097	2098	2099	2100	2101	2102	2103	2104	2105	2106	2107	2108	2109	2110	2111	2112	2113	2114	2115	2116	2117	2118	2119	2120	2121	2122	2123	2124	2125	2126	2127	2128	2129	2130	2131	2132	2133	2134	2135	2136	2137	2138	2139	2140	2141	2142	2143	2144	2145	2146	2147	2148	2149	2150	2151	2152	2153	2154	2155	2156	2157	2158	2159	2160	2161	2162	2163	2164	2165	2166	2167	2168	2169	2170	2171	2172	2173	2174	2175	2176	2177	2178	2179	2180	2181	2182	2183	2184	2185	2186	2187	2188	2189	2190	2191	2192	2193	2194	2195	2196	2197	2198	2199	2200	2201	2202	2203	2204	2205	2206	2207	2208	2209	2210	2211	2212	2213	2214	2215	2216	2217	2218	2219	2220	2221	2222	2223	2224	2225	2226	2227	2228	2229	2230	2231	2232	2233	2234	2235	2236	2237	2238	2239	2240	2241	2242	2243	2244	2245	2246	2247	2248	2249	2250	2251	2252	2253	2254	2255	2256	2257	2258	2259	2260	2261	2262	2263	2264	2265	2266	2267	2268	2269	2270	2271	2272	2273	2274	2275	2276	2277	2278	2279	2280	2281	2282	2283	2284	2285	2286	2287	2288	2289	2290	2291	2292	2293	2294	2295	2296	2297	2298	2299	2300	2301	2302	2303	2304	2305	2306	2307	2308	2309	2310	2311	2312	2313	2314	2315	2316	2317	2318	2319	2320	2321	2322	2323	2324	2325	2326	2327	2328	2329	2330	2331	2332	2333	2334	2335	2336	2337	2338	2339	2340	2341	2342	2343	2344	2345	2346	2347	2348	2349	2350	2351	2352	2353	2354	2355	2356	2357	2358	2359	2360	2361	2362	2363	2364	2365	2366	2367	2368	2369	2370	2371	2372	2373	2374	2375	2376	2377	2378	2379	2380	2381	2382	2383	2384	2385	2386	2387	2388	2389	2390	2391	2392	2393	2394	2395	2396	2397	2398	2399	2400	2401	2402	2403	2404	2405	2406	2407	2408	2409	2410	2411	2412	2413	2414	2415	2416	2417	2418	2419	2420	2421	2422	2423	2424	2425	2426	2427	2428	2429	2430	2431	2432	2433	2434	2435	2436	2437	2438	2439	2440
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no	name	country	year	area (km ²)	pop.
1	Algeria	Algeria	1990	238140	10 500 000
2	Algeria	Algeria	2000	238140	12 500 000
3	Algeria	Algeria	2005	238140	13 500 000
4	Algeria	Algeria	2010	238140	14 500 000
5	Algeria	Algeria	2015	238140	15 500 000
6	Algeria	Algeria	2020	238140	16 500 000
7	Algeria	Algeria	2025	238140	17 500 000
8	Algeria	Algeria	2030	238140	18 500 000
9	Algeria	Algeria	2035	238140	19 500 000
10	Algeria	Algeria	2040	238140	20 500 000
11	Algeria	Algeria	2045	238140	21 500 000
12	Algeria	Algeria	2050	238140	22 500 000
13	Algeria	Algeria	2055	238140	23 500 000
14	Algeria	Algeria	2060	238140	24 500 000
15	Algeria	Algeria	2065	238140	25 500 000
16	Algeria	Algeria	2070	238140	26 500 000
17	Algeria	Algeria	2075	238140	27 500 000
18	Algeria	Algeria	2080	238140	28 500 000
19	Algeria	Algeria	2085	238140	29 500 000
20	Algeria	Algeria	2090	238140	30 500 000
21	Algeria	Algeria	2095	238140	31 500 000
22	Algeria	Algeria	2100	238140	32 500 000
23	Algeria	Algeria	2105	238140	33 500 000
24	Algeria	Algeria	2110	238140	34 500 000
25	Algeria	Algeria	2115	238140	35 500 000
26	Algeria	Algeria	2120	238140	36 500 000
27	Algeria	Algeria	2125	238140	37 500 000
28	Algeria	Algeria	2130	238140	38 500 000
29	Algeria	Algeria	2135	238140	39 500 000
30	Algeria	Algeria	2140	238140	40 500 000
31	Algeria	Algeria	2145	238140	41 500 000
32	Algeria	Algeria	2150	238140	42 500 000
33	Algeria	Algeria	2155	238140	43 500 000
34	Algeria	Algeria	2160	238140	44 500 000
35	Algeria	Algeria	2165	238140	45 500 000
36	Algeria	Algeria	2170	238140	46 500 000
37	Algeria	Algeria	2175	238140	47 500 000
38	Algeria	Algeria	2180	238140	48 500 000
39	Algeria	Algeria	2185	238140	49 500 000
40	Algeria	Algeria	2190	238140	50 500 000
41	Algeria	Algeria	2195	238140	51 500 000
42	Algeria	Algeria	2200	238140	52 500 000
43	Algeria	Algeria	2205	238140	53 500 000
44	Algeria	Algeria	2210	238140	54 500 000
45	Algeria	Algeria	2215	238140	55 500 000
46	Algeria	Algeria	2220	238140	56 500 000
47	Algeria	Algeria	2225	238140	57 500 000
48	Algeria	Algeria	2230	238140	58 500 000
49	Algeria	Algeria	2235	238140	59 500 000
50	Algeria	Algeria	2240	238140	60 500 000
51	Algeria	Algeria	2245	238140	61 500 000
52	Algeria	Algeria	2250	238140	62 500 000
53	Algeria	Algeria	2255	238140	63 500 000
54	Algeria	Algeria	2260	238140	64 500 000
55	Algeria	Algeria	2265	238140	65 500 000
56	Algeria	Algeria	2270	238140	66 500 000
57	Algeria	Algeria	2275	238140	67

100 100 10000 100 100

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1000

Figure 1

1000 1000 1000

Abstract

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code	code	code	code
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2000	2001	2002	2003	2004	2005	2006	2007	2008	2009	2010	2011	2012	2013	2014	2015	2016	2017	2018	2019	2020	2021	2022	2023	2024	2025	2026	2027	2028	2029	2030	2031	2032	2033	2034	2035	2036	2037	2038	2039	2040	2041	2042	2043	2044	2045	2046	2047	2048	2049	2050	2051	2052	2053	2054	2055	2056	2057	2058	2059	2060	2061	2062	2063	2064	2065	2066	2067	2068	2069	2070	2071	2072	2073	2074	2075	2076	2077	2078	2079	2080	2081	2082	2083	2084	2085	2086	2087	2088	2089	2090	2091	2092	2093	2094	2095	2096	2097	2098	2099	2100	2101	2102	2103	2104	2105	2106	2107	2108	2109	2110	2111	2112	2113	2114	2115	2116	2117	2118	2119	2120	2121	2122	2123	2124	2125	2126	2127	2128	2129	2130	2131	2132	2133	2134	2135	2136	2137	2138	2139	2140	2141	2142	2143	2144	2145	2146	2147	2148	2149	2150	2151	2152	2153	2154	2155	2156	2157	2158	2159	2160	2161	2162	2163	2164	2165	2166	2167	2168	2169	2170	2171	2172	2173	2174	2175	2176	2177	2178	2179	2180	2181	2182	2183	2184	2185	2186	2187	2188	2189	2190	2191	2192	2193	2194	2195	2196	2197	2198	2199	2200	2201	2202	2203	2204	2205	2206	2207	2208	2209	2210	2211	2212	2213	2214	2215	2216	2217	2218	2219	2220	2221	2222	2223	2224	2225	2226	2227	2228	2229	2230	2231	2232	2233	2234	2235	2236	2237	2238	2239	2240	2241	2242	2243	2244	2245	2246	2247	2248	2249	2250	2251	2252	2253	2254	2255	2256	2257	2258	2259	2260	2261	2262	2263	2264	2265	2266	2267	2268	2269	2270	2271	2272	2273	2274	2275	2276	2277	2278	2279	2280	2281	2282	2283	2284	2285	2286	2287	2288	2289	2290	2291	2292	2293	2294	2295	2296	2297	2298	2299	2300	2301	2302	2303	2304	2305	2306	2307	2308	2309	2310	2311	2312	2313	2314	2315	2316	2317	2318	2319	2320	2321	2322	2323	2324	2325	2326	2327	2328	2329	2330	2331	2332	2333	2334	2335	2336	2337	2338	2339	2340	2341	2342	2343	2344	2345	2346	2347	2348	2349	2350	2351	2352	2353	2354	2355	2356	2357	2358	2359	2360	2361	2362	2363	2364	2365	2366	2367	2368	2369	2370	2371	2372	2373	2374	2375	2376	2377	2378	2379	2380	2381	2382	2383	2384	2385	2386	2387	2388	2389	2390	2391	2392	2393	2394	2395	2396	2397	2398	2399	2400	2401	2402	2403	2404	2405	2406	2407	2408</
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Year	1990	1995	2000	2005	2010
Population (millions)	1.2	1.5	1.8	2.1	2.4
GDP (billions of dollars)	0.5	1.0	1.5	2.0	2.5
Life expectancy (years)	55	60	65	70	75

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Barber

Outline of Barber's Review

Lecture:

Subject:

- 1 review of critical phenomena: order parameter, correlation length critical exponents, universality, scaling(homogeneous functions) and exponent relations; spin correlation function $\hat{\Gamma}$ and FT $\hat{\Gamma}$.
- 2 Basics of renormalization group: effective Hamiltonian and the operation R_b scale parameter is b . Look for fixed point H^* . Rule for how $\hat{\Gamma}$ transforms, the exponent γ related to η , determined by R_b .
- 3 Renorm group near a fixed point: linearization of R_b to L_b , use of eigenoperators Q_i and eigenvalues b^{γ_i} . Express H near H^* using these. Relevant vs. irrelevant critical operators Q . The basic one Q_E . ($\gamma_E = 1/\nu$)
Notion of critical surface where "fields" $\phi_i = 0$ of relevant operators. H on critical surface goes to H^* under repeated L_b application. Critical dimension of operator Q_i defined. Scaling laws and derivation of basic exponent relations.
- 4 Wilson's rough recursion formula: an attempt to find H^* for spin system. Gives $\eta = 0$. You get a recursion formula for Q_L which determines H_L which approaches H^* . In $d=3$ gives $\nu = .609$ for 3-D Ising, good.
- 5 Analysis for $d = 4 - \epsilon$. For $d=4$ get classical gaussian model and $Q^*=0$. Assume $Q = rs^2 + us^4$ and expand everything in powers of ϵ . Exponents improve from their classical values.
- 6, 7 Feynman expansions: Since s is like ϕ , a field, and $\hat{\Gamma}(x, x')$ is like a VEV of some fields, can use Feynman-Dyson expansion in graphs to get $\hat{\Gamma}$ to given power of ϵ [coupling constant is u_0 of $u_0 s^4$, is order ϵ] Hence improved calculation of exponents in powers of ϵ .
- 8 Alternative expansion for exponents as $(1/n)$. This is perturbation around the $n = \infty$ spherical model. Similar results to the ϵ expansions.
- 9 Attempt to improve calculation of the fixed point H^* and R_b . (ie, derive exact recursion relation)
- 10 Special calculations for spin systems with $d=2$, same idea, various methods. Exponents don't come out very well in general here.



Michael Barber

82 pages.

"Intro to the Renorm. Group"

Physica Reports 29,1 (Jan 1977)

preface: other earlier reviews are:

Fisher RMP 46 (74)

Wilson, Kogut PR 12,1 (74)

Ma RMP 45 (73)

1. Introduction and Basic Review

1.1 Introduction, History: vander Waals Thesis ~ (1860)

Onsager and Ising model [1] (1944)

unable to compute exponents $\rightarrow$ Wilson's renorm group (1971)

$n = D$ of Stanley
= dim of the spin (etc)

② Order Symmetry. (Spontaneous): order parameter is $\Psi(T)$. In magnet it is the residual magnetization. In other cases $\Psi =$ a wavefunction deal. Let $n =$ the dimension of Ψ (eg, $n=1$ in Ising with any # of spatial dimensions). An "XY-system" has $\vec{M}$ confined to a plane, $n=2$; whereas "Heisenberg magnet" has the full freedom, $n=3$. Notice that a symmetry is broken by direction of spont. mag. You can even have $n=-2$!

"Long-range order" is a related idea. Eg, $\Gamma(\vec{r}, \vec{r}', t) \equiv \langle S_i^z S_{i'}^z \rangle_{\text{ensemble}}$ describes correlation between z -spins at $\vec{r}$ and $\vec{r}'$. In spontaneous cases you find result: $\lim_{|\vec{r}-\vec{r}'| \rightarrow \infty} \Gamma(\vec{r}, \vec{r}', t) \neq 0$! rather $\Gamma(\infty) \sim M_0^2$. Above T_c both senses of order go away (spont., long-range cor.).

1.3 Critical Exponents; Universality. It is an experimental fact that, near critical Temp [let $t \equiv (T-T_c)/T_c$], the spontaneous order [like magnetization] vanishes in some way. Not surprising that it is usually a power:

$$M_0 \approx A |t|^\beta \quad \text{where } \beta \approx .38 \text{ exp, } = \frac{1}{2} \text{ in "classical theory"}$$

Similarly, due to large fluctuations in M_0 near $t=0$, χ_0 [suscept] diverges:

$$\chi_0 \sim t^{-\gamma} \quad \text{where } \gamma \approx 1.2 \quad \left. \begin{array}{l} \gamma = 1 \text{ classical theory} \\ \text{for specific heat divergence} \end{array} \right\}$$

$$C \sim A t^{-\alpha} \quad \text{where } \alpha \approx .1 \quad \left. \begin{array}{l} \alpha = 0 \text{ for Ising } d=2, n=1 \\ \text{other classical theory} \end{array} \right\} \quad (\log)$$

These powers are dimensionless, "critical exponents". We shall see that they are theoretically connected to a certain "correlation fctn" $\xi(T)$ which is correlation length, and

$$\xi \sim t^{-\nu} \quad \nu \approx \frac{2}{3} \quad \text{blows up too.}$$

Read this after reading your much cleaner...

$$= \frac{-1/\xi}{r^{d-2+\eta}} \quad \eta=0 \text{ in Ornstein-Zernike}$$

Now, $P(\vec{r}, T)$ is a correlation fun telling just how the correlation drops off with distance. $\hat{P}(\vec{k}, T)$ is the F.T.

These exponents seem to be nearly the same for widely varying physical critical problems: "universality". [see notes added]

1.4 Scaling + Exponent Relations Recall: Here it is speculated that various functions [like $\hat{P}(\vec{k}, H, T)$] exhibit "scaling" behavior,

$$M \sim |t|^\beta$$

$$\chi_0 \sim t^{-\gamma}$$

$$C \sim t^{-\alpha}$$

$$\xi \sim t^{-\nu}$$

$$P(r, H, T) \approx (r)^{-(d-2+\eta)} f_{\pm}(h/|t|^{\phi}, |t|/\xi)$$

3 args

two args

$$\hat{P} \sim k^{-(2-\eta)} \text{ at } T=T_c$$

I am not clear what this does for you.

To remember: one hopes that all exponents are in fact functions of $d = \text{dim of system}$, $n = \text{dim of symmetry}$.

$$K_T(\alpha \chi_T) \doteq (N - \langle n \rangle)^2 \doteq \int d^d \vec{r} G(\vec{r}), \text{ hence (1.16) Barker}$$

2. Definition of the Renormalization Group.

$$\hat{P}|^{FT} = \frac{c}{k^2 + \xi^{-2}} \text{ based on } \frac{-1/\xi}{k^2 + \xi^{-2}} \text{ like pole at } k=0$$

2.1 Basic Idea

You want to transform a problem involving large correlation lengths ξ [hard to deal with large volume system] to a simpler problem involving small ξ . Recall that $\xi \rightarrow \infty$ at C.P. so makes it hard to work at. Kadanoff did this for "spin blocks". Will follow Ma closely.

2.2 Model + Notation

At each lattice site in space of d dim, put a real spin vector S^{α} where $\alpha = 1, 2, \dots, n$. Thus $S^{\alpha}(\vec{r})$. In k -space this is α_k . We will claim that:

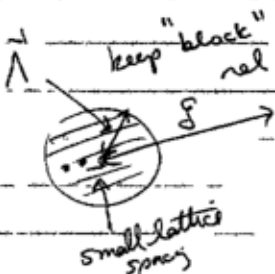
[Why say this is k -space rather than $\vec{r}$ -space?]

$$P\{\alpha_k\} = \frac{1}{Z} \int d\alpha_k e^{-\beta \mathcal{H}(\alpha_k)}$$

The more energy you give to degree of freedom α_k , the less likely that value α is. There are nL^d variables: $\{\alpha_k^{\alpha}\}$.

2.3 Effective Hamiltonian

We want to average away/wash out short distance information since not relevant. I.e., wash out large k . Just pre-sum over large k 's



$$P_{eff} = \prod_{\alpha=1}^n \prod_{|k| > 1/\xi} \left(\int d\alpha_k e^{-\beta \mathcal{H}(\alpha_k)} \right) \equiv \frac{1}{Z} e^{-\beta \mathcal{H}_{eff}(\Lambda)}$$

so we wash out all degrees of freedom which relate to irrelevant short distances. Odd to do this because $\mathcal{H}_{eff}(\Lambda)$ ought to be a mess.

2.4 The Renorm Group. Let $-\beta\mathcal{H}_b = \mathcal{H}$, so $P \sim e^{+\mathcal{H}}$. Now we have, in k -space:



cross-hatch shows the region of k -space that is "integrated out".

- 1) increase cutoff domain to that shown in red.
- 2) ~~rescale~~ ^{label} the σ_k^a by factor b . $[\sigma_k^a]^{new} = \sigma_k^a b$
- 3) let $\sigma_k^a \rightarrow \sigma_k^a \xi_b$... $\uparrow$ (ie, (ie, distance rescale))

Now do three things:
This set of operations defines a new P :

$$P' = \left[\prod_{\substack{a=1 \\ \Delta k < 1}}^n \prod_{\substack{k \\ \Delta k < 1}} \left(d\sigma_k^a [P(\text{orig})] = e^{+\mathcal{H}} \right) \right] \text{ then } \sigma_k^a \rightarrow \xi_b \sigma_{bk}^a = e^{+\mathcal{H}'}.$$

Certainly I can do this. This defines an operation $\mathcal{H}' = R_b \mathcal{H}$. The first step obviously reduces the # of degrees of freedom you deal with. To, we in effect decrease Λ , increase Λ . Thus, larger sphere in $\vec{r}$ -space is now averaged out. But then this sphere is shrunk again by rescaling: $k \rightarrow bk$, $r \rightarrow r/b$. Not clear why this idiotic procedure is at all helpful.

R_b is just a scale transformation but with cutoff held fixed at 1! In order for the R_b to form a semigroup, you will need

$$\xi_b = b^y \quad \text{where } y = \text{arbitrary for the moment. (Satisfies a multiplier rep.)}$$

2.5 The fixed point. Do some iterations of the above operation: ie, keep including more lattice points in real space, $\mathcal{H}_2 = (R_b)^2 \mathcal{H}_0$. Eventually it seems possible to me that further iterations make no difference. Ie, as $b \rightarrow \infty$, $\mathcal{H}_b \rightarrow \mathcal{H}^*$ and $R_b \mathcal{H}^* = \mathcal{H}^*$. This $\mathcal{H}^*$ is called a fixed-point of R_b for obvious reasons. Regard this as an eigenvalue equation and number y as eigenvalue: $\text{w/ say } y \Rightarrow \text{no fixed point, maybe Universality:}$ The fixed Ham. $\mathcal{H}^*$ is determined more by the operation R_b than by the physics entering $\mathcal{H}_{micro}$. Different initial $\mathcal{H}^0$ will go to the same $\mathcal{H}^*$. Could be several fixed pts.

2.6 How the correlation fctns transform. Recall $\langle S^z(0) S^z(\vec{r}) \rangle_P$. This is a mean value. For large $\vec{r}$ and no external $\mathcal{H}$, you might expect this to be $\langle S^z(0) \rangle \langle S^z(\vec{r}) \rangle$ and each factor is zero. Actually, maybe $\langle ab \rangle_{CP} \equiv \langle ab \rangle - \langle a \rangle \langle b \rangle$, Barker does not say. Regardless, let

$$\prod_{i=1}^{d_1+d_2+\dots+d_n} (\vec{k}_1, \vec{k}_2, \dots, \vec{k}_n) = \text{m.f.t.} \left\{ \langle S^{\alpha_1}(0) S^{\alpha_2}(\vec{r}_2) \dots S^{\alpha_n}(\vec{r}_n) \rangle_P \right\}$$

= momentum space correlation function. Note $\vec{k}_1 = -\vec{E}_{rest}$.

This is a tricky point, so simplest case:

$$\hat{\Gamma}^{\alpha\alpha}(\vec{k}; \mathcal{H}) \equiv \int d\vec{r} \langle S^{\alpha}(0) S^{\alpha}(\vec{r}) \rangle_{\mathcal{H}} e^{-i\vec{k} \cdot \vec{r}}$$

Digression: normally you would say

$$\hat{\Gamma}^{\alpha_1 \alpha_2}(\vec{k}_1, \vec{k}_2) \equiv \int d\vec{r}_1 \int d\vec{r}_2 e^{-i\vec{k}_1 \cdot \vec{r}_1} e^{-i\vec{k}_2 \cdot \vec{r}_2} \langle S^{\alpha_1}(\vec{r}_1) S^{\alpha_2}(\vec{r}_2) \rangle$$

But, you know that $\langle S^{\alpha_1}(\vec{r}_1) S^{\alpha_2}(\vec{r}_2) \rangle = \langle S^{\alpha_1}(\vec{r}_1 + \vec{r}) S^{\alpha_2}(\vec{r}_2 + \vec{r}) \rangle$ by translation invariance. Therefore

$$\hat{\Gamma}^{\alpha_1 \alpha_2}(\vec{k}_1, \vec{k}_2) = \int d\vec{r}_1 \int d\vec{r}_2 e^{-i\vec{k}_1 \cdot \vec{r}_1} e^{-i\vec{k}_2 \cdot \vec{r}_2} \langle S^{\alpha_1}(\vec{r}_1 + \vec{r}) S^{\alpha_2}(\vec{r}_2 + \vec{r}) \rangle$$

But you can then choose $\vec{r} = -\vec{r}_1$ to get

$$= \int d\vec{r}_1 \int d\vec{r}_2 e^{-i\vec{k}_1 \cdot \vec{r}_1} e^{-i\vec{k}_2 \cdot \vec{r}_2} \langle S^{\alpha_1}(0) S^{\alpha_2}(\vec{r}_2 - \vec{r}_1) \rangle$$

But now let $\vec{r}_2 - \vec{r}_1 = \vec{r}'$. Then

$$= \int d\vec{r}_1 \int d\vec{r}' e^{-i\vec{k}_1 \cdot [\vec{r}_1]} e^{-i\vec{k}_2 \cdot [\vec{r}' + \vec{r}_1]} \langle S(0) S(\vec{r}') \rangle$$

$$\rightarrow \underbrace{\delta(\vec{k}_1 - \vec{k}_2)}_{\delta_{\vec{k}_1, \vec{k}_2}} \int d\vec{r}' e^{-i\vec{k}_2 \cdot \vec{r}'} \langle S(0) S(\vec{r}') \rangle$$

$$= \hat{\Gamma}^{\alpha\alpha}(\vec{k}_2)$$

In other words, just as in particle physics greens functions, because these little correlation functions are translationally invariant, you are forced to conclude that the $\vec{k}$'s add up to zero. Then easier to extract a delta function and define the residual thing as your fourier transform of interest. Now in the special case you get:

$$\hat{\Gamma}^{\alpha\alpha}(\vec{k}; \mathcal{H}) = \int d\vec{r} \langle S^{\alpha}(0) S^{\alpha}(\vec{r}) \rangle e^{-i\vec{k} \cdot \vec{r}} = \sum_{\vec{k}'} \langle \sigma_{\vec{k}}^{\alpha} \sigma_{\vec{k}'}^{\alpha} \rangle_{\mathcal{H}}$$

$$= \langle \sigma_{-\vec{k}}^{\alpha} \sigma_{\vec{k}}^{\alpha} \rangle = \langle |\sigma_{\vec{k}}^{\alpha}|^2 \rangle \quad \text{since } f_{-k} = f_k^* \text{ for real } f.$$

Now (2 hrs later) back to the question: what does R do to Γ ?

$$\langle |\sigma_{\vec{k}}^{\alpha}|^2 \rangle_{\mathcal{H}} \xrightarrow{(1)} \langle |\sigma_{\vec{k}}^{\alpha}|^2 \rangle_{\mathcal{H}'} \xrightarrow{(2)} \langle |\sigma_{b\vec{k}}^{\alpha}|^2 \rangle_{\mathcal{H}'} \xrightarrow{(3)} b_{\vec{k}}^2 \langle |\sigma_{b\vec{k}}^{\alpha}|^2 \rangle_{\mathcal{H}'}$$

$\uparrow$ with new cutoff. $\uparrow$ rescale here $\uparrow$ add factor

Eg, $\hat{\Gamma}^{\alpha\alpha}(k, \mathcal{K}) \rightarrow b^{2\gamma} \hat{\Gamma}^{\alpha\alpha}(bk, R_b \mathcal{K})$

↖ here is rule that takes $P \rightarrow P'$

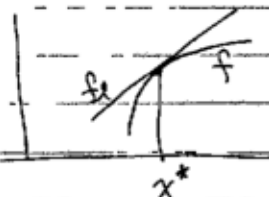
I think (2.21) is also OK.

Note that $L \rightarrow L/b$ as part of the scaling. or L has

3. Renorm Group near Fixed Point

3.1 Linearization Assume that $\mathcal{K} = \mathcal{K}^* + \delta\mathcal{K}$ where $R_b \mathcal{K}^* = \mathcal{K}^*$. Let $\mathcal{K}' = R_b \mathcal{K}$ so $\mathcal{K}^* + \delta\mathcal{K}' = R_b(\mathcal{K}^* + \delta\mathcal{K})$. But R_b is not linear, so beware. Maybe RHS = $R_b \mathcal{K}^* + \delta\mathcal{K} \cdot (DR_b)$ so $\delta\mathcal{K}' = (DR_b) \delta\mathcal{K}$.

Analogy needed: suppose $f(x^*) = x^*$ where $f =$ some non-linear function. Near $x = x^*$, we can always approximate f by a linear function $f_L(x)$. Of course $f_L(x^*) = x^*$ also. What is f_L !



$$f_L(x) = f(x^*) + (x - x^*) \left. \frac{df}{dx} \right|_{x=x^*} = x^* + \delta x (Df)$$

So, similarly, for $\mathcal{K}$ near $\mathcal{K}^*$, we replace R_b with L_b , a linear operator.

Now assume that L_b , as an operator, acts on a space of operators. Barber is confusing on this point. Let's think of

$L_b, R_b =$ operators

$\delta\mathcal{K}, \mathcal{K}^*, Q_i =$ vectors

Assume then that $L_b Q_i = \lambda_i Q_i$ so $Q_i =$ eigenvectors of L_b . Since L_b form a group representation of $\mathbb{R}$, must have $\lambda_i = b^{\gamma_i}$. Claim: The set of vectors Q_i is complete enough to expand the vector $\delta\mathcal{K}$. So

$$\delta\mathcal{K} = \sum_i \mu_i Q_i$$

↑ "field" ↑ "density"

Then $\delta\mathcal{K}' = L_b \delta\mathcal{K} = \sum_i \mu_i b^{\gamma_i} Q_i = \sum_i (\mu_i') Q_i$

So $\mu_i' = \mu_i b^{\gamma_i}$

Basically, $\mu_i =$ coeff's of $\delta\mathcal{K}$. Some kind of tangent space stuff.
 $\mu_i' =$ coeff's of $\delta\mathcal{K}'$

3.2 Relevant vs Irrelevant Q_i 's

$$\text{So } \mathcal{H} = \mathcal{H}^* + \delta\mathcal{H} = \mathcal{H}^* + \sum_i \mu_i Q_i$$

$$(\mathcal{L}_b)^l \mathcal{H} = ? \quad \mathcal{L}_b \mathcal{H} = \mathcal{H}^* + \sum_i \mu_i b^{y_i} Q_i$$

$$(\mathcal{L}_b)^2 \mathcal{H} = \mathcal{H}^* + \sum_i \mu_i (b^{y_i})^2 Q_i \Rightarrow [(\mathcal{L}_b)^l \mathcal{H} = \mathcal{H}^* + \sum_i \mu_i b^{ly_i} Q_i]$$

For those Q_i for which $y_i > 0$, $(\mathcal{L}_b)^l \mathcal{H} \not\rightarrow \mathcal{H}^*$ as $l \rightarrow \infty$. These operators we must worry about, so they are "relevant". $Q_i \ni y_i < 0$ clearly have correct convergence. $y_i = 0 = \text{marginal}$.

At "criticality", relevant operators must therefore vanish. A "critical" $\mathcal{H}$ is written:

$$\mathcal{H}_c = \mathcal{H}^* + \sum_i \mu_i^{(c)} Q_i \quad \text{where } \mu_i^{(c)} = 0 \text{ for relevant.}$$

There are many such $\mathcal{H}_c$ [one for each choice of $\{\mu_i^{(c)}\}$]. The locus of such $\{\mathcal{H}_c\}$ is called "critical surface."

Among the relevant operators one is called Q_E . Its "field" or coefficient μ_E is $\propto (T - T_c)/T_c$, which is seen to vanish at critical T_c as required. Q_E related somehow to energy. ✓

["critical dimension"] Another operator is Q_0 : $R_b Q_0 = b^d Q_0$, so $y_0 = d$.
 On this special operator. Consider the combination
 $(R^{w_i} Q_i) \leftarrow$ arrange for this to transform like Q_0 .
 The w_i that does this is the "crit. dim."

$Q_0 = L^d = \text{volume in coord space, hence}$

where $L = \text{length}$. Under a RFT we get

$$R^{w_i} Q_i \rightarrow R^{w_i} Q'_i = (R \mathcal{L}_b)^{w_i} \mathcal{L}_b Q_i = R^{w_i} b^{w_i} b^{y_i} Q_i = b^{(y_i + w_i)} (R^{w_i} Q_i)$$

If this $[R^{w_i} Q_i]$ scales as does Q_0 [i.e., with b^d], then $y_i + w_i = d$.
 So w_i tells you the "dimension" (a misnomer) of Q_i .

$$y_i = d - w_i$$

If this "crit. dim" $w_i < d$, Q_i is relevant, etc. Just another way to classify.

$$[y_i > 0 \Leftrightarrow \text{relevant} \Leftrightarrow w_i < d.] \quad \checkmark \text{ paused here.}$$

(Barber, cm'd)

3.3 Scaling-laws in 1st Order Theory. Consider $\hat{\Gamma}^{\text{ex}}(k; \mathcal{H})$. In our linear theory we may completely characterize $\mathcal{H}$ by the set of coefficients a_i fields μ_i , since $\mathcal{H} = \mathcal{H}^* + \sum \mu_i Q_i$ where $\mathcal{H}^*$ and the Q_i are presumably known. Thus, write $\Gamma(k; \mu_E, \{\mu_i\})$, where we assume only one "relevant" operator Q_E . In section 2.6, (2.19) we showed that

$$\hat{\Gamma}^{\text{ex}}(k; \mathcal{H}) = b^{\frac{1}{2}y} \hat{\Gamma}^{\text{ex}}(bk; R_b \mathcal{H})$$

Choose $b = \frac{1}{k}$ to get $\hat{\Gamma}^{\text{ex}}(k; \mathcal{H}) = (k)^{-\frac{1}{2}y} \hat{\Gamma}^{\text{ex}}(1; R_b \mathcal{H})$. But we know that as $k \rightarrow 0$, $\Gamma(k; \mathcal{H})$ is supposed to go as $k^{-(2-\eta)}$ at T_c . [one def. of η]. Compare to see that $\frac{1}{2}y = 2-\eta$. Thus $y = 1-\eta/2$.

Now we also saw that $\mu'_i = \mu_i b^{y_i}$ where μ'_i describes $\mathcal{H}' = R_b \mathcal{H}$. Thus we rewrite the above as

$$\hat{\Gamma}(k; \mu_E, \{\mu_i\}) = b^{-2+\eta} \hat{\Gamma}(bk; \mu_E b^{y_E}; \{\mu_i b^{y_i}\})$$

This is a "generalized homogeneity"! Choose $b = 1/2k$: irrelevant

$$\hat{\Gamma}(k; \mu_E, \{\mu_i\}) = k^{-2+\eta} (1/2)^{-2+\eta} \hat{\Gamma}(1/2; \mu_E (1/2k)^{y_E}; \{\mu_i (1/2k)^{y_i}\})$$

$\downarrow$ $b \rightarrow 0$

Examine as $k \rightarrow 0$. $(1/2k)^{y_i} = (\infty)^{y_i} = 0$ since $y_i < 0$. So second $\hat{\Gamma}$ is

$$\hat{\Gamma}_{\text{NS}} = f[\mu_E k^{-y_E}] = \tilde{f}[\mu_E^{-\frac{1}{y_E}} k] = f_{\pm}[|\mu_E|^{\frac{1}{y_E}} k].$$

Thus we have proven that: (our first scaling law).

$$\hat{\Gamma}(k; \mu_E, \{\mu_i\}) = k^{-2+\eta} \hat{D}_{\pm}(k |\mu_E|^{\frac{1}{y_E}}) \quad \text{as } k \rightarrow 0.$$

Postulate: except for scale, $\hat{\Gamma}(k, T)$ depends on ξ only via $k\xi$ (a. S/h in FT). Thus assume

$$\hat{\Gamma}(k, T) = k^{-2+\eta} \hat{D}_{\pm}(k\xi) \quad \text{same } \hat{D}$$

But let's compare these: fctns must be same, and $\xi = |\mu_E|^{-\frac{1}{y_E}} = (t)^{-\nu}$ [recall that $\mu_E = t$, known]. Thus, $y_E = \nu^{-1}$

In Sec I we assumed this; now we have proved it!

Next: suppose we have an ext H field ($h = \tilde{\mu} H / kT$). Over antiferro to para then:

$$\chi_1 \equiv h \left(\int S^z(\vec{r}) d^d \vec{r} \right) / i.e. \tilde{H} \cdot \tilde{\mu} / kT.$$

So $\chi_1 = h \cdot L^d \cdot \sigma_0^\alpha$ acc. to F.T. But $k=0$ is unaffected by R_b (commens only affected). So all we do is take $\sigma_0^\alpha \rightarrow b^{(1-\gamma/2)} \sigma_0^\alpha$

$$\begin{aligned} \chi_1 &\Rightarrow h L^{\frac{d}{2}-\alpha} \rightarrow h \cdot (L/b)^{\frac{d}{2}} \left[b^{+\frac{\gamma}{2}} \sigma_0^\alpha \right] \\ &= h' L^{\frac{d}{2}-\alpha} \quad \text{where} \quad h' = h \cdot \underbrace{b^{+\frac{1}{2}(\gamma+d)}}_{\text{so } \gamma_h = 1 - \frac{1}{2}(\gamma+d)} \end{aligned}$$

ie, γ_h is the scaling exponent for magnetic field h

$$\chi'_1 = R_b \chi_1 \quad h' = b^{\gamma_h} h.$$

Now recall (Stoner) that (the integral of G)

$$\chi^\alpha(h, T) = \hat{\Gamma}(k=0, T, h; \mu \in, \{ \mu_i \})$$

something to remember

$$\text{So: } \chi^\alpha(h, \mu \in \{ \mu_i \}) = \hat{\Gamma}(k=0, T, h; \mu \in, \{ \mu_i \})$$

$$\begin{aligned} &= b^{2-\gamma} \hat{\Gamma}(b k=0; \mu \in b^{\gamma \mu \in}, \{ \mu_i b^{\gamma \mu_i} \}, h = b^{\gamma_h} h) \\ &= b^{2-\gamma} \chi^\alpha(h b^{\gamma_h}; \mu \in b^{\gamma \mu \in}, \{ \mu_i b^{\gamma \mu_i} \}) \end{aligned}$$

Now, χ^α is geo. homogeneous. Set $b = |\mu \in|^{-\frac{1}{\gamma \mu \in}} = |\mu \in|^{-\frac{2}{\gamma}}$.

$$\chi^\alpha(h, \mu \in \{ \mu_i \}) = |\mu \in|^{-(2-\gamma)^2} \chi^\alpha(h / |\mu \in|^{\gamma \mu \in}; \pm 1; \{ \mu_i / |\mu \in|^{\gamma \mu_i} \})$$

Now define $\Delta = y_h / y_\infty = \nu \gamma_h$. Let $\mu \in = t$ and let $t \rightarrow 0$.

$$\text{Then } \chi^\alpha(h; \mu \in \{ \mu_i \}) = (t)^{-(2-\gamma)^2} \chi^\alpha(h/(t)^\alpha; \mu \in = \pm 1; \{ \mu_i \}) \quad \text{scaling}$$

Now, $(2-\gamma)^2 = \chi$. ie, we have derived one of those exponent relations using homogeneity induced by R.G. idea. knowing how geo $\chi = -2\gamma^2 \gamma_h$ (and now geo h) we conclude also that

$$\tilde{F}_3(h, \mu \in \{ \mu_i \}) \approx (t)^{2-\alpha} \tilde{Q}(\pm(h/t^\alpha)) \quad \text{as advertised}$$

Also $2-\alpha = d\nu$, so "[two-exponent hyperscaling]" obtains.

3.4. connection to scaling One means instead of setting $\mu_i/\mu_i^{(0)} = 0$, if you are near $T = T_c$ so $k = \mu_c = s_{small}$, you should do a power series to get correction. I.e., away from criticality, the irrelevant fields will have some small effects.

3.5. higher order Discussion of how you go beyond the lowered d approximation to d . See Wegner & am. happy to stick with d in m .

4. Wilson's Approximate Recursion Formula.

One will be an attractor or actually converging expansion.

4.1. Effective Isotropic Short-Range Hamiltonian The original $\mathcal{H}$ or contains only the $\vec{S}(\vec{r}-\vec{r}')$ function and a "weight" function (eg, which gives $|S| = +1, -1$, say) When you process the $\mathcal{H}$, you get

$$\mathcal{H}_0 = -\frac{1}{2} K_2 \sum_{\vec{r}} \sum_{\vec{r}'} |\vec{S}(\vec{r})|^2 - \sum_{\vec{r}} \vec{r} \cdot \vec{P}_0[\vec{S}(\vec{r})] \quad P_0 = W(\vec{S}) - K_1(\vec{S})|S|$$

where K_1, K_2 are derivatives of T or something. One form is called "standard London-Gringberg". I.e., this $\mathcal{H}_0$ approximate $\mathcal{H}$ exact for long wavelength.

4.2. Recursion Relation. Decoupage $S(\vec{r}) = S_0(\vec{r}) + S'(\vec{r})$ (We set $m=1$ so no α index). (also $\Lambda=1$). $\leftarrow$ small k $\leftarrow$ large, small.

Now, $D = \sum_{\vec{r}} \vec{S}^x$ where we sum over all possible spin states. But we are integrate to make error. At each lattice point $\vec{r}$ we integrate over all values $S(\vec{r})$ can have. One

$$D = \sum_{\vec{r}} \int dS(\vec{r}) e^{\mathcal{H}[\vec{S}]} / \text{like a path integral} \quad \leftarrow \text{except sum over } m.$$

Trick: replace $\int dS(\vec{r})$ by $\sum_m S_m$ where $S_m = \int d\vec{r} \psi(\vec{r}-\vec{r}_m) \vec{S}(\vec{r})$ $\leftarrow$ connected near subcell m .

Now $\sum_{\vec{r}} |\vec{S}(\vec{r})|^2 \rightarrow \sum_m \sum_{m'} (S_m')^2$, again. Now comes a messy set of changes. We end up with a renormal-group sequence of Hamiltonians:

$$\mathcal{H}_1 = -\frac{1}{2} \sum_{\vec{r}} |\vec{D} S(\vec{r})|^2 - \sum_{\vec{r}} \vec{r} \cdot \vec{Q}_2(S) \quad \leftarrow (4.31)$$

$\alpha_2, Q_2 \rightarrow Q_2^*, \mathcal{H}_1 \rightarrow \mathcal{H}^*$

"recursion formula" is given to express Q_{k+1} in terms of Q_k (Nobis just S_0 is. One $\mathcal{H}_k$ keeps going with a new one of the same form, i.e., just Q_k must be obtained each time.

4.3 Experiment 2. I think we have now made an explicit model on R_b . I.e., $\chi_i = R_b \chi_0$, etc. On this model +

$$\rho(r^2) = \langle S_0(r) S_0(r^2) \rangle = b^{-2d} \langle S_1(r) S_1(r/b) \rangle = b^{-2d} \int_1^{r/b} (r/b)$$

I.e., this is how ρ "moves" under R_b . But the scaling of ρ changes: η :

$$\int_1^{r/b} (r/b) = b^{-2d-d} \int_1^{r/b} (r/b) = b^{-2d-d} \int_1^{r/b} (r/b)$$

So Wilson's formula says $\eta=0$. Not a drastic failure because η small from exp.

To get other exponents we need numerics.

4.4 Numerics on the recursion for ρ

Our recursion relation in effect says

$$\rho_{b^2}(y) = f[\rho, y]$$

First, by computer Wilson found a "fixed point function" ρ^*

$$\rho^*(y) = f[\rho^*, y]$$

It has the simple form

$$\rho^*(y, r_0) = \rho^*(y)$$

$$\rho(y, r) = \rho^*(y) \leftarrow (r - r_0) (b^d) \rho^*(y)$$

I.e., that ρ^* like we treated χ^* earlier. Each iteration gives a

(b?) But why is it $y \in \mathbb{R}$? ρ^* is something like ρ in

Somehow a fit is made and $y \approx y^* = 0.609$. Nothing clear!

Knowledge.

3-D Ising model: $(n=1)$

Wilson $\chi^2 = 0.609$ $\chi = 1.218$

3-D XY and Heisenberg $\chi = 1.29$

Heisenberg $\chi = 1.36$

2-D XY $\chi = 1.32$ not bad.

2-D Ising $\chi = 1.38$

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5. Critical Phenom in 4-ε dimensions. Section opens with sum of last

5.1 Trivial Fixed Point. It happens that $Q_c^* \equiv 0$ also works in the recursion; this yields

$$\chi^* = -\frac{1}{2} \int |\nabla S(\vec{r})|^2 d\vec{r} \leftarrow \begin{matrix} \text{classical values} \\ \text{"gaussian model"} \end{matrix}$$

with no longer a second term.

(With such an χ^* you get $\eta=0, \nu=\frac{1}{2}, \gamma=1$).

Baker proceeds to show, however, that this trivial fixed point is not "normal" in some sense [unless $d \geq 4$]; it should be discarded. I.e., if you make a little δQ , the ϵ -group iteration pushes you away from the fixed point $Q_c^* = 0$, i.e. there are relevant ($\gamma > 0$) operators which fail to vanish at criticality.

5.2. The non-trivial fixed point near $d=4$. They seem to think that the numerically determined, non-trivial fixed point function $Q_c^*(d)$ coincides with the trivial $Q_c^* = 0$ when $d=4$. So, look near $d=4$, so that Q_c^* is near $Q_c^* = 0$. Try

$$Q(y) = \xi y^2 + \eta y^4 \leftarrow \begin{matrix} \text{5-penny} \\ \text{ie, } \epsilon \end{matrix} \text{ where } \xi, \eta \text{ are } O(\epsilon) \quad \epsilon = d-4$$

You can construct easily the recursion formulas for ξ and η . Solve these for ξ^* and η^* . You get an answer:

$$\begin{aligned} \eta^* &= \frac{1}{3} \epsilon \ln b + O(\epsilon^2) \\ \xi^* &= \frac{1}{3} \left[\frac{b^2 \ln b}{1-b^2} \right] \epsilon + O(\epsilon^2) \end{aligned}$$

Some number junko then gives

$$\nu = \frac{1}{2} + \frac{1}{12} \epsilon + O(\epsilon^2)$$

$$\gamma = 1 + \frac{1}{6} \epsilon$$

$$\beta = \frac{1}{2} - \frac{1}{6} \epsilon$$

$$\alpha = 0 + \frac{1}{6} \epsilon$$

ie, all move a little from the gaussian $d=4$ model. These numbers look good.

Recall $\eta=0, \nu=\frac{1}{2} + \frac{1}{12} \epsilon \rightarrow$ all other exponents.

Thing is generalized for $n=1$ to $n=n$, then $\gamma = 1 + \left(\frac{n+2}{2(n+8)} \right) \epsilon$ etc. As $n \rightarrow \infty$, you get

$$\eta=0 \quad \gamma=2\nu = 1 + \frac{1}{2} \epsilon$$

This agrees exactly with the (calculable) spherical model when you set $d=4-\epsilon$ in that model.

Comments at this point: it is not that anybody is really interested in $d = 3.9$ dimensions; it is just that when d is near 4 and you use the simple gaussian fixed point $Q^* = 0$ to perturb around, in this case you can actually compute the exponents. The model with $d=4$ itself gives the classical exponents which we know are not quite right. In an entirely separate and numerical calculation, Wilson computed the $d=3$ Ising exponents; of course no one knows what these are because the model has never been solved, but his results agree with hi-temp expansion calculations.

Basically, Wilson takes a ferromagnetic hamiltonian with interacting spins, and he processes it in such a way that it can be seen that the thing does in fact scale into its same functional form under "renorm-group transformations" i.e. fancy scalings. By requiring that H goes to a fixed point H^* , which is the same as Q going to Q^* , he is able to compute the exponents. The way he does this I think is to assume that there is only one "relevant" operator, the Q_E , which has as its exponent y_E which somewhere along the line was shown to be $1/\nu$. But y_E and therefore ν appears in the Q_L recursion relations, and is thus findable numerically. The exponent η is zero. Since Renorm Group method has two-exponent scaling, all exponents are determined by your computation of ν . For $d=3$ Ising he got $\nu = .609$, for example, whereas for near-4 he gets $\nu = .5 + \epsilon/12$.

add notes. Note that
$$\mathcal{H} = -\frac{1}{2} \int d^d r [\vec{\nabla} s(r)]^2 - \int d^d r Q(s)$$

where $Q(s) = \frac{1}{2} s^2 + u s^4$. This is the form of Q that Wilson uses, I don't really know why, except that it worked. This is called an " s^4 -Hamiltonian".

6. Feynman - Graph Expansions for Critical Exponents

Now take the $r s^2$ term as $\mathcal{H}_0$, and the $u s^4$ as $\mathcal{H}_I$.

6.1 Correlation Function. $\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_I$

$$\mathcal{H}_0 = -\frac{1}{2} \sum_{\alpha=1}^m \sum_{\vec{q}} (c_0 + q^2) \sigma_{\vec{q}}^{\alpha} \sigma_{-\vec{q}}^{\alpha} = \text{F.T. of the "gaussian" hamiltonian.}$$

Is, if you assume $u_0=0$, $r_0^{\text{cut}}=0$ and $\mathcal{H}^{\text{K}} = \text{gaussian}$. Set of like a mass term in QFT lagrangian. Only gets interesting if $u \neq 0$, like a ϕ^4 interaction.

$$\begin{aligned} \mathcal{H}_I &= -\frac{1}{8} u_0 L^d \sum_{\alpha, \beta} \sum_{\vec{q}_1, \vec{q}_2, \vec{q}_3, \vec{q}_4} \delta(\sum \vec{q}_i) \sigma_{\vec{q}_1}^{\alpha} \sigma_{\vec{q}_2}^{\alpha} \sigma_{\vec{q}_3}^{\beta} \sigma_{\vec{q}_4}^{\beta} \\ &= \text{FT} \left[-\frac{1}{8} u_0 \int d^d r s^4(r) \right]. \end{aligned}$$

Now if $d=4-\epsilon$, ϵ small, then $u_0 \sim \epsilon$ also small, so very much like a perturbation field theory. Recall:

$$\langle A \rangle = \frac{1}{Z} \int \prod_k d\phi_k^a A \in \mathcal{Z}[\phi_k^a]$$

$\uparrow$ some object of interest

path integral where time replaced by $\vec{k}$??

Comment: What is this? This is just a statistical sum with usual Boltzmann factor to get the mean value of something, here A . You sum over all the "spin states", just as if you were computing any other partition function Z . BUT, on the other hand, this thing looks to me exactly like the path integral expression for the vacuum expectation value of some operators A , ie, this might be a greens function. Then you can think of H as a usual particle physics interaction lagrangian. Then you can use usual path integral or regular perturbation expansion if L_I parameter is small!!!! Amazing how these things are so similar. Is it more than coincidence? In field theory this exponential is just the action, in stat mech it is the Boltzmann factor. Note that there is, however, a momentum cutoff in the k -integral... which removes the divergence

So write $e^{Z_2} = \sum_{j=0}^{\infty} \frac{1}{j!} (Z_2)^j$ + do as you always do.

6.2 "Wick Theorem" Just the idea of grabbing things in pairs, just as in usual QFT. Note:

"gaussian average" $\langle \sigma_{k_1}^{\alpha_1} \sigma_{k_2}^{\alpha_2} \dots \sigma_{k_n}^{\alpha_n} \rangle$

$\swarrow$ spinfields in "gaussian model", ie, only s^2 term in "pot."

$\searrow$ free fields

$\leftrightarrow$ "vacuum expectation value" $\langle 0 | \phi^{\alpha_1}(k_1) \dots \phi^{\alpha_n}(k_n) | 0 \rangle$

The "free propagator" is $\langle \sigma_{k_1}^{\alpha_1} \sigma_{k_2}^{\alpha_2} \rangle = \delta_{\alpha_1 \alpha_2} \delta(k_1 + k_2) \hat{P}_0(k_1)$

$\hat{P}_0(k_1) = \frac{1}{r_0 + k^2}$

$\uparrow$ like usual propagator.

$\nwarrow$ the gaussian (free) 2-spin correlation fcn.

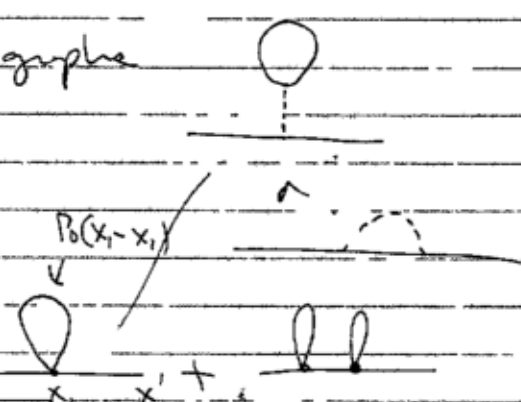
6.3 Diagrams

$P(\vec{x}-\vec{x}') = \langle S^{\alpha}(\vec{x}) S^{\alpha}(\vec{x}') \rangle_0 =$ series of graphs

$\uparrow$ "full" correlation function

note: $Z_2 = \int d\mathbf{x} S^{\alpha}(\mathbf{x}) = \times$

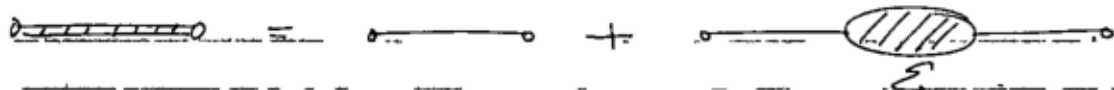
Phase: $P(\mathbf{x}-\mathbf{x}') = \text{series of graphs} = \text{graph 1} + \text{graph 2} + \dots$



6.4 In momentum space. $\hat{\Gamma}_0(k)$ replaces $\Gamma(x_1-x_2)$

6.5 Self-energy Standard. Dyson's Equation is:

$$\Gamma(x-x') = \Gamma_0(x-x') + \int d^4x_1 d^4x'_1 \Gamma_0(x-x_1) \Sigma(x_1, x'_1) \Gamma(x'_1-x')$$



Dyson equation for the propagator

$$\hat{\Gamma}(k) = \frac{1}{\Gamma_0 + k^2 - \hat{\Sigma}(k)}$$

Let $\Gamma = [\hat{\Gamma}(k)]^{-1}|_{k=0} = \Gamma_0 - \hat{\Sigma}(0)$. Then:

$$[\hat{\Gamma}(k)]^{-1} = \Gamma + k^2 - \hat{\Sigma}(k) + \hat{\Sigma}(0) \quad \Gamma = \text{"renormalized mass."}$$

Note $\hat{\Gamma}(0) = \chi_0(T)$, so $\Gamma = \frac{1}{\chi_0(T)} = 0$ at critical point.

$\hat{\Sigma}(0) =$ like δm , a counterterm.

7. ϵ -expansions of critical exponents.

Wilson's strategy is getting clearer

all the time. In the last section we computed the spin-spin correlation function only to lowest order in ϵ . But now with the Feynman graph interpretation, we can compute in principle to any order in ϵ . If we are lucky, we can use this expansion for large ϵ , for example, $\epsilon = 3$, in which case we are talking $d=3$. Then we can compute the critical exponents for $d=3$ and see if answer is the same as the high-temperature expansions for the Ising model.

7.1 Momentum integrals in d dimensions. Instead of giving a general theory of how to do this, it turns out that only the most symmetric two integrals will be needed, and each of these I know how to do, even when d = general number. (remember for later reference).

7.2 Self-energy at critical point. Note: no UV divergences since cutoff Λ is always there! Now

$$[\hat{\Gamma}(k, r)]^{-1} = r + k^2 - [\hat{\Sigma}(k, r) - \hat{\Sigma}(0, r)] \quad r \rightarrow 0 \Rightarrow T \rightarrow T_c$$

$$\equiv W(k, r).$$



compute this to order ϵ^2 adding

Thus, we in this way compute $\hat{\Gamma}(k, T_c)$ for small k . Thus, we will have computed η ! Result so far is

$$k^2 \hat{\Gamma}(k) \approx 1 + \frac{(n+2) u_0^2}{5/2\pi^4} \ln(k) + O(\epsilon^2) \quad / k \text{ small.}$$

But what is u_0 ? It is the bare ϕ^4 compl. It gets renormalized to u_R as usual

$$u_R = \text{diagram} = \text{diagram} + \text{diagram} + \text{etc.}$$

They get

$$-(u_R/u_0) = 1 + (32\pi^2)^{-1} (n+8) u_0 \ln(r) + O(\epsilon^2).$$

Now why do this? Reason: we have information re $u_R \equiv \hat{\Gamma}_4(0,0,0;r)$ near $r=0$ (critically), because we know how it transforms. Thus we find $\phi u_R = 1 + \frac{1}{2} \epsilon \ln r$. Ergo,

$$u_0 = \frac{16\pi^2 \epsilon}{(n+8)} + O(\epsilon^2)$$

I guess we did not know how the bare $\hat{\Gamma}_0$ transforms! The R -group works on full objects.

7.4. Now compute η to order ϵ^2 . An interesting detail here:

$$k^2 \hat{\Gamma}(k) \approx 1 + \frac{(n+2)}{2(n+8)^2} \epsilon^2 \ln k + O(\epsilon^3) \approx k^{\eta} \quad \text{what is } \eta?$$

$$\eta = 0 + b\epsilon^2 \quad / \text{because earlier we showed } \eta = 0 \text{ to order } \epsilon.$$

$$\text{So } k^{\eta} = e^{\eta \ln k} \approx 1 + \eta \ln k + \frac{1}{2} (\eta \ln k)^2 + \dots$$

$$\Rightarrow \left[\eta = \frac{1}{2} \frac{(n+2)}{(n+8)^2} \epsilon^2 + O(\epsilon^3) \right] \leftarrow \text{answer.}$$

$\leftarrow \eta \text{ to order } \epsilon^2$

7.5. Exponent γ to order ϵ . We connect to γ via $\chi_T \sim (\epsilon)^{-\gamma}$ but this is same as $(r)^{-\gamma}$ and $\chi_T = \hat{\Gamma}(k=0)$, so look for

$$\hat{\Gamma}(k \rightarrow 0) \sim (r)^{-\gamma} \quad \text{Not quite. } \chi_T = [\hat{\Gamma}(0)]^{-1} = r \sim (r_0 - r_{0,c})^{-\gamma}.$$

What is γ ? Ah, but

$$r = (r_0 - r_{0,c}) = \hat{\Sigma}(0, r) + \hat{\Sigma}(0, 0) \Rightarrow \gamma = 1 + \frac{(n+2)\epsilon}{2(n+8)} + O(\epsilon^2)$$

$\leftarrow \text{calculate to order } \epsilon \quad \leftarrow \gamma \text{ to order } \epsilon^2$

7.6 Higher Order. OK. Things have been done to ϵ^3 for γ , and ϵ^2 for all other exponents.

7.7 Discussion. The hi-temp series calcs and the ϵ -expansion agree quite well to second order, at least for GAMMA exponent. Then in next order it seems to get worse. They say, well, maybe it is an "asymptotic" series which has a best point and then diverges. But the best point looks very good. This comment refers to usage of the epsilon-expansion for $\epsilon = 1$ which is rather large, not surprising that thing does not converge.

8. Expansions involving $1/n$. Amazing similarities to Chew's topological expansion. Here, you use the same formalism as above except you assume that $u_0 = \text{order}(1/n)$ instead of $\text{order}(\epsilon)$. I.e., now instead of expanding around the classical gaussian model with $d=4$, you are expanding around the classical spherical model with $n = \infty$. I think this spherical model has the same classical exponents as the $d=4$ gaussian model.

So you go through the same deal, calculating various Feynman diagrams as needed. Interestingly, closed loops give factor of n as you can sum over spin degrees of freedom of the fields in a closed loop (just like quarks). This lets you know which graphs to throw away.

Barber proceeds to review the computation of the exponents γ and δ in the $1/n$ method for general d . Then for $d=3$ they are seen to agree very well with the ϵ -expansion calculated exponents, thereby strengthening ones belief in both methods.

8.5 Scaling Laws and the $1/n$. The point made here is that the $1/n$ method never makes use of the homogeneity/scaling stuff and can therefore be used to test the scaling ideas. Looks OK except there is some trouble ("anomaly") for exponent α for certain values of d , including $d=3$, which casts some doubt on the formula $d\gamma = 2 - d$. Recall that this is the only exponent relation that contains d .

9. Derivation of Exact Recursion Relation. The problem Wilson wants to do - The R-group problem - is calculate:

$$\exp[R_b \chi] = \left[\prod_{\alpha=1}^{\infty} \prod_{1/b' < k < 1} \int d\sigma_k^{\alpha} \exp(\chi) \right]_{\sigma_k^{\alpha} \rightarrow b^y \sigma_{bk}^{\alpha}} \quad y = 1 - \frac{1}{2} \epsilon$$

Then when you have calculated this, you know what R_b is explicitly and you can then find the fixed point χ^* + compute exponents.
Rewrite the above as

$$\exp[R_b \chi[\sigma_k^{\alpha}, k < 1]] = \left[\text{function of } \sigma_k^{\alpha} \text{ where } k < \frac{1}{b} \right]_{\sigma \rightarrow b^y \sigma} \quad \text{i.e., } bk < 1 \quad \text{and } k' = bk \text{ somehow.}$$

Shorthand:

$$\exp[R_b \chi] = [T_b(\exp \chi)]_{\sigma_k^{\alpha} \rightarrow b^y \sigma_{bk}^{\alpha}}$$

In this section a more precise attempt is made [Wilson/Kogut] to compute this operation in the $1S^2 + 4S^4$ model. An improvement over the "Wilson's approximate recursion relation" of lecture 4.

Conclusion of this section [which I will read in Wilson/Kogut] is this: "for isotropic spin system with $d \approx 4$, χ^* [the fixed-point χ] is the $1S^2 + 4S^4$ χ we have been discussing, where $b_0, b_1 \sim \epsilon$."

I think the problem is that irrelevant variables proliferate as you iterate? Hard to go beyond order ϵ in an exact analysis.

First mention is made here of renorm group equations, which involve δb small b change. Summary of recent work.

10. Special attacks on $d=2$ systems. This is sort of Kadanoffic. Start with general d spin system, spins $\sigma_i = \pm 1$, $\chi = \chi(a)$, lattice is Ω with spacing a . χ described by parameters K (n.n. coupling, etc). Idea: let Ω' be isomorphic lattice to Ω but upscaled by b so $a' = ba$. New spins called $\mu_i = \pm 1$. The new params are $K' = R_b K$. Game is to find $K^* = R_b K^*$, i.e., χ^* . Note: $Z_N = e^G Z_{N'}$ is has 2 scales.

So go ahead and linearize R_b to L_b . Expect L_b to have the usual two relevant operators + eigenvalues:

$$Q_E \Rightarrow \frac{y_E}{b} = b^{1/2} \quad \text{/energy}$$

$$Q_H \Rightarrow \frac{y_H}{b} = b^{1+(d-2)/2} \quad \text{/external field}$$

Now, $\mathcal{H}$ near $\mathcal{H}^*$ is characterized either by parameters K or by the relevant "fields". [Irrelevant ones are decoupled].
So here we would say

$$\mathcal{H} = \mathcal{H}^* + \sum \mu_i Q_i = + u_E Q_E + u_H Q_H + \text{irrelevant}.$$

$$\mathcal{H}_l = \mathcal{H}^* + \sum \mu_i (b^{y_i})^l Q_i = u_E (b^{y_E})^l Q_E + u_H (b^{y_H})^l Q_H \rightarrow \mathcal{H}^*.$$

Critical Surface is $\{\mu_i\} \ni \mathcal{H} \rightarrow \mathcal{H}^*$ at critical. So, critical surface is determined by $\mu_E = 0$ and $\mu_H = 0$, i.e., $f(K) = 0$ [a surface in parameter-space].

Any point on the critical surface corresponds to a set of values $\{K\}$ and therefore to some Ham. $\mathcal{H}$. For any such $\mathcal{H}$, you know that $\mathcal{H} \rightarrow \mathcal{H}^*$ under iteration of R_b (at least $\mathcal{H}$).

One point on this surface corresponds to n.n. interactions only ($K_2 \neq 0$, $K_{\text{else}} = 0$). Is the Ising model.

This section goes on to apply this direct method to several $d=2$ lattices, using various approximations. Clearly you have to restrict the set of parameters $\{K\}$ which is infinite for $d > 1$.

Various buzzwords:

"decimation", "linear block", "non-linear block"

These are ways to do the scaling

"finite lattice", "cumulant", "cluster"

These are computer methods to make things doable.

10.4 Results Some give exponents, but many give awful exponents. [χ and η really go with y_E and $y(\sigma_K)$].
This world is a testing ground.

. end of review.

Wilson Kogut

/ rec June 20 1978

p77- 1. Introduction.

1.1-Philosophy. To deal with a physical problem usually you only have to deal with the degrees of freedom which describe a piece of the system smaller than the correlation length which is usually only a few atomic diameters. The properties of the bulk can be reduced to the properties of a small piece of the bulk, and this piece has only a few degrees of freedom and therefore can be handled.

However, critical phenomena are those described by a very large correlation length; there are then so many degrees of freedom within the correlation length that you don't know what to do! The renormalization group is a method telling you what to do. You try to thin out the number of degrees of freedom systematically. You start with a small block of matter described by Hamiltonian H_0 with handleable number of degrees of freedom. Then you move up to larger scale, say twice, and try to compute H_1 in terms of the bulk parameters invented by studying H_0 . I.e., you compute H_1 in terms of H_0 and this step or iteration hopefully involves again a finite number of degrees of freedom. Then you keep on iterating until you get to the correlation length. This is infinite at critical point, so iterate many times. The assumption here is really that the goings on at the scale of H_1 say are determined by the goings on only at the preceding scale H_0 (which includes the effect of all smaller scales). Thus you are assuming a sort of scale-locality. Underlying microscopic H must therefore be local itself. Analogy made to hydrodynamics where you deal with density $\rho(x)$ and you simply ignore fluctuations on the microscopic level. Thus you have reduced the degrees of freedom to something you can do.

Hopefully the effective Hamiltonian H_0 you start with will lead to a fixed point H^* . Then the large-scale or cooperative phenomena is described not so much by the microphysics H_0 as by the nature of the transformation R_0 of which H^* is a solution. Violently different systems with different H_0 's might all have the same limit H^* and therefore have the same type of critical behavior: universality. There might be several competing fixed points.

1.1a) History of the Renorm Group method. In 1930 Landau proposed sort of hydronic theory. In 1954 Gell-Mann Low showed that many renormalized charges e go to the same unique bare charge e_0 in QFT. By the way, QFT has lots of degrees of freedom because $\phi(x)$ is infinite number of objects for x in a finite volume. Connection between QFT and Crit Phen not yet clear to me. Bogoliubov and Shirkov summarized things. Then in 1960's Kadanoff did spin-block thing without a "shred" of justification. This was the basic idea though.

Current ideas which look like but are not the renorm group are:

a) Migdal-Polyakov bootstrap approach to crit phenom.

b) Johnson-Baker-Willey formulation of QED.

They make no attempt to reduce the number of degrees of freedom. Finally, Callan-Symanzik is a recent development to look into.

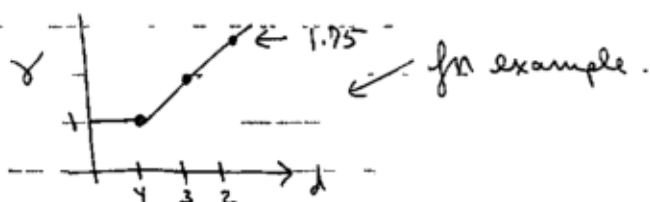
1.2 Current references: a few given on the connection between crit phen and QFT. Wilson himself has a review on QFT applications of the RG. Some comments about non-integer dimensionality d: t-Hooft Veltman also use it.

Finally: "particle physicists find it takes a long time to understand all this stuff"

1.3. Basics of Ising model ~ nothing new

$S_x = \pm 1$ $n = \text{lattice point}$

2.1 Crit. Phen Basics. Exponents are introduced. Idea: for $d > 4$, the classical exponents (mean field) are correct, but for $d < 4$ they depend on d :



2.2 "Search for Analyticity". Critical point is non-analytic point, arises when $N \rightarrow \infty$ in 2. (the "thermodynamic limit"). Still, people look for analytic formulations

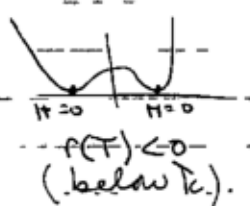
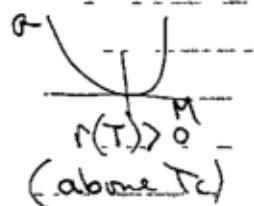
[2.2.1. mean field theory. Very good section (simple). Landau said: assume $G(M, T)$ is analytic at $T = T_c$. Then Super Simple Model

$$\text{ff. } G(M, T) = G(T) + r(T)M^2 + u(T)M^4 + \dots \quad \text{series in } M.$$

/note, G is even in M because up/down Ising symmetry/. Now

$$H = \partial G / \partial M = 2r(T)M + 4u(T)M^3.$$

Note that $H=0 \Rightarrow \text{slope} = 0$. Situation depends on sign of $r(T)$.



Reasonable to assume $r(T) \propto T - T_c$, since order goes away at $T = T_c$. ✓

Now, at $H=0$ we get $2(T - T_c) + 4u(T_c)M^2 = 0 \Rightarrow M \sim (T - T_c)^{\frac{1}{2}}$
 $\Rightarrow \beta = \frac{1}{2}$. And: $\chi_T = \frac{M}{H} = \frac{1}{2r(T)} \sim (T - T_c)^{-1} \Rightarrow \gamma = 1$

So already we have computed some exponents! And the plot of G vs M seems interesting too. This theory will be "zeroth order".

[2.2.2. Kadanoff Theory Let $\mathcal{H}_i = \sum_{n, i} K_i S_n^{(i)} S_{n+i}^{(i)}$ where K_i is the parameter which changes as scale changes. $S_n^{(1)}$ = individual spin, $S_n^{(2)}$ = block spin of 4 spin block, etc. Naively $K_1 = 2K_0$. But generally suppose $K_i = f(K_0)$, unknown. Now $\mathcal{F}_i = \mathcal{F}(K_i)$ also. Eg, $\mathcal{F}_1 = \mathcal{F}(K_1) = \frac{1}{2} \mathcal{F}(K_0)$ since we doubled block size. In new meter sticks, $\mathcal{F}$ is half what it was in old, shorter meter sticks.

One clearly obtain $[f(k)] = \pm \epsilon(k)$. At $k=0$, $f(k)=k$. Now suppose $f(k)$ is analytic near $k=0$. One "parametric transformation" can be analytic at $T=T_c$, although ϵ is not.

$$f(k) = f(k) + \lambda(k-k_c) \quad \lambda = \partial f / \partial k$$

Assume that $\epsilon(k) = (k-k_c)$ then $\epsilon[f(k)] = \left(\frac{f(k)-k_c}{f(k)-k_c} \right) \epsilon[k]$ obviously.

$$\frac{1}{2} = \lambda^{-1} \quad \lambda = \partial f / \partial k$$

If you know $f(k)$, you would know $\lambda = \partial f / \partial k$ + hence you would know λ !

$$\epsilon(k) = \left(\frac{1}{2} \right) \epsilon(k) \quad \text{the whole idea}$$

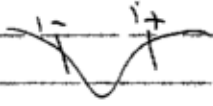
(for large enough n , $\epsilon(k_n)$ is small enough that we can compute it [you "guess"])

But we can compute $\epsilon(k_n)$ for $n=5000$, we at once know $\epsilon(k_0)$ which we need to calculate! The same analytically in $\epsilon(k)$ at $k=k_c$ arises from the iteration: $f(k)$ itself is analytic.

3. The Gaussian Model. How to reverse the sum Ising:

$$Z = \prod_{m=0}^{\infty} \int_{-\infty}^{\infty} ds_m \exp(-\frac{1}{2} s_m^2) \exp(-\sum_{m=0}^{\infty} s_m^2) = \text{Ising}$$

Now smooth out the delta to a gaussian.



$$\text{Ising: } s_m = +1, -1, < s_m = 0$$

Gaussian: $s_m = \text{gaussian}$ period of $6s_m = 0$.

$$Z_{\text{gaussian}} = \prod_{m=0}^{\infty} \int_{-\infty}^{\infty} ds_m \exp(-\frac{1}{2} s_m^2) \exp(-\sum_{m=0}^{\infty} s_m^2)$$

define gaussian model

This is a finite gaussian path integral which I know how to do. In present notation $s = \phi$, $\tilde{s} = \psi$ (source). One:

$$\prod_{m=0}^{\infty} \int_{-\infty}^{\infty} ds_m \exp(-\frac{1}{2} s_m^2) \exp(-\sum_{m=0}^{\infty} s_m^2)$$

$$= C \exp(-\sum_{m=0}^{\infty} s_m^2) = W[s]$$

just as in Coleman $C = 1/\sqrt{\det A}$

$$I(m_1, m_2, \dots, m_k) = \prod_{m=0}^{\infty} \int_{-\infty}^{\infty} ds_m \exp(-\frac{1}{2} s_m^2) \exp(-\sum_{m=0}^{\infty} s_m^2)$$

$$= \frac{1}{2} \frac{1}{2} \dots \frac{1}{2} W[s]$$

via VEV's

no global because no independent there!

—Nine—

point we have

$$2 \cdot \frac{1}{n} \sum_{i=1}^n S_i \hat{S}_{n+1} = \frac{1}{2} b \sum_{i=1}^n S_i^2 = 0$$

One gets:

$$\exp\left[-\frac{1}{2} \sum_{\vec{q}} (q^2 + r) \sigma_{\vec{q}} \sigma_{-\vec{q}}\right] \quad \text{where} \quad \sum_{\vec{q}} = \int d^d q (2\pi)^{-d}$$

$$r = \frac{b - 2dK}{K}$$

is the FT. Set $k=1$.

Now let $S(\vec{x}) \equiv \int_{\vec{q}} e^{i\vec{q} \cdot \vec{x}}$ but $0 < |\vec{q}| < 1$ (note) $\in (-\pi, \pi)$ (why?)

So $S(\vec{x})$ not same as $S(\vec{y})$ Anyway, result is

$$\exp(\mathcal{H}) = \mathcal{C}^{-\frac{1}{2}} \int \underbrace{(q^2 + r)}_{\text{propagator}} q \, dq = \text{the gaussian } \exp(\mathcal{H})$$

Next: compute correlation fch:

$$P(x) = \langle s(x)s(0) \rangle = \int [da] s(x)s(0) e^{\int da \cdot L[a(q)]}$$

Well hell. We have $\exp \left[-\frac{1}{2} \int d\tau \sigma_q^* (q^2 + m^2) \sigma_q \right] = \text{action } S[\sigma_q]$.

Obviously the

momentum space propagator is $\langle \sigma_1 \sigma_2 \rangle = \frac{1}{q^2 + r}$, inverse of $\square$

S. $P(x) = \text{F.T.} \left[\frac{1}{-q^2 + r} \right] \dots = P(q^2)$.

$$\therefore P(x) = \text{F.T.} \left[\frac{e^{-t}}{[-q^2 + r]} \right] \dots$$

S. gaussian model: $\Gamma(\vec{x}) = \langle s(\vec{x}) s(\vec{0}) \rangle = \int d^4 q \, e^{i \vec{q} \cdot \vec{x}} \frac{1}{[q^2 + r]}$

Thus, we have $P(|\vec{x}|) \sim e^{-|\vec{x}|^2/2}$ for large $|\vec{x}|$, so

$$\xi \sim (\sqrt{r})^L = r^{\frac{L}{2}}$$

But $r = \frac{b - 2dk}{T} = \frac{(b - 2d \frac{g}{KT})}{T} = \frac{bKT - 2dJ}{T^2} = \text{linear in } T$

So: $r \sim (I - T_c)$ $\frac{J}{kT}$ so $\gamma = \frac{1}{2}$ $H_c = \frac{2dJ}{b}$, $K_c = \frac{b}{2d}$!

great illustration

Now, let's do the first step of the 3-stage renm group process. Let's integrate $\Sigma[\sigma_q]$ over $\frac{1}{2} < |q| < 1$, i.e., we integrate over short wavelength fluctuations. How do you do this?

First look at Σ :

$$\Sigma = \int [d\sigma] \mathcal{C}^{-\frac{1}{2}} \int_0^1 dq \sigma_q (q^2 + r) \sigma_q$$

step 1: integrate out some short stuff.

where $\int_0^1 dq$ says we are summing over all states with $0 < |q| < 1$. But write this as

$$\Sigma = \int [d\sigma] \mathcal{C}^{-\frac{1}{2}} \left(\int_0^{\frac{1}{2}} dq + \int_{\frac{1}{2}}^1 dq \right) \sigma_q = \text{const.} \int [d\sigma] \mathcal{C}^{-\frac{1}{2}} \int_0^{\frac{1}{2}} dq \sigma_q$$

↑ only another way to write it.

So new Σ has same form except for const.

Now look at

$$\Sigma(q_1, q_2) P_q = \int [d\sigma] \sigma_{q_1} \sigma_{q_2} \mathcal{C}^{-\frac{1}{2}} \int_0^1 dq = \frac{2}{2j(q_1)j(q_2)} \int [d\sigma] \mathcal{C}^{-\frac{1}{2}} \int_0^1 dq \dots j(q) \sigma_q$$

If $q_1, q_2 < \frac{1}{2}$, can go ahead + do expo integral ... & see. so the new P_q is same form but $\int_0^{\frac{1}{2}}$ in place of $\int_0^1$. ✓

Second step: try this

step 2: rescale q so integral looks the same as before.

$$\sigma_q = \xi \sigma_{q'} \Rightarrow \mathcal{H} = \frac{1}{2} \int_0^{\frac{1}{2}} dq \sigma_q (q^2 + r) \sigma_q = ?$$

Change to $q' = 2q$ so $\int_0^{\frac{1}{2}} dq$ again! Thus: $\mathcal{H} = \left(\frac{1}{2}\right)^d \frac{1}{2} \int_0^1 dq' \sigma_{\frac{q'}{2}} \left(\left(\frac{q'}{2}\right)^2 + r\right) \sigma_{\frac{q'}{2}}$

Next, let $\sigma_q \rightarrow \xi \sigma_{q'}$ so $\sigma_{\frac{q'}{2}} \rightarrow \xi \sigma_{q'}$. We then have:

$$\mathcal{H} = \left(\frac{1}{2}\right)^d \frac{1}{2} \int_0^1 dq' \sigma_{q'} \left[\left(\frac{q'}{2}\right)^2 + r\right] \sigma_{q'} \cdot \xi^2$$

step 3: introduce new fields σ' with scale factor which you

Now finally, adjust ξ so $\mathcal{H}'$ has same coeff: adjust to get same coefficient overall for $\mathcal{H}$.

$$\mathcal{H} = \left[\left(\frac{1}{2}\right)^d \xi^2 \frac{1}{4}\right] \frac{1}{2} \int_0^1 dq' \sigma_{q'} [q'^2 + r'] \sigma_{q'} \quad \text{where } r' = 4r.$$

$$\stackrel{=}{=} \text{so } 2^{-d-2} \xi^2 = 1 \Rightarrow \xi^2 = 2^{d+2} \quad \xi = 2^{\frac{d+2}{2}} \quad \underline{\text{I see!}}$$

These two second stages are needed to make $\mathcal{H}'$ look exactly like $\mathcal{H}$.
Then the only change is: r becomes r' i.e., a "parameter transformation rule"

The σ_q rescaling is like Kadanoff rescale to ± 1 in S_n ✓

Next, $\Gamma_q = 4\Gamma'_{q'}$ where Γ' defined in obvious way. $\xi = (\Gamma') = (4r) = \frac{1}{2}\xi$.
But recall $r \sim (T - T_c)$ so $\nu = \frac{1}{2}$. In our previous language, $r' = f(r) = 4r$ / i.e., $r \equiv \text{scaled } k$. So $dt/dr = \frac{1}{4}$.

Then $\nu = \ln 2 / \ln 4 = \frac{1}{2}$ ✓

4. The S^4 model. Now

$\mathcal{H}[\sigma] = -\frac{1}{2} \int d^d q \cdot \sigma_q [q^2 + r] \sigma_{-q}$
 gaussian model.

small parameters.

$-u \int dx S^4(x)$

$-u \int d^d q_1 d^d q_2 d^d q_3 d^d q_4 \sigma_{q_1} \sigma_{q_2} \sigma_{q_3} \sigma_{q_4}$

Now write $\sigma_q = \sigma_{q,0} + \sigma_{q,1}$

piece exists only when $\frac{1}{2} < |q| < 1$; this piece will be integrated out

$Z = \int [d\sigma_0] [d\sigma_1] e^{\mathcal{H}[\sigma_0 + \sigma_1]} \rightarrow \int [d\sigma_0] e^{\mathcal{H}[\sigma_0]}$ / first step

$\rightarrow \int [d\sigma] e^{\mathcal{H}[\sigma]}$ / second + third steps, where $\sigma_{0,q} = \xi \sigma(2q)$

As before $\Gamma_q = \xi^{-2-d} \Gamma'_{(2q)}$ based on $\langle \sigma_q \sigma_{-q} \rangle$. And $\xi' = \frac{1}{2} \xi$ since $b=2$.

Now

$\Gamma_x \sim e^{-1/x^3} = \int d^d q \Gamma_q e^{iq \cdot x} \quad \Gamma_q \sim \int dx e^{-x/5} e^{-iq \cdot x}$

$\Gamma_q = \int dx P(x) e^{-iq \cdot x} \quad \frac{\partial \Gamma_q}{\partial q} = \int dx x P(x) e^{-iq \cdot x}$

$\left. \frac{\partial \Gamma_q}{\partial q} \right|_{q=0} \approx \int dx x P(x) \approx \int_0^\infty dx x e^{-x/5} \approx \frac{1}{(1/5)^2} = 5^2$

So roughly $\xi = \left[\frac{1}{\mu} \frac{\partial \Gamma}{\partial q^2} \right]^{1/2} \Big|_{q^2=0}$

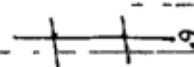
The idea now is this:

$e^{\mathcal{H}[\sigma]} = e^{\mathcal{H}[\sigma_0]} \int [d\sigma_1] e^{-\mathcal{H}[\sigma_1] + \mathcal{H}[\sigma_0 + \sigma_1]}$

You want to compute $\mathcal{H}'$. This requires doing a path integral over $[d\sigma_1]$. But this in turn is like computing $W[\sigma=0]$. So $\mathcal{H}' =$ sum of connected graphs where $\sigma_1 =$ the active field. Connected and no external σ_1 fields lines.

Graphs with N external σ_0 fields contribute to the term of order $(u')^N$ in $\mathcal{H}'$.

Thus

 generates a $(u')^2$ term in $\mathcal{H}'$, (there was none in $\mathcal{H}$!). ignore these. the free theory result

After all is done, you obtain these "parameter transformation relations":

$r' = 4 \left[r + \frac{3c u}{1+r} \right] + O(u^2)$
 $u' = 2^{1-d} \left[u - 9c \frac{u^2}{(1+r)^2} \right] + O(u^3)$

4.1 Simplification. The idea is to keep doing this, until there,

$$u_{n+1} = 4\{u_n + 3cu_n/(1+u_n)\}$$

4.2 The ϵ -expansion. Fixed point means r^*, u^* so that $r' = r$ $u' = u \Rightarrow r' = r$. Since $r = r(\text{params})$. But $r' = \frac{1}{2}r$. There:

$$(\text{fixed point of param iteration}) \Leftrightarrow (\text{criticality}, r(K^*) = \infty)$$

Trivial fixed point: $r^* = 0$ and $u^* = 0$. This is the gaussian model at extremity, recall that $r \equiv (T - T_c)$ $u = 0 \Rightarrow$ ignore interaction.

$$\text{Consider: } u_{n+1} = 2^{4-d} \{u_n - qc \cdot u_n^2 / (1+u_n)\}$$

If $d > 4$, then iteration forces $u \rightarrow 0$, i.e., force you to trivial fixed point!
If $d < 4$, let $\epsilon = 4 - d > 0$. Then you easily find a non-trivial fixed pt.:

$$u^* = \frac{1}{qc} \in \ln 2 \quad r^* = -\frac{4}{q} \in \ln 2$$

You end up here regardless of exact value of (u_n, r_n) . I.e. $\forall K_0$. So all K_0 on critical surface $\rightarrow K^*$.

4.3 Linearized Equations and V . At this point the recursion is $(r_n, u_n) = f(r_n, u_n)$, non-linear. If (r, u) close to (r^*, u^*) , you can linearize this baby [matrixize it]. Then

$$\begin{pmatrix} (r_{n+1} - r^*) \\ (u_{n+1} - u^*) \end{pmatrix} = \begin{pmatrix} M \end{pmatrix}^n \begin{pmatrix} r_1 - r^* \\ u_1 - u^* \end{pmatrix} = n \text{ iterations}$$

The idea here is that, essentially everything as $n \rightarrow \infty$ is controlled by the largest eigenvalue of M , just as in the Ising thing. The fact it turns out that $\lambda = \ln 2 / \ln 4$ and $\lambda_1 = 4 - (\frac{3}{2} \ln 2) \in$ so $\lambda = .5 + \frac{1}{2} \ln 2$. So you see how interaction pulls V away from its gaussian-model value.

5. The S^4 model continued. Question: suppose there exists some other parameter, say w , along with r and u . Suppose $w' = \frac{1}{4}w$, so w is "irrelevant". Can such a variable change the recursion involving r and u ? The principle yes, in this model no - (u_6) is such a variable here. This action shows that even considering all irrelevant, the original recursion for (r, u) is still valid to order ϵ .

5.1 irrelevant (see above)

5.2 complete β -group xfm. The idea here is that because most of the messiness of the exact recursion $\mathcal{H}'[0'] = \dots$ involves irrelevant activity, you can go ahead and compute Γ^*, u^* to higher orders in ϵ !

6. The approximate Recursion Formula. Include all powers of the spins, i.e., take into account $u_2, u_4, u_6, \dots, u_{\infty}$; but, neglect momentum dependence, i.e., $u_2(\vec{q})$.

6.1 Polyakov's derivation. We shall redo the preceding calculation [get $\mathcal{H}'[0']$ into same form as $\mathcal{H}(\sigma_i + \sigma_j)$ etc] but new assumptions are used in calculation.

$$\mathcal{H}[s] = -\frac{1}{2} \int dx \{ \mathcal{V}s(x) \}^2 = s_+ \int dx Q[\epsilon_2 s(x)] \quad \text{[note } |\mathcal{V}s|^2 \text{ comes from } q^2 \sigma_q \sigma_{-q} \text{ in } \mathcal{H}_0 \text{.]}$$

where $Q(s) = rs^2 + us^4 + ws^6 + \dots$. To make $\mathcal{H}'[s']$ look just like this, we need:

$$\bar{Q} \equiv f[Q] \text{ as in (6.5) = a mess.}$$

Too much detail.

here for my interest! The idea is to look for $Q \rightarrow Q^*$ by computer.

(p13) 7.1 More on the β -recursion. (a) Technical detail; a note that $\mathcal{V}(\epsilon)$ seems to be an asymptotic expansion only.

(b) A scaling law for $\Gamma(q_1, \dots, q_n)$ correlation function is quoted:

$$\Gamma(q_1, q_2, \dots, q_n) = \xi^{(n-1)d - nd_s} (\mathcal{I}_A)^n F(\xi q_1, \dots, \xi q_n) \quad (7.11)$$

where ξ is correlation length and $d_s \equiv \frac{1}{2}(d-2+\gamma)$. $\mathcal{I}_A$ = some constant. In field theory d_s = "anomalous dimension".

Up to now $\mathcal{W}^2$ have used $b=2$. If (discrete xfm) Now they let it be continuous again (as in Barker). So $q' = (\epsilon^t)q$, i.e., $b = \epsilon^t$. Then

$$\sigma_q = \xi(t) \sigma_{q'}. \quad \text{We then have more technical detail.}$$

7.2 Scaling Theorems for n-spin correlation fctns. Derivation of

8. Feynman Graph Calc. of Critical Exponents. Barker is clearer on this. Computes $u_R \equiv \mathcal{Q}$; somehow this is needed to get the exponents.

Comments on this review up to section 9: The opening section on the philosophy was pretty good, the introductory sections defining exponents were good. Kadanoff description was helpful, then the highlight was the discussion of the Gaussian Model. In Section 4 the authors ventured into the " s^4 model" where things suddenly became very complicated. At that point they went so heavily into technical detail that it was no longer profitable for me to study the paper carefully. They go into the recursion formula with the Q_1 and so on, and discuss technical details of the speed with which H goes to H^* fixed point. In Section 8 the idea for computing the critical exponents via the correlation functions to higher order in ϵ is discussed and the results are summarized.

The sections 9 and 10 still to come are supposed to say something about quantum field theory. These are the reasons I have born with this paper so far: I am hoping Wilson and Kogut will say something revelational about the connections to field theory. We shall see soon:

9. Dimensions of Tensor Operators in $4-\epsilon=d$. This section simply means nothing to me. Consider, they say, the expectation value of two spin operators $s(x)$ and $s(y)$ inside the $[\quad]$ along with some operator called T which is a double gradient sandwiched between two fields at $x=0$. This operator means nothing to me whatsoever. Regardless of what this operator T is, maybe I can figure out what the hell they are trying to say about it ?? They are unable to compute the "anomalous dimension" for this operator, called d_T . Since "anomalous dimension" also has no significance for me, this whole section is a total loss. Lets move on.

10. Connection Between Stat Mech and Field Theory. See Guber/Migdal and Polyakov, 1968-71. See also Moore. We will have to take field theory to a lattice to show connections

a) Lets show first how the transfer-matrix arises. Same as early description I read and $D = \text{tr}(V^N)$. Now, $V(S, S')$ is non-local, non-additive in S 's, so cannot be called a Ham. Txy. $H = -\ln V / \epsilon =$ a sum over lattice sites, clausi. Now look at a ϕ^4 field theory. Put it on a lattice of spacing a , so $\phi(x) \rightarrow \phi_m$. Lattice provides a UV cutoff, in effect, $\Lambda = a^{-1}$. Then you get

$$\langle \phi | e^{-H\tau} | \phi \rangle \propto \exp \sum_m \{ \text{stuff} \} / \text{for small } \tau \text{ you don't need full path integral}$$

Compare this to stat mech, what corresponds?

$$S_m \leftrightarrow \sum \phi_m$$

$$D_m(t) = \langle 0 | T \phi_m(t) \phi_0(0) | 0 \rangle$$

$$D_m(-i\tau) \xi^2 \leftrightarrow \Gamma_{m,m}$$

spin correl fun.

So there is some connection between the two ideas. As $a \rightarrow 0$, however, $\xi \rightarrow \text{finite}$. Then $\mu_R = \text{renorm mass} = 1/(\xi a) \rightarrow \infty$. Not very healthy.

Gaussian model has a double-pole $\Rightarrow$ indefinite metric. The research idea here is to construct an interesting (lattice) field theory starting from some stat mech theory.

11. Renorm Group Equations Completely opaque. Smooth boundary is made to replace sharp boundary in $0 \leq \frac{1}{2} \leq 1$ stuff. The method of "incomplete integration". Then t is the new continuous rescaling parameter, actually

$$\sigma_{q'} = e^{-dt/2} \sigma_q \quad \text{where} \quad q' = q e^t.$$

The renorm group differential equation seems to derive from saying that the 3-stage process of the RG cannot depend on t ; i.e., something is independent of t . General form

$$\frac{\partial \mathcal{H}_t}{\partial t} = \int dq' \left(\frac{1}{2} q' + \vec{q}' \cdot \vec{\nabla} q' \right) \frac{\delta \mathcal{H}_t}{\delta \sigma_{q'}} + \dots$$

Rest of section is tech. detail.

12.1 Topology of renorm group transformations

Wait: This says $\frac{\partial \mathcal{H}_t}{\partial t} = U[\mathcal{H}]$, just tells you how $\mathcal{H}_t$ changes with t in differential form. Integrate to get $\mathcal{H}_t = \dots$. The full result. Then see if $\mathcal{H}_{t+\infty} \rightarrow \mathcal{H}^*$. OK, no mystery about $\partial \mathcal{H}_t / \partial t = U[\mathcal{H}_t]$. $\mathcal{H}_t = \mathcal{H}^*$ means $U(\mathcal{H}^*) = 0$ so $\partial \mathcal{H}_t / \partial t = 0$.
i.e., $\partial \mathcal{H}^* / \partial t = 0$.

This topology section is more comprehensible than preceding sections. You consider the space of parameters which someone called K . This space is called S , and might have axes u and r for gaussianlike models. This is the space where things happen.

For example, each point in this space has some value of correlation length, so you could draw contours of constant correlation length through S . The one with $\xi = \infty$ is the surface of criticality, or simply the critical surface. (in S). Other interesting surfaces within S might be: the "canonical surfaces", i.e., the locations of simple standard models in this space. Or surfaces of some kind of symmetry.

The points in the space which are labelled by parameters are called "interactions" because a Hamiltonian H is labelled by its parameters, eg, $H(K)$ or $H(r, u)$. If you start at some point in the space, some H_0 , as you take t to infinity you might go along a trajectory and end up at a fixed point lying on curve $\xi = \infty$ or maybe $\xi = 0$.

Analogy is made with a rolling ball.

p/66. 12.2. Fixed Points, Subspaces, + Renormalization In this complicated discussion, the

act of renormalizing a field theory with a fixed physical mass is interpreted as motion through the space S . The idea is to end up at a point with infinite correlation length. You want to choose $\Lambda_0 = (\text{mass}) \times (\text{correlation length})$ holding that mass fixed. You end up with your "renormalized theory" at a point in the space S on the critical surface where $\xi = \infty$ and therefore $\Lambda_0 = \infty$. This final point must also lie along a "canonical surface" which is the form your simple field theory takes, model adjustment of its parameters. The act of renormalizing takes you along this canonical surface towards the desired point on the critical surface.

Thus, criticality is associated with taking the cutoff to infinity.

Next subject is: what happens if you have several fixed points. Which one does a starting theory H_0 evolve toward. Discussion of "tri-critical" and all that. Section ends with discussion of symmetry which I skip.

12.4 Fixed points and anomalous dimensions. Aha. "Anomalous dimension" refers to

the eigenvalues of the "critical operators" when you linearize your RG transformation.

I think what Wilson calls the anomalous dimension $d_m = d - \gamma_m$ where γ_m is the notation of Barber. Thus, if γ_m is negative, d_m is greater than d and you have irrelevant operator. Also, $b = e^{\epsilon}$. For each relevant operator in the theory, the fixed point has another degree of instability! Seems right. Remember that all the relevant operator "fields" have to vanish at the fixed point.

Section 13: This final section deals with this question: we have seen in the case of

ϕ^4 theory (gaussian model in stat mech) that there is a trivial fixed point in $d=4$

which corresponds to no interaction. When you start with a theory with slight interaction and you renormalized it, you are doing renormalization group transformations toward the renormalized theory with infinite cutoff. The triviality of the fixed point is that the ϕ^4 coupling vanishes, so you have no interaction.

Question: if you start with a large coupling constant, is it possible to end up at another, non-trivial fixed point where there is some interaction ϕ^4 ? The search for such a non-trivial fixed point has so far been futile.

14. Conclusion. A suggestion: just as criticality is a border between single phase

and dual phase region, maybe you should think of hadron physics in that way. To

learn about a phase, you need not be at critical point, ie, you can deal with a theory with a strong cutoff. Maybe hadrons and quarks are different phases of the same system and we have not been able to get to the quark phase because some parameters are not right.



Shang-keng Ma '73

Introduction to the Renormalization Group

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The basic idea of the renormalization group is introduced and illustrative examples are presented. Emphasis is put on the application to the theory of critical phenomena. This article is prepared for pedagogical purposes. It is written at a level that a second-year graduate student in physical sciences can understand. No previous knowledge of critical phenomena or field theory is needed. We make no attempt to survey the field or cover a wide range of subjects. On the contrary, we limit the scope to the most basic aspects. We choose to elaborate at length to make this basic idea clear and the definitions precise, and to go through the examples very carefully. We feel that once these basic aspects are understood, there will be no difficulty in confronting the rapidly expanding literature on this subject.

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I. INTRODUCTION

The basic idea of the renormalization group is easy to understand but the mathematical complication involved tends to cover it up. Early advances of the

renormalization group idea appeared in relativistic field theory [Gell-Mann and Low (1954)] and in the theory of critical phenomena [Kadanoff (1966)]. More recently, Wilson (1971-1973) has made important progress in bringing the idea into useful and concrete concepts and successfully applied them to different areas of physics. So far the most successful application has been to the theory of critical phenomena. Existing knowledge of critical phenomena has been very helpful in understanding the renormalization group as well. Work in this area has been expanding very rapidly. However, the name 'renormalization group' together with the mathematical complexity going with it has made the subject appear mysterious to many people. The purpose of this article is not to review what has been done. We attempt to introduce this subject at a very elementary level. We shall elaborate only on the most basic ideas and offer simple examples. A more complete coverage at an advanced level does exist [Wilson and Kogut (1972)]. Complete references on the renormalization group can be found there and will not be given here. At the end of this article, we shall give a short guide to the more recent work on this subject.

The conventional formulation of the renormalization group in relativistic field theory [see Bogoliobov and Shirkov (1959), for example] will not be discussed here. The picture and formulation introduced here is based on that put forth by Kadanoff and Wilson. The basic idea behind the two formulations is the same and justifies the same name, even though the two appear very different. The latter is more general and more easily visualized; in our opinion.

We want to do two things in this article. First, we introduce the basic idea and give a precise definition of the renormalization group. Second, we go through two examples. The basic idea is simple and a precise definition is not difficult although it takes a lot of words. The real difficulty is that the properties of the renormalization group are completely unclear from the definition and no classification scheme or rigorous theorem is available. Our general understanding is rather poor at

Ma

(ow)

25 pages

"Intro to the Renorm. Group."

copy made.

RMP 45, 589 (1973)

I. Intro. + Outline: They will accept critical phenomena. For a more advanced review, we are told to see a recent Physics Report (1972), by Wilson + Kogut of IAS. This thing is more complete + has full references.

There are really two ways to work:

a) formulation of Kadanoff + Wilson

b) rel. field theory form of, say, Bogoliubov

Ma sees a) as more general and clearly visualized.

Two examples will be worked through:

1) spherical model

2) small- ϵ case, with graphs ($d = 4 - \epsilon$)

A. The Role of Min. Length. The minimum length Λ^{-1} is introduced. Eg. if you are doing atomic physics, maybe $\Lambda^{-1} = \text{few fermis}$. Anything happening on a scale smaller than Λ^{-1} (nuclear physics) is summarized in some parameters in your equations (nuclear charge, mass, moments). The scale of atomic physics is then $\gg \Lambda^{-1}$. The object Λ^{-1} determines even the form of your equations of the physics. If you extend Λ^{-1} to 1. micron, then new equations are hydrodynamics, new parameters are like viscosity + depend on $< \Lambda^{-1}$ physics (atomics).

In these examples, Λ^{-1} was not quantitative and the behavior of the equations as functions of Λ^{-1} was discontinuous. In the crystal spins example, however, you consider a block of size Λ^{-1} that contains lots of spins. Somehow this block acts as a unit + interacts with other blocks, and physics is continuous in parameter Λ^{-1} .

B. Features of Critical Phenomena + Scaling. At critical points, certain physical quantities diverge. Example: ferromagnetism: At the critical temperature, the magnetic susceptibility, $\mu \equiv B/H_p$ goes infinite as the stuff spontaneously develops a B with no H. In fact it goes as $(T - T_c)^{-\gamma}$ where $\gamma \approx 1.3$. This γ is a critical exponent; there are lots of them.

The scaling idea is that there exists some correlation length ξ which is itself blowing up at T_c as, say, $|T - T_c|^{-\nu}$. One reason all other things blow up is that they are functions of this length ξ . Eg, in ferros, ξ is the length over which fluctuations in μ are observed.

Now, given that $\xi \sim |T - T_c|^{-\nu}$, how would we find the dependence on $|T - T_c|$ of some function $f(\xi)$?

Example: let $f(\xi)$ be amount of something per volume.

Thus:

$$F = \text{total amount in block} = f(\xi) \times (L^3).$$

We assume that this density f depends only on this length ξ .

Now if we simply rescale by factor s , no physics is changed and F is the same, but now:

$$F = f(\xi/s) (L/s)^3$$

$$\text{Thus: } f(s)/s^3 = f(\xi/s)/s^3 \rightarrow f(\xi) = s^3 f(\xi/s)$$

This equation comes merely from the fact that ξ is a length and f a density = sort of a dimensional analysis.

$$\text{Thus, if you scale by } \xi: f(\xi) = \xi^3 f(1)$$

$$\text{Therefore, near critical } T, f(\xi) \rightarrow \xi^3 = |T - T_c|^{-3\nu}$$

Quite amazing! This says that if ξ blows up, then any density inside depends only on ξ must vanish! Good.

C. Outline:

- II Formalism
 - III Large s and Critical Exponents
 - IV Example 1: large n (spherical model)
 - V Graph Expansion
 - VI Example 2: small ϵ
 - VII How to calculate those crit. expts in pert. theory
 - VIII Summary + Refs
- } all basic ideas.

II. The Renorm Group Defined. Again, the well accepted statistical mechanics type stuff, but quantum field is also possible. I suspect the classical field should come first.

A. Digression on Trivia. Well, suppose $P'(y_1) \equiv \alpha P(\alpha y_1)$. Then

$$\langle y_1 \rangle_P = \int dy_1 P(y_1) y_1 \quad \text{I don't yet really}$$

$$\langle y_1 \rangle_{P'} = \int dy_1 \alpha P(\alpha y_1) y_1 = \frac{1}{\alpha} \langle y_1 \rangle_P \quad \text{see why he added this.}$$

B. Model and Notation. A Lattice, dimension d , volume L^d . If $L=50$, then there are 50 spacings. Now, at each site, let's say there is a field or spin vector:

$$\vec{\phi}(x) = \begin{bmatrix} \phi_1(x) \\ \phi_2(x) \\ \vdots \\ \phi_n(x) \end{bmatrix}$$

hence, L^d points.

Each component can be further decomposed as:

$$\phi_i(x) = \frac{1}{L^{d/2}} \sum_k \phi_{ik} e^{ik \cdot x}$$

our variables

$$n \cdot L^d = \# \text{ variables}$$

The system has a Hamiltonian H_{micro} , and a temperature T . It is a function of all the variables $\{\phi_{ik}\}$.

Now somehow the field at $x=0$ overlaps the field at $x=x_1$, so define:

$$G(k) \delta_{ij} = \int dx \langle \phi_i(x) \phi_j(0) \rangle e^{-ik \cdot x} = \langle |\phi_{ik}|^2 \rangle \delta_{ij}$$

correlation fn. $\rightarrow$

I think that $P_{\text{micro}} \{\phi_{ik}\}$ is the probability that variables take on the values shown. My own example:

$$H_{\text{micro}} = \phi_{11}^2 + \phi_{12}^2 + \phi_{13}^2$$

$$P(\phi_{11}, \phi_{12}, \phi_{13}) \sim e^{-[\phi_{11}^2 + \phi_{12}^2 + \phi_{13}^2]/T}$$

In this case, the larger the spin components, the less likely they are. Averages:

$$\langle f \rangle = \int d\phi_{ik} P(\phi_{ik}) f$$

So maybe $\langle \phi_i \rangle$ = average value of 1 component of spin for the system.

Suppose H — a perturbation — is turned on

$$H_{\text{micro}} = H_0 = \int dx \phi_i(x) H$$

$$\text{Then: } \langle \phi_i \rangle = \int d\phi_{ik} P(\phi_{ik}) \cdot \sum_k \phi_{ik} e^{ik \cdot x} \text{ etc etc}$$

This stuff is a little vague so far.

C. The Idea of an Effective Hamiltonian. If we only care about large k^{-1} stuff, we should be able to bundle up the large k stuff into some parameters which then appear in an effective (for large k^{-1}) Hamiltonian.

We will look, then, for expectation values of things that do not depend on large k . So

$$\langle f \rangle = \int_{\text{small } k} \int_{\text{large } k} P(\text{small } k, \text{large } k) f$$

$$= \int_{\text{small } k} f \cdot \underbrace{\int_{\text{large } k} P(\text{small } k, \text{large } k)}_{\text{effective } P}$$

So, writing out P , we get

$$\prod_{k > \Lambda} \int d\phi_{ik} e^{-H_{\text{full}}/\pi} \equiv e^{-H(\Lambda)/T}$$

$\xrightarrow{\text{integrated out:}}$ $\xrightarrow{\text{applied } H(\Lambda)}$ $\xrightarrow{\text{here is our effective and continuously cutoff dependent Ham.}}$

Claim is that the "full" H , and H_{eff} look very different.

Any fluctuations shorter than Λ^{-1} do not appear at all in $H(\Lambda)$.

D. The Parameter Space. Now think more generally but still in terms of fields/variables ϕ_{ik} . But still, only $|k| \leq \Lambda$ appear in our $H(\Lambda)$. This is a sphere in k space.

Now for each possible distribution $P\{\phi_{ik}\}$, make a point μ in some "parameterspace". Example:

$$P \sim e^{-\mathcal{K}} \quad \mathcal{K} = \int d^d x \left[a(\nabla\phi)^2 + t_0\phi^2 + \frac{1}{2}\mu_4\phi^4 \right]$$

$$= \sum_{ik} |\phi_{ik}|^2 (t_0 + ak^2) + \frac{1}{L^d} \sum_{i,j,p,q} \mu_4 \phi_{i,p} \phi_{j,q} \dots$$

This distribution is completely characterized by 3 parameters, so here

$\mu = (a, t_0, \mu_4)$ [Note: so called "Landau-Ginzburg" form.]

One can write a marginal $\mathcal{K}$ as follows:

$$\mathcal{K} = \sum_{k_1} \sum_{k_2} \phi_{k_1 k_2} \times u_2(k_1)$$

$$+ \sum_{k_1, k_2, k_3} \phi_{k_1 k_2 k_3} \times u_3(k_1, k_2, k_3)$$

and so on

Comparing the samples from we see that:

$$u_2(k) = (1 + \alpha k^2) \delta_{k,0}$$

So even u_2 has 2 parameters. In general, you need all parameters:

$$\mu = (\mu_1, \mu_2, \mu_3, \dots)$$

each entry = a set of parameters

So now we know how to describe a distribution by its parameters

5. The Renormalization Group

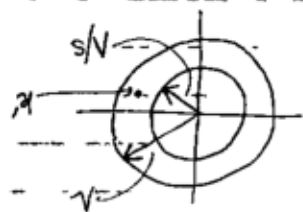
Now we want to do a transformation from one distribution $P(\mu)$ to another called $P'(\mu)$. Exactly how

will $\mathcal{M}$ do this depends on the sphere. It seems to be out-integrating some more

variables. We shrink the sphere, but we also rescale the fields

appearing in $\mathcal{K}$:

$$\phi_i k \rightarrow \alpha_i \phi_i k$$



The region where things are still integrated is

$$|k'| < \frac{1}{N} \quad \text{or} \quad s k' < 1$$

But the subscript on ϕ is raised to $\phi_i(x)$

Claim: inside the smaller sphere, both distributions are "equivalent"

up to const. factors:

$$P \approx e^{-\mathcal{K}(\phi)}$$

where $|k| < 1$

$$P' \approx e^{-\mathcal{K}'(\phi)}$$

where $|k| < \frac{1}{N}$

$$P' = e^{-\mathcal{K}'(\phi)} = \left[\sum_{k_1, k_2} \prod_i \phi_{k_1 k_2} \right] e^{-\mathcal{K}}$$

So

- 1) integrate out some more variables
- 2) rescale $k \rightarrow sk$
- 3) multiply all ϕ by α

spaces, this does not matter. Just forget about all variables outside, or in (2.2)

OK, now I see why he did that "trick". Compare:

$$\langle |\phi_{ik}|^2 \rangle_P = \alpha_s^2 \langle |\phi_{i(\alpha s)}|^2 \rangle_{P'}$$

Here, those fields which appeared in K as ϕ_{ik} , now appear in P' as $\phi_{i(\alpha s)}$.

$$\alpha_s^2 \langle |\phi_{i(\alpha s)}|^2 \rangle_{P'} = \alpha_s^2 \int d\phi_{ik} P'[\{\alpha_s \phi_{ik}\}] \times |\phi_{i(\alpha s)}|^2 \quad ?$$

↖ not clear here

Maybe should be:

$$\alpha_s^2 \langle |\phi_{i(\alpha s)}|^2 \rangle_{P'} = \alpha_s^2 \int d(\alpha_s \phi_{ik}) P'[\{\alpha_s \phi_{ik}\}] |\phi_{i(\alpha s)}|^2$$

$$= \int d\phi_{ik} P'[\phi_{ik}] |\phi_{ik}|^2 \quad \text{relabelling}$$

$$= \int d\phi_{ik} P[\{\phi_{ik}\}] |\phi_{ik}|^2$$

$$= \langle |\phi_{ik}|^2 \rangle_P \quad \text{↖ same prime since}$$

A bit hazy, but lets proceed anyway.

Recall that $G(k) = |\phi_{ik}|^2$

Let $G(k, \mu) = \langle |\phi_{ik}|^2 \rangle_P$ μ labels the dist. P

Then: $G(k, \mu) = \alpha_s^2 G(sk, R_{\alpha s} \mu) = \mu', \text{ describes } P'$

Note: P' is a distribution in fewer variables. [namely, fewer by the # in that spherical shell]

Now the set of operations $R_s: PS \rightarrow PS$ form a group. It has one parameter s whose range is $(1, \infty)$.

But if you want this to really be a group, and in particular a group with a multiplicative parameter.

$$R_s \circ R_{s'} = R_{s s'} \quad \text{↖ product}$$

Then you require $\alpha_s \alpha_{s'} = \alpha_{s s'} \rightarrow \alpha_s = s^y, y = \text{some power}$.

Then: $\phi_k \rightarrow S^y \phi_{sk}$ $y = \text{dimension of object } \phi_k$ ($y=3 \Rightarrow \phi = \text{volume, eg.}$)

Now define a fixed point in parameter space $\mu = \mu^*$ by

$$R_s \mu^* = \mu^* \quad \forall s!$$

A solution will only exist if the dimension y in $S^y = \phi_k$ is exactly right. Hence the idea that this is some kind of eigenvalue equation.

$$R_s(\mu^*) = \mu^* \quad \begin{matrix} \text{eigenvalue} \\ \text{eigenvector} \end{matrix}$$

Assume a solution exists for $y = 1 - \frac{1}{2}\epsilon$

Then:

$$G(k, \mu) = S^{2-\epsilon} G(sk, R_s \mu) \quad k < \frac{\Lambda}{s}$$

This is a "nothing" equation for the correlation function.

Now generalize the correl. fun:

$$G_{ij}(k, \mu) = \int d^d x \langle \phi_i(x) \phi_j(0) \rangle e^{-ik \cdot x}$$

$$G_{i_1 i_2}(k_1, \mu) = \int d^d x_1 e^{-ik_1 \cdot x_1} \langle \phi_{i_1}(0) \phi_{i_2}(x_1) \rangle_P$$

$$\begin{aligned} G_{i_1 i_2 i_3}(k_1, k_2, \mu) &= \int d^d x_2 d^d x_3 e^{-ik_2 \cdot x_2} e^{-ik_3 \cdot x_3} \langle \phi_{i_1}(0) \phi_{i_2}(x_2) \phi_{i_3}(x_3) \rangle_P \\ &= \int d^d x \langle \phi_{i_1}(0) \phi_{i_2}(x) \phi_{i_3}(x) \rangle_P \end{aligned}$$

Comment: The objects ϕ_k are not very clear to me. In this presentation they are abstract fields on the sphere $|k| < \Lambda$, and the i labels the components of the field. These correl coeffs. sure look like propagators & green's functions.

F: R_s as a "refined" scale transformation. In (2.2), when you do your rescaling $k \rightarrow sk$, you proceed this move by "shrinking the cutoff sphere". This "shrink" process means integrating over any extra variables

Example: $[0, 1, 2, \dots, 10] \leftarrow$ let this be initial range of k . We have already removed all $k > 10$ details. So, we have:

$$P = \int e^{-\mathcal{H}(k_1, k_2, \dots, k_{10})} \text{d}k_1 \dots \text{d}k_{10} \quad \text{or} \quad P = \int e^{-\mathcal{H}(k)} \text{d}k \quad \text{where } |k| < 10 = \Lambda$$

Now shrink the $\Lambda=10$ sphere to $\Lambda=5$ by using an $S=2$. This means

$$P' = \int \text{d}k_1 \text{d}k_2 \dots \text{d}k_5 e^{-\mathcal{H}(k_1, \dots, k_5)} = \int e^{-\mathcal{H}(k_1, \dots, k_5)} \text{d}k_1 \dots \text{d}k_5$$

start over.

New Example: Let $k = 1, 2, \dots, 10$ be discrete. Let y_k be some variables.

We start off with:

$$P = e^{-\mathcal{H}(y_1, y_2, \dots, y_{10})} = P(y_1, \dots, y_{10})$$

Now integrate out some by shrinking $|k|$ sphere down to $|k| \leq 5$:

$$P' = e^{-\mathcal{H}'(y_1, \dots, y_5)} = \int dy_6 dy_7 \dots dy_{10} e^{-\mathcal{H}(y_1, \dots, y_{10})}$$

Now we have $|k| \leq 5$. The cutoff has been changed + we have reduced the number of variables. Now scale all variables up in their index:

$$P' = e^{-\mathcal{H}'(y_2, y_4, y_6, \dots, y_{10})} \quad b' = sb$$

Now the new k is again $|k| \leq 10$. But we have removed some variables.

So, the multiple integral stage allows the rescaling to be done in a way that keeps the cutoff fixed. So

$$\mu' = R_s \mu \rightarrow \text{This operation is that mult int followed by rescale + relabel.}$$

This, μ' and μ are defined with the same cutoff!

5. Wilson's Recursion Formula: He says let $S = 2, 4, 8, 16, \dots, 2^l$ rather than continuous range. Now, to know any R_S , you only need R_2 since

$$R_S = R_2 \cdot R_2 \cdot R_2 \dots R_2 = (R_2)^l$$

The problem Wilson solved then was finding $R_2 \mu$ on an arbitrary μ . He used some fancy "wave packet" variables.

i.e., if you have an arbitrary wave:

$$\phi(x) = \sum_k \phi_k e^{ikx} \quad \phi_k = \int dx \phi(x) e^{-ikx}$$

There are $\{\phi_k\}$ as variables over k . Instead of going to continuous x , he went to a lattice x :

$$\phi(x_m) = \sum_k \phi_k e^{ikx_m} = \sum_k \int dx \phi(x) e^{ik(x-x_m)}$$

↑ summed over shell $\frac{1}{2}$ to 1.

Well, also know

4. Technical Remarks

1. Smoothed entropy - use our standard math problems which will be clear later in graph theory why so.
2. Real $\phi(x) \Rightarrow \phi_{i,k} = \phi_{i,k}^*$, so take as pairs.

III. Behavior at large s , and critical exponents

Claim: $\mu = \mu(T)$, since prob distribution depends on T . When T is near some T_c , the claim is that μ is near a fixed point μ^* (fixed for large s). Near μ^* , we will try to linearize things.

OK, suppose $\mu \approx \mu^* + \delta\mu$.

$\mu' \equiv \mu^* + \delta\mu'$. Then $\mu' = R_S \mu \rightarrow \delta\mu' = R_S \delta\mu$.

Now $\delta\mu$ is small, but has many components $\mu = (a_0, a_1, a_2, \dots)$. Thus, to first order, $R_S = R_S^L$, a matrix.

$$\begin{bmatrix} \delta\mu' \\ \vdots \end{bmatrix} = \begin{bmatrix} -R_S^L \\ \vdots \end{bmatrix} \begin{bmatrix} \delta\mu \\ \vdots \end{bmatrix}$$

Prop: Since $R_S R_S^L = R_S^L$ by group property, and if $R_S e_j(s) = \lambda_j(s) e_j(s)$, so $\lambda_j(s)$ are eigenvalues. Then:

$$\begin{aligned} R_{SS'} e_j(SS') &= \lambda_j(SS') e_j(SS') \\ &= R_S R_S' e_j(SS') \end{aligned}$$

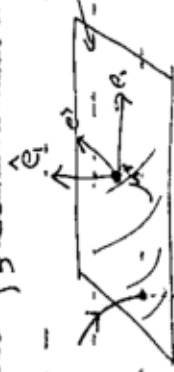
If the eigenvalues are s-independent, then $\lambda_j(SS') = \lambda_j(s) \lambda_j(s') \Rightarrow \lambda_j(s) = S_j^{y_j}$. So write:

$$\begin{aligned} \delta\mu' &= \sum_j t_j e_j \quad \text{Then} \quad R_S \delta\mu = \sum_j t_j (R_S e_j) = \sum_j t_j \lambda_j(s) e_j \\ \delta\mu' &= \sum_j t_j S_j^{y_j} e_j \approx t_1 S_1^{y_1} e_1 + O(S_1^{y_2}) \end{aligned}$$

If $y_1 > 0, y_2 < 0$, then? Now t_1 is one piece of $\delta\mu$.

$$\mu - \mu^* = \delta\mu = t_1 e_1 + t_2 e_2 + t_3 e_3 \dots$$

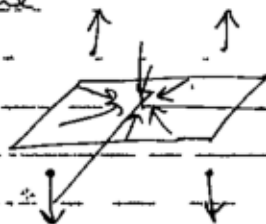
Assume $y_1 > 0, y_2 < 0$, then in the (e_1, e_2) space. [parameter space], we have:



Consider the "trajectory" parametrized by temperature T . As $T \rightarrow T_c, t_1 \rightarrow 0$ which means $\delta\mu \rightarrow 0 \Rightarrow \mu \rightarrow \mu^*$. Thus, at T_c , you have $\mu' \equiv R_S \mu = R_S(\mu^* + \delta\mu) = \mu^* + R_S \delta\mu$ "pushed" μ to μ^* .

The question here is: what is the action of R_i on the various μ which lie near μ^* : for large S , one has:

$$\delta\mu' \approx (t_1 s^{y_1}) \hat{e}_1$$



So, anything with $t_1 > 0$ has $\delta\mu' > 0$ and

$$\mu' = \mu^* + (t_1 s^{y_1}) \hat{e}_1 \quad \text{Such } \mu \text{ are pushed up}$$

If $t_1 < 0$, get pushed down. If $t_1 = 0$, then $\delta\mu' = 0$. All points on the critical surface are pushed into μ^* !

B. Critical Exponents and g . Consider a trajectory in param space $\mu(T)$ for a T near T_c . $\mu(T_c)$ has $t_1 = 0$ and $\mu(T_c)$ is close to μ^* . Then we expect

$$t_1(T) = 0 + A(T - T_c) + B(T - T_c)^2 + \dots$$

We know that $t_1(T_c) = 0$. Then,

$$\delta\mu'(T) = R_i \delta\mu(T) \approx t_1(T) s^{y_1} \hat{e}_1 \approx A(T - T_c) s^{y_1} \hat{e}_1$$

Now apply (2.31):

$$y_1 = \frac{1}{2}$$

$$G(k, \mu(T)) = S^{2-\gamma} G(sk, \mu^* + A(T - T_c) s^{1/2} \hat{e}_1 + \dots)$$

$$\text{Now choose } s = \frac{\Lambda}{2k} \quad \text{RHS} = \left(\frac{\Lambda}{2k}\right)^{2-\gamma} G\left(\frac{\Lambda}{2k}, \mu^* + A(T - T_c) \left(\frac{\Lambda}{2k}\right)^{1/2} \hat{e}_1 + \dots\right)$$

Then near T_c we have

$$G(k, \mu(T_c)) \sim k^{-\frac{\gamma+2}{2}} \quad (3.11)$$

This tells you how the correlation function depends on k near a critical temperature.

This approx is better as $k \rightarrow 0$. The region where 3.11 applies is a sphere in k -space, & the radius depends on y_2 . The second eigenvalue

$$\lambda_2(s) = S^{y_2}$$

$$y_1 > y_2 > y_3$$

The second largest eigenvalue

Now use (3.9) again with $s = t_1^{-\nu}$ and $k = 0$:

$$G(0, \mu(T_c)) = (t_1^{-\nu})^{2-\gamma} [G(t_1^{-\nu} 0, \mu^* + t_1^{-\nu} t_1^{(\gamma)} e_1)]$$

$$G(0, \mu(T)) = t_1^{-\nu(2-\gamma)} G(0, \mu^* + e_1) \sim t_1^{-\nu(2-\gamma)}$$

As $t_1 \rightarrow 0$, get $G(0, \mu(T)) = (T - T_c)^{-\nu(2-\gamma)} = (T - T_c)^{-\gamma}$ $\gamma = \nu(2-\gamma)$

For very small k , then, this $\uparrow$ blows up like so

Where is all this in the core lengths? Try

$$\xi = |t_1|^{-\nu} \quad (3.7) \text{ now reads: } \delta\mu'(T) = \xi^\nu s^{1/2} e_1 + \dots = (\xi/s)^{1/2}$$

$$\delta\mu \approx t_1 e_1 + \text{other} = (\xi)^{-1/2} e_1 + \dots \approx \emptyset$$

$$\delta\mu = (\xi)^{-1/2} e_1 + t_2 e_2 + \dots \quad (3.4) + (3.17)$$

$$R_s \delta\mu = \delta\mu' = (\xi/s)^{-1/2} e_1 + O(s^{1/2}) \dots \quad (3.18)$$

Thus: R_s reduces the core lengths by factor s .

How'bout a recap. We identify roughly $T - T_c \sim t_1 \sim (\xi)^{-1/2}$

As $T \rightarrow T_c$ from above, $\xi \rightarrow \infty$ and the core length blows up.

We also know how other mysterious things go.

C. The Free Energy

$$\prod_{0 < s_k < 1} \int d\phi_{sk} d\psi_{sk} [red]_{1,2}$$

4 1 4 44

