

FDR's of $SO(3,1) = SL(2, \mathbb{C})$

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Index:

1. What is not in this binder?

- a. material on infinite dim UIR's of Lorentz group
- b. notes on Liubarski Reading (see group books)
- c. spin and M-functions

2. Contents; References.

Lie, general:

Stuff involving only Lie algebras in abstract, could apply to either $SL(2, \mathbb{C})$ or $SO(3, 1)$

- a) classify IR's of LG
- b) Lie objects used by GG&V
- c) the C-H theorems

$SL(2, \mathbb{C})$ matrix:

Material about the 2×2 matrices (see also Both below)

- a) the generator matrices
- b) finite group matrices
- c) the exponent theorem

$SO(3, 1)$ matrix:

Material on the 4×4 boosts, rotations, LT 's etc.

- a) convention for active matrices
- b) facts about LT 's
- c) Schwēber chapter Xerox
- d) lightlike boosts and $E(2)$
- e) how to boost in arbitrary direction
- f) misc sheets

Both: the relation between $SL(2, \mathbb{C})$ and $SO(3, 1)$; see also notes in Bade.

Complex: brick wall to cms, etc.

Finite-dim IR's: Liubarski chapter Xerox plus notes ("Spinor reps.")

Bade-Jehle: the Spinor paper, 1953 plus notes

Diracia: material and old notes on the 4×4 world, inclu chp from Liu on invariant equations.

other notes

note: $SO(3, 1)$ and $SL(2, \mathbb{C})$ are FDR's of LG
 $\uparrow d=4$ $\uparrow d=2$
 $\mathbb{C}^{2 \times 2}$ $\mathbb{C}^{2 \times 1}$
 $SO(3, \mathbb{C})$

see also: Ruhl

Lie, general

The principal aim of these conferences is to present the techniques of invariance theory which are necessary in order to approach the study of elementary particle physics. However, they try equally to give a summary of the basis of current ideas on relativistic invariance. I thank and above all Bargmann and Wigner for some discussions and communications of unpublished results.

I. The Lorentz and Poincare Groups. The Lorentz Group L is the set of real linear transformations which leave invariant the bilinear form:

$$x \cdot y = x^\alpha y_\alpha = \vec{x} \cdot \vec{y} = x^\mu g_{\mu\nu} y^\nu$$

$$G = \{g_{\mu\nu}\} = \begin{Bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{Bmatrix} ; \quad \mu, \nu = 0, 1, 2, 3$$

(The repeated greek indices will be summed without explicit indication). We write

$$\hat{x}^\mu = \Lambda^\mu_\nu x^\nu$$

Then

$$\hat{x} \cdot \hat{y} = x \cdot y \quad \text{for all } x, y$$

is equivalent to

$$\Lambda^\kappa_\lambda g_{\kappa\mu} \Lambda^\mu_\nu = g_{\lambda\nu}$$

or, with $(\Lambda^\tau)^\mu_\nu = \Lambda^\mu_\nu$,

$$\Lambda^\tau G \Lambda = G. \quad (1.1)$$

In taking the determinant of (1.1) one sees that

$$\det \Lambda = \pm 1.$$

If $\det = +1$, one says that Λ is proper; if $\det = -1$, that Λ is improper. For example, the transformations

$$1 = \begin{Bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{Bmatrix} \quad \text{and} \quad -1$$

are proper, and the transformations G and $-G$ are improper. With the aid of (1.1) one sees also that

$$(\Lambda^0_0)^2 - \sum_{j=1}^3 (\Lambda^j_0)^2 = 1 \quad (1.2)$$

and consequently, $|\Lambda^0_0| \geq 1$.

If $\Lambda^0_0 \geq 1$, one says that Λ is orthochronous; if $\Lambda^0_0 \leq -1$, one says that Λ is non-orthochronous.

The product of two transformations is given by

$$(\Lambda M)^\mu_\nu = \Lambda^\mu_\kappa M^\kappa_\nu. \quad (1.3)$$

Evidently, the product of two proper Lorentz transformations is proper. The set of these transformations is a subgroup, the proper Lorentz group we call L_+ . The product of two orthochronous Lorentz transformations is also orthochronous:

$$(\Lambda M)^0_0 = \Lambda^0_0 M^0_0 - \sum_{j=1}^3 \Lambda^0_j M^j_0 \geq \Lambda^0_0 M^0_0 - \sqrt{\sum_j (\Lambda^0_j)^2} \sqrt{\sum_j (M^j_0)^2} \geq 0$$

The last inequality follows from

$$M^0_0 = \sqrt{1 + \sum_j (M^j_0)^2}$$

which is none other than (1.2), and

$$\Lambda^0_0 = \sqrt{1 + \sum_j (\Lambda^0_j)^2}$$

which follows from (1.2) and the fact that, if Λ is a Lorentz transformation, then so is $(\Lambda)^{-1}$, and, from (1.1), $(\Lambda)^{-1} = G \Lambda^T G$. The orthochronous Lorentz transformations constitute a subgroup, the orthochronous Lorentz group $L_+^\uparrow$.

The intersection of L_+ and $L_+^\uparrow$ is a subgroup, the restricted Lorentz Group $L_+^{\uparrow}$. We are going to determine its structure in detail. A physicist must know by heart two classes of the elements of $L_+^{\uparrow}$: the pure Lorentz transformations in the direction $\vec{n}$ and speed $v = c \tanh \chi$:

$$\begin{aligned} \hat{\chi}^0 &= \cosh \chi \, x^0 + \sinh \chi \, \vec{x} \cdot \vec{n} \\ \vec{\hat{\chi}} &= [\vec{x} - (\vec{x} \cdot \vec{n}) \vec{n}] + (\sinh \chi \, \vec{x} \cdot \vec{n} + \cosh \chi \, x^0) \vec{n} \end{aligned}$$

and the rotations around an axis $\vec{n}$ by an angle θ

$$\begin{aligned} \hat{\chi}^0 &= x^0 \\ \vec{\hat{\chi}} &= [\cos \theta \, 1 + (1 - \cos \theta) \vec{n} \otimes \vec{n} + \sin \theta \, \vec{n} \times] \vec{x} \end{aligned}$$

We have this

Theorem. Each element Λ of $L_+^{\uparrow}$ can be written $\Lambda = \Lambda_1 \Lambda_2$ where Λ_1 is a pure Lorentz transformation and Λ_2 is a rotation. Λ_1 and Λ_2 are uniquely determined by Λ .

We will prove this a little later using the polar decomposition. However, it is instructive to do a direct proof. For this, one first observes that the transformation Λ maps the vector $(1, 0, 0, 0)$ onto the vector $(\Lambda^0_0, \Lambda^1_0, \Lambda^2_0, \Lambda^3_0)$. If $(\Lambda^1_0, \Lambda^2_0, \Lambda^3_0) = 0$, one has $\Lambda^0_0 = 1$, and, from (1.1), $\Lambda^j_0 = 0$ ($j=1, 2, 3$). Thus, Λ already has the form of a rotation. If this same 3-vector does not vanish, let R_1 be a rotation which takes this vector into the plus x direction. Then one has:

$$R_1 \Lambda = \begin{pmatrix} \Lambda^0_0 & \Lambda^1_0 & \Lambda^2_0 & \Lambda^3_0 \\ \sqrt{\sum_j (\Lambda^j_0)^2} & \Lambda^1_1 & \Lambda^1_2 & \Lambda^1_3 \\ 0 & \Lambda^2_1 & \Lambda^2_2 & \Lambda^2_3 \\ 0 & \Lambda^3_1 & \Lambda^3_2 & \Lambda^3_3 \end{pmatrix}$$

where the 3-vectors,

$$(\Lambda_1^0, \Lambda_2^0, \Lambda_3^0), \quad (\bar{\Lambda}_1^2, \bar{\Lambda}_2^2, \bar{\Lambda}_3^2), \quad (\bar{\Lambda}_1^3, \bar{\Lambda}_2^3, \bar{\Lambda}_3^3)$$

are orthogonal. There exists a rotation R_3^{-1} which maps these 3-vectors onto

$$\sqrt{\sum (\Lambda_j^0)^2} (1, 0, 0), \quad (0, 1, 0), \quad (0, 0, 1).$$

We then have,

$$R_1 \Lambda R_3 = \Lambda_3 = \begin{Bmatrix} \Lambda_0^0 & a \\ a & \Lambda_0^0 \\ & & 1 \\ & & & 1 \end{Bmatrix} \quad a = \sqrt{\sum (\Lambda_j^0)^2}.$$

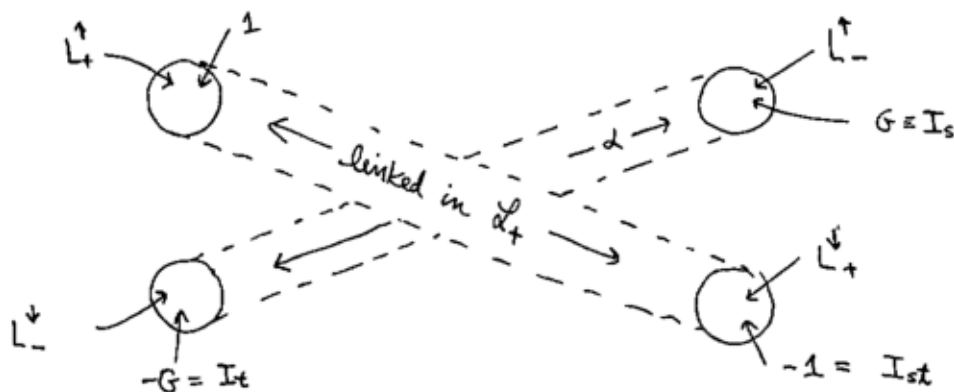
Then the Λ_1 and Λ_2 of the theorem are

$$\Lambda_1 = R_1^{-1} \Lambda_3 R_1 \quad \Lambda_2 = R_1^{-1} R_3^{-1}$$

(For the details of this proof one can consult M.A. HAUPMAK) (a book only in russian)

Exercise 1.1. Show that any orthogonal Lorentz transformation (ie, $\Lambda^T \Lambda = 1$) is of the form (1.5), or as well G times, $-G$ times, or -1 times one of the form (1.5)

From this theorem, and from formulas (1.4) and (1.5), it follows at once that the group L_+ is connected. This means that each pair of elements is linked by a continuous curve of elements. The group L has four connected components which are shown in the figure.



These connected components contain respectively

$L_+^{\uparrow}$	the identity	1
$L_-^{\uparrow}$	space inversion	I_s
$L_-^{\downarrow}$	time inversion	I_t
$L_+^{\downarrow}$	space-time inversion	I_{st}

There are three subgroups:

$$\begin{array}{ll} \text{orthochronous} & L^\uparrow = L_+^\uparrow \cup L_-^\uparrow \\ \text{proper} & L_+ = L_+^\uparrow \cup L_+^\downarrow \\ \text{(no name)} & L^\uparrow = L_+^\uparrow \cup L_-^\uparrow \end{array}$$

The last group has no name, although it is apparently the group which appears in nature.

We will also concern ourselves with the inhomogeneous Lorentz group, or, as we say, the Poincare Group (this designation has been introduced by Wigner, Bargmann, and myself; it is customary, and moreover, ~~more~~ correct from the historic viewpoint). The Poincare group is formed from transformations

$$x'^\mu = \Lambda^\mu_\nu x^\nu + a^\mu$$

which we denote (a, Λ) . The composition law is the following:

$$(a_1, \Lambda_1)(a_2, \Lambda_2) = (a_1 + \Lambda_1 a_2, \Lambda_1 \Lambda_2) \quad (1.6)$$

Like the Lorentz group L , the Poincare group has four connected pieces. One obtains them by considering (a, Λ) with a arbitrary and Λ in one or the other of the four Lorentz pieces already noted.

The Lorentz group is directly linked to $SL(2, \mathbb{C})$, the complex unimodular group of two dimensions. This group is formed of all linear transformations with unit determinant in a two-dimensional complex vector space. There is a 2:1 homomorphism from this group onto the group $L_+^\uparrow : \pm A \rightarrow \Lambda(A)$ given by

$$\Lambda(A)^\mu_\nu = \frac{1}{2} \text{tr} (\sigma_\mu A \sigma_\nu A^\dagger) \quad (1.7)$$

with the inverse (given in this form by H. Joos)

$$\begin{aligned} \pm A &= N^{-1} \left\{ \Lambda^\mu_\nu + (\Lambda^\mu_0 + \Lambda^\mu_j - i \epsilon_{jkl} \Lambda^k_l) \sigma_j \right\} \\ N &= [4 - \text{tr}(\Lambda^2) + (\text{tr} \Lambda)^2 - i \epsilon^{\kappa\lambda\mu\nu} \Lambda^\kappa_\lambda \Lambda^\mu_\nu]^\pm \end{aligned} \quad (1.8)$$

Here we have taken

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

One may simplify these formulas by using the quaternion formalism. With each 4-vector x^μ one identifies a 2x2 matrix

$$X = \sigma_\mu x^\mu \quad (1.9)$$

Clearly X is hermitian iff x is real. Moreover,

$$\det X = x \cdot x \quad x^\mu = \frac{1}{2} \text{tr} (\sigma_\mu X) \quad (1.10)$$

The homomorphism may then be written

$$X' = A X A^\dagger \quad x' = \Lambda(A) x \quad (1.11)$$

$$\Lambda(A) \in L_+^\uparrow \quad \text{cer} \quad \Lambda_0^0 = \frac{1}{2} \text{tr} (A A^\dagger) > 0, \quad \text{and}$$

$$\det \Lambda = \frac{1}{16} \det [\mathcal{L}(\bar{A} \otimes A) \tilde{\mathcal{L}}] = \frac{1}{16} \det \tilde{\mathcal{L}} \mathcal{L} \cdot \det \bar{A} \otimes A = 1$$

Here we have written

$$\sum_{\mu; \alpha\beta} (\sigma_\mu)_{\alpha\beta} \quad \sum_{\mu; \gamma\delta} (\sigma_\mu)_{\gamma\delta} \quad \left. \vphantom{\sum_{\mu; \alpha\beta}} \right\} \mu = 0, 1, 2, 3 \quad \alpha, \beta = 1, 2$$

and have used

$$\sum_{\mu=0}^3 (\sigma_\mu)_{\alpha\beta} (\sigma_\mu)_{\gamma\delta} = 2\delta_{\alpha\delta} \delta_{\beta\gamma}$$

and

$$\det(A \otimes B) = (\det A)^{\dim B} (\det B)^{\dim A}$$

Exercise 1.2. Verify that (1.8) satisfies (1.11). This is done directly but if one wants to find a formula like that which follows it can be useful.

Notation.

$$\begin{aligned} a) \quad \tau_j \tau_k &= \delta_{jk} + i \epsilon_{jkl} \tau_l \\ \tau_j \tau_k \tau_l &= \tau_j \delta_{kl} - \tau_k \delta_{jl} + \tau_l \delta_{jk} + i \epsilon_{jkl} \\ \tau_j \tau_k \tau_l \tau_m &= \dots \end{aligned}$$

Here, ϵ_{ijk} is the totally antisymmetric tensor with $\epsilon_{123} = 1$.

b) If $A = a^0 + \vec{a} \cdot \vec{\sigma}$, then

$$\begin{aligned} \Lambda(A)_k^j &= \delta_{jk} (|a|^2 - \vec{a} \cdot \vec{a}) + a_j \bar{a}_k + \bar{a}_j a_k + i a^0 \epsilon_{jkl} \bar{a}_l - i \bar{a}^0 \epsilon_{jkl} a_l \\ \Lambda(A)_0^0 &= |a|^2 + \vec{a} \cdot \vec{a} \\ \Lambda(A)_j^0 &= a^0 \bar{a}_j + \bar{a}_0 a_j + i \epsilon_{kjl} a_k \bar{a}_l \\ \Lambda(A)_0^j &= a^0 \bar{a}_j + \bar{a}_0 a_j - i \epsilon_{kjl} a_k \bar{a}_l \end{aligned}$$

The formulas imply $\Lambda_\mu^\mu = 4|a|^2$, and

$$\operatorname{Re} \left(\frac{\bar{a}_j}{a^0} \right) = \frac{\Lambda_j^0 + \Lambda_0^j}{\Lambda_\mu^\mu}; \quad \operatorname{Im} \left(\frac{\bar{a}_j}{a^0} \right) = - \frac{\epsilon_{jkl} \Lambda_l^k}{\Lambda_\mu^\mu}$$

Thus

$$A = N^{-1} [\Lambda_\mu^\mu + (\Lambda_0^0 + \Lambda_j^j) \tau_j - i \epsilon_{jkl} \Lambda_l^k \tau_j]$$

where N can be found from $(\det A = 1)$.

To calculate with $SL(2, \mathbb{C})$ is often more convenient than calculation with $L_+^{\uparrow}$. We illustrate this point. Each non-singular matrix A has a unique polar decomposition

$$A = UH \quad (1.12)$$

where U is unitary and H is hermitian positive (which means that the eigenvalues of H are positive). One shows directly, thanks to (1.12) that

$$A^\dagger A = H^2, \quad H = \sqrt{A^\dagger A} \quad U = A H^{-1}$$

The uniqueness of the decomposition (1.12) is a consequence of the uniqueness of the positive square root of a positive definite matrix. One can give an explicit form to these formulas:

If $A = a^0 + \vec{a} \cdot \vec{\tau}$ and $B = A^\dagger A = b^0 + \vec{b} \cdot \vec{\tau}$

one has

$$\begin{aligned} b^0 &= |a^0|^2 + |\vec{a}|^2 \\ \vec{b} &= a^0 \vec{a} + \vec{a}^0 \vec{a} + i \vec{a} \times \vec{a} \end{aligned} \quad (1.13)$$

and

$$\sqrt{B} = \frac{1}{\sqrt{2(b^0 + \sqrt{b^2})}} \left[\sqrt{b^2} + B \right] \quad (1.14)$$

To complete the proof of the above theorem, it suffices to show that $\Lambda(A)$ is a rotation if and only if A is unitary, and $\Lambda(A)$ is a pure Lorentz transformation if and only if A is hermitian. The first of these claims is a direct consequence of (1.10) and (1.11). One sees that, if

$$\begin{aligned} A A^\dagger &= 1 \\ \hat{\chi}^0 &= \frac{1}{2} \text{tr}(A \tau_3 A^\dagger) = \chi^0 \end{aligned}$$

and using the result of exercise (1.1), $\Lambda(A)$ is of the form (1.5). If A is hermitian one can, by a unitary transformation, bring it to the diagonal form

$$A = U \begin{pmatrix} e^{i\chi_1} & 0 \\ 0 & e^{-i\chi_2} \end{pmatrix} U^\dagger$$

An elementary calculation shows that $\begin{pmatrix} e^{i\chi_1} & 0 \\ 0 & e^{-i\chi_2} \end{pmatrix}$ corresponds to a Λ given by

$$\begin{aligned} \hat{\chi}^0 &= \cosh \chi \chi^0 + \sinh \chi \chi^3 \\ \hat{\chi}^1 &= \chi^1 \\ \hat{\chi}^2 &= \chi^2 \\ \hat{\chi}^3 &= \sinh \chi \chi^0 + \cosh \chi \chi^3 \end{aligned}$$

This is a pure Lorentz transformation along the z axis. $U \begin{pmatrix} e^{i\chi_1} & 0 \\ 0 & e^{-i\chi_2} \end{pmatrix} U^\dagger$ is a pure Lorentz transformation along an axis obtained by leaving the z -axis by a rotation corresponding to U .

If A is hermitian, $\Lambda(A)$ is symmetric, as (1.7) clearly shows. Reciprocally, if Λ is symmetric, (1.8) gives a hermitian matrix A .

There are two automorphisms of $SL(2, \mathbb{C})$ which are important in practice (recall that an automorphism is a bicontinuous one-to-one of the group onto itself which respects the law of multiplication):

$$A \rightarrow (A^T)^{-1} \quad \text{and} \quad A \rightarrow \bar{A}$$

The first has the property

$$(A^T)^{-1} = \xi A \xi^{-1} \quad \Rightarrow \quad A^T \xi A = \xi \quad \text{where} \quad \xi = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad (1.15)$$

This (1.15) is a simple consequence of the structure of the τ matrices:

$$\tau_j^T = -\tau_j \quad \tau_1^T = \tau_1 \quad \tau_2^T = -\tau_2 \quad \tau_3^T = \tau_3$$

$$\tau_j \tau_k = i \sum \epsilon_{jkl} \tau_l \quad j \neq k$$

In fact,

$$(\alpha^0 + \vec{\alpha} \cdot \vec{\tau})^T = \alpha^0 + \alpha_1 \tau_1 - \alpha_2 \tau_2 + \alpha_3 \tau_3$$

$$(\alpha^0 + \vec{\alpha} \cdot \vec{\tau})^{-1} = \alpha^0 - \vec{\alpha} \cdot \vec{\tau}$$

(do not forget that $(\alpha^0)^2 - \vec{\alpha}^2 = 1$ by hypothesis)
and

$$\tau_2 (\alpha^0 + \vec{\alpha} \cdot \vec{\tau})^{-1} \tau_2 = \alpha^0 - \alpha_1 \tau_1 + \alpha_2 \tau_2 - \alpha_3 \tau_3$$

The image of this automorphism in $L_+^\uparrow$ is

$$\Lambda \rightarrow \Lambda(\xi) \Lambda \Lambda(\xi^{-1})$$

with

$$\Lambda(\xi) = I_1 I_3 \quad (1.16)$$

the product of the reflection symmetries in the 3-planes orthogonal to the axes x and z respectively :

$$(I_j x)^j = -x^j \quad \text{but} \quad (I_j x)^k = +x^k, \quad k \neq j$$

There is no linear transformation ξ_1 having the property

$$\bar{A} = \xi_1 A \xi_1^{-1}$$

for all A. The image of the automorphism $A \rightarrow \bar{A}$ in $L_+^\uparrow$ is determined by

$$\Lambda(\bar{A})_\nu^\mu = \frac{1}{2} \text{tr}(\tau_\mu \bar{A} \tau_\nu A^T) = \frac{1}{2} \text{tr}(A \tau_\nu^T A^T \tau_\mu^T) = \frac{1}{2} \text{tr}(\tau_\mu^T A \tau_\nu^T A^T)$$

ie,

$$\Lambda(\bar{A}) = I_2 \Lambda(A) I_2$$

where I_2 is the symmetry which changes the sense of the axis x^2 .

The matrix ξ plays an important role in the spinor calculus. There, one constructs the bilinear form

$$a \cdot \xi \cdot b = \sum_i a_i \cdot \xi_{\alpha\beta} \cdot b_\beta \quad (1.18)$$

where a and b are vectors of two complex components. One has

$$(Aa) \cdot \xi \cdot (Ab) = a \cdot \bar{A}^T \xi A \cdot b = a \cdot \xi \cdot b = a^\alpha b_\alpha$$

where one defines some contravariant components related to a_β by

$$a^\alpha = \xi_{\alpha\beta} a_\beta$$

One can equally consider some more general quantities

$$a_1 a_2 \dots \beta_1 \beta_2 \dots$$

which transform according to $a \rightarrow \hat{a}$:

$$\hat{a}_1 \hat{a}_2 \dots \hat{\beta}_1 \hat{\beta}_2 \dots = A_{\alpha_1 \alpha_1'} A_{\alpha_2 \alpha_2'} \dots \bar{A}_{\beta_1 \beta_1'} \dots a_{\alpha_1'} a_{\alpha_2'} \dots \beta_1' \beta_2'$$

(the dotted indices transform with $\bar{A}$, the undotted with A). The matrix ξ is again employed to raise and lower indices. Under these conditions,

$$a_{\dot{\alpha}_1 \dot{\alpha}_2 \dots \dot{\alpha}_n} \dot{\beta}_1 \dot{\beta}_2 \dots \dot{\beta}_n b^{\alpha_1 \alpha_2 \dots \alpha_n} \beta_1 \beta_2 \dots \beta_n$$

is an invariant form. One is referred to the bibliography for more information on this formalism.

The introduction above to the group $SL(2, C)$ was not well motivated. The physical justification will appear farther down the road (sections 2 and 7), but here we shall introduce in a crude manner an important mathematical idea: that of a universal covering group of a connected group. This is defined as the "smallest" group which is simply connected and homomorphic to a given group. (A group is simply connected if all continuous closed curves can be continuously deformed to a point) (It is well known that the rotation group is not simply connected; and the same is true for the Lorentz and Poincare groups). More precisely, one considers the set of continuous curves on the group of the form

$$g(t) \quad 0 \leq t \leq 1 \quad \text{where } g(0) = \text{Identity}.$$

One introduces in this set an equivalence relation: two curves $g_1(t)$ and $g_2(t)$ are equivalent if there exists a continuous function $g(t, s)$ for $0 \leq t, s \leq 1$ such that $g_1(t) = g(t, 0)$ and $g_2(t) = g(t, 1)$, which is to say that the curve g_1 can be continuously deformed into the curve g_2 . The set of these equivalence classes constitutes the universal covering group. The multiplication law for this group is defined as follows: if $[g_1]$ and $[g_2]$ are two equivalence classes, one chooses some curves g_1 and $g_2(t)$ in these classes and forms the curve

$$g_1(t) g_2(t) \quad 0 \leq t \leq 1$$

which joins element for element $g_1(1) g_2(1)$. It belongs to an equivalence class which is, by definition, $[g_1][g_2]$. We refer to Pontrjagin's book for the proof of the fact that this definition does not depend on the classes of equivalence. One also finds there a discussion of the topology of this group.

For the case of a connected group, the universal covering group is determined in a unique manner. When the group has several disjoint connected components, one may have several non-isomorphic covering groups. Consider the case of the group L . Over the connected component of the identity, L_+ , we have constructed the group $SL(2, C)$. The other elements of L can be obtained by multiplying L_+ by the three following elements

$$I_s \quad I_t \quad I_{st}$$

such that

$$\begin{aligned} I_s x &= \xi \bar{x} \xi^{-1} \\ I_t x &= -\xi \bar{x} \xi^{-1} \\ I_{st} x &= -x \end{aligned} \tag{1.19}$$

(note: x is the 2×2 matrix I used to call X ; also, $\tau = \sigma$ Pauli).

One can write all the elements of L in the form

$$\Lambda \in L, \quad \Lambda x = \hat{x}$$

with

$$\hat{x} = C^{-1} A K^{\mu_2} x K^{\mu_1} A^{\dagger} \tag{1.20}$$

Here, μ_1 and μ_2 are mod-2 integers and K is the complex conjugation operator. One obtains the covering groups by introducing an associative law of multiplication in the set of triples (μ_1, μ_2, A) :

$$[\mu_1, \mu_2, A][\mu'_1, \mu'_2, A'] = [\mu_1 + \mu'_1, \mu_2 + \mu'_2, \epsilon(\mu_1, \mu_2; \mu'_1, \mu'_2) A K^{\mu_2} A' K^{\mu'_2}] \quad (1.21)$$

where $\epsilon(\mu_1, \mu_2; \mu'_1, \mu'_2)$ is a solution of

$$\begin{aligned} \epsilon(\mu_1, \mu_2; \mu'_1, \mu'_2) \epsilon(\mu_1 + \mu'_1, \mu_2 + \mu'_2; \mu''_1, \mu''_2) = \\ \epsilon(\mu_1, \mu_2; \mu'_1 + \mu''_1, \mu'_2 + \mu''_2) \epsilon(\mu'_1, \mu'_2; \mu''_1, \mu''_2) \end{aligned} \quad (1.22)$$

The solutions of (1.22) are

$$\begin{aligned} 1, \quad (-1)^{\mu_1 \mu'_1}, \quad (-1)^{\mu_2 \mu'_2}, \quad (-1)^{\mu_1 \mu'_1 + \mu_2 \mu'_2}, \quad (-1)^{\mu_1 \mu'_1 + \mu_2 \mu'_2} \\ (-1)^{\mu_2 \mu'_2 + \mu_1 \mu'_1}, \quad (-1)^{\mu_1 \mu'_1 + \mu_2 \mu'_2 + \mu_1 \mu'_2} \end{aligned} \quad (1.23)$$

Since all these solutions are different at least for the values of the squares of the elements above $I_{st} I_{st}$ (by definition, such an element has for a square ± 1 $SL(2, C)$) they give non-isomorphic groups. We will show that the significance of these divers possible groups is altogether different when one applies them in the theory of states or in the theory of fields. If one considers only I_{st} , one has two non-isomorphic covering groups which correspond to the two kinds of spinors introduced by Cartan.

One can give the analogous extensions of the Poincare group. One has, in this case,

$$\begin{aligned} \{a, \mu_1, \mu_2, A\} \{a', \mu'_1, \mu'_2, A'\} = \\ \{a + (-1)^{\mu_1} \Lambda(K^{\mu_2} A K^{\mu_2}) a', \mu_1 + \mu'_1, \mu_2 + \mu'_2, \epsilon(\mu_1, \mu_2; \mu'_1, \mu'_2) A K^{\mu_2} A' K^{\mu'_2}\} \end{aligned} \quad (1.24)$$

Finally, we consider the complex extensions of the Lorentz and Poincare groups. The elements of the complex Lorentz group $\mathcal{L}$ are the complex linear transformations satisfying (1.1). If one adds a complex translation a to Λ one has an element (a, Λ) of the complex Poincare group. As before, $\det \Lambda = \pm 1$, and the proper group $\mathcal{L}_+$ has those elements with $\det = +1$. It is a very important fact for applications (the TCP theorem) that $\mathcal{L}_+$ (contains $L_+^\uparrow$ and $L_+^\downarrow$) is connected. To show this, consider for example

$$\left\{ \begin{array}{cccc} \cosh \chi & 0 & 0 & \sinh \chi \\ 0 & \cos \phi & -\sin \phi & 0 \\ 0 & \sin \phi & \cos \phi & 0 \\ \sinh \chi & 0 & 0 & \cosh \chi \end{array} \right\} \in \mathcal{L}$$

for all χ , complex or real. If one varies (χ, ϕ) continuously from $(0, 0)$ to $(i\pi, \pi)$ one obtains a curve continuously linking 1 and -1 in L_+ (See figure 1.1).

The complex Lorentz group $\mathcal{L}_+$ has a universal covering group which one obtains by a simple generalization of (1.11). One can use the notation $\mathcal{Z}$ for a complex 4-vector z^μ . Then, for each pair of 2x2 unimodular matrices, the transformation

$$\hat{\mathcal{Z}} = A \mathcal{Z} B^T \quad (1.26)$$

is linear.

$\hat{Z} = \Lambda(A, B) Z$ and has the property $\hat{Z} \cdot \hat{Z}' = Z \cdot Z'$ since

$$Z \cdot Z' = \frac{1}{2} \{ \det(Z + Z') - \det Z - \det Z' \}$$

The same calculation made before in the case $B = \bar{A}$ shows that $\det \Lambda(A, B) = 1$. Evidently, the pairs (A, B) determine the same Λ . The multiplication law for pairs, dictated by (1.26), is simply $(A, B)(A', B') = (AA', BB')$ and, in consequence, their group is $SL(2, C) \times SL(2, C)$. The formula giving Λ in terms of (A, B) (analogous to (1.70)) is:

$$\Lambda(A, B)_{\mu}^{\nu} = \frac{1}{2} \text{tr} (\tau_{\mu} A \tau_{\nu} B^T) \quad (1.27)$$

The inverse is

$$\pm A = \frac{1}{N} \left[\text{tr} \Lambda(A, B) + \tau_j \left(\Lambda(A, B)_0^j + \Lambda(A, B)_j^0 + i \epsilon_{jkl} \Lambda(A, B)_l^k \right) \right] \quad \text{etc.} \quad (1.28)$$

Exercise 1.3 Demonstrate (1.28)

Exercise 1.4 Show that the pairs (A, B) arising from (1.25) by (1.28) give a continuous curve linking $(1, 1)$ to $(-1, 1)$.

In the following we will use the representations $D^{(j)}$ of the group $SU(2, C)$ and $D(j_1, j_2)$ of the Lorentz group obtained as follows: Let

$$X_j^m = \frac{S_1^{j-m} S_2^{j+m}}{\sqrt{(j-m)!(j+m)!}} \quad m = -j, \dots, j$$

with $j = \text{integer or half-integer}$. These X_j^m with j fixed form a basis for the homogeneous polynomials in S_1, S_2 of degree $2j$. If

$$\text{If } \begin{pmatrix} S_1 \\ S_2 \end{pmatrix} \rightarrow A \begin{pmatrix} S_1 \\ S_2 \end{pmatrix} \quad \text{then } X_j^m \rightarrow D_{mm'}^{(j)}(A) X_j^{m'} \quad (1.29)$$

Now let

$$X_{j_1 j_2}^{m_1 m_2} = \frac{S_1^{j_1-m_1} S_2^{j_1+m_1} S_3^{j_2-m_2} S_4^{j_2+m_2}}{\sqrt{(j_1-m_1)!(j_1+m_1)!(j_2-m_2)!(j_2+m_2)!}}$$

Then, if

$$\begin{pmatrix} S_1 \\ S_2 \end{pmatrix} \rightarrow A \begin{pmatrix} S_1 \\ S_2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} S_3 \\ S_4 \end{pmatrix} \rightarrow B \begin{pmatrix} S_3 \\ S_4 \end{pmatrix}$$

then

$$X_{j_1 j_2}^{m_1 m_2} \rightarrow D_{m_1 m'_1, m_2 m'_2}^{[j_1 j_2]}(A, B) X_{j_1 j_2}^{m'_1 m'_2} \quad (1.30)$$

This representation is defined for $SL(2, C) \times SL(2, C)$. For $SL(2, C)$ itself one must put $A = \bar{B}$. We might point out that these representations of the groups $SU(2, C)$ and $SL(2, C)$ can be extended to $GL(2, C)$, which is to say, to the group of all non-singular transformations. We will use this later.

There remains the γ -gymnastics. We want only to present the notation and some basic facts. The Dirac matrices (Clifford numbers) satisfy:

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = -2g^{\mu\nu} \quad (1.31)$$

$$(1.32)$$

The complex conjugate, adjoint, and transpose of ψ are related to ψ by

and one has also

with

The relations between $S(D)$ and A, B, C, γ^5 are

The fact that γ^5 commutes with $S(D)$ allows us to diagonalize these two matrices simultaneously. We arrive thereby at the notion of chirality. One says that the vectors $v_\pm$ satisfying

have a chirality $\pm$. A choice of appropriate basis leads to a form

and one sees that the notion of chirality conforms with that of undotted and dotted indices used in the spinor calculus.

Bibliographie

Spinor Calculus: Uhlenbeck and LaPorte Phys, Rev 37, 1380 (1931)
Bade Jehle

Representations of $SU(2, C)$ and $SL(2, C)$: some russian book
Cartan french reference

Universal covering group: Pontrjagin
Wigner book for rotations case

Covering groups: Shirkov "Space and Time reflections in Relativity theory" NP 15, 1 (1960)
also see JETP 36, 620 (1959)

Gamma gymnastique: Jauch Rorlick

Question: what are the irreps of the Lorentz group

① Schwesky is good. First, you identify the generators of your group. The homo LG [called?] has six generators $M_{\mu\nu}$, abstract group generators.

② If you want matrix representations of this group [ie, matrices will describe linear transformations of certain "vectors"], then seek sets of matrices $[M_{\mu\nu}]_{ij}$ that satisfy the algebra (30).

③ A matrix is finite in my mind. The RHLG (restricted homo, ie, the one we want) has reps that divide up: (ie, irreps).

finite infinite	no non trivial reps.	our interest lies here
	unitary	nonunitary

④ How do you construct the finite (nonunitary) reps?
a) find the casimirs of the group.

Interpretation of n_1 and n_2 in $SL(2, \mathbb{C})$.

① I will use my own generators $J_1, J_2, J_3, K_1, K_2, K_3$. Go to the 2x2 rep to get

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

$$e^{-i\phi J_3} \quad e^{-i\phi K_1}$$

$$e^{-i\phi J_3} \quad \text{or} \quad e^{-i\phi K_1}$$

	α	β	γ	δ
J_1	$\cos \phi/2$	$-i \sin \phi/2$	$-i \sin \phi/2$	$\cos \phi/2$
J_2	$\cos \phi/2$	$-\sin \phi/2$	$\sin \phi/2$	$\cos \phi/2$
J_3	$e^{-i\phi/2}$	0	0	$e^{i\phi/2}$
K_1	$\cosh \psi/2$	$\sinh \psi/2$	$\sinh \psi/2$	$\cosh \psi/2$
K_2	$\cosh \psi/2$	$-i \sinh \psi/2$	$i \sinh \psi/2$	$\cosh \psi/2$
K_3	$e^{\psi/2}$	0	0	$e^{-\psi/2}$

② Now let's construct all diff ops? No wait, my goal was to identify things. G.V. use these 6 generators:

$$A_0, A_+, A_-, \bar{A}_0, \bar{A}_+, \bar{A}_-$$

algebra: $[A_+, A_0] = A_+$ and same for $\bar{A}_i$
 $[A_-, A_0] = -A_-$
 $[A_+, A_-] = 2A_0$

③ Aside: compare to p. 289, 290 of Linbarster.

$$[A_i, A_j] = \epsilon_{ijk} A_k \quad \text{he says..}$$

$$\begin{aligned} A_+ &= iA_1 - A_2 \\ A_- &= iA_1 + A_2 \end{aligned} \Rightarrow [A_+, A_-] = [iA_1, A_2] + [-A_2, iA_1] = 2iA_3 \quad \text{damn.}$$

OK, it turns out that $\left[\begin{aligned} \Delta &= (n_1^2 - 1) \\ \bar{\Delta} &= (n_2^2 - 1) \end{aligned} \right\} \text{ of Linbarster} \quad \left[\begin{aligned} \Delta &= -2\hat{A}^2 \\ \bar{\Delta} &= -2\hat{B}^2 \end{aligned} \right]$

so $\left[\begin{aligned} n_1^2 - 1 &= -2j_1(j_1 + 1) \\ n_2^2 - 1 &= -2j_2(j_2 + 1) \end{aligned} \right\} \text{ say.}$

problem: how do we relate these two Lie Algebras for $SO(3,1)$

- 1) GGV
- 2) me...

① Start with J and K of me. Connect to Liu via:

$$\underline{J} = -i \underline{L} = -i \underline{I}^{Liu}$$

$$\underline{K} = -i \underline{B} = -i \underline{B}^{Liu} = -i \underline{J}^{Liu}$$

So:
$$\begin{cases} \underline{J} = -i \underline{I}^{Liu} \\ \underline{K} = -i \underline{J}^{Liu} \end{cases}$$

② Now get J, K in terms of A, B .

$$\begin{cases} A^1 = -\frac{1}{2}(\underline{I}^1 + i \underline{J}^1) = -\frac{1}{2}([i \underline{J}^1 + i [i \underline{K}^1]]) = -\frac{1}{2}(i \underline{J}^1 - \underline{K}^1) = \frac{1}{2}(\underline{K}^1 - i \underline{J}^1) \\ B^1 = -\frac{1}{2}(\underline{I}^1 - i \underline{J}^1) = \frac{1}{2}(-\underline{K}^1 - i \underline{J}^1) \end{cases}$$

③ But
$$\begin{cases} [A_i, A_j] = \epsilon_{ijk} A_k \\ [B_i, B_j] = \epsilon_{ijk} B_k \\ [A_i, B_j] = 0 \end{cases}$$

④ Now: how do we relate A^1, B^1 to the A^0 and A^2 ?

⑤ First, stick within the A^1 . Change to $\pm$. Assume $A_0^1 = A_3^1$.
Then ... solution is:

$$\begin{cases} A_{\pm} = \mp A_1 + i A_2 \\ A_0 = i A_3 \end{cases}$$

Then you get of GGV. $[A_+, A_0] = A_+$; $[A_-, A_0] = -A_-$; $[A_+, A_-] = -2A_0$.

⑥ So
$$\begin{cases} A_{\pm} = \mp \left\{ \frac{1}{2}(K_1 - i J_1) \right\} + i \left\{ \frac{1}{2}(K_2 - i J_2) \right\} \\ A_0 = i \left\{ \frac{1}{2}(K_3 - i J_3) \right\} \end{cases}$$

$$A_{\pm}^+ = \mp \left\{ \frac{1}{2}(-K_1 + i J_1) \right\} - i \left\{ \frac{1}{2}(-K_2 + i J_2) \right\}$$

⑦ Now, $J_i^+ = J_i$ but $K_i^+ = -K_i$ so

Restate:

$$\begin{aligned} 2A_{\pm} &= \mp (K_1 - iJ_1) + i(K_2 - iJ_2) \\ &= \mp i(-iK_1 - J_1) + (iK_2 + J_2) \\ &= \pm i(J_1 + iK_1) + (J_2 + iK_2) \end{aligned}$$

$$\boxed{2A_{\pm} = \pm i(J_1 + iK_1) + (J_2 + iK_2) \quad 2A_0 = (J_3 + iK_3)}$$

Note that: $A_{\pm}^{\dagger} = -A_{\mp}^{\dagger}$ and $A_0^{\dagger} = A_0$

⑧ I presume that $(\bar{A}_+, \bar{A}_-, \bar{A}_0) : (A_1, A_2, A_3)^{\dagger}$
or $(\bar{A}_+, \bar{A}_-, \bar{A}_0) : (A_1, A_2, A_3)^{\dagger}$

$$\text{Thus: } \boxed{2\bar{A}_{\pm} = \pm i(J_1 - iK_1) + (J_2 - iK_2) \quad 2\bar{A}_0 = (J_3 - iK_3)}$$

⑨ Short-hand: define ~~$Q_{\pm}$~~ $Q_i^{\pm} = (J_i \pm iK_i)$

These $Q_i^{\pm}$ are all hermitian. Then

$$Q_i^{\pm} = (J_i \pm iK_i)$$

Then A

⑩ 1) $Q^{\pm} = J \pm iK$

2) $A^{\text{lin}} = (-i/2) Q^+$ $B^{\text{lin}} = (-i/2) Q^-$

3) $2A_{\pm} = \pm i Q_1^{\pm} + Q_2^{\pm}$
 $2\bar{A}_{\pm} = \pm i Q_1^{\mp} + Q_2^{\mp}$
 $2A_0 = Q_3^+$
 $2\bar{A}_0 = Q_3^-$

⑪ Continue now to identify things:

$$\Delta = m_1^2 - 1 = 2A_0^2 + A_+ A_- + A_- A_+$$

$$\bar{\Delta} = m_2^2 - 1 = 2\bar{A}_0^2 + \bar{A}_+ \bar{A}_- + \bar{A}_- \bar{A}_+$$

So, calculate these guys:

$$\Delta = 2 \cdot \frac{1}{4} (\mathcal{J}_3 - i\mathcal{K}_3)^2 + \frac{1}{4} (\mathcal{J}_2 + i\mathcal{K}_2)$$

Use intermediaries:

$$\mathcal{J} + i\mathcal{K} = \mathcal{F} \Rightarrow A_+ = (\mathcal{F}_2 + i\mathcal{F}_1)^{\frac{1}{2}}$$

$$\mathcal{J} - i\mathcal{K} = \mathcal{G} \Rightarrow A_- = (\mathcal{F}_2 - i\mathcal{F}_1)^{\frac{1}{2}}$$

$$A_0 = (\mathcal{F}_3)^{\frac{1}{2}}$$

So

$$\begin{aligned} \Delta &= 2 \cdot \frac{1}{4} \mathcal{F}_3^2 + \frac{1}{4} (\mathcal{F}_2 + i\mathcal{F}_1)(\mathcal{F}_2 - i\mathcal{F}_1) + \frac{1}{4} (\mathcal{F}_2 - i\mathcal{F}_1)(\mathcal{F}_2 + i\mathcal{F}_1) \\ &= \frac{1}{2} \mathcal{F}_3^2 + \frac{1}{2} \mathcal{F}_2^2 + \frac{1}{2} \mathcal{F}_1^2 \end{aligned}$$

$$\Rightarrow \begin{aligned} \Delta &= \frac{1}{2} (\mathcal{F}_1^2 + \mathcal{F}_2^2 + \mathcal{F}_3^2) \\ \bar{\Delta} &= \frac{1}{2} (\mathcal{G}_1^2 + \mathcal{G}_2^2 + \mathcal{G}_3^2) \end{aligned}$$

$$\begin{aligned} A^1 &= (-i/2) \mathcal{F} \\ \tilde{B}^1 &= (-i/2) \mathcal{G} \end{aligned}$$

Thus:

$$[F_i, F_j] = (-i/2) \epsilon_{ijk} F_k$$

$$[G_i, G_j] = (-i/2) \epsilon_{ijk} G_k$$

$$[F_i, G_j] = 0$$

$$A_i^2 = -\frac{1}{4} F_i^2$$

$$\tilde{A}^2 = -\frac{1}{4} F^2$$

$$\Delta = \frac{1}{2} \mathcal{F}^2 = -2A^2$$

$$\bar{\Delta} = \frac{1}{2} \mathcal{G}^2 = -2\tilde{A}^2$$

⑫ So, Δ and $\bar{\Delta}$ are really $\sim j_1$ and j_2 .

$$[A_i, A_j] = \epsilon_{ijk} A_k \Rightarrow [iA_i, iA_j] = i\epsilon_{ijk} iA_k$$

So $\mathcal{J}$ is like iA . Then $\Delta = 2\mathcal{J}^2$ $\bar{\Delta} = 2\tilde{\mathcal{J}}^2$

Now this. $m_i^2 - 1 = 2j_i(j_i + 1) = (m_i + 1)(m_i - 1)$

⑬ $\Delta, \bar{\Delta}$ are really $2\mathcal{J}_A^2$ and $2\mathcal{J}_B^2$

⑭ Calculate in terms of $\underline{J}, \underline{K}$:

$$\Delta = \frac{1}{2}(\underline{E})^2 = \frac{1}{2}(\underline{J} + i\underline{K})^2 = \frac{1}{2}(\underline{J}^2 + i\underline{J} \cdot \underline{K} + i\underline{K} \cdot \underline{J} - \underline{K}^2)$$

$$\bar{\Delta} = \frac{1}{2}(\underline{G})^2 = \frac{1}{2}(\underline{J} - i\underline{K})^2 = \frac{1}{2}(\underline{J}^2 - i(\underline{J} \cdot \underline{K} + \underline{K} \cdot \underline{J}) - \underline{K}^2)$$

So,

$$\Delta + \bar{\Delta} = \underline{J}^2 - \underline{K}^2$$

$$\Delta - \bar{\Delta} = i[\underline{J} \cdot \underline{K} + \underline{K} \cdot \underline{J}]$$

Comments: so we have recovered the result that I had forgotten that the two casimirs of $SO(3,1)$ are $\underline{J}^2 - \underline{K}^2$ and $\underline{J} \cdot \underline{K}$. It turns out that Δ and $\bar{\Delta}$ as used by GGV correspond to the two "J-squared's" of the two separated algebras, that is, to what I have called F^2 and G^2 . These are proportional to the A^2 and B^2 of Liubarski. These were called P and Q I think by Liu.

However, I am

Summary:

- ① Given generators A and B , one quickly constructs:

$$B_0 = B \quad B_1 = [B, A] \quad B_2 = [B_1, A] \\ = [[B, A], A]$$

- ② If $B_2 = \pm B_0$, then we get a very simple result for:

$$e^{-iuA} B e^{+iuA} = \begin{cases} \cos u \cdot B_0 + i \sin u \cdot B_1 & \text{if } B_2 = +B_0 \\ \cosh u \cdot B_0 + i \sinh u \cdot B_1 & \text{if } B_2 = -B_0 \end{cases}$$

- ③ We found these results: ($i \neq j$)

A	$B = B_0$	iB_1	B_2/B_0	
J_i	J_j	$\epsilon_{ijk} J_k$	$+1$	} note: hyperbolics if $A = \text{a boost}$.
K_i	K_j	$-\epsilon_{ijk} J_k$	-1	
K_i	J_j	$\epsilon_{ijk} K_k$	-1	
J_i	K_j	$\epsilon_{ijk} K_k$	$+1$	

- ④ Thus, we obtained these theorems:

$e^{-iuJ_i} J_j e^{+iuJ_i}$	$=$	$J_j \cos u + \epsilon_{ijk} J_k \sin u$	✓
$e^{-iuK_i} K_j e^{+iuK_i}$	$=$	$K_j \cosh u - \epsilon_{ijk} J_k \sinh u$	✓
$e^{-iuK_i} J_j e^{+iuK_i}$	$=$	$J_j \cosh u + \epsilon_{ijk} K_k \sinh u$	✓
$e^{-iuJ_i} K_j e^{+iuJ_i}$	$=$	$K_j \cos u + \epsilon_{ijk} K_k \sin u$	✓

- ⑤ Algebra is:

$$[J_i, J_j] = i\epsilon_{ijk} J_k$$

$$[K_i, K_j] = -i\epsilon_{ijk} J_k$$

$$[J_i, K_j] = +i\epsilon_{ijk} K_k = [K_i, J_j]$$

Campbell-Hausdorff Theorem. and Loeutz group.

① It says:
$$e^{-A} B e^A = B + [B, A] + \frac{1}{2!} [[B, A], A] + \frac{1}{3!} [[[B, A], A], A] + \dots$$

Define: $B_0 = B \quad B_1 = [B, A] \quad B_2 = [[B, A], A] = [B_1, A]$
 $B_3 = [B_2, A]$

Thus, we define recursively: $B_n = [B_{n-1}, A]$ with $B_0 = B$.

Then:
$$e^{-A} B e^A = \sum_{n=0}^{\infty} \frac{B_n}{n!}$$

② Here are some alternate forms:

$$e^{-\alpha A} B e^{\alpha A} = \sum_{n=0}^{\infty} \frac{\alpha^n}{n!} B_n$$

$$e^{-iuA} B e^{iuA} = \sum_{n=0}^{\infty} \frac{(iu)^n}{n!} B_n$$

③ Application to JJ :

Consider: $e^{-iuJ_i} J_j e^{iuJ_i}$

$$A = J_i$$

$$B = J_j$$

If $i=j$, then $= J_i$, trivial.

If $i \neq j$, then: $B_0 = J_j \quad B_1 = [J_j, J_i] = i \epsilon_{jik} J_k$

$$B_2 = [B_1, J_i] = i \epsilon_{jik} [J_k, J_i] = i \epsilon_{jik} i \epsilon_{kil} J_l \quad / \text{sum on } k, l$$

$$= \sum_{k,l} \epsilon_{jik} \epsilon_{kil} J_l = \sum_l J_l \cdot \sum_k \epsilon_{jik} \epsilon_{kil} = \sum_l J_l [\delta_{ji} \delta_{il} - \delta_{il} \delta_{ji}]$$

$$= J_j - J_i \delta_{ji} = J_j \quad \text{since } i \neq j.$$

Thus $B_2 = B_0 = J_j$. Then $B_3 = [B_2, A] = [B_0, A] = B_1$,
 $B_4 = [B_3, A] = B_2 \dots$

Conclusion:

$$B_{2m} = J_j$$

$$B_{2m+1} = -i \epsilon_{jik} J_k$$

Thus:

$$e^{-iu\vec{J}_i} \vec{J}_j e^{+iu\vec{J}_i} = \overbrace{\vec{J}_j \left(\sum_{n=0,2,\dots} \frac{(iu)^n}{n!} \right)}^{\cos u} + \overbrace{(-i\epsilon_{ijk} \vec{J}_k) \left(\sum_{n=1,3,\dots} \frac{(iu)^n}{n!} \right)}^{i \sin u}$$

$$= \cos u \vec{J}_j + \sin u \epsilon_{ijk} \vec{J}_k$$

Complete J-J result:

$$e^{-iu\vec{J}_i} \vec{J}_j e^{+iu\vec{J}_i} = \vec{J}_j \cos u + \epsilon_{ijk} \vec{J}_k \sin u \quad i \neq j$$

④ Next, try: $e^{-iu\vec{K}_i} \vec{K}_j e^{+iu\vec{K}_i} \quad A = K_i \quad B = K_j$

$$B_0 = B = K_j \quad B_1 = [B_0, A] = [K_j, K_i] = i\epsilon_{ijk} \vec{J}_k$$

$$B_2 = [B_1, A] = i\epsilon_{ijk} [\vec{J}_k, K_i] = i\epsilon_{ijk} (i\epsilon_{kil} K_l)$$

$$= - \sum_l K_l \sum_k \epsilon_{ijk} \epsilon_{ilk}$$

$$= - \sum_l K_l [\delta_{il} \delta_{jl} - \delta_{il} \delta_{ij}] = -K_j = -B_0 \quad (i \neq j)$$

So $B_2 = -B_0 \quad ; \quad B_3 = [B_2, A] = -[B_0, A] = -B_1$

Then: $B_{2m} = (-1)^m B_0 \quad B_{2m+1} = (-1)^m B_1$

Thus:

$$\sum_{n=0}^{\infty} \frac{(iu)^n}{n!} B_n = \underbrace{\left(\sum_{n=0,2,\dots} \frac{(iu)^n}{n!} (-1)^{\frac{n}{2}} \right)}_{\cos u} B_0 + \underbrace{\left(\sum_{n=1,3,\dots} \frac{(iu)^n}{n!} (-1)^{\frac{n-1}{2}} \right)}_{i \sin u} B_1$$

$$= K_j \cos u + i \sin u (i \epsilon_{ijk} \vec{J}_k)$$

$$= K_j \cos u - \epsilon_{ijk} \vec{J}_k \sin u$$

Thus, the complete K-K result is:

$$e^{-i\mathbf{u}\cdot\mathbf{K}_i} K_j e^{+i\mathbf{u}\cdot\mathbf{K}_i} = K_j \cosh u - \epsilon_{ijk} J_k \sinh u$$

⑤ Next, consider

$$e^{-i\mathbf{u}\cdot\mathbf{K}_i} J_j e^{+i\mathbf{u}\cdot\mathbf{K}_i}$$

$$A = K_i$$

$$B = J_j$$

$$B_0 = B = J_j$$

$$B_1 = [B_0, A] = [J_j, K_i] = +i\epsilon_{jik} K_k$$

$$B_2 = [B_1, A] = i\epsilon_{jik} [K_k, K_i] = i\epsilon_{jik} (-i\epsilon_{kil} J_l)$$

$$= +\sum_l J_l \sum_k \epsilon_{jik} \epsilon_{ilk} = -\sum_l J_l \sum_k \underbrace{\epsilon_{ijk} \epsilon_{ilk}}_{\delta_{jl} = 0} = -J_j$$

$$\text{So } B_2 = -B_0 \Rightarrow B_3 = -B_1 \Rightarrow B_4 = -B_2 \dots$$

So whenever you get $B_2 = -B_0$, answer is:

$$= \cosh u B_0 + i \sinh u B_1$$

$$= \cosh u J_j + i \sinh u (i\epsilon_{jik} K_k) = J_j \cosh u + \epsilon_{ijk} K_k \sinh u$$

Then:

$$e^{-i\mathbf{u}\cdot\mathbf{K}_i} J_j e^{+i\mathbf{u}\cdot\mathbf{K}_i} = J_j \cosh u + \epsilon_{ijk} K_k \sinh u$$

⑥ Pause to quote Theorem:

$$B_2 = +B_0 \Rightarrow e^{-i\mathbf{u}\cdot\mathbf{A}} B e^{+i\mathbf{u}\cdot\mathbf{A}} = \cos u B_0 + i \sin u B_1$$

$$B_2 = -B_0 \Rightarrow e^{-i\mathbf{u}\cdot\mathbf{A}} B e^{+i\mathbf{u}\cdot\mathbf{A}} = \cosh u B_0 + i \sinh u B_1$$

$$\text{where: } B_0 = B, B_1 = [B_0, A], B_2 = [B_1, A]$$

② Last case: _____

$$e^{-iuJ_i} K_j e^{+iuJ_i} \quad A = J_i \quad B = K_j$$

$$B_0 = B = K_j \quad B_1 = [K_j, J_i] = +i \epsilon_{jik} K_k$$

$$B_2 = i \epsilon_{jik} [K_k, J_i] = i \epsilon_{jik} (i \epsilon_{kil} K_l)$$

$$= - \sum_l K_l \cdot \sum_k \epsilon_{jik} \epsilon_{kil} = + \sum_l K_l \cdot \sum_k \underbrace{\epsilon_{jik} \epsilon_{kil}}_{\delta_{ji}} = K_j$$

$$\text{So } B_2 = +B_0$$

$$\rightarrow = \cos u \cdot K_j + i \sin u (-i \epsilon_{jik} K_k)$$

$$\Rightarrow \boxed{e^{-iuJ_i} K_j e^{+iuJ_i} = K_j \cos u + \epsilon_{ijk} K_k \sin u}$$

Some other hyperbolic boost theorems of the Lorentz algebra.

$$\textcircled{1} \quad e^{-i\theta J_2} K_3 e^{i\theta J_2}$$

Comment: our basic Lorentz algebras are all cyclic. Eg. if $[K_1, K_2] = -iJ_3$, then $[K_2, K_3] = -iJ_1$. All statements are cyclically true. We may therefore simply extend our $O(2,1)$ results: we shall only be missing the pure $O(3)$ results:

$$e^{-i\theta J_1} J_2 e^{i\theta J_1} \quad [J_i, J_j] = i\epsilon_{ijk} J_k$$

$$= J_2 + [J_2, i\theta J_1]$$

$$\text{So } \textcircled{2} = [J_2, i\theta J_1] = i\theta (-iJ_3) = \theta J_3 \quad \swarrow \text{sign}$$

$$\text{So:} \quad = \cos\theta J_2 + \sin\theta J_3$$

Table:

$$\begin{cases} e^{-i\theta J_1} J_2 e^{i\theta J_1} = \cos\theta J_2 + \sin\theta J_3 \\ e^{-i\theta J_2} J_3 e^{i\theta J_2} = \cos\theta J_3 + \sin\theta J_1 \\ e^{-i\theta J_3} J_1 e^{i\theta J_3} = \cos\theta J_1 + \sin\theta J_2 \end{cases}$$

What about:

$$e^{-i\theta J_2} J_1 e^{i\theta J_2} \quad \text{Well,}$$

$$= J_1 \quad \textcircled{2} = [J_1, i\theta J_2] = i\theta i J_3 = -\theta J_3 \Rightarrow \theta \rightarrow -\theta$$

$$\begin{cases} e^{-i\theta J_2} J_1 e^{i\theta J_2} = \cos\theta J_1 - \sin\theta J_3 \\ e^{-i\theta J_3} J_2 e^{i\theta J_3} = \cos\theta J_2 - \sin\theta J_1 \\ e^{-i\theta J_1} J_3 e^{i\theta J_1} = \cos\theta J_3 - \sin\theta J_2 \end{cases}$$

2.2 $O(2,1)$ hyperbolic boost Theorems.

OBS.

5

⑧ Table of results:

$$\begin{aligned}[K_1, K_2] &= -i J_3 \\ [K_1, J_3] &= -i K_2 \\ [K_2, J_3] &= +i K_1\end{aligned}$$

$$e^{-i\theta K_2} J_3 e^{i\theta K_2} = \cosh \theta J_3 + \sinh \theta K_1$$

$$e^{-i\theta K_1} J_3 e^{i\theta K_1} = \cosh \theta J_3 - \sinh \theta K_2$$

$$e^{-i\theta K_1} K_2 e^{i\theta K_1} = \cosh \theta K_2 - \sinh \theta J_3$$

$$e^{-i\theta J_3} K_2 e^{i\theta J_3} = \cosh \theta K_2 - \sinh \theta K_1$$

$$e^{-i\theta K_2} K_1 e^{i\theta K_2} = \cosh \theta K_1 + \sinh \theta J_3$$

$$e^{-i\theta J_3} K_1 e^{i\theta J_3} = \cosh \theta K_1 + \sinh \theta K_2$$

check: 1) algebra as quoted is from my own "Lorentz" notes.

2) all checks at $\theta=0$.

3) appearing generators are all correct (i.e., last one is "third one").

4) infinitesimals all check out!

Hyperbolic Boost Theorems for $SO(2,1)$ standard algebra.

① Our algebra is:

$$\begin{aligned} [K_1, K_2] &= -i J_3 \\ [K_1, J_3] &= -i K_2 \\ [K_2, J_3] &= +i K_1 \end{aligned}$$

② First case:

$$\begin{aligned} e^{-i\eta K_2} J_3 e^{+i\eta K_2} &= J_3 + [J_3, i\eta K_2] + \dots \\ &= J_3 + (-i\eta) [K_2, J_3] + \dots = J_3 + (-i\eta)(+i K_1) \dots \\ &= J_3 + \eta K_1 + \frac{1}{2} [\eta K_1, i\eta K_2] \\ &= J_3 + \eta K_1 + \frac{1}{2} i \eta^2 (-i J_3) \\ &= J_3 + \eta K_1 + \frac{1}{2} \eta^2 J_3 \end{aligned}$$

① = J_3 ② = ηK_1 ③ = $\frac{1}{2} \eta^2 J_3$

④ = $\frac{1}{3} [\frac{1}{2} \eta^2 J_3, i\eta K_2] = \frac{i \eta^3}{3!} [-i K_1] = + \frac{\eta^3}{3!} K_1$

So far $J_3 [1 + \frac{\eta^2}{2!} + \dots] + K_1 [\eta + \frac{\eta^3}{3!} + \dots]$
 $= \cosh \eta J_3 + \sinh \eta K_1$

③ Second case: $e^{-i\eta K_1} J_3 e^{+i\eta K_1}$

① = J_3 ② = $[J_3, i\eta K_1] = i\eta (i K_2) = -\eta K_2$

③ = $\frac{1}{2} [-\eta K_2, i\eta K_1] = -\frac{1}{2} i \eta^2 (i J_3) = + \frac{1}{2} \eta^2 J_3$

④ = $\frac{1}{3} [\frac{1}{2} \eta^2 J_3, i\eta K_1] = \frac{i \eta^3}{3!} [i K_2] = -\frac{\eta^3}{3!} K_2$

So $J_3 [1 + \frac{1}{2} \eta^2 + \dots] - K_2 [\eta + \frac{\eta^3}{3!} + \dots]$

$= \cosh \eta J_3 - \sinh \eta K_2$

④ Next: $e^{-i\eta K_1} K_2 e^{i\eta K_1}$

① = K_2 ② = $[K_2, i\eta K_1] = i\eta (iJ_3) = -\eta J_3$

③ = $\frac{1}{2} [-\eta J_3, i\eta K_1] = -\frac{1}{2} i\eta^2 (iK_2) = \frac{1}{2} \eta^2 K_2$

④ = $\frac{1}{3!} \eta^3 [K_2, i\eta K_1] = +i\eta^3/3! (iJ_3) = -\eta^3/3! J_3$

Thus $= -J_3(\eta + \eta^3/3! + \dots) + K_2(1 + \frac{1}{2}\eta^2 + \dots)$
 $= \cosh \eta K_2 - \sinh \eta J_3$

⑤ Next: ~~$e^{-i\theta K_1} J_3 e^{i\theta K_1}$~~ $e^{-i\theta J_3} K_2 e^{i\theta J_3}$

① = K_2 ② = $i\theta [K_2, J_3] = i\theta (iK_1) = -\theta K_1$

I guess: $= \cos \theta K_2 - \sin \theta K_1$

③ = $\frac{1}{2} [-\theta K_1, i\theta J_3] = -\frac{i\theta^2}{2} [-iK_2] = \frac{\theta^2}{2} K_2$ ✓

④ = $\frac{1}{3!} (-\theta^3) [K_2, i\theta J_3] = \frac{-i\theta^3}{3!} iK_1 = \frac{\theta^3}{3!} K_1$ ✓

Why do these always come out so simply?

$K_2[1 - \frac{\theta^2}{2} + \dots] + K_1(\theta - \frac{\theta^3}{3!} + \dots)$

⑥ Next: $e^{-i\eta K_2} K_1 e^{i\eta K_2}$

① = K_1 ② = $[K_1, i\eta K_2] = i\eta (-iJ_3) = \eta J_3$

So $K_1[1 + \frac{\eta^2}{2!} + \dots] + J_3[\eta + \frac{\eta^3}{3!} + \dots]$

$= \cosh \eta K_1 + \sinh \eta J_3$

⑦ Last: $e^{-i\theta J_3} K_1 e^{i\theta J_3}$ ① = K_1 ; ② = $i\theta (-iK_2) = \theta K_2$

Thus: $\cos \theta K_1 + \sin \theta K_2$

⑧ See back of sheet 1 for results:

$SL(2, \mathbb{C})$ matrix

The parity problem in $SL(2, \mathbb{C})$.

1. We know that according to $X' = aXa^\dagger$, any element of $SL(2, \mathbb{C})$ maps into an element $\Lambda^\mu{}_\nu$ of the restricted Lorentz group, according to the usual formula. Since parity is not in this restricted Lorentz group, we know ahead of time that there will be no matrix a with $\det a = 1$ which will "do" the parity transformation x^μ to x_μ . But let's try to find an a anyway and see what happens.

$$\chi_\mu \sigma_\mu = a \chi^\mu \sigma_\mu a^\dagger \Rightarrow \begin{cases} \sigma_0 = a \sigma_0 a^\dagger \\ -\sigma_i = a \sigma_i a^\dagger = a \sigma_i a^{-1} \end{cases} \Rightarrow a \in SU_2.$$

But on data sheet I have shown that the only matrix which does this last is $a=0$. So, not only is there no $SL(2, \mathbb{C})$ matrix which does parity, there is no $GL(2, \mathbb{C})$ matrix either. There is no matrix period.

Theorem: there is no matrix a which does this:

$$(\chi_\mu \sigma_\mu) = a (\chi^\mu \sigma_\mu) a^\dagger$$

2. So we have a problem: how are we going to "fake up" parity in the world of 2×2 matrices? One suggestion (SW) is to first complex conjugate X , then apply some matrices.

$$\chi_\mu \sigma_\mu = a \chi^\mu \bar{\sigma}_\mu a^\dagger \quad \text{assume } \chi_\mu = \text{real}. \text{ Then } \sigma_\mu = a \bar{\sigma}_\mu a^\dagger$$

$$\Rightarrow \sigma_\mu = a [g_{\mu\nu} (\sigma_2 \sigma_\nu \sigma_2)] a^\dagger g_{\mu\nu} \Rightarrow \sigma_\mu = g_{\mu\nu} (a \sigma_2) \sigma_\nu (a \sigma_2)^\dagger \cdot g_{\mu\nu}$$

$$\Rightarrow \sigma_\mu = g_{\mu\nu} (b \sigma_\nu b^\dagger) \Rightarrow \begin{cases} \sigma_0 = b b^\dagger \Rightarrow b \in SU(2) \\ \sigma_i = + b \sigma_i b^\dagger \Rightarrow b = \text{identity} \end{cases}$$

$$\text{So: } (\chi_\mu \sigma_\mu) = a (\chi^\mu \sigma_\mu)^* a^\dagger \Rightarrow a = c \cdot \sigma_2.$$

$$\text{Now } \det a = c^2 \det \sigma_2 = c^2 \cdot (-1) = 1 \Rightarrow c = \pm i.$$

Theorem: if you agree to simulate parity by first complex conjugating,

$$(\chi_\mu \sigma_\mu) = a (\chi^\mu \sigma_\mu)^* a^\dagger$$

then you can find a 's that work. In general $a \propto \sigma_2$. In fact, if you want an $a \in SL(2, \mathbb{C})$, you can choose either $a = Y$ or Y^{-1} .

About the detail $B_{\hat{n}}(\alpha) = \frac{\mathbf{p} \cdot \boldsymbol{\sigma}}{m}$.

$$\boxed{Y L(P) Y^{-1} = [L(P)^{-1}]^T}$$

$$\boxed{Y^{-1} = Y^T = Y = -Y} \quad \rightarrow \text{proof m back.}$$

① We know that, in $SL(2, \mathbb{C})$

$$B_{\hat{n}}(\alpha) = e^{-i [\hat{n} \cdot \boldsymbol{\sigma}] [\alpha]} = e^{\frac{i}{2} \alpha \hat{n} \cdot \vec{\sigma}} = \cosh(\alpha/2) + \sinh(\alpha/2) (\hat{n} \cdot \vec{\sigma})$$

$$= X^\mu \sigma_\mu \quad \text{where } X^\mu = (\cosh \frac{\alpha}{2}, \sinh \frac{\alpha}{2} \hat{n})$$

② In $L_+^\uparrow$, we know that

$$B_{\hat{n}}(\alpha) \begin{pmatrix} m \\ 0 \\ 0 \\ 0 \end{pmatrix} = m \begin{pmatrix} \cosh \alpha \\ \sinh \alpha \hat{n} \end{pmatrix} \quad \text{or } B_{\hat{n}}(\alpha) \tilde{p} = p.$$

We shall mean by p a vector boosted from rest by $B_{\hat{n}}(\alpha)$, of mass m .

③ Note that $B_{\hat{n}}(\alpha)^\dagger = B_{\hat{n}}(\alpha)$ in $SL(2, \mathbb{C})$. Thus:

④ Now, let $L(p)$ be a shorthand for $B_{\hat{n}}(\alpha)$. I.e., $L(p)$ is a direct boost out to p .

⑤ Then:

$$L(p) L(p)^\dagger = L(p) L(p) = B_{\hat{n}}(2\alpha) = \tilde{X}^\mu \sigma_\mu$$

where $\tilde{X}^\mu = (\cosh \alpha, \sinh \alpha \hat{n}) = p^\mu / m$. Thus:

$$\boxed{L(p) L(p)^\dagger = L(p)^2 = \frac{p^\mu \sigma_\mu}{m}}$$

$$\boxed{L(p) = \sqrt{\frac{p \cdot \sigma}{m}}}$$

$$\boxed{L(p)^{-1} = L(sp)}$$

$$\boxed{L^2(-p) = -L^2(p)}$$

$$\boxed{L^\dagger(p) = L(p)}$$

⑥ Notice that $L(p) = L(p)^T$, Thus

$$D_{mm'}^{(0, \frac{1}{2})}(L(p)) = \sqrt{\frac{p^\mu (\sigma_\mu)_{mm'}}{m}}$$

$$\boxed{L^2(p) \cdot L \cdot L^2(p) = (L^\dagger)^{-1}}$$

or:

$$\boxed{D^{(0, \frac{1}{2})}(L(p)) = \sqrt{\frac{p \cdot \sigma}{m}}}$$

$$\boxed{D^{(0, \frac{1}{2})}(L(p) L(p)^\dagger) = D^{(0, \frac{1}{2})}(H(p) H(p)^\dagger) = \frac{p \cdot \sigma}{m}}$$

→ proof m back.

Proof: $X' = a X a^\dagger$

$$\Rightarrow p'^\mu \sigma_\mu = \bar{a} p^\mu \sigma_\mu a^\dagger \Rightarrow L^2(lp) = a L^2(p) a^\dagger$$

$$\Rightarrow L^2(lp) \cdot L \cdot L^2(p) = (a^\dagger)^{-1} \checkmark$$

Proof: since $L(p)$ has $\det=1$, we know for Pin that

$$\sigma_2 L(p) \sigma_2 = (L(p))^\top$$

$$\text{But } Y = R_3(\pi) = -i\sigma_2 \Rightarrow \sigma_2 = iY \sim \sigma_2 = -iY^\dagger$$

$$\text{so } Y L(p) Y^\dagger = ((L(p))^{-1})^\top \checkmark$$

Properties of the functions $D^{(0,s)}(a)$.

1. In agreement with Taylor, we agree that $D^{(0,\frac{1}{2})}(a) = a$. This shows at once that $D(a^T) = a^T = D(a)^T$ and the same for $*$ and $+$. Totally trivial for this representation. Even if we had taken $D(a) = a^*$ or $D(a) = a^T$ or a^+ , we would still get that $D(a^*) = D(a)^*$ where $*$ = T, *, or +.

2. We would now like to prove that $D(a^*) = D(a)^*$ for the representation $(0,s)$. I will do an induction proof making use of this rule:

$$D_{mm'}^{(0,s)}(a) = \sum_{m_1, m_2, m'_1, m'_2} \langle s, m | s - \frac{1}{2}, m_1; \frac{1}{2}, m_2 \rangle \langle s, m' | s - \frac{1}{2}, m'_1; \frac{1}{2}, m'_2 \rangle \\ \times D_{m, m_1}^{(0, s - \frac{1}{2})}(a) D_{m_2, m'_1}^{(0, \frac{1}{2})}(a)$$

We have already shown that $D(a^*) = D(a)^*$ for $s = \frac{1}{2}$. Now we use induction and show that if true for $s - \frac{1}{2}$, then true for s . First, since the Clebsch's are real, the result is transparent for $*$ = *. For the case $*$ = T, the result again follows because we can interchange the two Clebsches. Combining, we get true for $*$ = +.

QED. Conclusion:

$$D_{mm'}^{(0,s)}(a^*) = D_{mm'}^{(0,s)}(a)^* \quad * = T, *, +$$

3. Similarly, we can prove by induction that $D(-1) = (-1)^{2s}$. We know true for $s = \frac{1}{2}$. Thus:

$$D_{mm'}^{(0,s)}(-1) = (-1)^{2s} \delta_{m,m'}$$

4. The functions $D^{(s,0)}$ may be defined in this way, so no need to show anything:

$$D_{mm'}^{(s,0)}(a) = \overline{D_{mm'}^{(0,s)}(a)} = D_{mm'}^{(0,s)}(\bar{a})$$

4. The explicit functions can be obtained by a simple generalization of the rotation group functions, as done in Vilenkin. If you like, you can write everything as exponentiated generators. Thus, eg:

$$a = u \cdot h \text{ always possible} \Rightarrow a = e^{-i\vec{\theta} \cdot \vec{J}} e^{-i\vec{\varphi} \cdot \vec{K}} = e^{-i\vec{\theta} \cdot \vec{J}} e^{\vec{\varphi} \cdot \vec{J}} \\ \Rightarrow D^{(0,s)}(a) = e^{-i\vec{\theta} \cdot \vec{J}} e^{\vec{\varphi} \cdot \vec{J}} \text{ where } \vec{J} = (2s+1) \times (2s+1) \text{ generat.}$$

Realizations of the finite-dimensional generators of $SL(2, \mathbb{C})$.

1. Here is the algebra we want to satisfy:

$$[J_i, J_j] = i \epsilon_{ijk} J_k$$

$$[J_i, K_j] = i \epsilon_{ijk} K_k$$

$$[K_i, K_j] = -i \epsilon_{ijk} J_k$$

Now, suppose you find a realization $\tilde{J}_i$ which satisfies the $SU(2)$ algebra. I.e., $\tilde{J}_i$ are some matrices. Then I'll bet that if you take $\tilde{K}_i = i\tilde{J}_i$, you can satisfy the full Lorentz algebra:

$$[\tilde{J}_i, \tilde{K}_j] = [\tilde{J}_i, i\tilde{J}_j] = i[\tilde{J}_i, \tilde{J}_j] = i(i\epsilon_{ijk}\tilde{J}_k) = -\epsilon_{ijk}\tilde{J}_k$$

$$[\tilde{K}_i, \tilde{K}_j] = [i\tilde{J}_i, i\tilde{J}_j] = -[\tilde{J}_i, \tilde{J}_j] = -i\epsilon_{ijk}\tilde{J}_k$$

Up to now I had only visualized this for the $d=2$ representation, but now I see that obviously it also works for any finite-dimensional matrix representation (as noted in Taylor).

Of course, such representations exist of all dimensions since we know rep's of $SU(2)$ exist for all dimensions. None of these representations can be unitary, since the boost matrices are always hermitian rather than unitary.

2. Conclusion: $K_i = iJ_i$ and J_i allow you to trivially construct a realization of $SL(2, \mathbb{C})$ of any finite dimension, not just $d=2$.

3. Further comment: We know that every element of $SL(2, \mathbb{C})$ can be written as a boost times a rotation like so:

$$g \equiv h \cdot u \quad \text{where } h = \sqrt{g g^\dagger}$$

Thus, we recognise the $D^{(0,s)}(g)$ representations of $SL(2, \mathbb{C})$ as being just those mentioned above:

$$D^{(0,s)}(g) = D^{(0,s)}(h \cdot u) = e^{i\vec{h} \cdot \vec{K}} e^{i\vec{u} \cdot \vec{J}}$$

The six $SL(2, \mathbb{C})$ generators.

4

① First, we state the algebra:

$$[J, J] = +i\epsilon J$$

$$[K, K] = -i\epsilon J$$

$$[K, J] = [J, K] = +i\epsilon K$$

We seek to represent the six generators by 2×2 matrices. For starters, we take:

$$J_1 = \frac{1}{2} \sigma_1$$

$$J_2 = \frac{1}{2} \sigma_2$$

$$J_3 = \frac{1}{2} \sigma_3$$

$$K_1 = +iJ_2$$

$$K_2 = +iJ_3$$

$$K_3 = +iJ_1$$

Check:

$$[K_1, K_2] = [iJ_2, iJ_3] = -[J_2, J_3] = -iJ_1 \checkmark$$

$$[K_2, K_3] = [iJ_3, iJ_1] = -[J_3, J_1] = -iJ_2 \times$$

or

$$[K_i, K_j] = [iJ_i, iJ_j] = -[J_i, J_j] = -i\epsilon_{ijk} J_k$$

But:

$$[K_i, J_j] = [iJ_i, J_j] = i[J_i, J_j] = i(i)\epsilon_{ijk} J_k$$

$$= -i\epsilon_{ijk} (iJ_k) = i\epsilon_{ijk} K_k \checkmark$$

So, here are six nice generators:

$$J_1 = \sigma_1/2$$

$$K_1 = iJ_2 = \frac{1}{2} i\sigma_2$$

$$J_2 = \sigma_2/2$$

$$K_2 = iJ_3 = \frac{1}{2} i\sigma_3$$

$$J_3 = \sigma_3/2$$

$$K_3 = iJ_1 = \frac{1}{2} i\sigma_1$$

So:

$$J_1 = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$K_1 = \frac{1}{2} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

$$J_2 = \frac{1}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$K_2 = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$J_3 = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$K_3 = \frac{1}{2} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

② What are the finite forms: note

$$\sigma_i^2 = 1$$

$$\text{So: } e^{-i\phi T_3} = e^{-i\phi(\frac{1}{2}\sigma_3)} = e^{-i\frac{\phi}{2}\sigma_3}$$

$$= \sum_{n=0}^{\infty} \frac{(-i\frac{\phi}{2})^n \sigma_3^n}{n!}$$

$$= \sum_{n=0,2,4} \frac{(-i\frac{\phi}{2})^n}{n!} \cdot 1 + \left(\sum_{n=1,3,5} \frac{(-i\frac{\phi}{2})^n}{n!} \right) \sigma_3$$

$$= \cosh(-i\frac{\phi}{2}) + \sinh(-i\frac{\phi}{2}) \sigma_3$$

$$= \cos \frac{\phi}{2} - i \sin \frac{\phi}{2} \sigma_3$$

$$= \begin{pmatrix} \cos \frac{\phi}{2} - i \sin \frac{\phi}{2} & 0 \\ 0 & \cos \frac{\phi}{2} + i \sin \frac{\phi}{2} \end{pmatrix}$$

$$= \begin{pmatrix} e^{-i\phi/2} & 0 \\ 0 & e^{+i\phi/2} \end{pmatrix}$$

Verify: $e^{-i\phi T_3} \approx 1 - i\phi T_3 = \begin{pmatrix} e^{-i\phi/2} & 0 \\ 0 & e^{+i\phi/2} \end{pmatrix} \approx 1 + \begin{pmatrix} -i\phi/2 & 0 \\ 0 & i\phi/2 \end{pmatrix}$

$$\text{So } -i\phi T_3 = \begin{pmatrix} -i\phi/2 & 0 \\ 0 & i\phi/2 \end{pmatrix} \Rightarrow T_3 = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

③ Next:

$$e^{-i\phi T_2} = e^{-i\phi \frac{1}{2} \sigma_2} = \cos \frac{\phi}{2} - i \sin \frac{\phi}{2} \sigma_2$$

$$= \begin{pmatrix} \cos \frac{\phi}{2} & -\sin \frac{\phi}{2} \\ \sin \frac{\phi}{2} & \cos \frac{\phi}{2} \end{pmatrix}$$

Check: $1 - i\phi T_2 = 1 + \begin{pmatrix} 0 & -\phi/2 \\ \phi/2 & 0 \end{pmatrix}$

$$\Rightarrow -i T_2 = \begin{pmatrix} 0 & -1/2 \\ 1/2 & 0 \end{pmatrix} \Rightarrow T_2 = \begin{pmatrix} 0 & -i/2 \\ i/2 & 0 \end{pmatrix}$$

④ Finally,

$$e^{-i\phi T_1} = \cos \frac{\phi}{2} - i \sin \frac{\phi}{2} \sigma_1$$

$$= \begin{pmatrix} \cos \frac{\phi}{2} & -i \sin \frac{\phi}{2} \\ -i \sin \frac{\phi}{2} & \cos \frac{\phi}{2} \end{pmatrix}$$

Check Ans: $1 - i\phi J_1 = 1 + \begin{pmatrix} \phi - i\phi/2 & \\ & -i\phi/2 - \phi \end{pmatrix}$

$$\Rightarrow J_1 = \begin{pmatrix} 0 & -1/2 \\ 1/2 & 0 \end{pmatrix} \checkmark$$

⑤ Summarize:

$$e^{-i\phi J_i} = \cos \frac{\phi}{2} - i \sin \frac{\phi}{2} \sigma_i \checkmark$$

⑥ Now do the boosts:

$$e^{-i\nu K_i} = e^{-i\nu(iJ_i)} = e^{-i(i\nu)J_i}$$

So its as if $\phi \rightarrow i\nu$. Thus

$$e^{-i\nu K_i} = \cos \frac{i\nu}{2} - i \sin \frac{i\nu}{2} \sigma_i$$

$$= \cosh(\nu/2) + \sinh(\nu/2) \cdot \sigma_i \checkmark$$

Thus:

$$e^{-i\nu K_1} = \begin{pmatrix} \cosh \nu/2 & \sinh \nu/2 \\ \sinh \nu/2 & \cosh \nu/2 \end{pmatrix}$$

$$e^{-i\nu K_2} = \begin{pmatrix} \cosh \nu/2 & -i \sinh \nu/2 \\ i \sinh \nu/2 & \cosh \nu/2 \end{pmatrix}$$

$$e^{-i\nu K_3} = \begin{pmatrix} -\cosh \nu/2 + \sinh \nu/2 & 0 \\ 0 & \cosh \nu/2 + \sinh \nu/2 \end{pmatrix} = \begin{pmatrix} e^{\nu/2} & 0 \\ 0 & e^{-\nu/2} \end{pmatrix}$$

⑦ Most general $SL(2, \mathbb{C})$ element:

$$e^{-i[aJ + bK]} = e^{-i/2[a + ik] \cdot \underline{\sigma}} = e^{-(\nu/2) \underline{\alpha} \cdot \underline{\sigma}}$$

where $\underline{\alpha} = 3$ complex numbers.

Note: $\underline{\alpha} \cdot \underline{\sigma} = \begin{pmatrix} \alpha_3 & \alpha_1 - i\alpha_2 \\ \alpha_1 + i\alpha_2 & -\alpha_3 \end{pmatrix}$

What is:

$$(\underline{\alpha} \cdot \underline{\sigma})^2 = \begin{pmatrix} \alpha_3 - \alpha_1 - i\alpha_2 & \\ \alpha_1 + i\alpha_2 & -\alpha_3 \end{pmatrix} \begin{pmatrix} \alpha_3 - \alpha_1 - i\alpha_2 & \\ \alpha_1 + i\alpha_2 & -\alpha_3 \end{pmatrix}$$

$$= \begin{pmatrix} \alpha_3^2 + \alpha_1^2 + \alpha_2^2 & \alpha_1\alpha_3 - i\alpha_2\alpha_3 - \alpha_1\alpha_3 + i\alpha_2\alpha_3 \\ 0 & \alpha_1^2 + \alpha_2^2 + \alpha_3^2 \end{pmatrix}$$

$$\left[(\underline{\alpha} \cdot \underline{\sigma})^2 = (\alpha_1^2 + \alpha_2^2 + \alpha_3^2) \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \alpha_i \text{ all complex.} \right]$$

↑ denote this by $\alpha^2 \equiv |\alpha|^2$.

$$\text{So: } e^{-(i/\hbar)(\underline{\alpha} \cdot \underline{\sigma})} = \sum_{n=0}^{\infty} \frac{(-i/\hbar)^n}{n!} (\underline{\alpha} \cdot \underline{\sigma})^n$$

$$= \sum_{n=0,2,\dots} \frac{(-i/\hbar)^n}{n!} (\alpha^2)^{n/2} + \sum_{n=1,3,5,\dots} \frac{(-i/\hbar)^n}{n!} (\underline{\alpha} \cdot \underline{\sigma}) (\alpha^2)^{(n-1)/2}$$

$$\text{Recall } (\underline{\alpha} \cdot \underline{\sigma})^5 = \alpha^4 (\underline{\alpha} \cdot \underline{\sigma}) \text{ etc.}$$

$$\text{So } = \sum_{n=0,2,\dots} \frac{(-i/\hbar \alpha)^n}{n!} + \frac{\underline{\alpha} \cdot \underline{\sigma}}{\alpha} \sum_{n=1,3,5,\dots} \frac{(-i/\hbar \alpha)^n}{n!}$$

$$= \cosh\left(-\frac{i}{\hbar} \alpha\right) + \frac{1}{\alpha} (\underline{\alpha} \cdot \underline{\sigma}) \sinh\left(-\frac{i}{\hbar} \alpha\right)$$

$$= \cos\left(\frac{\alpha}{\hbar}\right) - i (\hat{\underline{\alpha}} \cdot \underline{\sigma}) \sin\left(\frac{\alpha}{\hbar}\right)$$

$$\text{where } \hat{\underline{\alpha}} \equiv \frac{\underline{\alpha}}{\alpha} \quad \alpha = (\alpha_1^2 + \alpha_2^2 + \alpha_3^2)^{1/2} \\ = \text{a complex \#}$$

$$\text{Now: } \alpha^2 = \underline{\alpha} \cdot \underline{\alpha} = \alpha_1^2 + \alpha_2^2 + \alpha_3^2 \\ = (a + ib)^2 = a^2 - b^2 + 2ia \cdot b$$

$$= \begin{pmatrix} \cos(\frac{\alpha}{\hbar}) + \frac{\alpha_3}{\alpha} \sin(\frac{\alpha}{\hbar}) & [-i(\frac{\alpha_1}{\alpha} - \frac{i\alpha_2}{\alpha}) \sin \frac{\alpha}{\hbar}] \\ -i(\frac{\alpha_1}{\alpha} + \frac{i\alpha_2}{\alpha}) \sin(\frac{\alpha}{\hbar}) & \cos(\frac{\alpha}{\hbar}) - \frac{\alpha_3}{\alpha} \sin(\frac{\alpha}{\hbar}) \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

③ Consider: $x^\mu \rightarrow \begin{pmatrix} \bar{x} + \bar{z} & x - iy \\ x + iy & \bar{x} - \bar{z} \end{pmatrix}$

Now try say a z -boost:

$$\begin{aligned} x'^\mu &= \begin{pmatrix} \bar{x}' + \bar{z}' & x' - iy' \\ x' + iy' & \bar{x}' - \bar{z}' \end{pmatrix} = \begin{pmatrix} \bar{e}^{\nu/2} & 0 \\ 0 & e^{\nu/2} \end{pmatrix} \begin{pmatrix} \bar{x} + \bar{z} & x - iy \\ x + iy & \bar{x} - \bar{z} \end{pmatrix} \begin{pmatrix} e^{\nu/2} & 0 \\ 0 & \bar{e}^{\nu/2} \end{pmatrix} \\ &= \begin{pmatrix} \bar{e}^{\nu/2}(\bar{x} + \bar{z}) & \bar{e}^{\nu/2}(x - iy) \\ e^{\nu/2}(x + iy) & e^{\nu/2}(\bar{x} - \bar{z}) \end{pmatrix} \begin{pmatrix} e^{\nu/2} & 0 \\ 0 & \bar{e}^{\nu/2} \end{pmatrix} \\ &= \begin{pmatrix} \bar{x} + \bar{z} & \bar{e}^{\nu}(x - iy) \\ e^{\nu}(x + iy) & \bar{x} - \bar{z} \end{pmatrix} \end{aligned}$$

So: $\left. \begin{aligned} \bar{x}' + \bar{z}' &= \bar{x} + \bar{z} \\ x' - iy' &= x - iy \\ \bar{x}' - \bar{z}' &= \bar{x} - \bar{z} \\ x' + iy' &= x + iy \end{aligned} \right\} \text{Cannot be a } x \text{ boost!!}$

$2x' = 2x \Rightarrow$

Perhaps $(x') = (g)^\dagger (x) (g)$

And $(g)^\dagger = g$ for this particular g . Then

$$\begin{aligned} x'^\mu &= \begin{pmatrix} e^{\nu/2} & 0 \\ 0 & \bar{e}^{\nu/2} \end{pmatrix} \begin{pmatrix} (\bar{x} + \bar{z}) e^{\nu/2} & (x - iy) \bar{e}^{\nu/2} \\ (x + iy) e^{\nu/2} & (\bar{x} - \bar{z}) \bar{e}^{\nu/2} \end{pmatrix} \\ &= \begin{pmatrix} (\bar{x} + \bar{z}) e^{\nu} & x - iy \\ x + iy & (\bar{x} - \bar{z}) e^{-\nu} \end{pmatrix} \leftarrow \text{This looks much better!!} \end{aligned}$$

So $\bar{x} = x'$ $y = y'$ and:

$$\begin{aligned} (\bar{x} + \bar{z}) e^{\nu} &= \bar{x}' + \bar{z}' \\ (\bar{x} - \bar{z}) e^{-\nu} &= \bar{x}' - \bar{z}' \end{aligned}$$

$$\begin{aligned} \bar{x} \frac{(e^{\nu} + \bar{e}^{\nu})}{2} + \bar{z} \frac{(e^{\nu} - \bar{e}^{\nu})}{2} &= \bar{x}' = \frac{\cosh \nu \cdot \bar{x} + \sinh \nu \cdot \bar{z}}{\cosh \nu} \\ \bar{x} \frac{(e^{\nu} - \bar{e}^{\nu})}{2} + \bar{z} \frac{(e^{\nu} + \bar{e}^{\nu})}{2} &= \bar{z}' = \frac{\sinh \nu \cdot \bar{x} + \cosh \nu \cdot \bar{z}}{\cosh \nu} \end{aligned}$$

Compare to:

$$\begin{pmatrix} \cosh & \sinh \\ \sinh & \cosh \end{pmatrix} \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \cosh t + \sinh z \\ x \\ y \\ \sinh t + \cosh z \end{pmatrix} = \begin{pmatrix} t' \\ x' \\ y' \\ z' \end{pmatrix}$$

This agrees exactly !!!

Thus, the correct rule is:

$$X = t + x i \sigma_i = \begin{pmatrix} t+z & x-iy \\ x+iy & t-z \end{pmatrix}$$

And $X' = G^\dagger X G$

where G is the 2×2 matrix. We do get the usual active 4×4 matrices!!

The Generators and One Parameter Subgroups of $SL(2, C)$.

1. The Lie Algebra of the Lorentz group $= SO(3, 1)$ $SL(2, C)$ is as follows:

$$\begin{aligned} [J_i, J_j] &= i\epsilon_{ijk} J_k \\ [K_i, K_j] &= -i\epsilon_{ijk} J_k \end{aligned} \quad \text{six generators: } J_1, J_2, J_3, K_1, K_2, K_3$$

$$[K_i, J_j] = [J_i, K_j] = i\epsilon_{ijk} K_k$$

2. We can easily form a two dimensional realization of the six generators as follows:

$$J_i = \frac{1}{2} \sigma_i \quad K_i = \frac{1}{2} i \sigma_i$$

where σ_i are the hermitian Pauli matrices. Notice that in this particular realization the three rotation generators are hermitian, whereas the three boost generators are anti-Hermitian.

3. In my usual minus sign convention I can now calculate the finite one-parameter subgroups. Thus, I am constructing a two dimensional representation of the group $SL(2, C)$. However, some of the group operators will be unitary, (the rotations), whereas other operators will be anti-unitary (the boosts).

$$e^{-i\phi J_i} = \cos \frac{\phi}{2} - i \sin \frac{\phi}{2} \cdot \sigma_i \quad \checkmark$$

$$e^{-i\nu K_i} = \cosh \frac{\nu}{2} + \sinh \frac{\nu}{2} \cdot \sigma_i \quad \checkmark = e^{+\nu J_i} = e^{+\nu/2 \sigma_i}$$

$$e^{-i\phi J_1} = \begin{pmatrix} \cos \phi/2 & -i \sin \phi/2 \\ -i \sin \phi/2 & \cos \phi/2 \end{pmatrix} \quad e^{-i\nu K_1} = \begin{pmatrix} \cosh \nu/2 & \sinh \nu/2 \\ \sinh \nu/2 & \cosh \nu/2 \end{pmatrix}$$

$$e^{-i\phi J_2} = \begin{pmatrix} \cos \phi/2 & -\sin \phi/2 \\ \sin \phi/2 & \cos \phi/2 \end{pmatrix} \quad e^{-i\nu K_2} = \begin{pmatrix} \cosh \nu/2 & -i \sinh \nu/2 \\ i \sinh \nu/2 & \cosh \nu/2 \end{pmatrix}$$

$$e^{-i\phi J_3} = \begin{pmatrix} e^{-i\phi/2} & 0 \\ 0 & e^{+i\phi/2} \end{pmatrix} \quad e^{-i\nu K_3} = \begin{pmatrix} e^{\nu/2} & 0 \\ 0 & e^{-\nu/2} \end{pmatrix}$$

4. Now we can express an arbitrary $SL(2, \mathbb{C})$ element as follows:

$$U(g) = e^{-i[a \cdot \underline{\sigma} + b \cdot \underline{k}]} = e^{-i[a + ib] \cdot \underline{\sigma}} \\ = e^{-i \alpha \cdot \underline{\sigma}}$$

In other words, we just let our three original rotation parameters take complex values; this must cover all of $SL(2, \mathbb{C})$. In fact we find:

$$e^{-i \alpha \cdot \underline{\sigma}} = \cos(\frac{1}{2} \alpha) - i \underline{\alpha} \cdot \underline{\sigma} \sin(\frac{1}{2} \alpha) \quad (\underline{\alpha} \cdot \underline{\sigma}) \quad \checkmark$$

$$\text{where } \alpha = [\underline{\alpha} \cdot \underline{\alpha}]^{1/2} = (\alpha_1^2 + \alpha_2^2 + \alpha_3^2)^{1/2} \quad \checkmark \\ = ([a + ib]^2)^{1/2} = (a^2 - b^2 + 2i a \cdot b)^{1/2} \quad \checkmark$$

where a and b are real.

5. Let's check to see if our group element so constructed is unimodular as desired: (i.e., $\exists$ 6 independent traceless 2×2 matrices)

$$\text{tr} \left(e^{-i \alpha \cdot \underline{\sigma}} \right) = e^{\text{trace}(-i \alpha \cdot \underline{\sigma})} = e^0 = 1 \quad \checkmark$$

6. Exactly which representation of the Lorentz group have we constructed here?

We have already noted that it is not a unitary representation (these are all infinite-dimensional). Recall in Liubarski that you can "separate" the Lie algebra into two non-overlapping sets (direct product). The reps there were labelled $\mathcal{D}_{\pm}$. Since we have represented all our generators by Pauli matrices, we must in fact be talking about the representation $\mathcal{D}_{\frac{1}{2}, \frac{1}{2}}$ which, we recall, is the "vector" representation. The action on vectors in the 2×2 $SL(2, \mathbb{C})$ space can be found in this way: $\checkmark$

7. In fact, the standard 4×4 matrix representation of these (active) group operations may be simply constructed from this complex 2×2 representation in the following manner (I have only checked the z boost): *but see elsewhere.*

$$X' = G^\dagger X G \quad \text{where} \quad X = \underline{x} \cdot \underline{\sigma} + t \cdot \underline{1} \\ = \begin{pmatrix} t+x & x-iy \\ x+iy & t-x \end{pmatrix} \\ G = (2 \times 2).$$

SL(2,C) exponentiation theorem.

1. First I thought this was true, then I thought it was false, now I think it is true again. The only confusion was with complex three vectors.

2. Lemma: $(\vec{a} \cdot \vec{\sigma})^2 = \alpha^2 \cdot 1$ where $\alpha^2 = \alpha_1^2 + \alpha_2^2 + \alpha_3^2$.
even if $\vec{a}$ is complex.

proof A: $(\vec{a} \cdot \vec{\sigma})^2 = \begin{pmatrix} \alpha_3 & \alpha_1 - i\alpha_2 \\ \alpha_1 + i\alpha_2 & -\alpha_3 \end{pmatrix}^2 = \alpha^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ explicitly

proof B: $(\vec{a} \cdot \vec{\sigma})^2 = (\vec{a} \cdot \vec{\sigma} + i\vec{b} \cdot \vec{\sigma})^2 = \vec{a}^2 - \vec{b}^2 + i[(\vec{b} \cdot \vec{\sigma}), (\vec{a} \cdot \vec{\sigma})] + 2i\vec{a} \cdot \vec{b}$
 $= \vec{a}^2 - \vec{b}^2 + 2i\vec{a} \cdot \vec{b}$

But $\alpha^2 = (\alpha_1 + i\alpha_2)^2 + \alpha_3^2 = \vec{a}^2 - \vec{b}^2 + 2i\vec{a} \cdot \vec{b} = \alpha^2$

So main results:

$$\boxed{\begin{aligned} (\vec{a} \cdot \vec{\sigma})^2 &= \alpha^2 & (\hat{a} \cdot \vec{\sigma})^2 &= 1 \\ \vec{a} = \vec{a} + i\vec{b} &\Rightarrow \alpha^2 = \vec{a}^2 - \vec{b}^2 + 2i\vec{a} \cdot \vec{b} \end{aligned}}$$

3. Now consider:

$$\begin{aligned} e^{(\vec{a} \cdot \vec{\sigma})} &= \sum_{n=0}^{\infty} \frac{(\vec{a} \cdot \vec{\sigma})^n}{n!} = \sum_{n=0,2,4}^{\infty} \frac{\alpha^n}{n!} + \sum_{n=1,3,5}^{\infty} \frac{\alpha^n}{n!} (\hat{a} \cdot \vec{\sigma}) \\ &= \cosh \alpha + \sinh \alpha (\hat{a} \cdot \vec{\sigma}) \end{aligned}$$

Thus:

$$\boxed{e^{\vec{a} \cdot \vec{\sigma}} = \cosh \alpha + \frac{1}{\alpha} \sinh \alpha (\vec{a} \cdot \vec{\sigma}) \quad \vec{a} = \text{general complex}}$$

$$\alpha = \pm \sqrt{\alpha_1^2 + \alpha_2^2 + \alpha_3^2} = \text{a complex } \#$$

Notice that there is ambiguity in finding complex α because if α is found to be the square root, then $-\alpha$ is also a square root. But this minus sign is seen to have no effect on the theorem as stated in the box because $-- = +$.

4. Here is a case of the above just rewritten:

$$\boxed{\begin{aligned} e^{i\vec{\beta} \cdot \vec{\sigma}} &= \cos(\beta) + i\vec{\beta} \frac{\sin(\beta)}{\beta} (\vec{\beta} \cdot \vec{\sigma}) & \vec{\beta} &= \text{general complex} \\ \beta &= \pm \sqrt{\beta_1^2 + \beta_2^2 + \beta_3^2} = \text{complex.} \end{aligned}}$$

5. Now, since $\vec{J} = \frac{1}{2} \vec{\sigma}$, we get

$$e^{-i\vec{\alpha} \cdot \vec{J}} = \cos\left(\frac{\alpha}{2}\right) - i \vec{\alpha} \sin\left(\frac{\alpha}{2}\right) (\vec{\sigma} \cdot \vec{\alpha})$$
$$\alpha = +\sqrt{\alpha_1^2 + \alpha_2^2 + \alpha_3^2} = \text{complex}$$

Here, if $\vec{\alpha} = \vec{a} + i\vec{b}$, then $\alpha = \sqrt{a^2 - b^2 + 2i\vec{a} \cdot \vec{b}}$

$so(3,1)$ matrix

Loeute Transformations

- ① Starting with any one of the four matrices a_{μ}^{ν} , you can generate all the others by use of $g_{\mu\nu}$ to raise or lower things:

$$g^{\mu\nu} a_{\mu}^{\nu} = a^{\mu\nu}$$

$$g_{\mu\nu} g^{\mu\alpha} a_{\alpha}^{\nu} = a^{\nu}_{\mu}$$

$$g_{\mu\nu} a_{\mu}^{\nu} = a_{\mu\beta}$$

- ② Of course we have these properties of $g_{\mu\nu}$:

$$g_{\mu\nu} = g_{\nu\mu}$$

$$g^{\mu\alpha} g_{\mu\nu} = g^{\alpha}_{\nu} = \delta^{\alpha}_{\nu} = \delta_{\alpha\nu}$$

$$g_{\mu\nu} = g^{\mu\nu}$$

$$g_{\mu\nu} = \begin{bmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{bmatrix}$$

- ③ Usually, one chooses the matrix a^{μ}_{ν} to effect a transformation of a physical vector, taken as contravariant:

$$x^{\mu} = (x^0, x^1, x^2, x^3) \quad (x^1, x^2, x^3) = \vec{x}$$

$$x^2 = x x = g_{\mu\nu} x^{\mu} x^{\nu} = (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2 = (x^0)^2 - (\vec{x})^2$$

$$x'^{\mu} = a^{\mu}_{\nu} x^{\nu} \leftarrow$$

- ④ The condition of a^{μ}_{ν} comes by insisting that

$$x'y' = xy \Rightarrow g_{\mu\nu} x'^{\mu} y'^{\nu} = g_{\mu\nu} a^{\mu}_{\alpha} x^{\alpha} a^{\nu}_{\beta} x^{\beta} = g_{\alpha\beta} x^{\alpha} x^{\beta}$$

$$\Rightarrow g_{\mu\nu} a^{\mu}_{\alpha} a^{\nu}_{\beta} = g_{\alpha\beta} \quad \text{or} \quad g^{\mu\nu} a_{\mu}^{\alpha} a_{\nu}^{\beta} = g^{\alpha\beta}$$

$$\text{or } a_{\nu\alpha} a^{\nu\beta} = g^{\beta}_{\alpha} = \delta^{\beta}_{\alpha} = \delta^{\alpha}_{\beta} = a^{\nu\alpha} a_{\nu\beta} \Rightarrow (\tilde{a}^{\alpha})^{\mu}_{\nu} = a^{\mu}_{\nu}$$

- ⑤ To show that $\det(a^{\mu}_{\nu}) = \pm 1$, use above condition:

$$a_{\nu\alpha} a^{\nu\beta} = g^{\beta}_{\alpha} = a_{\nu\alpha} g^{\lambda\nu} a_{\lambda\beta} \Rightarrow \tilde{a}_{\alpha\nu} g^{\nu\lambda} a_{\lambda\beta} = g^{\beta}_{\alpha}$$

$$\text{This is standard matrix form, so: } \det(\tilde{a}) \det(g) \det(a) = \det(g)$$

$$\Rightarrow (\det(a))^2 = 1 \Rightarrow \det(a) = \pm 1$$

- ⑥ Note: Bjorken Drell, Messiah, Hagen, Schwesky all use a^{μ}_{ν} in this way, call it Λ^{μ}_{ν} or Ω^{μ}_{ν} or whatever.

$$\text{But Ruhl uses: } \Lambda^{\mu}_{\nu}(\text{Ruhl}) \equiv a^{\mu}_{\nu}$$

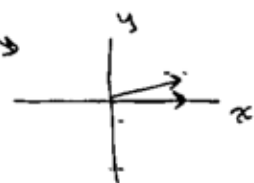
A sensible convention for active Lorentz Transformations.

① The "active" boosts are, e.g., $B_x(\alpha) = \begin{bmatrix} \cosh \alpha & \sinh \alpha & 0 & 0 \\ \sinh \alpha & \cosh \alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. No question here about signs.

② The "active" rotations rotate a vector positively in right hand sense, whereas active boost boosts vector in $+x$ direction $\Rightarrow +x$ component.

Consider:

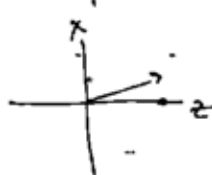
$$R_x(\theta) = \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos \theta \\ \sin \theta \\ 0 \end{bmatrix} \leftarrow \text{give new vector a positive } y \text{ comp!}$$



$$R_x(\theta) \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos \theta \\ \sin \theta \\ 0 \end{bmatrix} \leftarrow \text{pos } z \text{ comp!}$$



$$R_y(\theta) = \begin{bmatrix} \cos \theta & 0 & +\sin \theta \\ 0 & 1 & 0 \\ -\sin \theta & 0 & \cos \theta \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \sin \theta \\ 0 \\ \cos \theta \end{bmatrix} \leftarrow \text{pos } x \text{ comp!}$$



note: y is different sign.

③ Thus, The active Lorentz transformations are:

$$R_x(\theta) = \begin{bmatrix} \cos \theta & -\sin \theta & 0 & 0 \\ \sin \theta & \cos \theta & 0 & 0 \\ 0 & 0 & \cosh \alpha & -\sinh \alpha \\ 0 & 0 & \sinh \alpha & \cosh \alpha \end{bmatrix}$$

$$R_y(\theta) = \begin{bmatrix} \cos \theta & 0 & +\sin \theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin \theta & 0 & \cos \theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_z(\theta) = \begin{bmatrix} \cos \theta & -\sin \theta & 0 & 0 \\ \sin \theta & \cos \theta & 0 & 0 \\ 0 & 0 & \cosh \alpha & -\sinh \alpha \\ 0 & 0 & \sinh \alpha & \cosh \alpha \end{bmatrix}$$

④ I define my generators in this way:

$$R_i(\theta) = e^{-i\theta J_i} \\ B_i(\alpha) = e^{-i\alpha K_i}$$

Therefore:

$$1 - i\theta J_i = 1 + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\theta \\ 0 & \theta & 0 & 0 \end{bmatrix} \Rightarrow$$

$$1 - i\alpha K_i = 1 + \begin{bmatrix} 0 & \alpha & 0 & 0 \\ \alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow$$

$$J_i = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & i & 0 & 0 \end{bmatrix} \\ K_i = \begin{bmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

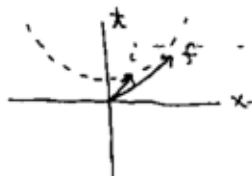
Connecting $SO(3,1)$ and $SL(2, \mathbb{C})$

I. Define active transformations of $SO(3,1)$

① We start with $SO(3,1)$. We shall define our finite x -boosts in this way:

$$B_x^{(a)}(u) \equiv \begin{bmatrix} \cosh u & \sinh u & 0 & 0 \\ \sinh u & \cosh u & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Thus, well, this is denoted "active" because it boosts a vector in the positive sense:



I.e., momentum arrow gets an increased x -component, if four vector is future-timelike.

$$\begin{bmatrix} \cosh u & \sinh u \\ \sinh u & \cosh u \\ & & 1 & 0 \\ & & 0 & 1 \end{bmatrix} \begin{bmatrix} t \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} t \cosh u + x \sinh u \\ x \cosh u + t \sinh u \\ y \\ z \end{bmatrix}$$

② Pause to check out action of $B_x^{(a)}(u)$ on various four vectors:



$$p_1 = (1, 0)$$

$$p_2 = (0, 1)$$

$$p_3 = (-1, 0)$$

$$p_4 = (0, -1)$$

$$p'_1 = (\cosh u, \sinh u)$$

$$p'_2 = (\sinh u, \cosh u)$$

$$p'_3 = (-\cosh u, -\sinh u)$$

$$p'_4 = (-\sinh u, -\cosh u)$$

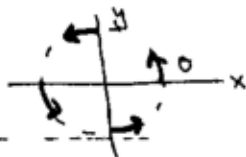
③ Thus, here is our definition:

"The active x -boost $B_x^{(a)}(u)$, where $u > 0$, is a boost which increases the x -component of a future-timelike four vector."

The action of $B_x^{(a)}(u)$, $u > 0$, on other 4-vectors is shown in sketch.

④ The other boosts are treated in exactly the same way.

⑤ Next, we wish to define an "active" rotation:



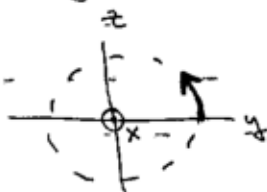
We illustrate the active z -rotation by $\theta > 0$: the active z -rotation $R_z^{(a)}(\theta)$, $\theta > 0$, increases the y -component of $(x, y) = (1, 0)$.

So:

$$R_z^{(a)}(\theta) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ +\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$

Better yet: $R_z^{(a)}(\theta)$ rotates a vector by angle θ in sense of usual right hand rule.

⑥ Next, consider $R_x^{(a)}(\theta)$



$$R_x^{(a)}(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

⑦ Finally, consider the analogous

$$R_y^{(a)}(\theta)$$



$$R_y^{(a)}(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} p_y \\ p_x \end{pmatrix} = \begin{pmatrix} p_{y'} \\ p_{x'} \end{pmatrix}$$

$$\text{So } \begin{pmatrix} p_{x'} \\ p_{y'} \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} p_x \\ p_y \end{pmatrix}$$

⑧ So, --

active rotations defined by right hand rule. ✓

II. define generators of $SO(3,1)$

① Define generators K_i, J_i in this way:

define J_i : $e^{-i\theta J_i} \equiv R_i^{(a)}(\theta)$

define K_i : $e^{-ia K_i} \equiv B_i^{(a)}(u)$

⑩ Therefore we can read off our 4×4 generators:

$$-i\alpha K_1 = \begin{bmatrix} 0 & \alpha \\ \alpha & 0 \end{bmatrix} \Rightarrow K_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \checkmark \quad K_2 = \begin{bmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ etc.}$$

$$-i\beta J_1 = \begin{bmatrix} 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow J_1 = \begin{bmatrix} 0 & 0 \\ 0 & -i \\ i & 0 \end{bmatrix} \checkmark \quad J_3 = \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

So here are the generators:

$$\left\{ \begin{array}{lll} K_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} & K_2 = \begin{bmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} & K_3 = \begin{bmatrix} 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ J_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & i & 0 \end{bmatrix} & J_2 = \begin{bmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{bmatrix} & J_3 = \begin{bmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \right\} \checkmark$$

This agrees with results obtained in "lightlike boosts" notes.

⑪ Note that, eg, K_2 really means: "a y-boost", + has nothing to do with x-boost. I.e. $x=y$.

⑫ arbitrary generator

$$-i(a \cdot J + b \cdot K) \equiv_{\nu} \begin{matrix} \nu=0 & \nu=1 & \nu=2 & \nu=3 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ \begin{bmatrix} 0 & b_1 & b_2 & b_3 \\ b_1 & 0 & -a_3 & a_2 \\ b_2 & a_3 & 0 & -a_1 \\ b_3 & -a_2 & a_1 & 0 \end{bmatrix} \end{matrix} \checkmark$$

⑬ Arbitrary $SO(3,1)$ element

$$e^{-i[a \cdot J + b \cdot K]} \in SO(3,1).$$

$$\text{I.e. } \Lambda_{\nu}^{\mu}(a,b) = (e^{-i[a \cdot J + b \cdot K]})^{\mu}_{\nu}$$

III. Relation to $SL(2, \mathbb{C})$

⑭ Define $\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ σ_i = usual Pauli's.

Then define $X = \sigma_\mu x^\mu = \sigma_0 t + \sigma_1 x + \sigma_2 y + \sigma_3 z$.

Define: $x'^\mu = \Lambda^\mu{}_\nu(a, b) x^\nu$

Define: $X' = \sigma_\mu x'^\mu = \sigma_\mu \Lambda^\mu{}_\nu x^\nu$

⑮ Claim: given any pair (a, b) defining an element of $SO(3, 1)$, we can produce a $g \in SL(2, \mathbb{C})$ such that

$$g X g^\dagger = X'$$

$$\text{I.e. } g \sigma_\mu x^\mu g^\dagger = \sigma_\mu \Lambda^\mu{}_\nu x^\nu$$

$$\text{I.e. } g \sigma_\nu g^\dagger = \sigma_\mu \Lambda^\mu{}_\nu$$

Basically, we merely solve this equation for g . The solution is:

$$g(a, b) = e^{-i[a \cdot \underline{J} + b \cdot \underline{K}]}$$

$$\text{where } \underline{J} = \frac{1}{2} \underline{\sigma} \quad \underline{K} = \frac{1}{2} i \underline{\sigma}$$

← arbitrary $SL(2, \mathbb{C})$ el.

← generators

⑯ Therefore, we conclude that the proper generators of $SL(2, \mathbb{C})$ are these.

⑰ So, we should identify $K_1 = \frac{1}{2} i \sigma_1$ with x-boost.

IV. Confusion: So why do I say things like $K_1 = -K_y$ and $K_2 = K_x$ when comparing me to CDM?!!
 ⑱ Think someone has goofed up.

Mukunda

① Mukunda discusses an algebra of J_0, J_1, J_2 and chooses:

$$J_0 = \frac{1}{2} \sigma_3$$

$$J_1 = \frac{1}{2} i \sigma_2$$

$$J_2 = -\frac{1}{2} i \sigma_1$$

However, he could have chosen:

$$J_0 = \frac{1}{2} \sigma_3 \quad \leftarrow \text{ie, these would still satisfy his algebra.}$$

$$J_1 = \frac{1}{2} i \sigma_1 = -J_2$$

$$J_2 = \frac{1}{2} i \sigma_2 = J_1$$

Check:

$$[J_0, \tilde{J}_1] = [J_0, -J_2] = -[J_0, J_2] = i J_1 = i \tilde{J}_2 \quad \checkmark$$

$$[J_0, \tilde{J}_2] = [J_0, J_1] = i J_2 = -i \tilde{J}_1 \quad \checkmark$$

$$[\tilde{J}_1, \tilde{J}_2] = [-J_2, J_1] = [J_1, J_2] \quad \checkmark$$

② So, suppose Mukunda had made this choice. What would change? Ie, does he ever use explicit 2×2 generators? No! Nor do MAR! In fact, no do CDM!

③ Idea: why not simply ignore Mukunda's paragraph on his "choices" (which also occurs in S3), and make this modified table:

<u>PR = CDM</u>	<u>MUK = LN = MAR</u>	<u>ME</u>	<u>BF</u>	<u>Barg</u>
K_x	J_1	K_1	L_{23}	H_1
K_y	J_2	K_2	L_{13}	H_2
J_z	J_0	J_3	L_{12}	H_0

④ Now an x-boost is an x-boost & there is no confusion.

The Lorentz Group

2a. Relativistic Notation

Before embarking upon a description of the relativistic one-particle equations, let us briefly introduce the relativistic notation that will be used throughout the book.

We shall denote the space-time co-ordinates by x^μ (which as a four-vector is denoted by a light-faced x) with the understanding that $x^0 = ct$, $x^1 = x$, $x^2 = y$ and $x^3 = z$; $x = \{x^\mu\}$. We shall use a metric tensor $g_{\mu\nu}$ with components

$$\begin{aligned} g_{00} &= -g_{11} = -g_{22} = -g_{33} = +1 \\ g_{\mu\nu} &= 0 \quad \text{for } \mu \neq \nu \end{aligned} \quad (1)$$

We must therefore distinguish between covariant and contravariant vectors. A contravariant vector (one which transforms like the co-ordinate vector x^μ) is denoted by v^μ , a covariant one (which transforms like the gradient) by v_μ , and similarly for tensors. In general, Greek indices will be used to denote the components (0, 1, 2, 3) of a space-time tensor whereas Latin indices will be used to denote spatial components only (1, 2, 3). The raising and lowering of indices is defined in terms of the metric tensor, with

$$v_\mu = g_{\mu\nu} v^\nu \quad (2a)$$

$$v^\mu = g^{\mu\nu} v_\nu \quad (2b)$$

where a repeated Greek index implies a summation over that index from 0 to 3, i.e.,

$$g_{\mu\nu} v^\nu = \sum_{\nu=0}^3 g_{\mu\nu} v^\nu \quad (3)$$

The tensor $g^{\mu\nu}$ is defined by the equation

$$g^{\mu\nu} g_{\mu\sigma} = \delta^\nu_\sigma \quad (4)$$

where δ^μ_ν is the Kronecker symbol: $\delta^\mu_\nu = 1$ if $\mu = \nu$ and $\delta^\mu_\nu = 0$ otherwise.

$$\begin{aligned} g^{\mu\nu} p_\mu x_\nu &= p_\mu x^\mu = p \cdot x \\ &= x_0 p_0 - \mathbf{x} \cdot \mathbf{p} = x^0 p^0 - \mathbf{x} \cdot \mathbf{p} \end{aligned} \quad (5)$$

A vector v is called time-like if $v \cdot v = v^2 > 0$, space-like if $v^2 < 0$ and a null vector if $v^2 = 0$.¹ The relativistic energy-momentum relation for a free particle, $E^2 = p^2 c^2 + m^2 c^4$ when written in terms of the (time-like) energy-momentum four-vector of the particle, $p = \{E/c, \mathbf{p}\}$, reads $p^2 = m^2 c^2$. The operator relation $E \rightarrow i\hbar \partial_t$, $\mathbf{p} \rightarrow -i\hbar \nabla$ can now be written as

$$p_\mu \rightarrow i\hbar \frac{\partial}{\partial x^\mu} = i\hbar \partial_\mu \quad (6)$$

In this connection it should be noted that since x^μ is a contravariant vector $\partial/\partial x^\mu = \partial_\mu$ is a covariant vector with

$$\partial_\mu = \frac{\partial}{\partial x^\mu} = \left\{ \frac{\partial}{\partial ct}, \nabla \right\} \quad (7a)$$

and

$$\partial^\mu = \frac{\partial}{\partial x_\mu} = \left\{ \frac{\partial}{\partial ct}, -\nabla \right\} \quad (7b)$$

The wave operator, or the D'Alembertian operator as it is often called,

$$\square = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2 \quad (8)$$

can thus be written as $\square = g_{\mu\nu} \partial^\mu \partial^\nu = \partial^\mu \partial_\mu$.

We shall have frequent occasion to make use of the step functions $\theta(a)$ and $\epsilon(a)$ which are defined as follows:

$$\begin{aligned} \epsilon(a) &= +1 \quad \text{if } a > 0 \\ &= -1 \quad \text{if } a < 0 \end{aligned} \quad (9a)$$

$$\begin{aligned} \theta(a) &= \frac{1}{2}(1 + \epsilon(a)) = 1 \quad \text{if } a > 0 \\ &= 0 \quad \text{if } a < 0 \end{aligned} \quad (9b)$$

Finally we shall almost always use natural units in which c , the velocity of light, and $\hbar$, Planck's constant divided by 2π , are set equal to one: $\hbar = c = 1$. In this system of units, energy, mass, inverse length, and inverse time all have the same dimension.

¹ Note that by v^2 we mean $v_0^2 - \mathbf{v}^2$.

2b. The Homogeneous Lorentz Group

We shall briefly recall in this section a few facts about homogeneous Lorentz transformations. For two frames in relative motion along the x^1 axis, the Lorentz transformation relating them is

$$\begin{aligned}x'^0 &= \gamma(x^0 - \beta x^1) = x^0 \cosh u - x^1 \sinh u \\x'^1 &= \gamma(x^1 - \beta x^0) = x^1 \cosh u - x^0 \sinh u \\x'^2 &= x^2 \\x'^3 &= x^3\end{aligned}\quad (10)$$

where

$$\gamma = (1 - \beta^2)^{-1/2} \quad \beta = v/c \quad (11a)$$

$$\tanh u = \beta \quad (11b)$$

and v is the relative velocity of the two frames. This Lorentz transformation leaves the quadratic form x^2 invariant. Note also that at time $x_0 = 0$ the origins of the two co-ordinate systems coincide. The most general homogeneous Lorentz transformation between two co-ordinate systems is a linear transformation

$$x'^\mu = \Lambda^\mu_\nu x^\nu \quad (12)$$

or in matrix form $x' = \Lambda x$, which leaves invariant the quadratic form $x_\mu x^\mu$, i.e., for which $x^2 = x'^2$. The transformation coefficients Λ^μ_ν are all real. The condition that the quadratic form x^2 be invariant requires that

$$\Lambda_\mu^\alpha \Lambda^\mu_\lambda = \Lambda^\alpha_\lambda = \delta^\alpha_\lambda \quad (13a)$$

We can write this last equation in the form

$$\Lambda^\mu_\nu g_{\mu\rho} \Lambda^\rho_\sigma = g_{\nu\sigma} \quad (13b)$$

or in matrix form as

$$\Lambda^T g \Lambda = g \quad (13c)$$

where the superscript T denotes the transposed matrix. It follows from this equation that $\det \Lambda = \pm 1$ and therefore that for every homogeneous Lorentz transformation there exists an inverse transformation. Since the product of two Lorentz transformations is again a Lorentz transformation, the set of all homogeneous Lorentz transformations form a group.

The group of Lorentz transformations contains a subgroup which is isomorphic to the three-dimensional rotation group. This subgroup consists of all the Λ^μ_ν of the form

$$\Lambda(R) = \begin{pmatrix} 1 & 0 \\ 0 & R \end{pmatrix} \quad (14)$$

where R is a 3×3 matrix with $RR^T = R^T R = 1$. We call such a Λ a spatial rotation. Every homogeneous Lorentz transformation can be decomposed as follows:

$$\Lambda = \Lambda(R_2) \Lambda(l_1) \Lambda(R_1)$$

where $\Lambda(R_1)$ and $\Lambda(R_2)$ are spatial rotations and $\Lambda(l_1)$ a Lorentz transformation in the x^1 direction.

If we set $\sigma = \nu = 0$ in Eq. (13b), we then obtain

$$(\Lambda^0_0)^2 = 1 + \sum_{i=1}^3 (\Lambda^i_0)^2 \geq 1 \quad (16)$$

so that $\Lambda^0_0 \geq 1$ or $\Lambda^0_0 \leq -1$. A Lorentz transformation for which $\Lambda^0_0 \geq 1$ is called an orthochronous Lorentz transformation. A Lorentz transformation is orthochronous if and only if it transforms every positive time-like vector into a positive time-like vector. The set of all orthochronous Lorentz transformations forms a group: the orthochronous Lorentz group. The set of all Λ can be divided into four subsets according to whether $\det \Lambda$ equals plus or minus one and Λ^0_0 is greater than one or less than minus one. The subset with $\det \Lambda = +1$ and $\Lambda^0_0 \geq 1$ is called the group of restricted homogeneous Lorentz transformations. The restricted homogeneous Lorentz group is a six-parameter continuous group. The other subsets can be obtained by adjoining to the restricted Lorentz group the following three transformations:

1. *Space inversion:* $x_0 \rightarrow x_0, \mathbf{x} \rightarrow -\mathbf{x}$

$$\Lambda(i_s) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad (17)$$

2. *Time inversion:* $x_0 \rightarrow -x_0, \mathbf{x} \rightarrow \mathbf{x}$

$$\Lambda(i_t) = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \quad (18)$$

3. *Space-time inversion:* $x \rightarrow -x$

$$\begin{aligned}\Lambda(i_{st}) &= \Lambda(i_s) \Lambda(i_t) \\ &= \begin{pmatrix} -1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} \end{aligned} \quad (19)$$

These subsets are disjoint and are not continuously connected.

As in the case of the rotation group, we can easily determine the form of the generators of an infinitesimal Lorentz transformation. For an infinitesimal Lorentz transformation

$$\Lambda^\mu_\nu = \delta^\mu_\nu + \epsilon \lambda^\mu_\nu \quad (20)$$

in order that Eqs. (13a, b, c) be satisfied, we must require that

$$\lambda^{\mu\nu} = -\lambda^{\nu\mu} \quad (21)$$

which is a necessary as well as sufficient condition for $\lambda^{\mu\nu}$ to correspond to an infinitesimal Lorentz transformation. The infinitesimal transformation which is the inverse of $\Lambda^{\mu\nu}$ is thus $\Lambda^{\nu\mu}$.

The explicit matrix representation of a restricted homogeneous Lorentz transformation in the x^1 direction (rotation in x^0x^1 plane) is given by

$$\Lambda(10, u) = \begin{pmatrix} \cosh u & -\sinh u & 0 & 0 \\ -\sinh u & \cosh u & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (22)$$

The infinitesimal generator $\mathfrak{M}^{10}$ for this rotation is defined as

$$\mathfrak{M}^{10} = \left. \frac{d}{du} \Lambda(10, u) \right|_{u=0} \quad (23a)$$

and is exhibited by

$$\mathfrak{M}^{10} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (23b)$$

Similarly the infinitesimal generators $\mathfrak{M}^{20}$ and $\mathfrak{M}^{30}$ for rotations in the "20" and "30" planes respectively, are exhibited by

$$\mathfrak{M}^{20} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \mathfrak{M}^{30} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \quad (24)$$

The infinitesimal generators for rotations in the x^i - x^j plane, i.e., spatial rotations, are

$$\mathfrak{M}^{12} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \mathfrak{M}^{23} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

$$\mathfrak{M}^{31} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad (25)$$

We define $\mathfrak{M}^{\mu\nu} = -\mathfrak{M}^{\nu\mu}$. An arbitrary infinitesimal Lorentz transformation can be written as

$$\Lambda(\omega) = I + \frac{1}{2} \omega^{\mu\nu} \mathfrak{M}_{\mu\nu} \quad (26)$$

where $\omega^{\mu\nu} = -\omega^{\nu\mu}$. A finite rotation in the $\mu\nu$ plane (in the sense μ to ν), is again obtained by exponentiation:

$$\Lambda(\mu\nu; u) = e^{u \mathfrak{M}^{\mu\nu}} \quad (27)$$

One verifies that the infinitesimal generators, $\mathfrak{M}_{\mu\nu}$, satisfy the following commutation rules:

$$[\mathfrak{M}_{\mu\nu}, \mathfrak{M}_{\rho\sigma}] = g_{\mu\rho} \mathfrak{M}_{\nu\sigma} + g_{\nu\sigma} \mathfrak{M}_{\mu\rho} - g_{\mu\sigma} \mathfrak{M}_{\nu\rho} - g_{\nu\rho} \mathfrak{M}_{\mu\sigma} \quad (28)$$

Thus if the four indices $\mu\nu\rho\sigma$ are all the same or all different the matrices commute. On the other hand if one index is common to both matrices, say $\sigma = \mu$, the right-hand side is proportional to $\mathfrak{M}_{\mu\nu}$.

If $D(\Lambda)$ is any representation of the restricted Lorentz group, we shall call the infinitesimal generators for that representation $M_{\mu\nu}$. Therefore if Λ is of the form given by Eq. (26)

$$D(\omega) = I + \frac{1}{2} \omega^{\mu\nu} M_{\mu\nu} \quad (29)$$

Since the $M_{\mu\nu}$ are representations of the generators of Lie algebra, they satisfy the same commutation rules as the $\mathfrak{M}_{\mu\nu}$, viz.,

$$[M_{\mu\nu}, M_{\rho\sigma}] = g_{\mu\rho} M_{\nu\sigma} + g_{\nu\sigma} M_{\mu\rho} - g_{\mu\sigma} M_{\nu\rho} - g_{\nu\rho} M_{\mu\sigma} \quad (30)$$

The problem of finding the representations of the restricted Lorentz group is equivalent to finding all the representations of the commutation rules (30). The following important fact about representations of the restricted homogeneous Lorentz group will be used in the sequel [see Van der Waerden (1932), Bargmann (1947), Naimark (1957)]. The group has both finite and infinite dimensional irreducible representations. However, the only finite *unitary* representation is the one-dimensional trivial representation $\Lambda \rightarrow 1$. The finite dimensional irreducible representation of the restricted group can be labeled by two discrete indices which can take on as values the positive integers, the positive half-odd integers, and zero. That this is so can be seen as follows. Let us define the operators

$$\mathbf{M}_i = (M_{32}, M_{13}, M_{21}) \quad (31)$$

$$\mathbf{N}_i = (M_{01}, M_{02}, M_{03}) \quad (32)$$

Their commutation rules are

$$[M_i, M_j] = \epsilon_{ijk} M_k \quad (33a)$$

$$[N_i, N_j] = -\epsilon_{ijk} M_k \quad (33b)$$

$$[M_i, N_j] = \epsilon_{ijk} N_k \quad (33c)$$

From these operators we can construct the operators $\mathbf{M}^2 = \mathbf{M} \cdot \mathbf{M} = \frac{1}{2} M_{\mu\nu} M^{\mu\nu}$ and $\frac{1}{2} \epsilon^{\mu\nu\rho\sigma} M_{\mu\nu} M_{\rho\sigma} = -\mathbf{M} \cdot \mathbf{N}$, which commute with all the M_i and N_i . They are therefore the invariants of the group and they are multiples of the identity in any irreducible representation. The representations can thus be labeled by the values of these operators in the given representation.

² $\epsilon^{\mu\nu\rho\sigma}$ is the antisymmetric tensor of rank four, which equals +1 if $\mu\nu\rho\sigma$ is an even permutation of 0123, equals -1 if $\mu\nu\rho\sigma$ is an odd permutation of 0123, and equals zero if the indices $\mu\nu\rho\sigma$ are not distinct.

To make the range of values of the label more transparent let us introduce the following operators:

$$J_i = \frac{1}{2}i(M_i + iN_i) \quad (34)$$

and

$$K_i = \frac{1}{2}i(M_i - iN_i) \quad (35)$$

which satisfy the following commutation rules:

$$[J_i, J_j] = i\epsilon_{ijk}J_k \quad (36)$$

$$[K_i, K_j] = i\epsilon_{ijk}K_k \quad (37)$$

$$[J_i, K_j] = 0 \quad (38)$$

It follows from these commutation rules that a finite dimensional irreducible representation space, $V^{jj'}$ can be spanned by a set of $(2j+1)(2j'+1)$ basis vectors $|jm; j'm'\rangle$ where j, m, j', m' are integers or half-odd integers, $-j \leq m \leq j$, $-j' \leq m' \leq j'$ and in terms of which the J and K operators have the following representation:

$$\begin{aligned} J_{\pm} |j, m; j', m'\rangle &= (J_1 \pm iJ_2) |j, m; j', m'\rangle \\ &= \sqrt{(j \mp m)(j \pm m + 1)} |j, m \pm 1; j', m'\rangle \\ J_3 |j, m; j', m'\rangle &= m |j, m; j', m'\rangle \end{aligned} \quad (39)$$

and

$$\begin{aligned} K_{\pm} |j, m; j', m'\rangle &= (K_1 \pm iK_2) |j, m; j', m'\rangle \\ &= \sqrt{(j' \mp m')(j' \pm m' + 1)} |j, m; j', m' \pm 1\rangle \\ K_3 |j, m; j', m'\rangle &= m' |j, m; j', m'\rangle \end{aligned} \quad (40)$$

The matrix representing any particular Lorentz transformation $D^{(jj')}(\Lambda)$ is now easily obtained from this representation and it will be noticed that it is not, in general, unitary. The representations we have obtained are all finite dimensional. More generally, the irreducible representations of the restricted Lorentz group can be specified by a pair of indices (j_0, ν) where j_0 is integer or half-integer and positive and ν is a complex number. If $\nu^2 = (j_0 + n)^2$ for some integer n , then the representation is finite dimensional. If, however, $\nu^2 \neq (j_0 + n)^2$ for any integer n , then the representation is infinite dimensional. We shall actually only be concerned with the finite dimensional representations since for physical applications we are interested in the classification of quantities which have a finite number of components and which transform according to a finite dimensional representation of the Lorentz group. The unitary infinite dimensional irreducible representations are, however, of relevance in the classification of the irreducible unitary representation of the inhomogeneous Lorentz group (see Sec. 2c). The reader interested in a unified derivation of all the irreducible representations of the restricted Lorentz group is referred to the very readable review article of Naimark (1957).

To summarize, there are a denumerable infinity of nonequivalent finite dimensional (in general nonunitary) irreducible representations. These can be labeled by two non-negative indices (j, j') where $j, j' = 0, \frac{1}{2}, 1, \frac{3}{2}, \dots$. The dimension of the representation is $(2j+1)(2j'+1)$ and $D^{(jj')}$ is single-valued if $j+j'$ is integer and double-valued otherwise. In an irreducible representation the value of the invariant $\frac{1}{2}(\mathbf{M}^2 - \mathbf{N}^2)$ is $-\{j(j+1) + j'(j'+1)\}$ times the unit matrix of dimension $(2j+1)(2j'+1)$ and the value of the second invariant is $i\{j(j+1) - j'(j'+1)\}$ times the unit matrix. The basis vectors spanning the representation space $D^{(\frac{n}{2}, \frac{m}{2})}$ and $D^{(\frac{m}{2}, \frac{n}{2})}$ can be so chosen that the representation matrices for $D^{(\frac{n}{2}, \frac{m}{2})}$ are just the complex conjugate of those for $D^{(\frac{m}{2}, \frac{n}{2})}$.

A quantity which transforms under $D^{(0,0)}$ is called a scalar, one which transforms under $D^{(1/2, 1/2)}$ a four-component vector, one which transforms under the $(\frac{1}{2}, 0)$ representation a two-component spinor. A quantity which transforms under $(0, \frac{1}{2})$ is called a conjugate spinor. For the $D^{(0, 1/2)}$ and the $D^{(1/2, 0)}$ representations an explicit matrix representation of the infinitesimal generators can be given in terms of the Pauli matrices with

$$M_j^{(1/2, 0)} = \frac{1}{2}i\sigma_j, \quad M_j^{(0, 1/2)} = \frac{1}{2}i\sigma_j \quad (41a)$$

$$N_j^{(1/2, 0)} = \frac{1}{2}\sigma_j, \quad N_j^{(0, 1/2)} = -\frac{1}{2}\sigma_j \quad (41b)$$

These are clearly inequivalent representations, since there exists no 2×2 matrix which anticommutes with all the σ_i . A two-component spinor, ξ , transforms under a spatial rotation as in the three-dimensional situation. For example, under an infinitesimal rotation about the l th axis,

$$\xi = \xi' (1 + \frac{1}{2}i\epsilon\sigma_l) \quad (42)$$

Under an infinitesimal Lorentz transformation in the x^i direction, this spinor transforms according to

$$\xi \rightarrow \xi' = (1 + \frac{1}{2}i\epsilon\sigma_i) \xi \quad (43)$$

Note, however, that the quantity $\xi^* \xi$ is not a scalar. This is a consequence of the fact that the representation $D^{(1/2, 0)}$ is not unitary. On the other hand if ξ' is a spinor which transforms according to $D^{(0, 1/2)}$, one verifies that $\xi^* \xi'$ is a scalar. Quite generally if the representations U and U^{*-1} are equivalent, this implies the existence of a nondegenerate matrix, B , such that $U = B^{-1}U^{*-1}B$. This in turn implies that if $|\psi\rangle$ is a vector which transforms according to U , namely, $|\psi\rangle \rightarrow |\psi'\rangle = U|\psi\rangle$, then $\langle\psi|B^*|\psi\rangle$ is a scalar invariant. The proof is as follows:

$$\langle\psi'|B^*|\psi\rangle = \langle\psi|U^*B^*U|\psi\rangle = \langle\psi|B^*U^{-1}U|\psi\rangle = \langle\psi|B^*|\psi\rangle \quad (44)$$

The converse is also true. For a unitary representation clearly $B = 1$.

We shall not consider the representations of the full Lorentz group including the inversion operations. A complete and simple discussion of the finite dimensional irreducible representations of the full Lorentz group may be found in Heine (1957) [see also Watanabe (1951, 1955) and Shirokov (1960a, b)]. We shall, however, at the appropriate places discuss the inversion properties of the relativistic wave functions and operators describing free particles. We here note that the commutation rules of the operators for the inversions I_s , I_t , I_{st} , with the generators are:

$$[I_s, M_i] = [I_t, N_i] = 0 \quad (45a)$$

$$[I_t, N_i]_+ = [I_s, M_i] = 0 \quad (45b)$$

$$[I_s, N_i]_+ = [I_t, M_i] = 0 \quad (45c)$$

To conclude this section, we briefly consider the vectors which span an irreducible representation of the orthochronous, improper (i.e., $\det \Lambda = -1$) Lorentz group. This is the group obtained by adjoining the space inversion operation to the elements of the restricted group. It follows from the commutation rules (45c) that

$$I_s K = JI, \quad (46a)$$

$$I_s J = KI, \quad (46b)$$

so that the basis vectors $|jm; j'm'\rangle$ for $j \neq j'$ which transform under a restricted Lorentz transformation according to $D^{(j,j')}$ do not transform into one another under I_s . In fact, in view of (46)

$$J_s(I_s |jm; j'm'\rangle) = m'(I_s |jm; j'm'\rangle) \quad (47)$$

so that $I_s |jm; j'm'\rangle$ behaves like a base vector $\lambda |j'm'; jm\rangle$ where λ is a constant which depends on j, j', m, m' . The vectors $|j, m; j', m'\rangle$ and $I_s |j, m; j', m'\rangle$ thus transform under restricted Lorentz transformations under different irreducible representations and are therefore orthogonal to one another. To obtain a vector space invariant under the improper orthochronous Lorentz group it is therefore necessary to take the $2(2j+1)(2j'+1)$ linearly independent vectors $|jm; j'm'\rangle$ and $|j'm'; jm\rangle$ together. We thus expect the vector space $V^{jj'} \oplus V^{j'j}$ to be an irreducible vector space for the representations of the improper orthochronous group. This is indeed the case for $j \neq j'$.

2c. The Inhomogeneous Lorentz Group

An inhomogeneous Lorentz transformation, $L = \{a, \Lambda\}$, is defined by

$$x'_\mu = (Lx)_\mu = \Lambda_\mu^\nu x_\nu + a_\mu \quad (48)$$

i.e., as the product operation of a translation by a real vector a_μ and a homogeneous Lorentz transformation, Λ , the translation being performed

after the homogeneous Lorentz transformation. It can conveniently be represented by the following matrix equation:

$$\begin{pmatrix} \Lambda^0_0 & \Lambda^0_1 & \Lambda^0_2 & \Lambda^0_3 & a^0 \\ \Lambda^1_0 & \Lambda^1_1 & \Lambda^1_2 & \Lambda^1_3 & a^1 \\ \Lambda^2_0 & \Lambda^2_1 & \Lambda^2_2 & \Lambda^2_3 & a^2 \\ \Lambda^3_0 & \Lambda^3_1 & \Lambda^3_2 & \Lambda^3_3 & a^3 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \\ 1 \end{pmatrix} = \begin{pmatrix} x'^0 \\ x'^1 \\ x'^2 \\ x'^3 \\ 1 \end{pmatrix} \quad (49)$$

where the last co-ordinate, 1, has no physical significance and is left invariant by the transformation. The product of two inhomogeneous Lorentz transformations $\{a_1, \Lambda_1\}$ and $\{a_2, \Lambda_2\}$ is given by

$$\{a_1, \Lambda_1\} \{a_2, \Lambda_2\} = \{a_1 + \Lambda_1 a_2, \Lambda_1 \Lambda_2\} \quad (50)$$

The inhomogeneous Lorentz transformations form a ten-parameter continuous group. The generators for infinitesimal translations are the Hermitian operators p_μ , and their commutation relations with the Hermitian generators² for "rotations" in the x^μ - x^ν plane, $M_{\mu\nu} = -M_{\nu\mu}$ are

$$[M_{\mu\nu}, p_\sigma] = i(g_{\sigma\nu} p_\mu - g_{\sigma\mu} p_\nu) \quad (51)$$

The commutation rules of these generators with themselves are

$$[p_\mu, p_\nu] = 0 \quad (52)$$

$$[M_{\mu\nu}, M_{\rho\sigma}] = -i(g_{\mu\rho} M_{\nu\sigma} - g_{\mu\sigma} M_{\nu\rho} + g_{\nu\rho} M_{\mu\sigma} - g_{\nu\sigma} M_{\mu\rho}) \quad (53)$$

The problem of classifying all the irreducible unitary representations of the inhomogeneous Lorentz group [Wigner (1939), Bargmann (1948), Shirokov (1958a, b)] can again be formulated in terms of finding all the representations of the commutation rules (51), (52), (53) by self-adjoint operators. The first task is to find all the invariants of the group. Clearly only scalar operators can be invariants of the group and we are thus confronted with the problem of constructing the scalar quantities which commute with p_μ and $M_{\mu\nu}$. Let us define the following quantities

$$\begin{aligned} v_{\mu\nu\rho} &= p_\mu M_{\nu\rho} + p_\nu M_{\rho\mu} + p_\rho M_{\mu\nu} \\ &= M_{\nu\rho} p_\mu + M_{\rho\mu} p_\nu + M_{\mu\nu} p_\rho \end{aligned} \quad (54)$$

and the pseudovector

$$w_\sigma = \frac{1}{2} \epsilon_{\sigma\mu\nu\lambda} M^{\mu\nu} p^\lambda \quad (55)$$

so that

$$(w^0, w^1, w^2, w^3) = (v^{231}, v^{230}, v^{310}, v^{120}) \quad (56)$$

or in vector notation

$$w^0 = \mathbf{p} \cdot \mathbf{M} \quad (57a)$$

$$\mathbf{w} = p_0 \mathbf{M} - \mathbf{p} \times \mathbf{N} \quad (57b)$$

² We have appended a factor $-i$ to our previous definition of the infinitesimal generators to make them Hermitian.

We note that

$$w_\mu p^\mu = 0 \quad (58)$$

The commutation rules of w_μ are:

$$[M_{\mu\nu}, w_\rho] = i(g_{\mu\rho}w_\nu - g_{\nu\rho}w_\mu) \quad (59)$$

$$[w_\mu, p_\nu] = 0 \quad (60)$$

$$[w_\mu, w_\nu] = i\epsilon_{\mu\nu\rho\sigma}w^\rho p^\sigma \quad (61)$$

One then verifies that the following scalar operators

$$P = p^\mu p_\mu \quad (62)$$

and

$$\begin{aligned} IV &= \frac{1}{8}v^{\mu\nu\rho\sigma}v_{\mu\nu\rho\sigma} = -w^\mu w_\mu \\ &= \frac{1}{2}M_{\mu\nu}M^{\mu\nu}p_\sigma p^\sigma - M_{\mu\sigma}M^{\mu\sigma}p_\nu p^\nu \end{aligned} \quad (63)$$

commute with all the infinitesimal generators, $M_{\mu\nu}$ and p_μ . They are therefore multiples of the identity for every irreducible representation of the inhomogeneous Lorentz group and their eigenvalues can be used to classify the irreducible representations.

It is convenient for the classification of the unitary representations of the inhomogeneous Lorentz group to choose a definite basis in the vector space on which the representations are defined. To define a basis, we select from among the infinitesimal operators of the group a complete set of commuting operators. Many different sets of commuting operators can, of course, be constructed. These different sets will then give rise to equivalent representations. We could, for example, choose as a complete set the operators $M_{\mu\nu}, M^{\mu\nu}, \epsilon^{\mu\nu\rho\sigma}M_{\mu\nu}M_{\rho\sigma}, M^2$ and M_3 but such a choice would not be translationally invariant. A complete commuting set that is translationally invariant consists of the operators p_μ and of one of the components of w_μ , say w_3 . We adopt this set for our subsequent discussion. The eigenvalue spectrum of these operators then specifies the range of the variables labeling the basis vectors. Furthermore we note that for an irreducible vector space, only three of the four momenta are independent since p^2 is an invariant of the group and has a constant value in an irreducible representation. The basis functions for an irreducible representation can thus be written as $|p'_0, p'_1, p'_2, p'_3; \zeta\rangle$ where p'^2 is equal to some constant and ζ is the variable corresponding to the w_3 eigenvalue. It is important to note that although we have chosen a complete set of commuting operators from among the operators of the group, this set will not in general be a complete set of commuting *observables* for a physical system. There will be in general other invariant operators (such as the total charge and nucleonic charge, for example) which commute with the group operators and whose eigenvalues together with p'_μ and ζ characterize *states* of the system. Therefore, the basis vectors of an irreducible representation more generally can be written as $|p'; \zeta; \alpha\rangle$ where α denotes

certain invariant parameters which in physical applications are the eigenvalues of those operators which must be added to the set (p_μ, w_3) to make it a complete set of observables. In what follows we often suppress the dependence on α of the basis vectors $|p; \zeta; \alpha\rangle$. Note that for such a basis the operation of translation is very simple. The set of all four-dimensional translations is a commutative subgroup of the inhomogeneous Lorentz group. Since it is commutative, the irreducible unitary representations of this subgroup are all one dimensional and are obtained by exponentiation. The operator corresponding to the translation by the four-vector a_μ is given by

$$U(a) = \exp(-ia_\mu p^\mu) \quad (64)$$

In an irreducible representation, the operation of translation by a thus corresponds to multiplying each basis vector $|p'; \zeta\rangle$ by $\exp(-ia_\mu p^\mu)$.

The irreducible representations of the inhomogeneous Lorentz group can now be classified according to whether p_μ is a space-like, time-like, or null vector, or p_μ is equal to zero. For this last case, $p_\mu = 0$, the complete system of unitary representations coincides with the complete system of (infinite dimensional) unitary representations of the homogeneous group [Bargmann (1947), Naimark (1957)]. They will not be considered further as they do not seem to have any correspondence with physical systems except for the important case of the (trivial) identity representation which is one dimensional.

The representations of principal interest for physical applications are those for which $p^2 = m^2 = \text{positive constant}$, and those for which $p^2 = 0$. Let us first discuss the case $p^2 = m^2$. In that case, $p_0/|p_0|$, the sign of the energy, commutes with all the infinitesimal generators and is therefore an invariant of the group. There are thus two irreducible representations for each value of P and W , one for each sign of $p_0/|p_0|$. An irreducible vector space, for $p_0 > 0$, is spanned by basis vectors all belonging to the same eigenvalue m^2 of p^2 and having $p_0 = +\sqrt{p^2 + m^2}$. We can therefore write $|p, \zeta\rangle$ as $|p, \zeta\rangle$.

In order to obtain the spectrum of w_3 in an irreducible representation we consider Eq. (61) within the manifold obtained from linear combinations of vectors $|p', \zeta\rangle$ with fixed p' . No difficulty or ambiguity arises since p^μ and w^μ commute. It is convenient, furthermore, to make a Lorentz transformation to the "rest frame" in which $p' = 0$, $p'_0 = m$. In the rest frame

$$\begin{aligned} w^\mu &= m(0, M_{23}, M_{31}, M_{12}) \\ &= m(0, S_1, S_2, S_3) \end{aligned} \quad (65)$$

with

$$[S_k, S_l] = i\epsilon_{klm}S_m \quad (66)$$

The S_i obey angular momentum commutation rules. The eigenvalues of S^2 are therefore $s(s+1)$ where $s = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \dots$, and the S_i are the

Lightlike Boosts

① Question: what is the invariance group of a lightlike vector?
 Eg, let solve this for a and k :

$$e^{-i[a \cdot J + k \cdot K]} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Go at once to generators:

$$[a \cdot J + k \cdot K] \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Be careful! $J_i = -i R_i = (-i)(-S^{jk}) = i S^{jk}$

So $(J_i)^\alpha_\beta = i (S^{jk})^\alpha_\beta = i [g^{j\alpha} g^k_\beta - g^{k\alpha} g^j_\beta]$

Thus, $(a \cdot J)^\alpha_\beta = \dots$ stop.

② I had better display my generator matrices!

$$[S^{23}]^\alpha_\beta = g^{2\alpha} g^3_\beta - g^{3\alpha} g^2_\beta = \begin{matrix} \alpha \rightarrow \\ \downarrow \end{matrix} \begin{bmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & -1 \\ \cdot & \cdot & +1 & \cdot \end{bmatrix}$$

(Use $g^{ii} = -1$ $g^{00} = 1$):

Thus, $(J_i)^\alpha_\beta = i (S^{23})^\alpha_\beta = i \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & -1 \\ \cdot & \cdot & +1 & \cdot \end{pmatrix}$

This agrees with my usual active stuff, so I can guess the other two:

③ Boosts?! $(K_i)^\alpha_\beta = (-i)(B_i) = (-i)[M^{i0}] = (-i)[g^{i\alpha} g^0_\beta - g^{0\alpha} g^i_\beta]$

$$\stackrel{\beta \rightarrow}{\downarrow} = (-i) \begin{matrix} \alpha \rightarrow \\ \downarrow \end{matrix} \begin{bmatrix} \cdot & -1 & \cdot & \cdot \\ -1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix} = \begin{bmatrix} \cdot & i & \cdot & \cdot \\ i & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

(i=1)

Similarly for other boosts.

④ So, here they are: $(J_i)_k^j = -i\epsilon_{ijk}$

$$(J_x)_{\beta}^{\alpha} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{bmatrix}$$

$$(K_x)_{\beta}^{\alpha} = \begin{bmatrix} 0 & i & 0 & 0 \\ i & & & \\ 0 & & \phi & \\ 0 & & & \end{bmatrix}$$

$$(J_y)_{\beta}^{\alpha} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{bmatrix}$$

$$(K_y)_{\beta}^{\alpha} = \begin{bmatrix} 0 & 0 & i & 0 \\ 0 & & & \\ i & & \phi & \\ 0 & & & \end{bmatrix}$$

$$(J_z)_{\beta}^{\alpha} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$(K_z)_{\beta}^{\alpha} = \begin{bmatrix} 0 & 0 & 0 & i \\ 0 & & & \\ 0 & & \phi & \\ i & & & \end{bmatrix}$$

⑤ Therefore:

$$a \cdot \underline{J} + b \cdot \underline{K} = (i) \begin{bmatrix} 0 & b_1 & b_2 & b_3 \\ b_1 & 0 & -a_3 & a_2 \\ b_2 & a_3 & 0 & -a_1 \\ b_3 & -a_2 & a_1 & 0 \end{bmatrix}$$

⑥ Now solve:

$$[a \cdot \underline{J} + b \cdot \underline{K}] \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} = 0$$

$$\begin{bmatrix} 0 & b_1 & b_2 & b_3 \\ b_1 & 0 & -a_3 & a_2 \\ b_2 & a_3 & 0 & -a_1 \\ b_3 & -a_2 & a_1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} b_3 \\ b_1 + a_2 \\ b_2 - a_1 \\ b_3 \end{bmatrix}$$

$$\Rightarrow b_3 = 0 \quad \begin{matrix} b_1 = -a_2 \\ b_2 = +a_1 \end{matrix}$$

Ans: $a_1 [J_1 + K_2] + a_2 [J_2 - K_1] + a_3 J_3$

① This shows that $(1, 0, 0, 1)$ is invariant under three kinds of transformations

$$e^{-i a_3 J_3}, \quad e^{-i a_1 (J_1 + K_2)}, \quad e^{-i a_2 (J_2 - K_1)}$$

② What about $(1, 0, 1, 0)$?

$$\begin{bmatrix} 0 & b_1 & b_2 & b_3 \\ b_1 & 0 & -a_3 & -a_2 \\ b_2 & a_3 & 0 & -a_1 \\ b_3 & -a_2 & a_1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} b_2 \\ b_1 - a_3 \\ b_2 \\ b_3 + a_1 \end{bmatrix} \Rightarrow \begin{aligned} b_2 &= 0 \\ b_1 &= a_3 \\ -b_3 &= a_1 \end{aligned}$$

So: $J_2, (J_1 + K_3), (J_3 + K_1)$ are invariance gens.

③ Finally $(1, 1, 0, 0)$?

$$\begin{bmatrix} b_1 \\ b_1 \\ b_2 + a_3 \\ b_3 - a_2 \end{bmatrix} \Rightarrow \begin{aligned} &J_1 \\ &(J_3 - K_2) \\ &(J_2 + K_3) \end{aligned}$$

⑩ Summary:

	<u>invariance vectors</u>	<u>rotation</u>	<u>light like boosts</u>
1.	$(1, 0, 0, 1)$	J_3	$J_1 + K_2 \quad J_2 - K_1$
2.	$(1, 0, 1, 0)$	J_2	$J_1 + K_3 \quad J_3 + K_1$
3.	$(1, 1, 0, 0)$	J_1	$J_3 - K_2 \quad J_2 + K_3$

⑪ What group is this. Use case 1.

Let $J_3 = J_3$

$L_1 = J_1 + K_2$

$L_2 = J_2 - K_1$

$$[L_2, J_3] = [J_2, J_3] - [K_1, J_3]$$

$$= i J_1 + i K_2$$

$$= i L_1$$

Calculate algebras:

$$[L_1, J_3] = [J_1, J_3] + [K_2, J_3]$$

$$= -i J_2 + i K_1$$

$$= -i (J_2 - K_1)$$

$$= -i L_2$$

$$[L_1, L_2] = [J_1 + K_2, J_2 - K_1]$$

$$= [J_1, J_2] - [J_1, K_1] + [K_2, J_2] - [K_2, K_1]$$

$$= i J_3 - i J_3 = 0$$

⑫ So, our lightlike algebra is: ---

$$[L_1, L_2] = 0 \quad X_1$$

$$[L_1, J_3] = -iL_2$$

$$[L_2, J_3] = +iL_1$$

~~Now let $J_3 = iX_3$ then $L_1 = X_1$,~~

~~Let $J_3 = -iR_3$ so~~

$$[L_1, L_2] = 0$$

$$[L_1, R_3] = L_2$$

$$[L_2, R_3] = -L_1$$

This is E_2 where L_1, L_2 are translations in the plane, R_3 is rotation.

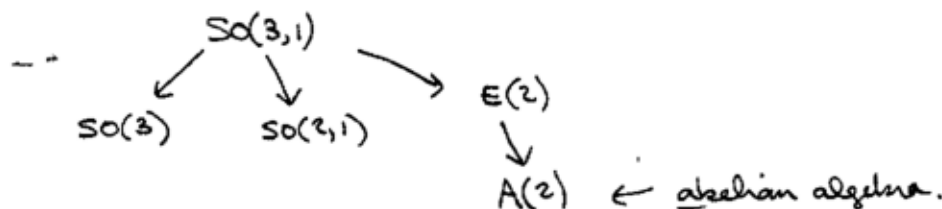
~~Then let~~

13. Conclusion. The invariance group of a lightlike 4-vector happens to be the group E_2 . The rotation of E_2 corresponds to the obvious rotation which does not change a particular lightlike vector. The two translations of E_2 do not correspond to translations but to lightlike boosts.

14. Further remarks: recall from Wybourne the definition of an invariant subalgebra (corresponding to invariant subgroup). (See page 44). The group $SO(3)$ has three one parameter subgroups, but none of these is an invariant subgroup, which is to say, $SO(3)$ has no proper ideals. Thus, $SO(3)$ is simple (and thus also semisimple). Same goes for $SO(2,1)$. Also, note that $SO(3)$ has no two-parameter subgroups at all.

15. Now look at the group E_2 . It has a two dimensional abelian subalgebra consisting of the two lightlike boost generators. Moreover, this subalgebra is invariant! Thus, the group E_2 is not even semisimple, much less simple.

16. Like $SO(3)$, E_2 is a subgroup of $SO(3,1)$. Thus, going through $E(2)$, we find finally a two dimensional subgroup of $SO(3,1)$:



fast way to get arbitrary direction boost

① Let $B_{\hat{n}}(u) \equiv e^{-iu \hat{n} \cdot \vec{K}} = \sum_{n=0}^{\infty} \frac{(u)^n}{n!} [-i \hat{n} \cdot \vec{K}]^n$

Define: $A \equiv [-i \hat{n} \cdot \vec{K}]$. Let $n_i =$ direction cosines.

Then: $B_{\hat{n}}(u) = \sum_{n=0}^{\infty} \frac{u^n}{n!} A^n$

② From data, clear that

$$A = \begin{bmatrix} 0 & n_1 & n_2 & n_3 \\ n_1 & \phi & & \\ n_2 & & \phi & \\ n_3 & & & \phi \end{bmatrix}$$

③ Square this to find:

$$A^2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & n_1 n_1 & n_1 n_2 & n_1 n_3 \\ 0 & n_2 n_1 & n_2 n_2 & n_2 n_3 \\ 0 & n_3 n_1 & n_3 n_2 & n_3 n_3 \end{bmatrix}$$

④ Compute A^3 . Turns out that $A^3 = A$. Therefore:

$$A = A^3 = A^5 = \dots \quad A^2 = A^4 = A^6 = \dots$$

$$\begin{aligned} \text{Thus: } B_{\hat{n}}(u) &= 1 + \left(\sum_{n=2,4,6,\dots} \frac{(u)^n}{n!} \right) A^2 + \left(\sum_{n=1,3,5,\dots} \frac{u^n}{n!} \right) A \\ &= 1 - A^2 + \cosh A^2 + \sinh A \\ &= (\cosh + 1) A^2 + \sinh A + 1 \end{aligned}$$

As a 4×4 matrix:

$$B_{\hat{n}}(u) = \begin{bmatrix} \cosh & n_1 \sinh & n_2 \sinh & n_3 \sinh \\ n_1 \sinh & (\cosh - 1)n_1^2 + 1 & (\cosh - 1)n_1 n_2 & (\cosh - 1)n_1 n_3 \\ n_2 \sinh & (\cosh - 1)n_2 n_1 & (\cosh - 1)n_2^2 + 1 & (\cosh - 1)n_2 n_3 \\ n_3 \sinh & (\cosh - 1)n_3 n_1 & (\cosh - 1)n_3 n_2 & (\cosh - 1)n_3^2 + 1 \end{bmatrix}$$

⑤ The space matrix elements appear as:

$$B_{\hat{n}}(u)_{ij} = (\cosh - 1)n_i n_j + \delta_{ij}$$

The Symmetric Boost ($\hat{x}$ direction)

6

- ① We know a symmetric, active x -boost matrix. We know what it does to $(m, 0, 0, 0)$:

$$\begin{bmatrix} \cosh \sigma & \sinh \sigma & 0 & 0 \\ \sinh \sigma & \cosh \sigma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} m \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} m \cosh \sigma \\ m \sinh \sigma \\ 0 \\ 0 \end{bmatrix}$$

Changing all but the first column of this boost matrix will result in the same action on $(m, 0, 0, 0)$.

- ② What is the most general boost matrix that accomplishes the above transform? The answer is that any pre-rotation is allowed + will make no difference. So,

$$\begin{bmatrix} \cosh \sigma & \sinh \sigma & 0 & 0 \\ \sinh \sigma & \cosh \sigma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & R_{11} & R_{12} & R_{13} \\ 0 & R_{21} & R_{22} & R_{23} \\ 0 & R_{31} & R_{32} & R_{33} \end{bmatrix} = \begin{bmatrix} \cosh \sigma & \sinh \sigma R_{11} & \sinh \sigma R_{12} & \sinh \sigma R_{13} \\ \sinh \sigma & \cosh \sigma R_{11} & \cosh \sigma R_{12} & \cosh \sigma R_{13} \\ 0 & R_{21} & R_{22} & R_{23} \\ 0 & R_{31} & R_{32} & R_{33} \end{bmatrix}$$

Of course R must be a valid 3-rotation matrix (rows orthonormal...)

- ③ Now we ask: what must R be for this matrix to be symmetric? Clearly $R_{11} = 1$; $R_{12} = 0$; $R_{13} = 0$. Then we have:

$$\begin{bmatrix} \cosh \sigma & \sinh \sigma & 0 & 0 \\ \sinh \sigma & \cosh \sigma & 0 & 0 \\ 0 & R_{21} & R_{22} & R_{23} \\ 0 & R_{31} & R_{32} & R_{33} \end{bmatrix}$$

Thus $R_{21} = 0$; $R_{31} = 0$. Then, $\begin{pmatrix} R_{22} & R_{23} \\ R_{32} & R_{33} \end{pmatrix} = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}$.

But this $\Rightarrow \theta = 0$.

→ Conclusion: $\exists$! one symmetric x -boost matrix, ① above.

Symmetric Boost ($\hat{n}$ direction, arbitrary)

6

① We conjecture that there is exactly one symmetric boost which takes $(m, 0, 0, 0)$ into (E, p_x, p_y, p_z) where $\hat{n}$ is a unit vector. Let's find this boost.

② In general, we will first prerotate by some $\tilde{R}$; then do a symmetric $\hat{x}$ boost; then we will do a specific rotation L to get the desired result.

After the sym $\hat{x}$ boost, we have $(E, p, 0, 0)$. We seek an L that takes this where we want:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & L_{11} & L_{12} & L_{13} \\ 0 & L_{21} & L_{22} & L_{23} \\ 0 & L_{31} & L_{32} & L_{33} \end{bmatrix} \begin{bmatrix} E \\ p \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} E \\ p_x \\ p_y \\ p_z \end{bmatrix} \Rightarrow \begin{pmatrix} L_{11} \\ L_{21} \\ L_{31} \end{pmatrix} = \begin{pmatrix} \gamma_x \\ \gamma_y \\ \gamma_z \end{pmatrix}$$

③ The rest of L seems arbitrary, but in fact it is determined up to one parameter, because we know that the 3 columns must form 3 mutually $\perp$ unit vectors, and $\hat{n}$ is one of them.

④ The most general $\hat{n}$ boost will now be:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \gamma_x & L_{12} & L_{13} \\ 0 & \gamma_y & L_{22} & L_{23} \\ 0 & \gamma_z & L_{32} & L_{33} \end{bmatrix} \begin{bmatrix} \cosh \sigma & \sinh \sigma R_{11} & \sinh \sigma R_{12} & \sinh \sigma R_{13} \\ \sinh \sigma & \cosh \sigma R_{11} & \cosh \sigma R_{12} & \cosh \sigma R_{13} \\ 0 & R_{21} & R_{22} & R_{23} \\ 0 & R_{31} & R_{32} & R_{33} \end{bmatrix}$$

$$= \begin{bmatrix} \cosh \sigma & \sinh \sigma R_{11} & \sinh \sigma R_{12} & \sinh \sigma R_{13} \\ \sinh \sigma \gamma_x & \boxed{P} \\ \sinh \sigma \gamma_y & \\ \sinh \sigma \gamma_z & \end{bmatrix}$$

Here, P is a 3×3 matrix which we write as:

$$P = L R + L \cdot (\cosh \sigma - 1) \cdot \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

So far, R is completely arbitrary.

(symmetric boost, $\hat{n}$)

6

Now choose $R = \tilde{L} (= \tilde{L}^T)$. Then:

$$P = \tilde{L} \tilde{L}^T + (\text{ch}\sigma - 1) \tilde{L} \begin{bmatrix} n_x & n_y & n_z \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\text{But, } \tilde{L} \begin{bmatrix} \cdot \\ \cdot \\ \cdot \end{bmatrix} = \begin{bmatrix} n_x & \cdot & \cdot \\ n_y & \cdot & \cdot \\ n_z & \cdot & \cdot \end{bmatrix} \begin{bmatrix} n_x & n_y & n_z \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} n_x^2 & n_x n_y & n_x n_z \\ n_y n_x & n_y^2 & n_y n_z \\ n_z n_x & n_z n_y & n_z^2 \end{bmatrix}$$

Call this the N -matrix [it is symmetric!].

Thus:

$$P = I + (\gamma - 1)N$$

So we have constructed a symmetric P too!

Thus, the (unique) $\hat{n}$ symmetric boost is:

$$P = \begin{bmatrix} 1 + (\gamma - 1)n_x^2 & (\gamma - 1)n_x n_y & (\gamma - 1)n_x n_z \\ (\gamma - 1)n_x n_y & 1 + (\gamma - 1)n_y^2 & (\gamma - 1)n_y n_z \\ (\gamma - 1)n_z n_x & (\gamma - 1)n_z n_y & 1 + (\gamma - 1)n_z^2 \end{bmatrix}$$

$$\text{boost} = \begin{bmatrix} \text{ch}\sigma & n_x \text{sh}\sigma & n_y \text{sh}\sigma & n_z \text{sh}\sigma \\ n_x \text{sh}\sigma & \text{ch}\sigma & 0 & 0 \\ n_y \text{sh}\sigma & 0 & \text{ch}\sigma & 0 \\ n_z \text{sh}\sigma & 0 & 0 & \text{ch}\sigma \end{bmatrix} \begin{bmatrix} P \\ 0 \\ 0 \\ 0 \end{bmatrix} = B_{\text{sp}}$$

We see how this reduces to known special cases
[eg, $\hat{n} = (1, 0, 0)$].

By the way:

$$N_{ij} = n_i n_j$$

$$B_{\text{sp}} = B^{u,0}$$

So,

$$\begin{bmatrix} B_{00} = \text{ch}\sigma \\ B_{0i} = \text{sh}\sigma \cdot n_i \\ B_{ij} = (\text{ch}\sigma - 1)n_i n_j + \delta_{ij} \end{bmatrix}$$

} compact way to say it.

See page 29 of Weinberg.

$$\text{ch}\sigma = \gamma$$

$$\text{sh}\sigma = \gamma\beta$$

$$\text{ch}\sigma = \gamma$$

What is the significance of the symmetric boost;
Why is it unique.

6

- ① The symmetric boost is the one boost that leaves all $\hat{n}$ - $\perp$ spatial vectors unchanged.

Eg, let $a^\mu = (a^0, \vec{a})$ where $\vec{a} \perp \hat{n}$ so $\vec{a} \cdot \hat{n} = 0$.
Then, consider

$$a'^\mu = B^\mu_\nu a^\nu = B^\mu_\nu (\delta^\nu_0 + \gamma \hat{n}^\nu) a^\nu$$

$$\text{Well, } a'^0 = B^0_0 a^0 + B^0_i a^i = \cosh \cdot a^0 + \sinh \hat{n} \cdot \vec{a} \\ = a^0 \cdot \cosh$$

$$\text{And, } a'^i = B^i_0 a^0 + B^i_j a^j \\ = \sinh a^0 + \cosh a^i$$

$$= \sinh \cdot \hat{n}_i a^0 + (\cosh - 1) \hat{n}_i \hat{n}_j a^j + a^i$$

$$\Rightarrow \vec{a}' = \vec{a} + a^0 \sinh \hat{n}$$

But, if a^μ started out as a purely spatial vector $\perp \hat{n}$,
Then:

$$a'^0 = 0 \\ \vec{a}' = \vec{a}$$

no change

That is: $(0, \vec{a}) \xrightarrow[\text{boost}]{\text{symmetric}} (0, \vec{a})$ if $\vec{a} \perp \hat{n}$

- ② Since any other boost can be written as the symmetric boost with a pre-rotation, we see that any other $\hat{n}$ -boost will not have the above property.

- ③ When we say "a pure boost", we mean exactly this symmetric $\vec{a}$ -preserving boost.

Passive boosts:

$$\Lambda^\mu{}_\nu(10, u) = e^{u \overset{\text{Schublen } M_0}{M_0}{}^{\mu}{}_{\nu}} \approx \delta^\mu{}_\nu + u M_0{}^{\mu}{}_{\nu}$$

$$\Lambda(10, u) = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad M^{10} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\Lambda(20, u) = \begin{pmatrix} \gamma & 0 & -\gamma\beta & 0 \\ 0 & 1 & 0 & 0 \\ -\gamma\beta & 0 & \gamma & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad M^{20} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\Lambda(30, u) = \begin{pmatrix} \gamma & 0 & 0 & -\gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma\beta & 0 & 0 & \gamma \end{pmatrix} \quad M^{30} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

$$e^{+u M^{10}} = e^{-u M^{01}} = e^{+u N_1} = e^{-iu K_1} H$$

$$\begin{aligned} \tanh u &= \beta & E &= \gamma m & &= m \cosh u \\ \cosh u &= \gamma & p &= \beta E = \gamma \beta m & &= m \sinh u \\ \sinh u &= \gamma \beta \end{aligned}$$

$$\sinh \frac{u}{2} = \sqrt{\frac{E-m}{2m}}$$

$$\cosh \frac{u}{2} = \sqrt{\frac{E+m}{2m}}$$

$$\tanh \frac{u}{2} = \frac{p}{E+m} = \frac{\gamma-1}{\gamma\beta} = \frac{\gamma\beta}{\gamma+1}$$

$$\begin{aligned} B &= e^{\vec{u} \cdot \vec{N}} \\ R &= e^{\vec{\theta} \cdot \vec{M}} \\ N^1 &= M^{10} \\ M^1 &= M^{23} \end{aligned}$$

Passive rotations:

$$\Lambda(12, \theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \theta & \sin \theta & 0 \\ 0 & -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad M^{12} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\Lambda(23, \theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos \theta & \sin \theta \\ 0 & 0 & -\sin \theta & \cos \theta \end{pmatrix} \quad M^{23} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

$$\Lambda(31, \theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \theta & 0 & -\sin \theta \\ 0 & 0 & 1 & 0 \\ 0 & \sin \theta & 0 & \cos \theta \end{pmatrix} \quad M^{31} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$e^{\theta M^{12}} = e^{-\theta M^{21}} = e^{+i\theta J_3} H$$

Are Lorentz transformations unitary?

Pure rotations. Since $J = J^\dagger$, answer is yes:

$$R = e^{i\theta J}$$

$$R^\dagger = e^{-i\theta J^\dagger} = e^{-i\theta J} = R^{-1} \quad R^\dagger = R^{-1}$$

Pure boosts. Here, $K = -K^\dagger$, so not so:

$$B = e^{i\theta K}$$

$$B^\dagger = e^{-i\theta K^\dagger} = e^{i\theta K} = B$$

$$B^{-1} = e^{-i\theta K} \neq B, \neq B^\dagger \quad B^\dagger = B$$

relating to other sheets.

$$iJ = R \Leftrightarrow R = -R^\dagger$$

$$iK = B \Leftrightarrow B = B^\dagger$$

Since, in the 4-dim rep of the group, all elements are real, we see that

$$\tilde{R} = R^{-1} \quad \tilde{B} = B$$

The $\hat{n}$ -direction, active, pure (symmetric) boost is:

$$x^\mu = B^\mu{}_\nu x^\nu \quad B^\mu{}_\nu = B^\nu{}_\mu$$

$$B^0{}_0 = \cosh \sigma$$

$$B^0{}_i = \sinh \sigma \cdot n_i$$

$$B^i{}_j = (\cosh \sigma - 1) n_i n_j + \delta_{ij}$$

where $\vec{\beta} = \beta \hat{n}$ is velocity of particle which started out at rest.

Exercise: Derive these equations:

Appear on p322 of Martin; also in Wick.

$$\text{ch}\chi = \text{ch}\sigma \text{ch}\sigma' - \text{sh}\sigma \text{sh}\sigma' \cosh\omega$$

$$S\omega/\text{sh}\chi = S\sigma/\text{sh}\sigma' = S\sigma'/\text{sh}\sigma$$

1. First, understand why $R = \Lambda_p^{-1} \Lambda \Lambda_p$ is a y-rotation if $p' = \Lambda p$. Then, since Λ is a z-boost, we copy: $\Lambda = \Lambda_p R \Lambda_p^{-1}$, where $\Lambda_p = R_0 Z_p$. Always $\phi = 0$.

$$\Lambda_p = R_y(\theta) Z_p = \begin{pmatrix} \text{ch}\theta & 0 & 0 & \text{sh}\theta \\ \text{sh}\theta S_\theta & C_\theta & 0 & \text{ch}\theta S_\theta \\ 0 & 0 & 1 & 0 \\ \text{sh}\theta C_\theta & -S_\theta & 0 & \text{ch}\theta C_\theta \end{pmatrix}, \quad \Lambda_p^{-1} = \begin{pmatrix} \text{ch}\theta & -\text{sh}\theta S_\theta & 0 & -\text{sh}\theta C_\theta \\ 0 & C_\theta & 0 & -S_\theta \\ 0 & 0 & 1 & 0 \\ -\text{sh}\theta & \text{ch}\theta S_\theta & 0 & \text{ch}\theta C_\theta \end{pmatrix}$$

$$R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & C_\omega & 0 & S_\omega \\ 0 & 0 & 1 & 0 \\ 0 & -S_\omega & 0 & C_\omega \end{pmatrix}, \quad \Lambda = \begin{pmatrix} \text{ch}\chi & 0 & 0 & \text{sh}\chi \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \text{sh}\chi & 0 & 0 & \text{ch}\chi \end{pmatrix}$$

2. Now write out $(R\Lambda_p^{-1})$ and adjoin Λ_p ; set equal to Λ :

$$\begin{bmatrix} \text{ch}\chi & 0 & 0 & \text{sh}\chi \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \text{sh}\chi & 0 & 0 & \text{ch}\chi \end{bmatrix} = \begin{bmatrix} \text{ch}\theta & 0 & 0 & \text{sh}\theta \\ \text{sh}\theta S_\theta & C_\theta & 0 & \text{ch}\theta S_\theta \\ 0 & 0 & 1 & 0 \\ \text{sh}\theta C_\theta & -S_\theta & 0 & \text{ch}\theta C_\theta \end{bmatrix} \begin{bmatrix} \text{ch}\theta & -\text{sh}\theta S_\theta & 0 & -\text{sh}\theta C_\theta \\ -\text{sh}\theta S_\omega & C_\omega C_\theta + \text{ch}\theta S_\omega S_\theta & 0 & -S_\theta C_\omega + \text{ch}\theta C_\omega S_\omega \\ 0 & 0 & 1 & 0 \\ -\text{sh}\theta C_\omega & -S_\omega S_\theta + \text{ch}\theta C_\omega S_\theta & 0 & S_\theta S_\omega + \text{ch}\theta C_\theta C_\omega \end{bmatrix}$$

3. Write out ①: at once, $\text{ch}\chi = \text{ch}\theta \text{ch}\theta' - \text{sh}\theta \text{sh}\theta' \cosh\omega$. This is the first equation.

Now, write out ④ and ② together. Neither contains θ' .

Also, write out ⑤ and ③ together. Neither contains θ :

$$\textcircled{1} \quad \text{sh}\chi = A C_\theta + (\text{sh}\theta' S_\omega) S_\theta$$

$$\textcircled{2} \quad 0 = A S_\theta - (\text{sh}\theta' S_\omega) C_\theta$$

$\Downarrow$

$$\text{sh}\chi = \text{sh}\theta' \cdot \frac{S_\omega}{S_\theta}$$

$$\textcircled{3} \quad \text{sh}\chi = A' C_\theta' + (\text{sh}\theta S_\omega) S_\theta'$$

$$\textcircled{4} \quad 0 = A' S_\theta' - (\text{sh}\theta S_\omega) C_\theta'$$

$\Downarrow$

$$\text{sh}\chi = \text{sh}\theta \cdot \frac{S_\omega}{S_\theta'}$$

These comprise the second equations

Notation for Lorentz Rotations

First, Halpern's notation as used in Physics 230A:

$$U = \exp(i \omega_{\mu\nu} M^{\mu\nu}) \approx 1 + i \omega_{\mu\nu} M^{\mu\nu}$$

$$\chi'_\mu = \Lambda_\mu{}^\nu \chi_\nu \quad \Lambda_\mu{}^\nu = \delta_\mu{}^\nu - 2 \omega_\mu{}^\nu$$

$$\chi'_\mu = \chi_\mu - 2 \omega_{\mu\nu} \chi^\nu$$

Next, Bjorken and Drells notation for the same thing:

$$U = \exp(-i/2 \epsilon_{\mu\nu} M^{\mu\nu}) \approx 1 - i/2 \epsilon_{\mu\nu} M^{\mu\nu}$$

$$\chi'_\mu = a_\mu{}^\nu \chi_\nu \quad a_\mu{}^\nu = \delta_\mu{}^\nu + \epsilon_\mu{}^\nu$$

$$\chi'_\mu = \chi_\mu + \epsilon_{\mu\nu} \chi^\nu$$

The relationship between the two notations is very simple:

$$a_\mu{}^\nu = \Lambda_\mu{}^\nu$$

$$\epsilon_{\mu\nu} = -2 \omega_{\mu\nu}$$

Momentum:

$$i[P^\mu, \phi] = \partial^\mu \phi \quad \Rightarrow \quad \phi(x+a) = e^{iP \cdot a} \phi(x) e^{-iP \cdot a}$$

Discussion of The Lorentz generators.

In the "vector" rep, Schweber writes out all 6 L.G. generators. I wanted to sort of "fit" these with a magic formula:

$$[M^{\mu\nu}]^{\rho}_{\sigma} = g^{\mu\rho} g^{\nu}_{\sigma} - g^{\nu\rho} g^{\mu}_{\sigma}$$

This thing works for the boosts:

$$[M^{10}]^1_0 = g^{11} = -1$$

$$[M^{10}]^0_1 = -g^{00} = -1$$

But it screws up on the rotations; since

$$[M^{12}]^1_2 = g^{11} = -1$$

$$[M^{12}]^2_1 = -g^{22} = +1$$

There is some question as to exactly what Sch is showing us. ie, is he displaying

$$[M^{\mu\nu}]^{\rho}_{\sigma}, [M^{\mu\nu}]_{\rho\sigma}, \dots ?$$

Suppose he were really showing $[M^{\mu\nu}]_{\rho\sigma}$. Then we'd say

$$[M^{\mu\nu}]_{\rho\sigma} = \delta^{\mu}_{\rho} \delta^{\nu}_{\sigma} - \delta^{\nu}_{\rho} \delta^{\mu}_{\sigma}$$

And: $[M^{10}]_{10} = 1$

$[M^{10}]_{01} = -1$

Well, for sure he's not doing that.

So let's assume the $^{\rho}_{\sigma}$ form, which is the one which moves contravariant vectors.

Let's construct the generators from scratch. It would say:

$$e^{\mu} M^{\mu\nu} \sim e^{\mu} M^{10} \sim 1 + \mu M^{10}$$

$$[e^{\mu} M^{\mu\nu}]^{\rho}_{\sigma} = \left[\begin{array}{c} \text{stop} \end{array} \right]$$

We must identify our $M^{\mu\nu}$'s more carefully! What are they?

I understand Lubarsky's antihermitean I, J .

Here is Liu's algebra:

$$[I_i, I_j] = -\epsilon_{ijk} I_k$$

$$[J_i, J_j] = +\epsilon_{ijk} I_k$$

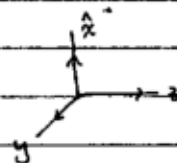
$$[I_i, J_j] = -\epsilon_{ijk} J_k$$

Here is the "usual" angular algebra:

$$[L_i, L_j] = i\epsilon_{ijk} L_k$$

Take a vector in $\hat{x}$: $(0, 0, 0, 1)$.

Apply this:



Here are Liu's generators for $\mathcal{Z}_{\pm\pm}$:

$$\alpha^{i,k} = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \cos\alpha & \sin\alpha \\ & & -\sin\alpha & \cos\alpha \end{bmatrix} =$$

$$\alpha^{i,k} \hat{x}^k = (0, 0, \alpha, 1)$$

Picks up a + y component, this.
This must be a passive by $\alpha > 0$.

Now we wish to say that

$$\alpha^{i,k} = [e^{\alpha I_1}]^i_{\cdot k} \approx 1^i_{\cdot k} + \alpha [I_1]^i_{\cdot k}$$

Thus:

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \alpha \\ & & -\alpha & 1 \end{bmatrix} = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} + \alpha \begin{bmatrix} & & & \\ & & & \\ & & & 1 \\ & & -1 & \end{bmatrix}$$

Conclude that $[I_1]^i_{\cdot k} = E_{23} - E_{32} = \delta^i_2 \delta^3_k - \delta^i_3 \delta^2_k$

If we more simply defined:

$$[S^{23}]^i_{\cdot k} = g^{2i} g^3_k - g^{3i} g^2_k$$

Then $[S^{23}]^2_{\cdot 3} = -1$ wrong sign, so

Let $I_1 = -S^{23}$. Then $[I_1]^i_{\cdot k} = [S^{32}]^i_{\cdot k}$

Better: define: $[S^{\mu\nu}]_{ik} = \delta^\mu_i \delta^\nu_k - \delta^\nu_i \delta^\mu_k$

Then: $[I_1] = -S^{23}$ i.e., $(I_1)^{\alpha}_{\cdot \beta} = -(S^{23})^{\alpha}_{\cdot \beta}$

Now do the ~~rotational~~ boost.

$$\alpha_{ik} = \begin{vmatrix} \cosh u & -\sinh u \\ -\sinh u & \cosh u \end{vmatrix} = \text{passive boost}$$

$$\approx \begin{vmatrix} 1 & -u \\ -u & 1 \end{vmatrix} = 1 + u \begin{vmatrix} -1 & \\ & -1 \end{vmatrix}$$

But, $\alpha_{ik} = e^{u J_i} \approx 1 + u J_i$. Therefore:

$$[J_i]^\alpha_\beta = \begin{vmatrix} 0 & -1 \\ 1 & 0 \end{vmatrix} \checkmark$$

Now, check: $[S^{0i}]^\alpha_\beta = g^\alpha_\beta$.

$$[S^{0i}]^\alpha_\beta = g^\alpha_\beta \delta^\alpha_\beta = +1 \quad \left. \begin{array}{l} [S^{0i}]^\alpha_\beta = -g^\alpha_\beta = +1 \end{array} \right\} \text{both wrong, thus conclude:}$$

$$[S^{0i}]^\alpha_\beta = -g^\alpha_\beta = +1$$

$$J_i = S^{i0}$$

Now the connection between J_i , I_i and $S^{\mu\nu}$ is clear:

boost $e^{\alpha I}$	$J_i = S^{i0}$
rot $e^{\alpha J}$	$I_i = -\epsilon_{ijk} S^{jk}$

$$\text{passive boost} = e^{\alpha J}$$

Now make the connection to Schaubers:

$$\begin{aligned} M &= -I \\ N &= J \end{aligned}$$

Thus, expect

$$\begin{aligned} N_i &= S^{i0} \\ M_i &= S^{ik} \end{aligned}$$

How do Sak's $M^{\mu\nu}$ fit in?

$$M_1 \equiv M_{32} = M^{32} = -M^{23} = S^{23} !$$

$$N_1 = M_{01} = -M^{01} = M^{10} = S^{10} \leftarrow \text{aha!}$$

I expect, therefore, that my $S^{\mu\nu}$ will not have that simple algebra.

How are $S^{\mu\nu}$ and $M^{\mu\nu}$ really related?

$$\begin{aligned} -M^{23} &= S^{23} = S_{23} \\ +M^{10} &= S^{10} = -S_{10} \end{aligned}$$

One could say that: $S_{\mu\nu} = -M^{\mu\nu}$ / but what means?!

$M^{\mu\nu}$ and $S^{\mu\nu}$ are two different tensors. How could I write an algebra for $S^{\mu\nu}$???

? $[S^{\mu\nu}, S^{\rho\sigma}] = -g_{\mu\rho} S^{\sigma\nu}$ etc. / makes no sense.

Well, we do know that, $[S^{i0}, S^{j0}] = -S^{ij}$ etc.

Can't we just convert (28)? No!

I'll bet $S^{\mu\nu} \neq$ tensor.

If you want to use the compact algebra, must use the M 's.

$$[M^{\mu\nu}]^\alpha_\beta = -[S_{\mu\nu}]^\alpha_\beta \quad ?$$

Well, $M^{ij} = -S^{ij}$
 $M^{i0} = S^{i0}$

So the boosts "work". So remember:

$$M^{i0} \leftarrow +$$

$$\begin{aligned} [M^{i0}]^\alpha_\beta &= + (g^{i\alpha} \delta^0_\beta - g^{0\alpha} \delta^i_\beta) \\ [M^{ij}]^\alpha_\beta &= - (g^{i\alpha} \delta^j_\beta - g^{j\alpha} \delta^i_\beta) \end{aligned}$$

cannot write it all in one formula?

What is $[M_{i0}]^\alpha_\beta = ?$ Well, $M^{i0} = -M_{i0}$, so

$$[M_{i0}]^\alpha_\beta = -[M^{i0}]^\alpha_\beta = - (g^{i\alpha} \delta^0_\beta - g^{0\alpha} \delta^i_\beta)$$

$$[M_{ij}]^\alpha_\beta = +[M^{ij}]^\alpha_\beta = - (g^{i\alpha} \delta^j_\beta - g^{j\alpha} \delta^i_\beta)$$

both

A Few Comments on the Relation between $SO(3,1)$ and $SL(2,C)$.

1. First, let's make sure we know what we mean by notation $SO(3,1)$. This is group of 4×4 matrices which preserve the usual form $x \cdot x$, and which have $\det = +1$. The "preservation" condition is the thing you usually see involving the metric tensor. Actually, once you have the "preservation of $x \cdot x$ " condition, $\det$ can only be $+1$ or -1 . The group $O(3,1)$ would allow both, but $SO(3,1)$ picks out $\det = +1$. This is the piece of $O(3,1)$ which is connected to the identity transformation.

2. Now $SO(3,1)$ is not the "proper Lorentz group". If you take only the piece of $SO(3,1)$ such that a_{00} is positive, then you have something I have called $SO(3,1)^+$. This object is the proper Lorentz group, called $\mathcal{L}_+$ by Ruhl, or just $\mathcal{L}_0$ by Messiah. So, the proper Lorentz group is $SO(3,1)^+$. Cannot change a future timelike vector into a past timelike vector, etc.

3. Now if you add to $SO(3,1)^+$ the parity operation s , then you would have both $\det$ s possible. I would call this group $O(3,1)^+$. Messiah calls it $\mathcal{L}_+ \cup s\mathcal{L}_+$. This group is called the "orthochronous Lorentz group". Just the proper plus parity.

4. Finally, or almost finally, we note that the group $SO(3,1)$ where you allow both signs of the a_{00} component has no name. It would be $\mathcal{L}_0 \cup s\mathcal{L}_0$.

5. The "complete" Lorentz group allows for both signs of $\det$, and for both signs of a_{00} . This group I would call just $O(3,1)$.

$$\left. \begin{array}{ll} \rightarrow \det = +, a_{00} > 0 & SO(3,1)^+ \\ \det = +, a_{00} < 0 & SO(3,1)^- \\ \rightarrow \det = -, a_{00} > 0 & s\mathcal{L}_+ \\ \det = -, a_{00} < 0 & s\mathcal{L}_- \end{array} \right\} = SO(3,1) \left\{ \begin{array}{l} \mathcal{L}_0 \\ s\mathcal{L}_0 \end{array} \right\} = O(3,1)$$

Orthochronous.

6. Now the question: to which of these groups does $SL(2,C)$ correspond? According to the connection that I use, $a_{00} = \frac{1}{2} \text{tr}(gg^+)$ since the trace of such a matrix is always positive (see matrix notes). Moreover, I cannot seem to generate a_{00}^u $(-1, 0, 0, 0)$. Thus, it seems that $SL(2,C)$ is related 2:1 to $SO(3,1)^+ = \text{proper } \mathcal{L}_0$.

Returning, we find:

$$G = e^{-(\frac{1}{2}(a+ib)\cdot\sigma)} = e^{(\frac{1}{2}b + i\frac{1}{2}a)\cdot\sigma}$$

$$= e^{-(\frac{i}{2})(a-ib)\cdot\sigma}$$

and $G^\dagger = e^{-(\frac{i}{2})(a+ib)\cdot\sigma}$

We have thus proven that:

$$e^{-(\frac{i}{2})(a+ib)\cdot\sigma} X e^{+(\frac{i}{2})(a-ib)\cdot\sigma} = X'$$

Why not rename things:

$$= g = G^\dagger$$

$$g^\dagger = G$$

Then $\boxed{g X g^\dagger = X'} = e$

If we then write:

$$g = e^{-i[a\cdot\vec{J} + b\cdot\vec{K}]}$$

we find:

$$\boxed{\vec{J} = \frac{1}{2}\sigma \quad \vec{K} = \frac{1}{2}i\sigma}$$

And

$$\boxed{X = \sigma_\mu X^\mu = \mathbb{1} + \sigma \cdot \underline{x}}$$

note.

$\sigma = \sigma_i$ whereas $\underline{x} = x^i$

And,

$$\boxed{x'^\mu = \Lambda^\mu_\nu x^\nu = \left(e^{-i[a\cdot\vec{J} + b\cdot\vec{K}]} \right)^\mu_\nu x^\nu}$$

Problem: show how $gXg^t = X'$ works out.

stated: suppose we have: $G^t X G = X'$, where all objects are 2×2 matrices, and where $\det G = 1 \Rightarrow \det G^t = 1 \Rightarrow \det X = \det X'$.
Let

$$X = \sigma_\mu X^\mu = \sigma_0 X^0 + \sigma_1 X^1 + \sigma_2 X^2 + \sigma_3 X^3 \quad \text{where } \sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Thus $X = \begin{pmatrix} x+z & x-iy \\ x+iy & x-z \end{pmatrix}.$

Assume that: $X'^\mu = \Lambda^\mu_\nu X^\nu = e^{-i[a \cdot \vec{x} + b \cdot \vec{k}]}_\mu \cdot X^\nu$

Question: can we solve for G ?

Solution: since G has $\det = 1$, we should be able to write

$$G = e^{+[\xi + i\eta] \cdot \vec{x}}$$

So the problem reduces to:
$$e^{(\xi - i\eta) \cdot \vec{x}} \sigma_\mu X^\mu e^{(\xi + i\eta) \cdot \vec{x}} = \sigma_\mu X'^\mu$$

$$= \sigma_\mu \left(e^{-i[a \cdot \vec{x} + b \cdot \vec{k}]} \right)_\mu^\nu X^\nu$$

Get rid of X^ν on both sides:

$$e^{(\xi - i\eta) \cdot \vec{x}} \sigma_\nu e^{(\xi + i\eta) \cdot \vec{x}} = \sigma_\mu \left(e^{-i[a \cdot \vec{x} + b \cdot \vec{k}]} \right)_\mu^\nu$$

Now use infinitesimals to get:

$$[1 + (\xi - i\eta) \cdot \vec{x}] \sigma_\nu [1 + (\xi + i\eta) \cdot \vec{x}] = \sigma_\mu \left[\delta_\mu^\nu - i(a \cdot \vec{x} + b \cdot \vec{k})_\mu^\nu \right]$$

Compare, assume all parameters very small:

$$\xi \cdot [\vec{x}, \sigma_\nu] - i\eta \cdot [\vec{x}, \sigma_\nu] = \sigma_\mu \cdot [-i(a \cdot \vec{x} + b \cdot \vec{k})_\mu^\nu]$$

If $\nu=0$, we get: $2\xi \cdot \vec{x} = \sigma_\mu [-i(a \cdot \vec{x} + b \cdot \vec{k})_\mu^0]$

But

$$-i[a \cdot \underline{x} + b \cdot \underline{k}] = \begin{bmatrix} 0 & b_1 & b_2 & b_3 \\ b_1 & 0 & -a_3 & a_2 \\ b_2 & a_3 & 0 & -a_1 \\ b_3 & -a_2 & a_1 & 0 \end{bmatrix} \quad \checkmark$$

→ lower index labels columns

Thus,

$$\sigma_\mu [-i(a \cdot \underline{x} + b \cdot \underline{k})]^\mu_{\cdot 0} = \sigma_0 \cdot 0 + \sigma_1 \cdot b_1 + \sigma_2 \cdot b_2 + \sigma_3 \cdot b_3$$

lower index

$$= \underline{\sigma} \cdot \underline{b} \quad \checkmark$$

Therefore, we must have: $2\underline{c} \cdot \underline{x} = \underline{x} \cdot \underline{b}$

$$\Rightarrow \boxed{\underline{c} = \frac{1}{2} \underline{b}} \quad \checkmark$$

Now, let $\underline{x} = \text{index } j$. Then

$$\underline{c} \cdot [\underline{x}, \underline{\sigma}]_+ - i \underline{d} \cdot [\underline{x}, \underline{\sigma}]_- = \sigma_\mu [-i(a \cdot \underline{x} + b \cdot \underline{k})]^\mu_{\cdot j}$$

$$\text{Or: } 2c_j + 2(\underline{x} \times \underline{d})_j = \sigma_\mu [\quad]^\mu_{\cdot j}$$

The terms on the right involving $\underline{b}$ all occur in the $\sigma_0 = 1$ term. In fact, if we take 2×2 trace of both sides we get

$$2c_j \cdot 2 = 2 \cdot [\quad]^\circ_{\cdot j} = 2 \cdot b_j$$

so we recover our $\underline{v} = 0$ result. So now we have: (set $\underline{b} = \underline{c} = 0$)

$$2(\underline{x} \times \underline{d})_j = \sigma_\mu [-i a \cdot \underline{x}]^\mu_{\cdot j} = \sigma_i [-i a \cdot \underline{x}]^i_{\cdot j}$$

$$\begin{aligned} \text{RHS} &= -i a_k \sigma_i (\underline{x} \cdot \underline{\sigma})^i_{\cdot j} = -i \sigma_i a_k (-i \epsilon_{kij}) = -\epsilon_{kij} \sigma_i a_k \\ &= -\epsilon_{jik} a_k \sigma_i \\ &= +\epsilon_{jik} a_k \sigma_i = (\underline{x} \times \underline{a})_j \end{aligned}$$

Thus $\boxed{\underline{d} = \frac{1}{2} \underline{a}}$

August 24, 1976

Note on the relation between $SO(3,1;R)$ and $SL(2;C)$

1. This was done to verify facts that I have been using for two years now.
2. The basic idea is this: there exists a 2:1 homomorphism between these two groups. One way to connect the two is to "embed" the usual Lorentz transformations into $SL(2,C)$ matrices in this way: $X' = gXg^+$. Here, $X = \sigma_\mu x^\mu = t + \sigma_1 x + \sigma_2 y + \sigma_3 z$. As a single matrix this is: $\begin{pmatrix} t+z & x-iy \\ x+iy & t-z \end{pmatrix}$. Clearly $\det X = x_\mu x^\mu$, the quantity we want to be preserved under Lorentz transformations. Since $\det g = 1$ we see that our embedding insures this fact. The fact that the right side is bilinear in g shows the 2:1 fact. In addition to preserving length, you want your LT to keep the components real. This means that X is unitary. That is why $X' = ggX$ is not acceptable, for example. I.e., unitarity of X must be preserved. If you put the dagger on the other side, you get the standard $SL(2,C)$ generators identified with passive Lorentz boosts. Of course this is all just convention.

3. Now on these sheets I have shown that assuming $X' = gXg^+$, you can easily solve for $\Lambda^\mu{}_\nu$ in terms of g :

$$\Lambda^\mu{}_\nu = \frac{1}{2} \text{tr}(\sigma_\mu g \sigma_\nu g^+)$$

4. Now I assume the standard generators for $SL(2,C)$ so that $J_i = \frac{1}{2}\sigma_i$ and $K_i = iJ_i$. Then general $SL(2,C)$ elements is $\exp(-i[a \cdot J + b \cdot K])$. If you put this in for g , you can then solve for Λ . On page 3 I find the infinitesimal result:

$$\Lambda^\mu{}_\nu = 1, \quad \Lambda^\mu{}_k = b_k, \quad \Lambda^i{}_j = \delta_{ij} - \epsilon_{ijk} a_k$$

5. Now you define $SO(3,1)$ generators in the same way as for $SL(2,C)$. Then you will find the standard forms which yield the standard active boosts and rotations. I.e., you get:

$$[a \cdot J^\mu + b \cdot K^\mu] = i \begin{bmatrix} 0 & b_1 & b_2 & b_3 \\ b_1 & 0 & -a_3 & a_2 \\ b_2 & a_3 & 0 & -a_1 \\ b_3 & -a_2 & a_1 & 0 \end{bmatrix}$$

It is clear that this duplicates the "senseible" generators which go with the active boosts.

6. Earlier, I did the reverse problem: given the standard $SO(3,1)$ generators, I solved for the $SL(2,C)$ generators. Got same answer!

$$① X' = g X g^\dagger$$

$$X = \sigma_\mu x^\mu = t + \sigma_1 x + \sigma_2 y + \sigma_3 z$$

$$X' = \sigma_\mu x'^\mu = \sigma_\mu \Lambda^\mu{}_\nu x^\nu$$

$$\Rightarrow \sigma_\mu \Lambda^\mu{}_\nu x^\nu = g \sigma_\mu x^\mu g^\dagger$$

$$\Rightarrow \boxed{\sigma_\mu \Lambda^\mu{}_\nu = g \sigma_\nu g^\dagger}$$

$$② \text{ Now } \sigma_\alpha \sigma_\mu \Lambda^\mu{}_\nu = \sigma_\alpha g \sigma_\nu g^\dagger$$

$$\Rightarrow \Lambda^\mu{}_\nu \text{ trace}(\sigma_\alpha \sigma_\mu) = \text{trace}(\sigma_\alpha g \sigma_\nu g^\dagger) = 2 \Lambda^\mu{}_\nu \delta_{\alpha\mu} = 2 \Lambda^\alpha{}_\nu$$

$$\Rightarrow \boxed{\Lambda^\mu{}_\nu = \frac{1}{2} \text{tr}(\sigma_\mu g \sigma_\nu g^\dagger)}$$

③ Now set:

$$g = g(\vec{a}, \vec{b}) = g(\vec{a}) = e^{-i\vec{a} \cdot \vec{\sigma}} = e^{-i[\vec{a} \cdot \vec{\sigma} + \vec{b} \cdot \vec{K}]}$$

$$\text{since } \vec{K} = i\vec{\sigma} \quad = \left[\cos\left(\frac{\alpha}{2}\right) - i\vec{a} \cdot \vec{\sigma} \sin\left(\frac{\alpha}{2}\right) \right] (\vec{a} \cdot \vec{\sigma}) = g$$

$$\text{where } \alpha = \sqrt{a^2 - b^2 + 2i\vec{a} \cdot \vec{b}}. \text{ Then: } \left[\cos\left(\frac{\alpha}{2}\right) + i\vec{a} \cdot \vec{\sigma} \sin\left(\frac{\alpha}{2}\right) \right] (\vec{a} \cdot \vec{\sigma}) = g^\dagger$$

④ Note that $\sigma_i^\dagger = \sigma_i$ but $\vec{a}_i = \vec{a}_i$. We obtain:

$$\Lambda^\mu{}_\nu = \frac{1}{2} \text{trace} \left[\sigma_\mu \left\{ \cos\left(\frac{\alpha}{2}\right) - i\vec{a} \cdot \vec{\sigma} \sin\left(\frac{\alpha}{2}\right) \right\} \sigma_\nu \left\{ \cos\left(\frac{\alpha}{2}\right) + i\vec{a} \cdot \vec{\sigma} \sin\left(\frac{\alpha}{2}\right) \right\} \right]$$

⑤ Simplify to

$$\begin{aligned} \Lambda^\mu{}_\nu &= \frac{1}{2} \left| \cos\left(\frac{\alpha}{2}\right) \right|^2 \text{tr}(\sigma_\mu \sigma_\nu) + \frac{1}{2} |\alpha|^2 \left| \sin\left(\frac{\alpha}{2}\right) \right|^2 \text{tr}(\sigma_\mu \vec{a} \cdot \vec{\sigma} \sigma_\nu \vec{a} \cdot \vec{\sigma}) \\ &\quad - \frac{i}{2} \alpha^{-1} \sin\left(\frac{\alpha}{2}\right) \cos\left(\frac{\alpha}{2}\right) \text{tr}(\sigma_\mu \vec{a} \cdot \vec{\sigma} \sigma_\nu) \\ &\quad + \frac{i}{2} \alpha^{-1} \sin\left(\frac{\alpha}{2}\right) \cos\left(\frac{\alpha}{2}\right) \text{tr}(\sigma_\mu \sigma_\nu \vec{a} \cdot \vec{\sigma}) \end{aligned}$$

⑥ Thus:

$$\begin{aligned} \Lambda^\mu{}_\nu &= \delta_{\mu\nu} \left| \cos\left(\frac{\alpha}{2}\right) \right|^2 + |\alpha|^2 \left| \sin\left(\frac{\alpha}{2}\right) \right|^2 \alpha_i \vec{a}_j \frac{1}{2} \text{tr}(\sigma_\mu \sigma_i \sigma_\nu \sigma_j) \\ &\quad + A_i \frac{1}{2} \text{tr}(\sigma_\nu \sigma_\mu \sigma_i) + \bar{A}_i \frac{1}{2} \text{tr}(\sigma_\mu \sigma_\nu \sigma_i) \\ \text{where } A_i &= -\frac{i}{\alpha} \alpha^{-1} \sin\left(\frac{\alpha}{2}\right) \cos\left(\frac{\alpha}{2}\right) \alpha_i \end{aligned}$$

⑦ For $\mu = \nu = 0$, $\frac{1}{2} \hbar (\sigma_i \sigma_j) = \delta_{ij}$

$$|\vec{\Lambda}^0_0|^2 = |\cos(\frac{\alpha}{2})|^2 + |\sin(\frac{\alpha}{2})|^2$$

If $\alpha = \text{real}$, this is $\cos^2(\frac{\alpha}{2}) + \sin^2(\frac{\alpha}{2}) = 1$.

If $\alpha = \text{imag}$, this is $\cosh^2(\frac{|\alpha|}{2}) + \sinh^2(\frac{|\alpha|}{2}) = \cosh|\alpha|$.

Otherwise, a mess.

⑧ Λ^0_k $\mu=0$ $\nu=k$ $\frac{1}{2} \hbar (\sigma_i \sigma_k \sigma_j) = i \epsilon_{ijk}$

$\frac{1}{2} \hbar (\sigma_k \sigma_i) = \delta_{ki}$ $\frac{1}{2} \hbar (\sigma_k \sigma_i) = \delta_{ik}$

$$|\Lambda^0_k|^2 = |\alpha|^2 |\sin \frac{\alpha}{2}|^2 \epsilon_{kij} \bar{\alpha}_i \alpha_j + 2 \text{Re}(A_k)$$

$$A_k = -\frac{i}{4} \bar{\alpha} \sin(\frac{\alpha}{2}) \cos(\frac{\alpha}{2}) \alpha_i$$

If $\alpha = \text{real}$... stop.

Let $\vec{\alpha} = (\alpha_1, 0, 0)$, $\alpha_1 = \text{real}$.

Then $A_k = -\frac{i}{4} \alpha_1 \sin(\frac{\alpha_1}{2}) \cos(\frac{\alpha_1}{2}) \delta_{k,1}$

$= -\frac{i}{4} \sin(\alpha_1) \delta_{k,1}$ $\text{Re}(A_k) = 0$

$\epsilon_{kij} \bar{\alpha}_i \alpha_j = \epsilon_{kij} \alpha_i \alpha_j = 0$

So $|\vec{\alpha} = (\alpha_1, 0, 0) \Rightarrow \Lambda^0_k = 0$

Now let $\vec{\alpha} = (i|\alpha|, 0, 0)$. Then $\epsilon_{kij} \bar{\alpha}_i \alpha_j = \epsilon_{k11} \dots = 0$.

~~Stop. If~~ But $A_k = \frac{i}{2} \sin(\frac{i|\alpha|}{2}) \cos(\frac{-i|\alpha|}{2}) (-i|\alpha|) \delta_{k,1}$

$A_k = -\frac{i}{4} \frac{1}{i} \sin(\frac{\alpha_1}{2}) \cos(\frac{\alpha_1}{2}) \delta_{i,1} = -\frac{1}{4} \sin(\frac{i|\alpha|}{2}) \cos(\frac{-i|\alpha|}{2}) \delta_{i,1}$

$= +\frac{1}{4} \sinh \frac{|\alpha|}{2} \cosh \frac{|\alpha|}{2} \delta_{i,1} = \frac{1}{8} \sinh|\alpha| \delta_{i,1}$

So $|\vec{\alpha} = (i|\alpha|, 0, 0) \Rightarrow \Lambda^0_k = \sinh|\alpha| \delta_{k,1}$

It all looks right so far!

⑨ Look at infinitesimals to get generators:

$$\Lambda_{\mu\nu}^{\mu} = \frac{i}{2} \hbar (\sigma_{\mu} g \sigma_{\nu} g^{\dagger})$$

$$\text{Set } g = e^{-i \vec{\alpha} \cdot \vec{\sigma}} = e^{-\frac{i}{2} \vec{\alpha} \cdot \vec{\sigma}} \approx 1 - \frac{i}{2} (\vec{\alpha} \cdot \vec{\sigma})$$

Then

$$\Lambda_{\mu\nu}^{\mu} = \frac{i}{2} \hbar \left[\sigma_{\mu} \left(1 - \frac{i}{2} \vec{\alpha} \cdot \vec{\sigma} \right) \sigma_{\nu} \left(1 + \frac{i}{2} \vec{\alpha} \cdot \vec{\sigma} \right) \right]$$

Keep only lowest order

$$\Lambda_{\mu\nu}^{\mu} = \delta_{\mu\nu} - \frac{i}{2} \alpha_i \frac{1}{2} \hbar [\sigma_{\mu} \sigma_i \sigma_{\nu}] + \frac{i}{2} \bar{\alpha}_i \frac{1}{2} \hbar [\sigma_{\mu} \sigma_{\nu} \sigma_i]$$

⑩ Now $\Lambda_{00}^0 = 1$

$$\begin{aligned} \Lambda_{0k}^0 &= -\frac{i}{2} \alpha_k + \frac{i}{2} \bar{\alpha}_k = -\frac{i}{2} (\alpha_k - \bar{\alpha}_k) = -\frac{i}{2} (2i \operatorname{Im}(\alpha_k)) \\ &= \operatorname{Im}(\alpha_k) \end{aligned}$$

$$\begin{aligned} \Lambda_{ij}^i &= \delta_{ij} - \frac{i}{2} \alpha_k \frac{1}{2} \hbar [\sigma_i \sigma_k \sigma_j] + \frac{i}{2} \bar{\alpha}_k \frac{1}{2} \hbar [\sigma_i \sigma_j \sigma_k] \\ &= \delta_{ij} - \frac{i}{2} \alpha_k i \epsilon_{ikj} + \frac{i}{2} \bar{\alpha}_k i \epsilon_{ijk} \end{aligned}$$

$$= \delta_{ij} + \frac{1}{2} \alpha_k \epsilon_{jik} - \frac{1}{2} \bar{\alpha}_k \epsilon_{ijk} = \delta_{ij} + \frac{1}{2} \epsilon_{ijk} [\bar{\alpha}_k + \alpha_k]$$

$$= \delta_{ij} + \epsilon_{ijk} \operatorname{Re}(\alpha_k)$$

⑪ Summarize:

$$\epsilon_{123} = \epsilon_{213} = +\epsilon_{231}$$

$$\begin{aligned} g &\equiv e^{-i \vec{\alpha} \cdot \vec{\sigma}} \Rightarrow \begin{aligned} \Lambda_{00}^0 &= 1 \\ \Lambda_{0k}^0 &= \operatorname{Im}(\alpha_k) \\ \Lambda_{ij}^i &= \delta_{ij} + \epsilon_{ijk} \operatorname{Re}(\alpha_k) \end{aligned} \end{aligned}$$

⑫ Now $\Lambda_{\mu\nu}^{\mu} = e^{-i \vec{\alpha} \cdot \vec{\sigma}} \approx 1 - i \operatorname{Re}(\alpha_k) \frac{1}{2} \hbar \epsilon_{ijk} - i i \operatorname{Im}(\alpha_k) K_k$

$$\Lambda_{\mu\nu}^{\mu} = e^{-i [\vec{a} \cdot \vec{J} + \vec{b} \cdot \vec{K}]} \approx \delta_{\mu\nu}^{\mu} - i \vec{a} \cdot \vec{J}^{\mu} - i \vec{b} \cdot \vec{K}^{\mu}$$

$$\Lambda_{\mu\nu}^{\mu} = \delta_{\mu\nu}^{\mu} = \begin{matrix} \nu \rightarrow \\ \mu \uparrow \end{matrix} \begin{bmatrix} 1 & b_1 & b_2 & b_3 \\ b_1 & 1 & -a_3 & +a_2 \\ b_2 & -a_3 & 1 & -a_1 \\ b_3 & +a_2 & -a_1 & 1 \end{bmatrix}$$

here $\nu=2 \quad \mu=1$

$$\Lambda_{12}^1 = \epsilon_{123} a_3 = +a_3$$

$$\Lambda_{13}^1 = \epsilon_{132} a_2 = +a_2$$

$$\Lambda_{23}^1 = \epsilon_{231} a_1 = +a_1$$

circled in

Complex LT

Incorporating parity into the complex Lorentz group.

1. We know what the parity operator does to real Lorentz transformations:

$$s(\underline{x}, 0) \bar{s}^{-1} = (\underline{x}, 0)$$

$$s(0, \underline{x}) \bar{s}^{-1} = (0, -\underline{x})$$

Here, s is a 4×4 matrix which is, I presume, $\text{diag}(1, -1, -1, -1)$. ✓ The commutation of s with the generators must be, in order to make the above work out,

$$s \underline{J} \bar{s}^{-1} = \underline{J}$$

$$s \underline{K} \bar{s}^{-1} = -\underline{K}$$

I.e., if we write a power series for the finite LT's, then this rule for the generators gives the above rule. Similarly, we can show for simultaneous rotation/boost LT that:

$$s(\underline{x}, \underline{x}') \bar{s}^{-1} = (\underline{x}, -\underline{x}')$$

$$\text{or } s \cdot e^{-i[\underline{x} \cdot \underline{J} + \underline{x}' \cdot \underline{K}]} \bar{s}^{-1} = e^{-i[\underline{x} \cdot \underline{J} - \underline{x}' \cdot \underline{K}]}$$

As a test, let's check to see if:

$$s \cdot \{-i[\underline{x} \cdot \underline{J} + \underline{x}' \cdot \underline{K}]\} \bar{s}^{-1} = \{-i[\underline{x} \cdot \underline{J} - \underline{x}' \cdot \underline{K}]\}$$

or:

$$\left[\begin{array}{c|c} 1 & \\ \hline & -1 \end{array} \right] \left[\begin{array}{c|ccc} 0 & \sigma_1 & \sigma_2 & \sigma_3 \\ \hline \sigma_1 & 0 & -\sigma_3 & \sigma_2 \\ \sigma_2 & \sigma_3 & 0 & -\sigma_1 \\ \sigma_3 & -\sigma_2 & \sigma_1 & 0 \end{array} \right] \left[\begin{array}{c} 1 \\ -1 \\ -1 \\ -1 \end{array} \right] \stackrel{?}{=} \left[\begin{array}{c|ccc} 0 & -\sigma_1 & -\sigma_2 & -\sigma_3 \\ \hline -\sigma_1 & 0 & \sigma_3 & -\sigma_2 \\ -\sigma_2 & \sigma_3 & 0 & \sigma_1 \\ -\sigma_3 & -\sigma_2 & \sigma_1 & 0 \end{array} \right] \quad \text{yes} \checkmark$$

$$\left[\begin{array}{c|ccc} 0 & -\sigma_1 & -\sigma_2 & -\sigma_3 \\ \hline \sigma_1 & 0 & \sigma_3 & -\sigma_2 \\ \sigma_2 & \sigma_3 & 0 & \sigma_1 \\ \sigma_3 & -\sigma_2 & \sigma_1 & 0 \end{array} \right] \left[\begin{array}{c} \text{same} \\ \text{minu} \end{array} \right]$$

2. Now, we know that the real Lorentz group is homo to $SL(2, \mathbb{C})$, and we know that rotations do this: $a^+ = a^{-1}$ whereas boosts do this: $a^+ = a$. Thus, in the world of $SL(2, \mathbb{C})$, we want an operator that does this:

$$s a s^{-1} = (a^+)^{-1}$$

I.e., rotations unchanged, boosts negated. However, as discussed in parity misc notes, there is no 2×2 matrix which performs this task (because a and a^{+-1} are not equivalent representations of $SL(2, \mathbb{C})$). Thus, we must here regard s not as a 2×2 matrix, but as an outside operator.

3. Now we can just let the parameters THETA and SIGMA go complex. The action of $s \dots s^{-1}$ on the 4×4 generators is unchanged, just variables go complex. Thus, we conclude that:

$$s(\underline{x}, \underline{x}') \bar{s}^{-1} = (\underline{x}, -\underline{x}') \quad \forall (\underline{x}, \underline{x}') = g \in L_+(\mathbb{C})$$

4. Now that we have established what s does to g , we can convert the result to (a, b) notation using facts recorded in complex LG section. We have:

$$\begin{aligned}
 s: (\mathbb{R}, \mathbb{R}) &\rightarrow (\mathbb{R}, -\mathbb{R}) \Rightarrow a = e^{-i(\mathbb{R}+i\mathbb{R}) \cdot \mathbb{Z}} \rightarrow e^{-i(\mathbb{R}-i\mathbb{R}) \cdot \mathbb{Z}} = (b^+)^{-1} \\
 &\Rightarrow b = e^{-i(\bar{\mathbb{R}}+i\bar{\mathbb{R}}) \cdot \mathbb{Z}} \rightarrow e^{-i(\bar{\mathbb{R}}-i\bar{\mathbb{R}}) \cdot \mathbb{Z}} = (a^+)^{-1} \\
 &\Rightarrow \boxed{s(a, b) \tilde{s}^{-1} = ((b^+)^{-1}, (a^+)^{-1})}
 \end{aligned}$$

5. Lets compare this to Toller's claim made in the Cozensa paper. Recall that Toller and I differ in that he chooses $\text{tr}(o \ a \ o \ \not{b}^{-1})$ whereas I choose $\text{tr}(o \ a \ o \ b^+)$. To keep track of things, I will here always slash his $\not{b}$. Obviously the relations between his stuff and mine are:

$$\not{a} = a \quad \text{and} \quad \not{b} = (b^+)^{-1}$$

Well, this does not agree with Toller's version which is $s(\not{a}, \not{b}) s^{-1} = (\not{b}, \not{a})$. I think the difference is perhaps that he seeks analyticity in a, b whereas I want it in THETA and SIGMA, the Lorentz group parameters.

6. We both agree, however, that:

$$\boxed{s(1, -1) \tilde{s}^{-1} = (-1, 1)}$$

If we denote:

$$\begin{aligned}
 (1, -1) &\equiv g_t \\
 (1, 1) &\equiv e \\
 (-1, -1) &\equiv e'
 \end{aligned}$$

Then:

$$\boxed{s g_t \tilde{s}^{-1} = e' g_t}$$

or, as Toller says,

$$\boxed{s t = e' t s}$$

$$7. \quad g_t^2 \rightarrow (1, -1)(1, -1) = (1, 1) = e \quad \text{so} \quad \boxed{g_t^2 = e}$$

$$\boxed{g g_t = g_t g}$$

Comparison of Complex Lorentz Group and $SL(2, \mathbb{C}) \times SL(2, \mathbb{C})$.

1. An element of the group $SL(2, \mathbb{C})^2$ will be called (a, b) . Each a and b is a unimodular 2×2 complex matrix. The meaning of a "direct product" group is contained in the multiplication rule: $(a, b)(c, d) = (ac, bd)$. Any representation of this group must have this multiplication rule.

2. Let us now construct a representation of this direct product group. The representation space will be the space of 2×2 complex matrices. The representation is defined on this space as follows:

$$R(a, b)X = X' = aXb^\dagger$$

To show that this is in fact a representation, we prove the group multiplication:

$$\begin{aligned} R(a, b)R(c, d)X &= R(a, b)[cXd^\dagger] = acXd^\dagger b^\dagger \\ &= (ac)X(bd)^\dagger = R(ac, bd)X \end{aligned}$$

Notice why the transpose was needed on b . I think this could equally well be an inverse or a dagger. These would give equivalent representations, related 1:1.

3. Now we connect to the Lorentz group. We begin by making a 1:1 map between the space of matrices X and the space of 4-vectors x^μ :

$$\begin{aligned} X^\mu &= \frac{1}{2} \text{tr}(X\sigma_\mu) & \det X &= x_\mu x^\mu \\ X &= x^\mu \sigma_\mu \end{aligned}$$

We see that the above mapping of X into X' defines a map of x^μ into x'^μ . Since it is obvious that $\det X' = \det X$ (since $\det a = \det a^T = \det a^{-1} = \det a^\dagger = 1$), we see that whatever this transformation does, it preserves the length of the 4-vector. Also, we have at once that:

$$x'^\mu = \frac{1}{2} \text{tr}(\sigma_\mu a \sigma_\nu b^\dagger) x^\nu$$

This shows that the transformation is linear. Finally, from the fact proved in subroutine 1, namely

$$\frac{1}{2} [\det(X_1 + X_2) - \det X_1 - \det X_2] = x_1 \cdot x_2$$

you can obviously show that in general $x \cdot y$ is preserved under the transformations in question. But this allows one to conclude at once that

$$g_{\mu\nu} \Lambda^\mu_\alpha \Lambda^\nu_\beta = g_{\alpha\beta}$$

Therefore, the transformation which maps x into x' is some kind of Lorentz transformation.

4. The elements of the group $SL(2, \mathbb{C})^2$ which we called (a, b) are clearly all connected. I.e., you can go from (a, b) to (c, d) smoothly. In fact we have mapped $SL(2, \mathbb{C}) \times SL(2, \mathbb{C})$ into the group called $L_+(C)$ which is one of the two connected pieces of the complex Lorentz group. We know we have gone into this piece because $(a, b) = (1, 1)$ means $X' = X$ means $x' = x$ which is the identity. Thus, we have the "proper complex Lorentz group". Since $\det \Lambda = +1$, we may deduce this perhaps non-obvious identity:

$$\det \left(\text{trace} [\sigma_\mu a \sigma_\nu b^\dagger] \right) = +2 \quad \text{if} \quad \det a = \det b = 1$$

I.e., this determinant of a 4×4 matrix is independent of a and b ! You can quickly check the norm by setting $a=b=e$ and using $\frac{1}{2} \text{tr}(\sigma_\mu \sigma_\nu) = \delta_{\mu\nu}$. Since $\det$ and summation do not commute, I think this identity is highly non-obvious.

5. Where is the real Lorentz group? So far, the elements of $\Lambda_{\cdot\nu}^{\cdot u}$ are in general complex numbers. You can show that if you want these elements to be real, then b and a must be related in a special way, namely $a = \bar{b}$ (complex conjugation).

Proof:

$$\begin{aligned} \frac{1}{2} \text{tr}(\sigma_\mu a \sigma_\nu b^\dagger) &= \frac{1}{2} \text{tr}(\sigma_\mu a \sigma_\nu b^\dagger)^* \\ \Rightarrow &= \frac{1}{2} \text{tr}(\bar{\sigma}_\mu \bar{a} \bar{\sigma}_\nu \bar{b}^\dagger) = \frac{1}{2} (b^\dagger a^\dagger \sigma_\mu^\dagger \sigma_\nu^\dagger) = \frac{1}{2} (b^\dagger a^\dagger \sigma_\mu) = \frac{1}{2} \text{tr}(\sigma_\mu b \sigma_\nu a^\dagger) \\ \Rightarrow &\text{tr}(\sigma_\mu a \sigma_\nu b^\dagger) = \text{tr}(\sigma_\mu b \sigma_\nu a^\dagger) \quad \forall a, b \\ \Rightarrow &a = b \quad \text{or} \quad a = -b \quad (\text{same sign}) \end{aligned}$$

If we choose $b = +a$, then we get $X' = a X a^\dagger$ and $\Lambda_{\cdot\nu}^{\cdot u} = \frac{1}{2} \text{tr}(\sigma_\mu a \sigma_\nu a^\dagger)$. Since this again contains the identity and is connected, it must be $L_+^\uparrow$. If we took the other choice, $b = -a$, we would get the piece of L containing $\Lambda = -1$, namely, $L_+^\downarrow$.

6. What have we shown in terms of analyticity? If $b = a$, the 6 Lorentz parameters are all real. As we move slowly away by slightly varying b , these parameters go complex. It would be nice to know the 6 usual parameters in terms of a and b .

7. We note here another non-trivial identity which follows from the above:

$$g_{\mu\nu} \text{tr}(\sigma_\mu a \sigma_\nu b) \text{tr}(\sigma_\alpha a \sigma_\beta b) = 4 g_{\alpha\beta} \quad \forall a, b \Rightarrow \det a = \det b = 1.$$

Subroutine 1

Show that $x_1 \cdot x_2 = \frac{1}{2} [\det(X_1 + X_2) - \det X_1 - \det X_2]$.

Direct calculation:

$$\det X_1 = x_1 \cdot x_1 \quad \det X_2 = x_2 \cdot x_2$$

$$\det(X_1 + X_2) = \begin{vmatrix} t_1 + t_2 + z_1 + z_2 & x_1 + x_2 - iy_1 - iy_2 \\ x_1 + x_2 + iy_1 + iy_2 & t_1 + t_2 - z_1 - z_2 \end{vmatrix}$$

$$= [(t_1 + t_2) + (z_1 + z_2)][(t_1 + t_2) - (z_1 + z_2)] \\ - [(x_1 + x_2) + i(y_1 + y_2)][(x_1 + x_2) - i(y_1 + y_2)]$$

$$= (t_1 + t_2)^2 - (z_1 + z_2)^2 - (x_1 + x_2)^2 - (y_1 + y_2)^2$$

$$= (t_1^2 - z_1^2 - x_1^2 - y_1^2) + (t_2^2 - z_2^2 - x_2^2 - y_2^2) \\ + 2(t_1 t_2 - z_1 z_2 - x_1 x_2 - y_1 y_2)$$

$$= x_1 \cdot x_1 + x_2 \cdot x_2 + 2x_1 \cdot x_2$$

$$\Rightarrow \frac{1}{2} [\det(X_1 + X_2) - \det X_1 - \det X_2] = \frac{1}{2} [2x_1 \cdot x_2] = x_1 \cdot x_2 \quad \checkmark$$

Conclusion:

$$\boxed{\frac{1}{2} [\det(X_1 + X_2) - \det X_1 - \det X_2] = x_1 \cdot x_2}$$

Finite-dimensional, irreducible representations of the complex Lorentz group.

1. Actually, by "Lorentz group" I here mean exactly $SL(2, \mathbb{C})$. This thing is 2:1 in relation to $L_+^\uparrow$. In the group $SL(2, \mathbb{C})$, the elements 1 and -1 are distinct.

2. By the phrase "complex Lorentz group" I mean for the moment $SL(2, \mathbb{C}) \times \overline{SL(2, \mathbb{C})}$. The matrix elements of the FDI representations are given by this formula:

$$D_{m_1 m_2, m_1' m_2'}^{(s_1, s_2)}(a, b) = D_{m_1 m_1'}^{(s_1, 0)}(a) D_{m_2 m_2'}^{(s_2, 0)}(\bar{b})$$

The functions appearing on the right are the standard FD $(s, 0)$ representations of the real Lorentz group, or $SL(2, \mathbb{C})$, used eg by Taylor. I will summarize the properties of these standard functions on another sheet. The only properties we will use here are these:

$$\begin{aligned} D^{(s, 0)}(1) &= 1 & D^{(s, 0)}(\bar{a}) &= \overline{D^{(s, 0)}(a)} \\ D^{(s, 0)}(-1) &= (-1)^{2s} 1 & &= D^{(0, s)}(a) \end{aligned}$$

3. Using the above form for the representation functions, it is easy to verify the $SL(2, \mathbb{C}) \times SL(2, \mathbb{C})$ multiplication group property, which is $(a, b) \cdot (a', b') = (aa', bb')$. Thus, we have at least verified that the above representation is in fact a group representation of $SL(2, \mathbb{C}) \times SL(2, \mathbb{C})$. The fact that it is irreducible is almost obvious but will be reviewed below.

4. The way you generate your complex Lorentz transformations in $L_+^\uparrow$ is as follows:

$$\Lambda(a, b)^\mu{}_\nu = \frac{1}{2} \text{trace}(\sigma_\mu a \sigma_\nu b^\dagger)$$

Notice that (a, b) and $(-a, -b)$ both map into the same complex Lorentz transformation. In order to show that this really is a LT, you must show this:

$$g_{\mu\nu} \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta = g_{\alpha\beta}$$

This is true, but easier to show indirectly, see other sheets. We find that we have mapped the group $SL(2, \mathbb{C}) \times SL(2, \mathbb{C})$ onto the group $L_+^\uparrow(\mathbb{C})$ in a 2:1 fashion.

5. Some special cases of the above formula:

$$D_{m_1 0, m_1' 0}^{(s, 0)}(a, b) = D_{m_1 m_1'}^{(s, 0)}(a, b) = D_{m_1 m_1'}^{(s, 0)}(a)$$

$$D_{0 m_2, 0 m_2'}^{(0, s)}(a, b) = D_{m_2 m_2'}^{(0, s)}(a, b) = D_{m_2 m_2'}^{(s, 0)}(\bar{b}) = D_{m_2 m_2'}^{(0, s)}(b)$$

Thus, when we go to the real Lorentz transformations we find:

$$D_{m_1 m'_1}^{(s_1, 0)}(a, a) = D_{m_1 m'_1}^{(s_1, 0)}(a)$$

$$D_{m_2 m'_2}^{(0, s_2)}(b, b) = D_{m_2 m'_2}^{(0, s_2)}(b)$$

6. We are now ready to prove the rule given in (1.). First, we use the direct-product property as given in Liubarski:

$$D_{m_1 m_2, m'_1 m'_2}^{(s_1, s_2)}(a, b) = D_{m_1 m'_1}^{(s_1, 0)}(a, b) D_{m_2 m'_2}^{(0, s_2)}(a, b)$$

Next, we observe that the rep $(s, 0)$ can be found in a certain product of lower reps. We write this as follows:

$$D^{(s_1, 0)}(a, b) = \pi(\text{clebsch}) \cdot \pi D^{(\frac{1}{2}, 0)}(a)$$

$$D^{(0, s_2)}(a, b) = \pi(\text{clebsch}) \cdot \pi D^{(0, \frac{1}{2})}(b)$$

These facts are just like those in $SU(2)$ and follow, for example, from:

$$(\frac{1}{2}, 0) \otimes (\frac{1}{2}, 0) \otimes (\frac{1}{2}, 0) \dots = \dots + (s_1, 0) + \dots$$

The point is that $D^{(s_1, 0)}(a, b)$ is independent of b , so we can set $b = a$ to get the regular real Lorentz transformation. The second line is independent of a so we set $a = b$. This then gives:

$$\begin{aligned} D^{(s_1, 0)}(a, b) &= D^{(s_1, 0)}(a, a) = D^{(s_1, 0)}(a) \\ D^{(0, s_2)}(a, b) &= D^{(0, s_2)}(b, b) = D^{(0, s_2)}(b) = D^{(s_2, 0)}(\bar{b}) \end{aligned}$$

We have thus proven formula (1). In addition, we find these extra properties:

$$\begin{aligned} D_{m_1 m_2, m'_1 m'_2}^{(s_1, s_2)}(\bar{a}, \bar{b}) &= \overline{D_{m_1 m_2, m'_1 m'_2}^{(s_1, s_2)}(a, b)} \\ D_{m_1 m_2, m'_1 m'_2}^{(s_1, s_2)}(b, a) &= D_{m_2 m_1, m'_2 m'_1}^{(s_2, s_1)}(a, b) \\ D_{m_1 m_2, m'_1 m'_2}^{(s_1, s_2)}(1, 1) &= \delta_{m_1, m'_1} \cdot \delta_{m_2, m'_2} \\ D_{m_1 m_2, m'_1 m'_2}^{(s_1, s_2)}(-1, 1) &= (-1)^{2s_1} \delta_{m_1, m'_1} \delta_{m_2, m'_2} \quad \text{etc.} \end{aligned}$$

How do you connect the 12 parameters of $L_+(C)$ to those of $SL(2,C)^2$?

1. We have a and b as our two $SL(2,C)$ matrices. Actually, we are using $SL(2,C) \times SL(2,C)$ -bar, since this makes for easier relations. Eg, a real Lorentz transformation then occurs when $a = b$. Let θ = rotation parameters and ϕ = boost parameters. Then:

$$a = e^{-i[\theta \cdot J + \phi \cdot K]} = \exp[-i[\theta_a + i\theta_b] \cdot J] = e^{-\frac{i}{2}[\theta_a + i\theta_b] \cdot \sigma}$$

Let $z \equiv \theta_a + i\theta_b$ = complex rotation parameters.

$$\text{Then: } a = e^{-\frac{i}{2}(z \cdot \sigma)} \text{ and } b = e^{-\frac{i}{2}(\bar{z} \cdot \sigma)} \Rightarrow b^\dagger = e^{\frac{i}{2}(\bar{z} \cdot \sigma)}$$

$$\text{So: } \Lambda^\mu{}_\nu = \frac{1}{2} \text{tr}(\sigma_\mu a \sigma_\nu b^\dagger)$$

$$\Rightarrow (e^{-i[\theta \cdot J + \phi \cdot K]})^\mu{}_\nu = \frac{1}{2} \text{tr} \left[\sigma_\mu \left(e^{-\frac{i}{2} z \cdot \sigma} \right) \sigma_\nu \left(e^{\frac{i}{2} \bar{z} \cdot \sigma} \right) \right]$$

We don't want to show that this is true because we already know it is true. All we want is the relation between the parameters, so we may for this purpose go to infinitesimals:

$$(1)^\mu{}_\nu - i[\theta \cdot J + \phi \cdot K]^\mu{}_\nu = \frac{1}{2} \text{tr} \left[\sigma_\mu \left(1 - \frac{i}{2} z \cdot \sigma \right) \sigma_\nu \left(1 + \frac{i}{2} \bar{z} \cdot \sigma \right) \right]$$

$$\Rightarrow -i[\theta \cdot J + \phi \cdot K]^\mu{}_\nu = -\frac{i}{4} z \cdot \text{tr}(\sigma_\nu \sigma_\mu \sigma) + \frac{i}{4} \bar{z} \cdot \text{tr}(\sigma_\mu \sigma_\nu \sigma)$$

Now special cases:

$$\begin{aligned} \sigma_i = -i[\theta \cdot J + \phi \cdot K]^\circ{}_i &= -\frac{i}{4} z \cdot \text{tr}(\sigma_i \sigma) + \frac{i}{4} \bar{z} \cdot \text{tr}(\sigma_i \sigma) \\ &= \frac{i}{2} [\bar{b}_i - a_i] \end{aligned}$$

Note: if $a=b$, then $z=b$ and $\bar{a}_i - a_i = -2a_i \text{Im} a_i = -2a_i \theta_i \Rightarrow \sigma_i = \sigma_i^a \checkmark$

Next:

$$\begin{aligned} \theta_i &= -\frac{1}{2} \epsilon_{ijk} (-i) [\theta \cdot J + \phi \cdot K]^j{}_k \\ &= -\frac{1}{2} \epsilon_{ijk} \left[-\frac{i}{4} z \cdot \text{tr}(\sigma_k \sigma_j \sigma) + \frac{i}{4} \bar{z} \cdot \text{tr}(\sigma_j \sigma_k \sigma) \right] \end{aligned}$$

$$\Rightarrow \Theta_i = \left(\frac{1}{2}\right) \epsilon_{ijk} \left(\frac{i}{4}\right) \left[\bar{b}_l h(\sigma_i \sigma_k \sigma_l) - a_l h(\sigma_k \sigma_j \sigma_l) \right]$$

$$= \left(\frac{i}{4}\right) \epsilon_{ijk} \left[\bar{b}_l i \underbrace{\epsilon_{jkl}}_{\epsilon_{ljk}} - a_l i \underbrace{\epsilon_{klj}}_{\epsilon_{lkj} = -\epsilon_{ljk}} \right]$$

$$\Theta_i = \left(+\frac{i}{4}\right) \epsilon_{ijk} \epsilon_{ljk} [\bar{b}_l + a_l] = \left(+\frac{i}{4}\right) \cdot 2 \delta_{il} [\bar{b}_l + a_l] = +\frac{1}{2} [\bar{b}_i + a_i]$$

$$\text{So } \Theta_i = +\frac{1}{2} [a_i + \bar{b}_i].$$

$$\text{Check in real case: } a=b \Rightarrow +\frac{1}{2} [a_i + \bar{a}_i] = +\frac{1}{2} [2 \operatorname{Re} a_i] = +\Theta_i^a \checkmark$$

Summary:

$$\Theta_i = -\frac{1}{2} \epsilon_{ijk} \left\{ (-i) [\mathcal{Q} \cdot \mathcal{I} + \mathcal{Q} \cdot \mathcal{K}] \right\}_{.k}^i$$

$$\sigma_i = \left\{ (-i) [\mathcal{Q} \cdot \mathcal{I} + \mathcal{Q} \cdot \mathcal{K}] \right\}_{.i}^0$$

From these, we find

$$\Theta_i = \frac{1}{2} [a_i + \bar{b}_i]$$

$$\sigma_i = \frac{1}{2i} [a_i - \bar{b}_i]$$

$$a_i = (\Theta^a)_i + i(\sigma^a)_i$$

$$b_i = (\Theta^b)_i + i(\sigma^b)_i$$

$$\operatorname{Re} \Theta_i = \frac{1}{2} [(\Theta^a)_i + (\Theta^b)_i]$$

$$\operatorname{Re} \sigma_i = \frac{1}{2} [(\sigma^a)_i + (\sigma^b)_i]$$

$$\operatorname{Im} \Theta_i = \frac{1}{2} [(\sigma^a)_i - (\sigma^b)_i]$$

$$\operatorname{Im} \sigma_i = -\frac{1}{2} [(\Theta^a)_i - (\Theta^b)_i]$$

$$\Theta_i^a = \operatorname{Re} \Theta_i - \operatorname{Im} \sigma_i$$

$$\sigma_i^a = \operatorname{Re} \sigma_i + \operatorname{Im} \Theta_i$$

$$\Theta_i^b = \operatorname{Re} \Theta_i + \operatorname{Im} \sigma_i$$

$$\sigma_i^b = \operatorname{Re} \sigma_i - \operatorname{Im} \Theta_i$$

Examples

$$\textcircled{1} \quad B_z(i\pi) \Rightarrow \Theta_3 = i\pi \rightarrow \det \sigma_3 = \pi \Rightarrow \begin{aligned} \Theta_3^a &= -\pi \\ \Theta_3^b &= +\pi \end{aligned}$$

$$\text{So: } \underline{\Lambda_{\nu}^{\mu} [B_z(i\pi)] \leftrightarrow (a, b) = (R_z(-\pi), R_z(+\pi))}$$

$$\textcircled{2} \quad \underline{\Lambda_{\nu}^{\mu} [R_z(\pi)] \leftrightarrow (a, b) = (R_z(\pi), R_z(\pi))}$$

$$\textcircled{3} \quad \text{Thus: } \begin{aligned} \Lambda_{\nu}^{\mu} [B_z(i\pi) R_z(\pi)] &\leftrightarrow (a, b) = (1, -1) & (a) \\ \Lambda_{\nu}^{\mu} [B_z(i\pi) R_z(-\pi)] &\leftrightarrow (a, b) = (-1, 1) & (b) \end{aligned}$$

Of course both of these are $\Lambda_{\nu}^{\mu} = -\delta_{\nu}^{\mu}$ and $\Lambda = -1$.

Here is a simple application of this last result: consider the M-functions of Taylor, functions of the helicities and 4-momenta. Lorentz invariance on $L_+(C)$ tells us that:

$$\boxed{M_{m_1, \dots} (p_1, \dots) = \pi D^{(0, s)}(a, b) M_{m_1, \dots} (\Lambda(a, b) p_1, \dots)}$$

$$= M_{m_1, \dots} (-p_1, \dots) \quad \text{by (a)}$$

$$= \pi (-1)^{\sum s_i} M_{m_1, \dots} (-p_1, \dots) \quad \text{by (b)}.$$

$$\Rightarrow \prod_{\text{all}} (-1)^{2s_i} = +1 \Rightarrow 2 \sum s_i = \text{even} \Rightarrow \boxed{\sum s_i = \text{integer}}$$

The number of half-integral spin particles must be even, or the sum of all spins must be an integer. Of course this is obvious, but amusing to check in this way.

Complex Lorentz Group and Analytic Continuation.

1. Let g_i be elements of the abstract real Lorentz group. This group has 6 generators called the J_i and the K_i , the rotations and boosts. Since no rotation can change the parity of a 3-vector, and since no LT can change the futureness of a 4-vector, the group generated by just these 6 generators is $SO(3,1)^+$ or $L_+^\uparrow$. This is the first of the four pieces of the real Lorentz group, sometimes called the restricted Lorentz group. This piece is 1:2 homomorphic with $SL(2, \mathbb{C})$.

We can label each g_i in $L_+^\uparrow$ with a set of 6 parameters called $p_i = (\underline{a}_i, \underline{b}_i)$. These parameters are real, three boost and three rotation parameters. The group multiplication when converted to these parameters is really a set of equations:

$$g_3 = g_1 g_2 \quad \text{is same as} \quad p_3 = f(p_1, p_2)$$

Ie, given any two group elements with parameters p_1 and p_2 , there is a function f which tells you the parameters of the new group element g_3 whose parameters are p_3 . Really this is 6 functions and each function is a function of 12 real parameters.

2. Analytics: if we simply continue these parameter equations so that they are functions of 12 complex parameters, we know what we mean by $g_3 = g_1 g_2$ where all g_i are now elements of the complex Lorentz group $L_+(\mathbb{C})$. Ie, given any points g_1 and g_2 , each described by 6 complex parameters, I can tell you the 6 complex parameters of the point g_3 by just computing them from the analytically continued equations shown above. This is what you mean by saying that $L_+(\mathbb{C})$ is just a complexification of $L_+^\uparrow$. Complexification and analytic continuation are the same thing.

3. The (a,b) notation. Any point g_i in the complex Lorentz group can be labelled either by the 6 complex parameters $p = (\underline{a}, \underline{b})$, or by a pair of $SL(2, \mathbb{C})$ matrices a and b . In other words, there is a 1:2 homomorphism between the groups $L_+(\mathbb{C})$ and $SL(2, \mathbb{C}) \times SL(2, \mathbb{C})$. If p corresponds to (a, b) , then it also corresponds to $(-a, -b)$, hence the 2:1 stuff. To show we have a homomorphism, we can say:

$$\underline{SL(2, \mathbb{C}) \times SL(2, \mathbb{C})}: \quad g_i \rightarrow (a_i, b_i)$$

$$g_1 g_2 = g_3 \rightarrow (a_1, b_1) \cdot (a_2, b_2) = (a_1 a_2, b_1 b_2) = (a_3, b_3)$$

$$\underline{L_+(\mathbb{C})}: \quad g_i \rightarrow \Lambda_{\mu\nu}^{\mu}(p)$$

$$g_1 g_2 = g_3 \rightarrow \Lambda_{\mu\nu}^{\mu}(p_1) \Lambda_{\mu\nu}^{\nu}(p_2) = \Lambda_{\mu\nu}^{\mu}(p_3)$$

Homomorphism:

$\rho_i \leftrightarrow (a_i, b_i)$ $a_i = a \text{ } 2 \times 2 \text{ matrix}$ $\leftarrow 6 \text{ complex \#s for } g_i$

$\rho_i = (\underline{a}_i, \underline{b}_i)$ $\rightarrow a_i = \exp \left[-i (\underline{a}_i + i \underline{b}_i) \cdot \underline{J} \right]$ $\underline{J} = \frac{1}{2} \underline{\sigma}$ *Pauli*

$\uparrow$ $3 \text{ complex \#s}$ $b_i = \exp \left[-i (\underline{a}_i^* + i \underline{b}_i^*) \cdot \underline{J} \right]$

$\left. \begin{array}{l} a_i = \exp \left[-i (\underline{a}_i + i \underline{b}_i) \cdot \underline{J} \right] \\ b_i = \exp \left[-i (\underline{a}_i^* + i \underline{b}_i^*) \cdot \underline{J} \right] \end{array} \right\} \text{note: } \underline{a}_i, \underline{b}_i \text{ real} \Rightarrow a_i = b_i$

To show homomorphism, not only must you make the 1:1 correspondence, but also you have to show that group multiplication corresponds:

$$\begin{aligned} \Lambda_{\nu}^{\mu}(\rho_3) &= \Lambda_{\nu}^{\mu}(a_3, b_3) = \frac{1}{2} \text{tr}(\sigma_{\mu} a_3 \sigma_{\nu} b_3^{\dagger}) = \frac{1}{2} \text{tr}(\sigma_{\mu} a_1 a_2 \sigma_{\nu} (b_1 b_2)^{\dagger}) \\ &= \Lambda_{\nu}^{\mu}(a_1, b_1) \Lambda_{\nu}^{\mu}(a_2, b_2) \quad / \text{by trace theorem identity} \\ &= \Lambda_{\nu}^{\mu}(\rho_1) \Lambda_{\nu}^{\mu}(\rho_2) \end{aligned}$$

So if $\rho_i \leftrightarrow (a_i, b_i)$, then

$$\begin{aligned} \rho_1, \rho_2 &\rightarrow \rho_3 \\ \rightarrow (a_1, b_1)(a_2, b_2) &\rightarrow (a_3, b_3) \checkmark \end{aligned}$$

4. Representation functions: Because we know the representations of $SL(2, \mathbb{C})$, we also know the representations of $SL(2, \mathbb{C}) \times SL(2, \mathbb{C})$. This means that, using the (a,b) parametrization, we already know the matrix elements of the complex Lorentz group. Recall that the representations are the same for complex LG as for real LG because the Lie algebra is the same. Roughly speaking, we had $L_+^{\uparrow} = SU_2 \times SU_2$ which told us how to make FDR's of real LG (diagonalize A_3 and B_3 etc). Now we extend things to $L_+(C) = SL(2, \mathbb{C}) \times SL(2, \mathbb{C})$.

In particular, in the (a,b) parameters we know that:

$$\text{basis vector} = |s_1, s_2; m_3^A, m_3^B\rangle$$

$$\begin{aligned} D_{m_3^A m_3^B, m_3^A m_3^B}^{(s_1, s_2)}(a, b) &= \langle s_1, s_2; m_3^A, m_3^B | U(a, b) | s_1, s_2; m_3^A, m_3^B \rangle \\ &= D^{(s_1, s_2)} \cdot D^{(s_1, 0)} \\ &= D \dots \text{as in appendix.} \end{aligned}$$

In general, we may replace (a,b) with abstract element g and write $D^{(s_1, s_2)}(g)$ with the understanding that if you are given g in terms of the p parameters, you must convert to the a,b form to see what you have. In particular:

$$D_{mm'}^{(0,s)}(g) = D_{mm'}^{(0,s)}(a,b) = D_{mm'}^{(0,s)}(a) \quad \text{where } a = e^{-i[\varrho + i\varrho] \cdot \mathbf{J}} \in SL(2, \mathbb{C}).$$

5. The Special LT for I_{st} .

Let's look at $\Lambda_{\nu}^{\mu}(g) = \Lambda_{\nu}^{\mu}(a,b) = \frac{1}{2} \text{tr}(\sigma_{\mu} a \sigma_{\nu} b^{\dagger})$.

If $a=b=\sigma_0$, we get $\Lambda_{\nu}^{\mu}(\sigma_0, \sigma_0) = \delta_{\nu}^{\mu}$.

But $\Lambda_{\nu}^{\mu}(\sigma_0, -\sigma_0) = \Lambda_{\nu}^{\mu}(-\sigma_0, \sigma_0) = -\delta_{\nu}^{\mu}$. But

$$a = \sigma_0 \Rightarrow \varrho = \varrho_1 \Rightarrow \varrho + i\varrho = 0 \Rightarrow \varrho = -i\varrho$$

$$b = -\sigma_0 \Rightarrow \varrho = 2\pi\hat{n} \Rightarrow \bar{\varrho} + i\bar{\varrho} = 2\pi\hat{n} \Rightarrow \varrho - i\varrho = 2\pi\hat{n}$$

$$\Rightarrow \varrho = \pi\hat{n} \quad \text{and} \quad \varrho = i\pi\hat{n}$$

To summarize:

$$(a,b) = (1,-1) \Rightarrow \Lambda_{\nu}^{\mu} = -1_{\nu}^{\mu} \quad \text{and} \quad (\varrho, \bar{\varrho}) = (\pi\hat{n}, i\pi\hat{n})$$

$$(a,b) = (-1,1) \Rightarrow \Lambda_{\nu}^{\mu} = -1_{\nu}^{\mu} \quad \text{and} \quad (\varrho, \bar{\varrho}) = (\pi\hat{n}, -i\pi\hat{n}).$$

$$SL(2, \mathbb{C})^2$$

$$L_+(\mathbb{C})$$

$$2$$

$$:$$

$$1$$

$$:$$

$$2$$

Another case:

$$(a,b) = (1,1) \Rightarrow \Lambda_{\nu}^{\mu} = 1_{\nu}^{\mu} \quad \text{and} \quad (\varrho, \bar{\varrho}) = (\varrho, \varrho)$$

$$(a,b) = (-1,-1) \Rightarrow \Lambda_{\nu}^{\mu} = 1_{\nu}^{\mu} \quad \text{and} \quad (\varrho, \bar{\varrho}) = (2\pi\hat{n}, \varrho) \text{ as a possible candidate}$$

so $(-1,-1) \leftrightarrow$ usual 2π rotation.

Comments on the Complex Lorentz Transformation Business.

1. Basically, there are two kinds of things I want to "explain" with these complex Lorentz transformations. First, I want to show why it is that the Breit frame kinematics look so much like cms kinematics. Secondly, I want to understand the great similarity between the single parameter subgroup matrix elements for $SO(3)$ and for $SO(2,1)$, namely, why they look the same except for the fact that $\cos\theta$ gets replaced with $\cosh\theta$.
2. I am now beginning to think that these two questions have essentially different types of answers. The first question I can explain by using my very special $SO(3,1;C) \times SU(1)$ transformation which connects a t-z cms frame to a t-z bws frame of two particles. This transformation shows clearly why things have the same form, and it really is some kind of complex, unphysical Lorentz transformation.
3. On the other hand, the second question may be answerable without going to these complex Lorentz transformations. We know that when we are dealing with the matrix elements of $SO(3)$ and $SO(2,1)$, we are really operating with $SU(2)$ and $SU(1,1)$ which are both subgroups of $SL(2,C)$. But, we already have shown that inside $SL(2,C)$ a real y rotation is equivalent in all respects to an imaginary y boost: ie, it is completely indistinguishable. Thus, if you are doing your usual $SO(3)$ with a real y rotation in there, and if you are doing your usual $SO(2,1)$ with a real y boost in there, then ~~that~~ anything $SL(2,C)$ involving these two (such as matrix elements) will have a very simple relation. Ie, if you let that y rotation go imaginary, then you have the same thing as a real y boost and you effectively go over from $SO(3)$ to $SO(2,1)$. !!!!!

"Complex" Lorentz Xforms and $BWS \leftrightarrow CMS$.

① Comment: you must do this in $SO(3,1)$ + because there is no room in $SL(2, \mathbb{C})$ for such complexification. I.e., in $SL(2, \mathbb{R})$, an "imaginary z -boost" is a z -rotation. Not so in $SO(3,1)$

② What is the name of complex Lorentz group? I know that $PLG = SO(3,1)$, perhaps we should call this $SO(3,1; \mathbb{R})$. Consider $SO(p, q; \mathbb{R})$. what are the conditions? This preserves:

$$x_1^2 + x_2^2 + \dots + x_q^2 - x_{q+1}^2 - \dots - x_{p+q}^2$$

So construct a metric tensor

$$g_{\mu\nu} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 & & \\ & & & & -1 & \\ & & & & & \ddots \\ & & & & & & -1 \end{pmatrix} \quad \text{Then} \quad g_{\mu\nu} X^\mu X^\nu = \text{invariant.}$$

So then the condition on a matrix becomes.

$$X \cdot Y = g_{\mu\nu} X^\mu Y^\nu = X' \cdot Y' = g_{\mu\nu} X'^\mu Y'^\nu = g_{\mu\nu} [M^\mu{}_\alpha X^\alpha] [M^\nu{}_\beta Y^\beta]$$

$$\text{or } g_{\alpha\beta} X^\alpha Y^\beta = g_{\mu\nu} M^\mu{}_\alpha M^\nu{}_\beta X^\alpha Y^\beta$$

$$\Rightarrow \boxed{g_{\alpha\beta} = g_{\mu\nu} M^\mu{}_\alpha M^\nu{}_\beta}$$

$$\boxed{SO(p, q; \mathbb{R})}$$

By the way, $\det(g_{\alpha\beta}) = (-1)^{p+q}$. This $\rightarrow$ This says $\det |M^\mu{}_\alpha| = \pm 1$.
Let $p+q = n$. Then this is n^2 equations, but symmetric, so only $(n^2 + n)/2 \equiv n(n+1)/2$ equations. Each equation is a condition on the parameters of $M^\mu{}_\alpha$. Thus, instead of all n^2 parameters, you are left with $n^2 - [n(n+1)/2] = n(n-1)/2$.

$$\boxed{n = p+q}$$

$$\boxed{n(n-1)/2 \text{ parameters.}}$$

③ Now, let all parameters go complex. Everything stays the same, the metric tensor, ~ but the number of real parameters doubles. Thus:

$$\boxed{\text{complex Lorentz Group} = SO(3,1; \mathbb{C})}$$

$$n(n-1) = 4 \cdot 3$$

$$= 12 \text{ params}$$

This group has 12 generators. In general, a complex L.T. does not change $p_\mu p^\mu$, so cannot by itself change a timelike to spacelike...

$\hat{G}$ = regular L.G. generators J_i, K_i

$\hat{G}$ = imaginary ones.

④ Consider: $B_z(i\frac{\pi}{2}) = \begin{bmatrix} \cosh(\pi/2) & -\sinh(\pi/2) \\ \sinh(\pi/2) & \cosh(\pi/2) \end{bmatrix} = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$

Now $B_z(i\frac{\pi}{2}) \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ i \end{pmatrix}$

So: $(-i) B_z(i\frac{\pi}{2}) \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

Maybe we are really talking $SO(3,1; \mathbb{C}) \times SU(1)$

The $SU(1)$ allows phases. Thirteen parameter group?

⑤ Can I rephase without the $SU(1)$. No! Otherwise, never change sign of metric!

$\tilde{O} = (-i) B_z(i\frac{\pi}{2}) = \begin{pmatrix} 0 & & & 1 \\ & -i & & \\ & & -i & \\ 1 & & & 0 \end{pmatrix}$

$\tilde{O} \hat{x} = \hat{z} \quad \tilde{O} \hat{z} = \hat{x}$

And I think $\tilde{O}^\dagger = \tilde{O}^{-1}$ so unitary... = $\tilde{O}^*$

$\begin{bmatrix} 1 & & & \\ & -i & & \\ & & -i & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & i & & \\ & & i & \\ & & & 1 \end{bmatrix} = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \checkmark$

Complex LT connects CMS to BWS.

8

What complex L.T. takes $(1, 0, 0, 0)$ to $(0, 0, 0, 1)$?

① Let $(1, 0, 0, 0) = \hat{x}$ and $(0, 0, 0, 1) = \hat{z}$. Now:

$$B_z(\xi) \hat{x} = (\cosh \xi, 0, 0, \sinh \xi)$$

Can we find ξ s.t.: $\cosh \xi = 0$ $\sinh \xi = 1$

Well: This means:

$$e^{\xi} + e^{-\xi} = 0$$

$$e^{\xi} = 1$$

$$e^{\xi} - e^{-\xi} = 2$$

$$e^{-\xi} = -1$$

There is no such ξ . Thus, a complex z -boost does not work.

② Let's try an x -boost:

$$B_x(\xi) \hat{x} = (\cosh \xi, \sinh \xi, 0, 0) \quad \text{So what!}$$

③ If we do an imaginary z -boost, $\xi = i|\xi|$, then

$$\cosh \xi = \cos |\xi|$$

$$\sinh \xi = i \sin |\xi|$$

$$\text{so } B_z(i|\xi|) \hat{x} = (\cos |\xi|, 0, 0, i \sin |\xi|)$$

The result is still timelike! But, if we choose $\xi = i\pi/2$, then:

$$B_z(i\pi/2) \hat{x} = (0, 0, 0, i) = i \hat{z}$$

That's close to $\hat{z}$. So:

$$\boxed{(-i) B_z(i\pi/2) \hat{x} = \hat{z}}$$

④ So define an operator $O = -i B_z(i\pi/2) = -i e^{-(i\pi/2) B_3}$.
Maybe this can take us from a cms to a bws!

Apply $\mathcal{O}$ to a c.m. kinematics.

① Suppose we have:

$$\vec{p} = (\sqrt{s}, 0, 0, 0)$$

$$\vec{p}_1 = (E_1, 0, 0, p_1)$$

$$\vec{p}_2 = (E_2, 0, 0, -p_1)$$

$$\vec{p}_3 = (E_3, \vec{p}_3)$$

$$\vec{p}_4 = (E_4, -\vec{p}_3)$$

Then: $\mathcal{O}p = \sqrt{s} \mathcal{O}\hat{x} = \sqrt{s} \hat{z} = (0, 0, 0, \sqrt{s})$

$$\mathcal{O}p_1 = (-i) \begin{bmatrix} \cos|\xi| & 0 & 0 & i \sin|\xi| \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ i \sin|\xi| & 0 & 0 & \cos|\xi| \end{bmatrix} \begin{bmatrix} E_1 \\ 0 \\ 0 \\ p_1 \end{bmatrix}$$

$$= (-i) \begin{bmatrix} E_1 \cos|\xi| + i p_1 \sin|\xi| \\ 0 \\ 0 \\ -i \sin|\xi| E_1 + \cos|\xi| p_1 \end{bmatrix}$$

But wait. $|\xi| = \pi/2$, so $\mathcal{O} = -i \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$

So: $\boxed{\mathcal{O} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}}$

② Now, $\mathcal{O}p_1 = (p_1, 0, 0, E_1)$
 $\mathcal{O}p_2 = (-p_1, 0, 0, E_2)$

③ Notice that $\mathcal{O}: p \rightarrow \tilde{p}$. $p \cdot p = s$, $\tilde{p} \cdot \tilde{p} = -s$. Thus, $\mathcal{O}$ does not preserve scalars. Define

$$\tilde{s} \equiv \tilde{p} \cdot \tilde{p} = -s$$

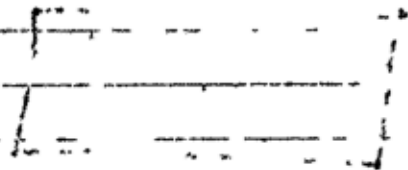
$$\tilde{s}_i \equiv \tilde{p}_i \cdot \tilde{p}_i = -s_i \text{ etc.}$$

Then: $\tilde{p} = (0, 0, 0, \sqrt{-\tilde{s}})$

$$\tilde{p}_i = \left(\frac{\sqrt{\Delta(-\tilde{s}, -\tilde{s}_1, -\tilde{s}_2)}}{2\sqrt{-\tilde{s}}}, 0, 0, \frac{(-\tilde{s}) + (-\tilde{s}_1) - (-\tilde{s}_2)}{2\sqrt{-\tilde{s}}} \right)$$

④ Thus, σ takes us from the cms situation to the $\Delta > 0$ b.w.s. Δ is invariant!

Thus, σ could take us from the cross channel cms frame to a direct channel b.w. frame with $X = (x) < 0$. But σ cannot take us to the $\Delta < 0$ region of the direct channel.



An idea on the meaning of $z = \cosh \xi$ in usual case.

① I have a feeling that: $\tilde{O}^\dagger B_x(i\beta) \tilde{O} = R_y(\beta)$
 Try to show via generators.

$$B_x(i\beta) = e^{-[i\beta]B_{01}}$$

$$R_y(\beta) = e^{-[\beta]R_2}$$

$$\tilde{O} = (-i) B_z(i\frac{\pi}{2}) = (-i) e^{(i\frac{\pi}{2})B_3}$$

↑ active

$$\tilde{O}^\dagger = (+i) e^{-(i\frac{\pi}{2})B_3}$$

So, we want to check this:

$$(+i) e^{+i\frac{\pi}{2}B_3} e^{-[\beta]B_{01}} (-i) e^{-i\frac{\pi}{2}B_3} = e^{+\beta R_2} \quad \checkmark$$

I.e.,
 Def:
$$\left[e^{i\frac{\pi}{2}B_3} e^{-i\beta B_{01}} e^{-i\frac{\pi}{2}B_3} = e^{+\beta R_2} \right] \quad (*)$$

omit

$$(1 + i\frac{\pi}{2}B_3)(1 - i\beta B_{01})(1 - i\frac{\pi}{2}B_3) = (1 - \beta R_2) \quad ?$$

$$1 + i\frac{\pi}{2}B_3 - i\beta B_{01} - i\frac{\pi}{2}B_3 + \frac{\pi}{2}\beta [B_3, B_{01}] \quad \text{stop}$$

Cannot use small expansion for $\pi/2$! Instead, use the boost theorem:

$$e^{i\theta B_3} B_{01} e^{-i\theta B_3} = \dots$$

② See elsewhere for proof of (*).

③ So $\tilde{O}^\dagger B_x(-i\beta) \tilde{O} = R_y(\beta)$. What is the interpretation?!

$$\text{If } \xi = -i\beta, \text{ then: } \tilde{O}^\dagger B_x(\xi) \tilde{O} = R_y(\beta).$$

Recall that operator $\tilde{O}$ takes you from your cms frame to a bws frame in which all masses squared are exactly negated. Thus, all timelike cms vectors become spacelike vectors in the bws. Thus, the above equation shows us that a real X-boost by ξ done in the bws appears as a purely imaginary Y-rotation when viewed from the cms system, $\beta = i\xi$. Thus, I now understand the meaning of $\cos(\rho) = z = \cosh \xi$. HOWEVER, we have only been able to connect the cms and bws when $\text{triangle}(t, t_1, t_2)$ is positive. This situation obtains in the conventional A_4 analysis with all masses² positive. In the sector of the bws where triangle is negative, there is no such simple connection between z and ξ . In fact I have showed that z connects to a combination of ξ and an azimuth.

F.D. IR's

Comments on Chapter on EDR's of Lorentz group.

1. Notation: the symbol $\mathcal{L}_1$ stands for the "proper" Lorentz group. When this group is combined with the inversion operator called I , you get the "full" Lorentz group called $\mathcal{L}$. As best as I can tell, Liubarskii is throwing out the time reversal operator from his discussion? Remember that "proper" just means $\det \Lambda = +1$ and includes the pieces $L_+^\uparrow$ and $L_+^\downarrow$ which are disconnected. So here Liu simply ignored the fact that $L_+ = \mathcal{L}_1$ happens to have two disconnected pieces.

Further notation: $I_i =$ rotation generators $= R_i = iJ_i$, my usual J_i
 $J_i =$ boost generators $= B_i = iK_i$, my usual K_i

This mixup adds to the general confusion.

The A_i and B_i are the generators which separate the algebra, and (P, Q) are the Casimirs which go with A and B . τ_{PQ} is a rep. of L_+ . Obvious. Then e_{pq} are supposed to be the basis vectors of such a rep. Sometimes u_{pq} and v_{pq} are used as basis vectors for reps. Symbols ζ_{pq} and η_{pq} are used as coefficients of a vector in some rep. I.e., they are coefficients of the basis vectors like e_{pq} .

2. Summary of the chapter:

1. Algebra is stated in terms of I, J and then in terms of A, B .
2. Reps are classified. The UIR's are mentioned in passing.
3. Clebsh decomposition rule is stated.
4. Complex conjugate rep is defined. One finds this equivalence: $\tau_{PQ}^* = \tau_{QP}$. This is the source of Bade Jehle stuff where $(0, \frac{1}{2})$ and $(\frac{1}{2}, 0)$ are complex conjugates.
5. Spinor algebra: general spinors with any number of dotted and undotted indices. Note that $(\frac{1}{2}, 0)$ is called first kind = undotted. Spinor "of rank $m+n$ " obvious. The metric spinor is called a metric tensor g_{ij} but it is not discussed at all.
6. Tensor algebra: the $(\frac{1}{2}, \frac{1}{2})$ basis vectors are connected to usual x^u basis vectors. Tensor of "rank n " is obvious. $g_{uv} = \text{diag}(1, -1, -1, -1)$ is actually stated, as opposed to metric spinor before, was not stated.

3. Last section is about parity. First, he shows that parity negates the boost generators, but leaves the rotation generators unchanged (as I have always said). Therefore, the operator I (parity) interchanges the A and B ! This is why parity converts T_{PQ} into T_{QP} . If e_{pq} is basis vectors of τ_{PQ} and u_{qp} basis vectors of τ_{QP} then parity does this: $I e_{pq} = u_{qp}$. This is why parity will be identified with complex conjugation, which also related the PQ to the QP .

For the (P, P) type reps like vector, something special happens. Either $u_{qp} = + e_{qp}$ or minus sign. This leads to τ_P^+ and τ_P^- which are scalar and pseudo.

So, to summarize this parity business, here are the representations of the full Lorentz group:

$$\begin{aligned}
 T_{PQ} &= \underbrace{\tau_{PQ} \oplus \tau_{QP}}_I \\
 T_{PP} &= \underbrace{\tau_{PP} \oplus \tau_{PP}}_I = \underbrace{\tau_P^+ \oplus \tau_P^-}_I
 \end{aligned}
 \left. \vphantom{\begin{aligned} T_{PQ} \\ T_{PP} \end{aligned}} \right\} \text{so } \begin{cases} T_{PQ} & P \neq Q \\ \tau_P^\pm & P = Q \end{cases}$$

For $P \neq Q$, the parity operator connects the τ_{PQ} to the τ_{QP} so that neither rep is a rep of L because neither piece is invariant under the operators of L (which include I). However, the sum of these two representations does make an invariant subspace. Thus, we call these reps of L by the name T_{PQ} .

This is why, for example, the Dirac particle is $(0, \frac{1}{2}) + (\frac{1}{2}, 0)$. A particle must be a representation of the full Lorentz group, not just the proper Lorentz group, as long as parity is a good symmetry of the physics of the particle. Similarly, ~~xx~~ a spin-1 particle must become $(1, 0) + (0, 1)$, if you are talking about a rep of the Lorentz group full.

When $P=P$ as in the vector case $(\frac{1}{2}, \frac{1}{2})$, it happens that the T_{PP} rep has two invariant subspaces which form the $+$ and $-$ reps as above. Thus, in the full Lorentz classification a particle is not just a vector. It is a regular vector (is, $\tau_{\frac{1}{2}}^-$) whose components change sign under parity, or it is an axial vector $\tau_{\frac{1}{2}}^+$. On the last page he shows that there are only two tensors of rank n , one is $+$ and one is $-$.

Liubarski does not talk about the direct product and reduction of full Lorentz group representations, but here is my guess for what happens:

$$\begin{aligned}
 \tau_0^{\sigma_1} \otimes \tau_P^{\sigma} &= \tau_P^{\sigma_1 \sigma} \\
 \tau_{\frac{1}{2}}^{\sigma_1} \otimes \tau_P^{\sigma} &= \tau_{P-\frac{1}{2}}^{\sigma_1 \sigma} \oplus T_{P-\frac{1}{2}, P+\frac{1}{2}} \oplus \tau_{P+\frac{1}{2}}^{\sigma_1 \sigma} \\
 \left[\underbrace{\tau_{\frac{1}{2}}^- \otimes \tau_{\frac{1}{2}}^-}_{\text{regular vect.}} = \underbrace{\tau_0^+ \oplus T_{01} \oplus \tau_1^+}_{\text{scalar}} \right] \\
 \tau_0^{\sigma_1} \otimes T_{PQ} &= T_{PQ} \\
 \tau_{\frac{1}{2}}^{\sigma_1} \otimes T_{PQ} &= T_{P+\frac{1}{2}, Q+\frac{1}{2}} \oplus T_{P+\frac{1}{2}, Q-\frac{1}{2}} \oplus 2 \text{ more.}
 \end{aligned}$$

Chapter IV

→ (finite-dimensional, nonunitary, irreducible —)
Representations of the Lorentz Group

65. The Lorentz Group

In the terminology of the theory of relativity the set consisting of the three Cartesian coordinates x, y, z and the time t is called an event "occurring" at the point x, y, z at the instant of time t . The numbers x, y, z and t are called the four-dimensional coordinates of the event.

According to the special theory of relativity the coordinates of the same event in different inertial systems of reference may be expressed linearly in terms of each other in such a way that the interval

$$S_{AB} = \sqrt{c^2(t_A - t_B)^2 - (x_A - x_B)^2 - (y_A - y_B)^2 - (z_A - z_B)^2}$$

between any two events

$$A(x_A, y_A, z_A, t_A) \text{ and } B(x_B, y_B, z_B, t_B)$$

remains the same in all such systems.

In order to arrive at the concept of the Lorentz group, we have to single out from all inertial reference systems those whose origins coincide at a certain instant of time and take this instant as the origin of time measurement for all of them. In the following, when speaking of inertial systems of reference we shall understand just such systems.

Evidently, an event with coordinates $(0, 0, 0, 0)$ in one inertial system has the same coordinates in any other inertial system. Hence the relation between the coordinates of a certain event in different inertial systems is not only linear but also homogeneous. Let us denote by $\alpha_{ij}(K'K)$ the matrix describing the transition from the coordinates in the system K to those in the system K' , so that

$$\left. \begin{aligned} x'_i &= \alpha_{ij}(K'K) x_j, \quad (i, j = 0, 1, 2, 3; \\ x_0 &= ct, \quad x_1 = x, \quad x_2 = y, \quad x_3 = z). \end{aligned} \right\} \quad (65,1)$$

If in the system K' the coordinates of the same event are denoted by x'_i ($i = 0, 1, 2, 3$) we can write:

$$x'_i = \alpha_{ij}(K'K) x_j = x'_i = \alpha_{ii}(K'K) x'_i$$

From these equations and from (65,1) it follows that

$$\alpha_{ij}(K'K) = \alpha_{ii}(K'K) \alpha_{ij}(K'K). \quad (65,2)$$

We see that the matrices α form a group with ordinary matrix multiplication as the law of composition. We have

$$\alpha(K'K) = \alpha(K'K') \alpha(K'K). \quad (65,3)$$

This group is called the full Lorentz group. We shall denote it by $\mathcal{L}$.

The full Lorentz group has as a subgroup the so-called proper Lorentz group. It is obtained if among all inertial systems we select only the left (or only the right) handed systems. The operators that correspond to transitions from one left handed system to another left handed system evidently form a group. This group is just the proper Lorentz group (or simply the Lorentz group) and is denoted by $\mathcal{L}_+$.

The full Lorentz group contains the inversion operator $\underline{I}$, defined by

$$Ix = -x, \quad Iy = -y, \quad Iz = -z, \quad It = t. \quad (65,4)$$

This operator makes the transition from an arbitrary system K to a system K' at rest with respect to it and having axes opposite in direction to the axes of K . Evidently the square of the inversion operator is equal to the unit operator. The operator $\underline{I}$ is represented by the matrix

$$I \sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \quad (65,5)$$

Let us give explicitly the matrix α_{ik} for a very important particular case. Assume that the axes of the systems K' and K are parallel and that K' is moving relative to K with a velocity V in the OX direction. As known, the transformation formulae in this case are:

$$x' = \frac{x - Vt}{\sqrt{1 - \frac{V^2}{c^2}}}, \quad y' = y, \quad z' = z, \quad t' = \frac{t - \frac{Vx}{c^2}}{\sqrt{1 - \frac{V^2}{c^2}}} \quad (65,6)$$

and are called Lorentz transformations. According to (65,6) the matrix α_{ik} in this case is equal to

not signs

$$x_{ik}(V) = \begin{pmatrix} \frac{1}{\sqrt{1-\beta^2}} & -\frac{\beta}{\sqrt{1-\beta^2}} & 0 & 0 \\ -\frac{\beta}{\sqrt{1-\beta^2}} & \frac{1}{\sqrt{1-\beta^2}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \left(\beta = \frac{V}{c} \right). \quad (65,7)$$

The Lorentz group $\mathcal{L}_4$ is a Lie group. As group parameters one can choose the three projections of the velocity of the system K' relative to the system K and any three parameters, describing the rotation to be performed on the axes of the system K in order that they should become parallel to the corresponding axes of the system K' . Thus, the total number of parameters of the Lorentz group $\mathcal{L}_4$ is six.

66. Infinitesimal Operators of the Lorentz Group

As we shall use Lie's infinitesimal method to find all finite dimensional irreducible representations of the group $\mathcal{L}_4$, we must find the commutation relations for the infinitesimal operators of this group.

Let us make a choice of parameters. The operator $\alpha(K'K)$ depends on nothing but the relative motion of the reference systems K and K' . This motion can be described completely by giving the three projections $\alpha_x, \alpha_y, \alpha_z$ of the rotation "vector" of the system K' with respect to the system K and the three projections v_x, v_y, v_z of the velocity v of the origin of the system K' in the system K .

We denote by

$$a_x, a_y, a_z, \beta_x = \frac{v_x}{c}, \beta_y = \frac{v_y}{c}, \beta_z = \frac{v_z}{c}.$$

the infinitesimal operators corresponding to the six parameters

rotations $i_1, i_2, i_3, j_1, j_2, j_3$ *boosts*

Since the elements of the Lorentz group are operators in a four-dimensional vector space, they themselves constitute a representation of the Lorentz group. This representation is called the vector representation.

The operator I_1 in the vector representation can be easily evaluated. To do this we differentiate the expression (65,7) with respect to β and put $\beta=0$. We have

$$I_1 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = -e_{11} - e_{22} \quad (66,1)$$

where $e_{ik}(i, k=0, 1, 2, 3)$ is the matrix in which the element at the intersection of the i -th row and k -th column is equal to 1 while all its other elements are zero. *note*

Performing a cyclic permutation of the indices 1, 2, 3 in (66,1) we obtain:

$$I_2 = -e_{22} - e_{33}, \quad I_3 = -e_{33} - e_{11}. \quad (66,2)$$

To obtain I_1 we shall start from the relations

$$x' = x, \quad y' = y \cos \alpha + z \sin \alpha, \quad z' = -y \sin \alpha + z \cos \alpha, \quad t' = t \quad (66,3)$$

between the coordinates of an event in the systems K and K' , the latter being rotated with respect to the former through an angle α about the OX axis. From (66,3) it follows that the matrix α_{ik} corresponding to such a rotation is equal to

$$\alpha_{ik} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos \alpha & \sin \alpha \\ 0 & 0 & -\sin \alpha & \cos \alpha \end{pmatrix}.$$

Differentiating this expression with respect to α and then putting $\alpha=0$, we obtain $I_1 = e_{23} - e_{32}$. The expressions for I_2 and I_3 can now be obtained by cyclic permutation. Thus we find

$$I_1 = e_{23} - e_{32}, \quad I_2 = e_{31} - e_{13}, \quad I_3 = e_{12} - e_{21}. \quad (66,4)$$

By means of (66,1), (66,2) and (66,4) we easily find all the commutation relations:

$$\begin{aligned} [I_1, I_1] &= [I_2, I_2] = [I_3, I_3] = 0, \\ [I_1, I_2] &= -I_3, \quad [I_2, I_3] = -I_1, \quad [I_3, I_1] = -I_2, \\ [I_1, j_1] &= j_2, \quad [I_2, j_1] = j_3, \quad [I_3, j_1] = -j_1, \\ [I_1, j_2] &= -j_3, \quad [I_2, j_2] = -j_1, \quad [I_3, j_2] = -j_2, \\ [I_1, j_3] &= j_1, \quad [I_2, j_3] = j_2, \quad [I_3, j_3] = j_3. \end{aligned} \quad (66,5)$$

Together with the operators I_k, j_k ($k=1, 2, 3$) it is convenient to use the operators

$$A_k = -\frac{1}{2}(I_k + j_k), \quad B_k = -\frac{1}{2}(I_k - j_k) \quad (k=1, 2, 3). \quad (66,6)$$

The commutation relations for these operators have an especially simple form:

$$[A_j, A_k] = e_{jk} A_l, \quad [B_j, B_k] = e_{jk} B_l, \quad [A_j, B_k] = 0 \quad (j, k, l=1, 2, 3); \quad \text{separate} \quad (66,7)$$

which follows from the commutation relations (66,5).

We stress that the commutation relations for the operators A_k ($k=1,2,3$) are the same as those for the infinitesimal operators of the 3-dimensional rotation group. This is also true for the operators B_k .

67. Classification of the Irreducible Representations of the Lorentz Group

To find all possible irreducible representations of the group $\mathcal{L}_4$, it is sufficient to find all forms of the operators A_k, B_k ($k=1,2,3$) that satisfy the commutation relations (66,5). This problem is equivalent to that of finding the operators A_k, B_k ($k=1,2,3$) that satisfy the considerably simpler commutation relations (66,7). To find these operators it is convenient to use the theorem of sec. 46.

Let L be a linear space that transforms according to a certain irreducible representation τ of the Lorentz group. Let us introduce the notation

$$\begin{aligned} A_+ &= A_1 + iA_2, & A_- &= A_1 - iA_2, & A_3 &= A_3, \\ B_+ &= B_1 + iB_2, & B_- &= B_1 - iB_2, & B_3 &= B_3. \end{aligned} \quad (67,1)$$

According to the above-mentioned theorem, there exists in L a vector e such that

$$A_+ e = Qe, \quad B_+ e = 0, \quad (67,2)$$

where Q is some integer or half integer. We denote by L_Q the subspace of all vectors in L that satisfy

(67,2). This subspace is invariant under the operators A_+, A_-, A_3 because they commute with the operators B_+, B_-, B_3 . Hence in the subspace L_Q there exist vectors

$$e_{pq} (p = -P, -P+1, \dots, P-1, P; \quad p = 0, \frac{1}{2}, 1, \dots).$$

that transform under the action of the operators A_+, A_-, A_3 according to the formulae

$$\begin{aligned} A_- e_{pq} &= a_{p,q} e_{p-1,q}, & A_+ e_{pq} &= a_{p,q}^* e_{p+1,q}, \\ A_3 e_{pq} &= -ip e_{pq}. \end{aligned} \quad (67,3)$$

Let us now construct the vectors

$$e_{pq} (p = -P, \dots, P; \quad q = -Q, \dots, Q) \quad (67,4)$$

by means of the recurrence relations

$$B_- e_{pq} = a_{p,q}^* e_{p,q-1}. \quad (67,5)$$

According to the theorem of sec. 46, we have:

$$B_+ e_{pq} = a_{p,q} e_{p,q+1}, \quad B_3 e_{pq} = -iq e_{pq}. \quad (67,6)$$

basis vectors
of τ_{PQ}

If we act on (67,3) with the operator $(B_-)^{Q-q}$, we obtain:

$$\begin{aligned} A_- e_{pq} &= a_{p,q}^* e_{p-1,q}, & A_+ e_{pq} &= a_{p,q}^* e_{p+1,q}, \\ A_3 e_{pq} &= -ip e_{pq}. \end{aligned} \quad (67,7)$$

Formulae (67,5), (67,6) and (67,7) show that the linear closure of the vectors (67,4) is invariant with respect to the operators A_+, B_+ . Hence, it coincides with the space L . The vectors (67,4) form a basis in L , while the relations (67,5), (67,6), (67,7) completely define the operators A_+, A_-, B_+, B_-, B_3 in it.

We see that the infinitesimal operators are completely defined if the numbers P, Q are given. Consequently to each pair of numbers P, Q ($P, Q = 0, 1/2, 1, \dots$) there corresponds one and only one irreducible representation of the Lorentz group. We shall denote this representation by τ_{PQ} . The representation τ_{PQ} has

$$(2P+1)(2Q+1)$$

dimensions. the only one-dimensional representation of the Lorentz group is τ_{00} . We shall call the pair of numbers P, Q the weight of the representation τ_{PQ} . The basis (67,4) is called the canonical basis.

It can be shown that the representation τ_{PQ} is single valued if $P+Q$ is a integer, but that if $P+Q$ is half an integer the representation τ_{PQ} is double-valued.

In conclusion we must mention the infinite-dimensional irreducible representations of the Lorentz group, found by I.M. Gel'fand and M.A. Naimark.

Every infinite-dimensional irreducible representation $\tau(k_0, k_1)$ of the Lorentz group $\mathcal{L}_4$ is defined by two indices k_0 and k_1 . The index k_0 can take on all integer and half integer values

$$k_0 = 0, \pm \frac{1}{2}, \pm 1, \dots$$

while the index k_1 can take on any value, including a complex one. But, if the difference $k_1 - k_0$ is an integer, this representation degenerates into a finite-dimensional representation. We shall assume in the following that the difference $k_1 - k_0$ is not an integer.

Choose the basis

$$e_p (p = k, k-1, \dots, -k; \quad k = |k_0|, |k_0|+1, \dots)$$

in the space L , in which the operators of the representation $\tau(k_0, k_1)$ act and define by means of it

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the infinitesimal operators of the Lorentz group. Instead of the infinitesimal operators I_i, J_i ($i=1,2,3$) it will be more convenient to use the six operators:

$$H^0 = I_3, \quad H^+ = I_1 + iI_2, \quad H^- = I_1 - iI_2$$

$$F^0 = -J_3, \quad F^+ = -J_1 - iJ_2, \quad F^- = -J_1 + iJ_2$$

The irreducible representation $\tau(k_0, k_1)$ is defined by the formulas

$$H^0 \tau_p^k = k \tau_p^k, \quad H^+ \tau_p^k = \sqrt{(k+p)(k-p+1)} \tau_{p-1}^{k+1},$$

$$H^- \tau_p^k = -\sqrt{(k+p+1)(k-p)} \tau_{p+1}^{k-1},$$

$$2F^0 \tau_p^k = -\sqrt{(k+p)(k-p)} B_{k+1} \tau_{p-1}^{k-1} + \sqrt{(k+p+1)(k-p+1)} B_{k+1} \tau_{p+1}^{k+1},$$

$$2F^+ \tau_p^k = \sqrt{(k+p)(k-p-1)} B_{k+1} \tau_{p-1}^{k-1} + \sqrt{(k+p+1)(k-p+1)} A_{k+1} \tau_{p+1}^{k+1} + \sqrt{(k-p+1)(k-p+2)} B_{k+1} \tau_{p+1}^{k+1},$$

$$2F^- \tau_p^k = \sqrt{(k-p)(k-p-1)} B_{k+1} \tau_{p-1}^{k-1} - \sqrt{(k+p+1)(k-p)} A_{k+1} \tau_{p+1}^{k+1} + \sqrt{(k+p+1)(k-p+2)} B_{k+1} \tau_{p+1}^{k+1},$$

where

$$A_k = \frac{2k k_0}{k(k+1)}, \quad B_k = \sqrt{\frac{(k^2 - k_0^2)(k^2 - k_1^2)}{k^2(k^2 - \frac{1}{4})}}.$$

The pairs of numbers k_0, k_1 and $-k_0, -k_1$ define equivalent representations. In all other cases we obtain non-equivalent representations.

The representation $\tau(k_0, k_1)$ is unitary, if the number k_1 is purely imaginary or if $k_0=0$ and k_1 real.

It can easily be seen that in these cases the operators I_i, J_i are anti-Hermitian, which proves that the representation is unitary (see problem IV, sec. 43).

68. Product of Irreducible Representations of the Lorentz Group

Consider the product of two irreducible representations $\tau_{p_1 q_1}$ and $\tau_{p_2 q_2}$ of the Lorentz group and decompose it into irreducible representations.

Let us denote by

$$u_{p,q}^{p_1,q_1} \quad \text{and} \quad v_{p,q}^{p_2,q_2}$$

the canonical bases in the spaces L_1 and L_2 which transform according to the representations $\tau_{p_1 q_1}$ and

$\tau_{p_2 q_2}$. The products

$$u_{p_1,q_1}^{p_1,q_1} v_{p_2,q_2}^{p_2,q_2}$$

form a basis in the product of these spaces.

Let us first consider the case $Q_1 = P_2 = 0$. We note that

$$C u_{p,q}^{p_1,q_1} v_{p,q}^{p_2,q_2} = (C u_{p,q}^{p_1,q_1}) v_{p,q}^{p_2,q_2} + u_{p,q}^{p_1,q_1} (C v_{p,q}^{p_2,q_2}), \quad \text{i.e., } C = C \otimes 1 + 1 \otimes C$$

where C is any infinitesimal operator or a linear combination of such operators. It can be easily seen by means of this formula that the matrices of the operators $A_0, A_{\pm}, B_0, B_{\pm}$ are identical in the basis

$$u_{p,q}^{p_1,q_1}$$

and in the basis

$$u_{p,q}^{p_1,q_1} v_{p,q}^{p_2,q_2}$$

This means that the product of the representations $\tau_{p_1 q_1} \times \tau_{p_2 q_2}$ is equivalent to the representation $\tau_{p,q}$.

The basis

$$u_{p,q}^{p_1,q_1} v_{p,q}^{p_2,q_2}$$

is of the same type as the basis e_{pq}^{pq}

$$u_{p,q}^{p_1,q_1} v_{p,q}^{p_2,q_2} = e_{pq}^{pq}. \quad (68,1)$$

This result reduces the problem of decomposing the products of the general type to the problem of expanding the products

$$\tau_{p_1 q_1} \times \tau_{p_2 q_2} \quad \text{and} \quad \tau_{0 q_1} \times \tau_{0 q_2}.$$

These decompositions are carried out exactly in the same way as the decompositions of the representations $D_p \times D_p$ of the rotation group. Hence, we can write at once:

$$\tau_{p_1 q_1} \times \tau_{p_2 q_2} = \sum_{|p_1-p_2| < p < p_1+p_2} \tau_{p q_1} \times \tau_{p q_2} = \sum_{|q_1-q_2| < q < q_1+q_2} \tau_{p q}.$$

The relation between the basic vectors is expressed by means of the Clebsch-Gordan coefficients:

$$\left. \begin{aligned} e_{p,q}^{p_1,q_1} &= \sum_{p_1, p_2, p} (p_1 p_2 p | p q) u_{p_1,q_1}^{p_1,q_1} v_{p_2,q_2}^{p_2,q_2} \\ e_{p,q}^{p_1,q_1} &= \sum_{q_1, q_2, q} (q_1 q_2 q | p q) u_{p_1,q_1}^{p_1,q_1} v_{p_2,q_2}^{p_2,q_2} \end{aligned} \right\} \quad (68,2)$$

Now we can easily multiply the representations $\tau_{p_1 q_1}$ and $\tau_{p_2 q_2}$. We have:

$$\text{i.e., } \left\{ \begin{aligned} \tau_{p_1 q_1} \times \tau_{p_2 q_2} &= \tau_{p_1 q_1} \times \tau_{0 q_2} \times \tau_{p_2 q_1} \times \tau_{0 q_2} = \sum_p \tau_{p q} \times \sum_q \tau_{p q} = \sum_{p,q} \tau_{p q} \\ \tau_{p_1 q_1} \times \tau_{p_2 q_2} &= \sum_{|p_1-p_2| < p < p_1+p_2} \sum_{|q_1-q_2| < q < q_1+q_2} \tau_{p q} \end{aligned} \right. \quad (68,3)$$

Hence it follows that the product $\epsilon_{P_1 Q_1} \times \epsilon_{P_2 Q_2}$ contains the unit representation only when

$$P_1 = P_2 \text{ and } Q_1 = Q_2$$

i.e. when the factor representations are equivalent.

The vectors

$$\epsilon_{PQ}^{PQ}$$

of the canonical basis in the subspace L_{PQ} which transform according to the representation ϵ_{PQ} , are expressed by means of

$$\epsilon_{PQ}^{PQ} \epsilon_{PQ}^{PQ}$$

as follows:

$$\epsilon_{PQ}^{PQ} = \sum_{P_1 P_2 P_3 | P} \sum_{Q_1 Q_2 Q_3 | Q} (P_1 P_2 P_3 | P) (Q_1 Q_2 Q_3 | Q) \epsilon_{P_1 Q_1}^{P_1 Q_1} \epsilon_{P_2 Q_2}^{P_2 Q_2} \epsilon_{P_3 Q_3}^{P_3 Q_3} \quad (68,4)$$

This follows from (68,1) and (68,2). Since the Clebsch-Gordan coefficients are unitary, (68,4) can be rewritten in the form

$$\epsilon_{PQ}^{PQ} \epsilon_{PQ}^{PQ} = \sum_{P_1 Q_1} (P_1 P_2 P_3 | P) (Q_1 Q_2 Q_3 | Q) \epsilon_{P_1 Q_1}^{P_1 Q_1} \quad (68,5)$$

$(P = P_1 + P_2, Q = Q_1 + Q_2)$

The formulae (68,3) and (68,4) fully solve the problem of decomposing the product of representations.

Let us consider some of the consequences of the relations we obtained.

Let

$$\epsilon_{PQ}^{PQ} \text{ and } \epsilon_{PQ}^{PQ}$$

be two arbitrary vectors in the spaces L_1 and L_2 . The vector

$$\epsilon_{PQ}^{PQ} \epsilon_{PQ}^{PQ}$$

can be represented in the form

$$\epsilon_{PQ}^{PQ} \epsilon_{PQ}^{PQ} = \sum_{P_1 Q_1} \epsilon_{PQ}^{PQ} (P_1 P_2 P_3 | P) (Q_1 Q_2 Q_3 | Q) \epsilon_{P_1 Q_1}^{P_1 Q_1}$$

Hence it follows that the expressions

$$\epsilon_{PQ}^{PQ} = \sum_{P_1 P_2 P_3 | P} (P_1 P_2 P_3 | P) (Q_1 Q_2 Q_3 | Q) \epsilon_{P_1 Q_1}^{P_1 Q_1} \quad (68,6)$$

are the coordinates of a certain vector of the space which transforms according to the representation ϵ_{PQ} .

We give the important particular case of (68,4) obtained when we put $P_1 = P_2, Q_1 = Q_2, P = Q = 0$:

$$\epsilon_{00}^{00} = \sum_{P_1 P_2 P_3 | 0} (P_1 P_2 P_3 | 0) (Q_1 Q_2 Q_3 | 0) \epsilon_{P_1 Q_1}^{P_1 Q_1} \epsilon_{P_2 Q_2}^{P_2 Q_2}$$

Recalling the expression (54,22) for

$$(P_1 P_2 P_3 | 00),$$

we have:

$$\sum_{P_1 Q_1} (-1)^{P_1 + Q_1 - P_1 - Q_1} \epsilon_{P_1 Q_1}^{P_1 Q_1} = \text{inv.} \quad (68,7)$$

Similarly, from the relation (68,6) it follows:

$$\sum_{P_1 Q_1} (-1)^{P_1 + Q_1 - P_1 - Q_1} \epsilon_{P_1 Q_1}^{P_1 Q_1} = \text{inv.} \quad (68,8)$$

In conclusion let us make clear the connection between the representations of the Lorentz group and those of the rotation group.

The rotation group is a subgroup of the Lorentz group. Hence every representation of the Lorentz group is at the same time a representation of the rotation group. Let us now consider the representation ϵ_{PQ} . In this representation the operators B_1, B_2, B_3 are zero and consequently $A_k = -I_k$ ($k=1,2,3$), see (66,6). The eigenvalues of the operator I_3 are $-P, \dots, P$. This means that the representation ϵ_{PQ} , considered as a representation of the rotation group, is equivalent to the representation D_P . In the same way it can be shown that the representation ϵ_{0Q} is equivalent to D_Q . Hence the representation

$$\epsilon_{PQ} = \epsilon_{PQ} \times \epsilon_{0Q}$$

is equivalent to the representation

$$D_P \times D_Q = \sum_{|P-Q| \leq J \leq P+Q} D_J$$

of the rotation group.

69. Complex-conjugate Representations

Let us consider the representation complex-conjugate to the representation ϵ_{PQ} . Let us denote it by ϵ_{PQ}^* . In the complex-conjugate basis the matrices of the infinitesimal operators are complex conjugate to the matrices of the corresponding infinitesimal operators of the original representation

$$(I^*)_{\mu} = \bar{I}_{\mu}, \quad (J^*)_{\mu} = \bar{J}_{\mu}.$$

Omitting the indices of the matrices, we have hence:

$$A_k^* = -\frac{1}{2} (\bar{I}_k + \bar{J}_k) = -\frac{1}{2} (\bar{I}_k - \bar{J}_k) = \bar{B}_k$$

$$B_k^* = -\frac{1}{2} (\bar{I}_k - \bar{J}_k) = -\frac{1}{2} (\bar{I}_k + \bar{J}_k) = \bar{A}_k.$$

Furthermore,

$$A_- = (A_1 + A_2) = (\bar{B}_1 + \bar{B}_2) = -(\bar{B}_1 + \bar{B}_2) = -\bar{B}_+.$$

The operators

$$A_+, B_-, B_+.$$

can be evaluated in the same way. As a result we have the following relations:

$$\left. \begin{aligned} A_+ &= \bar{B}_-, & A_- &= -\bar{B}_+, & A_+ &= -\bar{B}_-, \\ B_+ &= \bar{A}_-, & B_- &= -\bar{A}_+, & B_+ &= -\bar{A}_-. \end{aligned} \right\} \quad (69,1)$$

Let us denote by $X_{pq}^{(1)}$ the canonical basis in L_1 which transforms according to the representation τ_{pq} . The complex-conjugate vectors $\bar{e}_{pq}$ form a basis in $\bar{L}_1$ which transforms according to the representation $\bar{\tau}_{pq}$. By means of (67,5), (67,6) and (67,7) we can easily see how the operators (69,1) act upon the vectors e_{pq} . After simple operations we obtain:

$$\left. \begin{aligned} A_+ e_{pq} &= i q e_{pq}, & A_- e_{pq} &= -\frac{q}{2} e_{pq+1}, & A_+ e_{pq} &= -\frac{q}{2} e_{pq-1}, \\ B_+ e_{pq} &= i p e_{pq}, & B_- e_{pq} &= -\frac{p}{2} e_{pq+1}, & B_+ e_{pq} &= -\frac{p}{2} e_{pq-1}. \end{aligned} \right\} \quad (69,2)$$

The first column shows that the vectors e_{pq} are the eigen-vectors of the operators A_+ and B_+ . Hence they differ only by a constant factor from the vectors of the canonical basis u_{-q-p}

$$e_{pq} = \lambda_{pq} u_{-q-p}$$

We see that the first index takes on $(2Q+1)$ values and the second $(2P+1)$ values. Hence the representation τ_{pq} is equivalent to the representation τ_{qp} :

$$\tau_{pq} = \tau_{qp}$$

We determine the coefficients λ_{pq} using the second pair of relations (69,2).

$$\begin{aligned} \text{We have: } A_- e_{pq} &= \lambda_{pq} A_- u_{-q-p} = \lambda_{pq} \frac{q}{2} u_{-q-1-p} = \\ &= -\frac{q}{2} \lambda_{pq+1} u_{-q-1-p} = -\frac{q}{2} \lambda_{pq+1} u_{-q-1-p} \end{aligned}$$

and

$$\begin{aligned} B_- e_{pq} &= \lambda_{pq} B_- u_{-q-p} = \lambda_{pq} \frac{p}{2} u_{-q-p-1} = \\ &= -\frac{p}{2} \lambda_{pq+1} u_{-q-p-1} = -\frac{p}{2} \lambda_{pq+1} u_{-q-p-1} \end{aligned}$$

Hence it follows that

$$\frac{\lambda_{pq}}{\lambda_{pq+1}} = -\frac{q}{q+1}, \quad \frac{\lambda_{pq}}{\lambda_{pq+1}} = -\frac{p}{p+1}$$

and

$$\lambda_{pq} = (-1)^{Q+P-q-p}$$

Thus, finally

$$\bar{e}_{pq} = (-1)^{Q+P-q-p} e_{-q-p} \quad (69,3)$$

Recalling the relations (68,7) and (68,8), we have

$$\sum_{pq} \bar{e}_{pq} e_{pq} = \text{inv.} \quad (69,4)$$

$$\sum_{pq} \bar{e}_{pq} e_{pq} = \text{inv.} \quad (69,5)$$

70. Spinor Algebra

In contrast to the rotation group the proper Lorentz group has 2 two-dimensional representations τ_{10} and τ_{01} . The two-dimensional vectors that transform according to these representations are called the spinors of the first (τ_{10}) and the second (τ_{01}) kind. In spinor space the vectors of the canonical basis are usually denoted as follows:

$$\left. \begin{aligned} \text{first kind: } & \begin{pmatrix} e_{\frac{1}{2}0} = e_{-1} \\ e_{\frac{1}{2}1} = e_1 \end{pmatrix} \\ \text{second kind: } & \begin{pmatrix} e_{0\frac{1}{2}} = e_{-1} \\ e_{0\frac{1}{2}} = e_1 \end{pmatrix} \end{aligned} \right\} \quad (70,1)$$

We write out for further reference the formulae defining the action of the operators $A_{0\pm}, B_{0\pm}$ on these vectors:

$$\left. \begin{aligned} A_+ e_{-1} &= \frac{1}{2} e_{-1}, & A_+ e_1 &= -\frac{1}{2} e_1, \\ B_+ e_{-1} &= \frac{1}{2} e_{-1}, & B_+ e_1 &= -\frac{1}{2} e_1, \\ A_{0\pm} e_{-1} &= A_{0\pm} e_1 = B_{0\pm} e_{-1} = B_{0\pm} e_1 = 0, \\ A_+ e_1 &= A_- e_{-1} = B_+ e_1 = B_- e_{-1} = 0, \\ A_+ e_{-1} &= e_1, & B_+ e_{-1} &= e_1, \\ A_- e_1 &= e_{-1}, & B_- e_1 &= e_{-1}. \end{aligned} \right\} \quad (70,2)$$

The spinor components are denoted respectively by

$$(\xi_{-1}, \xi_1) \quad \text{and} \quad (\xi_{-1}, \xi_1)$$

The following relations play a fundamental role in the theory of spinors:

$$\left. \begin{aligned} e_1 e_{-1} - e_{-1} e_1 &= \text{inv.}, & e_1 e_{-1} - e_{-1} e_1 &= \text{inv.}, \\ \xi_1 \xi_{-1} - \xi_{-1} \xi_1 &= \text{inv.}, & \xi_1 \xi_{-1} - \xi_{-1} \xi_1 &= \text{inv.} \end{aligned} \right\} \quad (70,3)$$

They are particular cases of (68,7) and (68,8). Introducing the matrix

$$g = \begin{pmatrix} \xi_{-1} & \xi_1 \\ \xi_1 & \xi_{-1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad (70,4)$$

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we can rewrite (70,3) as

$$\left. \begin{aligned} g_{M_1} \varepsilon_{M_1} &= \text{inv.} & g_{M_1} \varepsilon_{M_1} &= \text{inv.} \\ g_{M_1} \varepsilon_{M_1} &= \text{inv.} & g_{M_1} \varepsilon_{M_1} &= \text{inv.} \end{aligned} \right\} \quad (70,5)$$

Any element of the space that transforms according to the representations

$$\left(\begin{smallmatrix} \tau_1 & 1 \\ 1 & \tau_1 \end{smallmatrix} \right)^{m_1} \left(\begin{smallmatrix} \tau_2 & 1 \\ 1 & \tau_2 \end{smallmatrix} \right)^{m_2}$$

is called a spinor of rank m_1+m_2 . Thus, for example, the element that transforms as the product

$$\varepsilon_{\alpha_1 \alpha_2}^{(1)} \varepsilon_{\beta_1 \beta_2}^{(2)}$$

is a spinor of rank 3+2. It is usually denoted by:

$$\varepsilon_{\alpha_1 \alpha_2 \beta_1 \beta_2}$$

In denoting by letters the indices of a spinor of rank m_1+m_2 the dots are not placed above the letters; instead of this, the first m_1 indices are denoted by the letters

$$\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_{m_1}$$

while the other m_2 indices are denoted by the letters

$$\beta_1, \beta_2, \dots, \beta_{m_2}$$

The product of two spinors of rank m_1+m_2 and n_1+n_2 is a spinor of rank $(m_1+n_1)+(m_2+n_2)$. Hence, if

$$C_{\alpha_1 \alpha_2 \dots \alpha_{m_1} \beta_1 \beta_2 \dots \beta_{m_2}}$$

and

$$C'_{\alpha'_1 \alpha'_2 \dots \alpha'_{m'_1} \beta'_1 \beta'_2 \dots \beta'_{m'_2}}$$

are the coefficients of two spinors, their products

$$C_{\alpha_1 \alpha_2 \dots \alpha_{m_1} \beta_1 \beta_2 \dots \beta_{m_2}} C'_{\alpha'_1 \alpha'_2 \dots \alpha'_{m'_1} \beta'_1 \beta'_2 \dots \beta'_{m'_2}} =$$

$$= C''_{\alpha_1 \alpha_2 \dots \alpha_{m_1} \alpha'_1 \alpha'_2 \dots \alpha'_{m'_1} \beta_1 \beta_2 \dots \beta_{m_2} \beta'_1 \beta'_2 \dots \beta'_{m'_2}}$$

are the coefficients of a spinor of rank $(m+m')+(n+n')$.

In addition to multiplication there is another operation with spinors—simplification, which we now consider. Let

$$C_{\alpha_1 \alpha_2 \dots \alpha_{m_1} \beta_1 \beta_2 \dots \beta_{m_2}}$$

be a spinor of rank $(m+n)$. It transforms as the product of two spinors one of first and the other of $\{(m-2)+n\}$ th rank:

$$\varepsilon_{\alpha_1 \alpha_2} \varepsilon_{\alpha_3 \alpha_4} \dots \varepsilon_{\alpha_{m-1} \alpha_m} C_{\alpha_{m+1} \alpha_{m+2} \dots \alpha_{m+n}}$$

Hence, the expression

$$\varepsilon_{\alpha_1 \alpha_2} C_{\alpha_3 \alpha_4 \dots \alpha_{m-1} \alpha_m \alpha_{m+1} \alpha_{m+2} \dots \alpha_{m+n}}$$

transforms just as

$$\varepsilon_{\alpha_1 \alpha_2} \varepsilon_{\alpha_3 \alpha_4} \dots \varepsilon_{\alpha_{m-1} \alpha_m} C_{\alpha_{m+1} \alpha_{m+2} \dots \alpha_{m+n}} = \text{inv.} \cdot C_{\alpha_1 \alpha_2 \dots \alpha_{m-2} \alpha_{m-1} \alpha_m \alpha_{m+1} \alpha_{m+2} \dots \alpha_{m+n}}$$

i.e. as a spinor of $\{(m-2)+n\}$ the rank. The transition from a spinor

$$C_{\alpha_1 \alpha_2 \dots \alpha_{m-2} \alpha_{m-1} \alpha_m \alpha_{m+1} \alpha_{m+2} \dots \alpha_{m+n}}$$

to a spinor

$$C_{\alpha_1 \alpha_2 \dots \alpha_{m-2} \alpha_{m-1} \alpha_m \alpha_{m+1} \alpha_{m+2} \dots \alpha_{m+n}}$$

is called simplification of spinors over the indices α_1, α_2 . The simplification can also be carried out over any other pair of indices α or any pair of indices β . As a result of simplification the rank of a spinor is decreased by two.

The matrix $\varepsilon_{\alpha\beta}$ is called the metric tensor. There exists a simple relation between spinors

$$C_{\alpha_1 \alpha_2 \dots \alpha_{m-2} \alpha_{m-1} \alpha_m \alpha_{m+1} \alpha_{m+2} \dots \alpha_{m+n}}$$

and vectors, that transform according to the representation τ_{mn} which is defined by the following theorem:

Theorem. The subspace of spinors

$$C_{\alpha_1 \alpha_2 \dots \alpha_{m-2} \alpha_{m-1} \alpha_m \alpha_{m+1} \alpha_{m+2} \dots \alpha_{m+n}}$$

that are symmetrical with respect to all permutations of the indices

$$\alpha_1, \dots, \alpha_m$$

and all permutations of the indices $\beta_1, \dots, \beta_n$.

transforms according to the representations τ_{mn} .

This theorem is proved in the same way as that of sec. 50.

71. Tensor Algebra

Just as spinors of rank $(m+n)$ can be constructed from the representations $\tau_{\frac{1}{2}0}$ and $\tau_{0\frac{1}{2}}$, tensors of rank n can

be constructed by use of the four-dimensional vector representation. Let us consider this representation more closely. As a representation of the rotation group, it evidently decomposes into D_1 and D_0 corresponding to

the splitting of the components of a four-dimensional vector into its three spatial components and one time component. Since neither of these subspaces $(x, y, z, 0)$ and $(0, 0, 0, t)$ is invariant with respect to the Lorentz group, the vector representation is irreducible and, consequently, it coincides with the representation $\tau_{\frac{1}{2}\frac{1}{2}}$.

+) The two other four-dimensional irreducible representations $\tau_{\frac{1}{2}0}$ and $\tau_{0\frac{1}{2}}$ cannot be equal to the vector representations e.g. because they are irreducible representations of the rotation group.

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Let us set up the relation between the canonical basis

$$e_{\frac{1}{2}, \frac{1}{2}}$$

of the representation $\mathcal{C}_{\frac{1}{2}, \frac{1}{2}}$ and the "natural" basis

$$e_0 = (1 \ 0 \ 0 \ 0), \quad e_1 = (0 \ 1 \ 0 \ 0), \quad e_2 = (0 \ 0 \ 1 \ 0), \quad e_3 = (0 \ 0 \ 0 \ 1). \quad (71,1)$$

To do this, let us find in the natural basis some common eigen-vector of the operators $\underline{I}_3$ and $\underline{J}_3$.

Using (66,2) and (66,4) we write:

$$\begin{aligned} I_3 e_0 &= 0, \quad I_3 e_1 = -e_1, \quad I_3 e_2 = e_2, \quad I_3 e_3 = 0, \\ J_3 e_0 &= -e_1, \quad J_3 e_1 = J_3 e_2 = 0, \quad J_3 e_3 = -e_0. \end{aligned}$$

Hence it follows that

$$I_3(e_2 - e_1) = I_3(e_1 - e_2), \quad J_3(e_2 - e_1) = 0$$

and

$$A_0(e_2 - e_1) = -\frac{1}{2} I_3(e_2 - e_1), \quad B_0(e_2 - e_1) = -\frac{1}{2} I_3(e_2 - e_1).$$

Comparing these relations with (67,6) and (67,7) we see that the vector $e_2 - e_1$ is parallel to the vector $e_{\frac{1}{2}, \frac{1}{2}}$. Hence, we put

$$e_{\frac{1}{2}, \frac{1}{2}} = \frac{1}{\sqrt{2}}(e_2 - e_1).$$

By operating on this expression with $\underline{A}$ and $\underline{B}$ we can easily express the other vectors of the canonical basis in terms of the vectors of the natural basis. As a result, we have

$$\left. \begin{aligned} e_{\frac{1}{2}, -\frac{1}{2}} &= \frac{1}{\sqrt{2}}(e_2 + e_1), \quad e_{-\frac{1}{2}, \frac{1}{2}} = \frac{1}{\sqrt{2}}(e_0 + e_3), \\ e_{-\frac{1}{2}, -\frac{1}{2}} &= \frac{1}{\sqrt{2}}(-e_0 + e_3), \quad e_{\frac{1}{2}, -\frac{3}{2}} = \frac{1}{\sqrt{2}}(-e_1 + e_2). \end{aligned} \right\} \quad (71,2)$$

Let us also write down the expression for the vectors of the natural basis in terms of the vectors of the canonical basis:

$$\left. \begin{aligned} e_1 &= \frac{1}{\sqrt{2}}(e_{\frac{1}{2}, -\frac{1}{2}} - e_{-\frac{1}{2}, \frac{1}{2}}), \quad e_2 = \frac{1}{\sqrt{2}}(e_{\frac{1}{2}, \frac{1}{2}} + e_{-\frac{1}{2}, -\frac{1}{2}}), \\ e_3 &= -\frac{1}{\sqrt{2}}(e_{-\frac{1}{2}, \frac{1}{2}} + e_{\frac{1}{2}, -\frac{1}{2}}), \quad e_0 = -\frac{1}{\sqrt{2}}(e_{\frac{1}{2}, -\frac{1}{2}} - e_{-\frac{1}{2}, -\frac{1}{2}}). \end{aligned} \right\} \quad (71,3)$$

Thus, the transition from the natural basis to the canonical basis is done by means of the formula

$$e_i = \sum_k U_{ik} e_k$$

where

$$i=0 \quad 1 \quad 2 \quad 3$$

$$P = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & -1 \\ -1 & 0 & 0 & -1 \\ 0 & -1 & 1 & 0 \end{pmatrix} \quad \begin{aligned} k &= \frac{1}{2}, & l &= \frac{1}{2} \\ k &= \frac{1}{2}, & l &= -\frac{1}{2} \\ k &= -\frac{1}{2}, & l &= \frac{1}{2} \\ k &= -\frac{1}{2}, & l &= -\frac{1}{2} \end{aligned} \quad (71,4)$$

We now define a tensor of rank n .

Every element of the space L_n , which transforms according to the representation $\mathcal{C}_{\frac{1}{2}, \frac{1}{2}}^n$ of the Lorentz group, is called a tensor of rank n . A basis in the space L_n , which is of the same type as the basis

$$e_{\frac{1}{2}, \frac{1}{2}}^{(0)}, \dots, e_{\frac{1}{2}, \frac{1}{2}}^{(n)} \quad (\mu_i = 0, 1, 2, 3; i = 1, 2, \dots, n),$$

will be called a natural basis and will be denoted by

$$e_{\mu_1 \dots \mu_n}. \quad (71,5)$$

Every tensor

$$e = C_{\mu_1 \dots \mu_n} e_{\mu_1 \dots \mu_n}$$

is fully defined by the set of its coordinates

$$C_{\mu_1 \dots \mu_n}. \quad (71,6)$$

which is also called a tensor of rank n . Let us denote by $\alpha_{ik}(g)$ ($g \in \mathcal{L}_4$, $i, k = 0, 1, 2, 3$) the matrix of the operator $\mathcal{C}_{\frac{1}{2}, \frac{1}{2}}(g)$ in the natural basis. It is evident that under the action of the operator $\mathcal{C}_{\frac{1}{2}, \frac{1}{2}}^n(g)$ the vectors of the natural basis in L_n transform according to the formula

$$\mathcal{C}_{\frac{1}{2}, \frac{1}{2}}^n(g) e_{\mu_1 \dots \mu_n} = \alpha_{\mu_1 \nu_1}(g) \dots \alpha_{\mu_n \nu_n}(g) e_{\nu_1 \dots \nu_n}. \quad (71,7);$$

In accordance with this, the coordinates of the tensor $\mathcal{C}_{\frac{1}{2}, \frac{1}{2}}^n(g) e$ may be expressed in terms of the coordinates of the tensor e by the formula

$$C'_{\mu_1 \dots \mu_n} = \alpha_{\mu_1 \nu_1}(g) \dots \alpha_{\mu_n \nu_n}(g) C_{\nu_1 \dots \nu_n}. \quad (71,8)$$

And, conversely, every set of 4^n numbers, which transforms according to the formula (71,8) when one goes over to the new system of coordinates, represents a tensor of rank n . Hence the product of two tensors of ranks

$$e_{pq} \quad (-P < p < P, -Q < q < Q)$$

be the canonical basis in H_1 . Consider now the vectors

$$\text{del } e'_{pq} = \tau^{-1}(I) e_{pq} \quad \text{partly mixed basis}$$

We have:

$$A'_{pq} = A_{pq} \tau^{-1}(I) e_{pq} = \tau^{-1}(I) B_{pq} e_{pq} = -I q \tau^{-1}(I) e_{pq} = -I q'_{pq}$$

and

$$B'_{pq} = -I p'_{pq}$$

Carrying out similar calculations we obtain:

$$\left. \begin{aligned} A'_{pq} &= -I q'_{pq}, & A'_{pq} &= \alpha_{q+1}^0 e'_{pq+1}, & A'_{pq} &= \alpha_{q-1}^0 e'_{pq-1}, \\ B'_{pq} &= -I p'_{pq}, & B'_{pq} &= \alpha_{p+1}^0 e'_{pq+1}, & B'_{pq} &= \alpha_{p-1}^0 e'_{pq-1} \end{aligned} \right\} \quad (72,6)$$

The relations (72,6) show that the subspace H_1 , being a linear closure of vectors e'_{pq} , transforms according to the representation τ_{qp} , the vectors

$$\text{del } \tau_{qp} e_{pq} = \tau^{-1}(I) e_{pq} \quad (72,7)$$

forming the canonical basis.

The linear closure of

$$\frac{2(2P+1)(2Q+1)}{2}$$

vectors e_{pq} and u_{qp} is invariant with respect to the elements of the proper Lorentz group. Besides, on account of the relations

$$\tau(I) e_{pq} = u_{qp} = \tau(I) u_{qp} = e_{pq} \quad (72,8)$$

which follow from (72,7), this closure is also invariant with respect to the inversion operator. Hence, it is invariant with respect to the full Lorentz group and, consequently, it coincides with the space L , in which the representation τ acts. Let us consider now two separate cases.

1. $P \neq Q$. In this case the representations τ_{pq} and τ_{qp} are not equivalent and, consequently, the vectors e_{pq} and u_{qp} are linearly independent. Hence, they form a basis in the space L . We shall call this the canonical basis. The operator $\tau(I)$ is defined by the equalities (72,8). We shall denote the representation τ by $\tau_{pq} = \tau_{qp}$. The dimensionality of the representation $\tau_{pq}(P \neq Q)$

$$\frac{2(2P+1)(2Q+1)}{2}$$

↑

now a rep
of full L

Considered as a representation of the proper Lorentz group, it decomposes into the representations τ_{pq} and τ_{qp} .

2. $P=Q$. Now one cannot say that the vectors e_{pq} and u_{qp} constitute a linearly independent system of vectors. On the contrary, it can be easily proved that

$$e_{qp} = \pm e_{pq}$$

To show this, let us consider the set of vectors

$$\text{del } e_{pq} = u_{pq} \pm e_{pq} \quad (-P < p, q < P).$$

According to (67,3) and (72,6) we have:

$$A_0 e_{pq} = -I p v_{pq}, \quad A_+ e_{pq} = \alpha_{p+1}^0 v_{pq+1}, \quad A_- e_{pq} = \alpha_{p-1}^0 v_{pq-1}$$

We have analogous formulae for the operators B_0 . Besides, from (72,8) we see that

$$\tau(I) v_{pq} = \tau(I) (u_{pq} \pm e_{pq}) = e_{qp} \pm u_{qp} = \pm v_{qp}$$

Let us denote by H_+ and H_- the linear closures of the vectors u_{pq}^+ and u_{pq}^- respectively. The relations obtained show that the subspaces H_+ and H_- are invariant with respect to all the elements of the full Lorentz group. Hence, either $H_+ = L$ and $H_- = 0$ or $H_- = L$ and $H_+ = 0$. In the first case $u_{pq} = e_{pq}$ and $\tau(I) e_{pq} = e_{qp}$. In the second case $u_{pq} = -e_{pq}$ and $\tau(I) e_{pq} = -e_{qp}$. Thus there exist two different representations

$$\tau_p^+ \text{ and } \tau_p^-$$

of the Lorentz group, each of which, if considered as a representation of the proper Lorentz group, coincides with the representation τ_{pp} . The operator corresponding in this representation to the inversion is defined by

$$\tau_p^+(I) e_{pq} = e_{qp}, \quad \tau_p^-(I) e_{pq} = -e_{qp} \quad (72,9)$$

The dimensionality of the representations τ_p^+ and τ_p^- is $(2P+1)^2$.

The representations τ_p^+ and τ_p^- are called scalar and pseudo-scalar respectively; quantities that transform under these representations are called scalars and pseudo-scalars. The representation $\tau_{1/2}$ is called a vector representation in accordance with the fact that the ordinary vectors of four-dimensional space-time transform according to it; the representation $\tau_{1/2}$

(call in Dirac ψ, ψ^c)

Lucht

The Application of Group Theory in Physics

G.Ya. Lyubarskii

Translated from the Russian

by

Steven Dedijs

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PREFACE

This book is based on a course of lectures given by the author over a number of years at the Gorki University in Kharkov. It is written for physicists specializing in theoretical physics. The author had three aims in writing the book: to present in detail, consistently and as concisely as possible those parts of the theory of representations of finite and continuous groups that are most important in application; to consider groups of interest in theoretical physics; and, finally, to demonstrate the principles according to which the abstract concepts and the theorems of representation theory are applied in theoretical physics. At the end of the book tables are given for the detailed description of the 230 space groups and for the characters of certain groups. The book contains a considerable number of problems.

A knowledge of the elements of linear algebra is assumed at the level given in the first two chapters of the book Lessons in Linear Algebra by I.M. Gel'fand.

The author has included in the book chapters on the representations of the rotation group and the Lorentz group, although an article by I.M. Gel'fand and Z.Ya. Shapiro on the first subject and one by M.A. Naimark on the second, have recently appeared. This has been done for the convenience of the reader and for the sake of completeness.

In conclusion the author wishes to thank N.Ya. Vilenkina, I.M. Gel'fand, M.G. Krein and E.M. Lifshits for their interest during the preparation of the book and for many useful suggestions. The author is also grateful to O.V. Kovalev, whose remarks have helped to eliminate many errors.

Bade + Jēhte

How do you find the metric spinor?*start on back side*

1. Start with the contra and covariant first kind spinor reps:

$$(1) \quad \chi^\mu = \Lambda^\mu{}_\nu \chi^\nu \quad \Lambda^\mu{}_\nu = D_{\nu\mu}^{(0\frac{1}{2})}(a) \equiv (a)_{\mu\nu} \quad (3)$$

$$(2) \quad \chi_\mu = \Lambda_\mu{}^\nu \chi_\nu \quad \Lambda_\mu{}^\nu = D_{\nu\mu}^{(0\frac{1}{2})}(a^{-1}) = (a^{-1})_{\mu\nu} \quad (4)$$

We know that the co. is related to the contra. in this way because that is the only way which will cause $x \cdot y$ to be a scalar. :

2. Now assume there exists a lowering "metric spinor" G which does this:

$$\chi_\mu = G_{\mu\nu} \chi^\nu \quad (5)$$

If we insert this assumed fact into both sides of (2), and then use (1), and (3) and (4) for the matrix elements, we get:

$$G a = (a^\dagger)^{-1} G$$

Here, all indeices are lower and there is no confusion about what is meant by matrix multiplication. Having made a study of this little equation, we know that the solution is: $G = z \sigma_2$ where $z = \text{any complex number}$. This means G is anti-symmetric! Thus, to lower an index you must apply G from the left, or you will be off by a sign. So:

$$G_{\mu\nu}^L = \alpha_L \cdot (\sigma_2)_{\mu\nu} \quad \alpha_L \in \mathbb{C}$$

3. Next, lets look for a raising metric spinor. For now I will call it H_{uv} . I want to keep indices down for the moment to avoid confusing matrix mult. Then:

$$\chi^\mu = H_{\mu\nu} \chi_\nu \quad (6)$$

Repeating the above procedure, we get an equation for H :

$$H (a^\dagger)^{-1} = a H$$

$$\Rightarrow G_{\mu\nu}^R = \alpha_R \cdot (\sigma_2)_{\mu\nu}$$

4. Now lets use up/down notation and here is what we have so far:

$$G_{\mu\nu} = \alpha_L \cdot (\sigma_2)_{\mu\nu}$$

$$G^{\mu\nu} = \alpha_R \cdot (\sigma_2)_{\mu\nu}$$

5. Now we apply a further constraint, namely, that

This tells us:

$$\chi^\alpha = G^{\alpha\nu} \chi_\nu = G^{\alpha\nu} [G_{\nu\beta} \chi^\beta] \Rightarrow$$

$$G^{\alpha\nu} G_{\nu\beta} = \delta^\alpha{}_\beta \Rightarrow \alpha_L \alpha_R = +1$$

5. So here are the conclusions so far: if you define your raising and lowering metric spinors to do their thing applied both before a ~~vector~~ spinor (which is not the choice made by Bade J), then you conclude that:

$$G^{\mu\nu} = \alpha_R \cdot (\sigma_z)_{\mu\nu}$$

$$G_{\mu\nu} = (\alpha_R)^{-1} \cdot (\sigma_z)_{\mu\nu}$$

I think it is important to realize that the decision of where to put your metric is also a convention. I think Taylor's convention is to take both after. BJ take one before and the other after. I think I will try to use these conventions:

- a) raise or lower by putting metric before spinor of interest
- b) I also want $G^{uv} = G_{uv}$ so there will be no confusion there.

It seems to me that the obvious choice to make is:

$$G^{\mu\nu} = G_{\mu\nu} = [\sigma_z]_{\mu\nu}$$

6. However, I suddenly see why people don't do this. The reason is that now the elements of G are imaginary, and that means ... well no, I think it would all work if you took: well, G is supposed to be a rank-2 spinor of the first kind, and to get to the object of the second kind you should complex conjugate. Otherwise the thing will not transform right. But this would imply:

$$G^{\mu\nu} = G_{\mu\nu} = -(\sigma_z)_{\mu\nu}$$

Taylor does not really face this problem.

7. So, lets start over with a new starting convention:

to Raise, put the metric before the vector

to Lower, put the metric after the vector.

Now lets see where this convention leads us. As before, equations (1) thru (4) are still valid. Now however,

$$\left\{ \begin{array}{l} \chi^\mu = G^{\mu\nu} \chi_\nu \quad (5') \Rightarrow G''(\alpha^T)^{-1} = \alpha G'' \\ \chi_\nu = \chi^\mu G_{\mu\nu} \quad (6') \Rightarrow \\ (1) \hat{\chi}^\mu = (\alpha)_{\mu\nu} \chi^\nu \\ (2) \hat{\chi}_\mu = (\alpha^T)^{-1}_{\mu\nu} \chi_\nu \end{array} \right.$$

Slow down. Do this out in detail for both cases:

I.

$$\left. \begin{array}{l} \hat{x}^\mu = G^{\mu\nu} \hat{x}_\nu \\ \downarrow \quad \downarrow \\ (a)_{\mu\beta} x^\beta \quad (a^{-1})_{\nu\alpha} x_\alpha \\ \downarrow \quad \downarrow \\ G^{\beta\alpha} x_\alpha \end{array} \right\} \Rightarrow (a)_{\mu\beta} G^{\beta\alpha} = G^{\mu\nu} (a^{-1})_{\nu\alpha}$$

$$\Rightarrow a \cdot G' = G' (a^{-1})$$

$$\Rightarrow \underline{G^{\mu\nu} = \alpha_R \cdot (\sigma_2)_{\mu\nu}}$$

II.

$$\left. \begin{array}{l} \hat{x}_\nu = \hat{x}^\mu G_{\mu\nu} \\ \downarrow \quad \downarrow \\ (a^{-1})_{\nu\alpha} x_\alpha \quad (a)_{\mu\beta} x^\beta \\ \downarrow \quad \downarrow \\ x^\beta G_{\beta\alpha} \end{array} \right\} \Rightarrow G_{\beta\alpha} (a^{-1})_{\nu\alpha} = (a^{-1})_{\beta\mu} G_{\mu\nu}$$

$$\Rightarrow G \cdot a^{-1} = a^T \cdot G$$

$$\Rightarrow \underline{G_{\mu\nu} = \alpha_L \cdot (\sigma_2)_{\mu\nu}}$$

Whatever they are, we know for sure that $G^{\mu\nu}$ and $G_{\mu\nu}$ are both anti-symmetric. Now we add the fact that if you raise then lower you have to recover the same thing:

$$x^\mu = G^{\mu\nu} x_\nu = G^{\mu\nu} (x^\alpha G_{\alpha\nu}) = G^{\mu\nu} G_{\alpha\nu} x^\alpha$$

$$\Rightarrow G^{\mu\nu} G_{\alpha\nu} = \delta^\mu_\alpha \Rightarrow \boxed{G^{\mu\nu} G_{\nu\alpha} = -\delta^\mu_\alpha}$$

We may regard the first metric as raising the first index of the second (it proceeds, which is correct), or the second metric as lowering the second index of the first (it follows) which is correct. Then we must conclude that:

$$G^\mu{}_\alpha = -\delta^\mu_\alpha \quad \text{and} \quad \underline{\alpha_R \alpha_L = -1}$$

Also $-\delta^\mu_\alpha = G_{\alpha\nu} G^{\nu\mu} = -G^{\nu\mu} G_{\nu\alpha} \dots$ mmt.

Now, if we impose a condition of convenience, namely, that $G_{\mu\nu} = G^{\mu\nu}$ we conclude that $\alpha_R = \alpha_L = \pm i$, take your choice. The choice made by JB is $\pm i$ so that:

convention:

$$\boxed{\begin{array}{l} G_{\mu\nu} = G^{\mu\nu} = i(\sigma_2)_{\mu\nu} \equiv \delta_{\mu\nu} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\ G^{\mu\nu} = G_{\mu\nu} = G^{\mu\nu} = G^{\mu\nu} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\ x^\mu = G^{\mu\nu} x_\nu \quad x_\mu = x^\nu G_{\nu\mu} \\ \uparrow \quad \uparrow \\ \text{before to raise} \quad \text{after to lower.} \end{array}}$$

8. Here then are the conventions we made and the non-conventions:

- A) we first define the meaning of raising and lowering metric spinors (in this convention, raise metric goes before, lower metric goes after, nearest indices match up, all is conventional definition of the raising and lowering operators).
- B) we define the undotted spinor rep (again, we use the convention that x^u goes with $(a)_{uv}$, but we could have used a^*_{uv} or anything else)
- C) from the requirement that $x_u x^u$ be a scalar, we deduce what the dotted spinor must do under transformation (if the undotted is $(a)_{uv}$, then the dotted must go as $(a^T - 1)_{uv}$).

So far, then, we have developed equations (1), (2), (5'), (6)' of last page.

- D) we demand consistency of these equations and we conclude that G_{uv} and G^{uv} must be (possibly different) multiples of the matrix σ_2 . At this point we know that G_{uv} and G^{uv} as well as the dotted counterparts must be antisymmetric, in contrast to the usual metric tensor g_{uv} associated with 4-vectors. The fact that G^{uv} and G_{uv} are anti-symmetric is not a convention. It has to do with the fact that the solution of $G a G^{-1} = (a^T)^{-1}$ is $G = \text{multiple of } \sigma_2$ and is therefore antisymmetric.
- E) next, we use another consistency: raise then lower, must get same thing. This then forces the product $\epsilon \epsilon = -1$. This is not a convention.
- F) Next, for convenience, we try to make $G_{uv} = G^{uv}$, which is a convention. This forces $\epsilon \epsilon = -1$ so that $\epsilon^2 = -1$ and thus $\epsilon = \pm i$.
- E) arbitrarily, we choose $\epsilon = +i$ so that the results come out as stated.
- F) since $G_{uv} = G^{uv}$ are real, the dotted objects are just the same, another bonus.

References for Spinors and Related material.

1. Bade and Jehle RMP article.
2. Liubarski chapter of book.
3. Barut appendix 2, book.
4. Taylor, section of paper.
5. Ruhl, somewhere in book.
6. Wightman, section in weWill Gmnes book Disp. Rels. and E. particles (frnech)
7. Corson, book in libe about 1953.
8. Barut, Muzinich, Williams paper 1962.

Finite-Dim Reps of Lorentz Group ; Meaning of Pauli matrices as spinors.

1. There is much convention going on here. Combining notations of Liubarski and Baade-Jehle I might say this:

$\tau_{\frac{1}{2}0}$ = first kind spinor rep.

$(\xi^a) =$ first kind spinor, contravariant

$$\xi^a = \Lambda^a_b \xi^b$$

$$\Lambda^a_b = D^{(\frac{1}{2}0)}(g)_{ab} \quad g \in SL(2, \mathbb{C})$$

$\tau_{0\frac{1}{2}}$ = second kind spinor rep.

$(\bar{\xi}_a) =$ second kind contravariant spinor

$$\bar{\xi}_a = \Lambda^{\bar{a}}_{\bar{b}} \bar{\xi}^{\bar{b}}$$

$$\Lambda^{\bar{a}}_{\bar{b}} = D^{(0\frac{1}{2})}(g)_{\bar{a}\bar{b}} = \bar{g} \in SL(2, \mathbb{C})$$

2. Comment: this is not quite the way Barut does things. He takes $g = D(0\frac{1}{2})$ and then $D(\frac{1}{2}0) = D(0\frac{1}{2})^{\dagger -1}$. The advantage of Barut is that for rotations which are unitary, you get $D(0\frac{1}{2}) = D(\frac{1}{2}0)$, whereas in the above scheme you got equivalence but not equality. On the other hand, the above scheme gives the much desired result: $\Lambda^{\bar{a}}_{\bar{b}} = \overline{\Lambda^a_b}$ which is a part of BJ's conventions.

3. In either spinor space, rather obvious how you might define basis spinors.

4. Now, according to LIU, we know that the vector representation is this:

$$\tau_{\frac{1}{2}\frac{1}{2}} = \tau_{\frac{1}{2}0} \otimes \tau_{0\frac{1}{2}} \quad \dim = 4$$

An object transforming according to this rep can be viewed as the outer-product of first and second kind spinors, ie, something like this:

$$\begin{aligned} M \in \tau_{\frac{1}{2}\frac{1}{2}} &\Rightarrow (M')^{ab} = \underbrace{\Lambda^a_{a'} \Lambda^{\bar{b}}_{\bar{b'}}}_{D^{(\frac{1}{2}\frac{1}{2})}(g)} (M)^{a'b'} \\ &= D^{(\frac{1}{2}\frac{1}{2})}(g)_{a'a'}^{b\bar{b}'} = D^{(\frac{1}{2}0)}(g)_{a'a'}^{b\bar{b}'} D^{(0\frac{1}{2})}(g)_{\bar{b}\bar{b}'} \\ &= (g)_{aa'} (\bar{g})_{\bar{b}\bar{b}'} \end{aligned}$$

I am not putting contr/covariant indices on the D functions.

5. Therefore, a Lorentz group vector is a 2×2 matrix with one index dotted and the other index undotted. The reason is that a vector is the direct product of the two different spinor reps.

6. Recall this fact: for the Lorentz group, the two spinor reps are not equivalent because you cannot connect matrices g to matrices $\bar{g}$ for g in $SL(2, C)$. For g just in $SU(2)$, you can make such a connection, so from the point of $SU(2)$ alone, the two spinor reps are the same. But not in terms of the Lorentz group (proper).

7. Just as one might imagine the two objects $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ forming a set of basis spinors in one of the spinor spaces (dotted or undotted, first or second kind), you might imagine forming a basis in the $\begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}$ space, like $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ etc. Obviously you need 4 basis objects. These are what the Pauli matrices do:

$$(\tau_\mu)^{ab} \quad \mu=0,1,2,3$$

The four Pauli matrices form a basis in $\mathbb{H}(\frac{1}{2}, \frac{1}{2})$ space. Any vector in this space must of course have the right spinor index structure, and these Pauli's do as I have just labelled them.

8. Obviously a Lorentz transformation will mess up the Pauli's just as the same would shift up the little spinors like $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. In any given frame, you take paulis as the basis, that's all.

8. Based on the above discussion, one might take as Lorentz group irreducible finite dim rep matrices (non unitary of course):

$$D^{(\frac{1}{2}, \frac{1}{2})}(g)_{aa'bb'} = D^{(\frac{1}{2})}(g)_{aa'} \overline{D^{(\frac{1}{2})}(g)}_{bb'}$$

$$\text{where } D^{(\frac{1}{2})}(g)_{aa'} = e^{-i[\vec{x} + i\vec{y}] \cdot \vec{\tau}}_{aa'} \quad \text{eq.}$$

Or, you might break up g into its so-called polar decomposition $g = hu$ where

$$h = e^{-i[\vec{x} \cdot \vec{\tau}]} = e^{i\vec{k} \cdot \vec{\tau}} = h^\dagger \quad \text{since } K_i^\dagger = -K_i \rightarrow h = \text{hermitean}$$

$$u = e^{-i\vec{a} \cdot \vec{\tau}} \Rightarrow u^{-1} = u^\dagger \rightarrow u = \text{unitary}$$

$$\cancel{gg^\dagger} \quad gg^\dagger = hu u^\dagger h^\dagger = hh = h^2 \Rightarrow h = \sqrt{gg^\dagger}$$

$$\rightarrow u = h^{-1}g$$

With the polar decomposition, a convenient way to define the D functions is:

$$D^{(j)}(g) = D^{(j,0)}(g) = D^{(j,0)}(\bar{h}u) = D^{(j,0)}(\bar{h}) D^{(j,0)}(u)$$

$$= e^{-i\frac{1}{2}\vec{K} \cdot \vec{\alpha}} e^{-i\vec{K} \cdot \vec{\beta}} \quad K = iJ$$

Basically, we see that we are just letting the Euler angles go complex to get 6 parameters instead of 3. The boost generators K are always just iJ , no matter how large the dimension of the rep. As noted above, to get the full $D(j_1 j_2)$ rep for some g in $SL(2, C)$, you just multiply but then complex conjugate the second function.

9. The objects which belong to the rep $(j_1 j_2)$ will resemble rectangular matrices. The matrix indices are no longer spinor indices because, naturally, you are no longer in the spinor rep.

10. Lets now return to the vector rep. So far a vector is a 2×2 matrix. But, when we expand such a vector matrix onto the basis consisting of the Pauli matrices, the coefficients in this expansion may be identified with the usual Minkowski vector which is the lowest tensor. Thus:

$$M^{ab} = \sum_{\mu} (\sigma_{\mu})^{ab} M^{\mu} \quad M^{\mu} = M^{ab} (\sigma^{\mu})_{ab}$$

basis "vectors" coefficients

From the way in which the M^{ab} transform under LTs (ie, $M' = g M g^+$), we can figure out how the M^{μ} transform. This will give the usual 4×4 Lorentz matrices. In more detail:

$$(M')^{ab} = (\Lambda^a_{\alpha'} (\Lambda^b_{\beta'}) (M)^{\alpha' \beta'})$$

$$\text{But } \Lambda^b_{\beta'} = \bar{g}_{\beta' b} ; \quad \Lambda^{\alpha'}_{a'} = \Lambda^{\alpha'}_{a'} = g_{a a'} ; \quad M^{\alpha' b} = M_{ab}$$

$$\text{Thus this reads, in matrix notation, } M'_{ab} = g_{a a'} \bar{g}_{b b'} M^{\alpha' \beta'}$$

$$\text{or } M' = g M g^+ \text{ as desired}$$

Thus, we arrive at the famous form $M' = g M g^+$ in a very systematic way. Note that you must dot the second index to identify with matrix to get this to work. I have shown elsewhere how the Lorentz matrices follow from this identification.

11. More on spinors: the way I have arranged things, we have:

$$(\xi')^a = \Lambda^a_b \xi^b \quad \text{where} \quad \Lambda^a_b = g_{ab} \quad g \in SL(2, \mathbb{C}) \quad \text{or} \quad g$$

$$(\xi')^{\dot{a}} = \Lambda^{\dot{a}}_{\dot{b}} \xi^{\dot{b}} \quad \text{where} \quad \Lambda^{\dot{a}}_{\dot{b}} = \bar{g}_{\dot{a}\dot{b}} \quad \text{or} \quad \bar{g}$$

In other words, the contravariant first and second kind spinors go with g and g^* .

How do the covariant spinors transform? Forget to say this:

$$(\xi')_a = \Lambda_a^b \xi_b \quad \Lambda_a^b = (\gamma)^{bb'} \Lambda_{a'b'} (\gamma)_{a'a} = (\gamma^T)_{aa'} \Lambda^{a'}_{b'} (\gamma)_{b'b}$$

$$\text{So} \quad \Lambda_a^b = \gamma^T g \gamma = \gamma g \gamma = -(\bar{g}')^T$$

$$\Rightarrow (\xi')_a \text{ transforms with } (\bar{g}')^T$$

$$\text{and } (\xi')_{\dot{a}} \text{ transforms with } (\bar{g}')^T = (\bar{g}')^T$$

An Introduction to Spinors*

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- VIII. Lorentz Transformations and Spin Transformations.
- IX. The Maxwell-Lorentz and Dirac Equations.
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IN order to give a satisfactory account of the Lorentz covariance of Dirac's equation for the electron, it is necessary to attribute certain transformation properties to the four-component wave function appearing there. In 1929, van der Waerden (20)¹ devised an algebra of spinors which plays a role in the transformation theory of such wave functions analogous to that played in special relativity theory by tensor algebra. Four years later, in 1933, van der Waerden and Infeld (14) presented a spinor analysis which liberated the formalism from the restrictions of special relativity theory and permitted the employment of spinors against the background of a general Riemannian metric. In addition, many other contributions to the subject appeared.

The purpose of the present article is to offer an exposition of spinor calculus which will be more elementary and more accessible than those which constitute the original literature.

I. INTRODUCTION

As a preliminary to description of the spinor formalism, the following brief discussion is intended to show how the concept of spinors arises. This discussion will also serve to define the notations to be used in this article.²

* The authors wish to express their indebtedness to the University of Nebraska Research Council, which generously supported this work.

¹ Arabic numerals enclosed by parentheses and inserted into the text refer to items in the List of References at the end of this paper.

² The fundamental mathematical discoveries underlying the theory of spinors were made by Cartan (13) and Weyl (12), Chap. III, Sec. 8c.

³ The spinor notation used is taken from the paper of Infeld and van der Waerden (14). Gaussian units are used for electromagnetic quantities; the charge of a particle is denoted by q .

Spinor analysis necessitates the choice (I 2) for the signature of the special relativistic metric, unless one would load some of the

Schrödinger's representation in nonrelativistic quantum mechanics is characterized by the equations

$$H\psi = i\hbar\partial\psi/\partial t, \quad P_x\psi = -i\hbar\partial\psi/\partial x, \dots \quad (I 1)$$

Our choice for the metric of special relativity theory is

$$g_{00} = 1, \quad g_{11} = g_{22} = g_{33} = -1, \quad g_{kl} = 0 \quad (k \neq l). \quad (I 2)$$

With the four-velocity $u^k = dx^k/ds$ we can form the free particle world momentum

$$mcu_k = (E/c, -P_x, -P_y, -P_z). \quad (I 3)$$

We choose the sign of the four potential in such a way as to retain the conventional form of the field equations (IX 2) and (IX 4):

$$\phi_k = (-V, A_x, A_y, A_z). \quad (I 4)$$

When a charged particle is in an electromagnetic field, its total energy is $H = E + qV$. The world momentum of a particle in a field should have the property that its time-like component is H/c . This consideration³ leads to

$$P_k = mcu_k - (q/c)\phi_k. \quad (I 5)$$

With the relativistic generalization of (I 1), $P_k\psi = i\hbar\partial_k\psi$ (where $\partial_k \equiv \partial/\partial x^k$), one can construct a single second-order wave equation:

$$g^{kl}[i\hbar\partial_k + (q/c)\phi_k][i\hbar\partial_l + (q/c)\phi_l]\psi = m^2c^2\psi, \quad (I 6)$$

by virtue of $g^{kl}u_ku_l = 1$. Equation (I 5) is linearized, following Dirac, to⁴

$$\Gamma^k(\partial_k + i\epsilon\phi_k)\psi = \mu\psi, \quad (I 7)$$

with the conditions

$$\Gamma^k\Gamma^l + \Gamma^l\Gamma^k = -2g^{kl}. \quad (I 8)$$

For abbreviation, we have set

$$\epsilon = -q/(hc), \quad \mu = mc/\hbar.$$

It is well known that (I 7) can be satisfied with matrices having four rows and columns, but not with matrices having fewer rows and columns.

formulas of Sec. VII with minus signs in a very ugly way. This would be most disturbing in Eqs. (VII 6) and (VII 11). The signature (I 2), however, implies the inconvenience that spatial contravariant and covariant vector components differ in sign, whereas in nonrelativistic quantum theory we work with a spatial metric +++ which makes it possible to equate a covariant to a contravariant vector: $mdx/dt = P_x = -i\hbar\partial/\partial x$. We make the transition to a relativistic metric (I 2) by identifying these P_x, P_y, P_z with the contravariant components.

³ For a more precise way of arriving at (I 4), consult Corben and Stehle (3), p. 349.

⁴ See Hill and Landshoff (43), and the numerous references given there.

However, the c

(asterisk denotes certain resemblance to two-row matrices

$$\sigma^0 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\sigma^2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Since (I 7) can be

because of (I 8), as a set of two tw

where the ψ of (I

Now, the covar

View (A). The form as components can assume that of a contravariant are covariant with. However, taking violates the spirit for the following vector, the σ^k with (I 9) only in certain quence, the four written out in full frames.

View (B). The fixed matrices; the Eqs. (I 10), it will transformation pr When this process of an appropriate functions ξ and η

The matrices a;

⁵ In the special ca. to the two-component H. Jehle, Phys. Rev. For comparison of Waerden, see footnote

Comments:

1. This paper was referenced by Barut, Muzinich, Williams as a source of information on spinor analysis. Mainly, the notion of covariant and contravariant spinors both of dotted and undotted type, how you raise and lower, and the analogy with usual "world" tensors.

2. Summary of this paper:-

Section I: a setup of the Dirac equation and notion of constant gamma matrices.

Section II: review of tensor analysis

Section III: Basic spinor data. Transformation matrix is called Λ^M , which is of course a 2x2-matrix. "Dotting" is identified with complex conjugation, although this need not be done, they note. Contracted indices must match. Very similar to tensor analysis in that you make invariants by contracting, etc.

Section IV: the (antisymmetric) metric spinor which does raising and lowering, is called $\delta^{\mu\nu}$, do not confuse with gamma matrices. General form.

Section V: properties of spinors: eg, $\chi, \bar{\chi} = 0$ and same for odd rank contraction.

Section VI: special case: Lorentz transformations, $\det = +1$. Then the metric spinor takes simple form.

Section VII: relation between 4-vectors and rank-2 hermitian spinors. This is the usual homomorphism of $SL(2, \mathbb{C})$ with $SO(3, 1)$, but these words are not used.

Section VIII: Here is the trick for "keeping the Pauli sigma matrices constant" when you do a Lorentz transformation. To get $\sigma = \sigma'$ you have to compensate the action of the LT in the tensor sense on the vector index of σ^{μ} with a contrived spinor action on the two spinor indices. For any given LT Λ^M , the necessary spinor transformation is determined by this requirement, up to a sign.

Thus, look at the Dirac equation presented in form I.11. Each equation has a match-up of spinor and tensor character on the two sides, so you know right away that such an equation is covariant. In order that the new-frame σ be the same σ in old frame, you must apply the "correspondent" spinor transformation to all spinor objects, including the Dirac-2-spinors. Thus, covariance is obvious.

Much of Section 8 deals with showing that Dirac is also covariant under "inversions", ie, LT which include inversion and so have $\det = -1$. The Section ends with some unrelated remarks about specific forms in $SL(2, \mathbb{C})$ for boosts and rotations, all of which agrees with my formalism.

Section 9: shows how you can convert Maxwell's Equations and the Dirac equation to full spinor form. This is the first time I have seen something like this. Not too sure what its usefulness is.

Section 10: spinor conversion of general relativity equations. Discussion of how you check invariance of such equations involving Christoffels, affine connections, and so on. Again, unique discussion.

References: lots.

Base, spin raise and lower:

$$(\bar{A})^{\dot{a}b} = (\gamma)^{bb'} (A)^{\dot{a}}_{b'} \quad \text{/ raise second}$$

$$(\bar{A})^{\dot{a}b} = (\gamma)^{\dot{a}\dot{a}'} (A)^{\dot{a}'}_{\dot{a}}{}^b \quad \text{/ raise first}$$

$$(\bar{A})^{\dot{a}}_{\dot{b}} = (\gamma)^{\dot{a}\dot{a}'} (A)^{\dot{a}'}_{\dot{a}}{}^b \quad \text{/ raise first}$$

$$(\bar{A})^{\dot{a}}_{\dot{b}} = (\bar{A})^{\dot{a}b'}_{\dot{b}} (\gamma)_{b'b} \quad \text{/ lower second}$$

$$(\bar{A})^{\dot{a}}_{\dot{b}} = (\gamma)^{bb'} A^{\dot{a}b'}_{\dot{b}} \quad \text{/ raise second}$$

$$(\bar{A})^{\dot{a}}_{\dot{b}} = A^{\dot{a}b'}_{\dot{b}} (\gamma)_{\dot{a}\dot{a}'} \quad \text{/ lower first}$$

$$(\bar{A})_{\dot{a}b} = A^{\dot{a}'}_{\dot{b}} \gamma_{\dot{a}\dot{a}'} \quad \text{/ lower first}$$

$$(\bar{A})_{\dot{a}b} = A^{\dot{a}b'}_{\dot{a}} \gamma_{b'b} \quad \text{/ lower second}$$

double raise:

$$(\bar{A})^{\dot{a}b} = (\gamma)^{\dot{a}\dot{a}'} (\gamma)^{bb'} A^{\dot{a}'}_{\dot{a}}{}^b$$

double lower:

$$(\bar{A})_{\dot{a}b} = A^{\dot{a}'}_{\dot{a}} \gamma_{\dot{a}\dot{a}'} \gamma_{b'b}$$

Rule: do lower, put metric after
do raise, put metric in front

[or in front with = sign]

Some important facts about relating Badé/ Jehle to Lucht notation.

1. One decision which must always be made is: what do you mean by matrix multiplication? I.e., which tensor/spinor "matrix" do you want to identify with the matrix array of numbers. For tensors, one always chooses this:

$$\Lambda_{\mu\nu}^{\alpha\beta}(\xi) = \begin{bmatrix} \alpha\xi_\mu - \alpha\xi_\nu & 0 & 0 \\ \beta\xi_\mu - \beta\xi_\nu & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$(\Lambda_1 \Lambda_2)^{\mu\nu} = (\Lambda_1)^{\mu\alpha} (\Lambda_2)^{\alpha\nu}$$

There is a certain sense to this choice because when you multiply matrices, you are attaching indices in a covariant way.

2. However, for spinors the convention seems to be different. I think this difference follows from the traditional way of treating Pauli matrices as just regular matrices with no mind given to contra versus covariant.

In the following, I will use the symbol τ_i to indicate the usual Pauli matrices. Thus,

$$(\tau_1)_{ij} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad (\tau_3)_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

The symbol σ is reserved for Badé's objects with true spinor-indices. The connection between our objects is this:

$$(\sigma^\mu)^{i\dot{j}} = (\tau_\mu)_{ij} \cdot \frac{1}{\sqrt{2}}$$

To check hermiticity (knowing that the τ are all hermitian):

$$\begin{aligned} (\sigma^\mu)^{i\dot{j}} (\sigma^\mu)^{\dot{a}b} &= (\tau_\mu)_{ab} / \sqrt{2} = (\tau_\mu^\dagger)_{ab} / \sqrt{2} = (\tau_\mu)_{ba}^* / \sqrt{2} \\ &= (\sigma^\mu)^{ba*} = (\sigma^\mu)^{\dot{b}a} \quad \checkmark \end{aligned}$$

Now, for spinors of rank 2 with first index dotted, by matrix mult we will mean the linking of upper indices. Thus:

$$(AB)^{\dot{a}b} = A^{\dot{a}c} B^{\dot{c}b}$$

System 1

(One one I like)

$$(\sigma_\mu)^{ab} = (\tau_\mu)_{ab}$$

$$\bar{A}^{ab} = (\sigma_\mu)^{ab} \bar{A}^\mu$$

OR SIMPLY

$$A = \sigma_\mu \bar{A}^\mu$$

$$\bar{A}^{ab} A_{ab} = 2 \det A = 2 A^\mu A_\mu$$

$$A' = g A g^\dagger \quad g \in SL(2, \mathbb{C})$$

$$\Rightarrow \det A' = \det A \Rightarrow A^\mu A_\mu = A'^\mu A'_\mu$$

$$\Rightarrow \Lambda^\mu{}_\nu = \frac{1}{2} \text{tr} [\tau_\mu g \tau_\nu g^\dagger]$$

$$= \frac{1}{2} (\tau_\mu)_{ab} (g)_{bc} (\tau_\nu)_{cd} (g^\dagger)_{da}$$

Comments on Baade Jehle regarding convention for choosing explicit form for objects σ^μ .

1. On sheet one top, I show the three conditions which the objects σ^μ must satisfy. On sheets 2 and 3 I show how these conditions arise from the desired requirement that (in BJ form):

$$A^{ab} B_{ab} = A^\mu B_\mu$$

This fact gives two of the properties. You can almost prove property II from I (as labelled on sheet 3), but not quite. This proof ends up with:

$$[(\sigma^\mu)_{ab} (\sigma^\mu)^{cd} - \delta_a^c \delta_b^d] A^{cd} B_{ab} = 0$$

But you cannot claim bracket = 0 because it is not obvious that you can independently set the 16 numbers which multiply I

2. I have then shown that BJ's choice for the σ^μ objects does in fact satisfy all three conditions. To prove condition (I), I reduce the proof to a statement about Pauli matrices that I know is true, namely:

$$\frac{1}{2} [\tau_\mu \tau_\nu] = \delta_{\mu\nu}$$

To prove property (II), I had to explicitly check all 16 terms, see sheet 5.

-----This shows that BJ's convention works: they say:-----

$$(\sigma^\mu)^{ab} = (\tau_\mu)_{ab} \cdot \frac{1}{\sqrt{2}}$$

3. I have shown also on sheet 4 that the following alternative choice also satisfies all the conditions on the sigmas:

$$(\sigma^\mu)^{ab} = (\tau_\mu)_{ab} \cdot \frac{1}{\sqrt{2}}$$

Although BJ do not use this second convention, I like it better because it agrees with the standard method of converting to $SL(2, C)$, namely:

$$A^{ab} = (\sigma^\mu)^{ab} A^\mu$$

4. There is always the factor of $\sqrt{2}$ to think about. It has to appear somewhere, different authors cause it to appear in different places.

$$(7.5) \quad (\sigma^\mu)^{\dot{a}b} (\bar{\sigma}^\nu)_{\dot{a}b} = g^{\mu\nu}$$

$$\text{or } (\gamma)^{\dot{a}b} (\gamma)^{\dot{c}d} (\sigma^\mu)_{bd} (\bar{\sigma}^\nu)_{ac} = g^{\mu\nu}$$

} same.

①

$$(7.3) \quad (\sigma^\mu)_{\dot{a}b} = -(\sigma^\mu)_{ba}$$

②

$$(7.10) \quad g_{\mu\nu} (\sigma^\mu)^{\dot{a}b} (\bar{\sigma}^\nu)^{\dot{c}d} = (\gamma)^{\dot{a}c} (\gamma)^{\dot{b}d}$$

③

These properties must be satisfied by objects $(\sigma^\mu)^{\dot{a}b}$

① Can we prove ③ from ①? Mult both sides of ① by $(\sigma_\mu)^{\dot{A}B} (\bar{\sigma}_\nu)^{\dot{C}D}$

$$\text{①} \Rightarrow g^{\mu\nu} (\sigma_\mu)^{\dot{A}B} (\bar{\sigma}_\nu)^{\dot{C}D} = (\gamma)^{\dot{A}b} (\gamma)^{\dot{C}d} (\sigma^\mu)_{bd} (\bar{\sigma}^\nu)_{ac} (\sigma_\mu)^{\dot{A}B} (\bar{\sigma}_\nu)^{\dot{C}D}$$

No.

Condition ① is caused by identifying.

$$A^\mu \bar{A}_\mu = A^{\dot{a}b} \bar{A}_{\dot{a}b}$$

Of course you can do this in two ways, hence getting ①, ③.

② What do we mean by trace? When (σ^μ) appears w.a. indices, how do you know which one you have!

I would like to see this. (σ^μ) means $(\sigma^\mu)^{\dot{a}b}$ because this looks like a basis for $\mathbb{R}^{4 \times 4}$.

How about $(\sigma_\mu)^{\dot{a}b} = (\tau_\mu)_{ab}$, not the way they do it.

Check properties:

$$\text{Prop ②} \quad (\sigma_\mu)^{\dot{a}b} = (\tau_\mu)_{ab}$$

$$(\sigma_\mu)^{\dot{b}a} = (\sigma_\mu)^{\dot{b}a*} = (\tau_\mu)_{ba}^* = (\tau_\mu^\dagger)_{ab} \quad \checkmark$$

Logic for conditions on Bada's objects $(\sigma^\mu)_{ab}$

① Following (1,2) we define:

$$A_{ab} \equiv (\sigma_\mu)_{ab} A^\mu \quad (1)$$

② We assume that the $(\sigma_\mu)_{ab}$ satisfy this property:

$$(\sigma^\mu)_{ab} (\sigma^\nu)^{ab} = g^{\mu\nu} \quad (2)$$

③ Define B_{ab} in terms of B^μ as above. Then, from definitions and (2) we can prove that

$$A^{ab} B_{ab} = A^\mu B_\mu \quad \forall A, B \quad (3)$$

proof:

$$\begin{aligned} A^{ab} B_{ab} &= (\sigma_\mu)^{ab} A^\mu (\sigma_\nu)_{ab} B^\nu = [(\sigma^\mu)^{ab} (\sigma^\nu)_{ab}] A_\mu B_\nu \\ &= g^{\mu\nu} A_\mu B_\nu = A_\mu B^\mu \quad \checkmark \end{aligned}$$

So, from (2) we have proven (3).

④ Apply to (1) the object $(\sigma^\nu)^{ab}$ to get

$$(\sigma^\nu)^{ab} A_{ab} = (\sigma_\mu)_{ab} (\sigma^\nu)^{ab} A^\mu = g^{\mu\nu} A^\mu = A^\nu$$

Thus we have inverted:

$$A^\mu = (\sigma^\mu)^{ab} A_{ab} \quad (4)$$

⑤ Assuming (2), we know that (3), (4) are true. Now combine 3, 4:

$$\begin{aligned} A^{ab} B_{ab} &= [(\sigma^\mu)^{cd} A_{cd}] [(\sigma_\mu)^{ef} B_{ef}] = (\sigma^\mu)^{cd} (\sigma_\mu)^{ef} A_{cd} B_{ef} \\ &= (\sigma^\mu)_{cd} (\sigma_\mu)^{ef} A^{cd} B_{ef} \end{aligned}$$

Therefore, since we can independently vary the entries in A, B , we conclude

$$(\sigma^\mu)_{cd} (\sigma_\mu)^{ef} = \delta_c^e \delta_d^f \quad \uparrow \text{not true!}$$

(start over)

- ① Let $A^{ab} =$ arbitrary 2×2 matrix.
 Let $B^{ab} =$ another arbitrary 2×2 matrix.

- ② Let $(\sigma^\mu)^{ab}$ be 4 2×2 matrices which satisfy one condition:

$$(\sigma^\mu)_{ab} (\sigma^\nu)^{ab} = g^{\mu\nu} \quad (\text{I})$$

/note: σ_μ or A_{ab} etc are all given in terms of the above objects by standard metric tensor $g_{\mu\nu}$ and metric spin δ_{ab} . So, all related tensors and spinors are defined.

- ③ Construct objects $A^\mu = (\sigma^\mu)_{ab} A^{ab}$ $B^\mu = (\sigma^\mu)_{ab} B^{ab}$

Components of A^μ are complex #s.

Aside: if $(\sigma^\mu)^{ab}$ and A^{ab} are both hermitian, then the A^μ would be real:

$$\bar{A}^\mu = \overline{(\sigma^\mu)_{ab}} \bar{A}^{ab} = (\sigma^\mu)_{ab} A^{ab} = (\sigma^\mu)_{ba} A^{ba} = A^\mu$$

- ④ From condition (I) and definitions ③, easy to show that

$$A^{ab} B_{ab} = \bar{A}^\mu \bar{B}_\mu$$

- ⑤ In order to invert ③, we must assume all $(\sigma^\mu)_{ab}$ are invertible 2×2 's. X.
 In order to invert ③, we need an orthogonality rule:

$$(\sigma_\mu)^{cd} (\sigma^\mu)_{ab} = \delta^c_a \delta^d_b \quad (\text{II})$$

Then you will get:

$$(\sigma_\mu)^{cd} A^\mu = A^{cd}$$

- ⑥ Conclusion: you cannot prove property I from property II, nor can you do it vice versa. Thus, you must really check that your candidates satisfy both properties!

Verify that Bode choice $(\gamma, 1)$ satisfies properties (I) and (II).

① Bode says:

$$(\sigma^\mu)^{ab} = (\tau_\mu)_{ab}$$

So rewrite (I) as:

$$(\gamma)_{aa'} (\gamma)_{bb'} (\sigma^\mu)^{a'b'} (\sigma^\nu)^{ab} = g^{\mu\nu} \quad (I)$$

$$\Rightarrow \delta_{aa'} \delta_{bb'} (\tau_\mu)_{a'b'} (\tau_\nu)_{ab} = g^{\mu\nu}$$

$$\rightarrow \delta_{aa'} (\tau_\mu)_{a'b'} (\gamma^T)_{b'b} (\tau_\nu)^T_{ba} = g^{\mu\nu}$$

$$\Rightarrow \text{trace} [\gamma \tau_\mu \gamma^T \tau_\nu^T] = g^{\mu\nu}$$

$$\Rightarrow g^{\mu\nu} = \text{trace} [\sigma_2 \tau_\mu \sigma_2 \tau_\nu] = \text{tr} [\sigma_2 \tau_\nu \sigma_2 \tau_\mu] = g^{\nu\mu} \text{tr} [\tau_\nu \tau_\mu] \\ = g^{\mu\nu} \delta_{\mu\nu} \text{ mod } 2 \checkmark$$

② Comment: Had we started with $(\sigma^\mu)^{ab} = (\tau^\mu)_{ab}$, we'd have ~~end~~ we'd end up with

$$(\sigma^\mu)^{ab} = (\tau_\mu)_{ab} \Rightarrow \text{same result since } g^{\mu\nu} = g^{\nu\mu}$$

③ Property II is harder:

$$(\sigma^\mu)^{ab} = (\tau_\mu)_{ab} \text{ mod } \sqrt{2}$$

$$II \Rightarrow g_{\mu\nu} (\sigma^\mu)^{cd} (\sigma^\nu)^{ab} = \gamma^{ca} \gamma^{db}$$

$$\Rightarrow g_{\mu\nu} (\tau_\mu)^{cd} (\tau_\nu)_{ab} = \gamma^{ca} \gamma^{db}$$

But this is true, as shown directly on added sheet!

④ Comment: Had we said $(\sigma_\mu) = (\tau_\mu)$, we'd end up with

$$g^{\mu\nu} \tau_\mu \tau_\nu \text{ which } \rightarrow \text{same result!}$$

$$(X =) g_{\mu\nu} (\tau_\mu)_{ab} (\tau_\nu)_{a'b'} = (\tau_0)_{ab} (\tau_0)_{a'b'} - (\tau_1)_{ab} (\tau_1)_{a'b'} \text{ etc.}$$

ab, a'b'	$(\tau_0)_{ab}$	$(\tau_1)_{ab}$	$(\tau_2)_{ab}$	$(\tau_3)_{ab}$	$(\tau_0)_{a'b'}$	$(\tau_1)_{a'b'}$	$(\tau_2)_{a'b'}$	$(\tau_3)_{a'b'}$	X	aa' bb'	$\delta_{aa'}$	$\delta_{bb'}$	y
11 11	1	0	0	1	1	0	0	1	0	11 11	0	0	0
12 11	0	1	-i	0	0	0	0	1	0	11 21	0	-1	0
21 11	0	1	i	0	1	0	0	1	0	21 11	-1	0	0
22 11	1	0	0	-1	1	0	0	-1	2	21 21	-1	-1	2
11 12	1	0	0	-1	0	1	-i	0	0	11 12	0	1	0
12 12	0	1	-i	0	0	1	-i	0	0	11 22	0	0	0
21 12	0	1	i	0	0	1	-i	0	-2	21 12	-1	1	-2
22 12	1	0	0	-1	0	1	-i	0	0	21 22	-1	0	0
11 21	1	0	0	1	0	1	i	0	0	12 11	1	0	0
12 21	0	1	-i	0	0	1	i	0	-2	12 21	1	-1	-2
21 21	0	1	i	0	0	1	i	0	0	22 11	0	0	0
22 21	1	0	0	-1	0	1	i	0	0	22 21	0	-1	0
11 22	1	0	0	1	1	0	0	-1	2	12 12	1	1	2
12 22	0	1	-i	0	1	0	0	-1	0	12 22	1	0	0
21 22	0	1	i	0	1	0	0	-1	0	22 12	0	1	0
22 22	1	0	0	-1	1	0	0	-1	0	22 22	0	0	0

$$(Y =) \delta_{aa'} \delta_{bb'} \cdot 2$$

Here, $(\tau_\mu)_{ab} \equiv$ usual Pauli's

$$X = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

We have thus proved that:

$$g^{\mu\nu} (\tau_\mu)_{ab} (\tau_\nu)_{a'b'} = 2 (\gamma)_{aa'} (\gamma)_{bb'}$$

Dirac 4×4

is called pseudo-vector representation. The corresponding quantities are called vectors and pseudo-vectors.

Quantities that transform according to the representations

$$(\tau_k)^m (\tau_k)^n$$

are called tensors of rank $(m+n)$. It can be easily seen that

$$(\tau_k)^2 = (\tau_k)^2$$

Hence, there are only two types of tensors of rank n : tensors of one type transform according to the representation

$$(\tau_k)^n$$

and those of the other according to

$$\tau_k (\tau_k)^{n-1}$$

The latter are called pseudo-tensors to distinguish them from the quantities of the first kind, which are called simply tensors.

Note also that

$$\tau_k \times \tau_k = \tau_k^2 \text{ and } \tau_k \times \tau_k^2 = \tau_k^3.$$

If we consider a scalar as a tensor of rank 0, then we can formulate the following obvious rule: the product of two pseudo-tensors is a tensor, while the product of a tensor and a pseudo-tensor is a pseudo-tensor.

Chapter XVI

Relativistically Invariant Equations

73. The Wave Function

According to the classical theory of elementary particles (electrons, various kinds of mesons and others) the state of every free elementary particle can be described by means of a wave function $\psi(x) = \psi(x_0, x_1, x_2, x_3)$. The function ψ can have one, several or even an infinite number of components. If the number of its components is finite, we shall denote that number by a . In terms of the wave function the Lagrangian of the particle is a quadratic functional:

$$L = L(\psi).$$

The form of this functional is in general different for various particles. The integral of the Lagrangian over the whole of four-dimensional space-time is called action S. In this theory the principle of least action is valid, according to which the wave function satisfies the equation $\delta S = 0$. This equation reduces to a system of linear homogeneous differential equations, which can always be reduced to the form

$$A_{ik} \frac{\partial \psi_k}{\partial x_j} + x_{ik} \psi_k = 0 \quad (i, k = 1, 2, \dots, a; j = 0, 1, 2, 3). \quad (73,1)$$

On account of the homogeneity of space and time, the coefficients A_{ik} and x_{ik} depend neither on the coordinates nor on time.

With the aid of the Lagrangian the energy and the momentum of the particle (or their values averaged over time) can be expressed in terms of the wave function ψ .

In mechanics and electrodynamics quantities characterizing the motion of matter (velocity, energy, electric and magnetic fields, etc.) take on different values in different coordinate systems. In complete accordance with this, the state of an elementary particle is described in different systems of coordinates by different wave functions. It is of a fundamental importance that the transformation undergone by the wave function in passing from one coordinate system to another is linear. More precisely, this means the following: Consider any point, say $x=y=z=0, t=0$. Let K' be any system of coordinates whose origin at the time $t=0$ coincides with the origin of the original system K . If we denote by

ψ_i the components of the wave function in the system K' , the linear character of the transformation is expressed by the relations

$$\psi_i(0,0,0,0) = T_{ik}(g) \psi_k(0,0,0,0). \quad (73,2)$$

Here $T_{ik}(g)$ is a square matrix of σ rows and σ columns, g is the element of the full Lorentz group which corresponds to the transition from the system K to the system K' .

Let us prove that the matrices $T_{ik}(g)$ ($g \in L$) realize a representation of the full Lorentz group. To do this, we introduce another system of coordinates K'' , whose origin also coincided at $t=0$ with the origin of the system K . We denote by g_1 the transition from the system K' to the system K'' and by h the transition from the system K to K'' . Then by the definition of the Lorentz group we have $h = g_1 g$. On the other hand, we can write down the expressions

$$\psi'_i = T_{ij}(g_1) \psi_j \text{ and } \psi'_i = T_{ik}(h) \psi_k.$$

for the components ψ'_i of the wave function in the system K'' . Comparing these expressions with (73,2), we have:

$$T_{ik}(h) = T_{ij}(g_1) T_{jk}(g), \text{ i.e. } T(g_1 g) = T(g_1) T(g).$$

This relation shows that the set of σ numbers $\psi_i(x)$ ($i=1,2,\dots,\sigma$) for a fixed x , can be considered as the set of components of a certain vector $\psi(x)$ in a σ -dimensional space E , that realizes the representation T of the full Lorentz group.

Of course, the choice of coordinates for the vector ψ is not unique: by changing the basis in the space E we can go over from old to new coordinates that are linear combinations of the old ones. In applications it is natural and convenient to choose a basis related to the decomposition of E into irreducible subspaces

You must be able to decompose E in this way.

$$E = \sum_s E_{\mathfrak{r}s} \quad (73,3)$$

Here $\mathfrak{r}$ is an irreducible representation of the group L , that transforms the subspace $E_{\mathfrak{r}s}$. The index s serves to label the subspaces, which transform according to the same representation $\mathfrak{r}$. In each space $E_{\mathfrak{r}s}$ let us now single out a canonical basis

The set of all vectors $e_{\mathfrak{r}s,i}$ forms a basis in E .

In the following we shall continue to use the coor-

comp's on ψ
dinates $\psi_{\mathfrak{r}s,i}$ referring to this basis as the components of the wave function.

74. Relativistically Invariant Equations

By virtue of the principle of equivalence of all inertial systems the field equations (73,1) should be of the same form (and consequently should have the same coefficients) in all inertial coordinate systems. Equations that have this property are called relativistically invariant. In this section we derive the general form of such equations.

We note first of all that in as much as the coefficients of the equation (73,1) do not depend upon the coordinates x, y, z, t its invariance at one point, say $(0,0,0,0)$, has as a consequence its invariance at all other points. We shall therefore consider the left hand side of the equation (73,1) only at the point $(0,0,0,0)$.

For simplicity we shall consider that the matrix χ_{ik} has an inverse. In this case we may, without any further loss of generality, put

$$\chi_{ik} = \chi_{ik}^0 \quad (i \neq 0).$$

Thus we consider the system

$$A_{ik} \frac{\partial \psi_i}{\partial x_j} + \chi_{ik} \psi_i = 0, \quad (74,1)$$

or, in the new notation

$$A_{\mathfrak{r}s, \mathfrak{r}'s'} \frac{\partial \psi_{\mathfrak{r}'s'}}{\partial x_j} + \chi_{\mathfrak{r}s} \psi_{\mathfrak{r}'s'} = 0. \quad (74,2)$$

We consider first what requirements are imposed by invariance with respect to the proper Lorentz group. Let us see how the numbers

$$\frac{\partial \psi_{\mathfrak{r}'s'}}{\partial x_j}$$

transform in a transition from one inertial system of coordinates to another. From (73,2) we have:

$$\frac{\partial \psi_{\mathfrak{r}'s'}}{\partial x_j} = \psi_{\mathfrak{r}s}(g) \frac{\partial \psi_{\mathfrak{r}s}}{\partial x_j} \cdot \frac{\partial x_j}{\partial x'_j} = \psi_{\mathfrak{r}s}(g) a_{jj'}(g) \frac{\partial \psi_{\mathfrak{r}s}}{\partial x'_j}.$$

We see that the 4σ numbers

$$\frac{\partial \psi_{\mathfrak{r}'s'}}{\partial x_j}(\mathfrak{r}', s')$$

($\mathfrak{r}', s'$ are fixed) transform as the coordinates in the space realized by the representation $\mathfrak{r}' \times \mathfrak{r}_1 \frac{1}{2} \frac{1}{2}$.

If the representation $\mathfrak{r}' \times \mathfrak{r}_1 \frac{1}{2} \frac{1}{2}$ does not contain the representation $\mathfrak{r}$, then no linear combination of the

quantities $\frac{\partial \psi_{\nu\mu}}{\partial x_j}$ transforms as the $\psi_{\nu\mu}$. Hence if τ is not contained in $\tau' \times \tau''$, those equations of the system (74,2), which on their right hand side contain $\psi_{\nu\mu}$ (τ is fixed) do not contain the derivatives $\frac{\partial \psi_{\nu\mu}}{\partial x_j}$ on the left hand side.

We shall say that a representation τ is linked with a representation τ' if the product $\tau\tau'$ contains τ . The result obtained can now be formulated as follows: if a representation τ' does not link with a representation τ , then all the coefficients $A_{\nu\mu}^{\tau\tau'}$ are equal to zero.

From what has been said this conclusion follows: if a representation τ is contained in T , then at least one of the representations linked with τ is contained in T . In fact, were this not true the left hand sides of the equations (74,2) corresponding to a given τ would include no terms and we would arrive at the absurd conclusion that $\psi_{\nu\mu} = 0$.

If the representation τ' is linked with τ , then the representation $\tau\tau'$ contains the representation τ only once, as follows from (68,3). Hence apart from a factor independent of l and l' there exists for fixed τ' and τ only one set of linear expressions

$$P_{\nu\mu}^{\tau\tau'} \frac{\partial \psi_{\nu\mu}}{\partial x_j} \quad (l=1, 2, \dots, s), \quad (74,3)$$

that transforms as the set of components $\psi_{\nu\mu}$ ($l=1, 2, \dots, s$). Hence relativistically invariant equations can be written in the form

$$\sum_{\nu\mu} Z_{\nu\mu} P_{\nu\mu}^{\tau\tau'} \frac{\partial \psi_{\nu\mu}}{\partial x_j} + \psi_{\nu\mu} = 0, \quad (74,4)$$

where the coefficients $Z_{\nu\mu}$ are arbitrary, if it is not required that the equations be invariant with respect to inversion.

According to formula (68,3) two representations τ_{PQ} and $\tau_{P'Q'}$ are linked if

$$|P-P'| = \frac{1}{2} \quad \text{and} \quad |Q-Q'| = \frac{1}{2}.$$

In other words, if $P \neq 0, Q \neq 0$, four representations

$$\tau_{P+\frac{1}{2}, Q+\frac{1}{2}}, \tau_{P+\frac{1}{2}, Q-\frac{1}{2}}, \tau_{P-\frac{1}{2}, Q+\frac{1}{2}}, \tau_{P-\frac{1}{2}, Q-\frac{1}{2}}$$

are linked with the representation τ_{PQ} . Only two representations

$$\tau_{P+\frac{1}{2}, Q+\frac{1}{2}}, \tau_{P+\frac{1}{2}, Q-\frac{1}{2}},$$

link with the representation τ_{PQ} ($Q \neq 0$). And similarly the representations

$$\tau_{P+\frac{1}{2}, \frac{1}{2}}, \quad \text{and} \quad \tau_{P-\frac{1}{2}, \frac{1}{2}}$$

link with the representation τ_{P0} ($P \neq 0$). Finally only one representation $\tau_{\frac{1}{2}, \frac{1}{2}}$ links with the unit representation τ_{00} .

If two representations τ and τ' are linked, the sums of their indices $P+Q$ and $P'+Q'$ are simultaneously either integers or half integers.

Let us evaluate the coefficients $P_{\nu\mu}^{\tau\tau'}$. We saw that the operators

$$\frac{\partial}{\partial x_j} \quad (j=0, 1, 2, 3)$$

transform as the components of a four-vector in the natural basis. Using the matrix (71,4) for the transformation from the natural to the canonical basis, we can easily obtain the operators

$$\left. \begin{aligned} \frac{\partial}{\partial x_0} - \frac{\partial}{\partial x_3} &= i \frac{\partial}{\partial x_1} + i \frac{\partial}{\partial x_2}, & \frac{\partial}{\partial x_1} - i \frac{\partial}{\partial x_2} &= i \frac{\partial}{\partial x_0} + i \frac{\partial}{\partial x_3}, \\ \frac{\partial}{\partial x_1} + i \frac{\partial}{\partial x_2} &= -i \frac{\partial}{\partial x_0} + i \frac{\partial}{\partial x_3}, & \frac{\partial}{\partial x_0} + i \frac{\partial}{\partial x_3} &= -i \frac{\partial}{\partial x_1} + i \frac{\partial}{\partial x_2}, \end{aligned} \right\} \quad (74,5)$$

that transform in the same way as the components of a vector in the canonical basis. Equation (74,3) can now be rewritten as

$$P_{\nu\mu}^{\tau\tau'} \frac{\partial \psi_{\nu\mu}}{\partial x_j} = P_{\nu\mu}^{\tau\tau'} \partial_{\nu\mu} \psi_{\nu\mu}. \quad (74,6)$$

According to (68,4) the coefficients $P_{\nu\mu}^{\tau\tau'}$ are equal to

$$P_{\nu\mu}^{\tau\tau'} = \left(\frac{1}{2} P' m p' | P p \right) \left(\frac{1}{2} Q' n q' | Q q \right), \quad (74,7)$$

where P, Q is the weight of the representation τ ; P', Q' the weight of the representation τ' ; p, q the indices of the basic vector $\psi_{\nu\mu}$; p', q' the indices of the basic vector $\psi_{\nu'\mu'}$.

Substituting (74,6) into (74,4) and using (74,5) and (74,7) we obtain:

$$\sum_{\nu\mu} Z_{\nu\mu} P_{\nu\mu}^{\tau\tau'} \left(\left(\frac{1}{2} P' m p' | P p \right) \left(\frac{1}{2} Q' n q' | Q q \right) \left(-i \frac{\partial}{\partial x_1} + i \frac{\partial}{\partial x_2} \right) + \left(\frac{1}{2} P' m p' | P p \right) \left(\frac{1}{2} Q' n q' | Q q \right) \left(-i \frac{\partial}{\partial x_0} + i \frac{\partial}{\partial x_3} \right) + \right.$$

$$\begin{aligned}
& + \left(\frac{1}{2} P' - \frac{1}{2} P' | P P \right) \left(\frac{1}{2} Q' - \frac{1}{2} Q' | Q Q \right) \left(i \frac{\partial}{\partial x_2} + i \frac{\partial}{\partial x_3} \right) + \\
& + \left(\frac{1}{2} P' - \frac{1}{2} P' | P P \right) \left(\frac{1}{2} Q' - \frac{1}{2} Q' | Q Q \right) \left(i \frac{\partial}{\partial x_1} + i \frac{\partial}{\partial x_3} \right) \times \\
& \times \Phi_{P'Q'P''Q''} + \Phi_{P'Q'P''Q''} = 0. \quad (74,8)
\end{aligned}$$

The Clebsch-Gordan coefficients appearing here were evaluated in sec. 54. Substituting the values obtained there into (74,8) we finally have:

$$\begin{aligned}
& \sum_{P'Q'} \frac{2P'Q'}{\sqrt{(P'+1)(Q'+1)}} \left\{ \sqrt{(P'+P+1/2)(Q'+Q+1/2)} \times \right. \\
& \quad \times \left(-i \frac{\partial}{\partial x_1} + i \frac{\partial}{\partial x_2} \right) \Phi_{P'Q'P''Q''} + \frac{1}{2} + \\
& + \sqrt{(P'+P+1/2)(Q'+Q+1/2)} \left(-i \frac{\partial}{\partial x_2} + i \frac{\partial}{\partial x_3} \right) \Phi_{P'Q'P''Q''} + \frac{1}{2} + \\
& + \sqrt{(P'-P+1/2)(Q'+Q+1/2)} \left(i \frac{\partial}{\partial x_2} + i \frac{\partial}{\partial x_3} \right) \Phi_{P'Q'P''Q''} + \frac{1}{2} + \\
& + \sqrt{(P'-P+1/2)(Q'-Q+1/2)} \left(i \frac{\partial}{\partial x_1} + i \frac{\partial}{\partial x_3} \right) \Phi_{P'Q'P''Q''} + \frac{1}{2} + \\
& \quad \left. + \Phi_{P'Q'P''Q''} = 0. \right. \quad (74,9)
\end{aligned}$$

where $\mu = \text{sign}(P-P')$ and $\nu = \text{sign}(Q-Q') +$

If we put $x=0$ in this equation it will, of course, remain relativistically invariant.

Let us note certain symmetry properties of the coefficients $P_{P'Q'}^{P''Q''}$. From (74,7) it follows that $P_{P'Q'}^{P''Q''}$ does not change as a result of the substitution

$$P \leftrightarrow Q, P' \leftrightarrow Q', P'' \leftrightarrow Q', P''' \leftrightarrow Q''.$$

This can be written as:

$$P_{P'Q'}^{P''Q''} = P_{Q'Q'}^{P''P'} \quad (74,10)$$

where

$$P = P_{PQ}, \quad P' = P_{P'Q'}, \quad P'' = P_{P''Q''}, \quad P''' = P_{P'''Q''} \text{ etc.}$$

If in (74,6) we substitute the explicit forms (74,5) for ∂_{mn} and then equate the coefficients of the derivatives $\frac{\partial}{\partial x_j}$, we obtain:

$$\begin{aligned}
P_{P'Q'}^{P''Q''} &= i P_{P'Q'}^{P''Q''} - i P_{P'Q'}^{P''Q''}, & P_{P'Q'}^{P''Q''} &= i P_{P'Q'}^{P''Q''} - i P_{P'Q'}^{P''Q''}, \\
P_{P'Q'}^{P''Q''} &= P_{P'Q'}^{P''Q''} + P_{P'Q'}^{P''Q''}, & P_{P'Q'}^{P''Q''} &= i P_{P'Q'}^{P''Q''} + i P_{P'Q'}^{P''Q''}.
\end{aligned} \quad (74,11)$$

If in these equations we make the substitution

$$P \rightarrow \bar{P}, P' \rightarrow \bar{P}', P'' \rightarrow \bar{P}'', P''' \rightarrow \bar{P}'''$$

and take into account the symmetry of the coefficients $P_{P'Q'}^{P''Q''}$, we obtain:

$$P_{P'Q'}^{P''Q''} = -P_{\bar{P}'\bar{Q}'}^{\bar{P}''\bar{Q}''}, \quad P_{P'Q'}^{P''Q''} = i P_{\bar{P}'\bar{Q}'}^{\bar{P}''\bar{Q}''} \quad (J=1,2,3). \quad (74,12)$$

Furthermore, the Clebsch-Gordan coefficients have the following property (see (55,6)):

$$(J_1 J_2 - m_1 - m_2 | J - m) = (-1)^{J_1 + J_2 - J} (J_1 J_2 m_1 m_2 | J m).$$

Hence

$$P_{P'Q'}^{P''Q''} = (-1)^{P'-P+Q-Q'+1} P_{\bar{P}'\bar{Q}'}^{\bar{P}''\bar{Q}''}.$$

Returning to (74,11), we find hence that

$$\begin{aligned}
P_{P'Q'}^{P''Q''} &= (-1)^{P'-P+Q-Q'+1} P_{\bar{P}'\bar{Q}'}^{\bar{P}''\bar{Q}''} \quad (\alpha=0, 1), \\
P_{P'Q'}^{P''Q''} &= -(-1)^{P'-P+Q-Q'+1} P_{\bar{P}'\bar{Q}'}^{\bar{P}''\bar{Q}''} \quad (\beta=2, 3).
\end{aligned} \quad (74,13)$$

For further use we need another general relation among the coefficients A_{jk}^{ik} of equation (74,1). On account of the relativistic invariance of this equation the function

$$\phi_k = T_M(g) \phi_k$$

must satisfy the equation

$$A_{ik}^{ik} \frac{\partial \phi_k}{\partial x_i} + x \phi_i = 0 \quad (x_i = x_j(g) x_j)$$

or

$$A_{ik}^{ik} T_M(g) \frac{\partial \phi_k}{\partial x_i} + x T_M(g) \phi_i = 0. \quad (74,14)$$

On the other hand, equation (74,1) can be rewritten in the form

$$A_{ik}^{ik} \frac{\partial \phi_k}{\partial x_i} x_j(g) + x \phi_i = 0.$$

Substituting here 1 for l and multiplying by $T_{1l}(g)$, we have:

$$T_M(g) A_{ik}^{ik} \frac{\partial \phi_k}{\partial x_i} + x T_M(g) \phi_i = 0.$$

Comparing this with (74,14) we see that

$$A_{ik}^{ik} T_M(g) = T_M(g) A_{ik}^{ik}.$$

or, omitting the matrix indices,

$$T(g^{-1}) A T(g) = A^2(g). \quad (74,15)$$

In particular, for $\gamma=0$

$$T(g^{-1}) A T(g) = A^2(g).$$

^{*)} Sign $x=1$, if $x>0$ and sign $x=-1$, if $x<0$

We differentiate this expression with respect to $\beta_x = -v/c$ and then put $g=e$. Taking (66,1) into account, we obtain:

$$-J_1 A^0 + A^0 J_1 = -A^1.$$

Since the x, y, z axes are equivalent, we can write down a more general relation:

$$A^k = [J_k A^0] = J_k A^0 - A^0 J_k \quad (k=1, 2, 3). \quad (74,16)$$

Let us see now under what conditions equation (74,1) is invariant with respect to inversion.

Note that the inversion operator carries any basic vector $e_{\tau pq}$ that transforms according to the representation $\tau = \tau_{PQ}$ of the proper Lorentz group, into a basic vector $u_{\tau qp}$, that transforms according to the representation $\tilde{\tau} = \tilde{\tau}_{QP}$. If $P=Q$, then the representation $\tilde{\tau}$ is equivalent to the representation τ . In this case the inversion operator is defined by the equation

$$I e_{\tau pq} = e_{\tau qp} \quad \text{or by} \quad I e_{\tau pq} = -e_{\tau qp}.$$

Hence it follows that the space E of the wave functions contains the same number of subspaces, transforming according to the representations τ and $\tilde{\tau}$. Let us agree to label them in such a way that the subspaces E and $E_{\tilde{\tau}}$ with the same label s transform into each other under the action of the inversion. Thus if $\tau \neq \tilde{\tau}$, then

$$I e_{\tau pq} = e_{\tilde{\tau} qp}.$$

But if $\tau = \tilde{\tau}$, then evidently

$$I e_{\tau pq} = e_{\tau qp}.$$

or

$$I e_{\tau pq} = -e_{\tau qp}.$$

These three expressions can be given a common form:

$$I e_{\tau pq} = \text{sign } \tau e_{\tau qp} \quad (74,17)$$

if we introduce the notation

$$\text{sign } \tau = \text{sign } \tilde{\tau} = \begin{cases} 1, & \text{if } \tau = \tau_{PP} \quad P \neq Q, \\ 1, & \text{if } \tau = \tau_P^P, \\ -1, & \text{if } \tau = \tau_P^P. \end{cases} \quad (74,18)$$

As its derivation shows, (74,15) should also be valid for $g=I$, if the equation (74,1) is invariant with re-

spect to inversion of the spatial coordinates. Since

$$a_{00}(I) = 1, a_{11}(I) = a_{22}(I) = a_{33}(I) = -1,$$

then for $g=I$ and $\gamma=0,1,2,3$ (74,15) reduces to

$$I^{-1} A^k I = A^k, \quad I^{-1} A^0 I = -A^0 \quad (k=1, 2, 3). \quad (74,19)$$

In other words, in order that the equation (74,1) be invariant with respect to inversion it is necessary and sufficient that the matrix A^0 should commute with the inversion operator and the other three A^1, A^2 and A^3 should anticommute with it.

Let us see how the conditions (74,19) are related to the coefficients $Z_{\tau}^{\tau'}$. Note that the last three relations (74,19) follow from the condition $A^0 I = I A^0$. Actually, it was shown in sec. 72 that the inversion operator anticommutes with the infinitesimal operators J_1, J_2, J_3 .

Hence, $A^k = (J_k A^0 - A^0 J_k) I = -I (J_k A^0 - A^0 J_k) = -I A^k \quad (k=1, 2, 3).$

Let us consider the condition $A^0 I = I A^0$. Acting with both sides of this operator equation on the basic vector $e_{\tau s l}$ and taking into account (74,17) we get:

$$\text{sign } \tau A^0 e_{\tau s l} = I A^0 e_{\tau s l}.$$

Hence, since

$$A^0 e_{\tau s l} = Z_{\tau' \tau}^0 P_{\tau' s l} e_{\tau' s l},$$

it follows that:

$$\text{sign } \tau Z_{\tau' \tau}^0 P_{\tau' s l} e_{\tau' s l} = \text{sign } \tau' Z_{\tau' \tau}^0 P_{\tau' s l} e_{\tau' s l}.$$

Equating coefficients of $e_{\tau' s l}$ and taking into account that $P_{\tau' \tau}^0 = -P_{\tau \tau'}^0$, we find the following condition for the invariance of the equation (74,1) with respect to inversion:

$$\text{sign } \tau Z_{\tau' \tau}^0 = -\text{sign } \tau' Z_{\tau' \tau}^0. \quad (74,20)$$

75. The Lagrangian

According to the theory of elementary particles, the Lagrangian of an elementary particle has the following form

$$L = -\left(\Gamma^k \frac{\partial \psi}{\partial x_k} \right) \cdot \psi + \tau(\psi, \psi). \quad (75,1)$$

where ψ is the wave function of the particle and the

Γ^k ($k=0,1,2,3$) are σ -dimensional matrices. The expression (ψ, ψ) is a bilinear form which can be represented as follows:

$$(\varphi, \psi) = B_{ij} \varphi_i \bar{\psi}_j \quad (75,2)$$

(summation over the indices i, j from 1 to σ is assumed).

Let us see what the matrices Γ^k and the coefficients B_{ij} must be in order that the Lagrangian be invariant and that the principle of least action

$$\delta S = \delta \int L dx_0 dx_1 dx_2 dx_3 = 0$$

be equivalent to the equations (74,1).

Let us calculate δS . We have

$$\begin{aligned} \delta S &= \delta \int L d^4x = \\ &= - \int \left\{ \left(\Gamma^k \frac{\partial \psi}{\partial x_k} \right) \cdot \varphi + \left(\Gamma^k \frac{\partial \varphi}{\partial x_k} \right) \cdot \psi + \varphi (\psi, \varphi) + \varphi (\psi, \psi) \right\} d^4x = \\ &= - \int \frac{\partial}{\partial x_k} \left(\Gamma^k \psi, \varphi \right) d^4x + \int \left\{ \left(\psi, \Gamma^k \frac{\partial \varphi}{\partial x_k} \right) - \right. \\ &\quad \left. - \left(\Gamma^k \frac{\partial \varphi}{\partial x_k} \right) \cdot \psi + \varphi (\psi, \psi) \right\} d^4x, \end{aligned}$$

where Γ^k is a matrix conjugate to the matrix Γ^k with respect to the bilinear form (75,2).⁺

The first integral on the right hand side of the last expression is zero when $\delta \psi$ becomes zero outside a certain region of four-dimensional space-time, and we obtain:

$$\delta S = \int \left(\psi, \Gamma^k \frac{\partial \varphi}{\partial x_k} - \varphi \psi \right) d^4x - \int \left(\Gamma^k \frac{\partial \varphi}{\partial x_k} + \varphi \psi, \psi \right) d^4x = 0.$$

Since $\delta \psi$ is arbitrary, it follows that:

$$\left(\psi, \Gamma^k \frac{\partial \varphi}{\partial x_k} - \varphi \psi \right) = 0,$$

$$\left(\Gamma^k \frac{\partial \varphi}{\partial x_k} + \varphi \psi, \psi \right) = 0.$$

From these relations the differential equations (74,1) follow, provided the bilinear form (75,2) is non-degenerate⁺⁺ and if

$$\Gamma^k = -\Gamma^{k*} = A^k.$$

⁺ In other words, the expression $(\Gamma^k \varphi, \psi) = (\varphi, \Gamma^{k*} \psi)$ is valid regardless of what the σ -dimensional vectors φ and ψ are.

⁺⁺ The form (φ, ψ) is said to be non-degenerate, if for any vector $\varphi \neq 0$ one can choose ψ in such a way that $(\varphi, \psi) \neq 0$.

Thus, we see that only such differential equations (74,1) can be derived from a principle of least action, in which the coefficient-matrices A^k are anti-Hermitian with respect to the bilinear form (75,2):

$$A^{k*} = -A^k. \quad (75,3)$$

Consider now the conditions that ensure that the Lagrangian (75,1) is invariant. We note, first of all, that the expression

$$\Gamma^k \frac{\partial \psi}{\partial x_k} = A_k \frac{\partial \psi}{\partial x^k}$$

transforms in the same way as the function ψ . This is ensured by the relativistic invariance of (74,1). Hence, if the operators A^k are chosen as indicated in the preceding section, then in order that the Lagrangian be invariant it is sufficient that the bilinear form (75,2) be invariant.

Let us write down the form (75,2) in more detail:

$$(\varphi, \psi) = \sum_{r,s} B_{rs} \varphi_r \overline{\psi_s}.$$

The components $\overline{\psi_s}$ transform according to the representation conjugate to τ . Hence (see sec. 69) out of the products of the form $\varphi_r \overline{\psi_s}$ (r, s

(τ, τ being fixed) we can form an invariant expression only in the case when $\tau = \tau$. We stress that the functions φ and ψ should transform according to the same representation not only of the proper but also of the full Lorentz group.

In sec. 69 it was shown that the sum

$$\sum_{r,s} \varphi_r \overline{\psi_s}$$

is such an invariant expression. Hence the invariant bilinear form can be represented as follows:

$$(\varphi, \psi) = \sum_{r,s} \sum_{\tau} B_{rs} \varphi_r \overline{\psi_s} \quad (75,4)$$

where the coefficients B_{rs} are completely arbitrary.

The condition that the form (φ, ψ) be non-degenerate is that there should exist for each τ and s such s'_0 that

$$B_{s'_0 s} \neq 0. \quad (75,5)$$

In order that the action S should take on only real values, the form (φ, ψ) must be Hermitian, i.e. it must satisfy the condition $(\varphi, \psi) = (\psi, \varphi)$. Hence the coefficients B_{rs} are also Hermitian

$$B_{\mu\nu} = \overline{B_{\mu\nu}}. \quad (75,6)$$

In conclusion, let us consider the condition that the form (φ, ψ) be invariant with respect to inversion. Taking into account (74,17), we obtain:

$$\begin{aligned} (\varphi, \psi) &= \sum_{\mu} \sum_{\nu} B_{\mu\nu} \sum_{\alpha} (i\varphi)_{\alpha\mu} (\overline{i\psi})_{\alpha\nu} = \\ &= \sum_{\mu} \sum_{\nu} B_{\mu\nu} \sum_{\alpha} \varphi_{\alpha\mu} \overline{\psi_{\alpha\nu}}. \end{aligned}$$

Changing the summation indices, we find:

$$(\varphi, \psi) = \sum_{\mu} \sum_{\nu} B_{\nu\mu} \sum_{\alpha} \varphi_{\alpha\nu} \overline{\psi_{\alpha\mu}}.$$

Hence it follows that the invariance of the form (φ, ψ) with respect to inversion is equivalent to the requirement that

$$B_{\nu\mu} = B_{\mu\nu}. \quad (75,7)$$

Equation (75,4) defines a Hermitian form which is non-degenerate and invariant with respect to the full Lorentz group, provided that the coefficients $B_{\mu\nu}$ satisfy the conditions (75,5)(75,6) and (75,7) but are otherwise arbitrary.

76. Conservation Laws

The fact that all inertial systems are equivalent has as a consequence that in any relativistically invariant theory there are at least eleven conservation laws. Not all of these laws are independent of each other. Thus, for example, the constancy of p_x, p_y and M_z is a consequence of the constancy of p_z, M_x and M_y (p is the momentum of a closed system, and M its angular momentum).

Let us try to establish the form of these laws for a field ψ described by the equation (74,1) and the representation $\underline{T}$.

To do this, we evaluate how the field ψ transforms in passing from one inertial system to another.

Let us start with translations. If a system K' is obtained from K by means of a translation $\underline{a}$ ($\underline{a}$ is a four-dimensional vector) the coordinates of the same

Note that if $\tau = \tau_{PQ}$, then $\underline{\tau}$ is understood to be the representation τ_{QP} .

event in the two systems are related by the expression

$$x'_i = x_i - a_i. \quad (76,1)$$

Hence the wave function $\psi(x)$ is transformed in the system K' into the function $\psi'(x')$, given by

$$\psi'(x') = \psi(x) = \psi(x' + a). \quad (76,2)$$

The translation group is evidently a four-parameter group. One can naturally choose for its parameters the components of the vector $\underline{a}$ that characterizes the translation $\underline{t}$. Let us evaluate the infinitesimal operators $\underline{Y}_k$ ($k=0,1,2,3$) that correspond to these parameters. The relations (76,2) can be rewritten in the form

$$T(\underline{a})\psi(x) = \psi(x' + \underline{a}).$$

Differentiating this expression with respect to a_k and then putting $\underline{a}=0$, we obtain

$$Y_k \psi(x) = \frac{\partial \psi(x)}{\partial x_k}. \quad (76,3)$$

Let us now consider "rotations" in four-dimensional space-time, i.e. the transformation of the full Lorentz group. The relation between the coordinates x'_k and x_k of the same event in the reference systems K' and K are expressed by

$$x'_k = a_{km}(K'K) x_m. \quad (76,4)$$

Let us evaluate the function $\psi'(x')$ in the system K' , into which the wave function $\psi(x)$ given in K is transformed. To do this let us consider any point x_k^0 in the system K . The coordinates of this same point in the system K' are

$$x'_k = a_{km}(K'K) x_k^0. \quad (76,5)$$

We now introduce two auxiliary reference systems K_0 and K'_0 , obtained from the systems K and K' , by means of the translations

$$\xi_i = x_i - x_i^0, \quad \xi'_i = x'_i - x'_i{}^0. \quad (76,6)$$

The transformation formulae from K_0 to K'_0 have the following form:

$$\xi'_i = a_{ij}(K'_0K_0) \xi_j.$$

They are obtained by subtracting (76,5) from (76,4). Let us denote by $\varphi(\xi)$ and $\varphi'(\xi')$ the wave functions in the systems K_0 and K'_0 . According to (76,2) and (76,6) we can write:

$$\psi_i(0) = \psi_i(x_0), \quad \psi'_i(0) = \psi'_i(x_0). \quad (76,7)$$

By the definition (73,2) of the representation $\underline{T}$ the relation between the quantities $\psi_i(0)$ and $\psi'_i(0)$ has the following form:

$$\psi'_i(0) = T_{ij}(K) \psi_j(0) \quad (76,8)$$

Combining the equations (76,6) and (76,8) and omitting the index 0, we get:

$$\psi'_i(x') = T_{ij}(K) \psi_j(x). \quad (76,9)$$

We denote by h that element of the Lorentz group, which realizes the transition from the system K to the system K' . Since $hx = x'$, this last equation can be rewritten in the form

$$\psi'_i(x') = T_{ij}(h) \psi_j(h^{-1}x') \quad (76,10)$$

or

$$T_{ij}(h) \psi_j(x') = T_{ij}(h) \psi_j(h^{-1}x'). \quad (76,11)$$

We see that the representation $\underline{T}_1$ which transforms the field ψ is in general not equal to the representation $\underline{T}$. As a simple illustration of a similar situation we give the vector field $f(\underline{x}) = \underline{x}$, which remains invariant under rotations of the coordinate system about its origin while its value at any fixed point $\underline{x}$ transforms under the rotations about this point according to the vector representation $\underline{D}_1$ of the rotation group.

Let $\underline{M}$ be any operator of the translations group or the full Lorentz group, which transforms the function ψ into ψ' according to (76,2) or (76,10). We write down the wave equation (74,1) satisfied by the function ψ in the vector form:

$$\Delta^2 \frac{\partial \psi}{\partial x_2} + \psi = 0.$$

Since all inertial systems are equivalent, this same equation is satisfied by $\psi' = \underline{M}\psi$

$$\Delta^2 \frac{\partial \psi'}{\partial x_2} + \psi' = 0.$$

From these two equalities the following important relation follows:

$$\frac{\partial}{\partial x_2} (\Delta^2 \underline{M} \psi) = 0. \quad (76,12)$$

In fact since $\underline{A}^{k*} = -\underline{A}^k$, we have

$$\begin{aligned} \frac{\partial}{\partial x_2} (\Delta^2 \underline{M} \psi) &= \left(\Delta^2 \frac{\partial}{\partial x_2} \underline{M} \psi \right) + \left(\Delta^2 \underline{M} \psi \frac{\partial \psi}{\partial x_2} \right) = \\ &= \left(\Delta^2 \frac{\partial}{\partial x_2} \underline{M} \psi \right) - \left(\underline{M} \psi \Delta^2 \frac{\partial \psi}{\partial x_2} \right) = \\ &= -\underline{x} (\underline{M} \psi, \psi) + \underline{x} (\underline{M} \psi, \psi) = 0. \end{aligned}$$

The operator $\underline{M}$ varies over a continuous set of operators corresponding to all the possible translations and transformations of the full Lorentz group, and we obtain an infinite set of relations of the form (76,12). All of these, however, are not independent. In order to obtain a finite number of relations of the type (76,12), from which all the others follow, we shall pass from the operators $\underline{M}$ to infinitesimal operators.

If we differentiate (76,12) with respect to one of the parameters α_s , on which the operator depends, we have:

$$\frac{\partial}{\partial \alpha_s} (\Delta^2 \underline{I}_s \psi) = 0. \quad (76,13)$$

where $\underline{I}_s$ is the infinitesimal operator that corresponds to the parameter α_s . Let us add to (76,13) the relation

$$\frac{\partial}{\partial x_2} (\Delta^2 \psi) = 0. \quad (76,14)$$

which is obtained, if in (76,12) we put $\underline{M} = \underline{1}$ (the factor 1 is included in order that the expressions

$$(\Delta^2 \psi) \psi$$

be real).

If we deal with the Lorentz group, it can be easily shown that all the relations (76,12) follow from the relations (76,13) and (76,14). In the case of the full Lorentz group we should add to (76,13) and (76,14) one more relation:

$$\frac{\partial}{\partial x_2} (\Delta^2 \underline{I} \psi) = 0. \quad (76,15)$$

where $\underline{I}$ denotes the inversion operator.

The relations (76,13), (76,14) and (76,15) are called conservation laws. They can be given the form of a continuity equation

$$\frac{\partial \rho}{\partial x_2} + \text{div } \underline{j} = 0. \quad (76,16)$$

where

$$\rho = \frac{1}{2} (\Delta^2 \psi, \psi), \quad \underline{j}_k = (\Delta^2 \underline{I}_k \psi, \psi) \quad (k=1, 2, 3). \quad (76,17)$$

$\underline{I}$ is any infinitesimal operator, the unit operator or the inversion operator.

The relation (76,16) has the following integral form:

$$\frac{d}{dt} \int p dV = - \int f dV. \quad (76,18)$$

Thus, the quantities $(A^k \psi, \psi)$ ($k=1, 2, 3$) can be considered as the components of the current density related to the density $\frac{1}{c} (A^0 \psi, \psi)$.

Note also that if in (76,18) V and S go to infinity, we have

$$\int p dV = \text{const.} \quad (76,19)$$

Consider now in more detail the conservation laws connected with the translation group. According to (76,3) equation (76,13) can be written as

$$\frac{\partial}{\partial x_k} (A^k \frac{\partial \psi}{\partial x_l}, \psi) = 0.$$

Introducing the notation

$$T_{kl} = \text{Re} i (A^k \frac{\partial \psi}{\partial x_l}, \psi). \quad (76,20)$$

we have:

$$\frac{\partial T_{kl}}{\partial x_k} = 0. \quad (76,21)$$

The quantities T_{kl} ($k, l=0, 1, 2, 3$) form a tensor, called the energy-momentum density tensor. The four-dimensional divergence of this tensor equated to zero expresses the conservation laws for energy and momentum.

To see that the quantities T_{kl} form a tensor we perform the transformation from the reference system K to K' . Taking into account (74,15), we obtain:

$$\begin{aligned} T'_{kl} &= \text{Re} i (A^k \frac{\partial \psi'}{\partial x'_l}, \psi') = \text{Re} i (A^k T(g) \frac{\partial \psi}{\partial x_l}, T(g) \psi) \alpha_{r1}(g) = \\ &= \alpha_{r1}(g) \text{Re} i (T(g^{-1}) A^k T(g) \frac{\partial \psi}{\partial x_l}, \psi) = \\ &= \alpha_{r1}(g) \alpha_{r2}(g) \text{Re} i (A^k \frac{\partial \psi}{\partial x_l}, \psi) = \alpha_{r1}(g) \alpha_{r2}(g) T_{kl}, \end{aligned}$$

which proves the tensor nature of the quantities T_{kl} .

Consider now the conservation laws related to the proper Lorentz group. In order to obtain these laws in the most symmetric form it is convenient to change the choice of the parameters describing the elements of the group $\mathcal{L}_4$.

Write the general Lorentz transformation in the form

$$x'_i = x_i + \gamma a^k_{ik} x_k. \quad (76,22)$$

where γ_{ik} is the matrix (71,9) and the ϵ_{ks} are parameters describing the given transformation. These parameters are connected by the condition

$$\gamma a^k_{ik} x'_k = \gamma a^k_{ik} x_k.$$

If the ϵ_{ks} are small quantities, then apart from higher order infinitesimals, the last condition indicates that

$$\epsilon_{ks} = -\epsilon_{sk}.$$

Thus, the number of independent parameters ϵ_{ks} is six, as was to be expected. We can take as these parameters any independent set of six numbers ϵ_{ks} . We denote by I_{ks} the infinitesimal operators corresponding to these parameters. Evidently

$$I_{ks} = -I_{sk} \quad (k, s=0, 1, 2, 3).$$

According to sec. 44 the expression $T(g^{-1}) I_{ks} T(g)$ is a linear combination of the infinitesimal operators I_{kl} , where the coefficients of I_{kl} do not depend on the representation T . Evaluating these coefficients for the vector representation $T_{\frac{1}{2}, \frac{1}{2}}$ we obtain the relation

$$T(g^{-1}) I_{ks} T(g) = \alpha_{sk}(g) I_{kr} \alpha_{r1}(g) \quad (g \in \mathcal{L}_4). \quad (76,23)$$

which is valid for any representation T of the proper Lorentz group.

The operators I_{ks} are connected with the operators I_i, J_i ($i=1, 2, 3$) as follows:

$$\begin{aligned} I_{01} &= -I_1, \quad I_{02} = -I_2, \quad I_{03} = -I_3, \\ I_{12} &= -J_3, \quad I_{23} = -J_1, \quad I_{31} = -J_2. \end{aligned}$$

This can be easily seen by direct calculation.

Let us now see how the infinitesimal operator I_{ks} acts on the wave field $\psi(x)$. To do this, let us differentiate (76,11) with respect to the parameter ϵ_{ks} of the element h^{-1} and then put $h=e$. According to problem V, sec. 43, we obtain:

$$-I_{ks} \psi(x) = -I_{ks} \psi(x) + \frac{\partial \psi}{\partial x_i} \frac{\partial \gamma_{ik}}{\partial \epsilon_{ks}} \Big|_{h=e}, \quad (76,24)$$

where I'_{ks} and I_{ks} are the infinitesimal operators related to the representations T_1 and $T, \gamma=h^{-1}x$. Let us calculate

$$\frac{\partial \gamma_{ik}}{\partial \epsilon_{ks}} \Big|_{h=e}.$$

We start from the identity

$$y_1 = x_1 + i\alpha_{12}(\Lambda^{-1})x_2$$

Differentiating with respect to $\frac{\partial}{\partial x_2}(\Lambda^{-1})$ we obtain:

$$\frac{\partial y_1}{\partial x_2(\Lambda^{-1})} = i\alpha_{12} - i\alpha_{12}x_2$$

Substituting this expression into (76,24), we find:

$$L_{12}\psi(x) = L_{12}\psi(x) - (i\alpha_{12}x_2 - i\alpha_{12}x_2)\frac{\partial\psi}{\partial x_1} \quad (76,25)$$

We can now write down the six conservation laws, connected with the Lorentz group $\mathcal{L}_4$. According to (76,13) we have

$$\frac{\partial J_{\mu\nu}}{\partial x_j} = 0, \quad (76,26)$$

where

$$J_{\mu\nu} = i(\Lambda^j{}_{\mu}\psi, \psi) - i(\Lambda^j{}_{\nu}\psi, \psi) + i(i\alpha_{12}x_2 - i\alpha_{12}x_2)\left(\Lambda^j{}_{\mu}\frac{\partial\psi}{\partial x_1}, \psi\right). \quad (76,27)$$

The numbers J_{jks} form a tensor of rank three. This follows directly from (74,15) and (76,23). The tensor J_{jks} is called the angular-momentum density tensor. It is anti-symmetrical with respect to the second and third indices

$$J_{jks} = -J_{jsk} \quad (76,28)$$

The relation (76,26) expresses the law of conservation of angular momentum in a relativistically invariant form.

The angular momentum tensor consists of two terms

$$J_{\mu\nu} = S_{\mu\nu} + M_{\mu\nu} \quad (76,29)$$

The first term is called the spin angular momentum density tensor and the second, the orbital angular momentum density tensor.

The conservation law, corresponding to inversion, evidently is of the form

$$\frac{\partial}{\partial x_1}(\Lambda^1{}_{\mu}\psi(x_1, -x_2, -x_3, \psi(x_1, x_2, x_3)) = 0.$$

It is called the law of conservation of parity.

77. Spin

The concept of spin is closely connected with the non-relativistic concept of angular momentum. As is well known, the components J_x, J_y, J_z of the angular momentum density vector $\underline{J}$ are equal to the components J_{023}, J_{031} and J_{012} of the four-dimensional tensor of

angular momentum density.

Taking into account (76,27), we can write:

$$J_x = i(\Lambda^0{}_{12}\psi, \psi) + i\gamma\left(\Lambda^0{}_{12}\frac{\partial\psi}{\partial x_1}, \psi\right) - i\gamma\left(\Lambda^0{}_{12}\frac{\partial\psi}{\partial x_2}, \psi\right). \quad (77,1)$$

The corresponding expressions for J_y and J_z are obtained from (77,1) by cyclic permutation of x, y, z .

Hence the vector of angular momentum density can be written in the form

$$\underline{J} = \underline{S} + \underline{M}, \quad (77,2)$$

where

$$\underline{S} = (\Lambda^0{}_{12}\psi, \psi), \quad \underline{M} = (\Lambda^0{}_{12}[\gamma\psi], \psi) \quad (77,3)$$

and

$$e_k = \frac{1}{i}I_k \quad (k=1, 2, 3; \quad p=i\frac{\partial}{\partial t}). \quad (77,4)$$

The vector $\underline{S}$ is called the density of the spin angular momentum of the particle, the vector $\underline{M}$ is the density of orbital angular momentum, provided the function ψ is normalized by

$$\int (\Lambda^0{}_{12}\psi, \psi) dV = 1. \quad (77,5)$$

Note that if the condition (77,5) is satisfied at a certain initial instant of time it is satisfied for all times. This follows from (76,19).

In accordance with the non-relativistic approach of this section we shall consider the representation $\underline{T}$ as a representation of the rotation group R . If $g \in R$ is a rotation, then it follows from (74,15) that

$$T(g^{-1})\Lambda^0 T(g) = \Lambda^0 \quad (g \in R). \quad (77,6)$$

We denote by E_0 the set of all vectors of E , which are transformed by the operator Λ^0 into zero:

$$\Lambda^0\psi = 0 \quad (\psi \in E_0).$$

From (77,6) it follows that the subspace E_0 is invariant with respect to the rotation group. Let us denote by $\underline{T}_0$ the representation of the rotation group under which E_0 transforms. Singling out from $\underline{T}$ the representation $\underline{T}_0$, we obtain:

$$\underline{T} = \underline{T}_0 + \underline{T}_1.$$

The subspace that transforms according to the represen-

tation T_g is denoted by E_1 .

Let us write ψ in the form $\psi = \psi_0 + \psi_1$ ($\psi_0 \in E_0$, $\psi_1 \in E_1$).

We have:

$$S = (A^0 \psi, \psi) = (A^0 \psi_1, \psi) = -(\psi_1, A^0 \psi) = \\ = -(\psi_1, A^0 \psi_1) = (A^0 \psi_1, \psi_1).$$

(77,7)

Thus, in the expression for the density of spin angular momentum we have only the component ψ_1 of the wave function ψ . Hence, the operator $\underline{g}$ can be replaced by the operator $\underline{g}_1$, related to the representation T_g .

The operator $\underline{g}_1$ is called the spin operator, while the operator $[\underline{r}, \underline{p}]$ is called the orbital angular momentum operator.

If the representation T_g contains only one irreducible representation $\mathcal{U}_P$ of the rotation group once or several times, then the weight P of the representation $\mathcal{U}_P$ is called the spin of the particle in question. If the representation T_g contains irreducible representations of different weights $P_1, P_2, \dots, P_k$ then these weights can naturally be considered as the possible values of the spin of the particle.*

If the wave function ψ belongs to one of the irreducible subspaces $E_P \subset E_1$, then it is said that the spin of the particle in the state ψ is equal to P , where P is the weight of the representation $\mathcal{U}_P$, that transforms the subspace E_P .

Let the wave function ψ be a plane wave with the four-dimensional momentum p , i.e.

$$\psi(x) = e^{i p x} \quad (p x = p_0 x_0 - p_1 x_1 - p_2 x_2 - p_3 x_3). \quad (77,8)$$

We consider the plane waves

$$\psi(x) = e^{i p x} \quad (77,9)$$

in which $p_1 = p_2 = p_3 = 0$. Substituting this expression in the equation (74,1), we have:

$$(A^0 p_0 + \alpha) \psi_0 = 0. \quad (77,10)$$

Denoting by $E(p_0)$ the subspace consisting of all vectors ψ_0 which satisfy the equation (77,10), it can be easily seen that the subspace $E(p_0)$ is invariant with respect to all rotations. In fact, taking into account

* It should be noted, however, that up to now no particles with variable spin have been observed.

(77,6) we have:

$$(A^0 p_0 + \alpha) T(g) \psi_0 = T(g) (A^0 p_0 + \alpha) \psi_0 = 0 \quad (g \in R).$$

Let us decompose the subspace $E(p_0)$ into irreducible subspaces. We assume that ψ_0 belongs to one such subspace $E_{\mathcal{U}} (\mathcal{U} \in \mathcal{U}_P)$. Then the wave function (77,9) describes a state of the particle in which the value of the spin is equal to P , and the four-dimensional angular momentum is equal to $(p_0, 0, 0, 0)$. Since the subspace E_P has $(2P+1)$ dimensions, there exist consequently $(2P+1)$ linearly independent states with the given momentum and the given value P of the spin.

78. The Relativistically Invariant Operation of Time Inversion and the Pauli Theorem

As is well known, the equations of mechanics are invariant with respect to time inversion, i.e. to the substitution of $-t$ for t . The equations of electrodynamics

$$\text{rot } E = -\frac{1}{c} \frac{\partial H}{\partial t}, \quad \text{rot } H = \frac{1}{c} \frac{\partial E}{\partial t} + \frac{4\pi}{c} p \psi,$$

$$\text{div } E = 4\pi \rho, \quad \text{div } H = 0, \quad F = eE + \frac{e}{c} [vH]$$

are invariant with respect to the substitution

$$t \rightarrow -t, \quad E \rightarrow E, \quad H \rightarrow -H,$$

which also contains time inversion.

Let us find the general form of the relativistically invariant linear operation

$$t \rightarrow -t, \quad \psi \rightarrow M\psi. \quad (78,1)$$

which contains time inversion.

Assume that the wave function ψ in the reference system K becomes in the system K'

$$\psi' = T(a, v) \psi \quad (78,2)$$

(the vector a describes the rotation of the system K' with respect to K , v is the velocity of the origin of the coordinates K' with respect to the system K). On account of the relativistic invariant of the operation (78,1), the function $M\psi$ should, in the system K' become the function $M\psi'$. After time inversion the vector does not change, while the velocity v reverses its direction, hence

$$M\psi' = T(a, -v) M\psi.$$

Substituting here ψ' from (78,2), we obtain

$$MT(a, a) = T(a, -a)M. \quad (78,3)$$

Differentiating this equation with respect to the parameters of the proper Lorentz group, we find:

$$M_A = I_A M, \quad M_B = -I_B M \quad (A=1, 2, 3). \quad (78,4)$$

From the formulae, connecting the operators $\underline{A}_k, \underline{B}_k$ with the operators $\underline{I}_k, \underline{J}_k$ ($k=1, 2, 3$), it now follows that

$$\underline{A}_k M = -\frac{1}{2}(\underline{I}_k + \underline{J}_k) M = -\frac{1}{2} M (\underline{I}_k - \underline{J}_k) = M \underline{B}_k,$$

i.e. $\underline{A}_k M = M \underline{B}_k$. In the same way the relation $\underline{B}_k M = M \underline{A}_k$ may be obtained. Furthermore,

$$\left. \begin{aligned} \underline{A}_1 M &= (A_1 - A_2) M = M (B_1 - B_2) = M \underline{B}_3, \\ \underline{A}_2 M &= M \underline{B}_1, \quad \underline{A}_3 M = M \underline{B}_2, \quad \underline{B}_1 M = M \underline{A}_2, \quad \underline{B}_2 M = M \underline{A}_3, \end{aligned} \right\} \quad (78,5)$$

Let $\psi_{\tau p q}$ be one of the components of the wave function ψ corresponding to the basic vector $e_{\tau p q}$ of the representation $\underline{U} = \underline{U}_{PQ}$ of the Lorentz group:

$$\psi = \sum_{\tau p q} \psi_{\tau p q} e_{\tau p q}$$

According to (78,5) we have:

$$\left. \begin{aligned} \underline{A}_1 M e_{\tau p q} &= M \underline{B}_3 e_{\tau p q} = -i q M e_{\tau p q}, \\ \underline{B}_1 M e_{\tau p q} &= M \underline{A}_2 e_{\tau p q} = -i p M e_{\tau p q}, \end{aligned} \right\}$$

We see that the vector $M e_{\tau p q}$ transforms as $u_{\tau q p}$, where $u_{\tau q p}$ is the vector of the canonical basis in the representation $\underline{U} = \underline{U}_{QP}$

$$M e_{\tau p q} = \lambda_{pq} u_{\tau q p}. \quad (78,6)$$

To determine the coefficients λ_{pq} we operate on this expression with $\underline{A}_+$ and $\underline{B}_+$. Since $\underline{A}_+ M = M \underline{B}_+$ we have

$$M \underline{B}_+ e_{\tau p q} = \lambda_{pq} \underline{A}_+ u_{\tau q p}$$

or

$$e_{\tau+1, p+1}^q u_{\tau+1, q+1} = \lambda_{pq} e_{\tau+1, q+1}^q u_{\tau+1, p+1}$$

Hence it follows that $\lambda_{pq+1} = \lambda_{pq}$, i.e. the numbers λ_{pq} do not depend upon q . They also do not depend upon p as it can be easily seen by acting on both sides of (78,6) with the operator $\underline{B}_+$. Therefore $\lambda_{pq} = \lambda$. Hence it follows from (78,6) that the linear operator $\underline{M}$ that

satisfies the condition (78,4) has the form:

$$M e_{\tau p q} = \sum_{\tau' p' q'} M_{\tau' p' q'} e_{\tau' p' q'} \quad (78,7)$$

where the $M_{\tau' p' q'}$ are completely arbitrary coefficients. Operating with $\underline{M}$ on the function ψ , we get:

$$\begin{aligned} M \psi &= \sum_{\tau' p' q'} \psi_{\tau' p' q'} M_{\tau' p' q'} e_{\tau' p' q'}, \\ \text{i.e.} \quad (M \psi)_{\tau' p' q'} &= \sum_{\tau p q} M_{\tau' p' q'} \psi_{\tau p q} \end{aligned} \quad (78,8)$$

Let us see whether the coefficients $M_{\tau' p' q'}$ can be so chosen that the relativistically invariant equation (74,4) is invariant also with respect to time inversion. Substituting the function

$$\sum_{\tau p q} M_{\tau' p' q'} \psi_{\tau p q} \gamma(-x_0, x_1, x_2, x_3)$$

on the left hand side of (74,4) in place of the function $\psi_{\text{rel}}(x_0, x_1, x_2, x_3)$ and using the relations

$$\text{sign } Z_{\tau' p' q'} = -\text{sign } Z_{\tau p q}$$

and

$$P_{\tau' p' q'}^j = -P_{\tau p q}^j, \quad P_{\tau' p' q'}^j = P_{\tau p q}^j \quad (j=1, 2, 3).$$

we easily find that (74,4) is invariant with respect to time inversion provided that

$$M_{\tau' p' q'} = M_{\tau p q} \quad \text{sign } M_{\tau' p' q'} = -\text{sign } M_{\tau p q}. \quad (78,9)$$

for any pair of linked representations.

Since any closed chain of linked representations always consists of an even number of representations, the condition (78,9) is never contradictory. From now on we shall consider it to be satisfied. Besides, we assume that the coefficient M_{τ} are equal to

$$M_{\tau} = \pm 1.$$

Let us see how the charge density $\rho = e(\underline{A}^0 \psi, \psi)$ and the energy density of the particle

$$z = (A^0 \frac{\partial \psi}{\partial x_0}, \psi)$$

change under time inversion. According to (75,4), we have:

$$\begin{aligned} M \rho &= e(A^0 M \psi, M \psi) = e \sum_{\tau' p' q'} B_{\tau' p' q'} (A^0 M \psi)_{\tau' p' q'} (M \psi)_{\tau' p' q'} = \\ &= e \sum_{\tau' p' q'} B_{\tau' p' q'} M_{\tau' p' q'} A_{\tau' p' q'}^0 \psi_{\tau' p' q'} \psi_{\tau' p' q'} \end{aligned}$$

Substituting here τ for τ' and recalling that $B_{\tau} = B_{\tau'}$

and $A_{\tau' p' q'}^0 = -A_{\tau p q}^0$, we get:

$$M_p = -e \sum_{\nu} B_{\nu} M_{\nu} M_{\nu}^* A_{\nu}^0 \psi_{\nu} \psi_{\nu}^* = -e \sum_{\nu} B_{\nu} (A_{\nu}^0)_{\nu} \psi_{\nu} \psi_{\nu}^* M_{\nu} \quad (78,10)$$

Let us evaluate the product

$$M_{\nu} M_{\nu}^* = \text{sign } \nu M_{\nu} \text{sign } \nu M_{\nu}^*$$

Since in going from any representation τ to the representation τ linked with it, the quantity $\text{sign } \tau M_{\nu}$ changes its sign, the product $M_{\nu} M_{\nu}^*$ is evidently equal to

$$M_{\nu} M_{\nu}^* = (-1)^{\nu}$$

where ν is the number of steps needed in order to pass from the representation τ to the representation τ . At each step the index P changes by $1/2$. Hence, the number of necessary steps is even if $P-Q$ is an integer and odd if $P-Q$ is a half integer. In other words, ν is even if the spin of the particle is an integer, and ν is odd if the spin of the particle is not an integer. Introducing the constant product $M_{\nu} M_{\nu}^*$ under the summation sign in (78,10), we obtain:

$$M_p = (-1)^{p+1} p \quad (78,11)$$

In a completely similar way we obtain:

$$M_s = (-1)^s$$

From these expressions the Pauli theorem follows:

The energy density of a particle with half integer spin can have both positive and negative values.

The charge density of a particle with integer spin can have both positive and negative values.

79. The Dirac Equation

If we consider the relativistically invariant equations satisfied by the wave function that transforms according to the representation $T = \tau_{\frac{1}{2}} + \tau_{\frac{1}{2}}$, we obtain the Dirac equations for the electron.

First of all let us find the form of these equations. To do this let us use formula (74,9). In the case we are considering each of the irreducible representations contained in T appears in T only once. Hence the index s , labelling the subspaces that transform according to the same representation is not necessary here. Only one representation $\tau_{\frac{1}{2}}$ is linked with the representation $\tau_{\frac{1}{2}}$. With $\tau_{\frac{1}{2}}$, also, only one representa-

tion is linked. Hence, in our case the summation over P, Q and s drops out. It is convenient to simplify the indices that describe the wave function components. We put

$$\psi_{\frac{1}{2}, \frac{1}{2}, 0} = \psi_1, \psi_{\frac{1}{2}, -\frac{1}{2}, 0} = \psi_{-1}, \psi_{\frac{1}{2}, \frac{1}{2}, 1} = \psi_2, \psi_{\frac{1}{2}, -\frac{1}{2}, 1} = \psi_{-2}$$

Let us now put $P = \frac{1}{2}, Q = 0, P' = 0, Q' = \frac{1}{2}, P = \frac{1}{2}, Q = 0$

in (74,9). We so obtain the following equation:

$$\frac{1}{\sqrt{2}} Z_{\frac{1}{2}, \frac{1}{2}, 0} \left\{ \left(-i \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} \right) \psi_{-1} - \left(-i \frac{\partial}{\partial x_3} + i \frac{\partial}{\partial x_4} \right) \psi_1 \right\} + \kappa \psi_1 = 0$$

In a similar way we obtain the other three equations:

$$\frac{1}{\sqrt{2}} Z_{\frac{1}{2}, -\frac{1}{2}, 0} \left\{ \left(i \frac{\partial}{\partial x_1} + i \frac{\partial}{\partial x_2} \right) \psi_{-1} - \left(i \frac{\partial}{\partial x_3} + \frac{\partial}{\partial x_4} \right) \psi_1 \right\} + \kappa \psi_{-1} = 0$$

$$\frac{1}{\sqrt{2}} Z_{\frac{1}{2}, \frac{1}{2}, 1} \left\{ \left(-i \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} \right) \psi_{-2} - \left(i \frac{\partial}{\partial x_3} + i \frac{\partial}{\partial x_4} \right) \psi_2 \right\} + \kappa \psi_2 = 0$$

$$\frac{1}{\sqrt{2}} Z_{\frac{1}{2}, -\frac{1}{2}, 1} \left\{ \left(-i \frac{\partial}{\partial x_1} + i \frac{\partial}{\partial x_2} \right) \psi_{-2} - \left(i \frac{\partial}{\partial x_3} + \frac{\partial}{\partial x_4} \right) \psi_2 \right\} + \kappa \psi_{-2} = 0$$

The condition (74,20) for the invariance of the equations with respect to inversion gives:

$$Z_{\frac{1}{2}, \frac{1}{2}, 0} = -Z_{\frac{1}{2}, -\frac{1}{2}, 0}$$

Hence we put

$$Z_{\frac{1}{2}, \frac{1}{2}, 0} = -\sqrt{2}, Z_{\frac{1}{2}, -\frac{1}{2}, 0} = \sqrt{2}$$

and obtain the following system of equations:

$$\left. \begin{aligned} -i \frac{\partial \psi_{-1}}{\partial x_1} - i \frac{\partial \psi_{-1}}{\partial x_2} + i \frac{\partial \psi_1}{\partial x_3} + \frac{\partial \psi_1}{\partial x_4} + \kappa \psi_1 &= 0 \\ i \frac{\partial \psi_{-1}}{\partial x_1} - i \frac{\partial \psi_{-1}}{\partial x_2} - i \frac{\partial \psi_1}{\partial x_3} + i \frac{\partial \psi_1}{\partial x_4} + \kappa \psi_{-1} &= 0 \\ -i \frac{\partial \psi_2}{\partial x_1} + i \frac{\partial \psi_2}{\partial x_2} - i \frac{\partial \psi_{-2}}{\partial x_3} - \frac{\partial \psi_{-2}}{\partial x_4} + \kappa \psi_2 &= 0 \\ -i \frac{\partial \psi_2}{\partial x_1} + i \frac{\partial \psi_2}{\partial x_2} - i \frac{\partial \psi_{-2}}{\partial x_3} - i \frac{\partial \psi_{-2}}{\partial x_4} + \kappa \psi_{-2} &= 0 \end{aligned} \right\} \quad (79,1)$$

Thus the matrices of the operator A^j ($j=0,1,2,3$) have the following form:

$$A^0 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad A^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \quad (79,2)$$

$$A^2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad A^3 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

(the rows and columns correspond to the following order of the indices: $-1, 1, -1, 1$)

In order to obtain the Dirac equations in the ordinary form

$$\gamma_0 \frac{\partial \psi}{\partial x_0} + \gamma_1 \frac{\partial \psi}{\partial x_1} + \gamma_2 \frac{\partial \psi}{\partial x_2} + \gamma_3 \frac{\partial \psi}{\partial x_3} + i p \psi = 0,$$

where $\gamma_0, \gamma_1, \gamma_2, \gamma_3$ are the Dirac matrices:

$$\gamma_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \gamma_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix},$$

$$\gamma_2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}, \quad \gamma_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

it is sufficient to change from the components

$$\psi_1, \psi_2, \psi_3, \psi_4$$

to their linear combinations

$$\varphi_1 = \psi_1 + \psi_2, \quad \varphi_2 = -\psi_1 - \psi_2, \quad \varphi_3 = -\psi_3 + \psi_4, \quad \varphi_4 = \psi_3 - \psi_4. \quad (79,3)$$

Let us now set up the invariant bilinear form (75,2). According to (75,4) it has here the following form:

$$(\varphi, \psi) = B_{\frac{1}{2}, \frac{1}{2}} (\varphi_1 \bar{\psi}_1 + \varphi_2 \bar{\psi}_2) + B_{\frac{1}{2}, \frac{1}{2}} (\varphi_3 \bar{\psi}_3 + \varphi_4 \bar{\psi}_4).$$

The condition that (75,6) be Hermitian indicates that $B_{\frac{1}{2}, \frac{1}{2}} = B_{\frac{1}{2}, \frac{1}{2}}^*$. As regards the condition of invariance with respect to the inversion (75,7), it reduces to the equality $B_{\frac{1}{2}, \frac{1}{2}} = B_{\frac{1}{2}, \frac{1}{2}}^*$. Simple verification shows that the matrices $A_{\frac{1}{2}, \frac{1}{2}}$ are anti-Hermitian with respect to the invariant form, as it should be.

Therefore,

$$(\varphi, \psi) = \varphi_1 \bar{\psi}_1 + \varphi_2 \bar{\psi}_2 + \varphi_3 \bar{\psi}_3 + \varphi_4 \bar{\psi}_4. \quad (79,4)$$

Let us consider certain properties of the electron that follow from the equations (79,1) and the form of the invariant expression (79,4). The representations $\mathcal{E}_{\frac{1}{2}, 0}$ and $\mathcal{E}_{0, \frac{1}{2}}$ considered as the representations of the rotation group, are equal to the representation $\mathcal{E}_{\frac{1}{2}}$. This corresponds to the fact that the spin of an electron is equal to $1/2$. Furthermore, according to (76,14) the four-dimensional current vector of the electron is

$$j_\mu = i(A^\mu \psi, \psi).$$

or in component form:

$$\rho = \frac{1}{c} j_0 = \frac{1}{c} (|\varphi_1|^2 + |\varphi_2|^2 + |\varphi_3|^2 + |\varphi_4|^2).$$

$$j_1 = \varphi_1 \bar{\psi}_2 + \varphi_2 \bar{\psi}_1 - \varphi_3 \bar{\psi}_4 - \varphi_4 \bar{\psi}_3.$$

$$j_2 = -i(\varphi_1 \bar{\psi}_3 - \varphi_3 \bar{\psi}_1 + \varphi_2 \bar{\psi}_4 - \varphi_4 \bar{\psi}_2).$$

$$j_3 = |\varphi_1|^2 - |\varphi_2|^2 - |\varphi_3|^2 + |\varphi_4|^2.$$

From (76,20) we calculate the energy density

$$e = T_{00} = \text{Re} \left(A^0 \frac{\partial \psi}{\partial x_0}, \psi \right) =$$

$$= i m \left\{ \frac{\partial \varphi_1}{\partial x_0} \bar{\psi}_1 + \frac{\partial \varphi_2}{\partial x_0} \bar{\psi}_2 + \frac{\partial \varphi_3}{\partial x_0} \bar{\psi}_3 + \frac{\partial \varphi_4}{\partial x_0} \bar{\psi}_4 \right\}.$$

In particular, if the wave function corresponds to a stationary state, i.e. if it has the following form:

$$\psi = \psi^0(x_1, x_2, x_3) e^{i E x_0},$$

then it follows from the last expression that:

$$e = E \rho.$$

Thus, E plays the role of energy per unit density.

By means of the formulae (76,20) and (76,27) we can calculate the other components of the energy momentum tensor and also all the components of the angular momentum tensor. We shall not, however, stop to do this.

Let us set up the spin operator.

To do this it is first of all necessary to single out in the space E the subspace E_0 of those vectors which become zero under the action of the operator $\hat{A}^0$ (see sec. 77). However the operator $\hat{A}^0$ is not a singular operator (its determinant is equal to one) and there is no subspace E_0 . The components of the spin operator are, consequently,

$$-i I_1, -i I_2 \text{ and } -i I_3.$$

where I_1, I_2 and I_3 are the infinitesimal operators connected with the representation $T = \tau_{\frac{1}{2}, 0} + \tau_{0, \frac{1}{2}}$.

We write down explicitly the operator for the z -component of the spin:

$$s_z = -i I_3 = \begin{pmatrix} -\frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}$$

Hence it follows that in the states $\begin{pmatrix} \psi_{-1} \\ 0 \\ \psi_{-i} \\ 0 \end{pmatrix}$ the z-component of the spin is $-1/2$, while in the states $\begin{pmatrix} 0 \\ \psi_1 \\ 0 \\ \psi_i \end{pmatrix}$ it is $1/2$.

Let us now consider the solutions of the system of equations (79,1) that have the form of plane waves:

$$\psi = \psi_0 e^{i(p_0 x - p_1 y - p_2 z)}$$

Substituting this into (79,1), we get:

$$\psi_{-1} + (p_0 - p_1)\psi_{-i} + (p_1 - ip_2)\psi_i = 0,$$

$$\psi_1 + (p_1 + ip_2)\psi_{-i} + (p_0 + p_1)\psi_i = 0,$$

$$(p_0 + p_1)\psi_{-1} + (-p_1 + ip_2)\psi_i + \psi_{-i} = 0,$$

$$-(p_1 + ip_2)\psi_{-1} + (p_0 - p_1)\psi_i + \psi_i = 0$$

The determinant of this system must be equal to zero:

$$\begin{vmatrix} x & 0 & p_0 - p_1 & p_1 - ip_2 \\ 0 & x & p_1 + ip_2 & p_0 + p_1 \\ p_0 + p_1 & -p_1 + ip_2 & x & 0 \\ -p_1 - ip_2 & p_0 - p_1 & 0 & x \end{vmatrix} = 0.$$

Expanding the determinant, we get the equation:

$$-(p_0^2 - p_1^2 - p_2^2 - p_3^2 - x^2) = 0.$$

Thus, if the state of an electron is described by a plane wave, then the following simple relation exists between its energy p_0 and its momentum $p = ip_1 + j p_2 + k p_3$:

$$p_0 = \sqrt{p^2 + x^2}.$$

From this relation it follows that the quantity x is equal to the rest energy, i.e. to the energy of an electron of zero momentum.

Chapter XVII

Nuclear Reactions

80. The Scattering Matrix

In this section we consider certain general properties connected with the isotropy of space for nuclear reactions of the type $A + P = B + Q$. At the same time no assumptions are made about the nature of the nuclear forces.

Let us consider the collision of two particles A and P (by a particle we mean either an elementary particle or a nucleus). As a result of the collision one or more particles may be formed. We limit ourselves to the case when two particles B and Q are formed. Such a process is usually denoted as follows:

$$A + P = B + Q. \quad (80,1)$$

Of course, if only the species of particles B and Q, produced in the reaction, are indicated, this does not by any means describe the reaction exhaustively. To describe the reaction fully we must determine the dependence of the wave function Ψ' of the system B+Q upon the wave function Ψ of the system A+P.

Since the Schrödinger equation is linear, Ψ' depends linearly upon Ψ . If we, therefore, introduce the orthonormalized basis Ψ_j in the space of the Ψ -function and the basis Ψ'_j in the space of the function Ψ' , the coordinates C_j of the function $\Psi (\Psi = \sum C_j \Psi_j)$ will be expressed in terms of the coordinates C'_j of the function $\Psi' (\Psi' = \sum C'_j \Psi'_j)$ by means of a matrix S:

$$C'_j = \sum S_{j\ell} C_\ell. \quad (80,2)$$

The S matrix completely describes the reaction (80,1) and is called the scattering matrix.

The isotropy of space places certain fundamental limitations upon the elements of this matrix (*). Below

(*) In the same way as the symmetry of a crystal places certain limitations upon the tensors that describe its physical properties (see sec. 41).

2-spinor rotation, passive:

$$T_j^{(1/2)}(\theta) = 1 \cos \frac{\theta}{2} + i \sigma_j \sin \frac{\theta}{2}$$

$$T_1 = \begin{pmatrix} \cos \frac{\theta}{2} & i \sin \frac{\theta}{2} \\ i \sin \frac{\theta}{2} & \cos \frac{\theta}{2} \end{pmatrix}$$

$$T_2 = \begin{pmatrix} \cos \frac{\theta}{2} & \sin \frac{\theta}{2} \\ -\sin \frac{\theta}{2} & \cos \frac{\theta}{2} \end{pmatrix}$$

$$T_3 = \begin{pmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2} \end{pmatrix}$$

$$\vec{\Sigma} = \begin{pmatrix} \sigma_x & 0 \\ 0 & \sigma_x \end{pmatrix}$$

S(1) spinor transformations for rotations:

$$S(R_j, \theta) = e^{\frac{i}{2} \theta \Sigma_j} = \cos \frac{\theta}{2} + i \Sigma_j \sin \frac{\theta}{2} = \begin{pmatrix} T_j^{1/2}(\theta) & 0 \\ 0 & T_j^{1/2}(\theta) \end{pmatrix}$$

S(1) for boosts:

$$S(L_j, u) = e^{-\frac{u}{2} \alpha^j} = \cosh \frac{u}{2} - \alpha^j \sinh \frac{u}{2}$$

$$= \cosh \frac{u}{2} \begin{pmatrix} 1 & -\sigma_j \tanh \frac{u}{2} \\ -\sigma_j \tanh \frac{u}{2} & 1 \end{pmatrix}$$

Jacob Wick active helicity rotation:

$$S(R(\theta, \phi, -\phi)) = S(R_3, -\phi) S(R_2, \theta) S(R_3, \phi) =$$

$$\left[\begin{array}{cc|c} \cos \frac{\theta}{2} & -\sin \frac{\theta}{2} e^{-i\phi} & & \circ \\ \sin \frac{\theta}{2} e^{i\phi} & \cos \frac{\theta}{2} & & \circ \\ \hline & & & \text{same} \end{array} \right]$$

The helicity spinors are:

$$u_{\pm}(p) = \sqrt{\frac{E+m}{2m}} \begin{bmatrix} \eta_{\pm}(\theta, \phi) \\ \pm \frac{p}{E+m} \eta_{\pm}(\theta, \phi) \end{bmatrix} \quad \text{where}$$

$$\eta_{+}(\theta, \phi) = \begin{pmatrix} \cos \frac{\theta}{2} \\ \sin \frac{\theta}{2} e^{i\phi} \end{pmatrix}$$

$$\eta_{-}(\theta, \phi) = \begin{pmatrix} -\sin \frac{\theta}{2} e^{-i\phi} \\ \cos \frac{\theta}{2} \end{pmatrix}$$

Notation comparisons:

$$(\alpha \cdot b)_P = -(\alpha \cdot b)_{BP}$$

$$(\gamma \cdot p)_P = i(\gamma \cdot p)_{BP}$$

$$\gamma^k|_{BP} = i\gamma_k|_P$$

Dirac

Now consider this thing:

$$\sum_{rs} \sigma_{rs}^i = g^{\mu r} g_s^i - g^{jr} g_s^i = [S^i]_{rs}$$

But "S" is supposed to do this:

$$\psi_r'(x') = S_{rs} \psi_s(x)$$

$$[S]_{rs} = \left[e^{-\frac{i}{2} \omega_{\mu\nu} \sigma^{\mu\nu}} \right]_{rs} \quad \text{for spinors.}$$

Problem: confusion because for "general case", they use $[S]_{rs}$, but for photons, use $[S]^{\mu\nu}_{rs}$!

Let's clarify it:

$$\phi_r'(x') = S_{rs}(\epsilon) \phi_s(x) \quad \text{general case.}$$

Now I think in general, BD would say:

$$U(\epsilon) = e^{-\frac{i}{2} \epsilon_{\mu\nu} \tilde{M}^{\mu\nu}} \quad (11.72)$$

$$\begin{aligned} \text{So in general, } [S]_{rs} &= \left[e^{-\frac{i}{2} \epsilon_{\mu\nu} \tilde{M}^{\mu\nu}} \right]_{rs} \\ &= S_{rs} - \frac{i}{2} \epsilon_{\mu\nu} [\tilde{M}^{\mu\nu}]_{rs} \end{aligned}$$

In the Dirac case (2.14) we see that

$$[\tilde{M}^{\mu\nu}]_{rs} = \frac{1}{2} [\sigma^{\mu\nu}]_{rs} = \frac{i}{4} [\gamma^\mu, \gamma^\nu]_{rs}.$$

Notice that $(\gamma^\mu)_{rs}$ does make sense.

$$\text{In vector theory,? Well, } \sum_{rs} \epsilon^{\mu\nu} = -i [\tilde{M}^{\mu\nu}]_{rs}.$$

So vector theory has

$$\begin{aligned} [\tilde{M}^{\mu\nu}]_{rs} &= i \sum_{rs} \epsilon^{\mu\nu} = i [g^{\mu r} g_s^\nu - g^{\nu r} g_s^\mu] \\ &= i [S^{\mu\nu}]_{rs} \quad \text{as I would call it.} \end{aligned}$$

Keep unravelling this sad confusion.

$$[\tilde{M}^{\mu\nu}]_{rs} = i \sum_{\mu\nu}^{\mu\nu} = i [S^{\mu\nu}]^r_s$$

We would then say:

$$\begin{aligned} \phi'_r(x') &= S_{rs} \phi_s(x) = \left[e^{-\frac{i}{2} \epsilon_{\mu\nu} \tilde{M}^{\mu\nu}} \right]_{rs} \phi_s \\ &\approx \left(\delta_{rs} - \frac{i}{2} \epsilon_{\mu\nu} [\tilde{M}^{\mu\nu}]_{rs} \right) \phi_s \\ &\approx \left(\delta_{rs} + \frac{1}{2} \epsilon_{\mu\nu} (S^{\mu\nu})^r_s \right) \phi_s \end{aligned}$$

Question: are r and s Lorentz indices or not?

So now we have: $[\tilde{M}^{\mu\nu}(\text{vector})]_{rs} = [S^{\mu\nu}]^r_s$

Let's guess that ϕ_s refers to contravariant, so really:

$$\phi'^r(x') = \left(\delta^r_s - \frac{i}{2} \epsilon_{\mu\nu} [\tilde{M}^{\mu\nu}]^r_s \right) \phi^s.$$

Then: $[\tilde{M}^{\mu\nu}]^r_s = i (\sum^{\mu\nu})^r_s = i (S^{\mu\nu})^r_s.$

Then the BD's generators are, up to i , my $S^{\mu\nu}$. This leads to a contradiction?? This suggests that

$$[\tilde{M}^{\mu\nu}]^r_s = i [S^{\mu\nu}]^r_s \quad / \text{vector.}$$

whereas $[\tilde{M}^{\mu\nu}]^r_s = -g^{\mu r} g^{\nu s} [S^{\mu\nu}]^r_s$

This suggests that:

$$[\tilde{M}^{\mu\nu}]^r_s = -i g^{\mu r} g^{\nu s} [\tilde{M}^{\mu\nu}]^r_s \quad \checkmark$$

Therefore, $\tilde{M}^{\mu\nu}$ cannot satisfy the same algebra!

Weinberg gets best of both worlds!

Let's compare Weinberg's algebra with Sch algebra.

$$[\sigma_{12}, \sigma_{34}] = g_{32} \sigma_{14} + g_{24} \sigma_{31} - g_{31} \sigma_{24} - g_{41} \sigma_{32}$$

$$[M_{12}, M_{34}] = -g_{32} M_{14} + g_{24} M_{13} + g_{13} M_{24} - g_{14} M_{23}$$

To repeat:

$$[\sigma_{12}, \sigma_{34}] = g_{23} \sigma_{14} - g_{24} \sigma_{13} - g_{13} \sigma_{24} + g_{14} \sigma_{23}$$

$$[M_{12}, M_{34}] = -g_{23} M_{14} + g_{24} M_{13} + g_{13} M_{24} - g_{14} M_{23}$$

Therefore, these are very close in fact, let

$$\sigma_{12} = i M_{12} \quad ? \quad \text{This would cause}$$

$$\rightarrow [M_{12}, M_{34}] = - \{ g_{23} i M_{14} \text{ etc } \}.$$

They are not equivalent up to i ! They are different!
That sign does make a difference. Wait, how about:

$$\sigma_{12} = -M_{12}$$

Then: $[M_{12}, M_{34}] = g_{23} (-M_{14}) \checkmark$

OK, trivial. So:

$\sigma_{\mu\nu} = -M_{\mu\nu}$ $\uparrow$ Wein. $\uparrow$ Schueb.
--

But, contradiction because,

$$[\sigma^{\alpha\beta}]^{\gamma}_{\delta} = [S^{\alpha\beta}]^{\delta}_{\gamma}$$

Example: $[\sigma^{\mu\nu}]^{\alpha}_{\beta}$

$$[\sigma^{23}]^2_3 = -1$$

$$[M^{23}]^2_3 = +1$$

Claim. That Schwebers' generators do not solve his algebra.

Proof:

$$M^{10} M^{12} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$M^{12} M^{10} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$[M^{10}, M^{12}] = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = M^{20} \text{ by see.}$$

His algebra says, however, that:

$$[M^{10}, M^{12}] = g^{11} M^{02} = M^{20} \text{ and, it works here!}$$

I want a relation that has only one boost.

$$\begin{aligned} \text{boost, boost} &= \text{rot} \\ \text{boost, rot} &= \text{boost} \\ \text{rot, rot} &= \text{rot} \end{aligned}$$

{ All have boosts twice, so algebra works for either sign on your boosts!!

mainly duacia
4x4 stuff

12

SCHWEBER DAY

Dec. 19, 1973.

1. The passive physical three dimensional rotation matrices and corresponding generators. This is the $n=3$ representation of the FRG.
2. The $n=2$ representation: passive spinor rotation matrices. Here, the hermitian generators are $\frac{1}{2}$ the usual Pauli matrices.
3. The boosts of the Lorentz group are examined:
 - a) the "sense" is indicated clearly (sign of lorentz angle); passive like others
 - b) the three generator matrices are given
 - c) the actual lorentz transformation boost matrices are stated; all three.
 - d) the infinitesimal transformation boost is stated also.
4. The rotations of the Lorentz group are examined. The actual matrices and generators can be cribbed from the work done in section 1 above.
5. Two notations are given for the finite general lorentz transformation.
6. The algebras are stated for the $M^{\mu\nu}$. The objects N, M, J, K are all listed.
7. The spinor transformation matrices S are given for rotations and boosts. Great care was taken to get the signs right here,.
8. Next, the boosts were practiced. Starting with the 1000 spinor (spin in the plus z), I first boosted to get the spinor for an electron going in the x and then in the z direction. Results match familiar ones off B+D.
9. The relativity algebra is fully worked out. See page 5. γ_5 , γ_5 , γ_5 ...
10. Next, we figure out how to actively rotate a spinor which starts with z momentum and x_z (z) spin quantization. We rotate this spinor in two steps to get a pure helicity state spinor at angles THETA, PHI. We do not yet prerotate by $-\phi$ in the Jacob Wick sense. The full rotation matrix is stated on p 7.
11. For practive, the PhL pure helicity spinors are found. Later, on p 11, the Jacob Wick prerotation convention is added and new helicity spinors are found. Result is a subtle differential rephasing of the two spinors for the electron. No positron calculations are done here.
12. The helicity amplitudes for PI-N scattering are then related to the A and B amplitudes as in Truman Wick introduction. This was actually the motivation for this entire paper. Results are obtained in both phase conventions.
13. The Pauli and Schweber gamma matrices and dot products are related cleanly.
14. Finally, by taking $\phi = 0$ we obtain, up to an overall minus sign, the exact results stated in Trueman and Wick.
15. Halpern's and Schwebers generator notations are compared.

Consider these equations on page 21:

$$R^{(3)}(\epsilon) = 1 + A_3 \epsilon + O(\epsilon^2)$$

$$R^{(3)}(\theta) = \lim_{n \rightarrow \infty} \left(1 + \frac{\theta}{n} A_3\right)^n = e^{\theta A_3} = \begin{pmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{matrix} | \\ | \\ | \end{matrix}$$

$$A_3 = \frac{d}{d\theta} R^{(3)}(\theta) \Big|_{\theta=0} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{matrix} | \\ | \\ | \end{matrix}$$

$$A_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \begin{matrix} | \\ | \\ | \end{matrix}, \quad A_2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{matrix} | \\ | \\ | \end{matrix}$$

↑ rotates coords. i.e., takes you to a new frame K' with x' and y' rotated $+\theta$ relative to x and y .

$$[A_i, A_j] = -\epsilon_{ijk} A_k$$

Note signs: $e^{\theta A_2} = \begin{pmatrix} \cos \theta & 0 & -\sin \theta \\ 0 & 1 & 0 \\ \sin \theta & 0 & \cos \theta \end{pmatrix} \begin{matrix} | \\ | \\ | \end{matrix}$ ← a peculiarity of ordering.

In order to obtain the conventional notation, replace A_i with iA_i everywhere. Also note that these are the generators for rotations of spatial vectors (x, y, z) , not for rotation of angular momentum eigenstates (j, m, l) . Recall that, for example,

$$J_x = \begin{pmatrix} 0 & \frac{\hbar}{2} & 0 \\ \frac{\hbar}{2} & 0 & \frac{\hbar}{2} \\ 0 & \frac{\hbar}{2} & 0 \end{pmatrix} \quad J_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Next, Schwinger distinguishes between operator (rotation) and representation.

$$R(\vec{\epsilon}) = 1 + \vec{\epsilon} \cdot \vec{A} \rightarrow e^{\vec{\epsilon} \cdot \vec{A}}$$

$$T(\vec{\epsilon}) = I + \vec{\epsilon} \cdot \vec{M} \rightarrow e^{\vec{\epsilon} \cdot \vec{M}}$$

$$T^{-1} = e^{-\vec{\epsilon} \cdot \vec{M}} = T^\dagger = e^{\vec{\epsilon} \cdot \vec{M}^\dagger} \Rightarrow \vec{M}^\dagger = -\vec{M} \quad \text{"skew hermitian"}$$

Define some hermitean operators:

$$\vec{J} = -i \vec{M} \quad \text{so that} \quad T(\vec{\epsilon}) = e^{i \vec{\epsilon} \cdot \vec{J}} \quad [\vec{J}_i, \vec{J}_m] = +i \epsilon_{lmk} J_k$$

Example:

$$M_i^{(j=\frac{1}{2})} = \frac{1}{2} i \sigma_i \quad \text{or} \quad J_i^{(j=\frac{1}{2})} = \frac{1}{2} \sigma_i$$

$$\int_1^1 T_3^{(1/2)}(\theta) = T^{(1/2)}(0,0,\theta) = e^{\frac{i}{2} \theta \sigma_3} = \begin{pmatrix} e^{i\frac{\theta}{2}} & 0 \\ 0 & e^{-i\frac{\theta}{2}} \end{pmatrix} = C_{\frac{\theta}{2}} + i \sigma_3 S_{\frac{\theta}{2}}$$

$$\int_1^1 T_1^{(1/2)}(\theta) = \begin{pmatrix} \cos \frac{\theta}{2} & i \sin \frac{\theta}{2} \\ i \sin \frac{\theta}{2} & \cos \frac{\theta}{2} \end{pmatrix} = C_{\frac{\theta}{2}} + i \sigma_1 S_{\frac{\theta}{2}}$$

$$\int_1^1 T_2^{(1/2)}(\theta) = \begin{pmatrix} \cos \frac{\theta}{2} & \sin \frac{\theta}{2} \\ -\sin \frac{\theta}{2} & \cos \frac{\theta}{2} \end{pmatrix} = C_{\frac{\theta}{2}} + i \sigma_2 S_{\frac{\theta}{2}}$$

Now we move on to the Lorentz group:

$$g_{00} = 1 = -g_{11} = -g_{22} = -g_{33}$$

$$g_{\mu\nu} = g^{\mu\nu}$$

$$g_{\mu\nu} g^{\mu\sigma} = \delta_\nu^\sigma$$

The homogeneous (no translations) Lorentz group:

$$\Lambda(10, u) = \begin{pmatrix} \cosh u & -\sinh u & 0 & 0 \\ -\sinh u & \cosh u & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \leftarrow \text{agree with Williams p489.}$$

takes a vector x^μ to x'^μ in new frame K' which is "rotated" relative to K , or which is moving at $+\beta$ relative to K . Again, the coords are rotated.

where:

$$\beta = \tanh u$$

$$\gamma\beta = \sinh u$$

$$\cosh^2 u - \sinh^2 u = 1$$

$$1 - \beta^2 = \frac{1}{\gamma^2}$$

$$\gamma = \cosh u = \frac{1}{1 - \beta^2}$$

We boosted by $\beta = \tanh u$ along the x -axis. The generator was:

$$M^{10} = \frac{d}{du} \Lambda(10, u) \Big|_{u=0} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Similarly:

$$M^{20} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\Lambda(20, u) = \begin{pmatrix} \cosh u & 0 & -\sinh u & 0 \\ 0 & 1 & 0 & 0 \\ -\sinh u & 0 & \cosh u & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$M^{30} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

$$\Lambda(30, u) = \begin{pmatrix} \cosh u & 0 & 0 & -\sinh u \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sinh u & 0 & 0 & \cosh u \end{pmatrix}$$

By the way, these matrices are the Λ^μ_ν that do the actual rotations: also M^μ_ν . For example:

$$x'^\mu = \Lambda^\mu_\nu x^\nu$$

$$\Lambda^\mu_\nu(u) = e^{u M^{(20)\mu}_\nu} \approx 1^\mu_\nu + u M^{(20)\mu}_\nu$$

$$\approx \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} + u \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\left. \begin{matrix} \sinh u \approx u \\ \cosh u \approx 1 \end{matrix} \right\} \text{small } u.$$

The real rotations are easy:

$$\Lambda(R) = \begin{pmatrix} 1 & 0 \\ 0 & R \end{pmatrix} \quad \Lambda(12, \theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & c_\theta & s_\theta & 0 \\ 0 & -s_\theta & c_\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$M^{12} = \frac{d}{d\theta} \Lambda(12, \theta) \Big|_{\theta=0} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\Lambda^{(12)}_{\mu\nu}(\theta) = e^{\theta M^{12}_{\mu\nu}}$$

$$M^{23} = \begin{pmatrix} \times & 0 \\ & 1 \\ & 0 & -1 & 0 \end{pmatrix} \quad M^{31} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

Remember that $\Lambda(3, \theta)$ has that sign change!

Now, the general notation:

$$\Lambda(\omega) = e^{\frac{1}{2} \omega_{\mu\nu} M^{\mu\nu}} \quad \Lambda(\mu\nu; u) = e^{u M^{\mu\nu}}$$

for obvious reason.

Now the algebras and operators: Note: see page 14 for more discussion on this ↓

$$[M_{\mu\nu}, M_{\rho\sigma}] = g_{\mu\sigma} M_{\nu\rho} + g_{\nu\rho} M_{\mu\sigma} - g_{\mu\rho} M_{\nu\sigma} - g_{\nu\sigma} M_{\mu\rho}$$

$$\vec{M} \leftarrow \text{rotation} \equiv (M_{32}, M_{13}, M_{21})$$

← boost

$$\vec{N} \equiv (M_{01}, M_{02}, M_{03})$$

$$R_i = e^{\theta M_i} \\ B_i = e^{u N_i}$$

$$[M_i, M_j] = \epsilon_{ijk} M_k$$

$$[N_i, N_j] = -\epsilon_{ijk} M_k$$

$$[M_i, N_j] = \epsilon_{ijk} N_k$$

$$\vec{M}^2 - \vec{N}^2 = \frac{1}{2} M_{\mu\nu} M^{\mu\nu}$$

$$-\vec{M} \cdot \vec{N} = \frac{1}{4} \epsilon^{\mu\nu\rho\sigma} M_{\mu\nu} M_{\rho\sigma} \quad \left. \begin{array}{l} \\ \end{array} \right\} \text{Casimirs}$$

$$(\epsilon^{0123} = +1) \quad \leftarrow \text{all upper}$$

$$\vec{N}^\dagger = \vec{N} \\ \vec{M}^\dagger = -\vec{M}$$

The objects $\vec{J}$ and $\vec{K}$ which we are more used to:

$$\vec{J} = \frac{1}{2} i (\vec{M} + i \vec{N})$$

$$\vec{J}^\dagger = \vec{J}$$

$$[J_k, J_\ell] = i \epsilon_{klm} J_m$$

$$\vec{K} = \frac{1}{2} i (\vec{M} - i \vec{N})$$

$$\vec{K}^\dagger = \vec{K}$$

$$[K_\ell, K_m] = i \epsilon_{lmn} K_n$$

$$[J_k, K_m] = 0$$

What comes is a good discussion of the irreducible representations of the restricted Lorentz group. Most reps are non-unitary. Good reps too, see p 43.

$$D^{(jj)}$$

$$D^{(00)} = \text{scalar}$$

$$D^{(1/2, 1/2)} = \text{spinor}$$

$$D^{(1, 1)} = \text{4-vector etc.}$$

Example problem: The connection to 4-spinors.

$$\psi'(x') = S(\Lambda) \psi(x) = S(\Lambda) \psi(\Lambda^{-1}x') \quad \partial_\mu = \Lambda^\nu_\mu \partial'_\nu$$

$$(-i \not{\partial} + m) \psi(x) = 0$$

$$S(\Lambda)^{-1} \gamma^\lambda S(\Lambda) = \Lambda^\lambda_\mu \gamma^\mu \quad (\text{same as BD 2.12})$$

The actual matrices are:

$$S(R_1) = e^{\theta T(R_1)} = e^{+\theta \cdot \frac{1}{2} \Sigma_1} = \cos \frac{\theta}{2} + i \Sigma_1 \sin \frac{\theta}{2}$$

not same as page 1.

$$\Sigma_1 = \begin{pmatrix} \sigma_1 & \\ & \sigma_1 \end{pmatrix}$$

$$S(R_1) = \begin{bmatrix} \cos \frac{\theta}{2} & i \sin \frac{\theta}{2} & 0 & 0 \\ i \sin \frac{\theta}{2} & \cos \frac{\theta}{2} & 0 & 0 \\ 0 & 0 & \cos \frac{\theta}{2} & i \sin \frac{\theta}{2} \\ 0 & 0 & i \sin \frac{\theta}{2} & \cos \frac{\theta}{2} \end{bmatrix}$$

← same matrix twice.

Recall that,

$$T_1^{(\frac{1}{2})}(\theta) = \begin{pmatrix} \cos \frac{\theta}{2} & i \sin \frac{\theta}{2} \\ i \sin \frac{\theta}{2} & \cos \frac{\theta}{2} \end{pmatrix}$$

Thus, all these rotation results can be written as:

$$S(R_i, \theta) = \begin{pmatrix} T_i^{(\frac{1}{2})}(\theta) & 0 \\ 0 & T_i^{(\frac{1}{2})}(\theta) \end{pmatrix} \quad \leftarrow \text{where the } T_i \text{ are given on page 1.}$$

For boosts,

$$S(L_i, u) = e^{-\frac{u}{2} \alpha_i^i} = \cosh \frac{u}{2} - \alpha_i^i \sinh \frac{u}{2}$$

note: $\alpha^i / \cosh = \gamma^i / \beta$, see BD p 23.

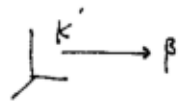
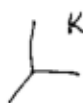
$$\alpha_i^i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix} = \text{Pauli matrix}$$

$$= \begin{pmatrix} \cosh \frac{u}{2} & -\sigma_i \sinh \frac{u}{2} \\ -\sigma_i \sinh \frac{u}{2} & \cosh \frac{u}{2} \end{pmatrix} = \cosh \frac{u}{2} \begin{pmatrix} 1 & -\sigma_i \tanh \frac{u}{2} \\ -\sigma_i \tanh \frac{u}{2} & 1 \end{pmatrix}$$

Now let's try these objects. Start with a spinor representing a part. at rest with spin "up", i.e. in the +z direction.

$$u(\vec{p}=0, \hat{s}=\hat{z}) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Now, go to a frame moving at β in the +z direction:



This interp is correct.

In this new frame, the electron has $\vec{p}' = -\beta \hat{z}$. What is the new u ?

$$u' = S(L_3, u) u$$

$$= \begin{bmatrix} \cosh \frac{u}{2} & 0 & -\sinh \frac{u}{2} & 0 \\ 0 & \cosh \frac{u}{2} & 0 & +\sinh \frac{u}{2} \\ -\sinh \frac{u}{2} & 0 & \cosh \frac{u}{2} & 0 \\ 0 & +\sinh \frac{u}{2} & 0 & \cosh \frac{u}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \cosh \frac{u}{2} \\ 0 \\ -\sinh \frac{u}{2} \\ 0 \end{bmatrix} = \cosh \frac{u}{2} \begin{bmatrix} 1 \\ 0 \\ -\tanh \frac{u}{2} \\ 0 \end{bmatrix}$$

Now some algebra

$$\gamma^2 \beta^2 = \gamma^2 - 1$$

$$E = \gamma m$$

$$p = \beta E = \gamma \beta m$$

$$E = m \cosh u$$

$$p = m \sinh u$$

$$\tanh u = \beta$$

$$\cosh u = \gamma$$

$$\sinh u = \gamma \beta$$

$$\sinh \frac{u}{2} = \sqrt{\frac{\gamma-1}{2}} = \sqrt{\frac{E-m}{2m}}$$

$$\cosh \frac{u}{2} = \sqrt{\frac{\gamma+1}{2}} = \sqrt{\frac{E+m}{2m}}$$

$$\tanh \frac{u}{2} = \frac{\gamma-1}{\gamma\beta} = \frac{\gamma\beta}{\gamma+1}$$

$$= \frac{E-m}{p} = \frac{p}{E+m}$$

Thus, the spinor representing an electron going in the $-\hat{z}$ direction with momentum $p > 0$ is:

$$u' = \sqrt{\frac{E+m}{2m}} \begin{bmatrix} 1 \\ 0 \\ -p/(E+m) \\ 0 \end{bmatrix}$$



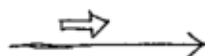
$$u'(-p_z) = \sqrt{\frac{E+m}{2m}} \begin{bmatrix} 1 \\ 0 \\ -p_z/(E+m) \\ 0 \end{bmatrix}$$

note: here $p = \beta E$.

~~if $p > 0$, $p_z > 0$ then $p = p_z$ and $p_z = p$. However, because the spin is negative, $p = -p_z$.~~

$$u(p_z) = \sqrt{\frac{E+m}{2m}} \begin{bmatrix} 1 \\ 0 \\ p_z/(E+m) \\ 0 \end{bmatrix}$$

Question: what about spin? Only has meaning in rest frame. If you boost to rest frame and spin there is in direction you came from, then + helicity!



$\rightarrow \hat{z}$

means, that as $|p| \rightarrow 0$, $\hat{s} \rightarrow \hat{p}$.

New example: start with same u , but boost by $-u$ in x to get $u(p_x)$:

$\rightarrow u > 0, -u < 0, p_x > 0.$
 $u(p_x) = S(L_1, -u) u(0)$

$$= \cosh \frac{u}{2} \left(\begin{array}{cc|cc} 1 & 0 & 0 & +\tanh \frac{u}{2} \\ 0 & 1 & +\tanh \frac{u}{2} & 0 \\ \hline 0 & +\tanh \frac{u}{2} & 1 & 0 \\ +\tanh \frac{u}{2} & 0 & 0 & 1 \end{array} \right) \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \cosh \frac{u}{2} \begin{pmatrix} 1 \\ 0 \\ 0 \\ +\tanh \frac{u}{2} \end{pmatrix} = \sqrt{\frac{E+m}{2m}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ +\frac{p_x}{E+m} \end{pmatrix}$$

Repeat last case:

$u(p_z) = S(L_3, -u) u(0)$

$$= \cosh \frac{u}{2} \left(\begin{array}{cc|cc} 1 & 0 & +\tanh \frac{u}{2} & 0 \\ 0 & 1 & 0 & -\tanh \frac{u}{2} \\ \hline +\tanh \frac{u}{2} & 0 & 1 & 0 \\ 0 & -\tanh \frac{u}{2} & 0 & 1 \end{array} \right) \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \cosh \frac{u}{2} \begin{pmatrix} 1 \\ 0 \\ +\tanh \frac{u}{2} \\ 0 \end{pmatrix} = \sqrt{\frac{E+m}{2m}} \begin{pmatrix} 1 \\ 0 \\ +\frac{p_z}{E+m} \\ 0 \end{pmatrix}$$

Both results agree with $B \neq 0$.

$u(p_x) = \begin{matrix} \uparrow \hat{x} \\ \Rightarrow \hat{z} \end{matrix} \quad \text{If } p_x \rightarrow 0, \text{ then } \hat{S} \rightarrow \hat{z}.$

Now, the next problem is this: want to rotate a spinor going in $\hat{z}$ to θ, ϕ !

$$u(\theta, \phi \pm) = R_z(-\phi) R_y(-\theta) u(p_z, \pm s) \\ = S(L_3, -\phi) S(L_2, -\theta) \begin{pmatrix} u_{\pm} \\ \frac{\sigma_z p_z}{E+m} u_{\pm} \end{pmatrix}$$

$\uparrow$ ie, rotate $\vec{p}$, not axes.

This should generate electron states with $\vec{p}$ along θ, ϕ , and of definite helicity:

$$S(L_3, -\phi) S(L_2, -\theta) = \left[\cosh \frac{\phi}{2} + \alpha^3 \sinh \frac{\phi}{2} \right] \left[\cosh \frac{\theta}{2} + \alpha^2 \sinh \frac{\theta}{2} \right]$$

$= \cosh \frac{\phi}{2}$

Errr. We want rotations, not boosts:

$$S(R_3, -\phi) S(R_2, -\theta) = \begin{pmatrix} T_3^{1/2}(-\phi) & 0 \\ 0 & T_3^{1/2}(-\phi) \end{pmatrix} \begin{pmatrix} T_2^{1/2}(-\theta) & 0 \\ 0 & T_2^{1/2}(-\theta) \end{pmatrix} \\ = \begin{pmatrix} T_3^{1/2}(-\phi) T_2^{1/2}(-\theta) & 0 \\ 0 & T_3^{1/2}(-\phi) T_2^{1/2}(-\theta) \end{pmatrix}$$

Write it out in detail:

$$T_3^{1/2}(-\phi) T_2^{1/2}(-\theta) = \begin{pmatrix} e^{-i\phi/2} & 0 \\ 0 & e^{+i\phi/2} \end{pmatrix} \begin{pmatrix} \cos \theta/2 & -\sin \theta/2 \\ \sin \theta/2 & \cos \theta/2 \end{pmatrix} = \begin{bmatrix} \cos \theta/2 e^{-i\phi/2} & -\sin \theta/2 e^{-i\phi/2} \\ \sin \theta/2 e^{i\phi/2} & \cos \theta/2 e^{i\phi/2} \end{bmatrix}$$

Thus, in full detail:

$$S(R_3, -\phi) S(R_2, -\theta) = \begin{bmatrix} \cos \theta/2 e^{-i\phi/2} & -\sin \theta/2 e^{-i\phi/2} & 0 & 0 \\ \sin \theta/2 e^{i\phi/2} & \cos \theta/2 e^{i\phi/2} & 0 & 0 \\ 0 & 0 & \cos \theta/2 e^{-i\phi/2} & -\sin \theta/2 e^{-i\phi/2} \\ 0 & 0 & \sin \theta/2 e^{i\phi/2} & \cos \theta/2 e^{i\phi/2} \end{bmatrix} = \tilde{S}(\theta, \phi)$$

Now apply to some known object:

$$\tilde{S}(\theta, \phi) \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos \theta/2 e^{-i\phi/2} \\ \sin \theta/2 e^{i\phi/2} \\ 0 \\ 0 \end{bmatrix} \leftarrow \text{just like rotating a spinor. I have done the "active" transform. Axes stayed fixed.}$$

Now for the payoff:

$$\tilde{S}(\theta, \phi) \sqrt{\frac{E+m}{2m}} \begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ 0 \end{pmatrix} = \sqrt{\frac{E+m}{2m}} \begin{bmatrix} \cos \theta/2 e^{-i\phi/2} \\ \sin \theta/2 e^{i\phi/2} \\ \frac{p_z}{E+m} \cos \theta/2 e^{-i\phi/2} \\ \frac{p_z}{E+m} \sin \theta/2 e^{i\phi/2} \end{bmatrix} = u_+(\theta, \phi)_R$$

Shorthand notation:

$$\chi_+ = \psi = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \psi(\theta, \phi) = \begin{pmatrix} \cos \theta/2 e^{-i\phi/2} \\ \sin \theta/2 e^{i\phi/2} \end{pmatrix} \equiv \chi_+(\theta, \phi)$$

$$\chi_- = \chi = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \chi(\theta, \phi) = \begin{pmatrix} -\sin \theta/2 e^{-i\phi/2} \\ \cos \theta/2 e^{i\phi/2} \end{pmatrix} \equiv \chi_-(\theta, \phi)$$

Then here is what we have found:

$$\tilde{S}(\theta, \phi) \sqrt{\frac{E+m}{2m}} \begin{pmatrix} \psi \\ \frac{p_z}{E+m} \psi \end{pmatrix} = \sqrt{\frac{E+m}{2m}} \begin{bmatrix} \psi(\theta, \phi) \\ \frac{p_z}{E+m} \psi(\theta, \phi) \end{bmatrix} \leftarrow \text{rotation only affects the spinors.}$$

We could see this from the start:

$$\begin{pmatrix} T_3(-\phi) T_2(-\theta) & 0 \\ 0 & T_3(-\phi) T_2(-\theta) \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{bmatrix} T_3(-\phi) T_2(-\theta) A \\ T_3(-\phi) T_2(-\theta) B \end{bmatrix}$$

Now, back to Trueman + Wick:

$$\bar{u}(p_2) \left[-A + \frac{1}{2\mu} \overset{(-1)}{\cancel{\gamma}} (\not{q}_1 + \not{q}_2) B \right] u(p_1)$$



To get helicity amplitudes, use:

$$u_{\pm}(p_1) = \sqrt{\frac{E_1+m_1}{2m_1}} \begin{bmatrix} \psi_{\pm} \\ \pm \frac{p_1}{E_1+m_1} \psi_{\pm} \end{bmatrix} \quad \propto \quad u_{\pm}(p_1) = \sqrt{\frac{E_1+m_1}{2m_1}} \begin{bmatrix} \chi_{\pm} \\ \pm \frac{p_1}{E_1+m_1} \chi_{\pm} \end{bmatrix}$$

$$u_{\pm}(p_2) = \sqrt{\frac{E_2+m_2}{2m_2}} \begin{bmatrix} \chi_{\pm}(\theta, \phi) \\ \pm \frac{p_2}{E_2+m_2} \chi_{\pm}(\theta, \phi) \end{bmatrix}$$

↑ helicity eigenspinors

$$u = \begin{pmatrix} a \\ b \end{pmatrix} \quad u^{\dagger} = (a^{\dagger} \ b^{\dagger}) \quad \bar{u} = (a^{\dagger} \ -b^{\dagger})$$

$$\bar{u}_{\pm}(p_2) = \sqrt{\frac{E_2+m_2}{2m_2}} \left[\chi_{\pm}^{\dagger}(\theta, \phi) \mp \frac{p_2}{E_2+m_2} \chi_{\pm}^{\dagger}(\theta, \phi) \right]$$

Thus:

$$G_{\pm\pm} = \sqrt{\frac{E_1+m_1}{2m_1}} \sqrt{\frac{E_2+m_2}{2m_2}} \left(\chi_{\pm}^{\dagger}(\theta, \phi) \mp \frac{p_2}{E_2+m_2} \chi_{\pm}^{\dagger}(\theta, \phi) \right) \left(-A I + \frac{1}{2\mu} \overset{(-1)}{\cancel{\gamma}} (\not{q}_1 + \not{q}_2) \gamma^{\mu} \right) \begin{pmatrix} \chi_{\pm} \\ \pm \frac{p_1}{E_1+m_1} \chi_{\pm} \end{pmatrix}$$

First, do one A term only:

$$G_{\pm\pm}(A) = \sqrt{\frac{E_1+m_1}{2m_1}} \sqrt{\frac{E_2+m_2}{2m_2}} (-A) \left(\chi_{\pm}^{\dagger}(\theta, \phi) \mp \frac{p_2}{E_2+m_2} \chi_{\pm}^{\dagger}(\theta, \phi) \right) \begin{pmatrix} \chi_{\pm} \\ \pm \frac{p_1}{E_1+m_1} \chi_{\pm} \end{pmatrix}$$

Need this fact:

$$\chi_{\pm}^{\dagger}(\theta, \phi) \chi_{\pm} = ?$$

$$\chi_{+}^{\dagger}(\theta, \phi) \chi_{+} = \begin{pmatrix} \cos \frac{\theta}{2} e^{i\phi/2} & \sin \frac{\theta}{2} e^{-i\phi/2} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \cos \frac{\theta}{2} e^{i\phi/2}$$

$$\chi_{+}^{\dagger}(\theta, \phi) \chi_{-} = \sin \frac{\theta}{2} e^{-i\phi/2}$$

$$\chi_{-}^{\dagger}(\theta, \phi) \chi_{+} = \begin{pmatrix} -\sin \frac{\theta}{2} e^{i\phi/2} & \cos \frac{\theta}{2} e^{-i\phi/2} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = -\sin \frac{\theta}{2} e^{i\phi/2}$$

$$\chi_{-}^{\dagger}(\theta, \phi) \chi_{-} = \cos \frac{\theta}{2} e^{-i\phi/2}$$

Then, holding this for a while,

$$G(A) = \sqrt{\frac{E_1+m_1}{2m_1}} \sqrt{\frac{E_2+m_2}{2m_2}} \chi^{\dagger}(\theta, \phi) \chi \left[1 + \frac{p_2 p_1}{(E_2+m_2)(E_1+m_1)} \right] (-A)$$

minus sign for ++, --
plus sign for +-, -+

In \$\pi\$-N scatter elastic case in one cms \$p_1 = p_2 \quad E_1 = E_2 \quad m_1 = m_2 = m\$.

Thus:

$$G(A) = \left(\frac{E_1+m}{2m} \right) \chi^{\dagger}(\theta, \phi) \chi \left[1 + \frac{p^2}{(E_1+m)^2} \right] (-A) = \frac{2E_1}{E_1+m} \left. \begin{matrix} \text{for } ++, -- \\ \oplus \\ \text{for } +-, -+ \\ \ominus \end{matrix} \right\}$$

Thus, the A term is very simple:

$$G_{\pm,\pm}(A) = \chi_{\pm}^{\dagger}(\theta, \phi) \chi_{\pm}(-A) \times \begin{bmatrix} E_1/m \sin & +, -, -+ \\ 1 & \sin & ++, -- \end{bmatrix}$$

$$G_{++}(A) = -A \cos \frac{\theta}{2} e^{i\phi/2}$$

$$G_{+-}(A) = -A \sin \frac{\theta}{2} e^{-i\phi/2} \cdot \frac{E_1}{m}$$

$$G_{-+}(A) = +A \sin \frac{\theta}{2} e^{i\phi/2} \cdot \frac{E_1}{m}$$

Back track a bit:

$$\tilde{S}(\theta, \phi) \sqrt{\frac{E+m}{2m}} \begin{pmatrix} 0 \\ 1 \\ 0 \\ -\frac{p_z}{E+m} \end{pmatrix} = \sqrt{\frac{E+m}{2m}} \begin{bmatrix} -s_{\frac{\theta}{2}} e^{-i\phi/2} \\ c_{\frac{\theta}{2}} e^{i\phi/2} \\ (-\frac{p_z}{E+m}) (-s_{\frac{\theta}{2}} e^{-i\phi/2}) \\ (-\frac{p_z}{E+m}) (c_{\frac{\theta}{2}} e^{i\phi/2}) \end{bmatrix} \equiv U_{-}(\theta, \phi) u_{-}$$

note minus sign

In the shorthand:

$$\tilde{S}(\theta, \phi) \sqrt{\frac{E+m}{2m}} \begin{pmatrix} \chi_{-} \\ -\frac{p_z}{E+m} \chi_{-} \end{pmatrix} = \sqrt{\frac{E+m}{2m}} \begin{bmatrix} \chi_{-}(\theta, \phi) \\ -\frac{p_z}{E+m} \chi_{-}(\theta, \phi) \end{bmatrix}$$

The general result is:

$$\tilde{S}(\theta, \phi) \sqrt{\frac{E+m}{2m}} \begin{pmatrix} \chi_{\pm} \\ \frac{p_z \sigma_z}{E+m} \chi_{\pm} \end{pmatrix} = \sqrt{\frac{E+m}{2m}} \begin{pmatrix} \chi_{\pm}(\theta, \phi) \\ \frac{p_z}{E+m} \sigma_z \chi_{\pm}(\theta, \phi) \end{pmatrix}$$

wrong. Use $\pm$.

Now try the B terms:

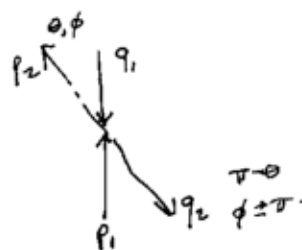
$$G_{\pm,\pm}(B) = \left(\frac{E+m}{2m}\right) \left[\frac{1}{2\mu} (q_1 + q_2)_{\mu} \right] \left(\chi_{\pm}^{\dagger}(\theta, \phi) \mp \frac{p_z}{E+m} \chi_{\pm}^{\dagger}(\theta, \phi) \right) \gamma^{\mu} \begin{pmatrix} \chi_{\pm} \\ \pm \frac{p_z}{E+m} \chi_{\pm} \end{pmatrix}$$

$$= \left(\frac{E+m}{2m}\right) \left[\frac{1}{2\mu} (q_1 + q_2)_{\mu} \right] \dots$$

First, the time piece. Since $|\vec{q}_1| = |\vec{q}_2|$, $E_1 = E_2 = \omega$. Also, $|\vec{q}_1| = p$ also!
The space piece is messy:

$$\vec{q}_1 = -p \hat{z}$$

$$\vec{q}_2 = R_{\text{active}}(\pi - \theta, \phi \pm \pi) (p \hat{z})$$



Go back to the time piece:

$$G_{\pm\pm}(B, \text{time}) = \frac{E+m}{2m} \cdot \frac{\cancel{\phi B}}{2\mu} \cdot 2\omega \cdot \left(\chi_{\pm}^{\dagger}(\theta, \phi) \mp \frac{p}{E+m} \chi_{\pm}^{\dagger}(\theta, \phi) \right) \begin{pmatrix} \chi_{\pm} \\ \mp \frac{p}{E+m} \chi_{\pm} \end{pmatrix}$$

$$= \frac{E+m}{2m} \cdot \frac{\cancel{\phi B}}{2\mu} \cdot 2\omega \cdot \chi_{\pm}^{\dagger}(\theta, \phi) \chi_{\pm} \left[1 \mp \frac{p^2}{(E+m)^2} \right]$$

note action of δ_0 here
now, minus for $+-, -+$.

$$G_{++}(B, \text{time}) = \frac{E+m}{2m} \cdot \frac{\cancel{\phi B}}{2\mu} \cdot 2\omega \cdot \left[\cos \frac{\theta}{2} e^{i\phi/2} \right] \frac{2E_1}{E_1+m}$$

$$= \frac{E}{m} \cdot \frac{\omega}{\mu} \cdot \cancel{\phi B} \cdot \left[\cos \frac{\theta}{2} e^{i\phi/2} \right]$$

$$G_{+-}(B, \text{time}) = \frac{E+m}{2m} \cdot \frac{\cancel{\phi B}}{2\mu} \cdot 2\omega \cdot \left[\sin \frac{\theta}{2} e^{-i\phi/2} \right] \frac{2m}{E+m}$$

$$= \frac{\omega}{\mu} \cdot \cancel{\phi B} \cdot \left[\sin \frac{\theta}{2} e^{-i\phi/2} \right]$$

This looks good so far except for (1) the i which maybe belongs with the δ , and also those damn phases ϕ . Maybe that's why Jacob & Wick use $(\phi, \theta, -\phi)$.

Go back to Jacob/Wick.

$$|p, \theta, \phi; \lambda\rangle = R_{\phi, \theta, -\phi} \psi_{p\lambda}$$

notice the pre-rotation.
active & presume.

Try this:

$$S(R_3, -\phi) S(R_2, -\theta) S(R_3, +\phi)$$

$$= \begin{pmatrix} T_3^{1/2}(-\phi) T_2^{1/2}(-\theta) & 0 \\ 0 & " \end{pmatrix} \begin{pmatrix} T_3(+\phi) & 0 \\ 0 & T_3(+\phi) \end{pmatrix}$$

$$= \begin{pmatrix} T_3(-\phi) T_2(-\theta) T_3(\phi) & 0 \\ 0 & T_3(-\phi) T_2(-\theta) T_3(\phi) \end{pmatrix}$$

where,

$$T_3(-\phi) T_2(-\theta) T_3(\phi) = \begin{bmatrix} c_{\theta/2} e^{-i\phi/2} & -s_{\theta/2} e^{-i\phi/2} \\ s_{\theta/2} e^{i\phi/2} & c_{\theta/2} e^{i\phi/2} \end{bmatrix} \begin{bmatrix} e^{i\phi/2} & 0 \\ 0 & e^{-i\phi/2} \end{bmatrix}$$

$$= \begin{bmatrix} c_{\theta/2} & -s_{\theta/2} e^{-i\phi} \\ s_{\theta/2} e^{i\phi} & c_{\theta/2} \end{bmatrix}$$

In this convention then we have:

$$S(R_3, -\phi) S(R_2, -\theta) S(R_3, +\phi) =$$

$$\begin{bmatrix} \cos\frac{\phi}{2} & -\sin\frac{\phi}{2} e^{-i\phi} & & \\ \sin\frac{\phi}{2} e^{i\phi} & \cos\frac{\phi}{2} & & \\ & & \cos\frac{\phi}{2} & -\sin\frac{\phi}{2} e^{-i\phi} \\ & & \sin\frac{\phi}{2} e^{i\phi} & \cos\frac{\phi}{2} \end{bmatrix}$$

Now apply this to known objects

$$\tilde{S}(\phi, -\theta, +\phi) \begin{bmatrix} 1 \\ 0 \\ p/E+m \\ 0 \end{bmatrix} = \begin{bmatrix} \cos\frac{\phi}{2} \\ \sin\frac{\phi}{2} e^{i\phi} \\ \frac{p}{E+m} \begin{pmatrix} \cos\frac{\phi}{2} \\ \sin\frac{\phi}{2} e^{i\phi} \end{pmatrix} \end{bmatrix} = U_+(\theta, \phi)_{SW}$$

The whole effect is, therefore, to change $\chi_{\pm}(\theta, \phi)$ to $\tilde{\chi}_{\pm}(\theta, \phi)$:

$$\tilde{\chi}_{\pm}(\theta, \phi) = \chi_{\pm}(\theta, \phi) = \begin{cases} \oplus \begin{pmatrix} \cos\frac{\phi}{2} \\ \sin\frac{\phi}{2} e^{i\phi} \end{pmatrix} \\ \ominus \begin{pmatrix} -\sin\frac{\phi}{2} e^{-i\phi} \\ \cos\frac{\phi}{2} \end{pmatrix} \end{cases} \quad \leftarrow \chi_{\pm}, \text{ Jacob Wick ans.}$$

In fact, the entire spinor is rephased by $e^{i\phi/2}$. That's all.

$$\tilde{S}(-\phi, -\theta, \phi) \begin{bmatrix} 0 \\ 1 \\ 0 \\ -\frac{p}{E+m} \end{bmatrix} = \begin{bmatrix} -\sin\frac{\phi}{2} e^{-i\phi} \\ \cos\frac{\phi}{2} \\ \frac{-p}{E+m} \begin{pmatrix} -\sin\frac{\phi}{2} e^{-i\phi} \\ \cos\frac{\phi}{2} \end{pmatrix} \end{bmatrix} = U_-(\theta, \phi)_{SW}$$

$$U_{\pm}(\phi) = \sqrt{\frac{E+m}{2m}} \begin{bmatrix} \chi_{\pm}(\theta, \phi) \\ (\pm \frac{p}{E+m}) \chi_{\pm}(\theta, \phi) \end{bmatrix}$$

But look:

$$\begin{aligned} e^{+i\phi/2} \times U_-(\theta, \phi)_{SW} &= U_-(\theta, \phi)_{RL} \\ e^{-i\phi/2} \times U_+(\theta, \phi)_{SW} &= U_+(\theta, \phi)_{RL} \end{aligned}$$

not the same sign.

The up and down spinors are rephased by different amounts! Thus:

$$\chi_+^\dagger(\theta, \phi) \chi_+ = (\cos\frac{\phi}{2} \quad \sin\frac{\phi}{2} e^{-i\phi}) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \cos\frac{\phi}{2}$$

$$\chi_+^\dagger(\theta, \phi) \chi_- = \sin\frac{\phi}{2} e^{-i\phi}$$

$$\chi_-^\dagger(\theta, \phi) \chi_+ = (-\sin\frac{\phi}{2} e^{+i\phi} \quad \cos\frac{\phi}{2}) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = -\sin\frac{\phi}{2} e^{i\phi}$$

$$\chi_-^\dagger(\theta, \phi) \chi_- = \cos\frac{\phi}{2}$$

Compare to page 8. These are not these $\times$ a constant phase! These are different.

So, using Jacob-Wick states and assuming that they use Pauli ϵ , we have found:

$$G_{++}(A) = -A \cos \frac{\theta}{2} \quad G_{+-}(A) = -A \sin \frac{\theta}{2} \cdot e^{-i\phi} \cdot \frac{\epsilon}{m}$$

$$G_{++}(B, \text{time}) = -B \cos \frac{\theta}{2} \cdot \frac{\epsilon}{m} \cdot \frac{\omega}{\mu} \quad G_{+-}(B, \text{time}) = -B \sin \frac{\theta}{2} \cdot e^{-i\phi} \cdot \frac{\omega}{\mu}$$

So, neglecting the space contribution to the B terms, we have:

$$\begin{aligned} G_{++} &\approx -\cos \frac{\theta}{2} \left[A + \frac{\epsilon}{m} \cdot \frac{\omega}{\mu} B \right] \\ G_{+-} &\approx -\sin \frac{\theta}{2} \left[A \frac{\epsilon}{m} + \frac{\omega}{\mu} B \right] e^{-i\phi} \end{aligned} \quad \left. \begin{array}{l} \text{Temp.} \\ \text{result.} \end{array} \right\}$$

Using Luchl $\chi_{\pm}$ we instead get

$$G_{++}(\chi) \approx -\cos \frac{\theta}{2} \left[A e^{i\phi/2} + \frac{\epsilon}{m} \cdot \frac{\omega}{\mu} B e^{i\phi/2} \right] = -\cos \frac{\theta}{2} \left[A + \frac{\epsilon}{m} \cdot \frac{\omega}{\mu} B \right] e^{i\phi/2}$$

$$G_{+-}(\chi) \approx -\sin \frac{\theta}{2} \left[A e^{-i\phi/2} \frac{\epsilon}{m} + \frac{\omega}{\mu} B e^{-i\phi/2} \right] = -\sin \frac{\theta}{2} \left[A \frac{\epsilon}{m} + \frac{\omega}{\mu} B \right] e^{-i\phi/2}$$

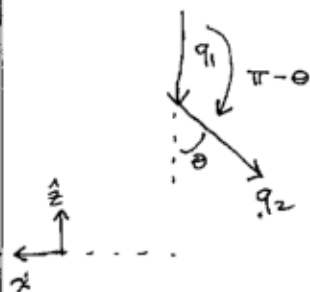
Same answers, just shifted phases. If $\phi=0$, no difference. Relative phase is $e^{-i\phi}$ in either case, so you can add amplitudes without worry.

Finally, let's pick up that space piece: raising k on the q 's.

$$\begin{aligned} G_{\pm\pm}(B, \text{space}) &= \left(\frac{E+m}{2m} \right) \left(-\frac{B}{2\mu} \right) (q_1+q_2)^k (-1) \left(\eta_{\pm}^+(\theta, \phi) \mp \frac{p}{E+m} \eta_{\pm}^+(\theta, \phi) \right) \gamma^k \left(\begin{array}{c} \eta_{\pm} \\ \pm \frac{p}{E+m} \eta_{\pm} \end{array} \right) \\ &= \frac{E+m}{2m} \cdot \frac{B}{2\mu} \cdot (q_1+q_2)^k \left(\eta_{\pm}^+(\theta, \phi) \mp \frac{p}{E+m} \eta_{\pm}^+(\theta, \phi) \right) \left(\begin{array}{c} \pm \frac{p}{E+m} \sigma^k \eta_{\pm} \\ -\sigma^k \eta_{\pm} \end{array} \right) \\ &= \frac{E+m}{2m} \cdot \frac{B}{2\mu} \cdot (q_1+q_2)^k \cdot \left(\pm \frac{2p}{E+m} \right) \cdot \eta_{\pm}^+(\theta, \phi) \sigma^k \eta_{\pm} \\ &= \pm \frac{pB}{2m\mu} (q_1+q_2)^k \cdot \eta_{\pm}^+(\theta, \phi) \sigma_k \eta_{\pm}, \quad \text{but } = 0 \text{ for } +-, -+. \end{aligned}$$

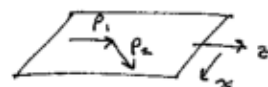
This is a lot of work. Let's choose $\phi=0$ so reaction occurs in the xz plane. Then:

$\hookrightarrow$ i.e., let $\vec{p}_2$ come out over the x axis.



$$\vec{q}_1 = -p \hat{z}$$

$$\vec{q}_2 = -p \cos \theta \hat{z} - p \sin \theta \hat{x}$$



$$(q_1+q_2)^1 = -p \sin \theta = -2p \sin \frac{\theta}{2} \cos \frac{\theta}{2}$$

$$(q_1+q_2)^3 = -p(1+\sin \theta) = -2p \cos^2 \frac{\theta}{2}$$

Next, we need:

$$\begin{aligned}
 \eta_+^\dagger(\theta\phi) \sigma_1 \eta_+ &= \eta_+^\dagger(\theta,\phi) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \eta_+^\dagger(\theta\phi) \eta_- = S\theta/2 \\
 \eta_+^\dagger(\theta\phi) \sigma_1 \eta_- &= \eta_+^\dagger(\theta\phi) \eta_+ = C\theta/2 \\
 \eta_+^\dagger(\theta,\phi) \sigma_3 \eta_+ &= \eta_+^\dagger(\theta\phi) \eta_+ = C\theta/2 \\
 \eta_+^\dagger(\theta\phi) \sigma_3 \eta_- &= -\eta_+^\dagger(\theta\phi) \eta_- = -S\theta/2
 \end{aligned}$$

Thus, put some pieces together:

$$G_{++}(B, \text{space}) = \frac{pB}{2m\mu} \left[(-2pS\frac{\theta}{2}C\frac{\theta}{2})S\theta/2 + (-2pC\frac{\theta}{2})C\theta/2 \right]$$

$$= -\frac{p^2 B}{m\mu} C\theta/2$$

$$G_{+-}(B, \text{space}) = 0 \quad \text{see p 13.}$$

Thus, the full G_{++} becomes:

$$G_{++} = -\cos\frac{\theta}{2} \left[A + \left(\frac{\epsilon\omega + p^2}{m\mu} \right) B \right] \quad \sim$$

At last, complete agreement except for overall sign. Had to assume $\phi=0$.

Compare Haldane and Schwinger notations:

$$\text{Haldane:} \quad U(a) = e^{i\omega^{\mu\nu} M_{\mu\nu}}$$

$$\text{Schwinger:} \quad U(\lambda) = e^{\frac{i}{2}\omega^{\mu\nu} M_{\mu\nu}}$$

Differences are:

$$1) \quad \frac{1}{2}\omega^{\mu\nu}|_S = \omega^{\mu\nu}|_H$$

$$2) \quad iM_{\mu\nu}|_H = M_{\mu\nu}|_S \Rightarrow M_{\mu\nu}|_H \text{ is hermitian}$$

Thus

$$iM_{i0}|_H = iK_i|_H = M_{i0}|_S = -M_{0i}|_S = -N_i|_S$$

$$\Rightarrow K_i|_H = iN_i|_S$$

$$K_H = iN_S$$

$K|_S$ means nothing to me

Similarly:

$$\left. \begin{aligned} i M_{12}|_H &= M_{12}|_S = -M_{21}|_S = -M_{31}|_S \\ &= i J_3|_H \end{aligned} \right\} \Rightarrow i J_H = -M_S$$

$$\boxed{J_H = i M_S}$$

Thus we might write:

$$U = e^{\frac{i}{2} \omega_{\mu\nu} M^{\mu\nu}} = e^{\frac{i}{2} (\vec{\theta} \cdot \vec{M} + \vec{u} \cdot \vec{N})} = e^{-i \vec{\theta} \cdot \vec{J} - i \vec{u} \cdot \vec{K}}$$

Specifically,

$$\Lambda_{\nu}^{\mu}(10, u) = e^{+u M_{S\nu}^{(20)\mu}} = e^{+iu K_H^2}.$$

or

$$\Lambda_{\nu}^{\mu}(12, \theta) = e^{+\theta M_{S\nu}^{(12)\mu}} = e^{+i\theta J_H^3}.$$

In either notation, the coordinate (passive) transformation for the positive sense angle has a plus exponential. The matrices are the same no matter how you annotate your generators.

Hunt for Sign Error

First, we have to determine whether or not there really is a sign error. This can be said for sure:

If

- 1) my boost operators mean what I think they mean
- 2) if Bjorken's results mean what I think they mean

Then

- 1) there is a sign error because the third component sign does not match.

Ideas?

- (1) Maybe B+D have some kind of different conventions on spinors?? No, they certainly agree, see page 84 of Schweber.
- (2) maybe boosts sense is not what I think. Yes, this might be so. Look at Bjorken page 29. There, they give the z-boost matrix and there are these minus signs, but the angle w is also negative, so all matrix entries are positive. This is in contradiction to my matrix signs. Go back to page 38 of Schweber where the x axis transformation matrix is given. Both sinh's appear with minus signs which is the same as in BD p 22. B and D confirm my interpretation that the primed frame moves in positive x direction so that w is positive. This idea is also confirmed by Williams in his appendix p 489.
- (3) The error could be in the step going from the lorentz transformation to the operator S . Remember that these two guys are not the same. Yes, here it is. Schweber and BD disagree on the sign of S for boosts.

$$\text{BD page 23: } S_L = e^{\frac{i}{2}(\omega/2)\alpha_1}$$

$$\text{Sch page 78: } S(L_1) = e^{\frac{\omega}{2}\alpha_1}$$

$$\text{Sch: } \alpha^k = \begin{pmatrix} 0 & \sigma^k \\ \sigma^k & 0 \end{pmatrix} \quad j^k = c \psi^\dagger \alpha^k \psi$$

Possibly the answer is hidden in this question: does the sign of ALPHA change as the index goes up and down? In Schweber, all the indices on the alphas are up until we come to the equation that is the point of confusion on page 78. There, for the first time in the entire book, alpha is written with a lower subscript. What a booboo. In Bjorken's book the alpha's are introduced always with lower subscripts. In fact, we have:

$$\left. \begin{array}{l} \text{B+D p. 17: } \gamma^i = \beta \alpha_i \\ \text{Sch p 69: } \gamma^i = \beta \alpha^i \end{array} \right\} \Rightarrow \alpha_i|_{\text{BD}} = \alpha^i|_{\text{Schw.}} \quad \text{both have same } \gamma_i, \gamma_i.$$

OK, problem is solved. My error was to misinterpret Schweber's alpha. The correct result is this:

$$S(L_1, \omega) = e^{-\frac{\omega}{2}\alpha^1} \quad \text{where } \alpha^1 = \begin{pmatrix} 0 & \sigma^1 \\ \sigma^1 & 0 \end{pmatrix} \quad \text{where } \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Exercise: try to show that. The generators $M^{\mu\nu}$ for the reducible, BD convention, Dirac spinor representation are:

$$M^{\mu\nu} = -\frac{i}{2} \sigma^{\mu\nu}$$

$\uparrow$ $\equiv$ $M^{\mu\nu}$'s. $\nwarrow$ their σ 's.
 not BD's.

Result: success except for some signs.

To get real S , need real h , need real (ϕ, ψ) . When is (ϕ, ψ) real?

$$(\phi, \psi) = (\psi, \phi) \Rightarrow \text{real if}$$

$$(\phi, \psi) = (\psi, \phi)$$

But this is true only if $B_{ss'} = B_{ss'}$ ~~is real~~
But $B_{ss'}$ are different B 's

In general, (ϕ, ψ) real $\Rightarrow$

$$(\phi, \psi) = \sum_s \sum_{s'} B_{ss'} \sum_l \phi_{3sl} \bar{\psi}_{3s'l}$$

$$(\psi, \phi) = \overline{(\phi, \psi)} = \sum_s \sum_{s'} \overline{B_{ss'}} \sum_l \bar{\phi}_{3sl} \psi_{3s'l}$$

But remember that $\phi_{3sl} = \bar{\phi}_{3s'l}$, so $\bar{\phi}_{3sl} = \phi_{3s'l}$
Now rename the sum indices:

$$(\psi, \phi) = \sum_s \sum_{s'} \overline{B_{ss'}} \sum_l \phi_{3sl} \bar{\psi}_{3s'l}$$

Well, try another approach:

$$(\phi, \psi) = \sum_{s, s'} B_{ss'} \sum_l \phi_{3sl} \bar{\psi}_{3s'l}$$

$$(\psi, \phi) = \sum_{s, s'} B_{ss'} \sum_l \psi_{3sl} \bar{\phi}_{3s'l}$$

$$= \bar{\psi}_{3s'l}$$

$$= \phi_{3sl}$$

$$= \sum_{s, s'} B_{ss'} \sum_l \phi_{3sl} \bar{\psi}_{3s'l}$$

Forget it! Enough on $B_{ss'}$. Need more precision in this notation; later.

26. Conservation Laws. Pause for questions. We have established that:

$$a) \quad \psi(x') = T \psi(x)$$

$$/BD's \quad S(a) = T$$

$$b) \quad \psi^\dagger(x') = T_I \psi^\dagger(x)$$

where $T \neq T_I$. T is the real $\mathcal{O}_1$ representation we have been using all along. For Dirac standard spinors

$T_{30} = \left[\begin{array}{c|c} \mathbb{C}_{20} & 0 \\ \hline 0 & \mathbb{I}_{20} \end{array} \right]$. This is for the canonical spinors.

$T = \mathbb{C}_{20} \oplus \mathbb{I}_{20}$. Now in any T_{30} , you can construct matrices for all the generators.

(Eg, for $\mathbb{C}_{20}$, the matrix $1 \times 1 = \text{"I"}$ represents all six generators. In $\mathbb{C}_{20}$, all 6 gens = 0, $e^{\partial \cdot \vec{C}} = e^0 = 1$ "trivial rep" ← error.)

Eg, for $\mathbb{C}_{\frac{1}{2}\pm}$, there are now 6 different 4×4 matrices; eg,

$$J_1 = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This is the ~~vector~~ vector rep that we discovered J_1 .

Now, what about $\mathbb{I}_{\frac{1}{2}\pm}$? What are the generators, $\vec{I}, \vec{J}$ for this rep? Well, for $\vec{A}$ this is $D^{(1)}$, so

$$A_k = \sigma_k$$

$$B_k = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

check the operation of $A+B$ to see that this is right! (It is).

$$A_k + B_k = -I_k \quad \checkmark$$

$$A_k - B_k = -iJ_k$$

So, $I_k = -\sigma_k$
 $iJ_k = -\sigma_k$

$I_k = -\sigma_k$
 $J_k = -\sigma_k$
 $J_k = -\sigma_k + i\sigma_k = i(\sigma_k)$
 $J_k = i\sigma_k$

Explicitly: $-I_k = 1 + \sigma_k$
 $iJ_k = 1 - \sigma_k$

$1 + \sigma_k = \text{why bother}$

$$\begin{matrix} I_k = -(\sigma_k + 1) \\ J_k = i(\sigma_k - 1) \end{matrix}$$

$$\begin{matrix} I_k = -\sigma_k \\ J_k = i\sigma_k \end{matrix}$$

These are the reps of the 6 gens for $\mathbb{I}_{\frac{1}{2}\pm}$.

For $\mathbb{I}_{\frac{1}{2}\pm}$, we get

$$A_k = 0$$

$$B_k = \sigma_k$$

$$-I_k = \sigma_k$$

 $-iJ_k = \sigma_k$

$$\begin{matrix} I_k = -(\sigma_k) \\ J_k = -(\sigma_k) \end{matrix}$$

for $\mathbb{I}_{\frac{1}{2}\pm}$

$$\begin{matrix} I_k = -\sigma_k \\ J_k = -i\sigma_k \end{matrix}$$

So, now we can explicitly construct the generators

$$T_{30} = \begin{bmatrix} \mathbb{I}_{\frac{1}{2}\pm} & \\ & \mathbb{I}_{\frac{1}{2}\pm} \end{bmatrix} \quad \text{Eg, } (I_k)_{30} = \begin{bmatrix} -(\sigma_k) & 0 \\ 0 & -(\sigma_k) \end{bmatrix} = -\Sigma^k$$

So far, then, we have found the canonical $\vec{I}, \vec{J}$ generators for the Dirac stuff.

$$\begin{pmatrix} \psi_1 \\ \psi_{-1} \\ \psi_i \\ \psi_{-i} \end{pmatrix}; \quad I_k = \begin{pmatrix} -(\sigma_k) & 0 \\ 0 & -(\sigma_k) \end{pmatrix}; \quad J_k = \begin{pmatrix} -i(\sigma_k) & 0 \\ 0 & -i(\sigma_k) \end{pmatrix}$$

In other words:

$$\hat{G}_{\frac{1}{2}0} \oplus \hat{G}_{0\frac{1}{2}} = \begin{bmatrix} \hat{G}_{\frac{1}{2}0} & 0 \\ 0 & \hat{G}_{0\frac{1}{2}} \end{bmatrix}$$

$$\boxed{\begin{aligned} I_k &= -\Sigma_k = -\gamma_0 \gamma^k \\ J_k &= i\gamma_0 \Sigma_k \end{aligned}}$$

There is some doubt as to whether you should use $-1, 1$ or $1, -1$ in the subspaces. My use of the Pauli's for $D^{(1)}$ suggests $(1, -1)$. Recall that -1 is raised to $+1$. So I may have things backwards.

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_{-1} \\ \psi_i \\ \psi_{-i} \end{pmatrix} \quad \psi_i = \psi_{\frac{1}{2}0}$$

OK, how do we use these $B_k + R_k$ generators? To transform!

$$\psi(x') = T(g) \psi(x) \quad \text{BD would call this } S.$$

Can we go to the natural Dirac basis?

$$I_k = -\Sigma_k \quad J_k = i\gamma_0 [\Sigma_k] \quad \Sigma_k = \begin{pmatrix} \sigma_k & 0 \\ 0 & \sigma_k \end{pmatrix}$$

$$\begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 1 \\ -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_{-1} \\ \psi_i \\ \psi_{-i} \end{bmatrix} \quad \phi_s = C_{ns} \psi_s$$

$\equiv C_{ns}$

So what are the generators in the new basis ϕ ?

$$\psi'_k(x') = T_{ns}(g) \psi_s(x) = (e^{i u B_i})_{ns} \psi_s(x) \quad \text{eg.}$$

$$\approx (\delta_{ns} + u (B_i)_{ns}) \psi_s(x)$$

$$\phi'_n(x') \approx (\delta_{ns} + u (B_i^\dagger)_{ns}) \phi_s(x)$$

$$\Rightarrow C_{ns} \psi'_s(x') \approx (\delta_{ns} + u (B_i^\dagger)_{ns}) C_{sq} \psi_q(x)$$

$$C_{rs} = \begin{bmatrix} 0 & 1 & 0 & 1 \\ -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & -1 & 0 \end{bmatrix} \quad \det C = ?$$

$$\det C = \det \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & -2 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & -1 & 0 \end{pmatrix} = -\det \begin{pmatrix} 1 & 0 & 1 \\ 0 & -2 & 0 \\ -1 & 0 & 1 \end{pmatrix} = -\det \begin{pmatrix} 1 & 0 & 1 \\ 0 & -2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$= -\det \begin{pmatrix} 1 & 0 & 1 \\ 0 & -2 & 0 \\ 0 & 0 & 2 \end{pmatrix} = -\det \begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix} = 4.$$

Since $\det C \neq 0$, C can be inverted

$$\text{adj } C_{rs} = ?$$

Guess: $C_{rs}^{-1} = \frac{1}{2} \begin{pmatrix} 0 & -1 & 0 & -1 \\ 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 \\ 1 & 0 & -1 & 0 \end{pmatrix}$

$$\text{adj } C_{rs} = (+1)(-1)(-1)(0) = 0$$

So $C_{rs}^{-1} = \frac{1}{2} (\tilde{C})_{rs} = \frac{1}{2} C_{sr}$ Now we had,

$$C_{rs} \Psi'_s(x') = (\delta_{rs} + u (G^\Phi)_{rs}) C_{sq} \Psi_q(x)$$

Mult then by $(\tilde{C})_{pr}^{-1} = \frac{1}{2} (\tilde{C})_{rp}$,

sum on r :

Change:

$$\text{LHS} = (\tilde{C})_{pr}^{-1} C_{rs} \Psi'_s = (\tilde{C}^{-1} C)_{ps} \Psi'_s = \delta_{ps} \Psi'_s$$

then sum on s :

$$= \Psi'_p(x')$$

$$\begin{aligned} \text{RHS} &= \sum_{rs} (\tilde{C})_{pr}^{-1} (\delta_{rs} + u (G^\Phi)_{rs}) C_{sq} \Psi_q(x) \\ &= \underbrace{(\tilde{C})_{pr}^{-1} \delta_{rs} C_{sq}}_{\delta_{ps}} \Psi_q + u \cdot \underbrace{(\tilde{C})_{pr}^{-1} (G^\Phi)_{rs} C_{sq}}_{(G)_{pq}} \Psi_q \\ &= \underbrace{C_{ps}^{-1} C_{sq}}_{\delta_{pq}} = (G)_{pq} \end{aligned}$$

Thus: $\Psi'_p(x') = (\delta_{pq} + u G_{pq}) \Psi_q$

So the rule is: $\boxed{G^\Psi = C^{-1} G^\Phi C}$ ← as we might have guessed.

This means some messy stuff!

$$C = \begin{pmatrix} i\sigma_y & i\sigma_y \\ -i\sigma_y & i\sigma_y \end{pmatrix}$$

Some nonline:

$$\begin{aligned}
 R_i^\phi &= +\frac{1}{2} \gamma^2 (1+\gamma_6) (\gamma_6 \gamma^i + 1) (1-\gamma_6) \gamma^2 \\
 &= \frac{1}{2} (\gamma_6 \gamma^i - \gamma^i + 1 + \gamma_6) (1-\gamma_6) \\
 &= \frac{1}{2} (\gamma_6 \gamma^i - \gamma^i + 1 + \gamma_6 - \gamma_6 \gamma^i \gamma_6 - \gamma^i \gamma_6 - \gamma_6 + 1) \\
 &= (2 + \gamma_6 \gamma^i - \gamma^i - \gamma_6 [\gamma_6 \gamma^i + 2g^{i0} \gamma_5] - [\gamma^i \gamma_6 + 2g^{i0} \gamma_5]) \\
 &= (2 + \cancel{\gamma_6 \gamma^i} - \gamma^i + \gamma^i - 2g^{i0} \gamma_6 \gamma_5 - \gamma^i \gamma_6 - 2g^{i0} \gamma_5) \\
 &= (2 + \cancel{2g^{i0} \gamma_6 \gamma_5} - 2g^{i0} (-\gamma_6 + \gamma_5)) = 2(1 + g^{i0} [\gamma_5 - \gamma_6])
 \end{aligned}$$

$$\begin{aligned}
 R_i^\phi &= \cancel{2} \gamma^2 (1 + \gamma_6 \gamma^i - \gamma^i - g^{i0} (\gamma_5 - \gamma_6)) \gamma^2 \\
 &= \cancel{2} (1 + \gamma_6 \gamma^i \gamma^2 - \gamma^i \gamma^2 - g^{i0} \gamma^2 (\gamma_5 - \gamma_6)) \gamma^2 \\
 &= \cancel{2} (1 + \gamma_6 \gamma^i - \gamma^i - g^{i0} (\gamma_5 - \gamma_6)) \gamma^2
 \end{aligned}$$

$$R_i^\phi = \gamma^2 (1 + g^{i0} (\gamma_5 - \gamma_6)) \gamma^2 = (-1 + g^{i0} (\gamma_5 - \gamma_6))$$

But $g^{i0} = 0$, so $R_i^\phi = -I$ ← strange indeed.

The generators are all the same??

$$\Sigma^i = \gamma_6 \gamma^i$$

Redo the calc. We know that:

$$R_i^\mu = -X - \Sigma_i^\mu = -(\Sigma_i^\mu)^\vee = -(\gamma_6 \gamma^i)^\vee$$

$$C = i(\gamma^2 + \Sigma^2)^\vee = i(\gamma^2 + \gamma_6 \gamma^2)^\vee = i\gamma^2(1 + \gamma_6)^\vee$$

$$C^{-1} = \frac{1}{2} i(\gamma^2 - \Sigma^2) = \frac{1}{2} i(\gamma^2 - \gamma_6 \gamma^2) = \frac{1}{2} i(1 - \gamma_6) \gamma^2$$

$$\text{Check: } CC^{-1} = -\frac{1}{2} \gamma^2 (1 + \gamma_6) (1 - \gamma_6) \gamma^2 = -\frac{1}{2} \gamma^2 \cdot 2 \gamma^2 = -(\gamma^2)^2 = +1 \checkmark$$

$$R_i^\phi = C R_i^\mu C^{-1} = +\frac{1}{2} \gamma^2 (1 + \gamma_6) (\gamma_6 \gamma^i) (1 - \gamma_6) \gamma^2$$

$$\begin{aligned}
 &= \frac{1}{2} \gamma^2 (1 + \gamma_6) \gamma_6 \gamma^i (1 - \gamma_6) \gamma^2 + \frac{1}{2} \gamma^2 (1 + \gamma_6) (1 - \gamma_6) \gamma^2 \\
 &= \frac{1}{2} \gamma^2 (1 + \gamma_6) (1 - \gamma_6) (\gamma_6 \gamma^i + 1) \gamma^2 = \gamma^2 (\gamma_6 \gamma^i) \gamma^2 = \gamma_6 [\gamma^2 \gamma^i \gamma^2] \\
 &= \boxed{\gamma_6 \gamma^i} = R_i^\phi \checkmark
 \end{aligned}$$

$$\boxed{R_i^\phi = \gamma_6 \gamma^i}$$

The boosts would be:

$$B_i^\mu = J_i = i\gamma^0 [\Sigma^i] = i\gamma^0 [\gamma_5 \gamma_0 \gamma^i] = -i [\gamma_5 \gamma^i]$$

$$B_i^\phi = C B_i^\mu C^{-1} = i\gamma^3 (1+\gamma_6) (-i) (\gamma_5 \gamma^i) \frac{1}{2} i (1-\gamma_6) \gamma^2$$

$$= -\frac{1}{2} i^3 \gamma^2 (1+\gamma_6) (\gamma_5 \gamma^i) (1-\gamma_6) \gamma^2$$

$$= +\frac{1}{2} i \left\{ \gamma^2 (1+\gamma_6) (\gamma_5 \gamma^i) \right\}$$

$$= \frac{1}{2} i \gamma^2 \left[(1+\gamma_6) \gamma_5 \gamma^i (1-\gamma_6) - (1+\gamma_6) \gamma_5 (1-\gamma_6) \right] \gamma^2$$

$$\left\{ \begin{aligned} \gamma_6 \gamma_5 &= \gamma_5 \gamma_6 = -\gamma_5 \gamma_5 \gamma_0 = -\gamma_5 \gamma_6 \\ \{\gamma_6, \gamma_5\} &= 0 \end{aligned} \right.$$

$$\frac{1}{2} i \gamma^2 \left[(1+\gamma_6) (1+\gamma_6) \gamma_5 \gamma^i - (1+\gamma_6) \gamma_5 (1-\gamma_6) \gamma^0 - (1-\gamma_6) \gamma_5 \gamma^0 \right]$$

$$\frac{1}{2} i \gamma^2 \left[2\gamma_6 \gamma_5 \gamma^i - 2\gamma_6 \gamma_5 \gamma^0 \right] \gamma^2$$

$$= i \gamma^2 \left[\gamma_6 \gamma_5 \gamma^i - \gamma_6 \gamma_5 \gamma^0 \right] \gamma^2$$

$$\gamma^6 \gamma^5 = \gamma_5 \gamma_0 \gamma_5 = -\gamma_6$$

$$= i \gamma^2 \left[-\gamma_0 \gamma^i - \gamma_0 \gamma^0 \right] \gamma^2 = -i \gamma^2 \left[\gamma_0 \gamma^i + \gamma_0 \gamma^0 \right] \gamma^2$$

$$= -i \left(-\gamma_0 \gamma_5 \gamma^i \gamma^2 - \gamma_0 \gamma_5 \gamma^0 \gamma^2 \right) = -i \left(-\gamma_0 \gamma^i + \gamma_0 \gamma^0 \right)$$

$$= i \left(\gamma_0 \gamma^i - \gamma_0 \gamma^0 \right) = i \gamma^5 \left(\gamma_5 \gamma_0 \gamma^i - \gamma_5 \gamma_0 \gamma^0 \right) = i \gamma^5 \left(\gamma_6 \gamma^i - \gamma_6 \gamma^0 \right)$$

$$B_i^\phi = i \gamma_5 \gamma_6 \gamma_i^*$$

Thus,

$$B_i^\phi = i \gamma^5 R_i^\phi \quad \checkmark$$

$$R_i^\phi = \gamma_6 \gamma^i \gamma_6$$

In theory, these are the generators of the L_1 group acting on BD Dirac spinors.

Can I relate these to the $\sigma^{\mu\nu}$?

$$\sigma^{12} = \frac{1}{2} [\gamma^1, \gamma^2] = \frac{1}{2} (\gamma^1 \gamma^2 - \gamma^2 \gamma^1) = \frac{1}{2} \gamma^0 \gamma^0 (\gamma^1 \gamma^2 - \gamma^2 \gamma^1) \gamma^3 \gamma^3 (-1)$$

$$\left(R_2^\phi = (\gamma_6 \gamma^3 - 1) \right)$$

$$\gamma_6 \gamma^3 \gamma_6 = 2\gamma^3 \gamma^0$$

$$= \frac{1}{2} \gamma^0 i \gamma^0 2\gamma^3 \gamma^0 \gamma^3 (-1) = \gamma^0 (i \gamma^0 \gamma^1 \gamma^2 \gamma^3) \gamma^3 (-1) = -\gamma^0 \gamma_5 \gamma^3$$

$$= \gamma_6 \gamma^3 \quad \text{Close, but where is the } -1!!$$

$$\sigma^{12} = i \gamma^1 \gamma^2 = \gamma^0 (i \gamma^0 \gamma^1 \gamma^2 \gamma^3) (-\gamma^3) = -\gamma^0 \gamma_5 \gamma^3 = \gamma_6 \gamma^3$$

$$\sigma^{03} = i \gamma^0 \gamma^3$$

$$i \gamma^5 (\gamma_6 \gamma^3) = i \gamma_0 \gamma^3 \quad \checkmark$$

OK, now corrections are all made + we have:

$$\begin{aligned} R_i^\phi &= \gamma_6 \gamma^{i*} = \gamma_5 \gamma_0 \gamma^{i*} = i \gamma^0 \gamma^1 \gamma^2 \gamma^3 \gamma_0 \gamma^{i*} \\ &= -i \gamma^1 \gamma^2 \gamma^3 \gamma^{i*} = -i \gamma^1 \gamma^2 \gamma^3 [\gamma^2 \gamma^i \gamma^2] \\ &= -i \gamma^1 \gamma^3 \gamma^i \gamma^2 \end{aligned}$$

Check: $R_1^\phi = -i \gamma^1 \gamma^3 \gamma^1 \gamma^2 = -i \gamma^3 \gamma^2 = +i \gamma^2 \gamma^3 = \sigma^{23}$

$R_2^\phi = -i \gamma^1 \gamma^3 \gamma^2 \gamma^2 = +i \gamma^1 \gamma^3 = \sigma^{13}$ ouch: $= -\sigma^{31}$

$R_3^\phi = -i \gamma^1 \gamma^3 \gamma^3 \gamma^2 = i \gamma^1 \gamma^2 = \sigma^{12}$

Some thing seems non cyclic here, but otherwise great!

$B_i^\phi = i \gamma_5 R_i^\phi = i \gamma_5 \gamma_6 \gamma^2 \gamma^i \gamma^2$

$= i \cancel{\gamma_5} \gamma_5 \gamma_0 \gamma^2 \gamma^i \gamma^2 = i \gamma_0 \gamma^2 \gamma^i \gamma^2$

So, $B_1^\phi = i \gamma^0 \gamma^2 \gamma^1 \gamma^2 = i \gamma^0 \gamma^1 = \sigma^{01}$

$B_2^\phi = i \gamma^0 \gamma^2 \gamma^2 \gamma^2 = -i \gamma^0 \gamma^2 = -\sigma^{02}$

$B_3^\phi = i \gamma^0 \gamma^2 \gamma^3 \gamma^2 = i \gamma^0 \gamma^3 = \sigma^{03}$

Do I believe this results which say that somehow 2 is not equivalent to 1 and 3?

The reason is that γ^2 appears in C ! This is because

C is chock full of σ_2 's! So I do find it reasonable.

After all, 2 is imag, 1 and 3 are not.

Am now confused by spins "ordering". Think about $D^{(\frac{1}{2})}$ with ψ_1 and ψ_{-1} .
The "objects" here are two comp vectors, and in the "usual" ~~app~~ we say:

$$\psi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } \psi_{-1} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

It does not mean anything to write $\begin{pmatrix} \psi_1 \\ \psi_{-1} \end{pmatrix}$?

Well, later we do consider

sets of functions $\begin{pmatrix} \psi_1(x) \\ \psi_{-1}(x) \end{pmatrix}$ — goes as a spin!

Does it matter which way you group these guys? Not really, but then for $D^{(\frac{1}{2})}$ of FRG, what means $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$? i.e., this is supposed to generate transformations, how do you know it's right?

Now I think the order is important.

$$e^{iR_z} = z \text{ rotation.}$$

We know that:

$$L_i = i I_i = i R_i = \frac{1}{2} \sigma_i \text{ in } N=2, D^{(\frac{1}{2})}$$

So $I_i = R_i = -\frac{1}{2} i \sigma_i$ and thus

$$e^{iR_z} = e^{\frac{1}{2} i \sigma_z} = \text{object that } z\text{-rotates (per) spinor.}$$

What about: $\psi = \psi_1 e_1 + \psi_{-1} e_{-1} = \psi_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \psi_{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \psi_1 \\ \psi_{-1} \end{pmatrix}$?

This makes sense also:

$$\hat{\psi}_1 = 1 e_1 = e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \text{"up" unit vector.}$$

$$\sigma_z e_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = + \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \text{ hence "up"}$$

So, under $D^{1/2}$ [or $\mathbb{R}^3$], it would seem that $\begin{pmatrix} \psi_1 \\ \psi_{-1} \end{pmatrix}$ is logical.
But be careful: Recall that:

$$A_0 C_{pq} = -i p C_{pq} \quad \text{where } [A_1, A_2] = A_3$$

$$\boxed{L_i = \frac{1}{2} \sigma_i}$$

We know that $\boxed{L_i = i A_i}$, so this says:

$$L_0 C_{pq} = p C_{pq} \quad \checkmark$$

$$\text{But } L_0 = \frac{1}{2} \sigma_z \text{ if } e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

The whole thing hinges on arbitrary: $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. You could choose $e_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Summary of Results on Last 9 pages.

- ① By comparing the A and B algebras, we deduce that, [these corrections not shown here].

$$\mathcal{Z}_{\pm 0}: \quad A_k = -\frac{i}{2} \sigma_k \quad R_k = +\frac{i}{2} \sigma_k \quad \text{note: } R_k^\dagger = -R_k \quad \text{as} \\ B_k = 0 \quad B_k = +\frac{i}{2} \sigma_k \quad B_k^\dagger = R_k \quad \text{desired.}$$

$$\mathcal{Z}_{0\pm}: \quad A_k = 0 \quad R_k = +\frac{i}{2} \sigma_k \\ B_k = -\frac{i}{2} \sigma_k \quad B_k = -\frac{i}{2} \sigma_k.$$

A 2-comp spin in one of these reps would be:

$$\mathcal{Z}_{\pm 0}: \quad \Psi = \psi_+ e_+ + \bar{\psi}_- e_- = \begin{pmatrix} \psi_+ \\ \bar{\psi}_- \end{pmatrix} \quad e_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\mathcal{Z}_{0\pm}: \quad \Psi = \psi_i e_i + \bar{\psi}_{-i} e_{-i} = \begin{pmatrix} \psi_i \\ \bar{\psi}_{-i} \end{pmatrix} \quad e_i = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

- ② For Dirac spins in canonical basis, we have

$$\Gamma = \mathcal{Z}_{\pm 0} \oplus \mathcal{Z}_{0\pm}$$

The generators in this 4-dim reducible representation are:

$$R_k = \begin{bmatrix} \frac{i}{2} \sigma_k & 0 \\ 0 & \frac{i}{2} \sigma_k \end{bmatrix} \quad B_k = \begin{bmatrix} \frac{i}{2} \sigma_k & 0 \\ 0 & -\frac{i}{2} \sigma_k \end{bmatrix}$$

The general spin is written:

$$\Psi = \psi_+ e_+ + \bar{\psi}_- e_- + \psi_i e_i + \bar{\psi}_{-i} e_{-i} = \begin{pmatrix} \psi_+ \\ \bar{\psi}_- \\ \psi_i \\ \bar{\psi}_{-i} \end{pmatrix}$$

because $e_+ \equiv \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ etc.

- ③ There are many ways to write these generators using BD "conventional" matrices. Eg:

$$R_k = i \gamma_0 B_k \quad \text{where } \gamma_0 = \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \gamma_4 \text{ of some.}$$

and,

$$B_k = -i \gamma_0 R_k$$

~~R_k~~

$$\text{where } R_k = \frac{i}{2} \begin{pmatrix} \sigma_k & 0 \\ 0 & \sigma_k \end{pmatrix}$$

Once more:

$$B_K = -i \gamma_0 R_K \quad \text{where}$$

$$R_K = \frac{i}{2} \Sigma_{ik} = \frac{i}{2} \Sigma^K = \frac{i}{2} \sigma^{ij} = \frac{i}{2} (i \gamma^i \gamma^j) = -\frac{1}{2} \gamma^i \gamma^j$$

Let's check this:

$$\Sigma^i = \gamma_0 \gamma^i = \gamma_0 \gamma_0 \gamma^i = i \gamma^0 \gamma^1 \gamma^2 \gamma^3 \gamma^0 \gamma^i = -i \gamma^1 \gamma^2 \gamma^3 \gamma^i$$

$$\Sigma^1 = +i \gamma^2 \gamma^3$$

$$\Sigma^2 = +i \gamma^3 \gamma^1$$

$$\Sigma^3 = i \gamma^1 \gamma^2$$

So, in the canonical basis,

$$R_K = \frac{i}{2} \sigma^{ij}$$

$$B_K = \frac{1}{2} \gamma_0 \sigma^{ij}$$

Is this B_K related to σ^{0k} ?

$$\gamma_0 \sigma^{ij} = \gamma_0 i \gamma^i \gamma^j$$

$$\sigma^{0k} = i \gamma^0 \gamma^k$$

But $\gamma^k \neq \gamma^i \gamma^j$, so answer is no! But

$$\sigma^{ij} = \gamma_0 \gamma^k, \text{ so}$$

$$B_K = \frac{1}{2} \gamma_0 \sigma^{ij} = \frac{1}{2} \gamma_0 \gamma_5 \gamma_0 \gamma^k = \frac{1}{2} \gamma_0 \gamma_0 \gamma^k = \frac{1}{2} \gamma_0^2 (\gamma^k) = -i \gamma_0 (i \gamma^0 \gamma^k)$$

$$= -i \gamma_0 \sigma^{0k}$$

So:

$$R_K = \frac{i}{2} \sigma^{ij}$$

$$B_K = -i \gamma_5 \gamma_0 \sigma^{0k}$$

④ Next, we wanted to convert to the "natural" basis, so we set up

$$\psi = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 1 \\ -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}$$

as defined by Liu. In shorthand,

$$\phi_i = C_{is} \psi_s$$

$$\text{Now, } C = i \begin{bmatrix} \sigma_3 & \sigma_2 \\ -\sigma_2 & \sigma_3 \end{bmatrix} = i \{ \Sigma^2 + \gamma^2 \} = i (\gamma_0 + 1) \gamma^2$$

We showed that

$$\tilde{\gamma}^2 = ((\gamma^2)^*)^\dagger = (-\tilde{\gamma}^2)^\dagger = -(\tilde{\gamma}^2)^\dagger = +\gamma^2 \checkmark$$

$$\tilde{C}^\dagger = \frac{1}{2} \tilde{C}$$

$$\tilde{\gamma}^0 = \gamma^0$$

$$\tilde{\gamma}_6 = \tilde{\gamma}_0 \tilde{\gamma}_5 = \gamma_0 \gamma_5 = -\gamma_6$$

$$\tilde{\gamma}_5 = \gamma_5$$

$$\text{Thus, } \tilde{C}^\dagger = \frac{1}{2} [i(\gamma_6 + 1)\gamma^2] = \frac{1}{2} \gamma^2 (-\gamma_6 + 1)i = \frac{i}{2} \gamma^2 (1 - \gamma_6)$$

So :

$$\begin{aligned} C &= i(1 + \gamma_6)\gamma^2 \\ \tilde{C}^\dagger &= \frac{i}{2}(1 - \gamma_6)\gamma^2 \end{aligned}$$

⑤ Next, we showed how a basis change C affects the generators:

$$G_{\text{new}} = C G_{\text{canon}} C^{-1}$$

⑥ Thus, we at once computed:

$$\begin{aligned} R_K^\dagger &= C R_K C^{-1} = i(1 + \gamma_6)\gamma^2 \frac{i}{2} \sigma^j \frac{i}{2} (1 - \gamma_6)\gamma^2 \\ &= \frac{i^3}{4} (1 + \gamma_6)\gamma^2 (1 - \gamma_6) \sigma^j \gamma^2 = \frac{i^3}{4} \cdot 2 \cdot \gamma^2 \sigma^j \gamma^2 \\ &= \frac{i^4}{2} \cdot \gamma^2 \sigma^j \gamma^2 = \frac{i^4}{2} \gamma^2 [\gamma_6 \sigma^j] \gamma^2 = -\frac{i}{2} \gamma_6 \sigma^{j*} \end{aligned}$$

More carefully:

$$\gamma^2 \sigma^{12} \gamma^2 = \gamma^2 [\gamma^1 \gamma^2] \gamma^2 = \sigma^{12}$$

$$\gamma^2 \sigma^{23} \gamma^2 = \sigma^{23}$$

but

$$\gamma^2 \sigma^{31} \gamma^2 = -\sigma^{31}$$

$$\text{So, } R_K^\dagger = -\frac{i}{2} (\gamma^2 \sigma^j \gamma^2) = \begin{cases} K=1 & -\frac{i}{2} \sigma^{23} \\ K=2 & +\frac{i}{2} \sigma^{31} \\ K=3 & -\frac{i}{2} \sigma^{12} \end{cases} = R_K \checkmark$$

There is that minus sign.

⑦ Next,

$$B_K^\dagger = C B_K C^{-1} = C [-i\gamma_0 R_K] C^{-1} \neq +i\gamma_0 C R_K C^{-1}$$

$$B_K^\dagger = i\gamma_0 R_K$$

$$\gamma_0 \gamma_6 \neq \gamma_6 \gamma_0$$

$$= -\frac{i}{2} \gamma_5 \gamma_0 (i\gamma^0 \gamma^1) = -\frac{i}{2} \gamma_6 \sigma^{01}$$

← even.

$$\text{So, } B_K^\dagger = \gamma_0 \cdot \frac{i^3}{2} \gamma^2 \sigma^j \gamma^2$$

$$B_1^\dagger = i\gamma_0 \cdot \frac{i^3}{2} \sigma^{23} = \frac{i}{2} \gamma_0 [i\gamma^2 \gamma^3] = -\frac{i}{2} \gamma_0 \gamma^1 \gamma^2 \gamma^3$$

$$= -\frac{i}{2} (i\gamma_0 \gamma^1 \gamma^2 \gamma^3) \gamma^1 = -\frac{i}{2} \gamma^0 \gamma^1 = -\frac{i}{2} \gamma^0 \gamma^1$$

To summarize,

$$\begin{aligned} R_1^\phi &= -\frac{i}{2} \sigma_{23} \\ R_2^\phi &= +\frac{i}{2} \sigma_{31} \\ R_3^\phi &= -\frac{i}{2} \sigma_{12} \end{aligned}$$

~~$$\begin{aligned} B_1^\phi &= \frac{i}{2} \gamma_6 \sigma^1 \\ B_2^\phi &= \frac{i}{2} \gamma_6 \sigma^2 \\ B_3^\phi &= \frac{i}{2} \gamma_6 \sigma^3 \end{aligned}$$~~

Thus, here is how you would rotate a standard BD spinor:

$$\bar{\psi} = \bar{\psi} e^{\theta R_1} = \bar{\psi} e^{-\frac{i}{2} \theta \sigma_{23}}$$

$$\psi = \psi e^{\omega B_1} = \psi e^{\frac{i}{2} \omega \gamma_6 \sigma^1}$$

✓ all by sign from (2.24) BD

We've got trouble! It seems that BD does not have that extra γ_6 .

Repair: I now think that Lya. chose $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ so that his transformation $\psi \rightarrow \phi$ is for this choice.

So we need a new $C = C_1$

$$\begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} \psi_{-1} \\ \psi_1 \\ \psi_{-i} \\ \psi_i \end{bmatrix}$$

$$\text{Now, } C_1 = \begin{bmatrix} \sigma_3 & \sigma_3 \\ -\sigma_3 & \sigma_3 \end{bmatrix} = (\Sigma^3 + \gamma^3)$$

The business of

$$G^\phi = C_1 G^\psi C_1^{-1} \quad \text{is still OK, but now we need } C_1^{-1}$$

My guess is: $C_1^{-1} = \frac{1}{2} \tilde{C}_1$, as before. So:

$$C_1 = \Sigma^3 + \gamma^3 \quad \tilde{C}_1 = \Sigma^3 - \gamma^3 = \Sigma^3 - \gamma^3$$

$$\Rightarrow C_1^{-1} = \frac{1}{2} (\Sigma^3 - \gamma^3)$$

$$\text{Check: } C_1 C_1^{-1} = \frac{1}{2} (\Sigma^3 + \gamma^3)(\Sigma^3 - \gamma^3) = \frac{1}{2} [(\Sigma^3)^2 - (\gamma^3)^2] = \frac{1}{2} [1 + 1] = 1 \checkmark$$

$$C_1^{-1} C_1 = \frac{1}{2} (\Sigma^3 - \gamma^3)(\Sigma^3 + \gamma^3) = 1 \checkmark$$

Now, what about the computed $G^{\dagger}$? Assume OK. Then:

$$R_K^{\dagger} = \frac{i}{2} \Sigma_K \quad \text{and} \quad R_K^{\dagger} = -i \gamma_0 R_K^{\dagger} = \frac{i}{2} \gamma_0 \Sigma_K$$

Now;

$$R_K^{\dagger} = C_1 R_K^{\dagger} C_1^{-1} \\ = \frac{i}{4} (\Sigma^3 + \gamma^3) \Sigma_K (\Sigma^3 - \gamma^3)$$

Need algebra of σ^5 :

$$\frac{\Sigma^3 \gamma^1}{\Sigma^3} = -\gamma^1 \Sigma^3 \\ \text{stop} \quad \text{good} \quad \gamma_0 \gamma^1:$$

$$R_K^{\dagger} = \frac{i}{4} (\gamma_0 + 1) \gamma^3 \gamma_0 \gamma^K \gamma^3 (\gamma_0 - 1) = \frac{i}{4} \gamma_0 \underbrace{(\gamma_0 + 1)(\gamma_0 - 1)}_{-2} \gamma^3 \gamma^K \gamma^3 \\ = -\frac{i}{2} \gamma_0 \gamma^3 \gamma^K \gamma^3$$

$$\text{Eg, } R_1^{\dagger} = -\frac{i}{2} \gamma_0 \gamma^3 \gamma^1 \gamma^3 = -\frac{i}{2} \gamma_0 \gamma^1 = -\frac{i}{2} \Sigma^1$$

$$R_2^{\dagger} = -\frac{i}{2} \Sigma^2 \quad R_3^{\dagger} = +\frac{i}{2} \Sigma^3 \quad R_i^{\dagger} = \cancel{(-1)^i} \frac{i}{2} \Sigma^i$$

There is now a sign change in 3. Next:

$$R_K^{\dagger} = C_1 R_K^{\dagger} C_1^{-1} = C_1 [-i \gamma_0 R_K^{\dagger}] C_1^{-1} = i \gamma_0 C_1 R_K^{\dagger} C_1^{-1} \\ = \cancel{i \gamma_0 (-1)^K} = i \gamma_0 (-1)^K \frac{i}{2} \Sigma^K = -\frac{1}{2} \gamma_0 (-1)^K \Sigma^K$$

Same problem as before.

$$R_K^{\dagger} = \cancel{(-1)^{K+1}} \frac{1}{2} \gamma_0 \Sigma^K$$

$$\gamma_0 \Sigma^K = \gamma_0 \gamma_0 \gamma^K = [\gamma_0 \gamma_0 - 2\gamma_5] \gamma^K = \gamma_0 \sigma_{\dot{c}} - 2\gamma_5 \gamma^K$$

Error $\gamma_0 \gamma_0 \neq \gamma_0 \gamma_0$.

Repair: use direct calc for boosts.

$$B_K^\phi = C B_K^\psi C^{-1} = \frac{1}{4} (\Sigma^3 + \gamma^3) (-i \gamma_0 R_K^\psi) \left(\frac{1}{2}\right) (\Sigma^3 - \gamma^3)$$

$$= -\frac{i}{2} (\Sigma^3 + \gamma^3) \gamma_0 \left[\frac{1}{2} \Sigma_K^\psi\right] (\Sigma^3 - \gamma^3)$$

$$= \frac{1}{4} (\Sigma^3 + \gamma^3) \gamma_0 (\Sigma^3 - \gamma^3)$$

$$= \frac{1}{4} \gamma^3 (\gamma_0 + 1) \gamma_0 \gamma^K (\gamma_0 - 1) \gamma^3$$

$$= \frac{1}{4} \gamma^3 \gamma_0 (-\gamma_0 + 1) \gamma_0 \gamma^K (\gamma_0 - 1) \gamma^3$$

$$= \frac{1}{4} \gamma^3 \gamma_0 \gamma^K \gamma^3 (1 - \gamma_0) \gamma_0 (\gamma_0 - 1)$$

$$= -\frac{1}{2} \gamma^3 \gamma_0 \gamma^K \gamma^3$$

$$\hookrightarrow (\gamma_0 + 1)(\gamma_0 - 1) = -1 - 1 = -2$$

$$B_K^\phi = +\frac{1}{2} \gamma_0 \cdot \gamma^3 \gamma^K \gamma^3 = (-1) \cdot \frac{1}{2} \gamma^0 \gamma^K$$

$$= \begin{bmatrix} + \\ + \\ - \end{bmatrix} \frac{1}{2} \gamma_0 \gamma^K = \begin{bmatrix} + \\ + \\ - \end{bmatrix} \frac{1}{2i} (i \gamma_0 \gamma^K) = -\frac{i}{2} \begin{bmatrix} + \\ + \\ - \end{bmatrix} \sigma^{0K}$$

New results are:

$$\boxed{R_K^\phi = -\frac{i}{2} \begin{bmatrix} + \\ + \\ - \end{bmatrix} \sigma^{ij}}$$

$$B_K^\phi = -\frac{i}{2} \begin{bmatrix} + \\ + \\ - \end{bmatrix} \sigma^{0K}$$

using C and

$$\begin{pmatrix} \psi_1 \\ \psi_1 \\ \psi_{-i} \\ \psi_i \end{pmatrix}$$

In the original C theory,

$$R_K^\psi = -\frac{i}{2} \begin{bmatrix} + \\ + \\ - \end{bmatrix} \sigma^{ij}$$

using C and

$$\begin{pmatrix} \psi_1 \\ \psi_{-1} \\ \psi_i \end{pmatrix}$$

$$\boxed{R_K^\phi = -\frac{i}{2} \begin{bmatrix} + \\ + \\ - \end{bmatrix} \sigma^{ij}}$$

$$B_K^\phi = +\frac{i}{2} \begin{bmatrix} + \\ + \\ - \end{bmatrix} \sigma^{0K}$$

$$B_K^\phi = C B_K^\psi C^{-1} = i (1 + \gamma_0) \gamma^2 \left[-i \gamma_0\right] \left[\frac{1}{2} \gamma_0 \gamma^K\right] \frac{i}{2} (1 - \gamma_0) \gamma^2$$

$$= \frac{-i^4}{4} (1 + \gamma_0) \gamma^2 \gamma_0 \gamma^K \gamma^2 (1 - \gamma_0) \gamma^2$$

$$= \frac{-1}{4} \gamma^2 (1 + \gamma_0) \gamma_0 \gamma^K \gamma^2 (1 - \gamma_0) \gamma^2 = -\frac{1}{4} \gamma^2 \gamma_0 \underbrace{(1 - \gamma_0) \gamma^K \gamma^2}_{-2} (1 + \gamma_0)$$

$$= \frac{1}{2} \gamma^2 \gamma_0 \gamma^K \gamma^2 = \frac{1}{2} \gamma_0 (\gamma^2 \gamma^K \gamma^2)$$

$$= -\frac{1}{2} \begin{bmatrix} + \\ + \\ - \end{bmatrix} \gamma_0 \gamma^K = -\frac{i}{2} \begin{bmatrix} + \\ + \\ - \end{bmatrix} \sigma^{0K}$$

How do you x-boost your spin? (Use C):

$$\psi'(x) = e^{wB_1} \psi(x) = e^{w \left[-\frac{i}{2} \sigma^{01} \right]} \psi(x) \quad \text{/off by sign from (2.23)}$$

Or, to do a z-rot,

$$\psi'(x) = e^{\theta R_3} \psi(x) = e^{\theta \left[+\frac{i}{2} \sigma^{12} \right]} \psi(x) \quad \text{/agrees with (2.24)}$$

Now going to the yellow sheet, we have:

$$-M^{ij} = R_K = -\frac{i}{2} \begin{bmatrix} + \\ - \end{bmatrix} \sigma^{ij}$$

$$-M^{0K} = B_K = -\frac{i}{2} \begin{bmatrix} + \\ - \end{bmatrix} \sigma^{0K}$$

$\Rightarrow$

$$M^{ij} = \frac{i}{2} \begin{bmatrix} + \\ - \end{bmatrix} \sigma^{ij}$$

$$M^{0K} = \frac{i}{2} \begin{bmatrix} + \\ - \end{bmatrix} \sigma^{0K}$$

We almost have:

$$M^{\mu\nu} = \frac{i}{2} \sigma^{\mu\nu}$$

This would imply:

$$M^{\mu\nu} = -\sum_m$$

So the desired result is:

$$M^{\mu\nu} = -\frac{i}{2} \sigma^{\mu\nu}$$

Four signs are wrong in my work. Very bothersome.

Rotation Group

Rotation Group : Show how to build more complicated reps from simpler ones.

1. For some reason this has taken a while to soak in. Suppose you know the $D^{(\frac{1}{2})}$ matrix elements. How would you construct the matrix elements of the higher representations? The idea is that you can do everything with Clebsches. These coefficients always couple two angular momenta at a time. Thus, eg, consider:

$$\frac{1}{2} \otimes \frac{1}{2} = 0 \oplus 1$$

$$\begin{aligned} D_{m_1 m_1'}^{(1)}(u) &= \langle 1, m | \mathcal{U} | 1, m' \rangle = \langle 1, m | \frac{1}{2}, m_1; \frac{1}{2}, m_2 \rangle \langle 1, m' | \frac{1}{2}, m_1'; \frac{1}{2}, m_2' \rangle \\ &\times \underbrace{\langle \frac{1}{2}, m_1; \frac{1}{2}, m_2 | \mathcal{U} | \frac{1}{2}, m_1'; \frac{1}{2}, m_2' \rangle}_{= D_{m_1 m_1'}^{(\frac{1}{2})}(u) D_{m_2 m_2'}^{(\frac{1}{2})}(u)} \end{aligned}$$

More generally: let $s_1 \otimes s_2 = \dots \oplus s \oplus \dots$

$$D_{m m'}^{(s)}(u) = \langle s, m | s_1, m_1; s_2, m_2 \rangle \langle s, m' | s_1, m_1'; s_2, m_2' \rangle D_{m_1 m_1'}^{(s_1)}(u) D_{m_2 m_2'}^{(s_2)}(u)$$

2. So whenever you have an s contained in the product of s_1 and s_2 , you can always get the matrix elements of s just by using clebsches. Think of the Clebsches as changing the basis in a Hilbert space, and the operator U stays the same.

3. These facts are stated in Martin Spearman appendix.

4. Notice that this same idea works for 'combineing' $(s_1, 0) \times (s_2, 0) = \dots + (s, 0) + \dots$ reps of $SL(2, \mathbb{C})$.

Induction and $D^{(s)}$ Matrix Elements.

① To get $D^{(s+\frac{1}{2})}$ from $D^{(s)}$: use M.S.

$$D_{M'M}^{(s)} = \sum_{m_1, m_2, m'_1, m'_2} \langle s-\frac{1}{2}, m_1; \frac{1}{2}, m'_2 | s M \rangle \langle s-\frac{1}{2}, m'_1; \frac{1}{2}, m_2 | s M' \rangle D_{m'_1 m_1}^{(s-\frac{1}{2})} D_{m'_2 m_2}^{(\frac{1}{2})}$$

② In one case $s = \frac{1}{2}$ we have:

$$D_{M'M}^{(\frac{1}{2})} = \sum_{m_1, m_2, m'_1, m'_2} \langle 0, m_1; \frac{1}{2}, m'_2 | \frac{1}{2} M \rangle \langle 0, m'_1; \frac{1}{2}, m_2 | \frac{1}{2} M' \rangle D_{m'_1 m_1} \cdot D_{m'_2 m_2}^{(\frac{1}{2})}$$

But $\langle 0, m_1; \frac{1}{2}, m'_2 | \frac{1}{2} M \rangle = \delta_{m_1, 0} \delta_{m'_2, M}$ so

$$= \sum_{m_1, m_2, m'_1, m'_2} \delta_{m_1, 0} \delta_{m'_1, 0} \delta_{m'_2, M} \delta_{m_2, M'} \cdot D_{m'_2 m_2}^{(\frac{1}{2})}$$

So it checks out in one case $s = \frac{1}{2}$.

③ Now consider this claim: to prove this by induction.

$$D_{M'M}^{(s)}(-e) = (-1)^{2s} \delta_{M', M} \quad \text{Want}$$

Assume true for $(s-\frac{1}{2})$ and $(\frac{1}{2})$. Then.

$$\begin{aligned} D_{M'M}^{(s)}(-e) &= \sum_{m_1, m_2, m'_1, m'_2} \langle s-\frac{1}{2}, m_1; \frac{1}{2}, m'_2 | s M \rangle \langle s-\frac{1}{2}, m'_1; \frac{1}{2}, m_2 | s M' \rangle \\ &\quad \cdot D_{m'_1 m_1}^{(s-\frac{1}{2})} (-1)^{2(s-\frac{1}{2})} \cdot D_{m'_2 m_2}^{(\frac{1}{2})} (-1)^{2 \cdot \frac{1}{2}} \\ &= \sum_{m_1, m_2} \langle s M' | s-\frac{1}{2}, m_1; \frac{1}{2}, m_2 \rangle \langle s-\frac{1}{2}, m_1; \frac{1}{2}, m_2 | s M \rangle \cdot (-1)^{2s} \\ &= (-1)^{2s} \cdot \delta_{M', M} \quad \square \end{aligned}$$

Graphical Building of $D^{(s)}(u)$ from $D^{(t)}(u)$:

① Simple example: you know that $|l, m\rangle = \langle l, m | \frac{1}{2}, m_1; \frac{1}{2} m_2 \rangle | \frac{1}{2}, m_1; \frac{1}{2} m_2 \rangle$

Thus, $\langle l, m' | U | l, m \rangle = \langle l, m' | \frac{1}{2} m'_1; \frac{1}{2} m'_2 \rangle \langle l, m | \frac{1}{2} m_1, \frac{1}{2} m_2 \rangle$
 $\times \langle \frac{1}{2} m'_1; \frac{1}{2} m'_2 | U | \frac{1}{2} m_1, \frac{1}{2} m_2 \rangle$

$$= \langle 1, m_1 | \frac{1}{2} m_1' ; \frac{1}{2} m_2' \rangle \langle \frac{1}{2} m_1' | 1 | \frac{1}{2} m_1 \rangle \langle \frac{1}{2} m_2' | 1 | \frac{1}{2} m_2 \rangle \langle 1, m_1 | \frac{1}{2} m_1, \frac{1}{2} m_2 \rangle$$

Ans: $D_{m_1 m_2}^1(u) = \langle 1, m_1 | \frac{1}{2} m_1; \frac{1}{2} m_2 \rangle \langle 1, m_2 | \frac{1}{2} m_1; \frac{1}{2} m_2 \rangle D_{m_1 m_1}^{\frac{1}{2}}(u) D_{m_2 m_2}^{\frac{1}{2}}(u)$

② I propose this graphical shorthand notation:

$$|1, m\rangle = |\frac{1}{2}, m_1; \frac{1}{2} m_2\rangle \langle \frac{1}{2}, m_1; \frac{1}{2} m_2 | 1, m\rangle$$

- (b) Each internal line is summed.
- (c) Each internal vertex is a blob.
- (d) Each $\circ$ gets a spin
- (e) Each $—$ gets an m helicity label.

Then we have:

$$\langle 1, m' | U | 1, m \rangle = \overset{m'}{\underset{\cdot}{\circ}} U \overset{m}{\underset{\cdot}{\circ}}$$

Above equation becomes:

ation becomes:

$$\frac{m}{\text{---}} \circ \text{---} \frac{m}{\text{---}} = \text{---} \frac{m}{\text{---}} \circ \text{---} \frac{m}{\text{---}}$$

③ Another notation might then be to combine $\frac{m_1'}{2} \cup \frac{2}{m_1} = \frac{m_1'}{2} \text{---} \frac{2}{m_1}$

$$u = \frac{m_1}{m_1} u^{\frac{1}{2}} m_1$$

Then:

The diagram shows a single node with input m' and output m being replaced by two parallel nodes. The top node has input m'_1 and output m_1 , and the bottom node has input m'_2 and output m_2 . The inputs m'_1 and m'_2 are connected to the original input m' , and the outputs m_1 and m_2 are connected to the original output m .

where now vertex = blob, circle = matrix element.

Or we might say:

$$m \text{ --- } \textcircled{u} \text{ --- } m' = m \text{ --- } \textcircled{u} \text{ --- } m'$$

The diagram on the right shows a diamond shape with two vertices labeled u . The top vertex has an incoming line from the left labeled m and an outgoing line to the right labeled m' . The bottom vertex has an incoming line from the left labeled m and an outgoing line to the right labeled m' . The two vertices are connected by a vertical line labeled $\frac{1}{2}$. The edges of the diamond are labeled m_1, m_2, m_3, m_4 in clockwise order starting from the top-left edge.

④ Identity: Clearly

$$\langle j, m | I | j, m' \rangle = \langle j, m | j, m' \rangle = \delta_{m, m'}$$

Notation:

$$\text{---} \textcircled{I} \text{---} = \text{---}$$

The diagram shows a horizontal line with a circle labeled I in the middle. The line continues to the left and right of the circle.

$$\text{or: } \text{---} \textcircled{I} \text{---} = \delta_{m, m'} \text{---}$$

The diagram shows a horizontal line with a circle labeled I in the middle. The line continues to the left and right of the circle.

We conclude from the above that:

$$\delta_{m, m'} = \text{---} \textcircled{I} \text{---}$$

The diagram shows a horizontal line with a circle labeled I in the middle. The line continues to the left and right of the circle.

This says:

$$\delta_{m, m'} = \sum_{m_1, m_2} \langle m | m_1, m_2 \rangle \langle m' | m_1, m_2 \rangle$$

But, to know what range to sum m_1 , you need to see the spin!

④ How about showing spin on the lines:

$$l, m \text{ --- } \textcircled{u} \text{ --- } l, m' = l, m \text{ --- } \textcircled{u} \text{ --- } l, m'$$

The diagram on the right shows a diamond shape with two vertices labeled u . The top vertex has an incoming line from the left labeled l, m and an outgoing line to the right labeled l, m' . The bottom vertex has an incoming line from the left labeled l, m and an outgoing line to the right labeled l, m' . The two vertices are connected by a vertical line labeled $\frac{1}{2}$. The edges of the diamond are labeled $\frac{1}{2}, m_1, \frac{1}{2}, m_2, \frac{1}{2}, m_3, \frac{1}{2}, m_4$ in clockwise order starting from the top-left edge.

$$\text{then } \text{---} \textcircled{I} \text{---} = \text{---} \delta_{m, m'}$$

The diagram shows a horizontal line with a circle labeled I in the middle. The line continues to the left and right of the circle.

$$\text{So } \text{---} \textcircled{I} \text{---} = \text{---}$$

The diagram shows a horizontal line with a circle labeled I in the middle. The line continues to the left and right of the circle.

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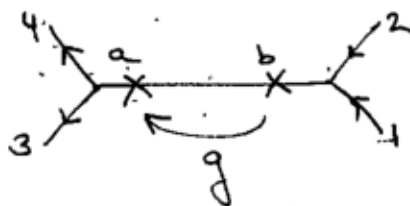
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1

Defining a regular CMS amplitude on $O(3)$.

Cal. U. L. 6 72 452 170

1. For some reason, this is slippery. First, set up a clear picture:



We have two standard reference frames in mind, a and b.

2. Is it correct to give a "ket" a frame label? I think this is reasonable on two examples. First, consider a rest frame for a single particle and we speak of $|jm\rangle^a$ as being a state in frame a. This means that in coordinate system called "a" the particle is in a J_3 eigenstate and 3 means the z axis. We could put the same particle in a different state, $|jm\rangle^b$ such that it is in an eigenstate of J_3 where now 3 means the new z axis in frame b, related to frame a by a rotation. Usually, we do not use "state-external labels" but I think they should be kept in mind.

Similarly, we are used to applying arbitrary LT's to helicity states (single as well as double). Thus, we may interpret $L(g)|1\rangle = |1\rangle'$ as giving us the same state observed in a new frame. Eg, $B_z(a)/\vec{p}0 = |p\rangle'$ could be read to mean: "a rest particle in frame S, when observed in frame S', has a momentum p."

3. The point of the above paragraph is to support our desire to characterize the above amplitude in this way:

$$(\text{amp})^2 = \langle 3, p_3, h_3; 4, p_4, h_4 | T | 2, p_2, h_2; 1, p_1, h_1 \rangle^2$$

$$\text{Then: } |2, 1\rangle^a = L(g) |2, 1\rangle^b$$

$$\text{But, } \langle 3, 4 | = \langle 3, 4 | \quad \text{and} \quad |1, 2\rangle^b = |1, 2\rangle^a, \text{ so}$$

$$HT(g) = \langle 3, 4 | T \cdot L(g) | 1, 2 \rangle = \langle 3, 4 | L(g) T | 1, 2 \rangle$$

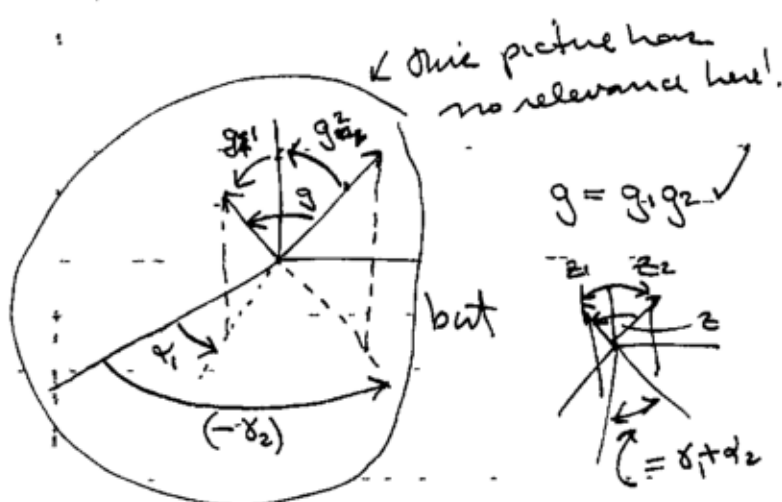
This last expression gives us the amplitude as "viewed" from frame b, so that the final particles appear at strange angles. This is the usual situation.

The addition theorem.

- First, set up your notations:
Use activities. Thus

$$g_i = R(\alpha, \beta, \gamma) \checkmark$$

$\Rightarrow$ as in the areas shown.



Also, we know: $z = z_1 z_2 + \sqrt{1-z_1^2} \sqrt{1-z_2^2} \cos(\alpha_1 + \alpha_2)$

- In general, the theory of group reps says:

$$D_{mn}^j(g) = D_{mn}^j(g_1 g_2) = \sum_s \bar{D}_{ms}^j(g_1) D_{sn}^j(g_2) \checkmark$$

In particular: $D_{00}^j(g) = \sum_s \bar{D}_{0s}^j(g_1) D_{s0}^j(g_2) \checkmark$

That is:

$$P_l(z) = \sum_s d_{0s}^l(z_1) d_{s0}^l(z_2) e^{-is(\alpha_1 + \alpha_2)} \checkmark$$

- ~~But, z is indep of α_1, α_2 . how can this be true??~~ See later sheet!

- Hunt error later, assume correct result is:

$$P_l(z) = \sum_s d_{0s}^l(z_1) d_{s0}^l(z_2) e^{-is(\alpha_2 + \alpha_1)}$$

- Note that $d_{0s}^l = d_{-s0}^l$, so $d_{0s}^l(z_1) d_{s0}^l(z_2) = d_{0-s}^l(z_1) d_{s0}^l(z_2)$

$$\text{So: } \left(\sum_{s=1}^l + \sum_{s=-l}^{-1} \right) = 2 \sum_{s=1}^l d_{0s}^l(z_1) d_{s0}^l(z_2) \cos(s\phi) \quad \phi = \alpha_2 + \alpha_1$$

Thus:

$$P_l(z) = P_l(z_1) P_l(z_2) + 2 \sum_{s=1}^l d_{0s}^l(z_1) d_{s0}^l(z_2) \cos(s\phi)$$

The $g=g_1 g_2$ anomaly

① I want a graphic interpretation of $g = g_1 g_2$ abstract $R(g) = R(g_1) R(g_2)$ as actual rotations; also, the representations of these rotations: $D^{(l)}_{mn}(g) = D^{(l)}_{ms}(g_1) D^{(l)}_{sn}(g_2) \sim$ ie, a matrix rep.

② I have always found it easiest to think in terms of $R(g)$ referring to the active rotation of an object. From Tinkham, I seem to gather that $D^{(l)}_{mn}(\alpha, \beta, \gamma)$ refers to the rep. of $R(\alpha, \beta, \gamma)$ which is $R_z(\alpha) R_y(\beta) R_z(\gamma)$, so the γ is done first! Also, the second index of D refers to γ , not α .

Example: $D^{(l)}_{00}(\alpha, \beta, \gamma) = \sum_{s=-l}^l D^{(l)}_{0s}(\alpha, \beta, \gamma_1) D^{(l)}_{s0}(\alpha_2, \beta_2, \gamma_2)$

$$\Rightarrow P_l(z) = \sum_s d^{(l)}_{0s}(z_1) e^{-is\gamma_1} e^{-is\alpha_2} d^{(l)}_{s0}(z_2)$$

③ We can in effect solve this equation for z , because $P_1(z) = z$.
Thus:

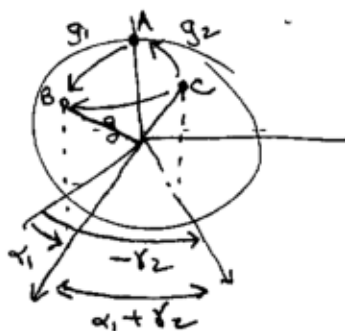
$$z = \sum_{s=-1,0,1} d^{(1)}_{0s}(z_1) d^{(1)}_{s0}(z_2) e^{-is(\gamma_1 + \alpha_2)}$$

But $d^{(1)}_{0s}(z_1) = (-1)^s d^{(1)}_{0,-s}(z_1)$, so $d^{(1)}_{0s}(z_1) d^{(1)}_{s0}(z_2) = d^{(1)}_{0,-s}(z_1) d^{(1)}_{-s,0}(z_2)$
Thus:

$$\begin{aligned} z &= z_1 z_2 + d^{(1)}_{01}(z_1) d^{(1)}_{10}(z_2) \{ e^{-i(\gamma_1 + \alpha_2)} + e^{+i(\gamma_1 + \alpha_2)} \} \\ &= z_1 z_2 + d^{(1)}_{01}(z_1) d^{(1)}_{10}(z_2) \cdot 2 \cos(\gamma_1 + \alpha_2) \\ z &= z_1 z_2 + \sqrt{1-z_1^2} \sqrt{1-z_2^2} \cos(\gamma_1 + \alpha_2), \quad d^{(1)}_{01}(\theta_1) = \frac{1}{\sqrt{2}} \sin \theta_1 \end{aligned}$$

④ Thus, there is no question: $\phi = \gamma_1 + \alpha_2$. How do we show ϕ in a picture?!

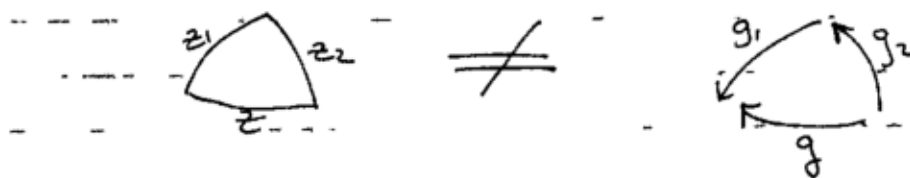
⑤ Consider this picture: we have defined here a certain spherical triangle. Although $\hat{A} \cdot \hat{B} = z_1$ and $\hat{A} \cdot \hat{C} = z_2$, it is not true that $\hat{B} \cdot \hat{C} = z$. In fact,



$$\hat{B} \cdot \hat{C} = z \equiv z_1 z_2 + \sqrt{1-z_1^2} \sqrt{1-z_2^2} \cos(\alpha_1 + \gamma_2)$$

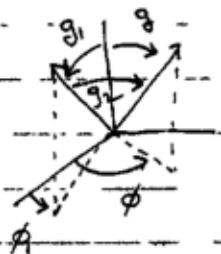
It is clear that $\tilde{z} \neq z$ because $g = R(\alpha, z, \delta)$ is a very messy function of z — z is a rotation about the y -axis + it is not the angle $\tilde{z}$ which appears.

⑥ However, there exists another, different spherical triangle which does display z, z_1, z_2 and the relation in ③. So



DERIVATION OF THE $P_j(z)$ FORMULA.

- ① Imagine 3 rotations g_1, g_2, g such that $g_1 g_2 = g$. Draw them like so:



subtious! see prev. page.

we know that:

$$z_2 = z_1 + \sqrt{1-z_1^2} \sqrt{1-z_2^2} \cos(\phi - \phi_1).$$

So, think of z_2 as the dependent z . Now, we think of

$$g_1 = (\alpha_1, \beta_1, \gamma_1).$$

But, γ_1 does nothing, so identify $\alpha_1 = \phi_1$; $\alpha = \phi$ $\alpha_2 = \phi_2$

- ② Now let $dg = d\alpha d\beta d\gamma$. Then, since $g_2 = g_1^{-1} g$,

$$\int dg_1 D_{00}^j(g_1) D_{00}^{j'}(g_2) = \int dg_1 D_{00}^j(g_1) \int D_{00}^{j'}(g_1^{-1}) D_{00}^{j'}(g)$$

$$= \frac{8\pi^2}{(2j+1)} \delta_{jj'} \times D_{00}^{j'}(g) = \delta_{jj'} \frac{8\pi^2}{2j+1} D_{00}^j(g)$$

But, $\int d\alpha = 2\pi$, so:

$$\int dz_1 d\alpha_1 P_j(z_1) P_j'(z_2) = \delta_{jj'} \frac{4\pi}{2j+1} P_j(z).$$

- ③ Thus, we have shown that:

$$\left\{ \int d\alpha_1 dz_1 P_j(z_1) P_j'(z_2 [\alpha_1, z_1]) = \frac{\delta_{jj'} 4\pi}{2j+1} P_j(z) \right\}$$

Explicit Verification of the Rearrangement Theorem for $O(3)$.

① Show: $\int_G dg_2 f(g_2) = \int_G dg f(\tilde{g}_1^{-1} g)$ where $g, g_2 = g$.

② Do the simplest case. Suppose $f(g) = f(\alpha, \beta, \gamma) = f(\beta)$ only. Then let $dg = d\alpha d\beta d\gamma$. Then want to show:

$$4\pi^2 \int_{-1}^1 dz_2 f(z_2) = \int d\alpha d\gamma \int_{-1}^1 dz f(z_2[\alpha, \gamma, r]).$$

We know that $g = g_1 g_2 \Rightarrow g_2 = \tilde{g}_1^{-1} g$. Thus, the appropriate azimuth in $z_2 = \dots$ is the Toller angle here, namely:

$$(-\alpha_1) + \alpha = \alpha - \alpha_1$$

Thus: ~~$z_2 = z_1 z_2 + \sin \theta \sin \theta_2 \cos(\alpha_1)$~~
 $z_2 =$

③ Well, let's get this right.

$$D(g_2) = D(\tilde{g}_1^{-1}) D(g)$$

$$P_l(z_2) = Z e^{-i(\alpha - \alpha_1)s} d_{00}^l(-\theta) d_{00}^l(+\theta)$$

So:

$$z_2 = z_1 z_2 - \sin(-\theta) \sin(\theta) \cos(\alpha - \alpha_1)$$

$$= z_1 z_2 + \sin \theta \sin \theta \cos(\alpha - \alpha_1)$$

Since $0 < \theta < \pi$, $\sin \theta > 0$, so: $z_2 = z_1 z_2 + \sqrt{1-z_1^2} \sqrt{1-z_2^2} \cos(\alpha - \alpha_1)$.

④ This depends on α but not γ , so:

$$4\pi^2 \int_{-1}^1 dz_2 f(z_2) = ? \quad 2\pi \int_0^{2\pi} d\alpha \int_{-1}^1 dz f(z_1 z_2 + \sqrt{1-z_1^2} \sqrt{1-z_2^2} \cos(\alpha - \alpha_1))$$

Now add $\int_{-1}^1 dz' \delta(z' - \tilde{z}_2)$ where $\tilde{z}_2 = z_1 z_2 + \sqrt{1-z_1^2} \sqrt{1-z_2^2} \cos(\alpha - \alpha_1)$

Thus, RHS:

$$\text{RHS} = 2\pi \int_0^{2\pi} d\alpha \int_{-1}^1 dz \int_{-1}^1 dz' f(z') \delta(z' - \tilde{z}_2)$$

$$= 2\pi \int dz dz' f(z') \underbrace{\int_0^{2\pi} d\alpha \delta(z' - \tilde{z}_2)}_{\frac{2\theta(-k)}{\sqrt{1-k(z', z, z_1)}}} = 4\pi \int dz' f(z') \underbrace{\int dz \frac{\theta(-k)}{\sqrt{1-k(z, z', z_1)}}}_{=\pi}$$

$$= 4\pi^2 \int dz' f(z')$$

⑤ So, for the simplest functions, I have checked the R-theorem. Notice that two non-trivial theorems were needed

- a) integral rep for $\sqrt{1-k}$
- b) z integral over $\sqrt{1-k}$

⑥ What about a general proof.

$$\int d\alpha_2 dz_2 d\gamma_2 f(\alpha_2, z_2, \gamma_2)$$

$$= \int d\alpha d\bar{z} d\bar{\gamma} f(\alpha_2[\dots], z_2[z, z_1, \alpha - \alpha_1], \gamma_2[\dots])$$

We cannot go anywhere without these

① Example: $D_{mn}^L(g_2) = \sum_s D_{ms}(g_1') D_{sn}(g)$

So $D_{0k}(g_2) = \sum_s D_{0s}(g_1') D_{sk}(g)$

$$\Rightarrow \frac{d'_{01}(\beta_2)}{\sqrt{2} \sin \theta_2} e^{-i\gamma_2} = \sum_s e^{-i[\gamma_1]s} d'_{0s}(\beta_1) d'_{s1}(\beta) e^{-is\alpha} e^{-i\gamma}$$

$$\text{Thus } e^{i(\gamma - \gamma_2)} = \frac{\sqrt{2}}{\sqrt{1-z_2^2}} \sum_s d'_{0s}(z_1) d'_{s1}(z) e^{-is(\alpha - \alpha_1)}$$

$\nwarrow \quad z_2 = z_2[z, z_1, \alpha - \alpha_1]$

This gives a real mess for $\mathcal{H}[\gamma, \alpha, z_1, z, \alpha_1]$. In principle, then, one could proceed to show how things work. I content myself with the simplest case only.

Constructing the Representations of $SU(2)$

We know what the generators of $SU(2)$ are, which is to say, we know the Lie Algebra for this group. In order to construct a matrix representation of order (dimension) n , say, we have to find 3 generators of this dimension.

But this problem has already been solved. The elements of the generator matrices are the matrix elements of the usual angular momentum operators between angular momentum eigenstates. We know all about this.

In particular, since J_z is hermitian, those matrices are all hermitian. Also, except for the trivial J_z , there are no diagonal elements. The only non-zero elements are one removed from the diagonal. Thus, to find the 6 -dimensional representation of J_y , you only have to use the formulae below 5 times. In general, $n-1$ times.

$$\begin{aligned} 2 \times (m+1/J_x/m) &= \text{SQRT}[(j-m)(j+m+1)] & k=1 \\ 2 \times (m+1/J_y/m) &= \text{SQRT}[(j-m)(j+m+1)](-i) = (-i) \times \text{the above element.} \\ 2 \times (m/J_z/m) &= 2m \end{aligned}$$

William's phase.

Of course, except for $j = \frac{1}{2}$, the matrix-squared will not be unity and the exponential expansions will be a little rough when you get around to looking for the representations generated by these generators.

Examples:

$$j = \frac{1}{2}; \quad (j-m)(j+m+1) = "(jm)"; \quad (\frac{1}{2}, \frac{1}{2}) = 1; \quad \langle \frac{1}{2} | X | -\frac{1}{2} \rangle = 1 \Rightarrow \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ \Rightarrow J_y = \frac{1}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$j = 1; \quad m = 0 \rightarrow (jm) = (10) = 2 \Rightarrow (1 | J_x | 0) = \sqrt{2} \Rightarrow \\ J_x = \frac{1}{2} \begin{bmatrix} 0 & \sqrt{2} & 0 \\ \sqrt{2} & 0 & \sqrt{2} \\ 0 & \sqrt{2} & 0 \end{bmatrix} \quad m = -1 \rightarrow (1-1) = 2 \Rightarrow (0 | J_x | -1) = \sqrt{2} \\ J_z = \frac{2}{2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad J_x = \frac{\sqrt{2}}{2} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad J_y = (-i) \frac{\sqrt{2}}{2} \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$

$$j = \frac{3}{2}: \quad (\frac{3}{2}, \frac{1}{2}) = 3 \quad (\frac{3}{2}, -\frac{1}{2}) = 4 \quad (\frac{3}{2}, -\frac{3}{2}) = 3 \\ J_x = \frac{1}{2} \begin{bmatrix} 0 & \sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & 2 & 0 \\ 0 & 2 & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{bmatrix}, \quad J_y = \frac{1}{2} \begin{bmatrix} 0 & \sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & 2 & 0 \\ 0 & -2 & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{bmatrix} (-i) \quad J_z = \frac{1}{2} \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix}$$

etc.

$$\begin{aligned} \left. \begin{matrix} (J_x) \\ i(J_y) \\ \hat{J}_x \end{matrix} \right\} &= \frac{1}{2} \left(\sqrt{(j-m)(j+m+1)} \delta_{\mu', \mu+1} + \sqrt{(j+m)(j-m+1)} \delta_{\mu', \mu-1} \right) \\ (J_z)_{\mu', \mu} &= \mu \delta_{\mu', \mu} \end{aligned}$$

● Some Group Theory Details

Let c be some q.n.'s denoting a particular rep, and let $\{X\}$ be a set of m.c.-ing operators. If S is an invariant operator, show that

1. Show that $\langle cx | S | cx' \rangle = \delta_{xx'} S(c, x)$

Start with invariance + end up with desired result:

$$0 = \langle cx | [S, X] | cx' \rangle = (x' - x) \langle cx | S | cx' \rangle$$

$$\Rightarrow \text{either } x' = x \text{ or } \langle cx | S | cx' \rangle = 0 \quad \text{QED.}$$

2. Show that $\langle cx | S | cx \rangle = S(c)$

For any x, x' there exists an operator such that

$$f^\dagger(K) | cx \rangle = | cx' \rangle \quad (\text{think of } J_+^2 \text{ or the like})$$

But, this operator commutes with S , so that

$$\langle cx | S | cx \rangle = \langle cx | f^\dagger S f | cx \rangle = \langle cx' | S | cx' \rangle$$

Thus, $\langle cx | S | cx \rangle$ is independent of x .

3. Show that $\langle cx | S | c'x \rangle = \delta_{cc'} S(c)$

The $\delta_{cc'}$ follows as in 1, since $[C, S] = 0$. But there is no $f \Rightarrow f | c \rangle = | c' \rangle$ for $c \neq c'$ because no operator can change c . This is because all operators commute with C .

$$\begin{aligned} f | c \rangle &= | c' \rangle \\ \Rightarrow C f | c \rangle &= c' | c' \rangle \\ \Rightarrow C c &= c' \end{aligned}$$

Rotation matrices

Orthogonality says:

$$(1) \int dg D_{mn}^j(g) D_{m'n'}^{j*}(g) = \frac{8\pi^2}{2j+1} \delta_{mn} \delta_{m'n'} \delta_{jj'}$$

where

$$\int dg \text{ means } \int_0^{2\pi} d\phi \int_{-1}^1 d(\cos\theta) \int_0^{2\pi} d\psi$$

This is M+S, not Gottfried.

The completeness relation is:

$$(2) \delta(g-g') = \sum_j N_j \sum_{mn} D_{mn}^j(g) D_{mn}^{j*}(g') \quad N_j = \frac{2j+1}{8\pi^2}$$

Notice that each representation contributes a little.

To check (2), integrate over g' to get:

$$1 = \sum_j N_j \sum_{mn} \int dg' D_{mn}^j(g) D_{mn}^{j*}(g')$$

Is this sensible? Instead, mult (1) by $\sum_{j'} N_{j'} \sum_{m'n'} D_{m'n'}^{j'}(g')$.

(1) then says:

$$\int dg \underbrace{\sum_{j'} N_{j'} \sum_{m'n'} D_{m'n'}^{j'}(g') D_{m'n'}^{j'*}(g)}_{\delta(g'-g)} D_{mn}^j(g) = \cancel{N_j} D_{mn}^j(g') \cdot \frac{1}{N_j}$$

$D_{mn}^j(g')$ ✓

Now consider a projection:

$$f_{mn}^j = \int dg D_{mn}^j(g) f(g)$$

To invert, go with $\sum_j N_j \sum_{mn} D_{mn}^{j*}(g')$. Result:

$$\sum_j N_j \sum_{mn} D_{mn}^{j*}(g') f_{mn}^j = \int dg \underbrace{\sum_j N_j \sum_{mn} D_{mn}^{j*}(g') D_{mn}^j(g)}_{\delta(g-g')} f(g)$$

$f(g')$

These are the results:

$$\begin{aligned} f_{mn}^j &= \int dg D_{mn}^j(g) f(g) \\ f(g) &= \sum_j N_j \sum_{mn} D_{mn}^{j*}(g) f_{mn}^j \end{aligned}$$

$\text{soc}(1,1)$

1 First, about $SO(2)$. By definition, this group consists of 2×2 matrices which have real elements, unimodular, and preserve $x^2 + y^2$. The word "rotation" need not be used. All such matrices may, if we like, be parametrized by the obvious angle and we may identify this parameter with a rotation if we like.

2. We may consider $SO(2)$ as a more abstract entity: an abstract group. Then the set of 2×2 matrices which are the original definition happen to form a two-dimensional representation of this abstract group. This Lie group has only one generator, thus the 2-dim rep must be reducible.

3. Suppose we ask: - what are the unitary representations of this abstract Lie group $SO(2)$. Abstractly, these are given by $e^{i\theta J}$, where J is the abstract generator. This abstract generator must be Hermitian, but such a statement, as well as unitarity, only have meaning when referred to some Hilbert space, so we assume the presence of some abstract Hilbert Space.

4. We could, eg, take as our Hilbert space E^2 . The elements of this space are two-vectors (x,y) . Any operator in this space is a 2×2 matrix. Thus, we may speak of unitary two dimensional representations if we can construct J as a hermitian 2×2 matrix. We could take J to be any of the 3 Pauli matrices, or the 2 dim unit matrix. Any choice gives us a unitary two dim rep of the Lie group $SO(2)$.

5. Of course we can form our unitary two dim rep by letting J be any two dim hermitian matrix. However, any such matrix can be formed from a real linear combination of the four matrices already mentioned.

6. Not all of these conform to the original narrow definition of $SO(2)$ as a matrix group. Eg, if you take J = unit matrix, then the elements are not real, and $\det \neq 1$. Nothing is wrong with this, but we just mention the old starting definition. Similarly, any Pauli matrix as J will give a $\det = 1$ rep (because Pauli's are traceless), but only " J_2 " gives 2×2 matrices with real elements.

7. Notice that the original definition of the group serves to provide us with the parametrization. Then we generalize, keeping the parametrization.

8. Clearly one can make a three dim rep unitaire in many ways. One could insist that such a rep have real elements and have $\det=1$. Still there would be more than one possibility.

9. As an alternative Hilbert space, one could take the single-valued functions defined on the unit circle, ie, functions of θ , the parameter. The scalar product is the obvious one. A potential realization of the hermitian generator would be $d/d\theta$. Such a Hilbert space is infinite dimensional unlike the ones considered above. But, if one restricts to functions of the form $e^{i m \theta}$, then one has a subspace that is invariant under the one generator. In order that such a function be single-valued, we must have $m = \text{integer}$. These functions are eigenfunctions of the generator in the rep, and they form a basis for the full hilbert space.

10. If we go into one of these subspaces, say the one labelled by m , we may compute the matrix element. It is just a single number. Thus, this UIR is one dimensional. In fact, the UIR's are $e^{im\theta}$ for group $SO(2)$. Notice that the huge $\inf \dim \text{hil}$ space contained each UIR one time.

11. If we project some function $f(\theta)$ onto these UIR's and also recover, we are talking about fourier series analysis.

12. Now lets try to repeat all these statements for the group $SO(1,1)$. By the formal definition, the matrix group is 2×2 matrices, real, $\det=1$, which preserve $x^2 - y^2$. Clearly, the way to parametrize this is $\cosh u, \sinh u$, etc where u runs over the entire real line. We may then, if we like, think of this as a boost.
13. Now extend to general lie group $SO(1,1)$. The rep above is just a 2 dimension rep of $SO(1,1)$ which happens to have real elements and $\det=1$. Obviously this two-dim rep is not unitary (transpose and star does not give inverse) and this we know in general because all finite reps of a noncompact group must in fact be non-unitary.
14. Next: what are the unitary reps of $SO(1,1)$? Abstractly, these are e^{uK} where u is on real axis (u =the parameter already know) and K must be an anti-hermitean operator in some Hilbert space. In the Hilbert Space $= E^2$, we may use any antihermitean matrix as generator. Traceless ones will give $\det=1$ reps (specials). Only if we use $K = iJ_1$ do we get the real rep of the original group $SO(1,1)$. Easy to make, say, an E^4 rep of $SO(1,1)$. Just put unity in the unused sectors.
15. We can construct an infinite dim Hilbert space consisting of functions of the parameter u in L^2 , ie, square integrable on the real line u . Again we use the obvious scalar product. We can choose to represent the anti-herm generator $K = d/du$. The eigenfunctions of K are just e^{su} with eigenvalue s . Off hand, I would say that the eigenvalues of an antiherm op must be pure imaginary, so we take $s = ik$ with k real, so fine. The functions e^{su} presume form a basis as s ranges over the im axis for this H space. Of course, if you restrict to the function e^{su} this forms an invariant subspace and suggests an irreducible rep. The larger H space contained all the UIR's once. The little hilbert space consists of just one function, e^{su} , labeled by s . There is now no single valued problem so s does not get quantized.
16. However, e^{su} must not be irreducible because then you would have a UIR of dimension one for a noncompact group which is no go. Don't know why this can be reduced. Lets call these guys: UR's of dim one of $SO(1,1)$.
17. If we project some function $f(u)$ onto these UR's, we get $F(k)$ and the corresponding recovery. This is the fourier integral analysis. Here k is real and we are talking:

$$\begin{aligned} \int_{-\infty}^{\infty} du f(u) e^{-iku} &= F(k) & 2\pi f(u) &= \int_{-\infty}^{\infty} dk F(k) e^{+iku} \\ \text{OR} & & & \\ \int_{-\infty}^{\infty} du f(u) e^{-su} &= \tilde{F}(s) & 2\pi f(u) &= \int_{-\infty}^{\infty} ds \tilde{F}(s) e^{su} \end{aligned}$$

When we write these things for UR's, we have in mind that k =real or s =imag. But in general no reason not to let s go complex, as long as things converge.

18. If we for some reason restrict this last transform to functions which by construction vanish for $u \rightarrow \infty$, we get the Laplace Transform. Then $F(s)$ is analytic suff far to the right in s -plane. For $u \rightarrow \infty$, we close inversion contour to right and recover this fact, that $f(u)$ vanishes there.
19. Finally, by changing variables from u to $x = e^u$, we get the Mellin Transform:

$$\int_0^{\infty} dx x^{-s-1} f(x) = \tilde{F}(s) \quad 2\pi \tilde{f}(x) = \int_{-\infty}^{\infty} ds x^s \tilde{F}(s)$$

Again, if we restrict $f(u)$ as before, the lower limit of the Mellin projection becomes 1 instead of 0. This allows more functions to be projectable, as we already know from the fourier to Laplace.

The two separate sectors of $SO(1,1)$ and the discrete index.

Recall that by definition the elements of $SO(1,1)$ can be parametrized in this way, where u runs over the real axis:

$$U(g) = \begin{pmatrix} \cosh u & \sinh u \\ \sinh u & \cosh u \end{pmatrix} \quad \det U = 1 \\ -\infty < u < \infty$$

It suddenly occurs to me that all elements of $SO(1,1)$ are not reachable via this particular param. In fact, $so(1,1)$ is a multi-sectored group. The elements of the other sector are given by:

$$\tilde{U}(g) = \begin{pmatrix} -\cosh u & -\sinh u \\ -\sinh u & -\cosh u \end{pmatrix} = -U(g).$$

Now it is eminently clear why one needs "the real line twice" to parametrize this group. Notice that both forms (with either sign) are $SO(1,1)$ because both preserve $x^2 - y^2$.

Recall the single hyperbolic generator K . To get unitary reps we needed K anti-hermitian, but let's forget that. Let's take K hermitian so we will be talking anti-unitary reps (of one dimension). Then, we do not (I don't think) need discrete labels in the eigenvectors, but we do have to use two separate params to get the matrix elements:

$$\mathbb{R} \quad K|x\rangle = x|x\rangle \quad x \in \mathbb{R}$$

$$D_+^x(u) = \langle x | e^{uK} | x \rangle = + e^{ux} \quad x \in \mathbb{R}, u \in \mathbb{R}$$

$$D_-^x(u) = \langle x | -e^{uK} | x \rangle = - e^{ux}$$

So complete anti-unitary wired? matrix elements are

$$D_a^x(u) = a \cdot e^{ux} \quad a = +, -$$

These are the one dim reps of $SO(1,1)$. The $+$ part is really the group of positive real numbers under multiplication, and the $-$ part is the negative reals under same. The full $SO(1,1)$ is the sum of these two pieces, i.e., all reals under mult.

9/96

Comments on the groups $SO(2)$ and $SO(1,1)$

1. In the pages that follow we have explicitly constructed two finite-dimensional unitary representations of $SO(2)$. Of course since this is an abelian group, only the one-dimensional unitary rep. is a UIR. The two-dim rep can be brought into block form by a similarity and then it is the direct product of two one dimensional reps. We have also constructed a rep of $SO(2)$ as operators in a Hilbert Space with an infinite discrete basis. This is indeed an infinite-dimensional representation unitaire of $SO(2)$, but looking at the matrix elements it is clearly highly reducible to a direct product of an infinite number of one-dimensional UIR's. The one-dim UIR is also known as $SU(1)$.
2. Next, we considered the group $SO(1,1)$ and found that this group has two parts only one of which contains the identity. Calling that part $SO(1,1)^+$, we were again able to construct representations in 1 and 2 dimensions. Neither of these reps was unitary, but no doubt the 2-dim rep was reducible to a pairs of ones, because the IR's of an abelian group are all of one-dimension. Then we proceeded to construct the first unitary rep of $SO(1,1)$ as operators in a Hilbert Space which an infinite and continuous basis. Probably this can be decomposed into a continuous direct sum of the one-dim reps, but again those are non-unitary because the generator is anti-hermitian. Thus you are stuck with your only UIR being of infinite dimension. In some sense this infinite dim UIR is really Irreducible in the correct sense. The one-dim IR of $SO(1,1)$ we also called R^+ the group of positive real numbers under multiplication. Clearly this group also translates functions so is also called $T(1)$. The fact that $SO(1,1)$ has the two "halves" introduces the notion of "parity", ~~xxx~~ by the way.

This type of analysis would be worth while for the higher groups. When we deal with $SO(2,1)$, we have really been dealing only with the $SO(2,1)^+$ part and have been ignoring the question of parity.

I do not think that this index on $SO(1,1)$ is that same index on the C^q matrix elements of $SO(2,1)_0$, but very tempting to associate them. Later!

Understanding SU(1) and SO(2).

① The elements of SO(2) are $\begin{pmatrix} C_\phi & -S_\phi \\ S_\phi & C_\phi \end{pmatrix} = e^{-i\phi J_3}$ $J_3 = (\sigma_y)$.

Proof:
$$e^{-i\phi \sigma_y} = e^{(-i\phi) \sigma_y} = \cosh(-i\phi) + \frac{1}{(-i\phi)} \sinh(-i\phi) \cdot [-i\phi \cdot \sigma_y]$$

since $\alpha = \sqrt{-1} = \alpha_2 = -i\phi$ $\therefore = \cos\phi - i \sin\phi \sigma_y$

$$= \begin{pmatrix} C_\phi & 0 \\ 0 & C_\phi \end{pmatrix} + \begin{pmatrix} 0 & -S_\phi \\ S_\phi & 0 \end{pmatrix} \checkmark$$

② Let $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ be two vector. Then let

$$x'^\mu = \Lambda^\mu_\nu x^\nu \text{ where } \Lambda^\mu_\nu = (e^{-i\phi J_3})^\mu_\nu = \begin{pmatrix} C_\phi & -S_\phi \\ S_\phi & C_\phi \end{pmatrix}.$$

$$\text{Let } X = \sigma_1 x_1 + \sigma_2 x_2 = \begin{pmatrix} 0 & x - iy \\ x + iy & 0 \end{pmatrix}.$$

Not: $\det(X) = -x^2 - y^2.$

If we use $X' = g X g^\dagger$, then g must be 2×2 , don't want this.

③ How about $X = \text{a number}$? Then $X' = g X g^\dagger \Rightarrow X' = g g^\dagger X$
 $\Rightarrow X' = X$ since g unitary $\Rightarrow$ no damn good.

④ Suppose we take $J_3 = 1$ as generator in SU(1). Then

$$g \in \text{SU}(1) \Rightarrow g = e^{-i\phi J_3} = e^{-i\phi} = \text{a number.} \\ = 1 \times 1 \text{ matrix.}$$

⑤ OK, now what is spectrum of J_3 ? First get a shift η

$$U(\phi) f(\phi) = f(\phi + \phi_1)$$

$$e^{-i\phi_1 J_3} f(\phi) = f(\phi + \phi_1);$$

~~But 2×2 matrix~~

$$[1 - i\phi_1 J_3] f(\phi) = f(\phi) + \phi_1 \frac{\partial}{\partial \phi}.$$

$$\Rightarrow \boxed{J_3 = i \frac{\partial}{\partial \phi}}$$

⑥ What is rep space? Any $f(\phi)$? ~~To be a~~ ...
~~single-valued representation of $SU(1)$~~ Sure.

Spectrum: reals: So, eigenfcts are

$$e^{-i\phi m} \quad m = \text{real}$$

$$J_3 e^{-i\phi m} = m e^{-i\phi m}$$

~~Thats it~~ In order to get a UIR, need J_3 hermitian wrt scalar product. What SP to choose?

$$(f, g) = \int_{-\infty}^{\infty} d\phi f(\phi)^* g(\phi) \quad ?$$

Then hermiticity of $J_3 \Rightarrow f$ in L_2 . OK. Then matrix elements:

$$\langle m | e^{-i\phi J_3} | m' \rangle = e^{-i\phi m} \delta(m - m')$$

⑦ Is this OK? The representations are not single-valued ∇ reps of $SU(1)$ because $\phi \rightarrow \phi + 2\pi \nrightarrow$ same m , but $\Rightarrow$ same $SU(1)$.

⑧ Thus, forced to take $\int_0^{2\pi} d\phi$ as SC, make $f(\phi)$ periodic, then spectrum of m is integers. no half-integers

⑨ Map $SU(1) \rightarrow SO(2)$ is isomorphism ||:

Understanding T(1) and SO(1,1).

① The $SO(2)$ matrices were "special orthogonal" in order to preserve $x^2 + y^2$. The $SO(1,1)$ preserve $x^2 - y^2$. Why? By def. also.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax+by \\ cx+dy \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix}. \quad x'^2 - y'^2 = (ax+by)^2 - (cx+dy)^2 \\ = (a^2 - c^2)x^2 + (b^2 - d^2)y^2 + 2(ab - cd)xy.$$

So we want $ab - cd = 0$ ✓ $ad - bc = 1$ ✓
 $a^2 - c^2 = 1$ ✓
 $b^2 - d^2 = -1$ ✓

Solve these: $a^2 - c^2 = 1$ ✓ $b^2 - d^2 = -1$ ✓ $(a^2 - 1)(b^2 + 1) = a^2 b^2 - b^2 + a^2 - 1$ ✓
 $\Rightarrow a^2 - b^2 = 1$ ✓ $\Rightarrow b^2 = c^2 \Rightarrow b = \pm c$ ✓
 $a^2 - c^2 = 1$ ✓

Assume $b = c$. Then $ab = cd \Rightarrow a = d$.
 $a^2 - c^2 = 1$, $d^2 - b^2 = 1$ are the same; $ad - bc = 1$ ✓
 $\Rightarrow a^2 - c^2 = 1 \Rightarrow a = \cosh, c = \sinh$.

So $\begin{pmatrix} \cosh & \sinh \\ \sinh & \cosh \end{pmatrix}$.

Assume $b = -c$. Then $ab = cd \Rightarrow a = -d$.
 $ad - bc = 1 \Rightarrow -a^2 + b^2 = 1 \Rightarrow b = \cosh, a = \sinh$ ✓
 $\Rightarrow a^2 - b^2 = -1$ ✓ $\Rightarrow$ contradiction ✓
 So two distinct possibilities.

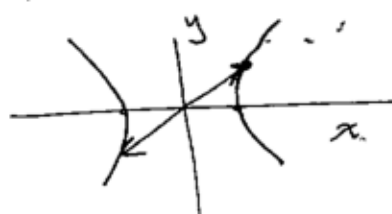
Also, $b = c \Rightarrow \begin{pmatrix} -\cosh & \sinh \\ \sinh & -\cosh \end{pmatrix}$ and $b = -c \Rightarrow \begin{pmatrix} \sinh & -\cosh \\ \cosh & -\sinh \end{pmatrix}$.

What are all these dudes?

$$\begin{pmatrix} \cosh & \sinh \\ \sinh & \cosh \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cosh x + \sinh y \\ \sinh x + \cosh y \end{pmatrix} \quad \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cosh^2 x + \sinh^2 y + 2\cosh x \sinh y \\ \sinh^2 x + \cosh^2 y + 2\sinh x \cosh y \end{pmatrix} \\ = x^2 (1 - c^2) + \dots \quad \text{Kuh?!?}$$

② OK, general matrix is $\begin{pmatrix} \cosh & \sinh \\ \sinh & \cosh \end{pmatrix}$ - $-\infty < \xi < \infty$

but also, $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ times this!



$x^2 - y^2 = \text{const}$ is hyperbola.

If const > 0, then $y = 0 \Rightarrow x = \pm \sqrt{\text{const}}$
 if $x = 0$, no go

The $\begin{pmatrix} \cosh & \sinh \\ \sinh & \cosh \end{pmatrix}$ more a vector

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ -y \end{pmatrix} = \text{parity}$$

What is this two-fold mess all about?? $SO(2)$ does not have this effect.

The "surface of motion" has two pieces.

But $O(1,1)$ has four pieces. Hence

$L_+^\uparrow =$ the $SO(1,1)$ with $\Lambda_0 > 1$, so do speak.

$O(1,1)$	4 pieces
$SO(1,1)$	2 pieces
one piece $\supset$ identity	

② So we will use $L_+^\uparrow$ piece.

$$e^{-i\epsilon K} = \begin{pmatrix} \cosh \epsilon & \sinh \epsilon \\ \sinh \epsilon & \cosh \epsilon \end{pmatrix}.$$

This is really $SO(1,1; \mathbb{R})^\uparrow$. Clearly, $K = 2K_1 = i\sigma_1$, anti-hermitean.

$$g \in SO(1,1)^\uparrow \Rightarrow g = e^{-i\epsilon K} = \begin{pmatrix} \cosh \epsilon & \sinh \epsilon \\ \sinh \epsilon & \cosh \epsilon \end{pmatrix}$$

and $K = i\sigma_1$.

③ Set up right shift rep:

$$U(s)f(s) = f(s+s)$$

$$f(s) \quad -\infty < s < \infty.$$

L_2

Then $e^{-i\epsilon \vec{K}} f(s) = f(s+\epsilon)$

$$-i\epsilon K = \epsilon \frac{\partial}{\partial s} \Rightarrow K = i \frac{\partial}{\partial s}$$

④ This is hermitean on L_2 with $(g, f) = \int_{-\infty}^{\infty} ds \bar{g} f$.

⑤ Spectrum: for K hermitean, $K|p\rangle = p|K\rangle$.

So $|p\rangle \rightarrow e^{-i\epsilon p} \Rightarrow (i \frac{\partial}{\partial s}) \rightarrow p \Rightarrow |p\rangle = e^{-\mu s}$.

Then invariant,

$$\langle \mu | e^{-i\epsilon K_1} | \mu' \rangle = e^{-\mu \epsilon} \delta(i\mu - i\mu')$$

⑥ Meanwhile, element of $\mathbb{R}^+$ is $\mathbb{R}^+$ ~ what is the corresp. OPLG?

Must be $SU(1)$ again: $e^{-i\phi}$. But then $p = \text{integers}$?

Before we said $g = e^{-i\phi}$ in $SU(1)$, so periodic 2π .

How about $R^+ = \mathbb{R}^+ = \text{real positive numbers}$.

Then $e^{-i\phi K} = e^{-\phi} \Rightarrow K = \frac{1}{i} = (-i)$.

So:

$$\begin{aligned} g \in \mathbb{R}^+ &\Rightarrow g = e^{-\phi} \quad -\infty < \phi < \infty \\ g &= e^{-i\phi K} \Rightarrow K = (-i) = \text{antihermitean} \\ &\Rightarrow g(\phi)^\dagger = g(\phi) \end{aligned}$$

⑦ Now, the UIR matrix elements: $e^{-i\phi p}$, no reason for p to be integer.