

$$K(z_1, z_1, z_2)$$

Phil Lucht

The Kinematic k -function and Related Formulae.

Contents:

1. Introduction
2. Geometric Interpretation
 - 2.1 Elliptic Case
 - 2.2 Hyperbolic Case
 - 2.3 Terminology
3. Some $\sqrt{-k}$ formulae.
 - 3.1 the $\sqrt{-k}$ integral representation.
 - 3.2 some integrals involving $\sqrt{-k}$
 - 3.2.1 The case $f(z) = 1/(z-a)$
 - 3.2.2 The case $f(z) = P_L(z)$
 - 3.3 diagonalizing elastic unitarity the hard way
 - 3.4 a sum representation for $\sqrt{-k}$
4. Some $\sqrt{+k}$ formulae.
 - 4.1 the $\sqrt{+k}$ integral representations
 - 4.2 some integrals involving $\sqrt{+k}$
 - 4.2.1 The case $f(z) = 1/(z-a)$
 - 4.3 relation of 4.2.1 to a famous integral
 - 4.4 the $Q_L(z)$ separation integral

1. Introduction The "k-function",

$$k(x, y, z) = x^2 + y^2 + z^2 - 2xyz - 1$$

is one of a small set of functions which appear quite commonly in particle kinematics calculations. Another of these unsung heroes of the kinematic high wire is the more commonplace "triangle function",

$$\Delta(x, y, z) = x^2 + y^2 + z^2 - 2xy - 2xz - 2yz,$$

so called because the area of a triangle of sides a, b , and c is equal to $\frac{1}{4} \sqrt{\Delta(a^2, b^2, c^2)}$. There seems to be no such glamorous interpretation for $k(x, y, z)$, so we just call it "the k-function."

Considered as a function of one of its variables, the k-function is just a non-descript quadratic form, hardly worthy of consideration. Nevertheless, its persistence in appearing all over the place~~s~~ has motivated the author to collect several formulas involving the k-function into one place.

None of the formulas presented here is new. In fact, most of them are given in an appendix of a 1962 paper of de Alfaro, Regge and Rossetti [1]. Some also appear in the brief appendix of reference (2). On the other hand, one does not find these results in such comprehensive works as Bateman (3) and Gradshteyn and Ryzhik (4).

2. Geometric Interpretation. The k-function is of course symmetric in its three arguments, which are taken here to be real, but we choose one argument z to classify things below. One is always interested in where the k-function vanishes, and where it is positive or negative. This information is best summarized graphically.

$$k(z, z_1, z_2) = z^2 + z_1^2 + z_2^2 - 2zz_1 - 2zz_2 - 1$$

2.1 the elliptic case. For fixed z , $|z| < 1$, the boundary $k(z, z_1, z_2) = 0$ appears as an ellipse in the z_1 - z_2 plane. This ellipse is rotated 45° relative to the axes and is inscribed in the "unit square" whose edge is actually two units. ~~XXXXXXXXXXXXXXXXXXXX~~ k is negative inside the ellipse, positive outside. In the figure (1) we have drawn this ellipse for $0 < z < 1$. Obviously it tilts the other way if $-1 < z < 0$. When $z=0$ we have a unit circle.

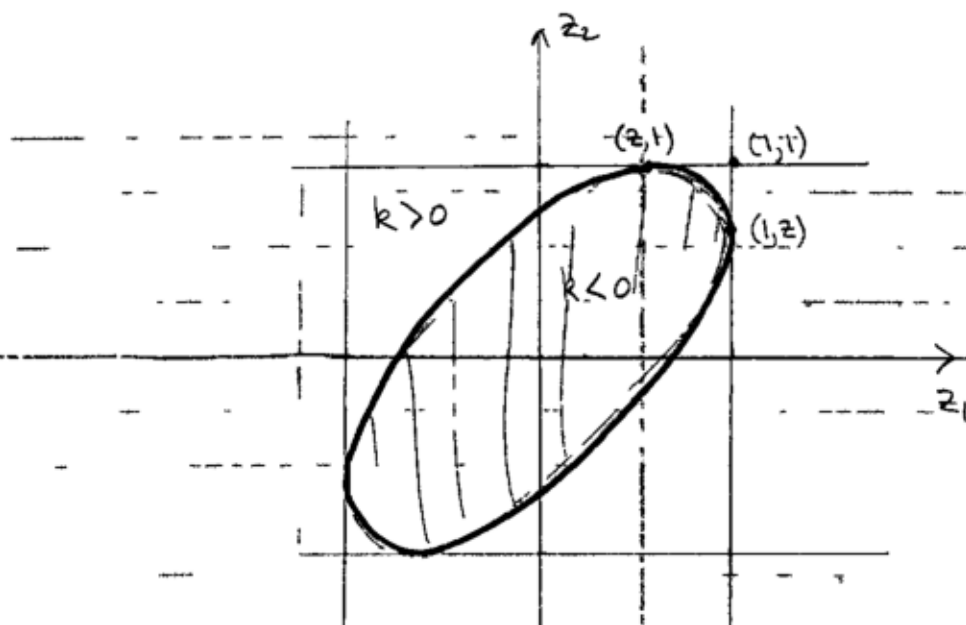


Figure 1

For fixed z , $|z| > 1$,

2.2 the hyperbolic case. Now the boundary $k(z, z_1, z_2) = 0$ is a hyperbola in the z_1 - z_2 plane as shown below. k is negative inside the hyperbola, positive outside. The sketch is shown for $z > 1$. For $z < -1$ the hyperbola tilts the other way. at $z = 1 + \epsilon$, the hyperbola is very thin and nearly touches the unit box, and as z goes to $1 - \epsilon$, it transforms into a thin ellipse inside the box, as above.

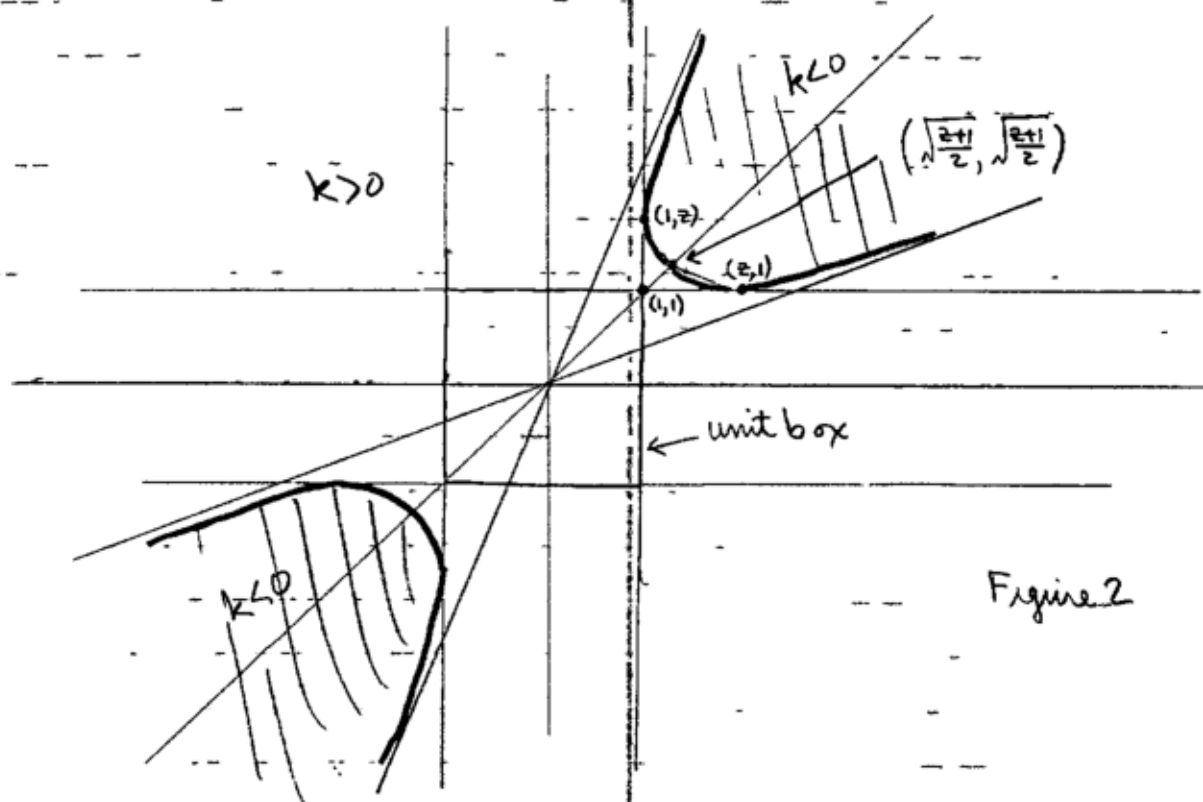
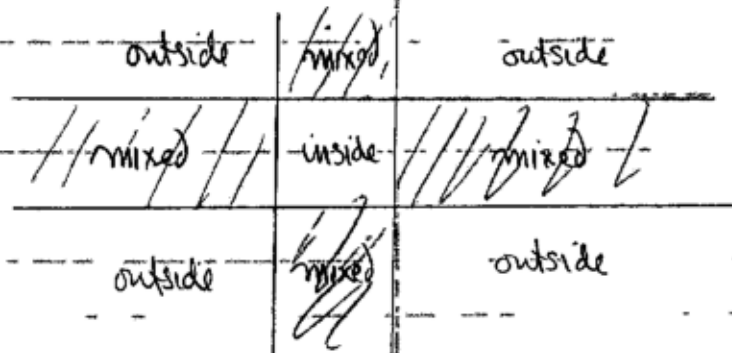


Figure 2

2.3 terminology Let us define some special terminology to describe the location of a point (z_1, z_2) in the z_1 - z_2 plane. We will say that (z_1, z_2) is "inside" if both $|z_1| \leq 1$ and $|z_2| \leq 1$. If $|z_1| > 1$ and $|z_2| > 1$, then (z_1, z_2) is "outside". Otherwise, (z_1, z_2) is "mixed." These regions are shown below.



3. Some $\sqrt{-k}$ formulas.

3.1 The $\sqrt{-k}$ Integral Representation. The usual "elastic unitarity" integral has the form,

$$a(z) = \int d\Omega'' b(z'') c(\tilde{z}') \quad (1)$$

where,

$$\tilde{z}' = f(z, \Omega'') = z z'' + \sqrt{(1-z^2)(1-z''^2)} \cos \phi''$$

By introducing $1 = \int_0^{2\pi} d\phi'' \delta(z' - \tilde{z}')$ one gets,

$$a(z) = \int_0^{2\pi} d\phi'' \int_0^{2\pi} d\phi' b(z'') c(z') \cdot \int_0^{2\pi} d\phi'' \delta(z' - \tilde{z}'). \quad (2)$$

We identify this last integral as a jacobian, but evaluate it directly:

$$J = \frac{\partial(\phi'')}{\partial(z')} = \int_0^{2\pi} d\phi'' \delta(z' - \tilde{z}')$$

$$= 2 \int_0^{\pi} d\phi'' \delta(z' - z z'' - \sqrt{(1-z^2)(1-z''^2)} \cos \phi'')$$

$$= \frac{2}{\sqrt{(1-z^2)(1-z''^2)}} \int_0^{\pi} \frac{d\phi''}{|\sin \phi''|} \delta(\phi'' - \tilde{\phi}'') = \frac{2 \Theta(|\cos \tilde{\phi}''| - 1)}{\sqrt{(1-z^2)(1-z''^2)} |\sin \tilde{\phi}''|}$$

where $\cos \tilde{\phi}'' = \frac{(z' - z z'')}{\sqrt{(1-z^2)(1-z''^2)}}$

A check shows that,

$$\Theta(|\cos \tilde{\phi}''| - 1) = \Theta(-k(z, z', z''))$$

$$\sqrt{(1-z^2)(1-z''^2)} |\sin \tilde{\phi}''| = \sqrt{-k(z, z', z'')}$$

Thus, we arrive at the following "integral representation" for $\sqrt{-k}$:

$$\frac{\Theta(-k(z, z_1, z_2))}{\sqrt{-k(z, z_1, z_2)}} = \int_0^\pi d\phi \delta(z + z_1 z_2 - \sqrt{(1-z_1^2)(1-z_2^2)} \cos \phi). \quad (3)$$

This derivation is valid for (z_1, z_2) either inside or outside, where the square-root is real. For (z_1, z_2) mixed, both sides vanish; the left side due to $\Theta(-k)$, and the right side because the square root goes imaginary and the delta function is not "hit." Thus, the integral representation ⁽³⁾ above is valid for all real z, z_1, z_2 . Two other equivalent integral representations are obtained by permuting the z 's in the integral. This is so because the integral is actually symmetric in the z 's, as the left side shows, together with the fact that the representation is valid for all z, z_1 and z_2 . Finally, we point out that (3) is also true with a plus sign in front of the square root on the right side.

3.2 Some integrals involving $\sqrt{-k}$. We shall be interested in certain integrals of the form,

$$I(z_1, z_2) = \int_{-\infty}^{\infty} dz f(z) \frac{\Theta(-k(z, z_1, z_2))}{\sqrt{-k(z, z_1, z_2)}}. \quad (4)$$

According to our discussion of k , the segments of the z integration which actually contribute to the z -integration are as shown below (now in the z - z_1 plane):
as solid lines.

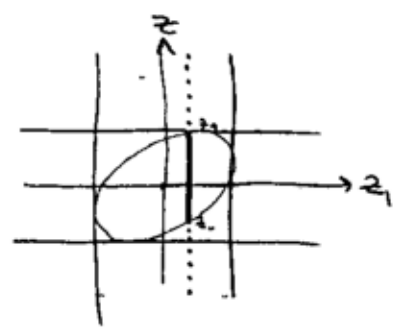


figure 3

(z_1, z_2) inside (shown: $z_1 z_2 > 0$)

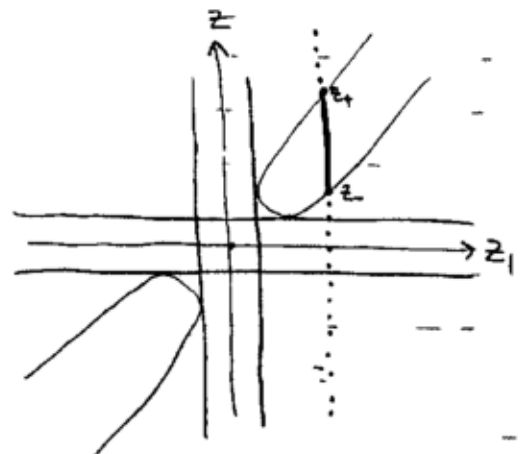


figure 4

(z_1, z_2) outside (shown: $z_1 z_2 > 1$)

The explicit endpoints $z_{\pm}$ are obtained from

$$k(z, z_1, z_2) = (z - z_+)(z - z_-)$$

$$z_{\pm} = z_1, z_2 \pm \sqrt{(1 - z_1^2)(1 - z_2^2)} \quad z_+ > z_-$$

For the case (z_1, z_2) mixed, the integral (4) vanishes because it has no supporting segments at all. In the other cases we have, by insertion of (3) into (4),

$$I(z, z_2) = \int_0^{\pi} d\phi f(\tilde{z}) \quad (5)$$

$$\text{where } \tilde{z} = z_1, z_2 \pm \sqrt{(1 - z_1^2)(1 - z_2^2)} \cos \phi$$

3.2.1 The case $f(z) = (z - a)^{-1}$ With the help of (5) and

$$\int_0^{\pi} \frac{dx}{a + b \cos x} = \frac{\pi}{\sqrt{a^2 - b^2}}$$

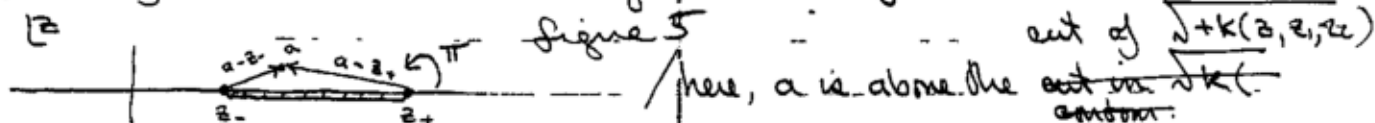
one may quickly show that

$$\int_{-\infty}^{\infty} \frac{dz}{z - a} \frac{\Theta(-k(z, z_1, z_2))}{\sqrt{-k(z, z_1, z_2)}} = \frac{\pi}{\sqrt{+k(a, z_1, z_2)}} \quad (6)$$

If the pole $z = a$ happens to lie on the integration contour, $z_- < a < z_+$, then $k(a, z_1, z_2)$ is negative and we get

$$\begin{aligned} \sqrt{k(a, z_1, z_2)} &= \sqrt{(a - z_+)(a - z_-)} \\ &= \sqrt{|a - z_+| e^{i\pi}} |a - z_-| e^{i0} \\ &= (\pm i) \sqrt{-k(a, z_1, z_2)}, \end{aligned}$$

The sign depending on whether a lies just above or just below the contour:



So, when $z_- < a < z_+$, we have:

$$\int_{-\infty}^{\infty} \frac{dz}{z-a} \frac{\Theta(-k(z, z_1, z_2))}{\sqrt{-k(z, z_1, z_2)}} = -i\pi \frac{1}{\sqrt{-k(a, z_1, z_2)}} \quad (7)$$

3.2.2. The case $f(z) = P_l(z)$: the P separation ^{integral} theorem.

We start with

$$I(z_1, z_2) \equiv \int_{-\infty}^{\infty} dz P_l(z) \frac{\Theta(-k(z, z_1, z_2))}{\sqrt{-k(z, z_1, z_2)}} = \int_0^\pi d\phi P_l(\tilde{z})$$

by equation (5). We now insert the usual Legendre addition theorem which has the form:

$$P_l(\tilde{z}) = P_l(z_1)P_l(z_2) + \sum_{m=1}^{\infty} f(m, z_1, z_2) \cos m\phi \quad (8)$$

which is valid, by the way, for arbitrary complex l . Only the leading term survives the ϕ integration, so

$$I(z_1, z_2) = \pi P_l(z_1) P_l(z_2). \quad (9)$$

We quote the result in two special cases: [see reference 5, p595].

$$(10) \quad \oint_{-1}^1 dz P_l(z) \frac{\Theta(-k(z, z_1, z_2))}{\sqrt{-k(z, z_1, z_2)}} = P_l(z_1) P_l(z_2) \quad (z_1, z_2) \text{ inside}$$

$$(11) \quad \int_1^\infty dz P_l(z) \frac{\Theta(-k(z, z_1, z_2))}{\sqrt{-k(z, z_1, z_2)}} = P_l(z_1) P_l(z_2) \quad z_1, z_2 > 1$$

3.3 Diagonalizing Elastic Unitarity, the hard way.

In section 3.1 we showed that elastic unitarity had

The sum

$$a(z) = \int d\Omega'' b(z'') c(z') = \int_{-1}^1 dz'' b(z'') \int_{-1}^1 dz' c(z') \frac{2 \Theta(-k(z, z', z''))}{\sqrt{-k(z, z', z'')}}.$$

Applying $\int_{-1}^1 dz P_\ell(z)$ to both sides and using our result (10) we find:

$$\left[\int_{-1}^1 dz P_\ell(z) a(z) \right] = 2\pi \left[\int_{-1}^1 dz'' P_\ell(z'') b(z'') \right] \times \left[\int_{-1}^1 dz' P_\ell(z') c(z') \right].$$

3.4 Another representation for $\sqrt{-k}$. The P_ℓ -separation integral (10) can be inverted by applying $\sum_{\ell} (2\ell+1) P_\ell(z_0)$ to both sides and by using the completeness of the Legendre polynomials,

$$\sum_{\ell=0}^{\infty} (2\ell+1) P_\ell(z_0) P_\ell(z) = 2 \delta(z_0 - z).$$

Thus we obtain [5]

$$(12) \quad \frac{1}{\sqrt{-k(z_0, z_1, z_2)}} = \frac{\pi}{2} \sum_{\ell=0}^{\infty} (2\ell+1) P_\ell(z_0) P_\ell(z_1) P_\ell(z_2),$$

valid for $-1 \leq z_0, z_1, z_2 \leq 1$ and $k(z_0, z_1, z_2) \leq 0$. (See figure 1).

Equation (12) is valid only for $-1 \leq z_0, z_1, z_2 \leq 1$ and $k(z_0, z_1, z_2) \leq 0$. (See ellipse in figure 1).

4. Some $\sqrt{+k}$ Formulas.

4.1. The $\sqrt{+k}$ Integral Representations. The integral representations for $\sqrt{+k}$ analogous to (3) are as follows:

$$\frac{1}{2} \int_{-\infty}^{\infty} d\phi \delta(z - z_1 z_2 - \sqrt{(z_1^2 - 1)(z_2^2 - 1)}) d\phi = \frac{\Theta(z - z_+)}{\sqrt{k(z, z_1, z_2)}} \quad (13)$$

$$\frac{1}{2} \int_{-\infty}^{\infty} d\phi \delta(z - z_1 z_2 + \sqrt{(z_1^2 - 1)(z_2^2 - 1)}) d\phi = \frac{\Theta(z - z_-)}{\sqrt{k(z, z_1, z_2)}} \quad (14)$$

Adding these Addition of these equations yields

$$\frac{\Theta(+k(z, z_1, z_2))}{\sqrt{+k(z, z_1, z_2)}} = \frac{1}{2} \sum_z \int_{-\infty}^{\infty} d\phi \delta(z - \tilde{z}) \quad (15)$$

where $\tilde{z} = z_1 z_2 + \tau \sqrt{(z_1^2 - 1)(z_2^2 - 1)} d\phi$. Unlike the case (3), these three representations are only valid for (z_1, z_2) unmixed.

4.2 Some $\sqrt{+k}$ integrals. These last results may be used to reduce certain integrals to simpler form, just as in section 3.2. Eg,

$$I(z_1, z_2) \equiv \int_{-\infty}^{\infty} dz f(z) \frac{\Theta(+k(z, z_1, z_2))}{\sqrt{+k(z, z_1, z_2)}} = \frac{1}{2} \sum_z \int_{-\infty}^{\infty} d\phi f(\tilde{z})$$

We can draw figures like 3 and 4 showing the contributing integration segments in the (z, z_1) plane; the active segments are just the complements of those shown in earlier.

Below, we shall consider a few examples using only the result (13). That is,

$$I(z_1, z_2) \equiv \int_{-\infty}^{\infty} dz f(z) \frac{\Theta(z - z_+)}{\sqrt{+k(z, z_1, z_2)}} = \frac{1}{2} \int_{-\infty}^{\infty} d\phi f(\tilde{z})$$

where $\tilde{z} = z_1 z_2 + \sqrt{(z_1^2 - 1)(z_2^2 - 1)} \cos \phi$.

4.2.1 The case $f(z) = (z-a)^{-1}$

By direct calculation using

$$\int_{-\infty}^{\infty} \frac{dx}{a+b \cosh x} = \frac{1}{\sqrt{a^2-b^2}} \ln \left[\frac{a + \sqrt{a^2-b^2}}{a - \sqrt{a^2-b^2}} \right]$$

one finds that

$$(16) \quad I_1^{(a, z_1, z_2)} \equiv \int_{-\infty}^{\infty} \frac{dz}{z-a} \cdot \frac{\Theta(z-z_+)}{\sqrt{k(z, z_1, z_2)}} = \frac{1}{\sqrt{k(a, z_1, z_2)}} \ln \left[\frac{z_1 z_2 - a + \sqrt{k(a, z_1, z_2)}}{z_1 z_2 - a - \sqrt{k(a, z_1, z_2)}} \right]$$

If we assume that $a < z_-$, then $k(a, z_1, z_2) > 0$ and ^{we may} choose the branch $\sqrt{k} > 0$ and the principal branch of the logarithm. For $z_- < a < z_+$ we have $k < 0$ and the answer can be shuffled to read,

$$I_1 = \frac{1}{\sqrt{-k(a, z_1, z_2)}} \left[\frac{\pi}{2} - \tan^{-1} \left(\frac{z_1 z_2 - a}{\sqrt{-k(a, z_1, z_2)}} \right) \right]$$

For $a > z_+$ the result (16) must be "corrected" by adding a term $\pm i\pi/\sqrt{k}$

depending on whether the integration contour runs above or below the pole. In this case the result (16) stands true as a principal part integral. Of course with the right interpretation of the branches of the logarithm, equation (16) is really valid in all cases.

4.3 Relation of (16) to a famous integral.

Whenever single variable dispersion relations are combined with elastic unitarity, the following special case of equation (1) tends to arise: (footnote: for example, in Mandelstam's analysis of the double spectral boundary. See Collins and Squires page 35. Note the ~~typographical~~ errors in their definition of the k function; See also Goldberger + Watson page 605, equations (218), (222) and in the sign of their result)

$$I_2^{(z, z_1, z_2)} \equiv \int d\Omega'' \frac{1}{z'' - z_1} \cdot \frac{1}{\tilde{z}' - z_2}$$

where $\tilde{z}' = z z'' + \sqrt{(1-z^2)(1-z''^2)} \cos \phi$.

This integral can be evaluated as follows:

$$\begin{aligned}
 I_2(z, z_1, z_2) &= \int \frac{d\Omega''}{(\hat{p} \cdot \hat{p}'' - z_1)(\hat{p}' \cdot \hat{p}'' - z_2)} \\
 &= \int_0^\infty dx \int \frac{d\Omega''}{[z_1 x + z_2 - \hat{p}'' \cdot (x \hat{p} + \hat{p}')]^2} \\
 &= \int_0^\infty dx \int \frac{d\tilde{\Omega}}{[z_1 x + z_2 - \sqrt{x^2 + 1} \cos \tilde{\theta}]^2} \\
 &= 4\pi \int_0^\infty \frac{dx}{(z_1^2 - 1)x^2 + 2(z_1 z_2 - z)x + (z_2^2 - 1)} \\
 &= + \frac{2\pi}{\sqrt{k(z, z_1, z_2)}} \ln \left[\frac{z_1 z_2 - z + \sqrt{k(z, z_1, z_2)}}{z_1 z_2 - z - \sqrt{k(z, z_1, z_2)}} \right]
 \end{aligned}$$

Therefore,

$$I_2(z, z_1, z_2) = 2\pi I_1(z, z_1, z_2) \quad (17)$$

The integral I_2 can be rewritten in a form that will be used below:

$$\begin{aligned}
 I_2(z, z_1, z_2) &= \int \frac{d\Omega''}{(z'' - z_1)(z'' - z_2)} \\
 &= \int_{-1}^1 \frac{dz''}{z'' - z_1} \int_{-1}^1 \frac{dz'}{z' - z_2} \int_0^{2\pi} d\phi'' \delta(z' - z'') \\
 &= \int_{-1}^1 \frac{dz''}{z'' - z_1} \int_{-1}^1 \frac{dz'}{z' - z_2} \frac{2\Theta(-k(z, z', z''))}{\sqrt{-k(z, z', z'')}} \quad (18)
 \end{aligned}$$

where use has been made of equation (3). Therefore, from (16), (17) and (18) we have:

$$\int_{-\infty}^{\infty} \frac{dz'}{z' - z} \frac{\Theta(z' - z_+)}{\sqrt{+k(z', z_1, z_2)}} = \frac{1}{\pi} \int_{-1}^1 \frac{dz''}{z'' - z_1} \int_{-1}^1 \frac{dz'}{z' - z_2} \frac{\Theta(-k(z, z', z''))}{\sqrt{-k(z, z', z'')}} \quad (19)$$

(19)

The point to notice here is that on the right side, the z -dependence is buried away as part of the definition of the boundary of integration in the $z'-z''$ plane -- (see ellipse in figure 1), whereas on the left side the z -dependence is extremely simple.

4.4 The Q - separation integral. We seek a result analogous to (10), so it seems reasonable to start directly with (10). (footnote: Abarbanel and Saunders used this method, although they have certain endpoint errors). For (z', z'') inside the unit box we have,

$$\int_{-1}^1 dz P_L(z) \frac{\Theta(-K(z, z', z''))}{\sqrt{-K(z, z', z'')}} = \pi P_L(z') P_L(z'')$$

Apply $\int_{-1}^1 \frac{dz''}{z''-z_1} \int_{-1}^1 \frac{dz'}{z'-z_2}$ to both sides and use

Neumann's Formula $\frac{1}{2} \int_{-1}^1 \frac{dz''}{z''-z_1} P_L(z'') = -Q_L(z_1)$ to get

$$Q_L(z_1) Q_L(z_2) = \frac{1}{4\pi} \int_{-1}^1 dz P_L(z) \cdot \int_{-1}^1 \frac{dz''}{z''-z_1} \int_{-1}^1 \frac{dz'}{z'-z_2} \frac{\Theta(-K(z, z', z''))}{\sqrt{-K(z, z', z'')}}.$$

From (19), the right hand side becomes

$$= \frac{1}{4} \int_{-1}^1 dz P_L(z) \cdot \int_{-\infty}^{\infty} \frac{dz'}{z'-z} \frac{\Theta(z'-z_+)}{\sqrt{+K(z, z_1, z_2)}}$$

Now the z -integration can be done via Neumann, and the result is:

$$\int_{-\infty}^{\infty} dz Q_L(z) \frac{\Theta(z-z_+)}{\sqrt{K(z, z_1, z_2)}} = 2 Q_L(z_1) Q_L(z_2) \quad (20)$$

where $z_+ = z_1 z_2 + \sqrt{(1-z_1^2)(1-z_2^2)}$. As shown by in reference (), this integral proves very ^{useful} helpful in diagonalizing a certain class of integral equations, just as equation (10) was useful in diagonalizing the "elastic unitarity" integral equation (1).

OLD NOTES

Modification of the +k integral rep.

① First, look at: $k(a, b, c) = a^2 + b^2 + c^2 - 2abc - 1$
 $= (a - a_+)(a - a_-)$

where

$$a_{\pm} = bc \pm \sqrt{(b^2-1)(c^2-1)}$$

② Now, consider:

$$I = \int_{-1}^{\infty} \frac{dt}{\sqrt{t^2-1}} \delta(a - bc - |\sqrt{b^2-1}\sqrt{c^2-1}|t)$$

Define: $\alpha = |\sqrt{(b^2-1)(c^2-1)}|$. (Now, assume α real, so so.)

$$I = \int_{-1}^{\infty} \frac{dt}{\sqrt{t^2-1}} \delta(a - bc - \alpha t) = \frac{1}{\alpha} \int_{-1}^{\infty} \frac{dt}{\sqrt{t^2-1}} \delta(\tilde{x} - t)$$

where $\tilde{x} = \left(\frac{a-bc}{\alpha}\right)$. Thus, $I = \frac{1}{\alpha} \frac{1}{\sqrt{\tilde{x}^2-1}} \Theta(\tilde{x}-1)$

But, $\alpha \sqrt{\tilde{x}^2-1} = \sqrt{1+k(a,b,c)}$ not true if we use $\alpha = 1$.

$$\Theta(\tilde{x}-1) = \Theta\left(\frac{a-bc}{\alpha} - 1\right) = \Theta(a - bc - \alpha) = \Theta(a - a_+)$$

③ Result: if $(b^2-1)(c^2-1) \geq 0$, then:

$$\int_{-1}^{\infty} \frac{dt}{\sqrt{t^2-1}} \delta(a - bc - \sqrt{(b^2-1)(c^2-1)}t) = \frac{\Theta(a - a_+)}{\sqrt{1+k(a,b,c)}}$$

④ Similarly, $\int_{-1}^{\infty} \frac{dt}{\sqrt{t^2-1}} \delta(a - bc + \alpha t) = \frac{1}{\alpha} \int_{-1}^{\infty} \frac{dt}{\sqrt{t^2-1}} \delta(t + \tilde{x}) = \frac{1}{\alpha \sqrt{\tilde{x}^2-1}} \Theta(-\tilde{x}-1)$

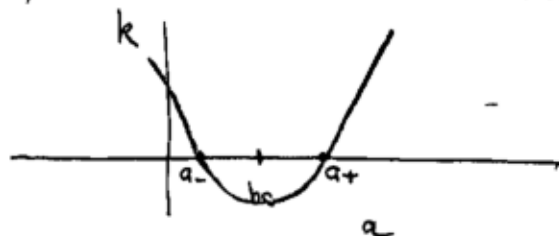
And: $\Theta(-\tilde{x}-1) = \Theta\left(-\frac{a-bc}{\alpha} - 1\right) = \Theta(-a + bc - \alpha) = \Theta(-a + [bc - \alpha]) = \Theta(-a + a_-)$
 ~~$\approx \Theta(-a + a_-)$~~ zero unless $a < (+a_-)$

So: $\int_{-1}^{\infty} \frac{dt}{\sqrt{t^2-1}} \delta(a - bc + \sqrt{(b^2-1)(c^2-1)}t) = \frac{\Theta(-a + a_-)}{\sqrt{1+k(a,b,c)}}$
 $\delta(a - \tilde{a})$
 $\tilde{a} = bc - \sqrt{1+k(a,b,c)}$

③ Now, we know that $k = (a - a_+)(a - a_-)$.

Thus,

$$\Theta(+k) = ?$$



$$\Theta(+k) = \underbrace{\Theta(a - a_+)}_{(a > a_+). \text{ OR. } (a < a_-)} + \Theta(-a + a_-)$$

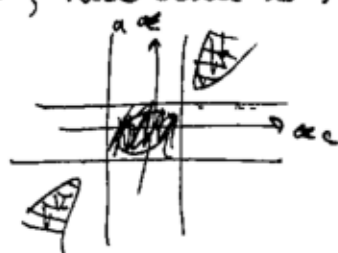
Therefore:

$$\sum_{\tau=\pm} \int_{-\infty}^{\infty} \frac{dt}{\sqrt{t^2-1}} \delta(a - bc + \tau \sqrt{(b^2-1)(c^2-1)}) = \frac{\Theta(+k(a,b,c))}{\sqrt{+k(a,b,c)}}$$

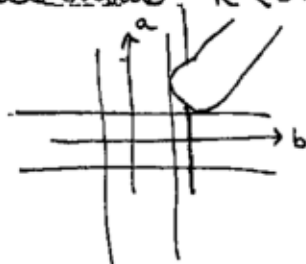
Since ⑥ So far, we assumed that $(b^2-1)(c^2-1) > 0$. This happens in two cases:

I: external case: $|b| > 1, |c| > 1$. Then $\Theta(+k) \Rightarrow |a| > 1$ also!

Thus, this rule is true when all 3 are "outside".



shades show $k < 0$ when $|b| > 1$.



⑦ Suppose b now falls below 1, $c > 1$. Then $\sqrt{(b^2-1)(c^2-1)} = \text{imag}$ and LHS "vanishes", RHS does not.

⑧ So, rule is true:

I. if $|b| > 1, |c| > 1, \forall a$

II. if $|b| < 1, |c| < 1, \forall a$.

⑨ Hyperbolic form: $t = \cosh \phi, \sqrt{t^2-1} = \sinh \phi$.

$$\text{Thus: } \int_{-\infty}^{\infty} d\phi \delta(a - bc - \sqrt{b^2-1} \sqrt{c^2-1} \cosh \phi) = \frac{\Theta(a - a_+)}{\sqrt{k(a,b,c)}}$$

$$\int_{-\infty}^{\infty} d\phi \delta(a - bc + \sqrt{b^2-1} \sqrt{c^2-1} \cosh \phi) = \frac{\Theta(-a + a_-)}{\sqrt{k}}$$

$$\sum_{\tau=\pm} \int_{-\infty}^{\infty} d\phi \delta(a - bc + \tau \sqrt{b^2-1} \sqrt{c^2-1} \cosh \phi) = \frac{\Theta(k)}{\sqrt{k}}.$$

August 4, '75

0. Recap of principle results of these 25 pages.

1. Origin of the famous k function.

2. Geometric analysis of the k function.

3. An integral representation for $\sqrt{1+k}$.

4. Conversion of above to hyperbolic functional form.

5. Exploration of alternative hyperbolic forms for 3 above.

6. First attempt to prove Abarbanel and Saunders $\sqrt{k}$ rep for I (failure).

7. Evaluations for the above.

8. More on the above attempt.

9. Exercise on the $a+bchx$ integral.

10. First sign of error in AS equation (A.7).

11. The sheet structure of $1 - z^2$.

12. Comments on the integral rep for $\sqrt{k}$.

13. A theorem using $\sqrt{k}$.

14. Where does the $a + b \cos(x)$ integral have validity?

15. A theorem relating $\sqrt{k}$ to $\sqrt{1+k}$.

16. Second attempt to prove AS equation (A.7). Failure.

17. Integral forms for the I function of Stanley.

18. Third attempt at equation (A.7). Partial success!

19. First attempt at "symmetry theorem" to finish (A.7) proof. Errors made here.

20. Example shows up error in 19 above.

21. A Counterproof of equation (A.7). It cannot be true!

22. Second attempt at symmetry theorem. Success.

23. Example verifies symmetry theorem.

24. Finally, the correct form of (A.7) is found, conditions for validity stated.

25. Improved version of the $\sqrt{1+k}$ integral rep, valid for all z_1 and z_2 . Thus, $A=I$.

Recap of principle results so far.

1. Our first success was a clear geometric understanding of the $k(x,y,z)$ function. See page 2 for ellipse, hyperbolas, regions where $k.GT.O$, and so on.
2. Next, we obtained an integral representation for $\Theta(-k)/\sqrt{-k}$, valid for all real values of the arguments. If any argument leaves "the unit box", both sides of this relation vanish, for different reasons. See page 1 and page 12. Result:

$$\frac{\Theta(-k(z_1, z_2, z_3))}{\sqrt{-k(z_1, z_2, z_3)}} = \int_0^\pi d\phi \delta(z - z_1 z_2 \pm \sqrt{(1-z_1^2)(1-z_2^2)} \cos \phi)$$

Of course this has symmetric variations, or you can run to 2π .

3. On page 13 we proved a little theorem involving an integral over $\sqrt{-k}$. It was:

$$\int_{-\infty}^{\infty} dz \frac{\Theta(-k)}{\sqrt{-k(z_1, z_2, z_3)}} = \begin{cases} \pi & \text{pure} \\ 0 & \text{mixed} \end{cases}$$

Also, some variations shown on page 13.

4. Next, we concocted an integral representation for $\Theta(+k)/\sqrt{+k}$. See pages 3,4,5 and improved version on page 25 valid for all z_1 and z_2 . Result:

$$\frac{\Theta(+k(z_1, z_2, z_3))}{\sqrt{+k(z_1, z_2, z_3)}} = \frac{1}{\tau} \int_0^\tau d\phi \delta(z - z_1 z_2 + \tau \sqrt{(1-z_1^2)(1-z_2^2)} \cos \phi)$$

5. On page 15 we showed a connection between a $-k$ and a $+k$. Result was:

$$\int_{-\infty}^{\infty} \frac{dz}{z-\mu} \frac{\Theta(-k(z, z_1, z_2))}{\sqrt{-k(z, z_1, z_2)}} = \begin{cases} \pi / \sqrt{+k(\mu, z_1, z_2)} & \text{pure} \\ 0 & \text{mixed} \end{cases}$$

6. Then we looked at the same integral with $+k$. See pages 17,18. Result was:

$$\begin{aligned} 2\pi \int_{-\infty}^{\infty} \frac{dz}{z-\mu} \frac{\Theta(+k(z, z_1, z_2))}{\sqrt{+k(z, z_1, z_2)}} &= I(z_1, z_2; \mu) \\ &= \int_{-1}^1 d\mu_1 \int_{-1}^1 d\mu_2 \frac{1}{z-\mu} \frac{1}{z_1-\mu_1} \frac{1}{z_2-\mu_2} \frac{2\Theta(-k(\mu, \mu_1, \mu_2))}{\sqrt{-k(\mu, \mu_1, \mu_2)}} \\ &\equiv \frac{2\pi}{\sqrt{K}} \ln \left[\frac{z_1 z_2 - \mu + \sqrt{K}}{z_1 z_2 - \mu - \sqrt{K}} \right] \quad K = K(\mu, z_1, z_2) \\ &= \int d\Omega_{\hat{q}} \frac{1}{(\hat{K}\hat{q}-z_1)(\hat{K}\hat{q}-z_2)} \quad \text{where } \mu = \hat{K} \cdot \hat{K}', \text{ when } |\mu| < 1 \\ &= 4\pi \int_0^\pi \frac{dx}{(z_1^2-1)x^2 + 2(z_1 z_2 - z)x + (z_2^2-1)} \quad \text{/ via Feynman param from } \int d\Omega_{\hat{q}} \\ &= 4\pi \int_0^\pi \frac{dx}{(z_1 z_2 - z) \pm \sqrt{1-z_1^2} \sqrt{1-z_2^2} \cos x} \quad \text{/ reduce dx integral to ln form.} \end{aligned}$$

7. Finally, on pages 22,23,24 we discussed a certain symmetry property of the integral rep in 6: for I , and we derived the result on page 605 of GW:

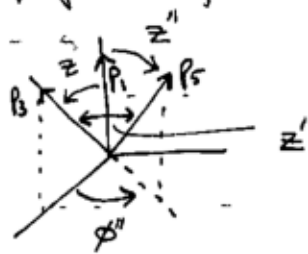
$$z_1, z_2 > 1; |\mu| < 1 \Rightarrow I = 4\pi \int_b^\infty \frac{dz}{z-\mu} \frac{1}{\sqrt{+k(z, z_1, z_2)}}$$

where $b = z_1 z_2 + \sqrt{(z_1^2-1)(z_2^2-1)}$

8. Thus, the Abarbanel and Saunders equation (A.7) is incorrect, since... (see p 24).

Origin of the famous $K(z, z_1, z_2)$ kinematic factor.

- ① For starters, consider page 594 of GW and consider the elastic unitarity situation:



As usual, we encounter:

$$a(z) = \int d\Omega'' b(z'') c(z')$$

where z' means $z'[z, z'', \phi''] = zz'' + \sqrt{1-z^2} \sqrt{1-z''^2} \cos \phi''$

② Introduce

$$1 = \int_{-1}^1 dz_1 \delta(z_1 - z')$$

$$1 = \int_{-1}^1 dz_2 \delta(z_2 - z'')$$

We know that z' and z'' are in range $(-1, 1)$. I.e., z'' is explicitly from $\int d\Omega''$, and z in range then $\Rightarrow z'$ in range. Thus:

$$a(z) = \int_{-1}^1 dz_1 \int_{-1}^1 dz_2 b(z_1) c(z_2) \times \int d\Omega'' \delta(z_1 - z'') \delta(z_2 - z').$$

③ Now calculate this jacobian:

$$J = \frac{\partial(z'', \phi'')}{\partial(z_1, z_2)} = \int_0^{2\pi} d\phi'' \int_{-1}^1 dz'' \delta(z_1 - z'') \delta(z_2 - zz'' - \sqrt{1-z^2} \sqrt{1-z''^2} \cos \phi'')$$

This is to say:

$$dz_1 dz_2 \cdot \frac{\partial(z'', \phi'')}{\partial(z_1, z_2)} = dz'' d\phi''$$

The z'' integral is trivial ~ no need really to introduce it. So:

$$J = \int_0^{2\pi} d\phi'' \delta(z_2 - zz_1 - \sqrt{1-z^2} \sqrt{1-z_1^2} \cos \phi'')$$

$$= \frac{\partial \phi''}{\partial z_1}$$

④ I.e., we are merely converting our ϕ'' integration variable to integral over z' .

So:
$$\left[\mathcal{I} = \int_0^{2\pi} d\phi'' \delta(z_2 - z z_1 - \sqrt{1-z^2} \sqrt{1-z_1^2} \cos \phi'') \right]$$

Recall that this thing "hits" twice, so

$$= 2 \cdot \frac{1}{\sqrt{1-z^2} \sqrt{1-z_1^2} \sin \tilde{\phi}} \underbrace{\Theta(1 - |\cos \tilde{\phi}|)}_{\Theta(\tilde{\phi}) \Theta(2\pi - \tilde{\phi})} \int_0^{2\pi} d\phi'' \delta(\phi'' - \tilde{\phi})$$

⑤ Now: $\cos \tilde{\phi} = \frac{(z_2 - z z_1)}{\sqrt{1-z^2} \sqrt{1-z_1^2}}$

Thus: $\sin^2 \tilde{\phi} = 1 - \cos^2 \tilde{\phi} = \frac{(1-z^2)(1-z_1^2) - (z_2 - z z_1)^2}{(1-z^2)(1-z_1^2)}$

$$= \frac{1 - z^2 - z_1^2 + z^2 z_1^2 - z_2^2 - z^2 z_1^2 + 2 z z_1 z_2}{(1-z^2)(1-z_1^2)} = \frac{-k(z, z_1, z_2)}{(1-z^2)(1-z_1^2)}$$

Thus: $\sin \tilde{\phi} = \frac{\sqrt{-k(z, z_1, z_2)}}{\sqrt{1-z^2} \sqrt{1-z_1^2}}$ $\nearrow$ k appears here!

⑥ What is the condition that $|\cos \tilde{\phi}| \leq 1$? Same as $|\sin \tilde{\phi}| \leq 1$. Same as $|\cos \tilde{\phi}|^2 \leq 1$. i.e. $(\cos \tilde{\phi})^2 \leq 1$.
 I.e., $1 - \sin^2 \tilde{\phi} \leq 1$ i.e. $\sin^2 \tilde{\phi} \geq 0$. So

$$\Theta(1 - |\cos \tilde{\phi}|) =$$

→ Actually, the condition that $|\cos \tilde{\phi}| \leq 1$ is the same as the condition that $\sin \tilde{\phi}$ be real. Thus, $\Theta(-k(z, z_1, z_2))$.

⑦ Therefore,
$$\left[\mathcal{I} = \frac{2 \Theta(-k(z, z_1, z_2))}{\sqrt{-k(z, z_1, z_2)}} \right]$$

and

$$a(z) = \int_{-1}^1 dz_1 \int_{-1}^1 dz_2 b(z_1) c(z_2) \cdot \frac{2 \Theta(-k(z, z_1, z_2))}{\sqrt{-k(z_1, z_2, z)}}$$

Geometric analysis of the k -function.

① $k(z_1, z_2) = z_1^2 + z_2^2 - 2z_1 z_2 + z^2 - 1$

② What is the nature of the curve, $k=0$ in (z_1, z_2) coords? From pages 211, 212 of Thomas we know this is a conic that is rotated by 45° since $\cot 2\theta_{\text{rot}} \propto (A-C) \propto (1-1)=0$. The discriminant is: $B^2 - 4AC = (z^2 - 1)^2$. Thus: $|z| < 1 \Rightarrow$ ellipse, $|z| > 1 \Rightarrow$ hyperbola. $z=1 \Rightarrow z_1 = z_2$, line.

③ The elliptic sector: assume $|z| < 1$; describe the ellipse. I hope to show it is inscribed inside the unit square:

a) when $z_1 = 1$, $k = (z - z_2)^2 \Rightarrow z_2 = z$.

b) OK, when $z_1 = 1$, $z_2 = z$, what is the slope dz_2/dz_1 ? If ∞ , then QED, it must be inscribed:

$$0 = z_1^2 + z_2^2 - 2z_1 z_2 + z^2 - 1 \Rightarrow 2z_1 + 2z_2 z_2' - 2z_1 z_2 - 2z_1 z_2' = 0$$

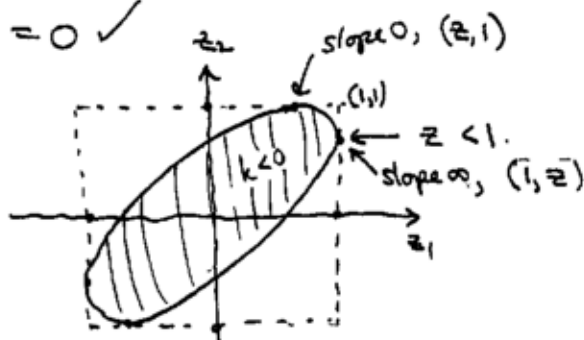
$$\Rightarrow z_2' = \frac{z z_2 - z_1}{z_2 - z_1}$$

at $(z_1, z_2) = (1, z)$, $z_2' = (z^2 - 1)/(z - 1) = \infty \checkmark$

at $(z_1, z_2) = (z, 1)$, $z_2' = (z(1-z)/(1-z^2)) = 0 \checkmark$

Thus we have proven

the picture:



If $|z| < 1$ and $z < 0$, ellipse is centered the other way.

If $z=0$, an inscribed circle.

Note: $(z_1, z_2) = (0,0) \Rightarrow k = z^2 - 1 < 0$. Thus, $k < 0$ is the inside of this ellipse. All of the interior is contained inside the $|z_1| < 1$, $|z_2| < 1$ box. Therefore:

$$\int_{-1}^1 \int_{-1}^1 dz_1 dz_2 f(z_1, z_2) \Theta(-k) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dz_1 dz_2 f(z_1, z_2) \Theta(-k)$$

ie, $|z| > 1$.

- ④ the hyperbolic sector. a) First, we want asymptotes: when $z_1 \dots$ is large, z_2 will also be large and we'll have:

$$z_1^2 + z_2^2 - 2z_1 z_2 \approx 0$$

$$\Rightarrow z_2 = (2z_1 \pm \sqrt{4z_1^2 - 4 \cdot 1 \cdot z_1^2})/2 = z_1 \{ z \pm \sqrt{z^2 - 1} \}$$

Asymptotes are $z_2 = z_1 \times (z \pm \sqrt{z^2 - 1})$, both + slope if $z > 1$. Both neg if $z < -1$.

b) again, when $(z_1, z_2) = (1, z)$, $k=0$ and slope is ∞ .
when $(z_1, z_2) = (z, 1)$, $k=0$ and slope is 0.

c) when $z_1 = z_2$, $z_1 = ?$

$$2z_1^2 - 2z_1 z_1^2 + z_1^2 - 1 = 0 \Rightarrow 2(1-z)z_1^2 + (z^2 - 1) = 0$$

$$\Rightarrow 2(z-1)z_1^2 = (z^2 - 1) \Rightarrow 2z_1^2 = (z+1) \Rightarrow z_1 = \pm \sqrt{\frac{z+1}{2}}$$

Note that $|z_1| > 1$.

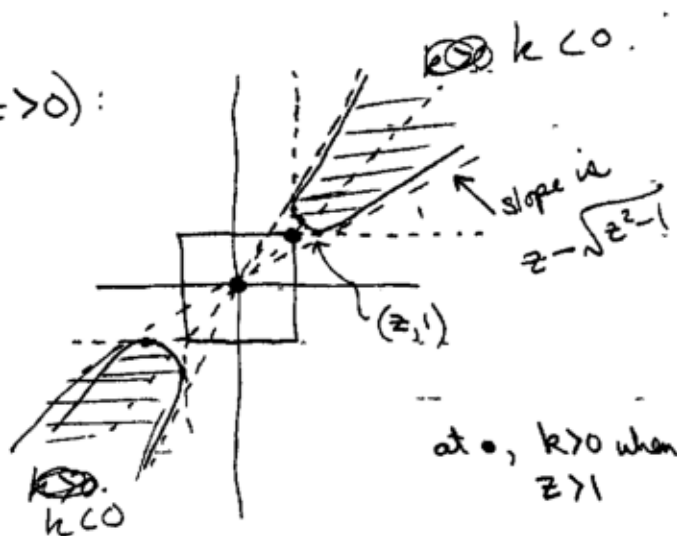
Therefore, this picture emerges: (for $z > 0$):

Note: $z < 0 \Rightarrow$ opposite cent.

For $z > 1$ as shown, hyperbolas are entirely within $z_1 > 1, z_2 > 1$ and $z_1 < -1, z_2 < -1$.

$$(z_1, z_2) = (0, 0) \Rightarrow k = (z^2 - 1) > 0$$

Therefore, inside of hyperbolas is $k < 0$.



As $z \rightarrow +1$, hyp. shrinks up to line at 45°

As $z \rightarrow \infty$, hyp. moves away + out.

An integral representation for $[+k(a,b,c)]^{-1/2}$

① Candidate:

$$[I = \int_{-1}^{\infty} \frac{dt}{\sqrt{t^2-1}} [\delta(c-ab - \sqrt{a^2-1}\sqrt{b^2-1}t) + \delta(c-ab + \sqrt{a^2-1}\sqrt{b^2-1}t)]]$$

② Define $\alpha = \sqrt{a^2-1}\sqrt{b^2-1}$. Assume $|a| > 1$ and $|b| > 1$, so $\alpha > 0$. Then:

$$I = \int_{-1}^{\infty} \frac{dt}{\sqrt{t^2-1}} [\delta(c-ab - \alpha t) + \delta(c-ab + \alpha t)]$$

③ Define: $\tilde{x}_1 \equiv \frac{c-ab}{\alpha}$ $\tilde{x}_2 \equiv \frac{c-ab}{-\alpha}$. Then

$$I = \int_{-1}^{\infty} \frac{dt}{\sqrt{t^2-1}} \frac{1}{|\alpha|} [\delta(t-\tilde{x}_1) + \delta(t-\tilde{x}_2)]$$

But $\tilde{x}_2 = -\tilde{x}_1$ so $I = \frac{1}{|\alpha|} \int_{-1}^{\infty} \frac{dt}{\sqrt{t^2-1}} [\delta(t-\tilde{x}_1) + \delta(t+\tilde{x}_1)]$

$$I = \frac{1}{|\alpha|} \cdot \frac{1}{\sqrt{\tilde{x}_1^2-1}} [\Theta(\tilde{x}_1-1) + \Theta(-\tilde{x}_1-1)]$$

④ Claim: $\Theta(x-a) + \Theta(-x-a) = \Theta(|x|-a)$ if $a > 0$.

$$= \Theta(|x|^2 - a^2)$$

$$= \Theta(+\sqrt{|x|^2 - a^2})$$

$$= \Theta(\sqrt{x^2 - a^2})$$

⑤ Therefore:

$$I = \frac{1}{|\alpha|} \cdot \frac{1}{\sqrt{\tilde{x}_1^2-1}} \Theta(\sqrt{\tilde{x}_1^2-1})$$

⑥ Now, $|\alpha| \sqrt{\tilde{x}_1^2-1} = \frac{\sqrt{(\tilde{x}_1 \alpha)^2 - \alpha^2}}{\alpha} = \frac{\sqrt{(c-ab)^2 - (\alpha^2-1)(b^2-1)}}{\alpha} = \sqrt{+k(a,b,c)}$

And: $\Theta(\sqrt{\tilde{x}_1^2-1}) = \Theta(|\alpha| \sqrt{\tilde{x}_1^2-1}) = \Theta(|\alpha|^2 (\tilde{x}_1^2-1)) = \Theta(+k)$.

⑦ Therefore, $[I = \Theta(+k) / \sqrt{k(a,b,c)}]$

⑧ Notice that this representation is only valid if $\alpha = \text{real}$. That is, if $|a|, |b|$ both > 1 , or both < 1 . In other cases, $\alpha = \text{complex}$ and delta functions are undefined.

Convert to hyperbolic functions

① So far we have shown that

$$\left[\frac{\Theta(k)}{\sqrt{k(a,b,c)}} = \sum_r \int_1^\infty \frac{dt}{\sqrt{t^2-1}} \delta(c-ab + \alpha r t) = I \right]$$

where $\alpha = \sqrt{a^2-1} \sqrt{b^2-1}$ must be real.

② Let $t = \cosh \phi$ $\frac{dt}{\sqrt{t^2-1}} = \sinh \phi d\phi$ ϕ runs 0 to $+\infty$.
 $\sqrt{t^2-1} = \sinh \phi$

Thus:

$$\left[I = \sum_r \int_0^\infty d\phi \delta(c-ab + \alpha r \cosh \phi) \right]$$

Alternative hyperbolic forms:

- ① Let $t = \coth \phi$. As t runs 1 to ∞ , ϕ runs ∞ to 0 .
As t runs -1 to $-\infty$, ϕ runs $-\infty$ to 0 .

- ② Now, $dt = -\operatorname{csch}^2 \phi d\phi$ and $t^2 - 1 = \operatorname{csch}^2 \phi$, so

$$\int_1^{\infty} \frac{dt}{\sqrt{t^2 - 1}} = \int_{\infty}^0 \frac{-\operatorname{csch}^2 \phi d\phi}{\operatorname{csch} \phi} = \int_0^{\infty} d\phi \cdot \operatorname{csch} \phi$$

$$\int_{-\infty}^{-1} \frac{dt}{\sqrt{t^2 - 1}} = \int_0^{-\infty} -\operatorname{csch} \phi d\phi = \int_{-\infty}^0 d\phi \cdot \operatorname{csch} \phi.$$

- ③ Earlier we had:

$$I = \left(\int_{-\infty}^{-1} + \int_1^{\infty} \right) \frac{dt}{\sqrt{t^2 - 1}} \delta(c - ab - \alpha t)$$

Now we have converted this to:

$$\alpha = \sqrt{a^2 - 1} \sqrt{b^2 - 1}$$

$$\left[I = \int_{-\infty}^{\infty} d\phi \cdot \operatorname{csch} \phi \cdot \delta(c - ab - \alpha \coth \phi) \right].$$

- ④ With this form on $\Theta(k)/\sqrt{k}$, I get

$$I' = \int d\ell \frac{1}{\dots} = \int_{-\infty}^{\infty} \frac{d\phi \cdot \operatorname{csch} \phi}{(z_1 z_2 - \mu) + \alpha \coth \phi} \Theta(z_1 z_2 + \alpha \coth \phi - 1)$$

- ⑤ Just for fun, try $x = \tanh \phi/2$ $dx = \operatorname{sech}^2 \frac{\phi}{2} \cdot \frac{1}{2} d\phi$

$$\text{But, } \operatorname{sech}^2 \frac{\phi}{2} = \frac{1}{\cosh^2 \phi/2} = \frac{2}{1 + \cosh \phi} = 1 - \tanh^2 \frac{\phi}{2} = (1 - x^2).$$

$$\text{So } dx = \frac{1}{2} (1 - x^2) d\phi$$

$$\text{Meanwhile: } \operatorname{csch} \phi = \frac{1}{\sinh \phi} = \frac{1}{2 \sinh \phi/2 \cosh \phi/2} = \frac{1}{x} \frac{(1 - x^2)}{x}$$

$$\text{Therefore: } d\phi \cdot \operatorname{csch} \phi = \frac{2 dx}{(1 - x^2)} \cdot \frac{(1 - x^2)}{x} = \frac{dx}{x} \checkmark$$

$$\text{Meanwhile, } \coth \phi = \frac{1 + x^2}{2x}$$

In terms of this variable X we have:

$$I' = 2 \int_{-1}^1 dx \frac{\Theta(z_1 z_2 + \alpha \left(\frac{1+x^2}{2x} \right) - 1)}{2x(z_1 z_2 - \mu) + \alpha(1+x^2)}$$

at least reminiscent of desired result.

Verify AS's integral for I, equation (A.7)

① Equation (A.7) reads:

$$I \equiv \int d\hat{k} \frac{1}{(\hat{q} \cdot \hat{k} - z_1)(\hat{q} \cdot \hat{k}' - z_2)} = 4\pi \int_1^\infty dz \frac{1}{z-\mu} \cdot \frac{\Theta(k)}{\sqrt{k(z, z_1, z_2)}} \leftarrow$$

where $\mu = \hat{k} \cdot \hat{k}'$.

② But we know that:

$$\alpha = \sqrt{z_1^2 - 1} \sqrt{z_2^2 - 1}$$

$$\frac{\Theta(k)}{\sqrt{k(z, z_1, z_2)}} = \sum_r \int_0^\infty d\phi \delta(z - z_1 z_2 + \alpha \tau \cosh \phi) \leftarrow$$

$$\begin{aligned} \text{Thus: } I &= 4\pi \int_1^\infty \frac{dz}{z-\mu} \sum_r \int_0^\infty d\phi \delta(z - z_1 z_2 + \alpha \tau \cosh \phi) \\ &= 4\pi \sum_r \int_0^\infty d\phi \cdot \int_1^\infty \frac{dz}{z-\mu} \delta(z - z_1 z_2 + \alpha \tau \cosh \phi) \end{aligned}$$

Define: $\tilde{z}_r = z_1 z_2 + \alpha \tau \cosh \phi$. Then:

$$I = 4\pi \sum_r \int_0^\infty d\phi \frac{1}{\tilde{z}_r - \mu} \Theta(\tilde{z}_r - 1)$$

③ Example: assume $|z_1|, |z_2| > 1$ both. Then $\tilde{z}_+ > 1$ for all ϕ .
But $\tilde{z}_- = z_1 z_2 - \alpha \cosh \phi \sqrt{z_1^2 - 1} \sqrt{z_2^2 - 1}$. This falls below 1 when ϕ increases to ϕ_m where

$$\cosh \phi_m = z_1 z_2 / \alpha$$

Then:

$$I = 4\pi \int_0^\infty d\phi \frac{1}{(z_1 z_2 - \mu) + \alpha \cosh \phi} + 4\pi \int_0^{\phi_m} d\phi \frac{1}{(z_1 z_2 - \mu) - \alpha \cosh \phi}$$

④ From Spiegel,

$$\int \frac{d\phi}{p \pm \alpha \cosh \phi} = \frac{1}{\sqrt{p^2 - \alpha^2}} \ln \left(\frac{\pm \alpha e^x + p - \sqrt{p^2 - \alpha^2}}{\pm \alpha e^x + p + \sqrt{p^2 - \alpha^2}} \right).$$

So:

$$\int_0^{\phi_m} \frac{d\phi}{p - \alpha \cosh \phi} = \frac{1}{\sqrt{p^2 - \alpha^2}} \ln \left[\frac{-\alpha e^{\phi_m} + p - \sqrt{p^2 - \alpha^2}}{-\alpha e^{\phi_m} + p + \sqrt{p^2 - \alpha^2}} \cdot \frac{-\alpha + p + \sqrt{p^2 - \alpha^2}}{-\alpha + p - \sqrt{p^2 - \alpha^2}} \right]$$

Evaluate the integrals

① since $d\phi_m = \frac{z_1 z_2}{\alpha}$, $\phi_m = \ln \left[\frac{z_1 z_2}{\alpha} + \sqrt{\left(\frac{z_1 z_2}{\alpha}\right)^2 - 1} \right]$

and $e^{\phi_m} = \frac{z_1 z_2}{\alpha} + \sqrt{\left(\frac{z_1 z_2}{\alpha}\right)^2 - 1} \Rightarrow \alpha e^{\phi_m} = z_1 z_2 + \sqrt{z_1^2 z_2^2 - \alpha^2}$

So: $\alpha e^{\phi_m} = z_1 z_2 + \sqrt{z_1^2 + z_2^2 - 1}$ $\rho = (z_1 z_2 - \mu)$

And: $\rho^2 - \alpha^2 = k$, so:

$$\int_0^{\phi_m} \frac{d\phi}{\rho - \alpha \cosh \phi} = \frac{1}{\sqrt{k}} \ln \left[\frac{(-z_1 z_2 - \sqrt{z_1^2 + z_2^2 - 1} + z_1 z_2 - \mu - \sqrt{k})(-\alpha + \rho + \sqrt{k})}{(+\sqrt{k})(-\alpha + \rho - \sqrt{k})} \right]$$

$$= \frac{1}{\sqrt{k}} \ln \left[\frac{-\sqrt{z_1^2 + z_2^2 - 1} - \mu - \sqrt{k}}{-\sqrt{z_1^2 + z_2^2 - 1} - \mu + \sqrt{k}} \cdot \frac{-\alpha + \rho + \sqrt{k}}{-\alpha + \rho - \sqrt{k}} \right]$$

② The second integral is:

$$\int_0^{\infty} \frac{d\phi}{\rho + \alpha \cosh \phi} = \frac{1}{\sqrt{k}} \ln \left[\frac{\alpha + \rho + \sqrt{k}}{\alpha + \rho - \sqrt{k}} \right]$$

③ Combine these to get

$$I = 4\pi \frac{1}{\sqrt{k}} \cdot \ln \left[\frac{(-\sqrt{z_1^2 + z_2^2 - 1} - \mu - \sqrt{k})(-\alpha + \rho + \sqrt{k})(\alpha + \rho + \sqrt{k})}{(+\sqrt{k})(-\alpha + \rho - \sqrt{k})(-\alpha + \rho - \sqrt{k})} \right]$$

④ Now consider last four factors:

$$\frac{[(\rho + \sqrt{k}) + \alpha][(\rho + \sqrt{k}) - \alpha]}{[(\rho - \sqrt{k}) + \alpha][(\rho - \sqrt{k}) - \alpha]} = \frac{(\rho + \sqrt{k})^2 - \alpha^2}{(\rho - \sqrt{k})^2 - \alpha^2} = \frac{\rho^2 - \alpha^2 + k + 2\rho\sqrt{k}}{\rho^2 - \alpha^2 + k - 2\rho\sqrt{k}} = \frac{k + \rho\sqrt{k}}{k - \rho\sqrt{k}}$$

$$= \frac{\rho + \sqrt{k}}{-\rho + \sqrt{k}}$$

Total answer is therefore:

$$\left[I = \frac{4\pi}{\sqrt{k}} \ln \left[\frac{-\sqrt{z_1^2 + z_2^2 - 1} - \mu - \sqrt{k}}{-\sqrt{z_1^2 + z_2^2 - 1} - \mu + \sqrt{k}} \cdot \frac{\sqrt{k} + \rho}{\sqrt{k} - \rho} \right] \right]$$

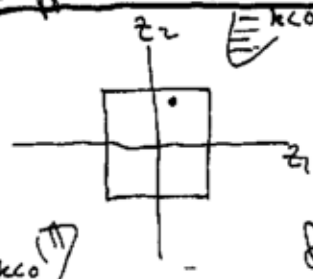
Does ϕ_m integral really hit?

① I.e., is $\chi(\phi_m) = \left(\frac{z_1 z_2}{2}\right) > 1$? If so, then you have to consider the ϕ_m integral.

$$z_1, z_2 >? \sqrt{z_1^2 - 1} \sqrt{z_2^2 - 1}$$

$$z_1^2 z_2^2 >? (z_1^2 - 1)(z_2^2 - 1), \text{ yes } \rightarrow \text{ of course.}$$

② Suppose z_1 and z_2 both lie in $-1, 1$. Assume both +. Then



If $z > 1$, then the $k < 0$ regions are outside, so $k > 0 \forall z_1, z_2 \in \text{box}$.

In this case, what do we know about $\tilde{z}$?

$$\tilde{z} = z_1 z_2 + \sqrt{1 - z_1^2} \sqrt{1 - z_2^2} \chi(\phi)$$

$$\tilde{z} >? 1 \rightarrow (z_1 z_2 - 1) >? -\sqrt{1 - z_1^2} \sqrt{1 - z_2^2} \chi(\phi)$$

$$(z_1 z_2 - 1)^2 >? (1 - z_1^2)(1 - z_2^2) \chi^2(\phi)$$

Test at $\phi = 0$: $\tilde{z} = z_1 z_2 - 2z_1 z_2 + 1 >? 1 - z_1^2 - z_2^2 + z_1^2 z_2^2$

$$-2z_1 z_2 >? -z_1^2 - z_2^2$$

$$z_1^2 + z_2^2 - 2z_1 z_2 >? 0 \text{ yes. But as } \phi \text{ increases this will no longer be true, so again, a } \phi_{\text{max}} \text{ situation?}$$

~~Yes! as ϕ inc, RHS goes more neg. Still true. (always).~~

$\tilde{z}_+ > 1$
(Never)

$$(z_1 z_2 - 1) >? + \sqrt{1 - z_1^2} \sqrt{1 - z_2^2} \chi(\phi)$$

③ Thus, in the case $0 \leq z_1, z_2 < 1$, only $\tilde{z}$ hits and:

$$I = 4\pi \int_0^\pi d\phi \frac{1}{z_1 z_2 + \sqrt{1 - z_1^2} \sqrt{1 - z_2^2} \chi(\phi) - \mu}$$

$$\rho = (z_1 z_2 - \mu) \\ + \alpha = \sqrt{1 - z_1^2} \sqrt{1 - z_2^2}$$

So

$$= 4\pi \cdot \frac{1}{\sqrt{K}} \ln \left(\frac{\alpha + \rho + \sqrt{K}}{\alpha + \rho - \sqrt{K}} \right) = \frac{2\pi}{\sqrt{K}} \ln \left(\frac{\alpha + \rho + \sqrt{K}}{\alpha + \rho - \sqrt{K}} \right)^2$$

$$= \frac{2\pi}{\sqrt{K}} \ln \left(\frac{z_1 z_2 - \mu + \sqrt{K}}{z_1 z_2 - \mu - \sqrt{K}} \right) = \frac{-2\pi}{\sqrt{K}} \ln \left(\frac{\mu - z_1 z_2 + \sqrt{K}}{\mu - z_1 z_2 - \sqrt{K}} \right)$$

yes! $\rightarrow$

→ This, in the region $|z_1|, |z_2| < 1$, I have validated the AS formula, via my integral representation for the $\overline{FK}$ function.

For other region of z_1, z_2 , I'm not sure about it. Hopefully, still true. The actual log form of the answer may differ, however.

Exercise on chx integrals

① On the one hand, we know:

$$\int_0^\infty d\phi \frac{1}{p+q \cosh \phi} = \frac{1}{\sqrt{k}} \ln \left[\frac{q+p+\sqrt{k}}{q+p-\sqrt{k}} \right]$$

$$k = p^2 - q^2$$

② But, integrand is even in ϕ , so

$$= \frac{1}{2} \int_{-\infty}^{\infty} d\phi \frac{1}{p+q \cosh \phi} = \frac{1}{\sqrt{k}} \ln \left[\frac{p+\sqrt{k}}{p-\sqrt{k}} \right]$$

③ Verify in the logarithms that this is in fact all true:

$$\ln \left(\frac{p+\sqrt{k}}{p-\sqrt{k}} \right) \stackrel{?}{=} \ln \left(\frac{q+p+\sqrt{k}}{q+p-\sqrt{k}} \right)^2$$

$$\frac{p+\sqrt{k}}{p-\sqrt{k}} \stackrel{?}{=} \frac{(q+p+\sqrt{k})^2}{(q+p-\sqrt{k})^2}$$

$$\left(\frac{q+p+\sqrt{k}}{q+p-\sqrt{k}} \right)^2 = \frac{q^2+p^2+k+2qp+2q\sqrt{k}+2p\sqrt{k}}{q^2+p^2+k+2qp-2q\sqrt{k}-2p\sqrt{k}}$$

$$= \frac{2p^2+2qp+2q\sqrt{k}+2p\sqrt{k}}{2p^2+2qp-2q\sqrt{k}-2p\sqrt{k}} = \frac{p(q+p)+\sqrt{k}(q+p)}{p(q+p)-\sqrt{k}(q+p)} = \frac{p+\sqrt{k}}{p-\sqrt{k}}$$

Yes, finally.

Trace this idea:

① We are comparing:

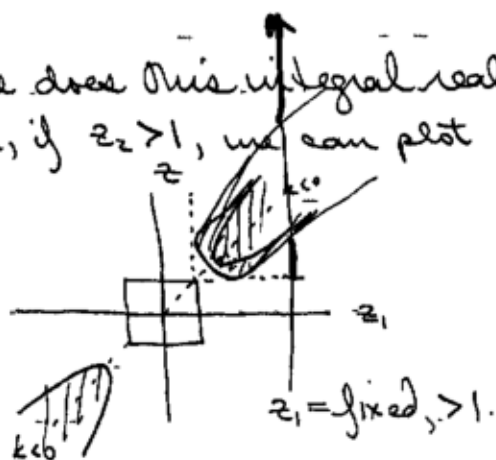
$$I_1 =$$

$$4\pi \int_0^{\infty} dx \frac{1}{(z_1^2 - 1)x^2 + 2(z_1 z_2 - \mu)x + (z_2^2 - 1)}$$

$$I_2 = 4\pi \int_1^{\infty} \frac{dz}{z - \mu} \frac{\Theta(k)}{\sqrt{k(z, z_1, z_2)}}$$

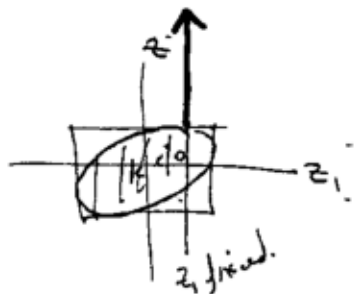
where does this integral really start? Assume $z_1 > 1, z_2 > 1$.

Well, if $z_2 > 1$, we can plot in (z_1, z) space.



Looks like a portion of integral is blanked out!

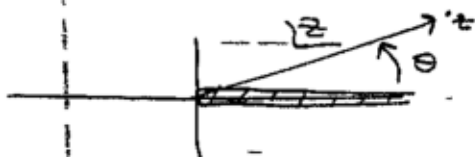
② If $z_2, z_1 < 1$, then let z_2 be < 1 . Then in (z_1, z) :



In this case, it really does go to ∞ .

The sheet structure of $\sqrt{1-z^2}$.

① First, consider the function $f(z) = \sqrt{-z}$. Draw cut to right so real analytic.



Now, the z plane has two sheets. We can denote these by:

$$(-z) = |z| e^{i\theta} \quad e^{\pm i\pi}$$

Thus, $f_{\pm}(z) = \sqrt{|z|} e^{i\theta/2} (\pm i)$

Now, when $\theta = +\pi$, you get $e^{i(\pi \pm \pi)/2} = \begin{Bmatrix} -1 \\ +1 \end{Bmatrix}$, so

$$f_{\pm}(z) = \mp \sqrt{|z|}$$

Usually, we deal with the $f_{-}(z)$ branch, since this is positive when $z < 0$. Then:

$$f(z) = \sqrt{|z|} e^{i(\theta - \pi)/2}$$

On this sheet, θ runs 0 to 2π . If $z > 0$ but on top of cut, then:

$$f(z) = \sqrt{|z|} e^{i(0 - \pi)/2} = (-i) \sqrt{|z|}$$

If $z > 0$ on bottom of cut, then: $f(z) = \sqrt{|z|} e^{i(2\pi - \pi)/2} = (+i) \sqrt{|z|}$.
Thus:

$$f(z+i\epsilon) - f(z-i\epsilon) = -2i \sqrt{|z|} \quad \text{when } z > 0.$$

② Now consider $g(z) = \sqrt{1-z}$. Choose this branch:

$$-(1-z) = |1-z| e^{i(\theta - \pi)}$$

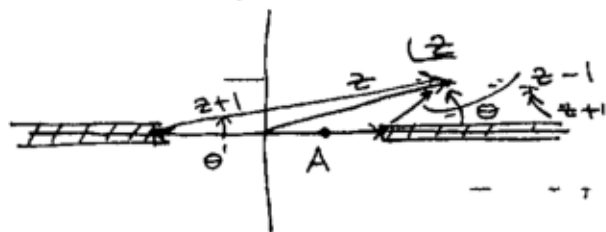
$$-(z-1) = |z-1| e^{i(\theta - \pi)}$$



Then $\theta = +\pi \Rightarrow g(z)$ is positive.

So: $g(z) = \sqrt{1-z} = \sqrt{|z-1|} e^{i(\theta - \pi)/2} = (-i) \sqrt{|z-1|}$ above cut.

③ Finally, go to $h(z) = \sqrt{1-z^2}$.



Again, let:

$$-(z-1) = |z-1| e^{i(\theta-\pi)}$$

Then also $z+1 = |z+1| e^{i\theta'}$ on this same sheet.

$$\begin{aligned} h(z) &= \sqrt{1-z^2} = \sqrt{(1+z)(1-z)} \\ &= \sqrt{|1+z||1-z|} \cdot e^{i(\theta'+\theta-\pi)/2} \end{aligned}$$

Example: if at A, then $\theta' = 0, \theta = +\pi$, $h(z) = \text{pos.}$
Note that θ goes $0, 2\pi$ but θ' goes $-\pi$ to $+\pi$.

④ So, we have carefully defined one branch of $h(z)$. I think there are only two sheets.
In particular, when $z > 1$, let's go above the cut.
Then $\theta' = 0, \theta = 0$ and:

$$h(z > 1) = \sqrt{z^2-1} \times (-i)$$

For $z < -1$, let's go below the cut. Then

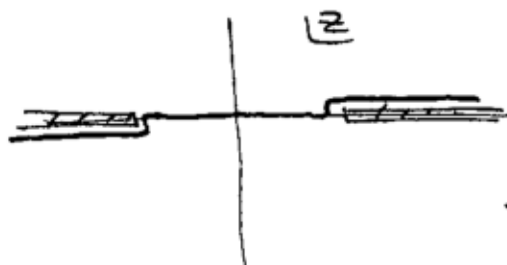
$$h(z < -1) = \sqrt{z^2-1} \times (-i) \text{ again}$$

since $\theta' = -\pi, \theta = \pi$, $(\theta'+\theta-\pi) = (-\pi+\pi-\pi) = -\pi$,
so get $(-i)$.

⑤ With this convention, we can say with conviction

$$h(|z| < 1) = +\sqrt{1-z^2}$$

$$h(|z| > 1) = (-i)\sqrt{z^2-1}$$

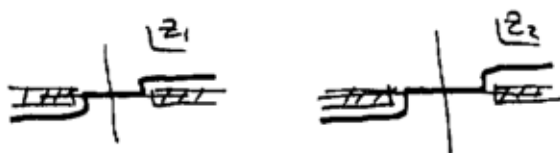


Comments on integral rep. for $\sqrt{-k(z, z_1, z_2)}$.

① First, re-derive the basic result:

$$I \equiv \int_0^{2\pi} d\phi \delta(z - z_1 z_2 - \sqrt{1-z_1^2} \sqrt{1-z_2^2} \cos \phi)$$

② As for the sheet structure in z_1 and z_2 , let's use the same convention for both. For example:



$$\sqrt{1-z_1^2} = (-i) \sqrt{z_1^2-1} \quad \text{if } |z_1| > 1.$$

$$\text{Thus: } \sqrt{1-z_1^2} \sqrt{1-z_2^2} = - \sqrt{z_1^2-1} \sqrt{z_2^2-1} \quad \text{if } |z_1| > 1 \text{ and } |z_2| > 1.$$

③ Define $\alpha(z_1, z_2) = \sqrt{1-z_1^2} \sqrt{1-z_2^2}$ with cuts as above. We see that if $|z_1| > 1$ and $|z_2| < 1$, or vice versa, then $I = 0$ because hit cannot occur at a complex point.

④ From now on, assume either $|z_1|, |z_2| > 1$, or $|z_1|, |z_2| < 1$. Then $\alpha(z_1, z_2) = \text{real}$, for sure.

$$I \equiv \int_0^{2\pi} d\phi \delta(z - z_1 z_2 - \alpha(z_1, z_2) \cos \phi)$$

Since arg is $f(\cos \phi)$, replace: $I = 2 \int_0^\pi d\phi \delta(z - z_1 z_2 - \alpha \cos \phi)$.
Note that $\sin \phi > 0$ in this range, hence $+\sqrt{1-\mu^2}$ below.

⑤ Define $\mu = \cos \phi$. Then: $I = 2 \int_{-1}^1 \frac{d\mu}{\sqrt{1-\mu^2}} \delta(z - z_1 z_2 - \alpha \mu)$

$$\text{Thus: } I = \frac{2}{|\alpha|} \frac{\Theta(1-|\tilde{\mu}|)}{\sqrt{1-\tilde{\mu}^2}} \quad \text{where } \tilde{\mu} \equiv (z - z_1 z_2) / \alpha$$

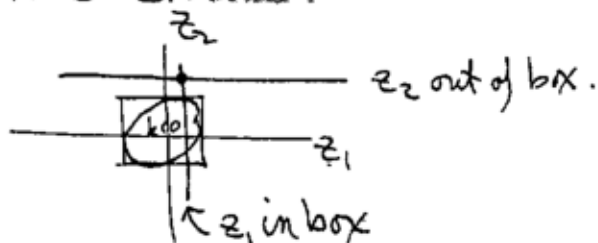
$$\text{But } \Theta(1-|\tilde{\mu}|) = \Theta(1-\tilde{\mu}^2). \text{ Also,}$$

$$1-\tilde{\mu}^2 = \frac{-k(z, z_1, z_2)}{\alpha^2} \Rightarrow \sqrt{1-\tilde{\mu}^2} = \sqrt{-k}/|\alpha|.$$

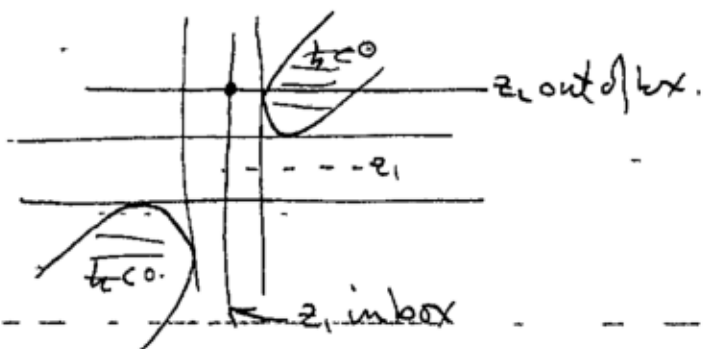
$$\text{Also, } \Theta(1-\tilde{\mu}^2) = \Theta(-k). \text{ So, } I = \frac{2 \Theta(-k(z, z_1, z_2))}{\sqrt{-k(z, z_1, z_2)}}$$

- ⑥ We have shown the equality when z_1, z_2 are both inside or both outside the unit box. When z_1, z_2 are "mixed", we showed that integral = 0 because δ cannot hit. Similarly, I can now show that $\Theta(-k)$ also vanishes under mixed condition:

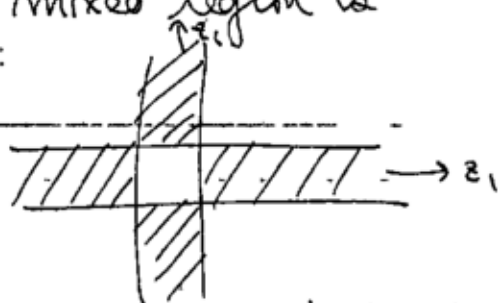
If $|z_1| < 1$, we have:
Thus, $k > 0$ at such a point, so $\Theta(-k) = 0$.



If $|z_2| > 1$, we have:
Again, $k > 0$ so $\Theta(-k) = 0$.



The "mixed" region is this:



In this region, k is always $+$, regardless of z !

- ⑦ Final result:

$$\left\{ \int_0^{2\pi} d\phi \delta(z - z_1 z_2 - \sqrt{1-z_1^2} \sqrt{1-z_2^2} \cos \phi) = \frac{2\Theta(-k)}{\sqrt{-k(z, z_1, z_2)}} \right\}$$

$+$, either OK.

- ⑧ Variations? Since the sign of α played no role in the derivation, either sign OK before $\sqrt{\cdot} \sqrt{\cdot}$.
I.e., you can write $\sqrt{z_1^2 - 1} \sqrt{z_2^2 - 1}$ if you like, but keep cuts as shown in ②

Secondly, since RHS is symmetric in any pair of z 's, you can rewrite the integral accordingly.

A theorem using $\sqrt{-k}$:

$$= \int_a^b \frac{dz}{\sqrt{-(z-a)(z-b)}}?$$

① What is $J \equiv \int_{-\infty}^{\infty} dz \cdot \frac{\Theta(-k)}{\sqrt{-k(z-z_1)(z-z_2)}}?$

We already know that $J=0$ if z_1, z_2 are "mixed", because $k > 0$ $\forall z$ in mixed regions. So, we need only consider the "pure" cases.

② $J = \int_{-\infty}^{\infty} dz \cdot \int_0^{\pi} d\phi \cdot \delta(z - z_1 z_2 - \alpha \cos \phi)$
 $= \int_0^{\pi} d\phi \cdot \int_{-\infty}^{\infty} dz \delta(z - \tilde{z})$
 $= \int_0^{\pi} d\phi \cdot 1 = \pi \leftarrow !$

/note: in pure regions, $\tilde{z}$ has to be real, so

③ Now we can divide into exterior + interior cases: For the interior case, we have:

$$\int_{-\infty}^{\infty} dz \frac{\Theta(-k)}{\sqrt{-k}} = \int_{-1}^1 dz \frac{\Theta(-k)}{\sqrt{-k}} = \pi \quad |z_1|, |z_2| < 1$$

I.e., in (z, z_1) -space, $|z_2| < 1 \Rightarrow$ ellipse and z only picks up contributions in $|z| \leq 1$.

For exterior case, we may say:

$$\int_{-\infty}^{\infty} dz \frac{\Theta(-k)}{\sqrt{-k}} = \left(\int_{-\infty}^{-1} + \int_1^{\infty} \right) dz \frac{\Theta(-k)}{\sqrt{-k}} = \pi \quad |z_1|, |z_2| > 1$$

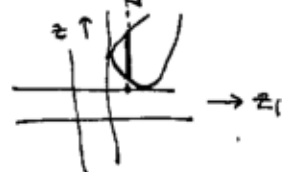
Again, $|z_2| > 1 \Rightarrow$ hyperbola in (z, z_1) space, etc.

④ As a slight refinement of the exterior case, we can say:

$$\int_{-\infty}^{\infty} dz \frac{\Theta(-k)}{\sqrt{-k}} = \left[\pi = \int_1^{\infty} dz \frac{\Theta(-k)}{\sqrt{-k}} \quad z_1, z_2 > 0 \right]$$

$$\left[\pi = \int_{-\infty}^{-1} dz \frac{\Theta(-k)}{\sqrt{-k}} \quad z_1, z_2 < 0 \right]$$

Easy to show this graphically, eg:
 Note that actual integration region is inside (red line).



Comments on the $a + b \cos x$ integral

① When $|a| > |b|$, $\int_0^{2\pi} d\phi \frac{1}{a + b \cos \phi} = \frac{2\pi}{\sqrt{a^2 - b^2}}$.

→ What happens if $|a| < |b|$? First, convert to $z = \cos \phi$,
 $dz = -\sin \phi d\phi$, so $\int_0^{2\pi} d\phi \frac{1}{a + b \cos \phi} = \int_{-1}^1 \frac{dz}{\sqrt{1-z^2}} \cdot \frac{1}{(a+bz)}$.
 Answer: naive result is correct.

$$\int_0^{2\pi} d\phi \frac{1}{a + b \cos \phi} = 2 \int_0^{\pi} d\phi \frac{1}{a + b \cos \phi} = 2 \int_{-1}^1 \frac{dz}{\sqrt{1-z^2}} \cdot \frac{1}{(a+bz)}$$

For the moment, let $a + bz = b(\frac{a}{b} + z) = b(-c + z)$
 where $c \equiv -a/b$. Then

$$L \equiv \int_0^{2\pi} d\phi \frac{1}{a + b \cos \phi} = 2 \cdot \frac{1}{b} \int_{-1}^1 \frac{dz}{\sqrt{1-z^2}} \cdot \frac{1}{(z-c)}$$

I am curious about:

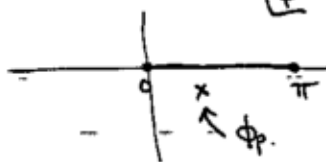
$$L' \equiv \int_{-1}^1 \frac{dz}{\sqrt{1-z^2}} \cdot \frac{1}{(z-c)}$$

② Or, we can use: integrand has pole when $\cos \phi = c = -a/b$.

$$\int_0^{\pi} d\phi \frac{1}{a + b \cos \phi}$$

$$\phi_p = \cos^{-1}(c) \quad c = \cos \phi_p$$

So.



What is the principle part, if ϕ_p lies on real axis between $0, \pi$?

$$= \frac{1}{b} \int_0^{\pi} d\phi \frac{1}{[\cos \phi - c]} = \frac{1}{b} \left(\int_0^{\phi_p - \epsilon} + \int_{\phi_p + \epsilon}^{\pi} \right) \frac{d\phi}{\cos \phi - c}$$

We may use Spiegel p 78:

$$\int \frac{d\phi}{\cos \phi - c} = \frac{1}{\sqrt{1-c^2}} \ln \left[\frac{\tan \frac{x}{2} + \sqrt{\frac{1-c}{1+c}}}{\tan \frac{x}{2} - \sqrt{\frac{1-c}{1+c}}} \right] = I(x).$$

Note that $I(0) = \frac{1}{\sqrt{1-c^2}} \ln(-1)$
 $I(\pi) = 0$

③ Assume $0 \leq \phi_p \leq \pi/2$, so that $\sin \phi_p$, $\cos \phi_p$, and $\tan \phi_p$ all +.

Note:

$$\tan \frac{\phi_p}{2} = \text{positive } \forall \phi_p \text{ in } 0, \pi$$

$$= + \sqrt{\frac{1 - \cos \phi_p}{1 + \cos \phi_p}} = + \sqrt{\frac{1 - c}{1 + c}}$$

$$\text{Now, } \tan\left(\frac{\phi_p \pm \epsilon}{2}\right) = \tan\left(\frac{\phi_p}{2} \pm \frac{\epsilon}{2}\right) = ?$$

$$\tan(x \pm \epsilon) = \tan x \pm \epsilon \sec^2 x$$

$$\sec^2 x = 1 + \tan^2 x$$

$$\sec^2 \frac{\phi_p}{2} = 1 + \tan^2 \frac{\phi_p}{2} = 1 + \frac{(1-c)}{(1+c)} = \frac{(1+c) + (1-c)}{1+c} = \frac{2}{1+c}$$

$$\text{Thus: } \tan\left(\frac{\phi_p \pm \epsilon}{2}\right) = \sqrt{\frac{1-c}{1+c}} \pm \frac{\epsilon}{2} \cdot \left(\frac{2}{1+c}\right)$$

④ We shall need $I(\phi_p - \epsilon) - I(\phi_p + \epsilon)$. That is:

$$\frac{1}{\sqrt{1-c^2}} \left\{ \ln \left[\frac{\sqrt{\frac{1-c}{1+c}} - \frac{\epsilon}{1+c} + \sqrt{\frac{1-c}{1+c}}}{\sqrt{\frac{1-c}{1+c}} - \frac{\epsilon}{1+c} - \sqrt{\frac{1-c}{1+c}}} \right] - \ln \left[\frac{\sqrt{\frac{1-c}{1+c}} + \frac{\epsilon}{1+c} + \sqrt{\frac{1-c}{1+c}}}{\sqrt{\frac{1-c}{1+c}} + \frac{\epsilon}{1+c} - \sqrt{\frac{1-c}{1+c}}} \right] \right\}$$

$$= \frac{1}{\sqrt{1-c^2}} \left\{ \ln \left[\frac{2\sqrt{\frac{1-c}{1+c}} - \epsilon/1+c}{-\epsilon/1+c} \right] - \ln \left[\frac{2\sqrt{\frac{1-c}{1+c}} + \epsilon/1+c}{+\epsilon/1+c} \right] \right\}$$

$$= \frac{1}{\sqrt{1-c^2}} \ln \left[\frac{2\sqrt{\frac{1-c}{1+c}} - \epsilon}{-\epsilon} \cdot \frac{+\epsilon}{2\sqrt{\frac{1-c}{1+c}} + \epsilon} \right]$$

$$= \frac{1}{\sqrt{1-c^2}} \ln \left[\frac{-2\sqrt{\frac{1-c}{1+c}} + \epsilon}{2\sqrt{\frac{1-c}{1+c}} + \epsilon} \right]$$

$$\text{As } \epsilon \rightarrow 0, \text{ this gives: } I(\phi_p - \epsilon) - I(\phi_p + \epsilon) = \frac{1}{\sqrt{1-c^2}} \ln(-1)$$

⑤ The entire principle part of this integral is:

$$\begin{aligned} \int_0^\pi d\phi \frac{1}{\cos \phi - c} &= I(\pi) - I(0) + I(\phi_p - \epsilon) - I(\phi_p + \epsilon) \\ &= 0 - \frac{1}{\sqrt{1-c^2}} \ln(-1) + \frac{1}{\sqrt{1-c^2}} \ln(-1) \\ &= 0 \end{aligned}$$

Amazing! Therefore:

$$\int_0^\pi d\phi \frac{1}{\cos \phi - c} = \pm i\pi \frac{1}{\sqrt{1-c^2}} \rightarrow \text{But this is just what the original formula says!}$$

Theorem relating $\sqrt{-k}$ to a $\sqrt{+k}$.

① Consider:

$$K = \int_{-\infty}^{\infty} dz \cdot f(z) \cdot \frac{\Theta(-k)}{\sqrt{-k(z, z_1, z_2)}}$$

As before, this must vanish in the mixed z_1, z_2 space. For pure z_1, z_2 we have:

$$K = \int_0^{\pi} d\phi f(\tilde{z}) \quad \tilde{z} = z_1 z_2 + \sqrt{1-z_1^2} \sqrt{1-z_2^2} \cos \phi.$$

② Special case: let $f(z) = \frac{1}{z-\mu}$. Then:

$$K = \int_0^{\pi} d\phi \frac{1}{(z_1 z_2 - \mu) + \sqrt{1-z_1^2} \sqrt{1-z_2^2} \cos \phi}$$
$$= \frac{1}{2} \int_0^{2\pi} d\phi \cdot \frac{1}{a + b \cos \phi} = \frac{\pi}{\sqrt{a^2 - b^2}} \quad \text{/ Spiegel p 96.}$$

$$= ? \quad a^2 - b^2 = (z_1 z_2 - \mu)^2 - (1-z_1^2)(1-z_2^2)$$
$$= \mu^2 + z_1^2 + z_2^2 - 1 - 2z_1 z_2 \mu = +k(\mu, z_1, z_2).$$

Therefore:

$$K = \left[\int_{-\infty}^{\infty} \frac{dz}{z-\mu} \cdot \frac{\Theta(-k(z, z_1, z_2))}{\sqrt{-k(z, z_1, z_2)}} = \frac{\pi}{\sqrt{+k(\mu, z_1, z_2)}} \right]$$

③ Be very careful with this. If $|b| > |a|$, we have shown that Spiegel's integral is still correct, so K goes imaginary. This gives with $k(\mu, z_1, z_2)$ going negative. Also, this gives with point μ lying on the integration contour. To show this: if z_1, z_2 are "interior", then $k(\mu, z_1, z_2) < 0 \Rightarrow \mu$ interior $\Rightarrow \mu$ lies on the pure integral. If z_1, z_2 exterior, then $k < 0 \Rightarrow |\mu| > 1 \Rightarrow$ again lies on contour + "causes" the sing.

④ When is formula true?

$\rightarrow$ a) z_1, z_2 must be "pure", both same nature.

$\rightarrow$ b) μ can take any real value.

c) actual integration is restricted as shown on previous page.

d) symmetry of $+k$ suggests other forms!

More play with (A.7).

① Start with $M \equiv 4\pi \int_1^\infty \frac{dz}{(z-\mu)} \frac{\Theta(+k(z, z_1, z_2))}{\sqrt{+k(z, z_1, z_2)}}$

Assume that z_1, z_2 are pure, so we can replace $\frac{1}{\sqrt{+k}}$:

$$M = 4\pi \int_1^\infty \frac{dz}{(z-\mu)} \cdot \Theta(+k(z, z_1, z_2)) \stackrel{1}{=} \frac{1}{\pi} \int_{-\infty}^\infty \frac{dx}{x-\mu} \frac{\Theta(-k(x, z_1, z_2))}{\sqrt{-k(x, z_1, z_2)}}$$

Next, move dz to the right

$$= -4 \cdot \int_{-\infty}^\infty dx \cdot \frac{\Theta(-k)}{\sqrt{-k(x, z_1, z_2)}} \cdot \left\{ \int_1^\infty \frac{dz \Theta(+k(z, z_1, z_2))}{(z-\mu)(z-x)} \right\}$$

Now evaluate:

$$f(z_1, z_2; \mu, x) = \int_1^\infty \frac{dz \Theta(+k(z, z_1, z_2))}{(z-\mu)(z-x)}$$

② First, assume z_1, z_2 are intern. Then $k < 0$ is entirely inside box and $\Theta(k)$ can be dropped.

$$\begin{aligned} &= \int_1^\infty \frac{dz}{(z-\mu)(z-x)} = \frac{1}{x-\mu} \ln \left[\frac{z-x}{z-\mu} \right] \Big|_1^\infty, \text{ Spiegel p. 62} \\ &= -\frac{1}{(x-\mu)} \ln \left(\frac{1-x}{1-\mu} \right) \end{aligned}$$

Then we have:

$$M = -4 \int_{-\infty}^\infty dx \left\{ \frac{\Theta(-k)}{\sqrt{-k(x, z_1, z_2)}} \cdot \frac{1}{x-\mu} \right\} \ln \left(\frac{1-x}{1-\mu} \right)$$

But intern $\Rightarrow x = \text{intern}$ also, so

$$M = 4 \cdot \int_{-1}^1 \frac{dx}{x-\mu} \ln \left(\frac{1-x}{1-\mu} \right) \cdot \frac{\Theta(-k)}{\sqrt{-k(x, z_1, z_2)}}$$

③ Now we could expand this $\uparrow$ to get

$$M = 4 \int_{-1}^1 \frac{dx}{x-\mu} \ln \left(\frac{1-x}{1-\mu} \right) \dots \text{use little } f(x) \text{ theorem.}$$

$$M = 4 \cdot \int_0^\pi d\phi \cdot \frac{\ln \left(\frac{1-\tilde{x}}{1-\mu} \right)}{\tilde{x}-\mu}, \text{ so what!}$$

What forms have I for I?

① One writes that $I = \int d\Omega \frac{1}{(\hat{q} \cdot \hat{k} - z_1)} \frac{1}{(\hat{q} \cdot \hat{k}' - z_2)}$

Thus, this thing is only defined for $|z| \leq 1$, where $z = \hat{k} \cdot \hat{k}'$.
For all z_1, z_2 , we know we can rewrite this as:

$$I = 4\pi \int_0^\infty \frac{dx}{(z_1^2 - 1)x^2 + (2)(z_1 z_2 + z)x + (z_2^2 - 1)}$$

This form then provides a continuation in z , if you like.

② Consider this denominator:

$Ax^2 + Bx + C$. Notice: $B^2 - 4AC = k(z, z_1, z_2) \times 4$

Spiegel p 21 tells us:

$$\int_0^\infty \frac{dx}{Ax^2 + Bx + C} = \frac{1}{\sqrt{B^2 - 4AC}} \ln \left[\frac{B + \sqrt{B^2 - 4AC}}{B - \sqrt{B^2 - 4AC}} \right]$$

It gives no restrictions. Thus,

$$\left\{ I = \frac{2\pi}{\sqrt{k(z, z_1, z_2)}} \ln \left[\frac{z_1 z_2 - z + \sqrt{k(z, z_1, z_2)}}{z_1 z_2 - z - \sqrt{k(z, z_1, z_2)}} \right] \right\}$$

③ I presume, then, that this is correct $\forall z_1, z_2$, and z , even when $k < 0$, with a prop. branch def. But, notice that if $|z| < 1$, then $k < 0$ only inside ellipse. So, for z_1, z_2 "exterior", k always > 0 . Also, if $|z| < 1$ and z_1, z_2 "mixed", still k always > 0 . But, z_1, z_2 "interior" runs you into $k < 0$, if $|z| < 1$.

④ From Spiegel, here is another rep for I:

$$I = 2\pi \int_{-\infty}^{\infty} \frac{dx}{(z_1 z_2 - z) \pm \sqrt{1 - z_1^2} \sqrt{1 - z_2^2} \cosh x} = 4\pi \int_0^\infty \dots$$

I see some other hyperkolic eps:

Another shot at (A.7)

① I am dead sure now that

- a) my hyperbolic rep. for I is correct, $\forall z_1, z_2, z$, certainly pure.
- b) my hyperbolic rep. for $\sqrt{k}$ is correct for pure.

② Now lets try to reduce (A.7) = I_7 down to I .

$$I_7 = \left(4\pi \int_1^\infty \frac{dz}{z-\mu} \frac{\Theta(k)}{\sqrt{k(z, z_1, z_2)}} \right)$$

$$= 4\pi \int_1^\infty \frac{dx}{x-z} \frac{\Theta(k(x, z_1, z_2))}{\sqrt{k(x, z_1, z_2)}}$$

If pure z_1, z_2 , then insert $\frac{\Theta}{\sqrt{k}} = \sum_{\tau} \int_0^\infty d\phi \delta(x - z_1 z_2 \pm \sqrt{1-z_1^2} \sqrt{1-z_2^2} \tau \cosh \phi)$

then

$$I_7 = 4\pi \int_0^\infty d\phi \cdot \sum_{\tau} \frac{1}{(\tilde{x}_\tau - z)} \cdot \Theta(\tilde{x}_\tau - 1)$$

where $\tilde{x}_\tau = z_1 z_2 + \sqrt{1-z_1^2} \sqrt{1-z_2^2} \tau \cosh \phi$ $\tau = \pm$.

③ In order to avoid this nasty $\Theta(\tilde{x}_\tau - 1)$, let's start instead with:

$$\tilde{I}_7 = 4\pi \int_{-\infty}^\infty \frac{dx}{x-z} \frac{\Theta[k(x, z_1, z_2)]}{\sqrt{k(x, z_1, z_2)}}$$

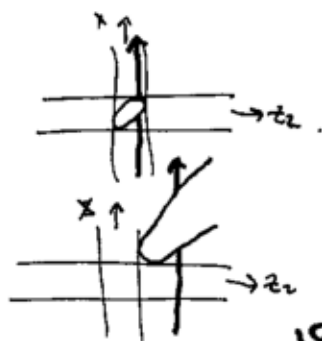
$$= 4\pi \int_0^\infty d\phi \cdot \sum_{\tau} \frac{1}{(z_1 z_2 - z) \pm \tau (\sqrt{1-z_1^2} \sqrt{1-z_2^2}) \cosh \phi}$$

$$= 2 \cdot I_- \quad \text{because same for either } \tau.$$

④ Thus, I have shown that in pure regime,

$$I = 2\pi \int_{-\infty}^\infty \frac{dx}{x-z} \frac{\Theta(+k(x, z_1, z_2))}{\sqrt{+k(x, z_1, z_2)}}$$

⑤ A geometric glance at this Θ shows this path of integration in the pure cases: (Say, $z_1 > 0$). So not obvious symmetry.

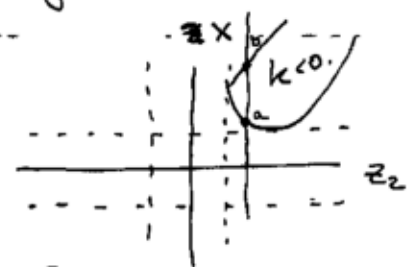
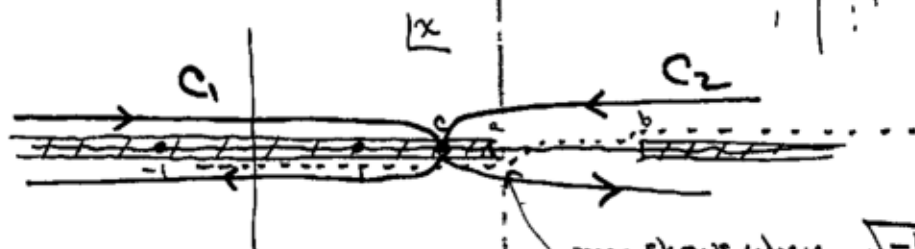


A symmetry property

- ① Let's assume the exterior situation. Then, if $z_1 > 0$ (and > 1), we have a $+45^\circ$ cut, as drawn.

Assume $z_2 > 1$, fixed.

Then, the cuts in the x -plane for $\sqrt{-k(x, z_1, z_2)}$ are as shown:



- ② In fact, on these cuts,

$$\text{disc}_x \frac{1}{\sqrt{-k}} = 2i \operatorname{Im} \frac{1}{\sqrt{-k}} = ?$$

I think $\sqrt{-k} = (-i)\sqrt{k}$ when $k > 0$ [ie, on cuts], on sheet where $\sqrt{-k}$ is positive between the cuts.

Thus, using this sheet:

$$\operatorname{Im} \frac{1}{\sqrt{-k}} = \operatorname{Im} \frac{i}{\sqrt{k}} = \frac{1}{\sqrt{k}}.$$

Thus:

$$\text{disc}_x \frac{1}{\sqrt{-k}} = 2i \frac{1}{\sqrt{k}}.$$

on Right Cut, = neg. on left cut. I use:

$$\text{disc} = f(x+i\epsilon) - f(x-i\epsilon).$$

- ③ Now consider:

$$N \equiv \int_{-\infty}^{\infty} dx f(x) \frac{\Theta(k)}{\sqrt{k(x, z_1, z_2)}}$$

Let $f(x)$ be some analytic fcn of x which ~~does not~~ $\rightarrow 0$ as $x \rightarrow \infty$. $\Theta(k)$ is another fcn, analytic except at $k=0$ where $\sqrt{k}$ has branch points anyway.

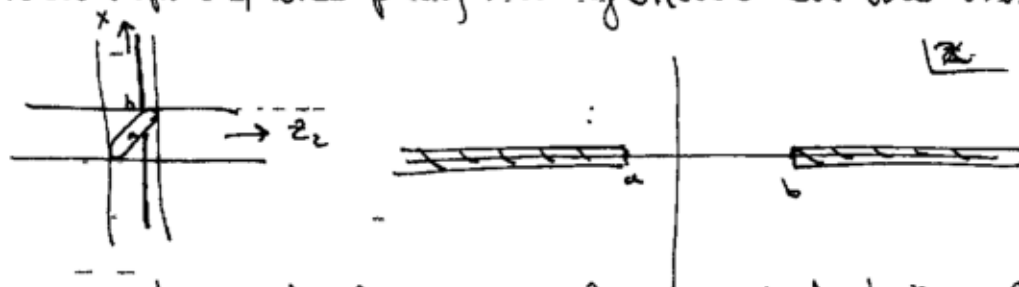
$$\begin{aligned} \text{Now, rewrite: } N &= \int_{C_1} dx f(x) \frac{\Theta(k)}{2i\sqrt{-k}} = - \int_{C_2} dx f(x) \frac{\Theta(k)}{2i\sqrt{-k}} \\ &= + \int_{C_1} dx f(x) \Theta(k) \frac{1}{2i} \text{disc}_x \frac{1}{\sqrt{-k}} \quad \text{Note: great circles vanish.} \\ &= \left(\int_{-\infty}^{\infty} + \int_{\infty}^{-\infty} \right) dx f(x) \frac{\Theta(k)}{\sqrt{k}} \end{aligned}$$

④ Therefore, we have shown:

$$\int_{-\infty}^c dx f(x) \frac{\Theta(k)}{\sqrt{k(x, z_1, z_2)}} = \int_c^{+\infty} dx f(x) \frac{\Theta(k)}{\sqrt{k(x, z_1, z_2)}}$$

True for any c .

⑤ In retrospect, the proof also follows for the "interval" z_1, z_2 .



The uncut region has merely shifted to the left.

⑥ Application:

$$\int_{-\infty}^1 dz \frac{1}{z-\mu} \frac{\Theta(k)}{\sqrt{k(x, z_1, z_2)}} = \int_1^{\infty} dz \frac{1}{z-\mu} \frac{\Theta(k)}{\sqrt{k(x, z_1, z_2)}}$$

Therefore

$$\int_{-\infty}^{\infty} dz \frac{1}{z-\mu} \frac{\Theta(k)}{\sqrt{k}} = 2 \cdot \int_1^{\infty} dz \frac{1}{z-\mu} \frac{\Theta(k)}{\sqrt{k}}.$$

I see no obvious way to show this otherwise!

⑦ Therefore, AS (A.7) is at last verified.

⑧ Near miraculous result:

$$\int_{-\infty}^{\infty} dx f(x) \frac{\Theta(k)}{\sqrt{k}} = 2 \int_c^{\infty} dx f(x) \frac{\Theta(k)}{\sqrt{k}}.$$

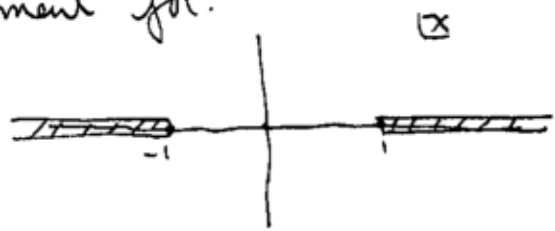
Thus, the integral is independent of c ! Amazing.
I don't believe it. That's right, aint true.

An explicit example of this Analytic Symmetry Theorem.

- ① I can repeat the previous argument for:

$$\tilde{N} = \int_{-\infty}^{\infty} dx \cdot \tilde{x}^{-2} \cdot \frac{1}{\sqrt{1-x^2}}$$

$$= 2 \int_c^{\infty} dx \cdot x^{-2} \cdot \frac{1}{\sqrt{1-x^2}}$$



This should therefore be independent of c . Let's see:
Spiegel p 69 has

$$\int dx \frac{1}{x^2} \frac{1}{\sqrt{1-x^2}} = - \frac{\sqrt{1-x^2}}{x}$$

Thus: $\int_c^{\infty} = -$ stop. I want $\sqrt{-(1-x^2)}$ to ^{appear} ~~have the cube~~.

② So try: $\tilde{N} = \int_{-\infty}^{\infty} dx \cdot \tilde{x}^{-2} \cdot \frac{1}{\sqrt{x^2-1}}$

Then: Spiegel p 68 says

$$\int_c^{\infty} dx \cdot x^{-2} \frac{1}{\sqrt{x^2-1}} = \frac{\sqrt{x^2-1}}{x} \Big|_c^{\infty} = 1 - \frac{\sqrt{c^2-1}}{c}$$

Also. Not indep of c !

③ Meanwhile, $\int_{-\infty}^c dx \cdot x^{-2} \frac{1}{\sqrt{x^2-1}} = \frac{\sqrt{x^2-1}}{x} \Big|_{-\infty}^c = \frac{\sqrt{c^2-1}}{c} - 1$

Answers are opposite — of course!

④ Try: $\int_{-\infty}^a = \int_{-1}^{\infty}$? $\int_{-\infty} = \int_{-1}^{\infty}$? not true.

$\int_{-\infty}^{-1} = \int_1^{\infty}$ yes, since integrand even.

⑤ Try $\int dx \cdot x^{-3} \frac{1}{\sqrt{x^2-1}}$. Well, $\int_{-\infty}^{-1} dx \cdot x^{-3} \frac{1}{\sqrt{x^2-1}}$
 $= \int_1^{\infty} (-dx) (-x^{-3}) \frac{1}{\sqrt{x^2-1}} = - \int_1^{\infty} dx \cdot x^{-3} \frac{1}{\sqrt{x^2-1}}$

Alternatives for A.7?

① Using $I_7 \equiv \int_1^\infty$, we end up with:
$$I_7 = \sum_{z=\pm} 4\pi \int_0^\infty d\phi \frac{\Theta(\tilde{x}_\tau - 1)}{\tilde{x}_\tau - z}$$

where $\tilde{x}_\tau = z_1 z_2 + \sqrt{1-z_1^2} \sqrt{1-z_2^2} \tau \cosh \phi$.

② Let's first assume the "interior" z_1, z_2 ; then where can $\tilde{x}_\tau$ be:

$$\tilde{x}_+ = z_1 z_2 + \sqrt{1-z_1^2} \sqrt{1-z_2^2} \cosh \phi$$

$$\tilde{x}_+(\min) = -|z_1 z_2| + \sqrt{1-z_1^2} \sqrt{1-z_2^2} > -1$$

$$\tilde{x}_+(\max) = +\infty$$

If $z_1 \approx 1-\epsilon$, then $\tilde{x}_+(\min) \approx -|z_2|$, which is less than 1. And $\tilde{x}_+(\max) = +\infty$.
Therefore, in the interior, $\tilde{x}_+$ requires a ϕ theta fn.

$$\tilde{x}_- = z_1 z_2 - \sqrt{1-z_1^2} \sqrt{1-z_2^2} \cosh \phi$$

$$\tilde{x}_-(\max) = |z_1 z_2| - \sqrt{1-z_1^2} \sqrt{1-z_2^2} \leq 1$$

$$\tilde{x}_-(\min) = -\infty$$

How large can $\tilde{x}_-(\max)$ be? Well, $(z_1 z_2 - 1) < \sqrt{1-z_1^2} \sqrt{1-z_2^2}$, since LHS is negative. Thus, $z_1 z_2 - \sqrt{1-z_1^2} \sqrt{1-z_2^2} < 1$. Thus, $\tilde{x}_-(\max) < 1$.
So, $\tilde{x}_-$ is $< 1 \forall \phi, \forall (z_1, z_2) \in \text{interior}$, so only $\tilde{x}_+$ "hits"

③ Therefore, for interior, we get not $\int_0^\infty d\phi$, but $\int_{\phi_{\min}}^\infty d\phi$, where $\cosh \phi_{\min} = -(1-z_1 z_2) / \sqrt{1-z_1^2} \sqrt{1-z_2^2}$.

But, this does not yield the I function, so: no go for z_1, z_2 intern.

④ Exterior? Define:
$$\begin{aligned} \tilde{x}_+ &= z_1 z_2 + \sqrt{z_1^2 - 1} \sqrt{z_2^2 - 1} \cosh \phi \\ \tilde{x}_+(\min) &= -|z_1 z_2| + \sqrt{z_1^2 - 1} \sqrt{z_2^2 - 1} \\ \tilde{x}_+(\max) &= +\infty \end{aligned}$$

Now, if $z_1 \approx 1+\epsilon$, then $\tilde{x}_+(\min) \approx -|z_2|$, certainly < 1 , so need a Θ .

And:

$$\tilde{x}_- = z_1 z_2 - \sqrt{z_1^2 - 1} \sqrt{z_2^2 - 1} \cosh \phi$$

$$\tilde{x}_-(\min) = -\infty$$

$$\tilde{x}_-(\max) = |z_1 z_2| - \sqrt{z_1^2 - 1} \sqrt{z_2^2 - 1}$$

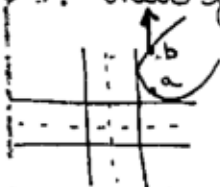
Well, $\tilde{x}_-(\max) > 1$, so Θ needed here too! If z_1 and $z_2 > 1$, need only this second piece, so here get not $\int_0^\infty d\phi$, but $\int_0^{\phi_{\max}} d\phi$ where $\cosh \phi_{\max} = (z_1 z_2 - 1) / \sqrt{z_1^2 - 1} \sqrt{z_2^2 - 1}$.

Again, this is no go.

The "(A.7)" of Goldberger Watson.

① They seem to be dealing with $z_1, z_2 > 1$. Their formula says:

$$I = \int_b^{\infty} (A.7) dz$$



② If I can show that $\int_{-\infty}^a = \int_b^{\infty}$, then my result will agree precisely with that of G.W., my result being $I = 2 \int_{-\infty}^a$. This will in turn shed lots of doubt on AS's use of 1 as a lower integration endpoint!

③ The question of $\int_{-\infty}^a = \int_b^{\infty}$ is intricately connected to the phase of $\sqrt{-k}$ on its cuts. Recall that, for $z_i > 1$, the cuts of $\sqrt{-k}$ are as shown here:



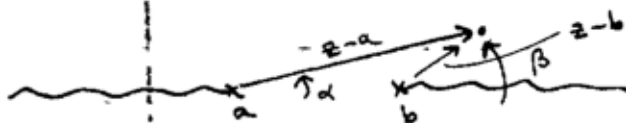
$$\begin{aligned} b &= z_1 z_2 + \sqrt{(z_1^2 - 1)(z_2^2 - 1)} \\ a &= z_1 z_2 - \sqrt{(z_1^2 - 1)(z_2^2 - 1)} \end{aligned}$$

← G.W. refer to this as " λ_{12} ".

④ We choose our sheet $\Rightarrow$ when $a \leq z < b$, $\sqrt{-k}$ is positive. The question is: at $z = c + i\epsilon$, $c > b$, what is the phase of this function! We need this phase in order to calc. Im. Thus re. an., this will then give us the disc.

$$k = (z-a)(z-b)$$

⑤ Standard analysis:



$$\left. \begin{aligned} z-a &= |z-a| e^{i\alpha} \\ z-b &= |z-b| e^{i\beta} \\ (-1) &= e^{\pm i\pi} \end{aligned} \right\}$$

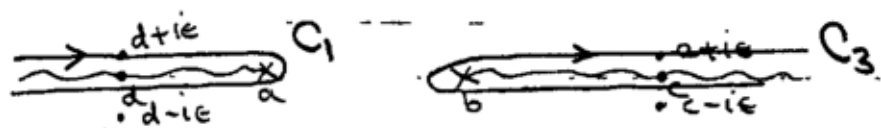
$$\sqrt{-k} = \sqrt{|z-a||z-b|} e^{i(\alpha+\beta-\pi)/2}$$

Sign chosen $\Rightarrow \alpha=0, \beta=\pi \Rightarrow \sqrt{-k}$ = positive. Thus,

$$\sqrt{-k} \Big|_{\substack{\text{on top} \\ \text{of right} \\ \text{cut}}} = \Big|_{\substack{\beta=0 \\ \alpha=0}} = e^{-i\pi/2} \sqrt{k} = (-i) \sqrt{k}$$

$$\sqrt{-k} \Big|_{\substack{\text{on top} \\ \text{of left cut}}} = \Big|_{\substack{\alpha=\pi \\ \beta=\pi}} = e^{+i\pi/2} \sqrt{k} = (+i) \sqrt{k}$$

⑥ This is in agreement with work on $\sqrt{z^2-1}$ etc. Now, what are the discontinuities of $\frac{1}{\sqrt{-k}}$ on these cuts?



We know that $\text{disc}_z g(z) \equiv g(z+ie) - g(z-ie)$
 $\quad \quad \quad = 2i \operatorname{Im} g(z+ie)$

Let $g(z) = \frac{1}{\sqrt{-k(z)}}$ $g(c+ie) = \frac{1}{(c-i)\sqrt{k}} = \frac{i}{\sqrt{k}}$

Thus, $\text{disc}\left(\frac{1}{\sqrt{-k}}\right)\Big|_c = 2i \frac{1}{\sqrt{k}}$

But, $g(d+ie) = \frac{1}{(d+i)\sqrt{k}} = \frac{-i}{\sqrt{k}}$ So

$\text{disc}\left(\frac{1}{\sqrt{-k}}\right)\Big|_d = -2i \frac{1}{\sqrt{k}}$

⑦ Now we repeat our former argument.

$$\begin{aligned} \int_{-\infty}^a dz f(z) \frac{1}{\sqrt{+k(z)}} &= \int_{-\infty}^a dz f(z) \frac{1}{(-2i)} \text{disc}\left(\frac{1}{\sqrt{-k}}\right) \\ &= \frac{1}{(-2i)} \int_{C_1} dz f(z) \frac{1}{\sqrt{-k}} = \frac{1}{(-2i)} \left(-\int_{C_3}\right) dz f(z) \frac{1}{\sqrt{-k}} \\ &= \int_{C_3} dz f(z) \left(\frac{1}{2i} \frac{1}{\sqrt{-k}}\right) = \int_b^{\infty} dz f(z) \frac{1}{2i} \text{disc} \frac{1}{\sqrt{-k}} \\ &= \int_b^{\infty} dz f(z) \frac{1}{\sqrt{k}} \end{aligned}$$

Some assumptions here, stated earlier.

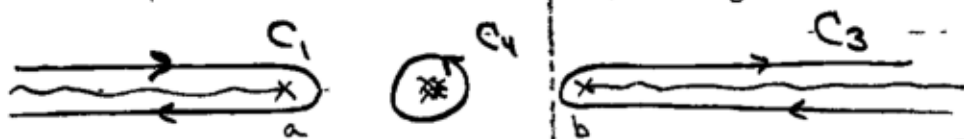
a) $f(z) \rightarrow 0$ as $z \rightarrow \infty$.

b) $f(z)$ analytic, except possibly on cuts.

⑧ Under these assumptions, we have shown that:

$$\int_{-\infty}^a dz f(z) \frac{1}{\sqrt{+k}} = \int_b^{\infty} dz f(z) \frac{1}{\sqrt{+k}}$$

⑨ However, if $f(z)$ has a singularity in the z plane somewhere, this is not quite true. Instead, we get:



$$\int_{-\infty}^a dz f(z) \frac{1}{\sqrt{z+k}} = \frac{1}{(-2i)} \int_{C_1} dz f(z) \frac{1}{\sqrt{-k}} = \text{old term} + \frac{1}{(-2i)} \int_{C_4} dz f(z) \frac{1}{\sqrt{-k}}$$

For example, let's say

there is a multiple pole

at e where $a < e < b$. Then:

$$\int_{-\infty}^a dz f(z) \frac{1}{\sqrt{z+k}} = \int_b^{\infty} dz f(z) \frac{1}{\sqrt{z+k}} - \frac{1}{2i} \oint_e dz f(z) \frac{1}{\sqrt{-k}}$$

⑩ Example: let $a = -1$, $b = +1$, $k = z^2 - 1$, $f(z) = z^{-3}$.
Thus, $e = 0$. Claim:

$$\int_{-1}^{-\infty} dz \cdot z^{-3} \cdot \frac{1}{\sqrt{1+z^2}} = \int_1^{\infty} dz \cdot z^{-3} \cdot \frac{1}{\sqrt{1+z^2}} - \frac{1}{2i} \oint_0 dz \cdot z^{-3} \cdot \frac{1}{\sqrt{1-z^2}}$$

Since integrand is odd in z , our prediction is:

$$2 \int_1^{\infty} dz \cdot z^{-3} \cdot \frac{1}{\sqrt{1+z^2}} = \frac{1}{2i} \oint_0 dz \cdot z^{-3} \cdot \frac{1}{\sqrt{1-z^2}}$$

⑪ To evaluate the LHS, Spiegel 14.215.

$$\int_1^{\infty} dz \cdot z^{-3} \cdot \frac{1}{\sqrt{z^2-1}} = \left(\frac{\sqrt{z^2-1}}{2z^2} + \frac{1}{2} \sec^{-1}|z| \right) \Big|_1^{\infty}$$

$$= \left(0 + \frac{1}{2} \cdot \frac{\pi}{2} \right) - \left(0 + \frac{1}{2} \cdot 0 \right) = \frac{\pi}{4} \Rightarrow \text{LHS} = \frac{\pi}{2}$$

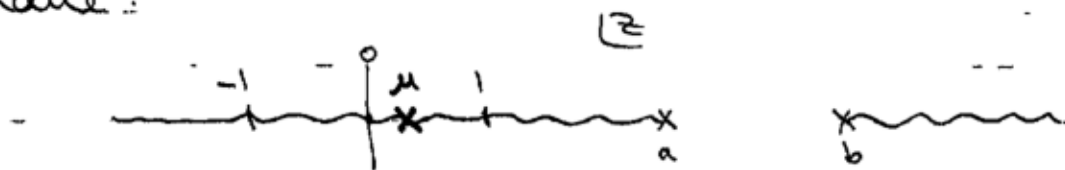
⑫ The RHS is:

$$\frac{1}{2i} \oint_0 \frac{dz}{\sqrt{1-z^2}} \frac{1}{(z-0)^3} = \pi \cdot \frac{1}{2!} \frac{d^2}{dz^2} \left(\frac{1}{\sqrt{1-z^2}} \right) \Big|_0^0$$

$$= \frac{\pi}{2} \frac{d}{dz} \left(z(1-z^2)^{-3/2} \right) \Big|_0^0 = \frac{\pi}{2} \left(6z^2(1-z^2)^{-5/2} + (1-z^2)^{-3/2} \right) \Big|_0^0 = \frac{\pi}{2}$$

✓ 23

- ⑬ Application to Abel + Saunders integral: If we are in the region exterior, then the break between the cuts does not overlap $(-1, 1)$. I.e., we have:



Now, A.S. use $f(z) = \frac{1}{z-\mu}$ where $|\mu| \leq 1$. Thus, their $f(z)$ has a pole at $z = \mu$. Since this lies on the cut, no problem, and we conclude:

$$\int_{-\infty}^a dz \frac{1}{z-\mu} \frac{1}{\sqrt{+k(z, z_1, z_2)}} = \int_b^{\infty} dz \frac{1}{z-\mu} \frac{1}{\sqrt{+k(z, z_1, z_2)}}$$

- ⑭ Now, my formula for I was:

$$I = \frac{1}{2} \cdot 4\pi \int_{-\infty}^{\infty} \frac{dz}{z-\mu} \frac{\Theta(+k(z, z_1, z_2))}{\sqrt{+k(z, z_1, z_2)}}$$

$$= \frac{1}{2} \cdot 4\pi \cdot \left(\int_{-\infty}^a + \int_b^{\infty} \right) \frac{dz}{z-\mu} \frac{1}{\sqrt{+k}}$$

$$I = 4\pi \int_b^{\infty} \frac{dz}{z-\mu} \frac{1}{\sqrt{+k(z, z_1, z_2)}}$$

$b = \text{appropriate point.}$

- ⑮ In particular, if $z_1 > 1$ and $z_2 > 1$, and $|\mu| \leq 1$, I have now proved conclusively that

$$I = 4\pi \int_b^{\infty} \frac{dz}{z-\mu} \frac{1}{\sqrt{+k(z, z_1, z_2)}}$$

where $b = z_1 z_2 + \sqrt{(z_1^2 - 1)(z_2^2 - 1)}$. My result agrees precisely with G.W. page 605. It is an error to write this as

$$4\pi \int_1^{\infty} \frac{dz}{z-\mu} \frac{\Theta(+k)}{\sqrt{+k}}$$

because $\int_1^a$ is added on!

⑩ Let's review the situation

a) we defined:

$$A \equiv I_{\frac{1}{2}} \equiv \text{The angular integral} \\ \equiv 2\pi \int_{-\infty}^{\infty} \frac{dx}{x-z} \frac{\Theta(k)}{\sqrt{k(x, z_1, z_2)}}$$

b) we showed that: $I = 2\pi \int_0^{\infty} \frac{dx}{(z_1 z_2 - z) \pm \sqrt{1-z_1^2} \sqrt{1-z_2^2} \cosh x} \quad \forall z_1, z_2$

c) we restricted z_1, z_2 to "pure", and inserted our rep:

$$\frac{\Theta(k)}{\sqrt{k(x, z_1, z_2)}} = \sum_r \int_0^{\infty} d\phi \delta(x - z_1 z_2 + r \sqrt{(1-z_1^2)(1-z_2^2)} \cosh \phi)$$

into A. The result was: $A = I$

d) Thus, our principle result was a representation for I:

$$I = 2\pi \int_{-\infty}^{\infty} \frac{dx}{x-z} \frac{\Theta(k(x, z_1, z_2))}{\sqrt{k(x, z_1, z_2)}}$$

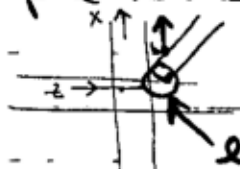
valid in "pure" regions. For "mixed" z_1, z_2 , A vanishes but I does not. (In fact, $I = \text{real in mixed regions}$)

e) is $I = A$ also in the "mixed" regions? Sure.

f) Next, we showed that if z does not lie between a and b , then:

$$I = 4\pi \int_b^{\infty} \frac{dx}{x-z} \frac{\Theta(k(x, z_1, z_2))}{\sqrt{k(x, z_1, z_2)}} \quad \leftarrow \text{This makes no diff.}$$

g) one example is: $z_1, z_2 > 1, |z| < 1$. Then this step f is OK! This is the G.W. result.



h) another example is $z_1 > 0$, but $z_1, z_2 < 1$. Then f is true if $|z| > 1$. Also, $b < 1$ in this case.



i) (A.7) as stated in AS is in error!

Improved integral rep for $\sqrt{1+k(a,b,c)}$

① Define $|\alpha| = |\sqrt{(a^2-1)(b^2-1)}|$

② Try: $I = \sum_r \int_1^\infty \frac{dt}{\sqrt{t^2-1}} \delta(c-ab + |\alpha| r t)$

③ Define: $\tilde{t}_1 = \frac{c-ab}{|\alpha|}$ $\tilde{t}_2 = -\tilde{t}_1$

Then:

$$I = \frac{1}{|\alpha|} \sum_r \int_1^\infty \frac{dt}{\sqrt{t^2-1}} \delta(t - r \tilde{t}_1) \leftarrow$$

But

$$\frac{1}{\tilde{t}_1} = \frac{1}{|\alpha|} \dots$$

Once you get here, result goes through as before.

not sure now!

④ Thus, we now have:

$$\frac{\Theta(k(z, z_1, z_2))}{\sqrt{k(z, z_1, z_2)}} = \sum_r \int_0^\infty d\phi \delta(z - z_1 z_2 + r \sqrt{(1-z_1^2)(1-z_2^2)} \cosh \phi)$$

↑ "improvement" ↑

This integral rep is now valid for any z_1, z_2 real.

⑤ Therefore, $A=I \forall z_1, z_2$

Proof of the Pl-Theorem:

with k-paper

4

① Want to show: $\left[\int_{-1}^1 d\mu P_\ell(\mu) \frac{\Theta(-k)}{\sqrt{-k(\mu, \mu_1, \mu_2)}} = \pi P_\ell(\mu_1) P_\ell(\mu_2) \right]$ ← only true for $|\mu_1| < 1$ $|\mu_2| < 1$

② Formally, we can start from the addition theorem:

$$P_\ell(z) = P_\ell(z_1) P_\ell(z_2) + 2 \sum_{m=1}^{\ell} d_{0m}^{\ell}(z_1) d_{m0}^{\ell}(z_2) \cos(\phi_m)$$

③ Now use integral rep for $\sqrt{-k}$ to get ↑ a piece of $D(z) = D(z_1) D(z_2)$

$$\begin{aligned} X &= \int_{-1}^1 d\mu P_\ell(\mu) \frac{\Theta(-k)}{\sqrt{-k(\mu, \mu_1, \mu_2)}} \\ &= \int_0^\pi d\phi P_\ell(\mu_1 \mu_2 + \sqrt{1-\mu_1^2} \sqrt{1-\mu_2^2} \cos\phi) = \int_0^\pi d\phi P_\ell(z[\phi]) \end{aligned}$$

④ Next, insert the addition theorem. Note that

$$\int_0^\pi d\phi \cos(\phi m) = 0 \quad \text{for } m = 1, 2, \dots, \ell$$

Thus

$$X = \int_0^\pi d\phi P_\ell(z_1) P_\ell(z_2) = \pi P_\ell(\mu_1) P_\ell(\mu_2) \quad \text{Q.E.D.}$$

⑤ By the way, now start with original theorem, mult both sides by $P_\ell(z)$, sum on $(2\ell+1)$:

$$\int_{-1}^1 d\mu \sum_{\ell=0}^{\infty} (2\ell+1) P_\ell(z) P_\ell(\mu) \frac{\Theta(-k)}{\sqrt{-k(\mu, \mu_1, \mu_2)}} = \pi \sum_{\ell=0}^{\infty} (2\ell+1) P_\ell(z) P_\ell(\mu_1) P_\ell(\mu_2)$$

Now use:

$$\sum_{\ell} (2\ell+1) P_\ell(z) P_\ell(\mu) = 2\delta(z-\mu)$$

$$\text{Thus: } \left[\frac{\Theta(-k(z, z_1, z_2))}{\sqrt{-k(z, z_1, z_2)}} = \frac{\pi}{2} \sum_{\ell=0}^{\infty} (2\ell+1) P_\ell(z) P_\ell(z_1) P_\ell(z_2) \right]$$

An unusual rep for $-k$.

On the Riemann.

4

① I think this is true for all $z_1, z_2 \geq 1$ and for all l :

$$\int_1^\infty dz \, P_l(z) \frac{\sigma(-k)}{\sqrt{-k(z, z_1, z_2)}} = \pi P_l(z_1) P_l(z_2)$$

in this paper.

Let's test it in lots of ways.

② First, apply $\int_{-\infty}^\infty dz_1$ to both sides, assume z_1, z_2 "pure" then

$$\text{LHS} = \int_1^\infty dz \, P_l(z) \cdot \pi \quad \text{RHS} = \pi \int_{-\infty}^\infty dz_1 \, P_l(z_1) \cdot P_l(z_2)$$

I suspect both sides diverge.

③ Apply $\int_1^\infty dz_1$ to both sides. LHS = $\frac{1}{2}\pi$, so:

$$\int_1^\infty dz \, P_l(z) \cdot \frac{1}{2}\pi = \int_1^\infty dz_1 \, P_l(z_1) \cdot P_l(z_2) \pi$$

Both sides still diverge, so not a viable test.

④ Try the case $l=0$:

$$\underbrace{\int_1^\infty dz \cdot 1}_{=\pi} \cdot \underbrace{\frac{\sigma(-k)}{\sqrt{-k}}}_{=\pi} = ? \underbrace{\pi \cdot 1 \cdot 1}_{=\pi} \quad \text{Yes!}$$

⑤ Try the case $l=1$. It also works!

$$\int_1^\infty dz \cdot z \cdot \frac{\sigma(-k)}{\sqrt{-k(z, z_1, z_2)}} = \int_{\lambda_-}^{\lambda_+} dz \cdot z \cdot \frac{1}{\sqrt{-(z-\lambda_+)(z-\lambda_-)}}$$

$$= \int_a^b \frac{z \cdot dz}{\sqrt{(b-z)(z-a)}}$$

where

$$a = z_1 z_2 \mp \sqrt{z_1^2 - 1} \sqrt{z_2^2 - 1}$$

$$b = z_1 z_2 + \sqrt{z_1^2 - 1} \sqrt{z_2^2 - 1}$$

OK, up to sign problem.

⑥ Look at Spiegel page 63 where he shows:

$$\int_a^b \frac{z \, dz}{\sqrt{(-z+b)(z-a)}} = \frac{\sqrt{(-z+b)(z-a)}}{(-1)} \Big|_a^b - \frac{(b \cdot 1 + 1 \cdot a)}{-2} \times \frac{2}{1} \tan^{-1} \sqrt{\frac{(-z+b)}{(z-a)}} \Big|_a^b$$

$$= (a+b) [\tan^{-1}(0) - \tan^{-1}(\pm\infty)] = -2z_1 z_2 \tan^{-1}(\pm\infty)$$

$$= \mp \pi z_1 z_2$$

⑦ However, the other result

$$\frac{\phi(-k)}{1-k} = \sum_{\ell} (\ell+1) P_{\ell}(z) P_{\ell}(\bar{z}_1) P_{\ell}(\bar{z}_2).$$

This cannot be true when $\bar{z}_1, \bar{z}_2 > 1$ because

$$\sum_{\ell=0}^{\infty} (\ell+1)^\ell \text{ diverges!}$$

① I believe that I have shown $\forall z, z_1, z_2$ that (also, $\forall z$).

$$I = 2\pi \int_{-\infty}^{\infty} \frac{dx}{x-z} \frac{\Theta(x)}{\sqrt{k(x, z_1, z_2)}}$$

Thus, this must be true $\forall z, z_1, z_2$:

$$\frac{1}{2} \int_{-\infty}^{\infty} dz Q(z) \frac{\Theta(x)}{\sqrt{k(z, z_1, z_2)}} = Q(z_1) Q(z_2)$$

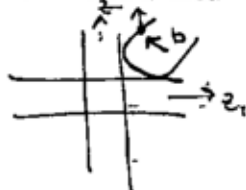
I was going to try to isolate $Q(z)/\sqrt{k}$, but there is no rule which says $\sum_z (z+1) Q(z) Q(z') = \delta(z-z')$. Maybe some complex integral would work.

② Due to "sym" properties, there are other forms for I , hence for this Q1 - Theorem:

$$\left. \begin{array}{l} z_1 \geq 1 \\ z_2 \geq 1 \\ -1 \leq x \leq 1 \end{array} \right\} \Rightarrow$$

$$I = 4\pi \int_b^{\infty} \frac{dx}{x-z} \frac{1}{\sqrt{k(x, z_1, z_2)}}$$

where $b = \lambda_1$ of G.W. p 605 = $z_1 z_2 + \sqrt{z_1^2 - 1} \sqrt{z_2^2 - 1}$



Thus, when $z_1 \geq 1, z_2 \geq 1$ we have:

$$\int_b^{\infty} dz Q(z) \frac{1}{\sqrt{k(z, z_1, z_2)}} = Q(z_1) Q(z_2)$$

For a derivation, see appendix of AS. For derivation of corrected (A.7) there, see separate appendix on k-function.

Is there a Q_l -theorem? with $\sqrt{-k}$

implicit in
big paper

5

① Well, $\int_1^\infty dx Q_l(x) \frac{2\Theta(-k)}{\sqrt{-k}(x, x_1, x_2)} = \int_0^{2\pi} d\phi \cdot Q_l(x_1 x_2 \pm \sqrt{x_1^2 - 1} \sqrt{x_2^2 - 1} \cos \phi)$

exactly as with P_l . If we assume that $x_1 > x_2$ for the moment, we get

$= 2\pi Q_l(x_1) P_l(x_2)$ and both > 1 .
based on Bateman p 69. (4). Check: ✓

$\left[\int_1^\infty dx Q_l(x) \frac{\Theta(-k)}{\sqrt{-k}(x, x_1, x_2)} = \pi \left[\begin{array}{l} \Theta(x_1 - x_2) Q_l(x_1) P_l(x_2) \\ + \Theta(x_2 - x_1) Q_l(x_2) P_l(x_1) \end{array} \right] \right. \quad \left. \begin{array}{l} x_1 > 1 \\ x_2 > 1 \end{array} \right]$

② Can we diagonalize with this?

LHS = $\int_0^\infty d(chg) Q_l(chg) a(chg)$

RHS = $\int_0^\infty d(chg_1) \int_0^\infty d(chg_2) b(chg_1) c(chg_2) \int_0^\infty d(chg) Q_l(chg) \frac{\Theta(-k)}{\sqrt{-k}}$
 $= \int_1^\infty dx_1 b(x_1) \int_1^\infty dx_2 c(x_2) \int_1^\infty dx Q_l(x) \frac{\Theta(-k)}{\sqrt{-k}}$

This is not going to work

$= \int_1^\infty dx_2 \int_{x_2}^\infty dx_1 b(x_1) c(x_2) \pi Q_l(x_1) P_l(x_2)$
 $+ \int_1^\infty dx_2 \int_1^{x_2} dx_1 b(x_1) c(x_2) \pi Q_l(x_2) P_l(x_1)$

Maybe can use: $\pi \cot(\pi l) P_l = i Q_l - Q_{l-1}$

③ If l happens to be conical at $\frac{1}{2} + i\sigma$, then

$i\pi \ln(\pi\sigma) P_{\frac{1}{2}+i\sigma} = Q_{\frac{1}{2}+i\sigma} - Q_{\frac{1}{2}-i\sigma}$

• If conical means $-l-1 = l^*$. So we can say:

$= \pi [\Theta(x_1 - x_2) Q_l(x_1) P_l(x_2) + \Theta(x_2 - x_1) Q_l(x_2) P_l(x_1)]$
 $= \tan(\pi l) Q_l(x_1) Q_l(x_2) - \tan(\pi l) [\Theta(x_1 - x_2) Q_l(x_1) Q_{l-1}(x_2) + \dots]$

So we have obtained:

$$\int_{-\infty}^{\infty} dx Q_l(x) \frac{\partial Q_l(x)}{\sqrt{-K(x_1, x_1, x_2)}} = \tan \pi l Q_l(x_1) Q_l(x_2)$$

$$+ (-1) \tan \pi l \left[\theta(x_1 - x_2) Q_l(x_1) Q_{l-1}(x_2) + \theta(x_2 - x_1) Q_l(x_2) Q_{l-1}(x_1) \right]$$

from 1.
general result, $x_1, x_2 > 1$. The first term does allow
diag:

$$\int_{-\infty}^{\infty} dx_1 b(x_1) Q_l(x_1) \equiv b_l$$

Thus, we have:

$$Q_l = \tan \pi l \cdot b_l \cdot Q_l + \underbrace{Q_l}_{\uparrow \text{function of } b, c}$$

④ What is this second piece?!

$$\tilde{Q}_l = (-1) \tan \pi l \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dx_1 dx_2 b(x_1) c(x_2) \left[\theta(x_1 - x_2) Q_l(x_1) Q_{l-1}(x_2) + \theta(x_2 - x_1) Q_l(x_2) Q_{l-1}(x_1) \right]$$

Does this have some nice sym. property?

$$\tan \pi(-l-1) = -\tan \pi l$$

$$\text{So } \tilde{Q}_{-l-1} = (+1) \tan \pi l \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dx_1 dx_2 b(x_1) c(x_2) \left[\theta(x_2 - x_1) Q_l(x_1) Q_{l-1}(x_2) + \theta(x_1 - x_2) Q_l(x_2) Q_{l-1}(x_1) \right]$$

I see no way to relate this to $\tilde{Q}_l$.

Thus, no way to unload it!

Thus, Q_l does not diagonalize, in spite of $A+S$

I do not believe their result!

⑤ What about conical case: $Q_{l^*}(x_1) = Q(l^*, x_1) = Q^*(l, x_1^*)$?

No help. Croenquist analysis
does not seem to help as
far as diag'ing is concerned!