

Kinematic Singularities

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Cms/bws/lab

CMS KINEMATICS

$$S = (P_1 + P_2)^2$$

$$t = (P_1 - P_3)^2$$

$$P_1^* = \frac{\sqrt{s}}{2} \left[1 - \frac{4m^2}{s} \right]^{1/2}, m_1 = m_2$$

$$= \frac{1}{2} [s - 4m^2]^{1/2}$$

$$E_1^* = \frac{s + m_1^2 - m_2^2}{2\sqrt{s}}$$

$$P_1^* = \frac{\lambda(s, m_1^2, m_2^2)}{2\sqrt{s}} \quad \lambda = \Delta^{1/2}$$

$$E_2^* = \frac{s + m_2^2 - m_1^2}{2\sqrt{s}}$$

$$P_1^* = \frac{\sqrt{s^2 - 2(m_1^2 + m_2^2)s + (m_1^2 - m_2^2)^2}}{2\sqrt{s}}$$

$$P_1^* = \frac{\sqrt{[s - (m_1 + m_2)^2][s - (m_1 - m_2)^2]}}{2\sqrt{s}}$$

CROSS SECTIONS (Jackson norm)

$$\text{flux} = \sqrt{(P_1 P_2)^2 - m_1^2 m_2^2} = m_2 P_1^{\text{LAB}} = \sqrt{s} P_1^* = \frac{\lambda(s, m_1^2, m_2^2)}{2}$$

$$d\phi_i = d^3\vec{p}_i / [(2\pi)^3 2E_i] \quad \rightarrow \frac{s}{2}$$

$$d\sigma = \frac{1}{4 \cdot \text{flux}} \left(\prod_f d\phi_f \right) (2\pi)^4 \delta^4(\text{ext}) |M|^2$$

$$\frac{d\sigma}{d\Omega_3^*} = \frac{|M|^2}{64\pi^2 s} \cdot \frac{P_3^*}{P_1^*}$$

$$\frac{P_3^*}{P_1^*} = \frac{\lambda(s, m_3^2, m_4^2)}{\lambda(s, m_1^2, m_2^2)}$$

$$= |f^*|^2 \cdot \frac{P_3^*}{P_1^*}$$

$$f^* = -\frac{1}{8\pi\sqrt{s}} M$$

$$\frac{d\sigma}{dt} = \frac{1}{64\pi (flux)^2} |M|^2 = \frac{1}{64\pi s P_1^{*2}} |M|^2$$

$$\sigma_K = -\frac{1}{2 \cdot \text{flux}} \delta m M_{ii} = \frac{4\pi}{P_1^*} \delta m f^*(0)$$

$$\frac{d\sigma}{dt}(t=0) = (1 + \beta^2) \frac{\sigma_K^2}{16\pi}$$

$$\frac{d\sigma}{dt} = -\frac{\pi}{P_1^* P_3^*} \frac{d\sigma}{d\Omega} \quad \text{all cm.}$$

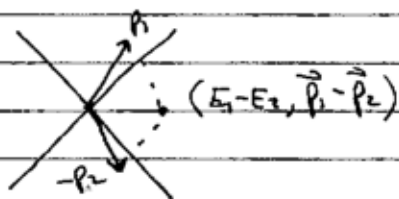
Note on cms. frames)

- ① Theorem: given two future timelike vectors $p_1 = (E_1, \vec{p}_1)$, $p_2 = (E_2, \vec{p}_2)$, it is always possible to find a frame in which $\vec{p}_1 + \vec{p}_2 = 0$.

proof: The sum of 2 F.T. vec's must be itself F.T., thus $(E_1 + E_2, \vec{p}_1 + \vec{p}_2)$ is F.T. But any F.T. vec can by LT. be brought to "rest" form, $(E' + E', \vec{p}' + \vec{p}' = 0)$. So, this gives the frame wherein $\vec{p}_1 + \vec{p}_2 = 0$.

- ② Theorem: given two F.T. vectors as above, it is not always possible to find a frame wherein $\vec{p}_1 = + \vec{p}_2$.

proof: The vector $p_1 - p_2 = (E_1 - E_2, \vec{p}_1 - \vec{p}_2)$ could be spacelike. If so, then you cannot "clean out" the 3-mom. components by doing L.T. In this case, $\exists$ no frame in which $\vec{p}_1 = \vec{p}_2$.



The Breit Frame (alias, the brick wall frame)

In a 2-to-2 reaction where $p_1 + p_2$ goes to $p_3 + p_4$, it is always possible to find a Lorentz frame in which the momentum of one of the final particles, say p_3 , is equal and opposite the momentum of one of the initials, say p_1 . In such a frame, particle one appears to bounce straight back and becomes particle 3. The reason such a frame is possible is this: Start with:

$$p_1^\mu = (a, b, c, d)$$

$$p_3^\mu = (e, f, g, h)$$

These are arbitrary but timelike 4-vectors. Now go to rest frame of p_1 :

$$\tilde{p}_1^\mu = (\tilde{a}, 0, 0, 0)$$

$$\tilde{p}_3^\mu = (\tilde{e}, \tilde{f}, \tilde{g}, \tilde{h})$$

Next, rotate so that p_3 has only a z-momentum. p_1 is unaffected:

$$\tilde{\tilde{p}}_1^\mu = (\tilde{\tilde{a}}, 0, 0, 0)$$

$$\tilde{\tilde{p}}_3^\mu = (\tilde{\tilde{e}}, 0, 0, \tilde{\tilde{h}})$$

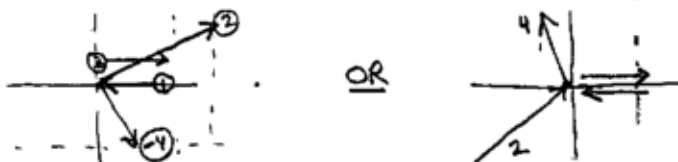
Now boost in the z direction (passive boost). This causes p_{3z} to decrease to a smaller positive value (assuming h is positive) and causes p_{1z} to go negative. Stop boosting when these momenta are equal and opposite. You have arrived in the Breit frame on the z axis:

$$^B p_1^\mu = (E_1, 0, 0, -z)$$

$$^B p_3^\mu = (E_3, 0, 0, +z)$$

By assuming that h was positive, we now have z positive. Of course, at the h stage we could have rotated 180 degrees around y, say, to get $-h$ instead of h , so in fact, z has no sign restriction.

For some weird mass configuration, then, the Breit frame gives:



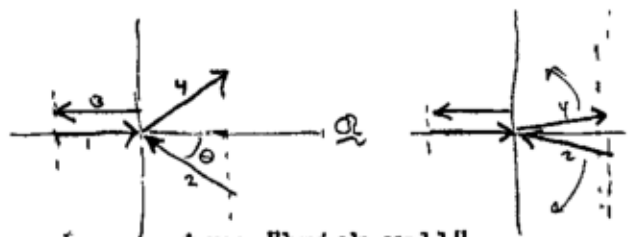
In general, 2 does not bounce off the brick wall in a very nice way to become 4.

However, if the mass structure is like PI-N or equal mass, then:

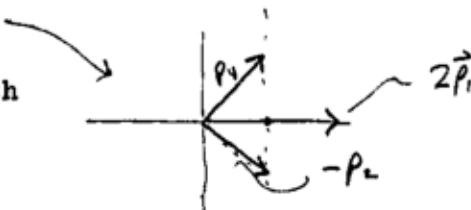
$$m_1 = m_3 \quad m_2 = m_4 \quad \vec{p}_1 = -\vec{p}_3 \Rightarrow E_1 = E_3 \Rightarrow E_2 = E_4 \Rightarrow |\vec{p}_2| = |\vec{p}_4|$$

$$\text{and } (\vec{p}_4) + (-\vec{p}_2) = \vec{p}_1 - \vec{p}_3 = (2\vec{p}_1)$$

The requirement that $\vec{p}_4$ and $-\vec{p}_2$ have equal length and equal transverse momenta restricts these vectors to have tips on the dotted line. Thus:

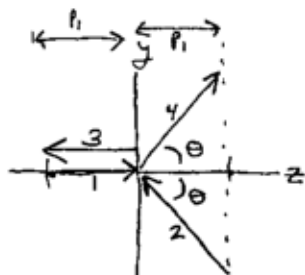


Now we have a true "brick wall".



Given the masses, the two degrees of freedom (s and t) are now the length of p_1 and the angle of p_2 .

These simple facts concerning the Breit frame are to be noted (here we are again in the PI-N mass structure or equal masses only):



$$\begin{aligned}\vec{p}_2 &= (-p_2 \cos \theta, p_2 \sin \theta) & p_2 > 0 & \quad p_2 \equiv |\vec{p}_2| \\ \vec{p}_1 &= (p_1, 0) & p_1 > 0 & \\ p_2 \cos \theta &= p_1 & \vec{p}_1 \cdot \vec{p}_2 &= -p_1 p_2 \cos \theta = -p_1^2\end{aligned}$$

The explicit expressions for s and t are:

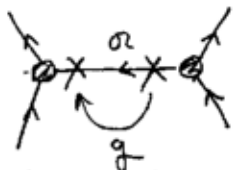
$$\begin{aligned}s &= (p_1 + p_2)^2 = m_1^2 + m_2^2 + 2 \left[\sqrt{m_1^2 + p_1^2} \cdot \sqrt{m_2^2 + \frac{p_1^2}{\cos^2 \theta}} + p_1^2 \right] = s(p_1, \theta) \\ t &= (p_3 - p_1)^2 = -4 p_1^2\end{aligned}$$

Unlike the cms expressions for s and t, here the angle appears only in s. Thus, a fixed-t process with s varying (energy cms varying) has a simple picture in the Breit frame. Changing s merely changes the angle θ . Now the range of θ is 0 to 90° . For a given t, the physical region starts at the value of s obtained with $\theta = 0$. Then as s gets infinite, θ approaches 90° .

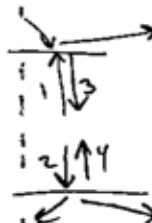
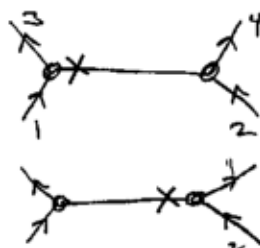
A convenient aspect of the Breit frame (again, in special mass case) is that the momentum transfer has a vanishing energy component:

$$\vec{Q} \equiv \vec{p}_3 - \vec{p}_1 = (0, 0, 0, -2p_1)$$

As discussed in AFFR, this makes some calculations easier. In passing, I notice that in the following special case, the Breit frame coincides with the frames used with the BCP variables ("Standard Toller frame")



In other words, if you do a Toller decomposition of a 2-to-2 amplitude, then the Toller frames (marked with x's) coincide with Breit frames.



By the way, in a general mass situation, the 2-2 momentum transfer is not always spacelike. See Mandelstam diagrams in Kibble, Omnes Froissart.

References:

- 1(AFFR page 400 , use of Breit frame
- 2(MS page 210, problem 6 on Breit frame
- 3(Omnes Froissart, p 68 and 70, why t not always spacelike

$$\Delta, \mathbb{Z}$$

About the two different z 's which you can choose for z_t .

Consider this situation:

$$\text{Let } z_t = \hat{p}_1 \cdot \hat{p}_3^{\text{cms}}$$

$$z_t' = \hat{p}_1 \cdot \hat{p}_4^{\text{cms}}$$

Compute and compare. But first look:

Obviously, $\theta_t + \theta_t' = \pi$

$$\Rightarrow z_t' = \cos \theta_t' = \cos(\pi - \theta_t) = -\cos \theta_t = -z_t$$

$$\boxed{z_t' = -z_t}$$

I seem to forget that this simple fact is independent of all the masses, etc.

Just for fun, let's compute them and compare:

$$S = (p_2 - p_4)^2 = m_2^2 + m_4^2 - 2[E_2 E_4 - \vec{p}_2 \cdot \vec{p}_4]$$

$$= m_2^2 + m_4^2 - 2[E_2 E_4 - p_2 p_4 z_t]$$

$$\Rightarrow \frac{S - m_2^2 - m_4^2 + 2 E_2 E_4}{2 p_2 p_4} = z_t$$

$$s + t + u = \Sigma$$

$$\text{But } E_2 = \frac{\sqrt{s} + m_2^2 - m_1^2}{2\sqrt{s}}$$

$$E_4 = \frac{\sqrt{s} + m_4^2 - m_3^2}{2\sqrt{s}}$$

$$p_2 = \frac{\lambda(\sqrt{s}, m_1^2, m_2^2)}{2\sqrt{s}}$$

$$p_4 = \frac{\lambda(\sqrt{s}, m_3^2, m_4^2)}{2\sqrt{s}}$$

$$\text{So } z_t = \frac{(S - m_2^2 - m_4^2) \sqrt{s} \cdot 2\sqrt{s} + 2(\sqrt{s} + m_2^2 - m_1^2)(\sqrt{s} + m_4^2 - m_3^2)}{2\lambda(\sqrt{s}, m_1^2, m_2^2)\lambda(\sqrt{s}, m_3^2, m_4^2)}$$

$$\Rightarrow z_t = \frac{\sqrt{s}^2 + \sqrt{s}(2s - \Sigma) + (m_2^2 - m_1^2)(m_4^2 - m_3^2)}{\lambda(\sqrt{s}, m_1^2, m_2^2)\lambda(\sqrt{s}, m_3^2, m_4^2)}$$

Now do $3 \leftrightarrow 4$

$$z_t' = \frac{\sqrt{s}^2 + \sqrt{s}(2u - \Sigma) + (m_2^2 - m_1^2)(m_3^2 - m_4^2)}{\lambda(\sqrt{s}, m_1^2, m_4^2)\lambda(\sqrt{s}, m_3^2, m_2^2)} = -z_t \quad \checkmark$$

Question: Does $k < 0$ always lie inside $\Delta < 0$?

① I.e., does $k < 0 \Rightarrow \Delta < 0$?

↑ unambiguous



$$z_1 = \cos \theta_{34} = \frac{s^2 + s(2t_1 - \Sigma_1) + \Delta_1}{4s q_{12} q_{56}}$$

$$z_2 = \cos \theta_{35} = \frac{s^2 + s(2t_2 - \Sigma_2) + \Delta_2}{4s q_{56} q_{34}}$$

where $\Delta_1 = (m_1^2 - m_2^2)(m_5^2 - m_6^2)$
 $\Delta_2 = (m_5^2 - m_6^2)(m_3^2 - m_4^2)$

$$\Sigma_1 = m_1^2 + m_2^2 + m_5^2 + m_6^2$$

$$\Sigma_2 = m_5^2 + m_6^2 + m_3^2 + m_4^2$$

$$q_{12} = \sqrt{\Delta(s, m_1^2, m_2^2)} / 2\sqrt{s}$$

$$q_{56} = \sqrt{\Delta(s, m_5^2, m_6^2)} / 2\sqrt{s}$$

$$q_{34} = \sqrt{\Delta(s, m_3^2, m_4^2)} / 2\sqrt{s}$$

$$z = \cos \theta_{31} = \frac{s^2 + s(2t - \Sigma) + \Delta}{4s q_{12} q_{34}}$$

$$\Delta = (m_1^2 - m_2^2)(m_3^2 - m_4^2)$$

$$\Sigma = m_1^2 + m_2^2 + m_3^2 + m_4^2$$

② $k < 0 \Rightarrow z^2 + z_1^2 + z_2^2 - 2zz_1z_2 - 1 < 0$

$$\Rightarrow \frac{(s^2 + s(2t_1 - \Sigma_1) + \Delta_1)^2}{16s^2 q_{12}^2 q_{56}^2} + \frac{(s^2 + s(2t_2 - \Sigma_2) + \Delta_2)^2}{16s^2 q_{56}^2 q_{34}^2} + \frac{(s^2 + s(2t - \Sigma) + \Delta)^2}{16s^2 q_{12}^2 q_{34}^2} - \frac{(s^2 + s(2t_1 - \Sigma_1) + \Delta_1)(s^2 + s(2t_2 - \Sigma_2) + \Delta_2)(s^2 + s(2t - \Sigma) + \Delta)}{32s^3 q_{12}^2 q_{56}^2 q_{34}^2} - 1 < 0$$

Define N_1, N_2, N obvious. Mult. by $32s^2 q_{12}^2 q_{56}^2 q_{34}^2$:

$$2q_{34}^2 N_1^2 + 2q_{12}^2 N_2^2 + 2q_{56}^2 N^2 - N_1 N_2 N / s - 32s^2 q_{12}^2 q_{56}^2 q_{34}^2 < 0$$

Now break up: $N_1 = 2st_1 + A_1$

$A_1 = s^2 - s\Sigma_1 + \Delta_1$ etc.

Then:

$$q_{34}^2 (2st_1 + A_1)^2 + q_{12}^2 (2st_2 + A_2)^2 + q_{56}^2 (2st + A)^2 - \frac{(2st_1 + A_1)(2st_2 + A_2)(2st + A)}{2s} - 16s^2 q_{12}^2 q_{56}^2 q_{34}^2 < 0 ?$$

Define instead: $B_1 = \left(\frac{s^2 - s\varepsilon_1 + \Delta_1}{2s} \right)$, etc.

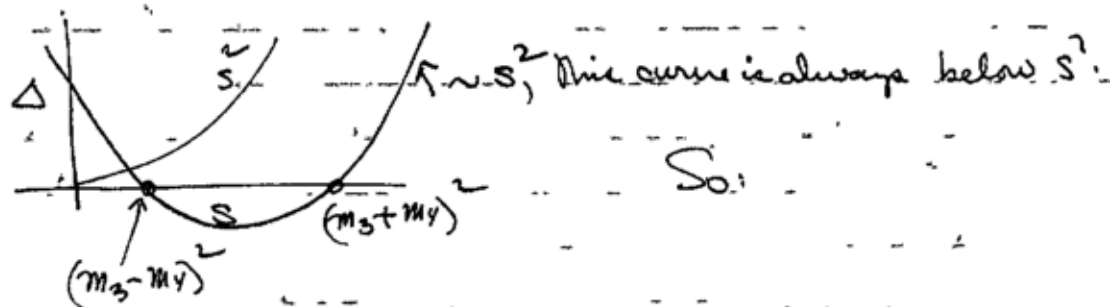
Then also: let $q_1 = q_{12}$, $q_5 = q_{56}$, $q_3 = q_{34}$. So:

$$q_3^2 (t_1 + B_1)^2 + q_1^2 (t_2 + B_2)^2 + q_5^2 (t + B)^2 - (t_1 + B_1)(t_2 + B_2)(t + B) / \cancel{4} - 4q_1^2 q_3^2 q_5^2 < 0$$

Now, divide each term by $(s/4)$.

Define $\alpha_3 \equiv \frac{q_3^2}{(s/4)} = \frac{\Delta(s, m_3, m_4)}{s^2}$, and α_1 and α_5 .

Notice that α_3 is < 1 ! Why?



Now, we are always going to be above all thresholds, therefore:

$$0 < \alpha_3 < 1$$

$$0 < \alpha_1 < 1$$

$$0 < \alpha_5 < 1$$

So: $\alpha_3 (t_1 + B_1)^2 + \alpha_1 (t_2 + B_2)^2 + \alpha_5 (t + B)^2 \Rightarrow$

$$- \frac{4}{s} (t_1 + B_1)(t_2 + B_2)(t + B) - \alpha_1 \alpha_3 \alpha_5 s^2 / 4 < 0$$

Rewrite using $\frac{2}{s} B_1 = 1 - \varepsilon_1/s + \Delta_1/s^2 \equiv 1 - C_1$
 $\alpha_3 + \beta_3 = 1$

$$\begin{aligned} & \Delta(t, t_1, t_2) + (2C_1 t_1 t_2 + 2C_2 t_1 t_1 + 2C_1 t_1 t_2) - \frac{4}{s} t_1 t_2 t \\ & - (4/s) (B_1 B_2 t + B_1 B t_2 + B_2 B t_1 + B B_1 B_2) \\ & - (\beta_3 t_1^2 + \beta_1 t_2^2 + \beta_5 t^2) \\ & + (2\alpha_3 t_1 B_1 + 2\alpha_1 t_2 B_2 + 2\alpha_5 t B) \\ & + (\alpha_3 B_1^2 + \alpha_1 B_2^2 + \alpha_5 B^2) \\ & - \alpha_1 \alpha_3 \alpha_5 s^2 / 4 < 0 \end{aligned}$$

Ignore up.

Some triangle function forms.

① define: $f(t, t_1, t_2) = + \sqrt{\frac{-\Delta(t, t_1, t_2)}{4t_1 t_2}}$

② Theorem: The arg of $\sqrt{}$ is positive if all real and
(a) $t, t_2 < 0 \Rightarrow \Delta > 0$
OR (b) $t, t_2 > 0$ and $\Delta(t, t_1, t_2) \leq 0$.

③ Theorem: if $t, t_2 > 0$ and $\Delta(t, t_1, t_2) \leq 0$, then

(a) $0 \leq f(t, t_1, t_2) \leq 1$

(b) can say $f = \sin \theta$, θ in quadrant 1.

(c) $\cos \theta = |t - t_1 - t_2| / 2\sqrt{t_1 t_2}$

$\sin \theta = \sqrt{-\Delta(t, t_1, t_2)} / 2\sqrt{t_1 t_2}$

(d) you could reverse $\sin \leftrightarrow \cos$ if you want.

④ Theorem: if $t, t_2 < 0$, then $\Delta > 0$ and.

(a) $f(t, t_1, t_2) \geq 1$

(b) can say $f = \cosh g$, with $g > 0$.

(c) $\sinh g = |t - t_1 - t_2| / 2\sqrt{-t_1 t_2}$

$\cosh g = \sqrt{\Delta(t, t_1, t_2)} / 2\sqrt{-t_1 t_2}$

⑤ define: $g(t, t_1, t_2) = + \sqrt{\frac{+\Delta(t, t_1, t_2)}{4t_1 t_2}}$

⑥ Theorem: (a) arg. of $\sqrt{}$ is pos. only if $t, t_2 > 0$ and $\Delta \geq 0$.

(b) Then, $g \geq 0 \Rightarrow g = \sinh g$, $g > 0$

(c) $\cosh g = |t - t_1 - t_2| / 2\sqrt{t_1 t_2} \geq 1$

$\sinh g = \sqrt{\Delta(t, t_1, t_2)} / 2\sqrt{t_1 t_2}$

Note:

Triangle Function

$$= (a+b+c)^2 - 4bc \quad \text{eg.}$$

$$\Delta(a,b,c) = a^2 + b^2 + c^2 - 2ab - 2ac - 2bc$$

$$\Delta(a,b,b) = a^2 - 4ab$$

$$\Delta(0,b,c) = (b-c)^2$$

area interpretation:



$$\text{Area} = \frac{1}{4} \sqrt{-\Delta(a^2, b^2, c^2)}$$

scaling:

$$x^2 \Delta\left(\frac{a}{x}, \frac{b}{x}, \frac{c}{x}\right) = \Delta(a,b,c)$$

momentum in cms:

$$p_i^* = \frac{1}{2\sqrt{s}} \Delta^{1/2}(s, m_1^2, m_2^2)$$

large s limit:

$$\Delta^{1/2}(s, b, c) \rightarrow (s - b - c) + \sqrt{\left(\frac{1}{s}\right)} = \left(-\frac{2bc}{s}\right)$$

other forms:

$$\Delta(a,b,c) = a^2 - 2a(b+c) + (b-c)^2$$

Fact:

$$|\cos \theta| = \left(\frac{a^2 - b^2 - c^2}{2bc} \right)$$

$$\Rightarrow |\sin \theta| = \frac{\sqrt{-\Delta(a^2, b^2, c^2)}}{2bc}$$

see another sheet

Or:

$$\cosh \eta = \frac{a^2 - b^2 - c^2}{2bc}$$

$$\Rightarrow |\sinh \eta| = \frac{\sqrt{+\Delta(a^2, b^2, c^2)}}{2bc}$$

Another case: if $a, b, c > 0$, then

$$\frac{\Delta^{1/2}(a, b, -c)}{2\sqrt{b} \sqrt{c}} = \cosh \theta$$

$$\left(\frac{a-b+c}{2\sqrt{b} \sqrt{c}} \right) = \sinh \theta$$

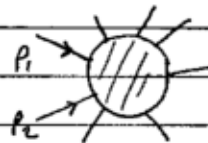
$$\cosh^2 \theta - 1 = \frac{\Delta(a, b, -c)}{4bc}$$

$$\frac{(a-b+c)^2 + 4bc - 4bc}{4bc} = \left(\frac{a-b+c}{2\sqrt{b} \sqrt{c}} \right)^2$$

$\Rightarrow$

The Triangle Function

- ① Let p_1 and p_2 be any timelike 4 vectors. ("momenta") connected to an S-matrix part:



- ② The physical region for the variables p_1^M and p_2^M is then restricted by the Gram determinantal condition:

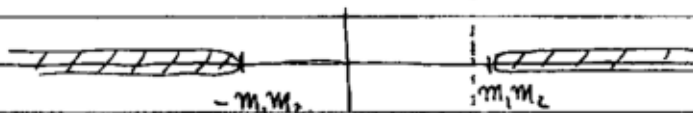
$$G \begin{pmatrix} p_1 & p_2 \\ p_1 & p_2 \end{pmatrix} \leq 0 \quad \text{OR} \quad \begin{vmatrix} p_1^2 & p_1 \cdot p_2 \\ p_1 \cdot p_2 & p_2^2 \end{vmatrix} \leq 0 \quad \text{OR} \quad p_1^2 p_2^2 \leq (p_1 \cdot p_2)^2$$

$$\text{OR} \quad m_1, m_2 \leq |p_1 \cdot p_2| \quad \begin{cases} \Rightarrow & \begin{array}{|l} p_1 \cdot p_2 \geq m_1 m_2 \\ p_1 \cdot p_2 \leq -m_1 m_2 \end{array} \\ \text{OR} \Rightarrow & \begin{array}{|l} p_1 \cdot p_2 \geq m_1 m_2 \\ p_1 \cdot p_2 \leq -m_1 m_2 \end{array} \end{cases}$$

↑ same sign ↓ opp sign

$\begin{array}{|l} p_1 \cdot p_2 \geq 0 \\ p_1 \cdot p_2 \leq 0 \end{array}$

In other words, the variable $p_1 \cdot p_2$ must lie in one of two regions which are separated, unless one or both masses $= 0$.



- ③ In fact, each of these two momenta is characterized by the sign of its energy component. I.e., each is future, or past. If both are future, then:

$$p_1 \cdot p_2 = E_1 E_2 - \vec{p}_1 \cdot \vec{p}_2 = |E_1| |E_2| - \vec{p}_1 \cdot \vec{p}_2 \geq |E_1| |E_2| - |\vec{p}_1| |\vec{p}_2|$$

$$\Rightarrow p_1 \cdot p_2 \geq +(\sqrt{p_1^2 + m_1^2})(\sqrt{p_2^2 + m_2^2}) - |\vec{p}_1| |\vec{p}_2|$$

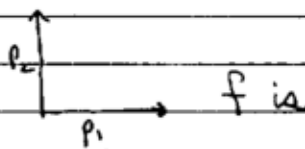
But, $\sqrt{p^2 + m^2} = p + \frac{1}{2} \frac{m^2}{p} + \dots$, terms have alt. signs.

- ④ Now consider mathematically:

$$F(p_1, p_2) = \sqrt{(p_1^2 + m_1^2)(p_2^2 + m_2^2)} - p_1 p_2 \quad f: \mathbb{R}^2 \rightarrow \mathbb{R}^1$$

What does this function look like?

(a) as long as $m_1 \neq 0, m_2 \neq 0$, over positive definite!



(b) if we fix p_2 + let $p_1 \rightarrow \infty$, $f \rightarrow +\infty$ since

$$\lim_{p_1 \rightarrow \infty} f(p_1, p_2) = p_1 [\sqrt{p_2^2 + m_2^2} - p_2] = |a| p_1 \rightarrow +\infty.$$

(c) Thus, we know that f takes its minimum value either on the $p_1=0$ or $p_2=0$ boundaries, or out in the middle.

(d) If out in the middle, then require:

$$\frac{\partial f}{\partial p_1} = 0 = \frac{p_1 (p_2^2 + m_2^2)}{\sqrt{p_1^2 + m_1^2}} - p_2$$

$$\frac{\partial f}{\partial p_2} = 0 = \frac{p_2 (p_1^2 + m_1^2)}{\sqrt{p_1^2 + m_1^2}} - p_1$$

This means:

$$\begin{aligned} p_2 &= p_1 \sqrt{p_2^2 + m_2^2} \\ p_1 &= p_2 \sqrt{p_1^2 + m_1^2} \end{aligned}$$

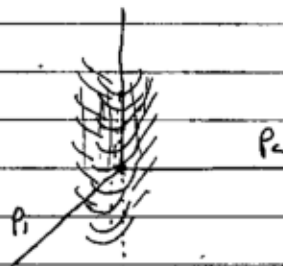
This is satisfied all along the line:

$$p_1 = p_2 \left(\frac{m_1}{m_2} \right)$$

And, in this case we find

$$f(p_1 = p_2 \frac{m_1}{m_2}, p_2) = m_1 m_2$$

(e) plotting this we get:



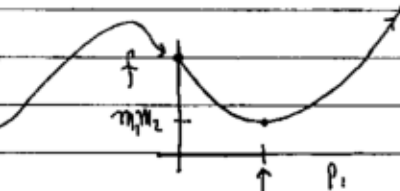
(f) Now consider $f(p_1, p_2)$ with p_2 fixed. Then

$$f = \sqrt{p_1^2 + m_1^2} - p_1 p_2$$

$$\frac{\partial f}{\partial p_1} = \frac{p_1 (p_2^2 + m_2^2)}{\sqrt{p_1^2 + m_1^2}} - p_2$$

Plot f vs. p_1 .

$$f(0, p_2) = m_1 \sqrt{p_2^2 + m_2^2}$$

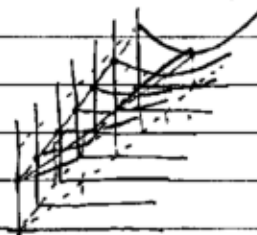


at $p_1=0$, slope is neg.
As p_1 increases, slope goes $\rightarrow +$. Eventually, slope is $\sqrt{p_2^2 + m_2^2}$.

Slope = 0 when $p_1 = \frac{m_1}{m_2} p_2$.

If $p_2=0$,

$$f = \sqrt{p_1^2 + m_1^2} m_2$$



We see the trough.

⑤ So, if both p_1, p_2 are future, or if both are past, then we know that $p_1 \cdot p_2 \geq m_1 m_2$. A faster way to show this would have been.

(a) gram says either $\geq m_1 m_2$ or $\leq -m_1 m_2$

(b) but $f(p_1, p_2)$ cannot be < 0 , this $\geq m_1 m_2$.

⑥ If p_1, p_2 have opposite signs of energy, then

$$-p_1 \cdot p_2 = |E_1| |E_2| + \vec{p}_1 \cdot \vec{p}_2 \geq |E_1| |E_2| - p_1 p_2 = f \geq 0.$$

Thus $p_1 \cdot p_2 \leq 0 \Rightarrow p_1 \cdot p_2 \leq -m_1 m_2$

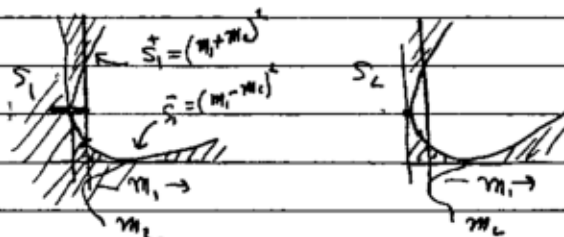
In fact, $p_1 \cdot p_2 = E_1 E_2 - \vec{p}_1 \cdot \vec{p}_2 = -|E_1| |E_2| - \vec{p}_1 \cdot \vec{p}_2$
 $\leq -|E_1| |E_2| + p_1 p_2$
 $\leq -f(p_1, p_2).$

But $f \geq m_1 m_2$, so $-f \leq -m_1 m_2 \Rightarrow p_1 \cdot p_2 \leq -m_1 m_2$ ✓

⑦ If we convert this gram condition to S_{ij} conditions, we get two equations:

$$\Delta(m_1^2, m_2^2, (p_1 + p_2)^2) \geq 0$$

$$\Delta(m_1^2, m_2^2, (p_1 - p_2)^2) \geq 0$$



Call these S_1 and S_2 .

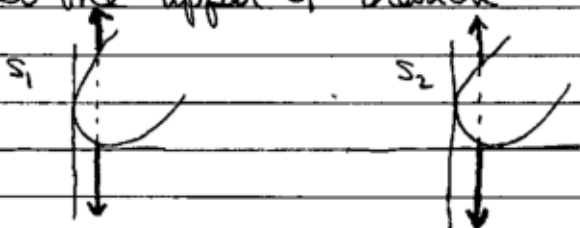
⑧ At some fixed m_1, m_2 (red lines), S_1 has two regimes, so does S_2 . But

$$S_1 \geq (m_1 + m_2)^2 \iff S_2 \leq (m_1 - m_2)^2$$

$$\updownarrow$$

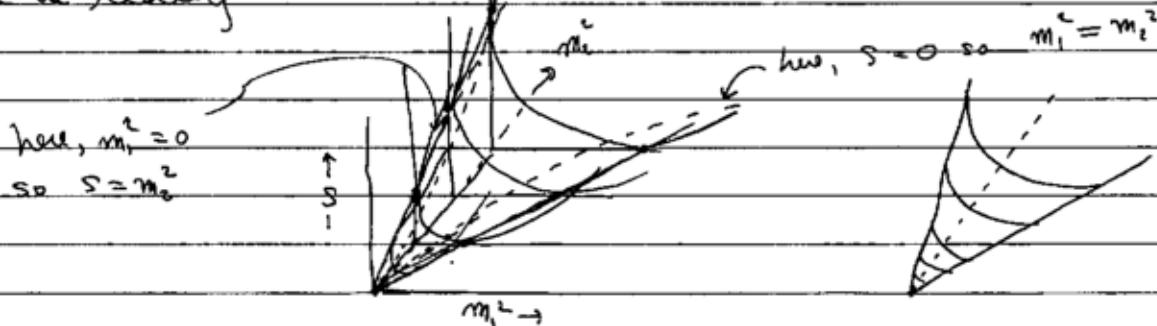
$$p_1 \cdot p_2 \geq m_1 m_2 \qquad -p_1 \cdot p_2 \leq -m_1 m_2$$

So the upper S_1 branch = the lower S_2 branch, + vice versa.



⑨ Now, depending on the signs of the energies of p_1, p_2 , only one of these pairs of restrictions counts. Eg, if both timelike, then purple only.

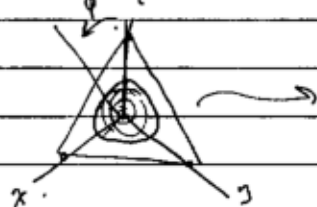
- ⑩ Now let's do a comprehensive plot of $\Delta(m_1^2, m_2^2, S) = 0$.
This is really:



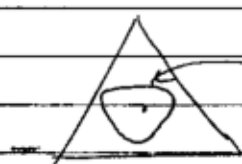
Set of a surface of parabolas. The line that touches the minima of all parabolas is the intersection of the Δ surface with $S = m_1^2$ plane. i.e., $m_2^4 - 4m_2^2 S = 0$ $m_2^2(m_2^2 - 4S) = 0$.
 $\Rightarrow S = \frac{1}{4} m_2^2$

- ⑪ Better to think of this in 3 variables at a time. Idea: what is the intersection of $\Delta(m_1^2, m_2^2, S)$ with the (1,1,1) plane?

$$\Delta(x, y, z) = 0$$



$$x + y + z = R \quad R = \sqrt{x^2 + y^2 + z^2}$$



I expect a symmetric curve here.

If the plane is at + distance R , center is $x = y = z \Rightarrow x = R/\sqrt{3}$.
 $\Rightarrow \sqrt{3x^2} = R \Rightarrow 3x^2 = R^2 \Rightarrow x = R/\sqrt{3}$.
Probably curve has a constant shape.

How far is (x, y, z) from $(R/\sqrt{3}, R/\sqrt{3}, R/\sqrt{3})$?

$$g^2 = (x - R/\sqrt{3})^2 + (y - R/\sqrt{3})^2 + (z - R/\sqrt{3})^2$$

$$\phi = \phi(x, y, z) = \text{ouch.}$$

- ⑫ Roughly, the $\Delta = 0$ surface is like a tetrahedron with rounded edges which comes to a point in the origin.

Some Triangle Calculations

① Let $F(a, b, c, d, e, f)$

$$\equiv \Delta(a, b, c) \Delta(d, e, f) - (a+b-c)^2 (d+e-f)^2$$

Easy to show that:

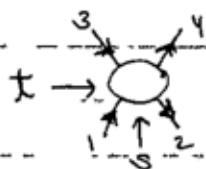
$$F_{abc, def} = -4 \left[ab(d^2 + e^2 + f^2 - 2de - 2ef) + de(a^2 + b^2 + c^2 - 2ac - 2bc) \right]$$

② A special case arises if $a = d$. Then:

$$F_{abc, aef} = -4a \left[(b+e)a^2 - 2(bf+ce)a + b(e-f)^2 + e(b-c)^2 \right]$$

Facts about z

① Consider:



First, compute $z_{12}^{(t)}$:

$$s = (p_1 - p_2)^2 = m_1^2 + m_2^2 - 2(E_1 E_2 - \mathbf{p}_1 \cdot \mathbf{p}_2 \cos \theta_{12}).$$

$$= m_1^2 + m_2^2 - 2 \frac{(t + m_1^2 - m_3^2)(t + m_2^2 - m_4^2)}{2\sqrt{t}} + 2\beta z_{12} \lambda(t, m_1, m_3) \lambda(t, m_2, m_4)$$

$$s = m_1^2 + m_2^2 - \frac{1}{2t} (t + m_1^2 - m_3^2)(t + m_2^2 - m_4^2) + \frac{1}{2t} z_{12} \lambda(t, 1, 3) \lambda(t, 2, 4)$$

$$\text{So: } s = m_1^2 + m_2^2 - \frac{t}{2} - \frac{1}{2}(1^2 - 3^2 + 2^2 - 4^2) - \frac{1}{2t}(1^2 - 3^2)(2^2 - 4^2) + \frac{1}{2t} z_{12} (\lambda(t, 1, 3) \lambda(t, 2, 4))$$

$$\Rightarrow 2ts = (1^2 + 2^2)2t - t^2 - t(1^2 - 3^2 + 2^2 - 4^2) - (1^2 - 3^2)(2^2 - 4^2) + z_{12} \lambda \lambda$$

$$\Rightarrow z_{12} = \frac{t^2 + t(2s - \Sigma) + \Delta}{\lambda \lambda}$$

$$\text{where } \Sigma = 1^2 + 2^2 + 3^2 + 4^2, \quad \Delta = (1^2 - 3^2)(2^2 - 4^2)$$

$$\textcircled{2} \text{ Specifically: } z_{12}^{(t)} = \frac{t^2 + t(2s - \Sigma) + (m_1^2 - m_3^2)(m_2^2 - m_4^2)}{\lambda(t, m_1, m_3) \lambda(t, m_2, m_4)}$$

One gives $z[s, t]$:

A nicer form uses $\lambda \lambda = 4t p_{13} p_{24} = 4t p_i p_f$. Then:

$$z_{12}^{(t)} = \frac{t + 2s - \Sigma + \Delta/t}{4 p_i p_f} = s/2 p_i p_f + \frac{(t - \Sigma + \Delta/t)}{4 p_i p_f}$$

$$= s/2 p_i p_f + d(t, \text{masses}).$$

This is the usual $z[s, t]$.

$$\textcircled{3} \text{ What is } s[z, t]? \quad s[z, t] = 2 p_i p_f (z - d).$$

④ What is $s[-z, t]$? Is it u ?

$$u = 1^2 + 4^2 - \frac{t}{2}$$

$$\Rightarrow z_{12}^{(t)} = \frac{t^2 + t(2u - \Sigma) + \Delta}{\lambda \lambda}$$

$$2tu = -t^2 + t\Sigma + (1^2 - 3^2)(2^2 - 4^2) + z \lambda \lambda$$

So we can compare:

$$\begin{array}{|l} z_{12} = \frac{t^2 + t(2s - \varepsilon) + \Delta}{\lambda\lambda} \quad \leftarrow \\ -z_{12} = \frac{t^2 + t(2u - \varepsilon) - \Delta}{\lambda\lambda} \end{array}$$

$$\begin{aligned} \Rightarrow & \frac{t(2s - \varepsilon) + \Delta}{\lambda\lambda} = \frac{-t(2u - \varepsilon) - \Delta}{\lambda\lambda} \\ \Rightarrow & -t\varepsilon + 2st + \Delta = -2ut + \Delta + t\varepsilon \Rightarrow \frac{2st + 2ut}{\lambda\lambda} = \frac{2t\varepsilon}{\lambda\lambda} \\ \Rightarrow & 2t(s - u) + 2\Delta = 0 \Rightarrow t(s - u) + \Delta = 0. \end{aligned}$$

Thus: $t^2 + 2ts - t\varepsilon + \cancel{\Delta} + t^2 + 2ut - t\varepsilon - \cancel{\Delta} = 0$

$$2t^2 + 2ts + 2tu - 2t\varepsilon = 0 \Rightarrow t + s + u - \varepsilon = 0 \checkmark$$

⑤ Now: solve both, one for s, one for u:

$$\frac{\lambda\lambda z - t^2 + t\varepsilon - \Delta}{2t} = s = s[z, t]$$

$$\frac{\lambda\lambda(-z) - t^2 + t\varepsilon + \Delta}{2t} = u$$

Therefore, $s[-z, t] = u$ only if $\Delta = 0$.

⑥ So, going back: $z = s/2\pi i f + d$

$$\Rightarrow s[z, t] = (z - d) \cdot 2\pi i f$$

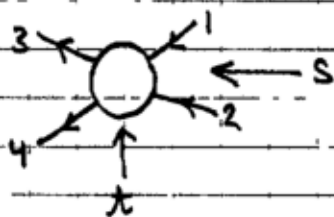
$$s[-z, t] = (-z - d) \cdot 2\pi i f$$

⑦ However, $\frac{s(-z, t)}{\lambda\lambda(-z) - t^2 + t\varepsilon - \Delta} = \frac{u - \Delta/t}{2t}$

And $u = -s - t + \varepsilon$, so

$$s[-z, t] = [-s - t + \varepsilon - \Delta/t]$$

The Relation between $\cos\theta$ and t



Using the usual notation and metric,

$$t = (p_3 - p_1)^2 = m_3^2 + m_1^2 - 2[E_3 E_1 - p_3 p_1 \cos\theta_{31}]$$

If we simply choose to work in the S-cms, then we know E_1, E_3, p_1, p_3 and result is:

$$\cos\theta_{13} = \frac{s^2 + s(2t - Z) + (m_1^2 - m_2^2)(m_3^2 - m_4^2)}{\Delta^{1/2}(s, m_1^2, m_2^2) \Delta^{1/2}(s, m_3^2, m_4^2)} \quad (1)$$

$= 4s q_{12} q_{34}$

where $Z = m_1^2 + m_2^2 + m_3^2 + m_4^2$. The denominator may also be written as

$$\Delta^{1/2} \Delta^{1/2} = 4s q_{12} q_{34} = 4s q_s^2$$

Notice that:

$$d(\cos\theta_{13}) = \left[\frac{+1}{2q_{s12} q_{s34}} \right] dt$$

Special Case (πN) $m_1 = m_3, m_2 = m_4$. This is simple because $E_1 = E_3$ etc throughout. In particular, from (1) above,

$$\begin{aligned} \cos\theta_{13} &= \frac{s^2 + s(2t - 2[m_1^2 + m_2^2]) + (m_1^2 - m_2^2)^2}{\Delta(s, m_1^2, m_2^2)} \\ &= \frac{2st + \Delta(s, m_1^2, m_2^2)}{\Delta(s, m_1^2, m_2^2)} = 1 + \frac{2st}{\Delta(s, m_1^2, m_2^2)} \\ &= 1 + \frac{t}{2q_s^2} \end{aligned}$$

Finally, if we make all masses equal, the result is the same and we can add that

$$\Delta(s, m^2, m^2) = 4s q_s^2 = s(s - 4m^2)$$

$$q_s^2 = \frac{1}{4}(s - 4m^2)$$

$$\cos\theta = 1 + \frac{2t}{s - 4m^2}$$

then the relation of $\cos\theta$ to t is:

Expand z in powers of $(1/s)$.

① We know that: $z = \underbrace{s^2}_{\lambda_1^{1/2}} + \underbrace{(2t-\varepsilon)s}_{\lambda_2^{1/2}} + \Delta$

② Suppose we find that:

$$(\lambda_1^{1/2})^{-1} = a_1 \left(\frac{1}{s}\right) + b_1 \left(\frac{1}{s}\right)^2 + c_1 \left(\frac{1}{s}\right)^3 + \dots$$

Then:

$$\begin{aligned} (\lambda_1^{1/2} \lambda_2^{1/2})^{-1} &= [a_1 \bar{s}^1 + b_1 \bar{s}^2 + c_1 \bar{s}^3] [a_2 \bar{s}^1 + b_2 \bar{s}^2 + c_2 \bar{s}^3] \\ &= (a_1 a_2) \bar{s}^2 + (a_1 b_2 + b_1 a_2) \bar{s}^3 + (a_1 c_2 + c_1 a_2 + b_1 b_2) \bar{s}^4 + \dots \end{aligned}$$

Then:

$$z = (s^2 + [2t-\varepsilon]s + \Delta) \left(\underbrace{[a_1 a_2] \bar{s}^2}_A + \underbrace{[a_1 b_2 + b_1 a_2] \bar{s}^3}_B + \underbrace{[a_1 c_2 + c_1 a_2 + b_1 b_2] \bar{s}^4}_C \right)$$

$$z = (s^2 + [2t-\varepsilon]s + \Delta) (A \bar{s}^2 + B \bar{s}^3 + C \bar{s}^4)$$

$$= (A) s^0 + (B + A(2t-\varepsilon)) \bar{s}^1 + (C + B(2t-\varepsilon) + A\Delta) \bar{s}^2 + \dots$$

So $z = F_0 s^0 + F_1 \bar{s}^1 + F_2 \bar{s}^2 + \dots$

where:

$$F_0 = A$$

$$F_1 = A(2t-\varepsilon) + B$$

$$F_2 = A\Delta + B(2t-\varepsilon) + C$$

where:

$$A = a_1 a_2$$

$$B = a_1 b_2 + b_1 a_2$$

$$C = a_1 c_2 + c_1 a_2 + b_1 b_2$$

③ So all we need are the a, b, c 's:

$$(\lambda_{12})^{-1} = [\lambda_{12}]^{-1/2} = [s^2 - 2(m_1^2 + m_2^2)s + (m_1^2 - m_2^2)^2]^{-1/2}$$

$$= \bar{s}^{-1} [1 - 2(m_1^2 + m_2^2) \bar{s}^1 + (m_1^2 - m_2^2)^2 \bar{s}^2]^{-1/2}$$

④ Now how to expand:

$$[1 + a\bar{x} + b\bar{x}^2]^{-1/2} = 1 - \frac{1}{2}(a\bar{x} + b\bar{x}^2) + \frac{(-\frac{1}{2})(-\frac{3}{2})}{2}(a\bar{x} + b\bar{x}^2)^2 + \dots$$

$$= 1 + (-\frac{1}{2}a)\bar{x} + (-\frac{1}{2}b + \frac{3}{8}a^2)\bar{x}^2 + \dots$$

↑ note extra piece.

⑤ Now, let $a = -2(m_1^2 + m_2^2)$
 $b = (m_1^2 - m_2^2)^2$
 $x = s.$

$$\text{Then: } (\lambda_{12})^{-1/2} = \bar{S}^{-1} \left[1 + (m_1^2 + m_2^2) \bar{S}^{-1} + \left\{ \frac{3}{2}(m_1^2 + m_2^2)^2 - \frac{1}{2}(m_1^2 - m_2^2)^2 \right\} \bar{S}^{-2} \right]$$

$$= \bar{S}^{-1} \left[1 + (m_1^2 + m_2^2) \bar{S}^{-1} + \{ (m_1^4 + m_2^4) + 2m_1^2 m_2^2 \} \bar{S}^{-2} \right]$$

$$= \bar{S}^{-1} \left[1 + (m_1^2 + m_2^2) \bar{S}^{-1} + (m_1^4 + m_2^4 + 4m_1^2 m_2^2) \bar{S}^{-2} \right].$$

⑥ Now, combine two of these:

$$(\lambda_{12})^{-1/2} (\lambda_{34})^{-1/2} = \bar{S}^{-2} \left[1 + (m_1^2 + m_2^2) \bar{S}^{-1} + (m_1^4 + m_2^4 + 4m_1^2 m_2^2) \bar{S}^{-2} \right] \left[1 + (m_3^2 + m_4^2) \bar{S}^{-1} + (m_3^4 + m_4^4 + 4m_3^2 m_4^2) \bar{S}^{-2} \right]$$

$$= \bar{S}^{-2} \left[1 + \Sigma \cdot \bar{S}^{-1} + \text{stop.} \right]$$

In 5 we showed that:

$$(\lambda_{12})^{-1/2} = 1 \cdot \bar{S}^{-1} + (m_1^2 + m_2^2) \cdot \bar{S}^{-2} + (m_1^4 + m_2^4 + 4m_1^2 m_2^2) \cdot \bar{S}^{-3}$$

Thus:

$$\begin{cases} a_1 = 1 \\ b_1 = m_1^2 + m_2^2 \\ c_1 = m_1^4 + m_2^4 + 4m_1^2 m_2^2 \end{cases} \quad \begin{cases} a_2 = 1 \\ b_2 = m_3^2 + m_4^2 \\ c_2 = m_3^4 + m_4^4 + 4m_3^2 m_4^2 \end{cases}$$

Thus:

$$A = a_1 a_2 = 1.$$

$$B = a_1 b_2 + a_2 b_1 = \Sigma$$

And, $C = (a_1 c_2 + c_1 a_2) + b_1 b_2 = c_2 + c_1 + b_1 b_2$

$$= (m_1^2 + m_2^2)^2 + 2m_1^2 m_2^2 + (m_3^2 + m_4^2)^2 + 2m_3^2 m_4^2 + (m_1^2 + m_2^2)(m_3^2 + m_4^2)$$

$$= (m_1^2 + m_2^2) [m_1^2 + m_2^2 + m_3^2 + m_4^2] + \dots \text{ so what.}$$

$$= (m_1^2 - m_2^2)^2 - 4m_1^2 m_2^2 + (m_3^2 - m_4^2)^2 - 4m_3^2 m_4^2 + (m_1^2 + m_2^2)(m_3^2 + m_4^2) \text{ so what.}$$

OK, C is a mess.

⑦ Now $F_0 = A = 1$.

$$F_1 = 1 \cdot (2t - \varepsilon) + \varepsilon = 2t$$

$$F_2 = 1 \cdot \Delta + \varepsilon(2t - \varepsilon) + C$$

⑧ F_2 needs simplification:

$$\Delta \varepsilon \varepsilon^2 + C = (m_1^2 - m_2^2)(m_3^2 - m_4^2) - (m_1^2 + m_2^2 + m_3^2 + m_4^2)^2 + C$$

$$= \Delta - \cancel{(m_1^2 + m_2^2)^2} - \cancel{(m_3^2 + m_4^2)^2} - 2(m_1^2 + m_2^2)(m_3^2 + m_4^2) + \cancel{(m_1^2 + m_2^2)^2} + \cancel{(m_3^2 + m_4^2)^2} + (m_1^2 + m_2^2)(m_3^2 + m_4^2) + 2m_1^2 m_2^2 + 2m_3^2 m_4^2$$

$$= (m_1^2 - m_2^2)(m_3^2 - m_4^2) - (m_1^2 + m_2^2)(m_3^2 + m_4^2) + 2m_1^2 m_2^2 + 2m_3^2 m_4^2$$

$$= -2m_1^2 m_2^2 - 2m_3^2 m_4^2 + 2m_1^2 m_2^2 + 2m_3^2 m_4^2$$

$$= 2(m_1^2 m_2^2 + m_3^2 m_4^2 - m_1^2 m_4^2 - m_2^2 m_3^2) \equiv 2G.$$

Thus: $F_2 = 2t\varepsilon + 2G$

⑨ End result:

$$Z = 1 + (2t)\varepsilon^1 + (2t\varepsilon + 2G)\varepsilon^2 + \dots$$

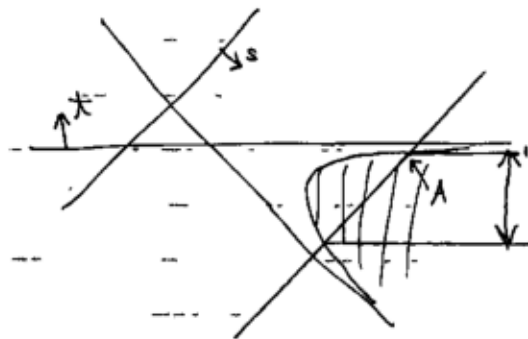
where $S' = m_1^2 + m_2^2 + m_3^2 + m_4^2$

$$G = m_1^2 m_2^2 + m_3^2 m_4^2 - m_1^2 m_4^2 - m_2^2 m_3^2$$

note sign. Z is $\frac{1}{1 - \varepsilon}$ not $\frac{1}{1 + \varepsilon}$

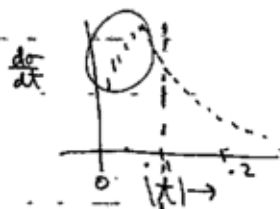
s-t-u

What is a "t-min effect"?



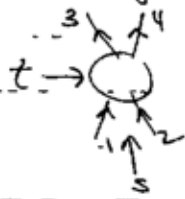
- ① Look at s.t.u for typical s-channel process:
Red line shows events in s-channel at some constant s. The range of physical t is shown as $\uparrow$. It does not come all the way up to $t=0$ except in equal mass case.

- ② Thus, in a DCS $\frac{d\sigma}{dt}$, the data will not go all the way to $t=0$ but will drop off at the kinematic boundary (point A), so you will see:



The dip at $t=0$ is merely a kinematic effect which is called "t-min effect".

- ③ Easy to calculate actual value of t-min



$$t = (p_3 - p_1)^2 = m_3^2 + m_1^2 - 2[E_1 E_3 - \vec{p}_1 \cdot \vec{p}_3]$$

$$t = m_3^2 + m_1^2 - 2 \left\{ \left(\frac{s + m_1^2 - m_2^2}{2\sqrt{s}} \right) \left(\frac{s + m_3^2 - m_4^2}{2\sqrt{s}} \right) - \left(\frac{\Delta^{1/2}(s, m_1^2, m_2^2)}{2\sqrt{s}} \right) \left(\frac{\Delta^{1/2}(s, m_3^2, m_4^2)}{2\sqrt{s}} \right) \cos \theta \right\}$$

t-min occurs at $\theta = 0$, so $\cos \theta = 1$, forward:

$$t_{min} = m_3^2 + m_1^2 - \frac{1}{2s} \left[(s + m_1^2 - m_2^2)(s + m_3^2 - m_4^2) - \Delta^{1/2}(s, m_1^2, m_2^2) \Delta^{1/2}(s, m_3^2, m_4^2) \right]$$

Go to large s limit where $\Delta^{1/2}(a, b, s) = s - a - b - \frac{2ab}{s} + \dots$

Then:

$$\begin{aligned} [] &= [s^2 + s(1^2 - 2^2 + 3^2 - 4^2) + (1^2 - 2^2)(3^2 - 4^2) - s^2 + s(1^2 + 2^2 + 3^2 + 4^2) + 2m_1^2 m_2^2 + 2m_3^2 m_4^2] \\ &\quad - (m_1^2 + m_2^2)(3^2 + 4^2) \\ &= 2s(m_1^2 + m_3^2) + 2(m_1^2 m_2^2 + m_3^2 m_4^2) - (m_1^2 + m_2^2)(m_3^2 + m_4^2) \end{aligned}$$

Then:

$$t_{min} = -\frac{1}{2s} \{ 2[m_1^2 m_2^2 + m_3^2 m_4^2] - (m_1^2 + m_2^2)(m_3^2 + m_4^2) \}$$

$$\text{So, } t(\theta_{\text{cm}}=0) = t_{\text{min}} \cong -\frac{1}{2s} \left[2(m_1^2 m_2^2 + m_3^2 m_4^2) - (m_1^2 + m_2^2)(m_3^2 + m_4^2) \right]$$

④ As we know, this is not always negative. (see s-t-u notes).

⑤ For this situation:



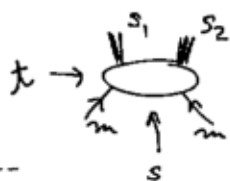
, t_{min} to order $\frac{1}{s}$ as

shown above vanishes.

t_{min} effect. I have not calculated this, but there is still a $\frac{1}{s^2}$

note on t_{min} effect and cluster separation.

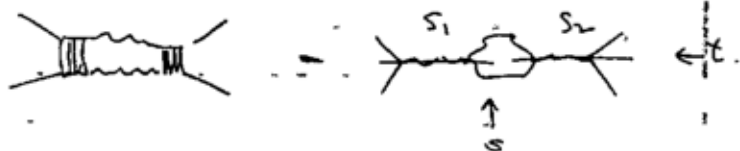
① Consider reaction, From sheet we know that



$$\Theta = 0 \Rightarrow t = t_{min} \approx - \frac{s_1 s_2}{s}$$

So given s_1, s_2, s you must have $t < t_{min} < 0$.

② In our cluster models, we have



The map $g(t, t_1, t_2)$ are assumed to cut off as $t \rightarrow \text{neg}$. Thus,
large $|t| \Rightarrow$ small amplitude.

③ Now the variable $s/s_1 s_2 \equiv s_m$ is the gap width:

$$\ln(s/s_1 s_2) = y - y_1 - y_2 = y_m$$

So

$$\ln|t_{min}| = \lim_{s \rightarrow 0} \ln\left(\frac{s_1 s_2}{s}\right) = -\ln(s/s_1 s_2) = -y_m$$

As gap grows smaller, $|t_{min}|$ increases.

$$|t_{min}| = e^{-y_m}$$

So consider:

$y_{gap} = y_m$
+10
+1
+0
-1
-2

$ t_{min} $
~ 0
$\sim 1/3$
~ 1
~ 3
~ 10

← start to lose here
}

④ Thus, as gap shrinks, $|t_{min}|$ grows + amp starts to die. Thus,
the t_{min} -effect makes clusters repel! This is an argument
for why you should use $y_{gap} > 0$.

An s-t-u kinematics problem:

"In general mass case, assume $(m_1+m_2)^2 \leq (m_3+m_4)^2$ so that $s = (m_3+m_4)^2$ is the physical threshold in the s-channel. At this threshold, what values are taken by t and u ?"



We can consider:

$z = +1$ boundary:

$$+\sqrt{\Delta(s, m_1^2, m_2^2) \Delta(s, m_3^2, m_4^2)} = s^2 + s(2t - z) + (m_1^2 - m_2^2)(m_3^2 - m_4^2)$$

$z = -1$ boundary:

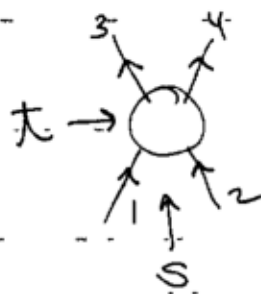
$$-\sqrt{\Delta(s, m_1^2, m_2^2) \Delta(s, m_3^2, m_4^2)} = "$$

At $s = (m_3+m_4)^2$, $\Delta(s, m_3^2, m_4^2) = 0$. Thus, both equations are satisfied at the same time, so easy to solve for t :

$$0 = (m_3+m_4)^4 + (m_3+m_4)^2(2t - z) + (m_1^2 - m_2^2)(m_3^2 - m_4^2)$$

$$2t(m_3+m_4)^2 = z(m_3+m_4)^2 - (m_3+m_4)^4 - (m_1^2 - m_2^2)(m_3^2 - m_4^2)$$

$$\Rightarrow 2t = z - (m_3+m_4)^2 - \frac{(m_1^2 - m_2^2)(m_3^2 - m_4^2)}{(m_3+m_4)^2}$$



$\Rightarrow$

$$2t = m_1^2 + m_2^2 + m_3^2 + m_4^2 - m_3^2 - m_4^2 - 2m_3m_4 - \frac{(m_1^2 - m_2^2)(m_3^2 - m_4^2)}{(m_3+m_4)}$$

$$= [(m_3+m_4)(m_1^2 + m_2^2 - 2m_3m_4) - (m_1^2 - m_2^2)(m_3^2 - m_4^2)] / (m_3+m_4) = 10 \text{ terms.}$$

$$= [m_3m_1^2 + m_3m_2^2 - 2m_3m_4 + m_4m_1^2 + m_4m_2^2 - 2m_3m_4 - m_3^2m_3 + m_1^2m_4 + m_2^2m_3 - m_2^2m_4] / (m_3+m_4)$$

$$= [2m_3m_2^2 + 2m_4m_1^2 - 2m_3^2m_4 - 2m_3m_4^2] / (m_3+m_4) = 2 [m_3m_2^2 - m_3m_4^2 + m_4m_1^2 - m_4m_3^2] / (\dots)$$

$$= 2 \frac{[m_3(m_2^2 - m_4^2) + m_4(m_1^2 - m_3^2)]}{(m_3+m_4)} = 2t$$

$$\Rightarrow t = \frac{m_3(m_2^2 - m_4^2) + m_4(m_1^2 - m_3^2)}{m_3+m_4}$$

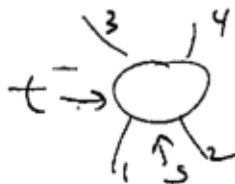
$$\text{So, } t = \frac{m_3(m_2^2 - m_4^2) + m_4(m_1^2 - m_3^2)}{(m_3 + m_4)} = S_{13}.$$

$$\text{Meanwhile, } u = -s - t - \Sigma = \frac{-s(m_3 + m_4) - t(m_3 + m_4) - \Sigma(m_3 + m_4)}{m_3 + m_4}$$

$$\begin{aligned} N &= - (m_3 + m_4)^3 - m_3(m_2^2 - m_4^2) - m_4(m_1^2 - m_3^2) - (m_1^2 + m_2^2 + m_3^2 + m_4^2)(m_3 + m_4) \\ &= \cancel{m_3^3} - (m_3 + m_4) [(m_3 + m_4)^2 + (m_1^2 + m_2^2 + m_3^2 + m_4^2)] \dots \times \end{aligned}$$

$$\begin{aligned} &= -m_3^3 - 3m_3^2 m_4 - 3m_3 m_4^2 - m_4^3 - m_3 m_2^2 + m_3 m_4^2 - m_4 m_1^2 + m_4 m_3^2 \\ &\quad - m_1^2 m_3 - m_2^2 m_3 - m_3^2 m_4 - m_4^2 m_3 - m_1^2 m_4 - m_2^2 m_4 - m_3^2 m_4 - m_4^2 m_4 \end{aligned}$$

Instead, by special cases:



① equal masses $\Rightarrow \underline{t = 0} \checkmark$

② π -N case $m_1 = m_3 = \mu$, $m_2 = m_4 = M$, again $\underline{t = 0} \checkmark$.

③ Whistler case: $m_3 = m_4 = m_2 = m$, but $m_1 = m_1$. so.

$$t = \frac{m(m_1^2 - m^2)}{2m} = \underline{\underline{\frac{1}{2}(m_1^2 - m^2)}} \checkmark$$

This agrees with White B39, I.

Some 4-pt fctn. kinematics

① There are several equations which determine the physical regions in the s - t - u plane. These equations are all the same!

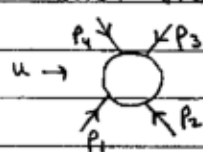
a) $G \begin{pmatrix} p_1, p_2, p_3 \\ p_1, p_2, p_3 \end{pmatrix} \geq 0$, or any other third order Gram condition

b) $\phi(s, t) \geq 0$, Kibble function [see M.S. p. 175~]

c) $|\tilde{z}^{(s)}| \leq 1$ or $|\tilde{z}^{(u)}| \leq 1$ or $|\tilde{z}^{(t)}| \leq 1$ (over $\rightarrow$)

② Normally, we think of $|\tilde{z}^{(s)}| \leq 1$ as determining only the s -channel physical region. In fact, when $\tilde{z}^{(s)}$ is written in terms of s, t, u , $|\tilde{z}^{(s)}| \leq 1$ determines all regions.

③ Specifically, let's choose $\tilde{z}^{(s)}$ to be z_{14} . Set up arrows all pointing in. Then:



$$u = (p_1 + p_3)^2 = m_1^2 + m_3^2 + 2p_1 \cdot p_3$$

Since, in the physical s -ch., (p_1) and $(-p_4)$ are future timelike, we may say:

$$p_1 \cdot p_4 = -[p_1 \cdot (-p_4)] = -[E_1^{(s)} E_4^{(s)} - p_1^{(s)} p_4^{(s)} \tilde{z}^{(s)}]$$

Here, $E_1^{(s)}, E_4^{(s)}, p_1^{(s)}, p_4^{(s)}$ are as usual positive. Thus:

$$u - m_1^2 - m_4^2 = -2E_1 E_4 + 2p_1 p_4 \tilde{z}$$

And: $\tilde{z}_{14}^{(s)} = \frac{u - m_1^2 - m_4^2 + 2E_1^{(s)} E_4^{(s)}}{2p_1^{(s)} p_4^{(s)}}$

Now: $|\tilde{z}_{14}^{(s)}| \leq 1 \Rightarrow [u - m_1^2 - m_4^2 + 2E_1^{(s)} E_4^{(s)}]^2 \leq 4[p_1^{(s)} p_4^{(s)}]^2$

In more detail:

$$\left(u - m_1^2 - m_4^2 + 2 \frac{(s + m_1^2 - m_2^2)(s + m_4^2 - m_3^2)}{2\sqrt{s}} \right)^2 \leq 4 \cdot \frac{\Delta(s, m_1^2, m_2^2)}{4s} \frac{\Delta(s, m_3^2, m_4^2)}{4s}$$

One could work this out in gory detail, but no need. This must be the same as the Kibble or Gram condition, because it is a L.I. inequality. We know one L.I. inequality works for all channels. Since this L.I. in. works for sure in the s , it must be the one condition that works everywhere.

$\rightarrow$ I have worked out the detail to find that $|\tilde{z}^{(s)}| \leq 1$ is Kibble for $s \geq 0$, but $|\tilde{z}| \geq 1$ is Kibble for $s < 0$!

Remark on $|z^{(s)}| \leq 1$ etc.

✗ I have explicitly computed out $|z^{(s)}|^2 \leq 1$. Since $z = N/D$, this becomes:

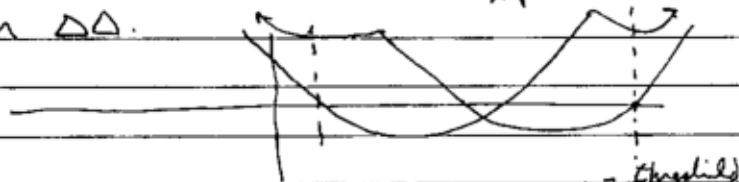
$$|N|^2 \leq |D|^2 \quad \text{here } |D|^2 = 4$$

① If one explicitly computes $|z^{(s)}|^2 \leq 1$, one gets

$$\frac{[2s(t - m_1^2 - m_2^2 + 2E_1^{(s)}E_2^{(s)})]^2}{|[2s(2p_1^{(s)}p_3^{(s)})]|^2} \leq 1$$

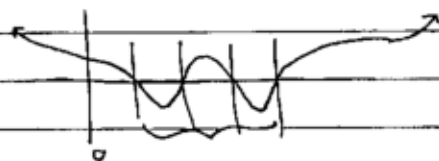
$$\text{i.e. } [2s(t - m_1^2 - m_2^2) + (s + m_1^2 - m_2^2)(s + m_3^2 - m_4^2)]^2 \leq |\Delta(s, m_1^2, m_2^2)\Delta(s, m_3^2, m_4^2)|$$

As long as you are not between the s-ch threshold ^{insert} pseudothreshold, you can replace $|\Delta\Delta|$ with $\Delta\Delta$.
Because both are > 0 .



$$\text{i.e. } \Delta\Delta = [s - (m_1 + m_2)^2][s - (m_1 - m_2)^2][s - (m_3 + m_4)^2][s - (m_3 - m_4)^2].$$

So a plot of $\Delta\Delta$ looks like:



As long as you are not between the bars, $\Delta\Delta = |\Delta\Delta|$. One or more bars could compress toward $s=0$.

② We know that the curves will not cross any "bars", so it is correct to say: in any physical region, $|\Delta\Delta| = \Delta\Delta$. In this case, if we work this out, we eventually get:

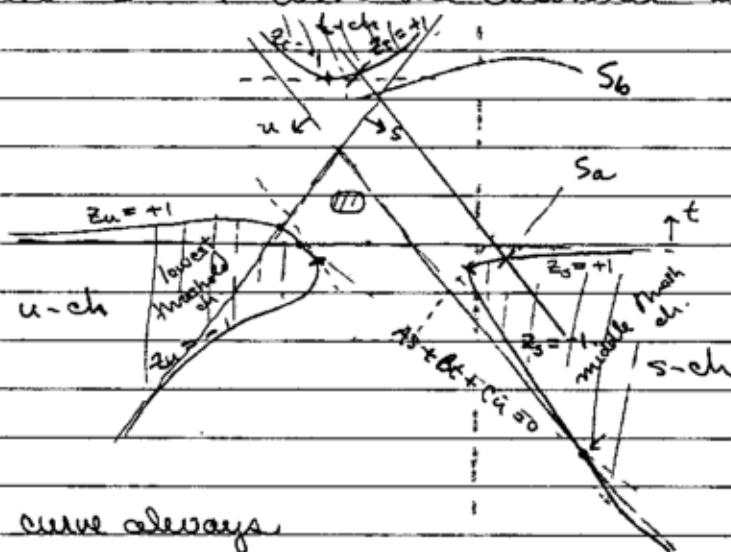
$$s\phi(s,t) \leq 0.$$

So, $s > 0 \Rightarrow$ Kibble. But, $s \leq 0$ requires $|z^{(s)}| \geq 1$ to get Kibble. Thus,

$$|z^{(s)}| \leq \frac{8}{15} : |z^{(s)}| \leq$$

③ So $|z^{(s)}| \begin{cases} \leq 1, & s > 0 \\ \geq 1 & s \leq 0 \end{cases}$ defines phys. region

- ④ No matter how complex the mass structure, we can still associate $z = \pm 1$ with the channel boundaries:



The $z = +1$ curve always meets the $z = -1$ curve at threshold in the channel of interest. This is obvious because only as $\vec{p}_3 \rightarrow 0$ can you have $z = +1$ and -1 at the same time.

- ⑤ Three aspects of the general diagram:

(a) The curves are always asymptotic to $s=0, t=0$, or $u=0$. Easy to show.

(b) The branch of the curve for the channel with the lowest threshold cuts each asymptote once.

(c) The branch with the middle threshold cuts one of its asymptotes only. That point lines up with the other two because:

(d) The branch with the highest threshold lies entirely inside its asymptotes (no cuts)!

$$As + Bt + Cu = 0$$

These facts are all illustrated in the above picture: ✓

- ⑥ Example: if the s-ch had the highest threshold of all three channels [i.e., no other thresholds are above the s threshold], then for the s-ch physical region $u \leq 0$ and $t \leq 0$.

- ⑦ Example: one can always replace $\Theta(1 - |z^{(u)}|)$ with $\Theta(1 - |z^{(u)}|)$ or $\Theta(1 - |z^{(t)}|)$. Often, one of these forms will be much more convenient than the others. ✓

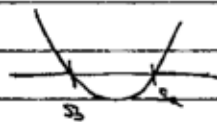
For example, suppose we are at some fixed $u \leq 0$ which lies in the s-channel. [see orange line above].

If we want to know the value of s where the physical s-ch begins [called S_a], we could find it by examining $\Theta(1 - |z^{(s)}|)$. From the picture we can see that

$$\frac{s}{|s|} |z^{(s)}| \leq 1 \Rightarrow s > S_a \text{ OR } s < S_b$$

But only $s > S_a$ gives you s-ch physical. I.e.,

$$(|z^{(s)}| - 1) \leq 0 \Rightarrow (s - S_a)(s - S_b) \geq 0$$



However, to reduce this to this requires an immense amount of algebra. If instead we use:

$$\frac{u}{|u|} |z^{(u)}| \leq 1 \quad \frac{u}{|u|} \left| \frac{s - m_1^2 - m_2^2 + 2E_1^{(u)} E_2^{(u)}}{2p_1^{(u)} p_2^{(u)}} \right| \leq 1 \quad |u| \geq 1 \Rightarrow a \geq 1 \text{ and } a \leq -1$$

We at once obtain: [since $u < 0$]

$$\Rightarrow \left[-2p_1 p_2 \leq s - m_1^2 - m_2^2 + 2E_1 E_2 \leq 2p_1 p_2 \right]$$

$$\text{OR, } \left[\underbrace{(m_1^2 + m_2^2 - 2E_1 E_2 - 2p_1 p_2)}_{\textcircled{S_b}} \leq s \leq \underbrace{(m_1^2 + m_2^2 - 2E_1 E_2 + 2p_1 p_2)}_{\textcircled{S_a}} \right]$$

really two separate equations.

Note $S_a > S_b$.

I.e., $s > S_a$ OR $s \leq S_b$.

Of course we have trivial forms for $E_i^{(u)}$ etc.

Thus:

$$S_a = m_1^2 + m_2^2 - 2E_1^{(u)} E_2^{(u)} + 2p_1^{(u)} p_2^{(u)} \quad / \text{see Abraham and Saunders.}$$

⑧ Note that there are always two ways to define $z^{(s)}$ etc. They are: (see picture):

$$a) \quad p_1 \cdot p_4 = -[p_1 \cdot (-p_4)] = -[E_1^{(s)} E_4^{(s)} - p_1^{(s)} p_4^{(s)} z_{14}^{(s)}] = (u - m_1^2 - m_4^2)/2$$

$$b) \quad p_1 \cdot p_3 = -[p_1 \cdot (-p_3)] = -[E_1^{(s)} E_3^{(s)} - p_1^{(s)} p_3^{(s)} z_{13}^{(s)}] = (t - m_1^2 - m_3^2)/2$$

Therefore:

$$z_{14}^{(s)} = \frac{u - m_1^2 - m_4^2 + 2E_1^{(s)} E_4^{(s)}}{2p_1^{(s)} p_4^{(s)}}$$

$$z_{13}^{(s)} = \frac{t - m_1^2 - m_3^2 + 2E_1^{(s)} E_3^{(s)}}{2p_1^{(s)} p_3^{(s)}}$$

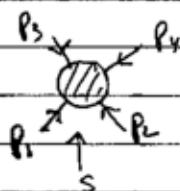
Of course $(|z_{14}^{(3)}| \leq 1) \Leftrightarrow (|z_{23}^{(3)}| \leq 1)$.

In general:

$$z_{ij}^{(2)} = \frac{S_{ij} - m_i^2 - m_j^2 + 2 E_i^{(2)} E_j^{(2)} + 2 p_i^{(1)} p_j^{(1)}}{2 E_i^{(2)} E_j^{(2)} + 2 p_i^{(1)} p_j^{(1)}}$$

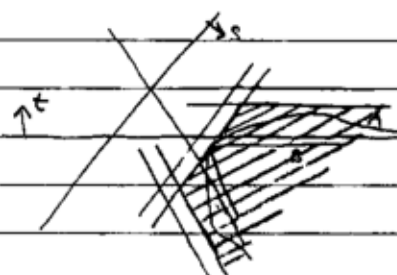
Here, (2) is a channel, st or u. And ij denote which vectors z is to be between. And, S_{ij} is $(p_i + p_j)^2$. All momenta point in here. Cannot have $S_{ij} = 0$, of course.

⑨ The future timelike stuff. In the general mass case, there is a simply defined region where p_1 and p_2 are F.T. and where p_3 and p_4 are P.T. This region contains [but does not coincide with] the physical s-channel.



This simple region is the one that respects all the gram second order requirements. I.e.,

$$\begin{aligned} p_1 \cdot p_2 &\geq m_1 m_2 \Rightarrow s \geq (m_1 + m_2)^2 & \Leftrightarrow G(1^2) \leq 0 \text{ etc.} \\ p_3 \cdot p_4 &\geq m_3 m_4 \Rightarrow s \geq (m_3 + m_4)^2 \\ p_1 \cdot p_3 &\leq -m_1 m_3 \Rightarrow t \leq (m_1 - m_3)^2 \\ p_2 \cdot p_4 &\leq -m_2 m_4 \Rightarrow t \leq (m_2 - m_4)^2 \\ p_1 \cdot p_4 &\leq -m_1 m_4 \Rightarrow u \leq (m_1 - m_4)^2 \\ p_2 \cdot p_3 &\leq -m_2 m_3 \Rightarrow u \leq (m_2 - m_3)^2 \end{aligned}$$

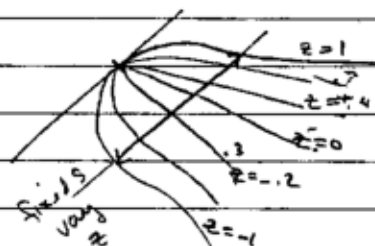


Thus, to crudely locate the "simple" region, we need only draw all six lines + form a polygonal area:

So, this simple region (orange hatch) satisfies the future timelike requirements. The actual region must also satisfy the Kibble or 3rd order Gram.

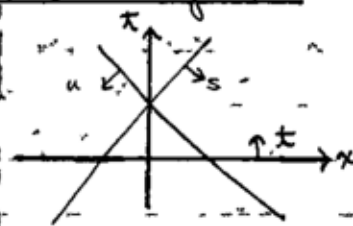
⑩ As you hit the $z = \pm 1$ curves, z becomes complex because some of the momenta become complex (Gram).

⑪ The z-distribution:



How to plot things on a Mandelstam Diagram.

- ① The general picture is:
Variable t serves as
the y-coordinate. For
an x-coord, we use



$$x = (s-u) \times \text{const?}$$

Since height of Δ is Σ ,

because $s+t+u=\Sigma$, we know $u=0, t=0 \Rightarrow x = \frac{1}{2}\Sigma$. But at
this point, $s=\Sigma$, so $x = \frac{1}{2}\Sigma = (\Sigma-0) \cdot \text{const} \Rightarrow \text{const} = \frac{1}{2}$.

$$x = \frac{1}{2}(s-u)$$

- ② We can express s or u as $f(x, t)$:

$$\left. \begin{array}{l} s-u=2x \\ t+s+u=\Sigma \end{array} \right\} \Rightarrow \begin{array}{l} s = x - \frac{1}{2}t + \frac{1}{2}\Sigma \\ u = -x - \frac{1}{2}t + \frac{1}{2}\Sigma \end{array}$$

Thus, the line $s=0$ must be: $t=2x+\Sigma$ i.e., $|\text{slope}|=2$.
the line $u=0$ must be: $t=-2x+\Sigma$

- ③ So in general, if someone gives you $f(s, t) = 0$ to plot,
replace s as shown to get $f(x, t) = 0$, then plot.

- ④ Example: let $\Sigma = 2m_1^2 + 2m_2^2$ as in πN . Then $Z_{13} = 1$ goes into
 $t=0$, but what is $Z_{13} = -1$?

$$-1 = 1 + \frac{2st}{\Delta(s, m_1^2, m_2^2)} \Rightarrow 2\Delta(s, m_1^2, m_2^2) + 2st = 0$$

$$\Rightarrow s^2 - 4s(\Sigma) + (m_1^2 - m_2^2)^2 + st = 0$$

$$\Rightarrow s[s - \Sigma + t] + \delta^2 = 0$$

$$\Rightarrow [x - \frac{1}{2}(t - \Sigma)][x + \frac{1}{2}(t - \Sigma)] + \delta^2 = 0$$

$$\Rightarrow X^2 - \left(\frac{t - \Sigma}{2}\right)^2 + \delta^2 = 0$$

OR: $\frac{(t - \Sigma)^2}{(2\delta)^2} - \frac{x^2}{\delta^2} = 1$

$$(t - \Sigma) = \pm 2x \Rightarrow t = \pm 2x + \Sigma$$

I.e., $s=0$ and $u=0$

hyperbola, asymptotes are

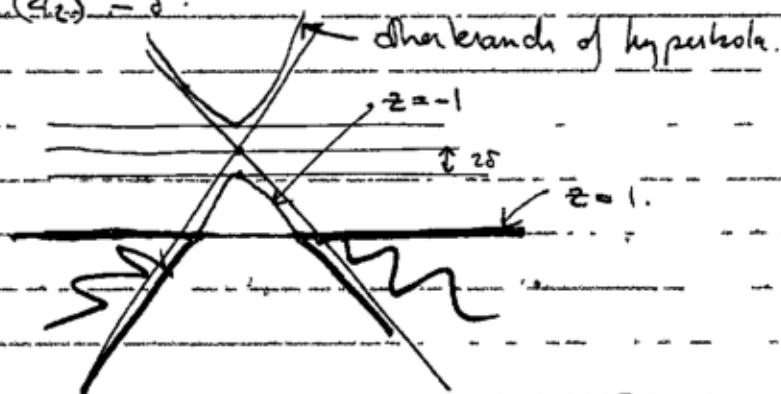
Also, $x=0 \Rightarrow t = \pm 2\delta =$

Now easy to show that $|2\delta| < 2$, so.

Also, since $su + \delta^2 = 0$, cannot hit $s=0$ or $u=0$.

And, $t=0 \Rightarrow x = \pm \sqrt{(z_1)^2 - \delta^2}$.

So, picture is this:



⑤ Now we ask: where is $z_{13} > -1$. Answer: $su + \delta^2 > 0$.

Or:

$$x^2 - \left(\frac{t-\varepsilon}{2}\right)^2 + \delta^2 > 0.$$

Answer: outside the hyperbola.

⑥ At the same time we want $z_{13} < 1 \Rightarrow t < 0$.

⑦ So, the intersection of ⑤ and ⑥ is shown in red. We can be in either the s or u channel.

⑧ Where is $z = z$? Or in general, $z = \text{constant}$?

$$0 = (1-z)\Delta + 2st \quad \Rightarrow \quad 0 = \Delta + st + st \left(\frac{1+z}{1-z}\right)$$

$$\text{We know: } \Delta + st = x^2 - \left(\frac{t-\varepsilon}{2}\right)^2 + \delta^2$$

$$\text{So: } 0 = x^2 - \left(\frac{t-\varepsilon}{2}\right)^2 + \delta^2 + \left(\frac{1+z}{1-z}\right) \cdot t \cdot \left(x - \left[\frac{t-\varepsilon}{2}\right]\right)$$

$$\Rightarrow 0 = x^2 + \left[\left(\frac{1+z}{1-z}\right)t\right]x + \frac{1}{4}t^2 \dots = \text{a rotated conic}$$

$$\Rightarrow 0 = x^2 + \left(\frac{1+z}{1-z}\right)xt + t^2 \left[\frac{z+3}{4(z-1)}\right] + t \left[\frac{3-\varepsilon}{2(1-z)}\right] + \left(\frac{\varepsilon^2}{4} + \delta^2\right)$$

A conic centered at $x=0, t=\varepsilon$ and rotated unless $z = -1, +1$.

So $z = \text{const}$ are asym.

⑨ Look back at $0 = (1-z)\Delta + 2st$. We know that $\Delta = 0$ at the s , t or threshold. So if $t = 0$ also, eqn. is satisfied. Thus, $\forall z$ this conic must pass thru those one points of threshold.

⑩ Asymptotes? As $s \rightarrow \infty$, $\Delta \rightarrow s^2$, so $0 = (1-z)s^2 + 2st$
 $\rightarrow s(1-z) + 2t = 0 \Rightarrow t = \frac{s(z-1)}{2} \rightarrow \infty$
 $z < 1 \Rightarrow t = \text{neg}$ $z > 1 \Rightarrow t \rightarrow +\infty$ Thus, not ellipse.

⑪ Let's now go to equal mass. Now:

$$z_s = 1 + \frac{2t}{(z-1)} \Rightarrow (z-1)(s-z) = 2t$$

$$\Rightarrow (z-1)(x - \frac{1}{2}t - \frac{1}{2}z) = 2t \Rightarrow t = \alpha(2x - z) \quad \alpha = \frac{z-1}{z+3}$$

So $z=1 \Rightarrow t=0$; $z=-1 \Rightarrow \alpha=-1$ so $t = -2x + z \Rightarrow u=0$
 In general, $z = \text{const}$ is a straight line!

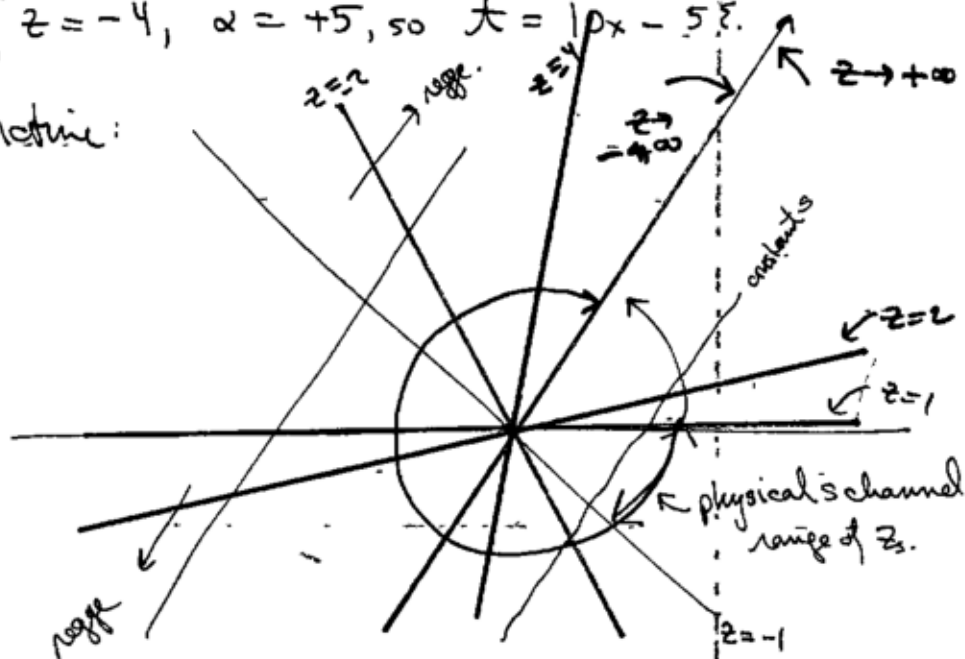
Eg, $z=2 \Rightarrow \alpha = \frac{1}{5} \Rightarrow t = \frac{2}{5}x - \frac{1}{5}z$

As $z \rightarrow +\infty$, $\alpha \rightarrow 1$, so $t \rightarrow (2x - z)$

If $z = -2$, $\alpha = -3$, so $t = -6x + 3z$

If $z = -4$, $\alpha = +5$, so $t = 10x - 5z$

Picture:



into (3.18). The spectral function A_{11} would be zero in this approximation.

The unitarity condition for the higher angular momentum states of the reaction I is no longer satisfied. However, the terms neglected appear by their form to arise from intermediate states of the reaction III with more than two particles, so that the approximation is in the spirit of the approximations already made. We have in fact made precisely this approximation in the unitarity condition for the S waves of the reaction III. The unitarity condition for the low angular momentum states of the reaction I, and in particular for the states with $j=1$ or 2 , is still satisfied, as it has been introduced separately. The present approximation treats the reactions I, II, and III on the same footing.

To summarize, then, our method of procedure will be the following: The first few angular momentum states of A_1 and A_2 are found on the assumption that each angular momentum state is an analytic function of the square of the center-of-mass energy except for the perturbation singularities and the cuts on the positive real axis. This calculation can be done exactly if the discontinuity across the cut along the positive real axis is determined by unitarity (complications arise, as the relations connecting a and b with A and B involve square roots of kinematical factors, but the methods can be modified accordingly). A_{11} and A_{21} are also found as just described. The analytic properties of the low angular momentum states are now determined from the analytic properties of the scattering amplitude given by (2.12). The singularities can be calculated in terms of A_1, A_2, A_3 . These absorptive parts can in turn be found from A_{11} and A_{21} by means of the dispersion relations (2.16), (2.17), (2.19), with subtraction terms which can be obtained from the low-angular-momentum states. In the next iteration, all the singularities of the low angular momentum states except that along the positive real axis are found from the quantities calculated in the first iteration, and the singularity along the positive real axis is redetermined from the unitarity condition. The iteration procedure is repeated until it converges. As in the calculations of Sec. 4, it is found necessary to cut off the absorptive parts A_1, A_2 and A_3 at high energies, before calculating the singularities of the low angular momentum states in the next iteration. However, the cutoff is only applied above the threshold for processes neglected in the unitarity condition, and in particular, above the threshold for pair production in the reaction I.

This approximation could be regarded as the first of a series of approximations in which more and more of the contributions to the spectral functions are included, until we ultimately reach a solution in which the unitarity condition in the one-meson approximation is satisfied for every angular momentum state. The higher approximations the spectral functions are no longer deter-

mined by perturbation theory, but, once the contribution from the crossing term enters, they will have to be recalculated after each iteration. However, it would be more worthwhile to go beyond the one-meson approximation at the same time as we took the higher contributions to the spectral functions into account. In other words, we continue to put the reactions I, II, and III on the same footing, bringing in the higher intermediate states of all three together. If the approximation scheme converged, the exact unitarity condition of the three reactions would finally be satisfied for all angular momentum states. Needless to say, one would not in practice be able to go beyond the first one or two approximations.

The number of angular momentum states for which the unitarity condition is applied separately will, as has been explained in the last section, depend on the behavior of A and B as t (or s) tends to infinity with s constant. However, in our first approximation, it should be sufficient to treat separately only states with $j=1$ and $j=2$, as the other angular momentum states will not be important below the threshold for pion production. If we went beyond the one-meson approximation we would probably have to treat some higher angular momentum states separately in any case, since, for instance, two pions both in a $(3,3)$ resonance state with 3 nucleon could form a D_1 state. For reaction III, one would have to treat separately S states and possibly P states as well.

If one neglected the nucleon-antinucleon intermediate state in the reaction III and only took the two-pion intermediate state into account, all three spectral functions A_{11}, A_{21} , and A_{31} would be zero, since they all begin above the threshold for processes which are being neglected. The entire scattering amplitude would then consist of "subtraction terms" for one or other of the dispersion relations. This may be the best first approximation from the point of view of the amount of work required and the accuracy of the result, as the nucleon-antinucleon intermediate state is a good deal heavier than multipion states which are being neglected. Though the spectral functions are not now brought in at all, it will of course be realized that the only justification for the approximation is that it is the first of a series of approximations which do involve the spectral functions. In this approximation, if the crossing term is neglected in the calculation of the pion-pion scattering amplitude, only intermediate S states occur in reaction III, so that the unitarity condition for the P states will not enter.

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Kinematics of General Scattering Processes and the Mandelstam Representation

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The kinematics of an arbitrary process involving two incoming and two outgoing particles is studied in terms of the invariants used in Mandelstam's representation, treating the three processes described by the same Green's function simultaneously. It is shown that the physical regions for these processes are bounded by a cubic curve in the plane of the two independent invariants. The unitarity conditions are discussed in the approximation of neglecting intermediate states of more than two particles. The formula for the spectral functions of the double dispersion relation is obtained explicitly in terms of the invariants chosen.

1. INTRODUCTION

MANDELSTAM¹ has recently proposed a representation of the scattering amplitude for meson-nucleon scattering, which is obtained from a plausible assumption about its behavior as an analytic function of two variables, the energy and momentum transfer. He has also been able to show,² for a more general process, that the representation is satisfied by the lower orders of the perturbation series, and that this series can actually be constructed from the representation and the unitarity relations,³ in a two-particle approximation. In this paper we shall discuss certain aspects, mainly kinematical, of the extension of this representation to a general process. We consider together the

three processes

$$I: 1+2 \rightarrow 3+4,$$

$$II: 1+\bar{3} \rightarrow \bar{2}+4,$$

$$III: 1+\bar{4} \rightarrow \bar{2}+3.$$

The complications of spin and isotopic spin will be ignored, and all the particles will be assumed to be stable.

In Sec. 2 we shall find the physical regions for the three scattering processes in terms of the three invariants s, t, u , whose sum is equal to the sum of squared masses of the four particles. These invariants may be regarded as homogeneous coordinates in a plane, and the physical regions are then bounded by a cubic curve in this plane. The curve has three branches corresponding to the physical regions for the three scattering processes, and also a closed branch within the stu -triangle. The interior of this closed curve would correspond to the physical region for the decay process

$$IV: 1 \rightarrow \bar{2}+3+4$$

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¹ S. Mandelstam, *Phys. Rev.* 112, 1344 (1958).

² S. Mandelstam, *Phys. Rev.* 115, 1741 (1959).

³ S. Mandelstam, *Phys. Rev.* 115, 1752 (1959).

If this were possible. The form of the double dispersion relation for such processes has been given by Mandelstam.²

In Sec. 3 we shall discuss the unitarity condition for a typical process in the two-particle approximation. By making use of the determinant of scalar products of the independent momenta, the unitarity condition can be written in the form of an integral over the invariants. From this we are able to derive the relation giving the spectral functions in terms of the absorptive parts, in a similar form. The boundaries of the regions where the spectral functions are nonzero will be given by the vanishing of the determinant.

2. KINEMATICS

For convenience, we shall assume that the masses of the four particles involved in the processes I, II, III satisfy the inequalities

$$m_1 \geq m_2 \geq m_3 \geq m_4. \quad (1)$$

If

$$m_1 > m_2 + m_3 + m_4, \quad (2)$$

then the decay process IV is also energetically possible (although in that case we assume that it has vanishing probability). We shall choose the momenta of the incoming and outgoing particles to be p_1 and $-p_1$, respectively, so that the conservation equation is always

$$p_1 + p_2 + p_3 + p_4 = 0. \quad (3)$$

The metric is chosen so that $p^2 = m^2$.

In addition to the masses, there are two independent scalar products. It is, however, convenient to use the three invariants

$$\begin{aligned} r &= (p_1 + p_2)^2 = (p_3 + p_4)^2, \\ s &= (p_1 + p_3)^2 = (p_2 + p_4)^2, \\ t &= (p_1 + p_4)^2 = (p_2 + p_3)^2, \end{aligned} \quad (4)$$

which satisfy

$$r + s + t = K = m_1^2 + m_2^2 + m_3^2 + m_4^2. \quad (5)$$

In the center-of-mass system for process I, the momenta are $p_1 = (E_1, \mathbf{q}_1)$, $p_2 = (E_2, -\mathbf{q}_1)$, $p_3 = (-E_3, -\mathbf{q}_2)$, $p_4 = (-E_4, \mathbf{q}_2)$. The invariant r is then the square of the total energy.

$$r = W^2, \quad W = E_1 + E_2 = E_3 + E_4.$$

The magnitudes of the spatial momenta are given by

$$\begin{aligned} 4q_1^2 &= [r - (m_1 + m_2)^2][r - (m_1 - m_2)^2], \\ 4q_2^2 &= [r - (m_3 + m_4)^2][r - (m_3 - m_4)^2], \end{aligned}$$

and the invariants s and t may be related to the scattering angle by

$$\begin{aligned} 2s &= K - r + 4q_1 q_2 \cos(\theta_1 - \theta_2) = (m_1^2 - m_2^2)(m_3^2 - m_4^2)/r, \\ 2t &= K - r - 4q_1 q_2 \cos(\theta_1 - \theta_2) = (m_1^2 - m_2^2)(m_3^2 - m_4^2)/r, \end{aligned} \quad (6)$$

where $\theta = \cos(\theta_1, \mathbf{q}_2)$.

The conditions for a physical scattering process may now be expressed in terms of r, s, t . The necessary condition

$$\pm p_1 \cdot p_2 > m_1 m_2$$

yields

$$r > (m_1 + m_2)^2 \quad \text{or} \quad r < (m_1 - m_2)^2, \quad (7)$$

and similar inequalities for other pairs of masses. The requirement that the scattering angle be real can be stated in the form

$$\begin{vmatrix} p_1^2 & p_1 \cdot p_2 & p_1 \cdot p_3 & p_1 \cdot p_4 \\ p_2 \cdot p_1 & p_2^2 & p_2 \cdot p_3 & p_2 \cdot p_4 \\ p_3 \cdot p_1 & p_3 \cdot p_2 & p_3^2 & p_3 \cdot p_4 \\ p_4 \cdot p_1 & p_4 \cdot p_2 & p_4 \cdot p_3 & p_4^2 \end{vmatrix} > 0. \quad ?$$

This may be written as a homogeneous inequality in r, s, t

$$rst > (r + s + t)^2 (ar + bs + ct) \quad (8)$$

where the dimensionless constants a, b, c are given by

$$\begin{aligned} Ka &= (m_1^2 m_2^2 - m_3^2 m_4^2)(m_1^2 + m_2^2 - m_3^2 - m_4^2), \\ Kb &= (m_1^2 m_3^2 - m_2^2 m_4^2)(m_1^2 + m_3^2 - m_2^2 - m_4^2), \\ Kc &= (m_1^2 m_4^2 - m_2^2 m_3^2)(m_1^2 + m_4^2 - m_2^2 - m_3^2). \end{aligned}$$

The variables r, s , and t may now be regarded as homogeneous coordinates in a plane, in which the line at infinity is $r + s + t = 0$. The region (8) is bounded by a cubic curve in this plane, whose asymptotes are $r = 0$, $s = 0$ and $t = 0$. Moreover, the curve intersects its asymptotes on the line

$$ar + bs + ct = 0. \quad (9)$$

The shape of the curve is shown in Fig. 1, in which the regions marked I, II, and III are the physical regions for the corresponding processes. We note that, by the assumed inequalities for the masses, the constants a, b , and c satisfy

$$a \geq b \geq c,$$

and that a and b are necessarily positive, although c may have either sign. If c is negative, the line (9) passes within the rst -triangle, and therefore the region

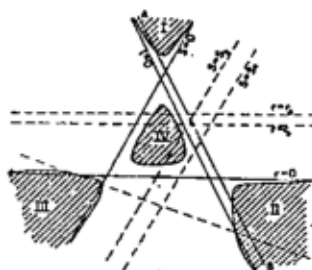


Fig. 1. The physical regions.

KINEMATICS OF GENERAL SCATTERING PROCESSES

III in Fig. 1 includes part of the triangle. In other words, all three invariants can be positive for this process.

The bounding lines of the region defined by (7) can be shown to be tangential to the curve, so that the entire boundary of each of the physical regions is a part of the curve.³ If the condition (2) is satisfied, then the excluded strips

$$\begin{aligned} (m_1 - m_2)^2 &< r < (m_1 + m_2)^2, \\ (m_3 - m_4)^2 &< r < (m_3 + m_4)^2 \end{aligned} \quad (10)$$

do not overlap, so that the region IV is allowed kinematically, as we should expect. If (2) is not satisfied, however, the strips (10) overlap, and the region IV is excluded.

A. MANDELSTAM REPRESENTATION AND UNITARITY CONDITIONS

We shall assume that for the processes in question the ordinary dispersion relations are valid. Thus for a fixed value of t , say, one can write a dispersion relation which will be an integral along a line such as AB in Fig. 1. The poles will occur on lines $r = r_0$ and $s = s_0$, and the continuous integrals will begin on lines $r = r_0$ and $s = s_0$, as indicated. Here r_0 and s_0 are the squared masses of the single particle⁴ and lowest two-particle intermediate states in the process I, respectively. Using the same assumptions as in his treatment of meson-nucleon scattering,¹ Mandelstam² has given a double-variable representation for these processes, involving three spectral functions, here denoted⁵ by A_{11} , A_{12} , and A_{22} . The function A_{11} , for example, will be nonzero in a region lying within the triangle defined by $r > r_0$, $s > s_0$. To find the exact boundary, C_{11} , of this region, as shown in Fig. 2, we must use the unitarity relation. We shall assume that intermediate states with three or

more particles may be neglected in this relation, and further that only one pair of particles contributes to the two-particle intermediate states. If there is more than one such pair, we have only to sum the contributions from each. In this approximation, the unitarity condition for the process I is

$$\begin{aligned} A_{11}(r_1) &= \frac{1}{2} (2\pi)^{-4} \int d^4 p d^4 p' \delta(p^2 - m_1^2) \delta(p'^2 - m_1^2) \\ &\times \delta(p + p' - p_1 - p_2 - p_3 - p_4) A_2^*(r_2) A_1(r_2), \end{aligned} \quad (11)$$

where A_1 , A_2 , and A_3 now refer to the processes

$$\begin{aligned} I_1: 1+2 \rightarrow 3+4, \\ I_2: 1+2 \rightarrow 5+6, \\ I_3: 3+4 \rightarrow 5+6, \end{aligned}$$

respectively. The invariants are defined⁶ as in Sec. 2. We now wish to convert (11) into an integral over invariants. To do this, we perform the p -integration using the δ -function, and convert the p' -integration into one over $p_1', (p_2 - a)^2, (p_3 - b)^2, (p_4 - c)^2$, where a, b and c are any three fixed timelike vectors. The Jacobian for this transformation is

$$J = \gamma_2 [-\Delta(a, b, c, p_1)]^{-1},$$

where

$$\Delta(a, b, c, p) = \begin{vmatrix} a^2 & a \cdot b & a \cdot c & a \cdot p \\ b \cdot a & b^2 & b \cdot c & b \cdot p \\ c \cdot a & c \cdot b & c^2 & c \cdot p \\ p \cdot a & p \cdot b & p \cdot c & p^2 \end{vmatrix}. \quad (12)$$

It is convenient to choose $a = p_1 + p_2$, $b = p_3$, $c = -p_4$ so that

$$(p_1 - a)^2 = p_1^2, \quad (p_2 - b)^2 = s_0, \quad (p_3 - c)^2 = s_0.$$

Then, using the definitions of invariants, we find

$$\Delta(a, b, c, p) = \frac{1}{2} \begin{vmatrix} 2r & r + m_1^2 - m_2^2 & r + m_3^2 - m_4^2 & r + m_1^2 - m_2^2 \\ r + m_1^2 - m_2^2 & 2m_2^2 & m_1^2 + m_3^2 - s_2 & m_1^2 + m_2^2 - s_1 \\ r + m_3^2 - m_4^2 & m_1^2 + m_3^2 - s_2 & 2m_3^2 & m_1^2 + m_2^2 - s_1 \\ r + m_1^2 - m_2^2 & m_1^2 + m_2^2 - s_1 & m_1^2 + m_2^2 - s_1 & 2m_2^2 \end{vmatrix} = \Delta(r; s_1, s_2), \quad (13)$$

This transformation is not one-to-one, since the scalar products are unaltered by changing the sign of the component of p_4 perpendicular to a, b and c . This introduces an extra factor of 2. Finally, we obtain

$$\begin{aligned} A_{11}(r_1) &= (1/64\pi^2) \int ds_1 ds_2 [-\Delta(r; s_1, s_2)]^{-1} \\ &\times A_2^*(r_2) A_1(r_2), \end{aligned} \quad (14)$$

where the integration is over the region where Δ is negative.

³ In the case of elastic scattering, the curve degenerates into a straight line and a hyperbola.

⁴ Of course there may be more than one such particle and hence more than one pole, or there may be none at all.

⁵ Mandelstam denotes the corresponding functions by A_{11} , A_{12} , and A_{22} . That notation would, however, be likely to cause confusion with the A_1 , A_2 , and A_3 introduced below.

⁶ Note that if we define the signs of the momenta in I_1 and I_2 according to the convention (2), then two of the momenta in I_3 have the "wrong" sign. Thus we must define, for example, $r_3 = (p_1 - p_2)^2$.

see BGKT for rest.

Double Spectral Boundary

① Stanley's procedure is as follows:

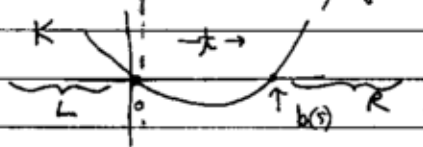
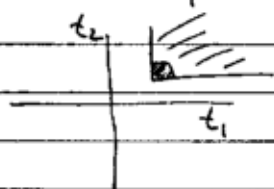
- write $D_s = \int A A$ at fixed $s < s_2$, $s > s_2$.
- disperse each A at fixed s
- compute the dR integral

Result so far is: $D_s = \int \frac{dt_1 dt_2}{\sqrt{+K}} [\dots] [\dots] \log \left[\frac{\alpha + \sqrt{K}}{\alpha - \sqrt{K}} \right] \Theta(K)$

The only t -dependence on RHS is in α and K . The dispersive yanked t out of AA + put it into this logarithm.

② $K > 0$ restricts the $dt_1 dt_2$ integration, so add $\Theta(K) \uparrow$

③ Examine in particular the piece of the double integral near $t_1 = t_2 = t_0$: There, we may plot:



We look for possible cuts in L and R region, since only these occur. I.e., $t = b/2 \Rightarrow K < 0$ so for such a t , there is no contribution from $\blacksquare$.

④ In L , $\text{disc. log} = 0$. In R , $\text{disc. log} = 4\pi i$. Thus, at $t = b(s)$, fst. begins .

⑤ Now consider some higher $t_1 t_2$ patch. At $t = 0$,

$$K(0, t_1, t_2, s) = [t_1^2 + t_2^2 - 2t_1 t_2] = (t_1 - t_2)^2 > 0.$$

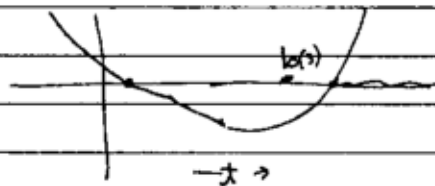
$$K(t, t_1, t_2, s) = [(t_1 - t_2)^2 - t(2t_1 + 2t_2 + \frac{t_1 t_2}{q_1^2} - t)]$$

Set $K = 0$ + solve for values of t :

$$K = 0 \Rightarrow (t_1 - t_2)^2 + t^2 - (2t_1 + 2t_2 + \frac{t_1 t_2}{q_1^2})t = 0$$

$$\Rightarrow t_0 = \left[(2t_1 + 2t_2 + \frac{t_1 t_2}{q_1^2}) \pm \sqrt{()^2 - 4(t_1 - t_2)^2} \right] / 2$$

Claim That this $t_0 > b(s)$. So this region of $\int dt_1 dt_2$ has $\text{fst. starting higher up}$.



⑥ Thus, the ~~to~~ region is most significant.

⑦ We conclude that the start of $\rho_{st}^{(s)}$ is where $k(t)$ becomes positive with $t_1 = t_2 = t_0$, the lowest cross. sh threshold!

$$z_1 = \frac{s^2 + 2s(t_0 - z) + \Delta}{\Delta^{1/2} \Delta^{1/2}}$$

Then, in general mass case, we look at

$$k=0 \Rightarrow \tilde{z}=? \Rightarrow \tilde{t}=?$$

⑧ Once you have $\rho_{st}^{(s)}$ and $\rho_{su}^{(s)}$ for s in (s_2, s_3) , plot em then also plot other 4.

⑨ In shaded regions, we have exact formulae for ρ_{st} .

$$\rho_{st}^{(s)} = f[D_t, D_u]. \quad D_t, D_u \text{ are at fixed } s.$$

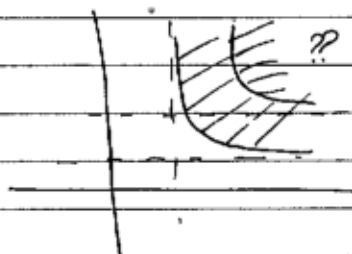
This merely relates data?

$$\begin{aligned} \text{disc}_s D_t &= f(D_t, D_u) \quad \text{for } s < s_3. \\ \text{disc}_s D_u &= g(D_t, D_u) \quad \text{for } s < s_3. \end{aligned}$$

The full amp must satisfy this. Well, all that went in was

- (a) s elastic v
- (b) anal in s, t, u .

⑩ One could in principle do this for three body elastic. One dk integrations $\rightarrow$ 5 variables!! Maybe? Then we'd have ρ_{st} boundary:



Stamley's ln analysis

- ① We take t as shown in red on last page from below $t=0$ to above $t=2$.

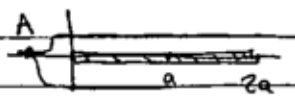
Let: $g(t) = \frac{1}{\sqrt{K}} \ln \left[\frac{f+1}{f-1} \right]$ where $f(t) = \frac{\alpha}{\sqrt{K}} = \frac{(t-a)}{\sqrt{t(t-2a)}}$

Where does $g(t)$ have a discontinuity? When $t < 0$,

$$|f| = \left| \frac{(t-a)}{\sqrt{t(t-2a)}} \right| = \left| \frac{(-1)(t-a)}{\sqrt{1+t-1-t-2a}} \right| = \left| \frac{(1+t+a)}{\sqrt{1+t} \cdot \sqrt{1+2a}} \right| > 1$$

In fact, $f < -1$, so arg of log is $+real$. Take this as principle branch.

- ② Now, also take " $\sqrt{K}$ " = $+\sqrt{t(t-2a)}$ for $t < 2$. As we make the trip as discussed earlier;

 we know that $\sqrt{K}$ goes to its other branch, and $\left[\frac{f+1}{f-1} \right]$ goes $\theta \rightarrow \theta + 2\pi$.

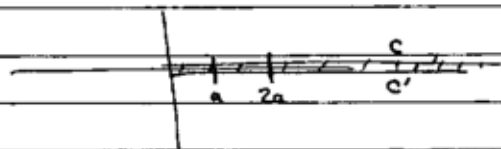
Thus: $g(A) = \frac{1}{+\sqrt{t(t-2a)}} \ln \left[\frac{f+1}{f-1} \right]$

But: $g(C) = \frac{1}{-\sqrt{t(t-2a)}} \left[\ln \left[\frac{f+1}{f-1} \right] - 2\pi i \right]$ $f = \frac{t-a}{-\sqrt{t(t-2a)}}$

$g(C') = \frac{1}{-\sqrt{t(t-2a)}} \left[\ln \left[\frac{f+1}{f-1} \right] + 2\pi i \right]$ $f = \frac{t-a}{-\sqrt{t(t-2a)}}$

So $g(C) - g(C') = \frac{4\pi i}{+\sqrt{t(t-2a)}}$

Thus, the cut for g extends beyond:



- ③ Result seems to be essentially:

$$\text{disc. } \ln \left[\frac{f+1}{f-1} \right] = 4\pi i$$

Then the gst starts at $2a$ since there, $K > 0$.

$$2a = 4t_0 + \frac{t_0^2}{g_s^2}$$

GRAM

4.3 Lorentz invariances + kinematics

(a) Lorentz invariants. Given a set of 4-moms $p_1^\mu, p_2^\mu, \dots, p_n^\mu$, what are the constructible invariants?

(i) $p_i \cdot p_j$, includes masses. By the way, if n lines, then there are $\frac{n(n-1)}{2}$ invariants of this form, excluding masses.

This is $\binom{n}{2} = \frac{n!}{(n-2)!2!} = \frac{n(n-1)}{2} = \# \text{ pair invariants}$.

In fact, all more complicated invariants are just linear combos of these pair invars.

Eg, $(p_1 + p_2 - p_4)^2 = 2p_1 \cdot p_2 + \dots$ ✓

(ii) The sign of p_i^0 for each mom is a L.I. If we draw lines like so: , then all these signs are +.

(iii) Pick 4 arbitrary momenta p_a, p_b, p_c, p_d . Then

$$\epsilon(p_a, p_b, p_c, p_d) \equiv \epsilon_{\mu\nu\alpha\beta} p_a^\mu p_b^\nu p_c^\alpha p_d^\beta = \text{scalar also.}$$

Besides $g_{\mu\nu}$, $\epsilon_{\mu\nu\alpha\beta}$ is the only other tensor available for contracting into vectors to get scalars. (a scalar density).

In matrix form, define

$$A_{\alpha\beta} = p_{\alpha}^{\beta} \quad \begin{matrix} \beta = \text{index label} \\ \alpha = \text{name} \end{matrix}$$

$$\det A = \begin{vmatrix} p_1^0 & p_1^1 & p_1^2 & p_1^3 \\ p_2^0 & p_2^1 & p_2^2 & p_2^3 \\ p_3^0 & p_3^1 & p_3^2 & p_3^3 \\ p_4^0 & p_4^1 & p_4^2 & p_4^3 \end{vmatrix} = \epsilon(p_1, p_2, p_3, p_4).$$

Now:

$$\epsilon(p_1, p_2, p_3, p_4) = ?$$

Consider: $\Lambda \begin{pmatrix} p_1^\mu & p_2^\mu & p_3^\mu & p_4^\mu \end{pmatrix} = \begin{pmatrix} p_1'^\mu & p_2'^\mu & p_3'^\mu & p_4'^\mu \end{pmatrix}$

Thus: $\underbrace{\det \Lambda}_{=1} \cdot \epsilon(p_1, p_2, p_3, p_4) = \epsilon(p_1', p_2', p_3', p_4')$

Thus, $\epsilon = \text{L.I.}$ Obvious. For parity operation, time reversal, $\det \Lambda = -1$, so ϵ changes sign.

Show (4.3.6).

$$\begin{aligned} \epsilon(p_1, p_2, p_3, p_4) \cdot \epsilon(q_1, q_2, q_3, q_4) &= \left| \begin{array}{cccc} p_1 & p_2 & p_3 & p_4 \end{array} \right| \cdot \left| \begin{array}{cccc} q_1 & q_2 & q_3 & q_4 \end{array} \right| \\ &= \det \left(\begin{array}{cccc} p_1 & p_2 & p_3 & p_4 \end{array} \right) \cdot \det \left(\begin{array}{cccc} q_1 & q_2 & q_3 & q_4 \end{array} \right) = \det \left(\begin{array}{cccc} p_1 & & & \\ & p_2 & & \\ & & p_3 & \\ & & & p_4 \end{array} \right) \\ &= \det \left[\left(\begin{array}{cccc} p_1 & p_2 & p_3 & p_4 \end{array} \right) \left(\begin{array}{cccc} q_1 & q_2 & q_3 & q_4 \end{array} \right) \right] = \det \left(\begin{array}{cccc} p_1 \cdot q_1 & p_1 \cdot q_2 & & \\ p_2 \cdot q_1 & & & \\ p_3 \cdot q_1 & & & \\ & & & \end{array} \right) \end{aligned}$$

Here we are using: $\det A \cdot \det B = \det \tilde{A} \cdot \det B = \det (\tilde{A} \cdot B)$
Be more careful!

$$\epsilon(p_1, p_2, p_3, p_4) = \left| \begin{array}{cccc} p_1^\mu & p_2^\mu & p_3^\mu & p_4^\mu \end{array} \right| = \epsilon_{\mu\nu\sigma\tau} p_1^\mu p_2^\nu p_3^\sigma p_4^\tau$$

All Lorentz indices are up. Now we can say:

$$p_1^\mu = g^{\mu\alpha} p_{1\alpha} \text{ etc, to get } = \underbrace{(g^{\mu\alpha} g^{\nu\beta} g^{\sigma\gamma} g^{\tau\delta} \epsilon_{\mu\nu\sigma\tau})}_{= - \epsilon^{\alpha\beta\gamma\delta}} \cdot p_{1\alpha} p_{2\beta} p_{3\gamma} p_{4\delta}$$

This is that old tricky minus sign, see p100 Weinberg.
Thus:

$$\begin{aligned} \epsilon(q_1, q_2, q_3, q_4) &= - \epsilon^{\alpha\beta\gamma\delta} q_{1\alpha} q_{2\beta} q_{3\gamma} q_{4\delta} \\ &= - \det \left(\begin{array}{cccc} q_{1\alpha} & q_{2\alpha} & q_{3\alpha} & q_{4\alpha} \end{array} \right) \end{aligned}$$

$$\text{So, } \epsilon(p_1, \dots) \epsilon(q_1, \dots) = - \det (p_i \cdot q_j) \quad \checkmark$$

Comment: if you know all your $p_i \cdot p_j$, then ϵ can be computed up to a sign. We only need one sign to know the sign of all the ϵ 's.

Similarly, if you know all $p_i \cdot p_j$, the sign of one p_a^α tells you all other signs in (ii).

(b) reconstruction of the momenta + the physical region.

The procedure is as in BCP.

$$\begin{aligned} p_1 &= (\epsilon(p_1) \sqrt{p_1^2}, 0, 0, 0) \\ p_2 &= (p_1 \cdot p_2 / \sqrt{p_1^2}, |p_2|, 0, 0) \\ p_3 &= (p_1 \cdot p_3 / \sqrt{p_1^2}, |p_{3x}|, |p_{3y}|, 0) \quad \text{etc.} \end{aligned}$$

Now consider $\det(p_i \cdot q_j) = \begin{vmatrix} p_1 \cdot q_1 & p_1 \cdot q_2 & \dots \\ \vdots & \vdots & \ddots \end{vmatrix} \equiv G \begin{pmatrix} p_1 \dots p_N \\ q_1 \dots q_N \end{pmatrix}$.

Note that this operation maps: $M^N \otimes M^N \rightarrow \mathbb{R}$

$$G(i) = \det x = G(p_i) = \det(p_i, p_i) = p_i^2, \text{ so } \sqrt{G(i)} = \sqrt{p_i^2}.$$

compute p_2 : $p_2 = (p_2^0, p_2^1, 0, 0)$.

$$p_1 \cdot p_2 = \sqrt{G(i)} p_2^0 \Rightarrow p_2^0 = \frac{G(i, 2)}{\sqrt{G(i)}}$$

$$p_2^1 = \sqrt{p_2^2 + p_2^0^2} = \sqrt{-G(i, 2) + \frac{[G(i, 2)]^2}{G(i)}} = \sqrt{\frac{\{G(i, 2)\}^2 - G(i)G(i, 2)}{G(i)}} = \sqrt{\frac{-G(i, 2)}{G(i)}}$$

But, $G(i, 2) = \begin{vmatrix} p_1 \cdot p_1 & p_1 \cdot p_2 \\ p_2 \cdot p_1 & p_2 \cdot p_2 \end{vmatrix} = (p_1 \cdot p_1)(p_2 \cdot p_2) - (p_1 \cdot p_2)^2 = G(i) - G(i)^2 = G(i) - G(i)G(i, 2)$.

compute p_3 : $p_3^0 \cdot \sqrt{G(i)} = p_3 \cdot p_1 = G(i, 3)$

$$p_2 \cdot p_3 = \frac{G(i, 2)G(i, 3)}{G(i)} - p_3^1 \cdot \sqrt{\frac{-G(i, 2)}{G(i)}} = G(i, 3)$$

$$\text{So, } p_3^1 = \left[\frac{G(i, 2)G(i, 3)}{G(i)} - G(i, 3) \right] / \sqrt{\frac{-G(i, 2)}{G(i)}} = \frac{-G(i, 3)}{\sqrt{-G(i)G(i, 2)}}$$

$$G(i, 3) = \begin{vmatrix} p_1 \cdot p_1 & p_1 \cdot p_3 \\ p_2 \cdot p_1 & p_2 \cdot p_3 \end{vmatrix} = G(i)G(i, 3) - G(i, 2)G(i, 3)$$

Note: $G \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{vmatrix} p_a \cdot p_a & p_a \cdot p_b \\ p_b \cdot p_a & p_b \cdot p_b \end{vmatrix} = G(i, a)G(i, b) - G(i, c)G(i, d)$.

You pair one from the top with one from the bottom.

OK, let's assume (4.3.12) is correct.

compute p_5 ? It must be that $p_5^m = \sum_{i=1}^4 a_i p_i^m$ thus

$$p_1 \cdot p_5 = \sum_{i=1}^4 a_i p_i \cdot p_5$$

In general, $p_j \cdot p_5 = \sum_i a_i p_i \cdot p_j \quad j=1, 2, 3, 4$

$$\text{So: } \begin{aligned} (p_1 \cdot p_1)a_1 + (p_1 \cdot p_2)a_2 + \dots &= p_1 \cdot p_5 & 4 \text{ in } 4 \\ (p_2 \cdot p_1)a_1 + (p_2 \cdot p_2)a_2 + \dots &= p_2 \cdot p_5 \\ \text{etc.} \end{aligned}$$

The solution is:

$$a_1^{(5)} = \frac{\begin{vmatrix} p_1 \cdot p_5 & 12 & 13 & \dots \\ p_2 \cdot p_5 & 22 & 23 & \dots \\ \vdots & \vdots & \vdots & \vdots \end{vmatrix}}{\begin{vmatrix} 11 & 12 & 13 & \dots \\ 21 & 22 & 23 & \dots \\ \vdots & \vdots & \vdots & \vdots \end{vmatrix}} = \frac{G(\begin{smallmatrix} 1 & 2 & 3 & 4 \\ 5 & 2 & 3 & 4 \end{smallmatrix})}{G(\begin{smallmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{smallmatrix})}$$

$$a_2^{(5)} = \frac{\begin{vmatrix} 11 & 15 & 13 & \dots \\ 21 & 25 & 23 & \dots \\ \vdots & \vdots & \vdots & \vdots \end{vmatrix}}{0} = \frac{G(\begin{smallmatrix} 1 & 2 & 3 & 4 \\ 1 & 5 & 3 & 4 \end{smallmatrix})}{G(\begin{smallmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{smallmatrix})}$$

So, we know the coeffs of the minors in terms of Gram det's. So G^5 set p_5 .

Compute p_6 : Now we could repeat the above to get p_6 in terms of p_1, p_2, p_3, p_4 . Eg,

$$a_1^{(6)} = \frac{G(\begin{smallmatrix} 1 & 2 & 3 & 4 \\ 6 & 2 & 3 & 4 \end{smallmatrix})}{G(\begin{smallmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{smallmatrix})} \quad \text{yes, but what is more interesting is this.}$$

1, 2, 3, 4, 5 are linearly dependent vectors.

Suppose we wrote: $a_1 p_1 + a_2 p_2 + a_3 p_3 + a_4 p_4 + a_5 p_5 = 0$.
lindep $\Rightarrow$ this is possible with at least one $a_i \neq 0$, at least 2 in fact.

$$\text{So: } \sum_{i=1}^5 a_i p_i \cdot p_j = 0 \quad \sum_{i=1}^5 (p_i \cdot p_j) a_i = 0 \quad j = 1, \dots, 5.$$

If $\det(p_i \cdot p_j) = 0$, then a_i must all be 0 $\Rightarrow$ lin indep.

$$\text{Eg. } \underbrace{V_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, V_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, V_3 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}}_{\text{are indep.}} \text{ are } M_{ji} a_i = 0 \quad M \vec{a} = 0.$$

$$\begin{vmatrix} V_1 \cdot V_1 & V_1 \cdot V_2 \\ V_2 \cdot V_1 & V_2 \cdot V_2 \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$$

$|M|$

Theorem: if $\{p_i\}$ is a set of lin. dependent vectors, then you can have $\sum_{i=1}^n a_i p_i = 0$ with not all $a_i = 0$.

$$\text{This } \Rightarrow: \sum_{i=1}^n (p_j \cdot p_i) a_i = 0 \quad \forall j.$$

$$\begin{vmatrix} 0 \\ 0 \\ 0 \end{vmatrix} \text{ not}$$

Treat this as a matrix equation. Then:

$$a_i = \frac{\det(p_i \cdot p_j)}{\det(p_i \cdot p_j)}$$

This $\Rightarrow$ all $a_i = 0$ unless $\det(p_i \cdot p_j) = 0$. True:

$$\text{linear dependence } \Rightarrow \det(p_i \cdot p_j) = 0.$$

Therefore, $G \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 6 \\ 1 & 2 & 3 & 4 & 6 \end{pmatrix} = 0$ | $G \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{pmatrix} = 0$ etc.
 etc.

So we now have a set of determinantal conditions.

a) $G(i) > 0$

c) $G(ijk) > 0$

b) $G(ij) < 0$

d) $G(ijkl) < 0$

} all labels are different

e) $G \begin{pmatrix} i & j & k & l & m \\ i & j & k & l & m \end{pmatrix} = 0$

f) $G \begin{pmatrix} i & j & k & l & m & n \\ i & j & k & l & m & n \end{pmatrix} = 0$ etc.

(c) choice of variables
 linear relations like:

Since $\sum_i p_i^m = \sum_f p_f^m$, we get

$\sum_i p_a \cdot p_i = \sum_f p_a \cdot p_f \quad a = \dots ?$

This is like $s+t+u = \Sigma$.

Now consider $G \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix} = 0 = \begin{vmatrix} p_1 \cdot p_1 & p_1 \cdot p_2 & p_1 \cdot p_3 & p_1 \cdot p_4 & p_1 \cdot p_5 \\ p_2 \cdot p_1 & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \end{vmatrix}$

Each term in this det seems to be a quintic in $p_i \cdot p_j$.

$\det(p_i \cdot p_j) = \det(M_{ij}) = \sum_{i_j} \epsilon_{i_j} M_{i_j} ?$

$= M_{1\alpha} M_{2\beta} M_{3\gamma} \dots M_{N\omega} \epsilon^{\alpha\beta\gamma\dots\omega} = 0$

So $\epsilon^{\mu_1 \mu_2 \dots \mu_N} (p_1 \cdot p_{\mu_1}) (p_2 \cdot p_{\mu_2}) (p_3 \cdot p_{\mu_3}) \dots (p_N \cdot p_{\mu_N}) = 0$.

Typically: Let $\mu_1 \mu_2 \dots \mu_N = N, N-1, \dots, 1$. Then this term is:

$(p_1 \cdot p_N) (p_2 \cdot p_{N-1}) (p_3 \cdot p_{N-2}) \dots$

Eg, if $N=6$ you get: $(p_1 \cdot p_6)^2 (p_2 \cdot p_5)^2 (p_3 \cdot p_4)^2$. This is not simply quadratic.

OK, the conditions (4.3.15) are all in terms of squares of $p_i \cdot p_j$, so quadratic in that sense. So from the conditions, there will be sign ambiguities.

$\pm p_3 \cdot p_6 = 1/2$.

(d) Cayley determinants. Define $(p_i - p_j)^2 = d_{ij}$. $i, j = 1, N$.

Also, define $d_{i, N+1} = p_i^2$. Now construct:

$$\begin{array}{c|cccc} 0 & 1 & 1 & 1 & 1 \\ \hline 1 & d_{11} & d_{12} & d_{13} & d_{14} \\ 1 & d_{21} & d_{22} & d_{23} & d_{24} \\ 1 & d_{31} & d_{32} & d_{33} & d_{34} \\ 1 & d_{41} & d_{42} & d_{43} & 0 \end{array} \quad \text{for } N=3.$$

Claim is that this relates to a Gram det:

To show this:

$$= (-1) \begin{vmatrix} 1 & d_{12} & d_{13} & d_{14} \\ 1 & d_{22} & d_{23} & d_{24} \\ 1 & d_{32} & d_{33} & d_{34} \\ 1 & d_{42} & d_{43} & 0 \end{vmatrix} + 3 \text{ other terms.}$$

$$= 1 \cdot \begin{vmatrix} d_{22} & d_{23} & d_{24} \\ d_{32} & d_{33} & d_{34} \\ d_{42} & d_{43} & 0 \end{vmatrix} + 3 \text{ other terms.}$$

$$d_{24} \begin{vmatrix} 22 & 23 \\ 32 & 33 \end{vmatrix} - d_{34} \begin{vmatrix} 22 & 23 \\ 42 & 43 \end{vmatrix}$$

Stop. - Notice that $d_{12} = d_{21}$ and $d_{11} = 0$.

Rewrite as:

$$(\pm) \begin{array}{c|cccc} 0 & 1 & 1 & 1 & 1 \\ \hline 1 & 0 & d_{12} & d_{13} & d_{14} \\ 1 & d_{12} & 0 & d_{23} & d_{24} \\ 1 & d_{13} & d_{23} & 0 & d_{34} \\ 1 & d_{14} & d_{24} & d_{34} & 0 \end{array}$$

This thing is symmetric Kill 1's in first col:

$$\begin{array}{c|cccc|c|cccc} 0 & 1 & 1 & 1 & 1 & & 1 & 1 & 1 & 1 \\ \hline 0 & -d_{14} & d_{12}-d_{24} & d_{13}-d_{34} & d_{14} & & -d_{14} & & & d_{14} \\ 0 & d_{12}-d_{14} & -d_{24} & d_{23}-d_{34} & d_{24} & = (\pm 1) & & -d_{24} & & d_{24} \\ 0 & d_{13}-d_{14} & d_{23}-d_{24} & -d_{34} & d_{34} & & & & -d_{34} & d_{34} \\ \hline 1 & d_{14} & d_{24} & d_{34} & 0 & & & & & \end{array}$$

$$d_{12} - d_{14} = (p_1 - p_4)^2 - (p_1 - p_2)^2 = p_1^2 - p_4^2 - 2p_1 p_2 - 2p_4 p_2$$

$$d_{12} = p_1^2 + p_2^2 - 2p_1 p_2 = d_{14} + d_{24} - 2p_1 p_2$$

see theorem which follows

On relating the Gram and Cayley Determinants

① For $N=2$, we want to prove that:

$$G\left(\begin{matrix} P_1 & P_2 \\ P_1 & P_2 \end{matrix}\right) = \underbrace{\begin{vmatrix} P_1 \cdot P_1 & P_1 \cdot P_2 \\ P_2 \cdot P_1 & P_2 \cdot P_2 \end{vmatrix}}_{\text{Gram}} = \frac{(-1)^3}{2^2} \underbrace{\begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & (P_1 - P_2)^2 & P_1^2 \\ 1 & (P_2 - P_1)^2 & 0 & P_2^2 \\ 1 & P_1^2 & P_2^2 & 0 \end{vmatrix}}_{\text{Cayley}}$$

The Cayley always has dimension $(N+2) \times (N+2)$, whereas Gram is $N \times N$.

② Idea: the Ghandour trick? Can't really see it.

③ Adding one col. to another? First, add $(-1) \times$ last col to cols 2 and 3:

$$G\left(\begin{matrix} P_1 & P_2 \\ P_1 & P_2 \end{matrix}\right) = ? \frac{(-1)^3}{4} \begin{vmatrix} 0 & 0 & 0 & 1 \\ 1 & -P_1^2 & P_2^2 - 2P_1 \cdot P_2 & P_1^2 \\ 1 & P_1^2 - 2P_1 \cdot P_2 & -P_2^2 & P_2^2 \\ 1 & P_1^2 & P_2^2 & 0 \end{vmatrix}$$

$$= \frac{(-1)^{N+1}}{2^N} \cdot (-1)^{N+2+1} \cdot \begin{vmatrix} 1 & -P_1^2 & P_2^2 - 2P_1 \cdot P_2 \\ 1 & P_1^2 - 2P_1 \cdot P_2 & -P_2^2 \\ 1 & P_1^2 & P_2^2 \end{vmatrix}$$

Notice the the sign picked up in this first stage is exactly right. X

④ Now, take $-P_1^2 \times$ 1st col + add to second.
 $-P_2^2 \times$ 1st col + add to third.

$$= \frac{1}{2^N} \begin{vmatrix} 1 & -2P_1^2 & -2P_1 \cdot P_2 \\ 1 & -2P_1 \cdot P_2 & -2P_2^2 \\ 1 & 0 & 0 \end{vmatrix} = \frac{1}{2^N} (-1)^{[N+2-1]} \cdot 2^2 \begin{vmatrix} P_1^2 & P_1 \cdot P_2 \\ P_2 \cdot P_1 & P_2^2 \end{vmatrix}$$

⑤ OK, now do general case in the same way. First, the signs will be

$$(-1)^{1+(N+2)} \cdots (-1)^{1+(N+1)} = (-1)^{2+2N+3} = (-1)$$

From the two det reductions, we shall pick up a -1 period. Later we'll get a $(-1)^N$, too!

⑥ General case:

0	1	1	1	...	1	1
1	0	$(p_1 - p_2)^2$	$(p_1 - p_3)^2$	...	$(p_1 - p_n)^2$	p_1^2
1	$(p_2 - p_1)^2$	0	$(p_2 - p_3)^2$	...	$(p_2 - p_n)^2$	p_2^2
1	$(p_3 - p_1)^2$	$(p_3 - p_2)^2$	0	...	...	...
...	...	...	$(p_4 - p_3)^2$	...	...	...
1	$(p_n - p_1)^2$	$(p_n - p_2)^2$	$(p_n - p_3)^2$	...	0	p_n^2
1	p_1^2	p_2^2	p_3^2	...	p_n^2	0

Take $(-1) \times$ last col + add to all others except first:

	○	○	○	...	○	1
=	1	$-p_1^2$	$p_2^2 - 2p_1 p_2$		$p_n^2 - 2p_1 p_n$	p_1^2
	1	$p_1^2 - 2p_1 p_2$	$-p_2^2$		$p_n^2 - 2p_2 p_n$	p_2^2
	1	$p_1^2 - 2p_1 p_3$	$p_2^2 - 2p_2 p_3$		.	.
	⋮	.	$p_2^2 - 2p_2 p_n$		.	⋮
	⋮	$p_1^2 - 2p_1 p_n$	$p_2^2 - 2p_2 p_n$		$-p_n^2$	.
	1	p_1^2	p_2^2		p_n^2	○

We now can drop one dimension to box. Now do $(-p_1) \times$ 1st + add to second, $-p_2 \times$ 1st + 3rd etc etc.

=	1	$-2p_1^2$	$-2p_2 p_1$	...	$-2p_1 p_n$	N x N
1	$-2p_1 p_2$	$-2p_2^2$	...	...	...	
...	$-2p_1 p_3$	$-2p_2 p_3$	...	...	...	
1	$-2p_1 p_n$	$-2p_2 p_n$	...	$-2p_n^2$	...	
1	0	0	...	0	...	
1	0	0	...	0	...	

Finally, this new box is: $(-2)^N$

$p_1 \cdot p_1$	...
$p_1 \cdot p_2$	...
...	...
$p_1 \cdot p_n$	...
$p_2 \cdot p_1$	...
$p_2 \cdot p_2$	...
...	...
$p_2 \cdot p_n$	...
...	...
$p_n \cdot p_1$	...
$p_n \cdot p_2$	...
...	...
$p_n \cdot p_n$	...

⑦ Overall result is therefore:

$$\text{Cayley} = (-2)^N \cdot (-1) \times \text{Gram}$$

$$= \frac{2^N}{(-1)^{N+1}} \times \text{Gram}$$

The Gram Rendering Theorem

① We want to show, eg, that:

$$G \begin{pmatrix} p_1 & p_2 \\ p_1 & p_2 \end{pmatrix} = G \begin{pmatrix} p_1 & p_{12} \\ p_1 & p_{12} \end{pmatrix} \quad \text{where } p_{12} = \pm p_1 \pm p_2$$

Well,

$$G \begin{pmatrix} p_1 & p_{12} \\ p_1 & p_{12} \end{pmatrix} = \begin{vmatrix} p_1 \cdot p_1 & p_1 \cdot p_{12} \\ p_1 \cdot p_{12} & p_{12} \cdot p_{12} \end{vmatrix} = \begin{vmatrix} p_1 \cdot p_1 & p_1 \cdot (\pm p_1 \pm p_2) \\ p_1 \cdot (\pm p_1 \pm p_2) & (\pm p_1 \pm p_2)^2 \end{vmatrix}$$

Well, write $p_{12} = \alpha p_1 + \beta p_2$ where α, β are signs. Then

$$= \begin{vmatrix} p_1^2 & \alpha p_1^2 + \beta p_1 \cdot p_2 \\ \alpha p_1^2 + \beta p_1 \cdot p_2 & p_1^2 + p_2^2 + 2\alpha\beta p_1 \cdot p_2 \end{vmatrix} = \begin{vmatrix} p_1^2 & \beta p_1 \cdot p_2 \\ \beta p_1 \cdot p_2 & p_2^2 + \alpha\beta p_1 \cdot p_2 \end{vmatrix}$$

Now take $-\alpha \times$ first col, add to second. Then $-\alpha \times$ first row, add to sec.

$$= \begin{vmatrix} p_1^2 & \beta p_1 \cdot p_2 \\ \beta p_1 \cdot p_2 & p_2^2 \end{vmatrix} = G \begin{pmatrix} p_1 & p_2 \\ p_1 & p_2 \end{pmatrix} \checkmark \quad \beta^2 = 1.$$

② Try next higher case:

$$G \begin{pmatrix} p_1 & p_{12} & p_{123} \\ p_1 & p_{12} & p_{123} \end{pmatrix} = \begin{vmatrix} p_1^2 & p_1 \cdot (\alpha p_1 + \beta p_2) & p_1 \cdot (\alpha p_1 + \beta p_2 + \delta p_3) \\ p_1 \cdot (\alpha p_1 + \beta p_2) & (\alpha p_1 + \beta p_2)^2 & (\alpha p_1 + \beta p_2 + \delta p_3)^2 \\ p_1 \cdot (\alpha p_1 + \beta p_2 + \delta p_3) & (\alpha p_1 + \beta p_2 + \delta p_3)^2 & (\alpha p_1 + \beta p_2 + \delta p_3)^2 \end{vmatrix}$$

Generalize notation: $p_{12 \dots n} = \alpha_1 p_1 + \alpha_2 p_2 + \dots + \alpha_n p_n = \sum_{i=1}^n \alpha_i p_i$

$$G \begin{pmatrix} p_1 & p_{12} & p_{123} & \dots & p_{12 \dots n} \\ p_1 & p_{12} & p_{123} & \dots & p_{12 \dots n} \end{pmatrix} = \begin{vmatrix} p_1^2 & p_1 \cdot \sum_{i=1}^2 \alpha_i p_i & p_1 \cdot \sum_{i=1}^3 \alpha_i p_i & \dots & p_1 \cdot \sum_{i=1}^n \alpha_i p_i \\ \sum_{i=1}^2 \alpha_i p_i \cdot p_1 & (\sum_{i=1}^2 \alpha_i p_i)^2 & (\sum_{i=1}^3 \alpha_i p_i)^2 & \dots & (\sum_{i=1}^n \alpha_i p_i)^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \end{vmatrix}$$

Redefine again.

$$= \begin{vmatrix} p_1^2 & p_1 \cdot s_{12} & p_1 \cdot s_{13} & \dots & p_1 \cdot s_{1n} \\ s_{12} \cdot p_1 & s_{12} \cdot s_{12} & s_{12} \cdot s_{13} & \dots & s_{12} \cdot s_{1n} \\ s_{13} \cdot p_1 & s_{13} \cdot s_{12} & s_{13} \cdot s_{13} & \dots & s_{13} \cdot s_{1n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ s_{1n} \cdot p_1 & s_{1n} \cdot s_{12} & s_{1n} \cdot s_{13} & \dots & s_{1n} \cdot s_{1n} \end{vmatrix}$$

③ Now take $-a_1 \times 1^{st}$ col, add to second:

$$(-a_1) \begin{pmatrix} p_1^2 \\ S_{12} \cdot p_1 \\ S_{13} \cdot p_1 \\ S_{1N} \cdot p_1 \end{pmatrix} + \begin{pmatrix} p_1 \cdot S_{12} \\ S_{12} \cdot S_{12} \\ S_{13} \cdot S_{12} \\ S_{1N} \cdot S_{12} \end{pmatrix} = \begin{pmatrix} p_1 \cdot S_{22} \\ S_{12} \cdot S_{22} \\ S_{13} \cdot S_{22} \\ S_{1N} \cdot S_{22} \end{pmatrix}$$

We could now do this for all other cols. Result is:

p_1^2	$p_1 \cdot S_{22}$	$p_1 \cdot S_{23}$	$p_1 \cdot S_{2N}$
$S_{12} \cdot p_1$	$S_{12} \cdot S_{22}$	$S_{12} \cdot S_{23}$	$S_{12} \cdot S_{2N}$
$S_{13} \cdot p_1$	$S_{13} \cdot S_{22}$	$S_{13} \cdot S_{23}$	$S_{13} \cdot S_{2N}$
$S_{1N} \cdot p_1$	$S_{1N} \cdot S_{22}$	$S_{1N} \cdot S_{23}$	$S_{1N} \cdot S_{2N}$

④ Now take $(-a_1) \times$ first row + add to second:

$$(-a_1) \begin{pmatrix} p_1^2 & p_1 \cdot S_{22} & p_1 \cdot S_{23} & p_1 \cdot S_{2N} \end{pmatrix} + \begin{pmatrix} S_{12} \cdot p_1 & S_{12} \cdot S_{22} & S_{12} \cdot S_{23} & S_{12} \cdot S_{2N} \end{pmatrix}$$

$$= \begin{pmatrix} S_{22} \cdot p_1 & S_{22} \cdot S_{22} & S_{22} \cdot S_{23} & S_{22} \cdot S_{2N} \end{pmatrix}$$

This raises all those 1's to 2's by cancelling first term:

Row	p_1^2	$p_1 \cdot S_{22}$	$p_1 \cdot S_{23}$	$p_1 \cdot S_{2N}$
	$S_{22} \cdot p_1$	$(S_{22})^2$	$S_{22} \cdot S_{23}$	$S_{22} \cdot S_{2N}$
	$S_{23} \cdot p_1$	$S_{23} \cdot S_{22}$	$S_{23} \cdot S_{23}$	$S_{23} \cdot S_{2N}$
	$S_{2N} \cdot p_1$	$S_{2N} \cdot S_{22}$	$S_{2N} \cdot S_{23}$	$S_{2N} \cdot S_{2N}$

But, $S_{22} = (a_1 p_1)^2 = p_1^2$, so upper square is as we want it.

⑤ Next stage: take $(-1/a_1) \times 2^{nd}$ col, add to 3rd, 4th ... Nth.

$$(-1/a_1) \begin{pmatrix} p_1 \cdot S_{22} \\ (S_{22} \cdot S_{22}) \\ S_{23} \cdot S_{22} \\ S_{2N} \cdot S_{22} \end{pmatrix} + \begin{pmatrix} p_1 \cdot S_{23} \\ S_{22} \cdot S_{23} \\ S_{23} \cdot S_{23} \\ S_{2N} \cdot S_{23} \end{pmatrix} = \begin{pmatrix} p_1 \cdot S_{33} \\ S_{22} \cdot S_{33} \\ S_{23} \cdot S_{33} \\ S_{2N} \cdot S_{33} \end{pmatrix}$$

This gives:

$S_{11} \cdot S_{11}$	$S_{11} \cdot S_{22}$	$S_{11} \cdot S_{23}$	$S_{11} \cdot S_{2N}$
$S_{22} \cdot S_{11}$	$S_{22} \cdot S_{22}$	$S_{22} \cdot S_{23}$	$S_{22} \cdot S_{2N}$
$S_{23} \cdot S_{11}$	$S_{23} \cdot S_{22}$	$S_{23} \cdot S_{23}$	$S_{23} \cdot S_{2N}$
$S_{2N} \cdot S_{11}$	$S_{2N} \cdot S_{22}$	$S_{2N} \cdot S_{23}$	$S_{2N} \cdot S_{2N}$

Here, replaced all f_i with S_{11} .

$S_{11} \cdot S_{11}$	$S_{11} \cdot S_{22}$	$S_{11} \cdot S_{33}$	$S_{11} \cdot S_{3N}$
$S_{22} \cdot S_{11}$	$S_{22} \cdot S_{22}$	$S_{22} \cdot S_{33}$	$S_{22} \cdot S_{3N}$
$S_{23} \cdot S_{11}$	$S_{23} \cdot S_{22}$	$S_{23} \cdot S_{33}$	$S_{23} \cdot S_{3N}$
$S_{2N} \cdot S_{11}$	$S_{2N} \cdot S_{22}$	$S_{2N} \cdot S_{33}$	$S_{2N} \cdot S_{3N}$

⑥ Now do $(-1) \times$ second row + 3rd, 4th, ... N^{th} . This raises all 2^{nd} to 3^{rd} in first index.

$S_{11} \cdot S_{11}$	$S_{11} \cdot S_{22}$	$S_{11} \cdot S_{33}$	$S_{11} \cdot S_{3N}$
$S_{22} \cdot S_{11}$	$S_{22} \cdot S_{22}$	$S_{22} \cdot S_{33}$	$S_{22} \cdot S_{3N}$
$S_{33} \cdot S_{11}$	$S_{33} \cdot S_{22}$	$S_{33} \cdot S_{33}$	$S_{33} \cdot S_{3N}$
$S_{3N} \cdot S_{11}$	$S_{3N} \cdot S_{22}$	$S_{3N} \cdot S_{33}$	$S_{3N} \cdot S_{3N}$

Now 9 elements are all set up.

⑦ Continue this process until matrix looks like:

$S_{11} \cdot S_{11}$	$S_{11} \cdot S_{22}$	$S_{11} \cdot S_{33}$	$S_{11} \cdot S_{44}$	$S_{11} \cdot S_{NN}$
$S_{22} \cdot S_{11}$	$S_{22} \cdot S_{22}$	$S_{22} \cdot S_{33}$	$S_{22} \cdot S_{44}$	$S_{22} \cdot S_{NN}$
$S_{33} \cdot S_{11}$	$S_{33} \cdot S_{22}$	$S_{33} \cdot S_{33}$	$S_{33} \cdot S_{44}$	$S_{33} \cdot S_{NN}$
$S_{44} \cdot S_{11}$	$S_{44} \cdot S_{22}$	$S_{44} \cdot S_{33}$	$S_{44} \cdot S_{44}$	$S_{44} \cdot S_{NN}$
$S_{NN} \cdot S_{11}$	$S_{NN} \cdot S_{22}$	$S_{NN} \cdot S_{33}$	$S_{NN} \cdot S_{44}$	$S_{NN} \cdot S_{NN}$

etc.

Now extract signs α_i from all of row 2, then col 2. $(\alpha_2)^2 = 1$
so the net result is all $S_{22} \rightarrow p_2$. Etc do get

$$= G = \begin{vmatrix} p_1 \cdot p_1 & p_1 \cdot p_2 & \dots \\ p_2 \cdot p_1 & \dots & \dots \end{vmatrix} = G \begin{pmatrix} p_1 & \dots & p_N \\ p_1 & \dots & p_N \end{pmatrix}$$

⑧ Note: These signs could not be pulled earlier!

⑨ So, for any system of signs, this theorem works.

Eg, $p_{12} = (p_1 + p_2)$ $p_{13} = (p_1 + p_2 + p_3)$ eq, all +.

⑩ Significance of Gram Ordering Theorem + Gram/Cayley:

The fancy Gram/Cayley will now say:

$$G(p_1 \dots p_N) = G \begin{pmatrix} p_1 & p_2 & p_3 & \dots & p_N \\ p_1 & p_2 & p_3 & \dots & p_N \end{pmatrix}$$

$$= \frac{(-1)^{N+1}}{2^N} \begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & p_1^2 & S_{23} & S_{24} & S_{25} \\ 1 & p_2^2 & 0 & 0 & 0 \\ 1 & S_{23} & p_3^2 & 0 & 0 \\ 1 & S_{234} & S_{34} & p_4^2 & 0 \\ 1 & p_1^2 & p_2^2 & p_3^2 & p_4^2 \end{vmatrix}$$

Since:

$$\begin{aligned} d_{21} &\rightarrow d_{12,1} \rightarrow (\alpha p_1 + \beta p_2 - \gamma p_1)^2 = p_1^2 & \text{Let } N=4. \\ d_{31} &\rightarrow d_{123,1} \rightarrow (\beta p_1 + \gamma p_2)^2 = S_{23} \\ d_{41} &\rightarrow d_{1234,1} \rightarrow S_{234} \\ d_{5,1} &\rightarrow p_1^2 \end{aligned}$$

$$\begin{aligned} \text{Also, } d_{12} &\rightarrow d_{1,12} = d_{12,1} \\ d_{3,2} &\rightarrow d_{123,2} \rightarrow p_2^2 \end{aligned}$$

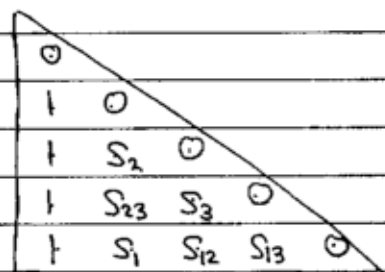
$$d_{4,2} \rightarrow d_{1234,2} \rightarrow$$

$$\text{In general } d_{n,m} \rightarrow d_{1\dots n, m} = (\pm p_m \pm p_{n+1} \dots p_n)^2 = S_{m, n+1, \dots, n}$$

⑪ Instructions for writing the Cayley of order N.

(a) fill in lower left triangle since whole thing is symmetric.

(b) The 3 boundaries are: 0^2 , 0111 , $S_1, S_{12}, S_{123} \dots$



The inwards are

$$\begin{aligned} d_{21} \\ d_{31} \quad d_{32} \\ d_{41} \quad d_{42} \quad d_{43} \\ \downarrow \end{aligned}$$

$$\begin{aligned} S_2 \\ S_{23} \quad S_3 \\ S_{234} \quad S_{34} \quad S_4 \end{aligned}$$

(12) So let's draw some typical Calley lower triangles:

$N=1 \Rightarrow$

0
1 0
1 S_1 0

/no mirrards at all.

$N=2 \Rightarrow$

0
1 0
1 S_2 0
1 S_1 S_{12} 0

$N=3 \Rightarrow$

0
1 0
1 S_2 0
1 S_{23} S_3 0
1 S_1 S_{12} S_{123} 0

$N=4 \Rightarrow$

0
1 0
1 S_2 0
1 S_{23} S_3 0
1 S_{234} S_{34} S_4 0
1 S_1 S_{12} S_{123} S_{1234} 0

(13) Full general case:

0
1 0
1 S_2 0
1 S_{23} S_3 0
1 S_{234} S_{34} S_4 0
1 S_{2345} S_{345} S_{45} S_5 0
1 0
1 S_1 S_{12} S_{123} . . . $S_{123 \dots N}$ 0

(14) Applications of The determinants

(a) What do we learn for $N=1$?

$$G(1) > 0 \Rightarrow \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & S_1 \\ 1 & S_1 & 0 \end{vmatrix} > 0 \quad \text{where } S_1 = p_1^2.$$

$$\Rightarrow \begin{vmatrix} 0 & 0 & 1 \\ 1 & -S_1 & S_1 \\ 1 & S_1 & 0 \end{vmatrix} = (+1) \cdot [S_1 + S_1] > 0 \Rightarrow S_1 > 0$$

$$\Rightarrow p_1^2 > 0$$

So the $N=1$ determinantal condition says: all momenta must be timelike, positive masses (regardless of sign of p_0). Of course this was assumed at the start.

[illegible]

[illegible]





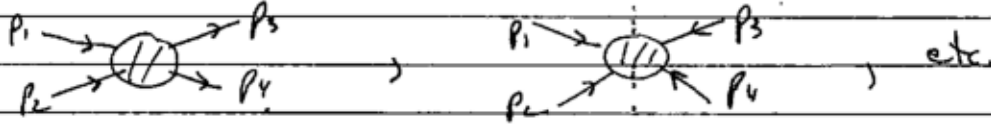
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1. Note on sign convention. The various gram inequalities are

① sign independent i.e.,

$$G\begin{pmatrix} p_1 & p_2 \\ p_1 & p_2 \end{pmatrix} = G\begin{pmatrix} -p_1 & p_2 \\ -p_1 & p_2 \end{pmatrix} \text{ etc.}$$

Thus, the inequalities are the same for any labelling:

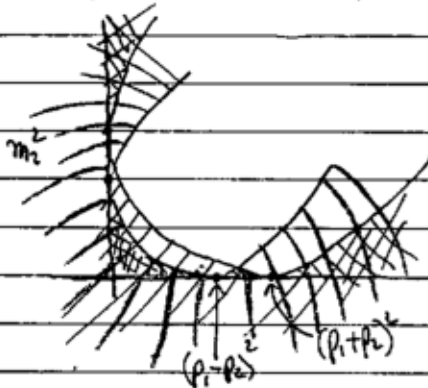


② Now it seemed to me that, when these gram det conditions are translated into S-conditions, the results are true for any system of signs, not just the one system Eden likes.

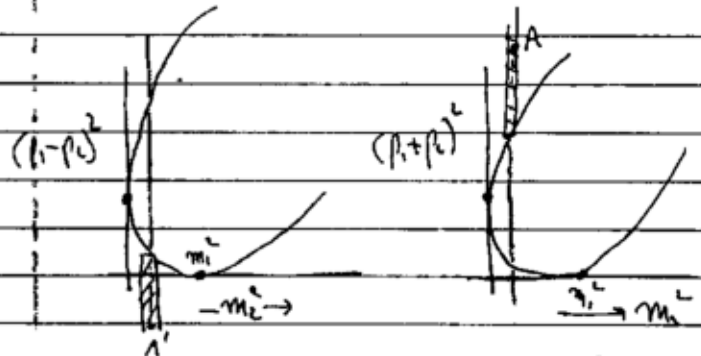
Thus, the $N=2$ condition is $\Delta(m_i^2, m_j^2, S_{ij}) > 0$ for any pair. This is only one of the equations implied by $N=2$ gram. The two are:

$$\begin{aligned} a) & \Delta(m_1^2, m_2^2, (p_1 + p_2)^2) > 0 \\ b) & \Delta(m_1^2, m_2^2, (p_1 - p_2)^2) > 0 \end{aligned}$$

If we plot both of these on m_1/m_2 plot:



As above shows regions allowed by both conditions.



Alternative plot method:

Suppose we have selected (m_1, m_2) , red line. + Suppose $(p_1 + p_2)^2 = S$. Then it makes sense. Suppose $\tilde{p}_1, \tilde{p}_2$ are selected $\Rightarrow$ you are at point A. Then, condition (1) is OK, but what about cond (2)?

If S is in region A, then $(p_1 + p_2)^2 = S \geq (m_1 + m_2)^2 \Rightarrow \tilde{p}_1 \cdot \tilde{p}_2 > m_1 m_2$. But $\tilde{p}_1 \cdot \tilde{p}_2 > m_1 m_2 \Rightarrow (p_1 - p_2)^2 \leq (m_1^2 - m_2^2)^2$, & this is region A'! Thus, $A \leftrightarrow A'$, & both are satisfied.

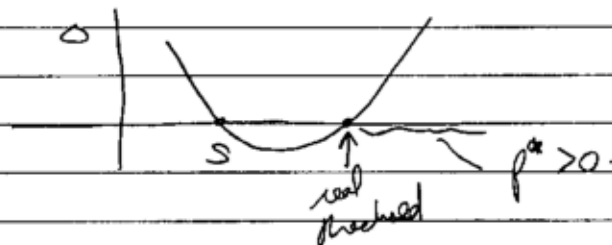
③ How did we know that $s \geq (m_1 + m_2)^2$, besides from the constraint? Well, threshold is one is at

$$\Delta(m_1^i, m_2^i, s) = 0 \quad \text{ie } p^* = 0.$$

So: $\Delta(m_1^i, m_2^i, s) = [s - (m_1 + m_2)^2][s - (m_1 - m_2)^2]$.

Now

$$= 0 \Rightarrow s = (m_1 \pm m_2)^2.$$



④ If both momenta are incoming, then physical region is $s = (p_1 + p_2)^2 \geq (m_1 + m_2)^2$
or $\tilde{s} = (p_1 - p_2)^2 \leq (m_1 - m_2)^2$

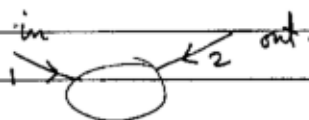
How do we rule out $s \leq (m_1 - m_2)^2$?!? ie, $p_1 \cdot p_2 < m_1 m_2$.

Well,

$$p_1 \cdot p_2 = E_1 E_2 - \vec{p}_1 \cdot \vec{p}_2 \geq E_1 E_2 - p_1 p_2 \geq 0.$$

So, if E_1 and $E_2 > 0$, you cannot be in $p_1 \cdot p_2 < m_1 m_2$.

⑤ Now consider this case:



Now the region is $s \leq (m_1 - m_2)^2$
 $\tilde{s} \geq (m_1 + m_2)^2$

because now $E_2 < 0$!

⑥ Summary: if 1 and 2 are on same side, then sensible variable is $s = (p_1 + p_2)^2$, and condition is: $s \geq (m_1 + m_2)^2$.
If 1 and 2 are on opposite sides, then sensible variable is $t = (p_1 - p_2)^2$, and condition is $t \leq (m_1 - m_2)^2$.

⑦ Plot the Δ inequality: $\Delta(m_1^i, m_2^i, s) > 0$.

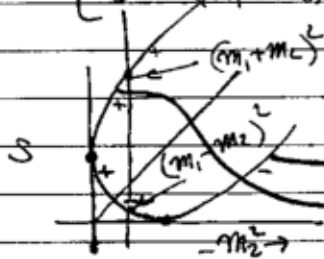
$$\Rightarrow [s - (m_1 + m_2)^2][s - (m_1 - m_2)^2] > 0.$$

$$\Rightarrow s^2 - s[2(m_1^2 + m_2^2)] + (m_1^2 - m_2^2)^2 > 0.$$

$$\Rightarrow s = \frac{1}{2}(m_1^2 + m_2^2) \pm \sqrt{(m_1^2 + m_2^2) - (m_1^2 - m_2^2)^2}$$

$$\Rightarrow s = m_1^2 + m_2^2 - [4m_1^2 m_2^2]^{1/2} = (m_1 - m_2)^2$$

$$\Rightarrow s_+ = (m_1 + m_2)^2.$$



this question cost mucho tiempo.
-9- see later notes

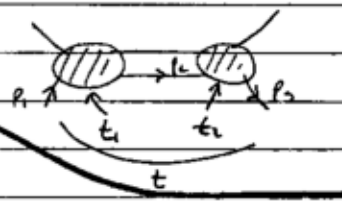
4.3

⑧ So, the $N=2$ Gram gives the standard conditions on subenergies and momentum transfers.

⑨ Can it be related to:

Let

$$\begin{aligned} t_1 &= (p_1 - p_2)^2 = (p_{12})^2 \\ t_2 &= (p_2 - p_3)^2 = (p_{23})^2 = (p_{33})^2 \\ t &= (p_3 - p_1)^2 = (p_{31})^2 \end{aligned}$$



?
yes, but
now all 3
momenta are
spacelike

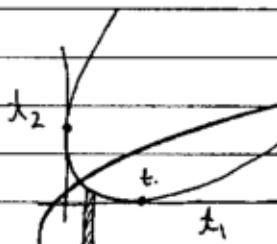
Notice that $p_1 + p_3 = p_2$. So

$$\begin{aligned} t_1 &= (p_{12})^2 \\ t_2 &= (p_{23})^2 \\ t &= (p_{12} + p_{23})^2 \end{aligned}$$

Well, since p_1, p_2, p_3 are supposed to be physical momenta [i.e., real components], the Gram $G(12) < 0$ does apply and we get:

$$\Delta(t_1, t_2, t) > 0$$

then:



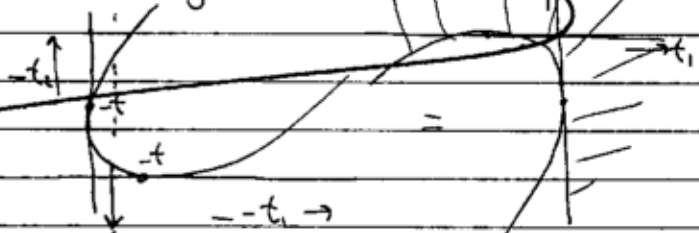
but I am used to $\Delta(t_1, t_2, t) < 0$!

Aha! I am plotting this as if t were > 0 .

Consider $[t - (t_1 + t_2)^2][t - (\sqrt{t_1} \dots$ Stop.

use: $\Delta(t_1, t_2, t) = \Delta(-t_1, -t_2, -t)$ to get:

$$\Delta(-t_1, -t_2, -t) > 0 \Rightarrow$$



Need a sign here!

⑩ Maybe I should redefine the gram det keeping $\epsilon(p^0)$ factor!

$$p_1^\mu = [\epsilon(p^0) \sqrt{\epsilon(p^3)} p_1^2, 0, 0, 0]$$

and allowing spacelikes?!

Stop. If p were spacelike, you could not do this. Things have to be redone, like:

$$p_1^\mu = (0, 0, 0, \epsilon(p^3) \sqrt{-p_1^2})$$

$$\begin{aligned} p_1 \cdot p_2 &= -\epsilon(p^3) \sqrt{-p_1^2} \cdot p_2^3 \\ \Rightarrow p_2^3 &= \frac{p_1 \cdot p_2}{-\epsilon(p^3) \sqrt{-p_1^2}} \end{aligned}$$

$$p_2^\mu = (p_2^0, 0, 0, \frac{p_1 \cdot p_2}{\epsilon(p^3) \sqrt{-p_1^2}}) \text{ no!}$$

⑪ OK, one thing at a time. If $p_1, p_2, \dots, p_N$ are arbitrary timelike ($p_i^0 > 0$ or < 0), then we have to modify the gram rules to take this into account. Eden uses all future timelike vectors. NO!

$$p_1^\mu = (\epsilon(p_1^0) \sqrt{p_1^2}, 0, 0, 0)$$

$$p_1 \cdot p_2 = (\epsilon(p_1^0) \sqrt{p_1^2}) \cdot p_2^0 \Rightarrow p_2^0 = \dots$$

$$\Rightarrow p_2^\mu = \left(\frac{p_1 \cdot p_2}{\sqrt{p_1^2}} \epsilon(p_1^0), p_2^1, 0, 0 \right)$$

$$p_1 \cdot p_3 = \dots \Rightarrow p_3^0 = \dots \frac{p_1 \cdot p_3}{\sqrt{p_1^2}} \epsilon(p_1^0)$$

$$p_2 \cdot p_3 = \left[\epsilon(p_1^0) \sqrt{p_1^2} \right] \left[\frac{p_1 \cdot p_3}{\sqrt{p_1^2}} \epsilon(p_1^0) \right] - p_2^1 p_3^1$$

$\Rightarrow$

$$p_3^1 = \frac{p_1 \cdot p_3 - p_2 \cdot p_3}{p_2^1}$$

$$\text{Finally, } p_3 \cdot p_3 = p_3^2 \Rightarrow -(p_3^2)^2 =$$

Stop: accept (4.3.12) and insert signs by hand:

$$G(1) = \det |p_i \cdot p_j| = p_1 \cdot p_1$$

If $p_3^\mu \rightarrow -p_3^\mu$, this does change sign.

$$G(1) = p_1 \cdot p_1, \text{ no sweat.}$$

$$G(12) = \begin{vmatrix} p_1 \cdot p_1 & p_1 \cdot p_2 \\ p_2 \cdot p_1 & p_2 \cdot p_2 \end{vmatrix} \text{ no effect.}$$

$$G(123 \dots N) = \begin{vmatrix} p_1 \cdot p_1 & p_1 \cdot p_2 & p_1 \cdot p_3 & \dots \\ p_2 \cdot p_1 & p_2 \cdot p_2 & p_2 \cdot p_3 & \dots \\ p_3 \cdot p_1 & p_3 \cdot p_2 & p_3 \cdot p_3 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{vmatrix}$$

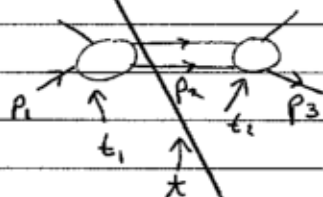
Conclusion

OK, these gram conditions are good for all energy sign conventions, as discussed before.

No corrections needed. (as long as all timelike)

Comment on Gram applied to elastic unitarity + $\Delta(t, t_1, t_2) \leq 0$.

① Recall the setup:



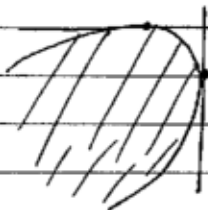
We recall that all we really know about these momenta is:

We know for sure that p_1, p_2, p_3 are all timelike (future).

But consider: $p_2 = (p_2 - p_1)$
 $p_{23} = (p_3 - p_2)$
 $p_{13} = (p_3 - p_1)$

$$\begin{aligned} p_{12}^2 = t_{12} &\leq (m_2 - m_1)^2 & t_1 \\ p_{23}^2 = t_{23} &\leq (m_3 - m_2)^2 & t_2 \\ p_{13}^2 = t_{13} &\leq (m_3 - m_1)^2 & t \end{aligned}$$

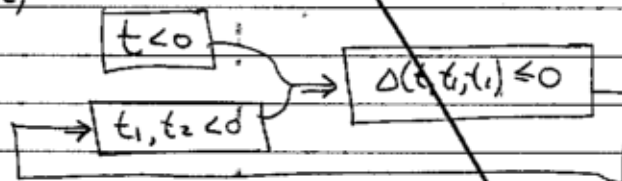
If masses are unequal, all variables t_1, t_2 , and t will have timelike ranges. Only if $m_1 = m_2 = m_3$, can we say for sure that all 3 are spacelike. And only then can we arrive at $\Delta(t, t_1, t_2) \leq 0$.



But the $\Delta \leq 0$ limit is also correct at $s \rightarrow \infty$ for any masses.

② However, if we are given that $t < 0$, then even if t_1, t_2 could go spacelike, such regions are unphysical because of the Δ condition? (NOT TRUE)

ie,



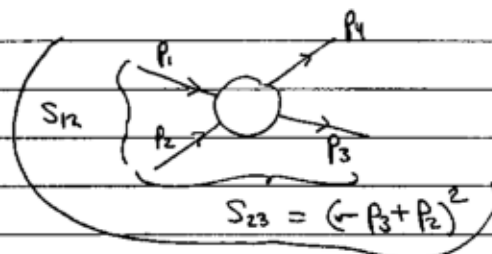
This is self consistent, but does not prove conclusively that $t_1 > 0$ is forbidden. I.e., if two needles, we need a theorem: ~~2cm~~

$$\left\{ \begin{array}{l} (p_3 - p_1) \text{ spacelike?} \\ + p_1, p_2, p_3 \text{ timelike future.} \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} (p_2 - p_1) \text{ spacelike?} \\ (p_3 - p_2) \text{ spacelike?} \end{array} \right.$$

This theorem is false. $\Delta(t, t_1, t_2) \leq 0$ is not the t_1, t_2 boundary for unequal masses even if $t < 0$.

The case $N=3$ Now we have 3 timelike momenta and the condition is:

$$\begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & & & \\ 1 & S_2 & 0 & & \\ 1 & S_{23} & S_3 & 0 & \\ 1 & S_1 & S_{12} & S_{123} & 0 \end{vmatrix} > 0$$



Apply this to:

$$S_{123} = (p_1 + p_2 - p_3)^2 = m_4^2$$

So we have

$$\begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & m_2^2 & t & m_1^2 \\ 1 & m_2^2 & 0 & m_3^2 & s \\ 1 & t & m_3^2 & 0 & m_4^2 \\ 1 & m_1^2 & s & m_4^2 & 0 \end{vmatrix} > 0$$

This defines the few physical region for s, t :

Mult out:

$$\begin{vmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & -m_1^2 & m_2^2 - m_1^2 & t - m_1^2 & \\ 1 & m_2^2 - s & -s & m_3^2 - s & \\ 1 & t - m_4^2 & m_3^2 - m_4^2 & -m_4^2 & \\ 1 & m_1^2 & s & m_4^2 & \end{vmatrix}$$

$$= + \begin{vmatrix} 1 & -2m_1^2 & m_2^2 - m_1^2 - s & t - m_1^2 - m_4^2 \\ 1 & -m_1^2 - s + m_2^2 & -2s & m_3^2 - s - m_4^2 \\ 1 & t - m_4^2 - m_1^2 & m_3^2 - m_4^2 - s & -2m_4^2 \\ -1 & 0 & 0 & 0 \end{vmatrix}$$

$$= - \left\{ (-2m_1^2) (4sm_4^2 - (s - m_3^2 + m_4^2)(s - m_3^2 + m_4^2)) \right.$$

$$= - \left\{ (-2m_1^2) (4sm_4^2 - (s - m_3^2 + m_4^2)^2) + (s + m_2^2 + m_1^2) (\right.$$

etc. This is the standard s/t Kibble function, includes Dalitz + Mandelstam regions.

The Painful Case of Two Spacelike Momenta

- ① Note: This subject was originally done wrong, the cost was ~ 6 hours
- ② Given q_1 and q_2 both spacelike, you can surely find a frame in which:

$$q_1 = (0, 0, 0, \sqrt{-t_1})$$

$$q_2 = (E'_2, X'_2, 0, Z'_2)$$

At this point we must distinguish between two cases.

- ③ Case I: if $|E'_2| \leq |X'_2|$, then $|Z'_2| \leq \sqrt{-t_2}$
you can boost away the energy to get: $q_2 = (0, X_2, 0, Z_2)$.

One quickly finds $Z_2 = -(q_1 \cdot q_2) / \sqrt{-t_1}$ and then

$$X_2 = \pm \sqrt{-Z_2^2 - t_2} \Rightarrow -(Z_2^2) - t_2 \geq 0 \Rightarrow \Delta(t, t_1, t_2) \leq 0$$

In this regime, $q = q_1 + q_2$ must be spacelike, i.e., $t \leq 0$.
In particular,

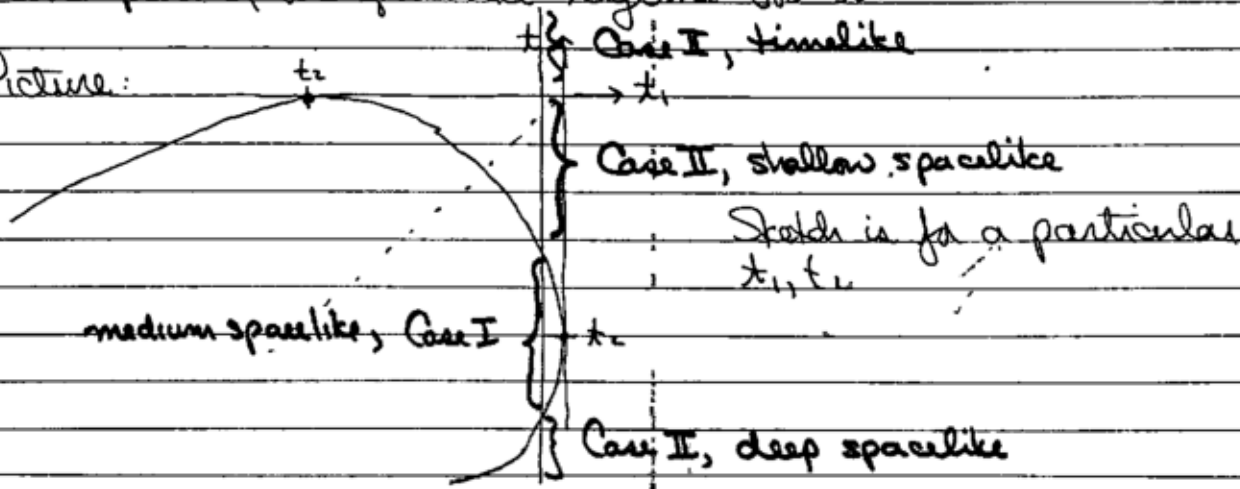
$$-[\sqrt{-t_1} + \sqrt{-t_2}]^2 \leq t \leq -[\sqrt{-t_1} - \sqrt{-t_2}]^2$$

- ④ Case II: if $|E'_2| \geq |X'_2|$, then $|Z'_2| \geq \sqrt{-t_2}$
you can boost away the X_2 to get: $q_2 = (E_2, 0, 0, Z_2)$.

Now, $Z_2 =$ same as ③, $E_2 = \pm \sqrt{Z_2^2 + t_2} \Rightarrow Z_2^2 + t_2 \geq 0$

$\Rightarrow \Delta(t, t_1, t_2) \geq 0$. This case thus includes the timelike and part of the spacelike regime for t .

⑤ Picture:

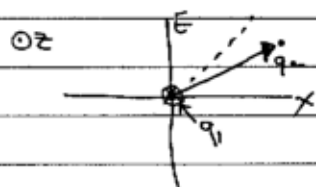


⑥ Pictorial interpretations:

Case 1: Here, $q_1 + q_2 = (0, x_2, 0, z_2 + \sqrt{-t_1})$.

This lies on the x - z plane, no question but that spacelike. Before boosting we had:

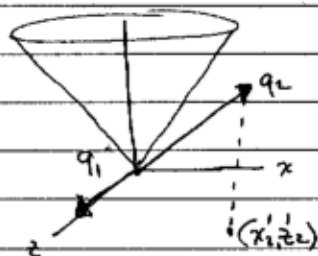
$$q_1 + q_2 = (E'_2, x'_2, 0, z_2 + \sqrt{-t_1}) \quad \text{with } |E'_2| \leq |x'_2|.$$



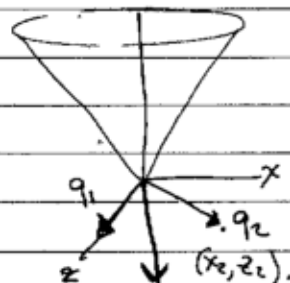
Add $q_1 + q_2$, get only spacelike.

$$q_1 = (0, 0, 0, \sqrt{-t_1}) \quad q_2 = (E'_2, x'_2, 0, z_2)$$

Try a better picture:
Even if q_2 goes behind
the light cone, $q_1 + q_2$
will still not penetrate
it.

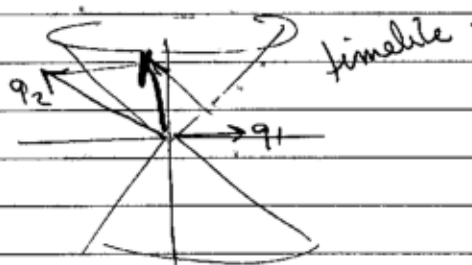


OR

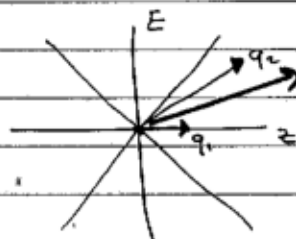


Case 2: Here, $q_1 = (0, 0, 0, z_1)$
 $q_2 = (E_2, 0, 0, z_2)$.

But:

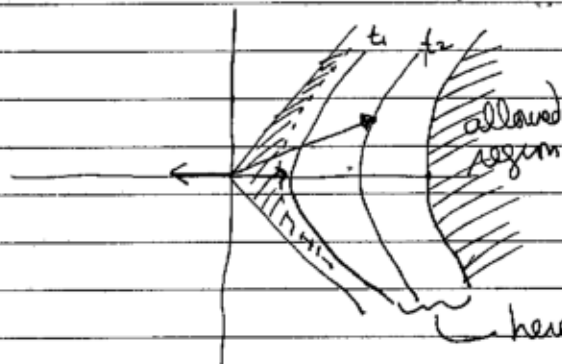


timelike.



spacelike.

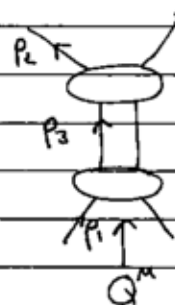
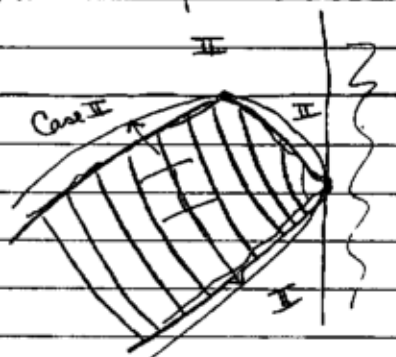
⑦ Why does Case II have absent that medium S.L. range?



here is that gap.

Elastic Unitarity : where is the physical region "by Gram"

- ① Set up f.t. p_1, p_2, p_3 , then form q_1, q_2, q_3 . For equal masses, q_1, q_2, q_3 all pure spacelike.
- ② Also consider q_1, q_2 , both spacelike. In general, [i.e., without considering higher constraints], both case I and case II are allowed. Thus, we really can make no claim like $\Delta \geq 0$ or $\Delta \leq 0$.
- ③ In fact, the exact analysis shows that the true physical region is in fact within case I, as drawn in red orange. This boundary is obtained by conversion of dR to ddR variables explicitly.



- ④ To find what the "higher constraint is", use:

$$\begin{aligned} t_1 &= (p_1 - p_3)^2 & u &= -p_3^2 \\ t_2 &= (p_2 - p_3)^2 & v &= (Q - p_3)^2 \end{aligned}$$

Then: $A = -2(p_1 - p_3)$ $B = -2(p_2 - p_3)$ $C = -2p_3$ $D = -2(Q - p_3)$

And: $J = \sqrt{-G}$ where $G = G \begin{pmatrix} A & B & C & D \\ A & B & C & D \end{pmatrix}$.

Here, $G = \begin{vmatrix} A \cdot A^+ & A \cdot B^+ & A \cdot C^+ & A \cdot D^+ \\ B \cdot A^+ & B \cdot B^+ & B \cdot C^+ & B \cdot D^+ \\ C \cdot A^+ & C \cdot B^+ & C \cdot C^+ & C \cdot D^+ \\ D \cdot A^+ & D \cdot B^+ & D \cdot C^+ & D \cdot D^+ \end{vmatrix} = 4^4 \tilde{G}$

$$\tilde{G} = \begin{vmatrix} (p_1 - p_3) \cdot (p_1 - p_3) & (p_1 - p_3) \cdot (p_2 - p_3) & (p_1 - p_3) \cdot p_3 & (p_1 - p_3) \cdot (Q - p_3) \\ (p_2 - p_3) \cdot (p_1 - p_3) & (p_2 - p_3) \cdot (p_2 - p_3) & (p_2 - p_3) \cdot p_3 & (p_2 - p_3) \cdot (Q - p_3) \\ p_3 \cdot (p_1 - p_3) & p_3 \cdot (p_2 - p_3) & p_3 \cdot p_3 & p_3 \cdot (Q - p_3) \\ (Q - p_3) \cdot (p_1 - p_3) & (Q - p_3) \cdot (p_2 - p_3) & (Q - p_3) \cdot p_3 & (Q - p_3) \cdot (Q - p_3) \end{vmatrix}$$

$$Q = (\sqrt{S}, 0, 0, 0)$$

By almost trivial manipulation, we find:

$$\tilde{G} = \begin{vmatrix} p_1 \cdot p_1 & p_1 \cdot p_2 & p_1 \cdot p_3 & p_1 \cdot Q \\ p_2 \cdot p_1 & p_2 \cdot p_2 & p_2 \cdot p_3 & p_2 \cdot Q \\ p_3 \cdot p_1 & p_3 \cdot p_2 & p_3 \cdot p_3 & p_3 \cdot Q \\ Q \cdot p_1 & Q \cdot p_2 & Q \cdot p_3 & Q \cdot Q \end{vmatrix} = \boxed{G(p_1, p_2, p_3, Q)} = \tilde{G}$$

↑ at last!

Thus, the condition $G \leq 0$ becomes:

$$G(p_1, p_2, p_3, Q) \leq 0.$$

This is the higher constraint which gives the boundary to the unitarity integral.

$$\text{I.e., } \int d^4 p_3 \cdot \delta^4(p_3^2 - m_3^2) \delta^4([Q - p_3]^2 - m_5^2) \cdot f \cdot \Theta(E_3) \Theta(\sqrt{S} - E_3)$$

$$\rightarrow \int dt_1 dt_2 du dv \cdot \Theta(-\tilde{G}) \cdot [\sqrt{-4\tilde{G}}]^{-1} f \cdot \delta(-u - m_3^2) \delta(v - m_5^2) \cdot N$$

In cases, no need to worry about three energy thresholds.

$$= \int dt_1 dt_2 \Theta(-\tilde{G}) [\sqrt{-4\tilde{G}}]^{-1} f \cdot N \quad \text{overcount factor.}$$

⑤ If we go to equal mass and cms, then:

$$p_1 \cdot Q = E_1 \sqrt{S} \equiv E \sqrt{S} = S/2$$

$$E = (S + \dots) = \sqrt{S}/2.$$

$$p^2 = \frac{(S - 4m^2)(S - 0)}{4S}$$

$$E^2 = \frac{(S)^2}{4S} = \frac{S}{4}$$

$$\tilde{G} = \begin{vmatrix} m^2 & p_1 \cdot p_2 & p_1 \cdot p_3 & E \\ p_2 \cdot p_1 & m^2 & p_2 \cdot p_3 & E \\ p_3 \cdot p_1 & p_3 \cdot p_2 & m^2 & E \\ E & E & E & (S/4) \end{vmatrix}$$

$$= \begin{vmatrix} m^2 - E^2 & p_1 \cdot p_2 - E^2 & p_1 \cdot p_3 - E^2 & 0 \\ p_2 \cdot p_1 - E^2 & m^2 - E^2 & p_2 \cdot p_3 - E^2 & 0 \\ p_3 \cdot p_1 - E^2 & p_3 \cdot p_2 - E^2 & m^2 - E^2 & 0 \\ 0 & 0 & 0 & (S/4) \end{vmatrix}$$

$$= (-S) \begin{vmatrix} m^2 - p^2 & p_1 \cdot p_2 - p^2 & p_1 \cdot p_3 - p^2 \\ p_2 \cdot p_1 - p^2 & m^2 - p^2 & p_2 \cdot p_3 - p^2 \\ p_3 \cdot p_1 - p^2 & p_3 \cdot p_2 - p^2 & m^2 - p^2 \end{vmatrix} + (p_1 \cdot p_3 - p^2) \begin{vmatrix} p_1 \cdot p_2 - p^2 & p_2 \cdot p_3 - p^2 \\ p_2 \cdot p_1 - p^2 & m^2 - p^2 \end{vmatrix} + \dots$$

So far we have:

$$\tilde{G} = + \begin{vmatrix} m^2 & p_1 \cdot p_2 & p_1 \cdot p_3 & 5/2 \\ p_1 \cdot p_2 & m^2 & p_2 \cdot p_3 & 5/2 \\ p_3 \cdot p_1 & p_3 \cdot p_2 & m^2 & 5/2 \\ 5/2 & 5/2 & 5/2 & 5 \end{vmatrix}$$

$$= \frac{1}{(-2)^4} \begin{vmatrix} -2m^2 & -2p_1 \cdot p_2 & -2p_1 \cdot p_3 & -5 \\ -2p_1 \cdot p_2 & -2m^2 & -2p_2 \cdot p_3 & -5 \\ -2p_1 \cdot p_3 & -2p_2 \cdot p_3 & -2m^2 & -5 \\ -5 & -5 & -5 & -20 \end{vmatrix} \sqrt{5} \sqrt{5}$$

Now pull out $\sqrt{5}$ from bottom row. No, instead do:

$$= \frac{1}{16} \begin{vmatrix} -2m^2 & -2p_1 \cdot p_2 & -2p_1 \cdot p_3 & -5/2 \\ 2p_1 \cdot p_2 & -2m^2 & -2p_2 \cdot p_3 & -5/2 \\ -2p_1 \cdot p_3 & -2p_2 \cdot p_3 & -2m^2 & -5/2 \\ -5 & -5 & -5 & -5 \end{vmatrix} \cdot 2 \checkmark$$

$$= \frac{1}{8} \begin{vmatrix} 5/2 - 2m^2 & 5/2 - 2p_1 \cdot p_2 & 5/2 - 2p_1 \cdot p_3 & -5/2 \\ 5/2 - 2p_1 \cdot p_2 & 5/2 - 2m^2 & 5/2 - 2p_2 \cdot p_3 & -5/2 \\ 5/2 - 2p_1 \cdot p_3 & 5/2 - 2p_2 \cdot p_3 & 5/2 - 2m^2 & -5/2 \\ 0 & 0 & 0 & -5 \end{vmatrix}$$

Now recall that $p^2 = (3 - 4m^2)/4 = \frac{1}{2}(5/2 - 2m^2)$.

Also: $5/2 - 2p_1 \cdot p_2 = 5/2 + t - 2m^2 = t + 2p^2$,

$$\text{So: } = \frac{1}{8} (s) \begin{vmatrix} 2p^2 & t+2p^2 & t_1+2p^2 \\ t+2p^2 & 2p^2 & t_2+2p^2 \\ t_1+2p^2 & t_2+2p^2 & 2p^2 \end{vmatrix} = \frac{s}{8} \begin{vmatrix} 2p^2 & t & t_1 \\ t+2p^2 & 2p^2 & t_2+2p^2 \\ t_1+2p^2 & t_2+2p^2 & 2p^2 \end{vmatrix}$$

$$= \frac{s}{8} \left\{ (2p^2)^3 + 2(t+2p^2)(t_1+2p^2)(t_2+2p^2) - 2p^2[(t+2p^2)^2 + (t_1+2p^2)^2 + (t_2+2p^2)^2] \right\}$$

$$= \frac{(s/e)}{8} \left\{ 8p^6 + 2t t_1 t_2 + 2p^2(t t_1 + t t_2 + t_1 t_2) + 4p^4(t + t_1 + t_2) - D \right\}$$

where $D = +2p^2 \{ (t^2 + t_1^2 + t_2^2) + 4p^2(t + t_1 + t_2) + 12p^4 \}$

So:

$$\tilde{G} = \frac{s}{8} \left\{ -16p^6 + 2t t_1 t_2 - (t^2 + t_1^2 + t_2^2)[2p^2] + 2(t t_1 + t t_2 + t_1 t_2)[2p^2] + (t + t_1 + t_2)[4p^4] \right\} = \frac{s}{8} \left\{ -16p^6 + 2t t_1 t_2 - 2p^2 \Delta(t, t_1, t_2) \right\}$$

$$\tilde{G} = -\Delta$$

⑥ Result seems to be:

$$\tilde{G} = G(p_1, p_2, p_3, Q) = (-1)^{\frac{p^2}{4}} \left[-\Delta(t, t_1, t_2) + \frac{t_1 t_2}{p^2} \right]$$

Now, I have already used $E_3 = +\sqrt{s}/2$ etc, so energy theta's are not important. The mass shell's can be done at once. Result:

$$\begin{aligned} I &= \int d^4 p_3 \delta^+(p_3^2 - m^2) \delta^+(p_5^2 - m^2) \cdot f \\ &= \int dt, dt_2 du dv \cdot \frac{\Theta(-G)}{\sqrt{-G}} \cdot \delta^+ \cdot \delta^+ \cdot f \\ &= \int dt, dt_2 \frac{\Theta(-G)}{\sqrt{-G}} \cdot f. \end{aligned}$$

$$\text{Now: } \frac{1}{\sqrt{-G}} = \frac{1}{16\sqrt{-G}} = \frac{1}{16 \left(\frac{p\sqrt{s}}{2} \right) \sqrt{-\Delta + \frac{t_1 t_2}{p^2}}} = \frac{1}{8p\sqrt{s}} \cdot \frac{\Theta(-\Delta + \dots)}{\sqrt{-\Delta + \dots}}$$

$$\text{Hence: } I = \frac{1}{8p\sqrt{s}} \cdot \int \frac{dt, dt_2 \Theta(-G)}{\sqrt{-\Delta + \frac{t_1 t_2}{p^2}}} \cdot f.$$

$$\text{Now: } 8p\sqrt{s} = 2\sqrt{s}p \cdot 4 = 4 \cdot \Delta^{1/2}(s, m^2, m^2)$$

⑦ I expect this result to be off by $\times 2$ because there is a $dR \rightarrow dt dt_1$ overcount, I think. But it was a good exercise!

(a) we see how the physical region is directly related to a gram determinant, of order 4.

(b) we can compute the jacobian fast in this way, easier than all those integrals.

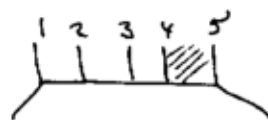
(c) we have tested the "special jacobian" method.

(d) typically, jacobian < 0 sets boundary.

$dQ_{2jel.u.}$

The cms nesting phase space formula.

1. Consider this situation:



$$dQ_5^{(1,2,3)} = C_3 \delta^4(\text{ext}) \cdot \prod_{i=1}^3 [\delta d^4 p_i \delta^4]$$

Now: $dQ_4^{(1,2,3,4)} = C_3 \delta^4(P_1 + P_2 + P_3 + P_4) \cdot \prod_{i=1}^4 [\delta d^4 p_i \delta^4]$

$$dQ_2^{(4,5)} = C_3 \delta^4(P_4 + P_5) \cdot \prod_{i=4}^5 [\delta d^4 p_i \delta^4]$$

Well, define (4,5) together as P_{45} . Then

$$dQ_4^{(1,2,3,4,5)} = C_3 \delta^4(P_1 + P_2 + P_3 + P_{45}) \cdot \prod_{i=1}^4 [\delta d^4 p_i \delta^4] \times [\delta d^4 P_{45} \delta^4(P_{45}^2 - s_{45})]$$

Claim:

$$\int dQ_5 = (C_3 \delta) \cdot \int dQ_4 \times$$

$$\int dQ_5 = (C_3 \delta) \int ds_{45} \cdot \int dQ_4 \int dQ_2$$

Again:

$$\int dQ_5(m_1^2, m_2^2, m_3^2, m_4^2, m_5^2) = \frac{1}{C_3 \delta} \cdot \int ds_{45} \cdot \int dQ_4(m_1^2, m_2^2, m_3^2, s_{45})$$

Again

$$\int dQ_5(s; m_1^2, m_2^2, m_3^2) = (C_3 \delta)^{-1} \int ds_{45} \int dQ_4(s; m_1^2, m_2^2, m_3^2, s_{45}) \int dQ_2(s_{45}; m_4^2, m_5^2)$$

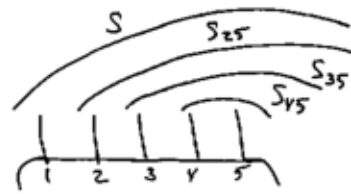
Integrate cms over the last two particles 4 and 5, then do a 4-body phase space, finally integrate over mass s_{45} . Here I am showing how you expose just the last cms phase space. Obviously I could continue this iteration some more:

$$\int dQ_4(s; m_1^2, m_2^2, m_3^2, s_{45}) = (C_3 \delta)^{-1} \int ds_{35} \int dQ_3(s; m_1^2, m_2^2, s_{35}) \int dQ_2(s_{35}; m_3^2, s_{45})$$

from

$$\int dQ_3(s; m_1^2, m_2^2, s_{35}) = (C_3 \delta)^{-1} \int ds_{25} \int dQ_2(s; m_1^2, s_{25}) \int dQ_2(s_{25}; m_2^2, s_{35})$$

So we find:



$$\int dQ_5 (s; m_1^2, m_2^2, m_3^2, m_4^2, m_5^2)$$

$$= \left[\frac{1}{8\pi^3} \right]^{5-2} \cdot \int ds_{45} ds_{35} ds_{25}$$

$$\times dQ_2 (s_{45}; m_4^2, m_5^2)$$

$$\times dQ_2 (s_{35}; m_3^2, s_{45})$$

$$\times dQ_2 (s_{25}; m_2^2, s_{35})$$

$$\times dQ_2 (s; m_1^2, s_{25})$$

the center-of-mass phase space

①  $dQ_2 = d^4 p_1 \delta^3(p_1^2 - m_1^2) d^4 p_2 \delta^3(p_2^2 - m_2^2) \delta^3(p_1 + p_2 - P)$
 $\delta^3(p^2 - m^2) = \Theta(p^0) \delta(p^2 - m^2)$
 $P = p_1 + p_2$

② Can show that in cms system [in any cms system].

$$dQ_2 = \frac{R}{4\sqrt{s}} d\Omega_1 \quad \text{where} \quad p_1 = \frac{\Delta^{1/2}(s, m_1^2, m_2^2)}{2\sqrt{s}}$$

$$d\Omega_1 = dz_1 d\phi_1$$

③ Also can show:

$$dQ_2 = \frac{1}{8\sqrt{s}} p_1 dt d\phi_1 \quad \text{where} \quad p_2 = \frac{\Delta^{1/2}(s, m_a^2, m_b^2)}{2\sqrt{s}}$$

Here, ϕ_1 can be azimuth of particle 1 in any cms frame.

④ Recall that $s = m_a^2 + m_b^2 + 2m_a m_b \cosh(Y) \quad Y = |y_a - y_b|$

Can show $\cosh Y = \frac{s - m_a^2 - m_b^2}{2m_a m_b} \Rightarrow \sinh Y = \frac{\Delta^{1/2}(s, m_a^2, m_b^2)}{2m_a m_b}$

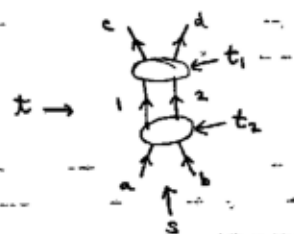
Thus: $dQ_2 = \frac{1}{4\Delta^{1/2}(s, m_a^2, m_b^2)} dt d\phi = \frac{dt d\phi}{8m_a m_b \sinh Y}$

(Y sometimes called η .)

⑤ Warnings: factors like $(2\pi)^3$, $(2\pi)^4$ etc. omitted from above definition of dQ_2 . Also, 1 and 2 are assumed non-identical, otherwise $\frac{1}{2}!$ phase space integral correction needed.

$$\Delta(a, b, c) = a^2 + b^2 + c^2 - 2ab - 2bc - 2ac$$

⑥ In elastic unitarity, this phase space appears in:



Notice: what was t is now t_2 .

Let: $z_1 = \frac{\hat{p}_1 \cdot \hat{p}_a}{\hat{p}_1 \cdot \hat{p}_c}$
 $z_2 = \frac{\hat{p}_1 \cdot \hat{p}_c}{\hat{p}_1 \cdot \hat{p}_b}$
 $z = \frac{\hat{p}_1 \cdot \hat{p}_c}{\hat{p}_1 \cdot \hat{p}_a}$

⑦ Then, can show that:

$$dQ_2 = dt_1 dt_2 \cdot \frac{1}{8\hat{p}_1 \hat{p}_2 \sqrt{s}} \cdot \frac{\Theta(-k(z, z_1, z_2))}{\sqrt{-k(z, z_1, z_2)}}$$

where $k(z, z_1, z_2) = z^2 + z_1^2 + z_2^2 - 2zz_1z_2 - 1$

and p_a, p_b, p_c are respective cms momenta, eg.

$$p_c = \Delta^{1/2}(s, m_c^2, m_d^2) / 2\sqrt{s} = \text{Lorentz Scalar.}$$

This form of dQ_2 is valid in all Lorentz frames, when z^i and p^i are written in terms of scalars.

Also: $dQ_2 = dz_1 dz_2 \cdot \frac{p_1}{2\sqrt{s}} \cdot \frac{\Theta(-k)}{\sqrt{-k}}$

⑧ If all masses are equal, can show:

$$\frac{1}{\sqrt{-k(z, z_1, z_2)}} = \frac{1}{4(s - 4m^2)} \frac{1}{\sqrt{-\Delta(t, t_1, t_2) + \frac{t_1 t_2}{(s/4 - m^2)}}}$$

⑨ Whether masses equal or not, if $s \gg$ all m_i^2 , then

$$k(z, z_1, z_2) \approx 4\bar{s}^2 \Delta(t, t_1, t_2) \quad \text{"large } s"$$

For large s , $dQ_2 = dt_1 dt_2 \cdot \frac{1}{2s} \cdot \frac{\Theta(-\Delta(t, t_1, t_2))}{\sqrt{-\Delta(t, t_1, t_2)}}$

How do you compute dQ_2 in a frame-independent manner?

① Use $dQ_2 = d^4 p_1 \delta^+(p_1^2 - s_1) \delta^+([P - p_1]^2 - s_2)$ $P^2 = s$. We shall not assume a sign for s . Our integration runs over real values of the p_i , so $p^2 \geq 0$ for sure, so can always say:

$$d^4 p_1 = \int_{-\infty}^{\infty} dE_1 \int_0^{\infty} p_1^2 dp_1 \int_{-2\pi}^{2\pi} dz_1 \int_0^{2\pi} d\phi_1 \quad p^2 \equiv p_1^2 + p_2^2 + p_3^2$$

② Now, once again do the p_1 integral to get:

$$d^4 p_1 \delta^+ \delta^+ = \int_{-\infty}^{\infty} dE_1 \int_{-2\pi}^{2\pi} dz_1 \int_0^{2\pi} d\phi_1 \frac{1}{2} \tilde{p}_1 \theta(\tilde{p}_1) \delta^+(s + s_1 - s_2 - 2P \cdot p_1) \{ \theta(E_1) \theta(P^0 - E_1) \}$$

For the time being, lets hold $\theta(E_1) \theta(P^0 - E_1)$ off on the RHS

③ Next, $P \cdot p_1 = P^0 E_1 - |P| |\tilde{p}_1| \tilde{z}_1$ so we have:

$$\tilde{p}_1 = +\sqrt{E_1^2 + s_1}$$

$$= \int_{-\infty}^{\infty} dz_1 \int_0^{2\pi} d\phi_1 \int_{-\infty}^{\infty} dE_1 \frac{1}{2} \tilde{p}_1 \theta(\tilde{p}_1) \theta(1 - |z_1|) \delta(s + s_1 - s_2 - 2P^0 E_1 + 2|P| |\tilde{p}_1| \tilde{z}_1) \theta(E_1) \theta(P^0 - E_1)$$

Do the $\int dE_1$ to pick up:

$$= \int_{-\infty}^{\infty} dz_1 \int_0^{2\pi} d\phi_1 \frac{(\frac{1}{2} \tilde{p}_1)^2}{|P^0 \tilde{p}_1 - \tilde{E}_1 |P| \tilde{z}_1|} \theta(\tilde{p}_1) \theta(1 - |z_1|) \theta(E_1) \theta(P^0 - E_1) \theta(\tilde{E}_1 \text{ real!})$$

and now $\tilde{E}_1 = f(p_1, s_1, s_2, P^0, |P|, \tilde{z}_1)$

④ Here, we could add $\int_{-\infty}^{\infty} dz_2 \delta(z_2 - \dots)$ + then do the $d\phi$ to get

$$= \int_{-\infty}^{\infty} dz_1 \int_{-\infty}^{\infty} dz_2 \frac{2\theta(-k)}{\sqrt{-k(z_1, z_2)}} \frac{(\tilde{p}_1)^2/4}{|P^0 \tilde{p}_1 - \tilde{E}_1 |P| \tilde{z}_1|} \theta(\tilde{p}_1) \theta(1 - |z_1|) \theta(E_1) \theta(P^0 - E_1) \}$$

Now if we were in a place where $s > 0$, then we could go to c.m.s. where $|P| = 0$ & $s = (s_1 + s_2)$. ③. The θ 's are O.K. If $|z_1| < 1$, fourth θ is O.K. Then get:

$$= \int_{-\infty}^{\infty} dz_1 \int_{-\infty}^{\infty} dz_2 \frac{2\theta(-k)}{\sqrt{-k}} \left(\frac{\tilde{p}_1^2}{4s} \right) \leftarrow \text{usual result.}$$

⑤ However, if $s < 0$, we could go to a frame where $P^0 = 0$, $|P| = \sqrt{-s}$. Then we have:

$$\int_{-\infty}^{\infty} dz_1 \int_{-\infty}^{\infty} dz_2 \frac{2\theta(-k)}{\sqrt{-k}} \frac{\tilde{p}_1^2 \theta(\tilde{p}_1)}{4\sqrt{-s} \tilde{z}_1 \tilde{E}_1} \theta(1 - |z_1|) \theta(\tilde{E}_1) \theta(-\tilde{E}_1) \}$$

$\leftarrow \text{you dead!}$

⑥ First thing to note: if you keep the $+$'s of $\vec{s}$, then you are stuck with $\Theta(E_1)\Theta(E_2)$. But in the Breit frame, $E_1 + E_2 = 0$, so can never both be plus. Thus:

$$\underline{dQ_2(s < 0) = 0}$$

⑦ Well, suppose you left off the $+$'s. Then your result is

$$d\tilde{Q}_2 = \int_{-1}^1 dz_1 \int_{-\infty}^{\infty} dz_2 \cdot \frac{2\Theta(-k)}{\delta\Theta - k} \cdot \frac{\tilde{p}_1^2 \Theta(\tilde{p}_1)}{4\sqrt{-s} \tilde{E}_1 |z_1|} \Theta(\tilde{E}_1 \text{ real})$$

where $\tilde{E}_1 =$ a mess. No, in Breit we can get $\tilde{p}_1$ and $\tilde{E}_1$:

$$(-s - s_1 + s_2) = 2\sqrt{-s} z_1 \tilde{p}_1 \quad \text{so} \quad \tilde{p}_1 = \frac{(-s - s_1 + s_2)}{2\sqrt{-s} z_1}$$

$$\text{And } \tilde{p}_1^2 + s_1 = \frac{(-s - s_1 + s_2)^2}{-4s z_1^2}$$

$$\Rightarrow \tilde{E}_1 = \frac{1}{2\sqrt{-s} z_1} \sqrt{s^2 + s_1^2 + s_2^2 - 2ss_2 - 2ss_1 + 2s_1s_2 - 4ss_1 z_1^2}$$

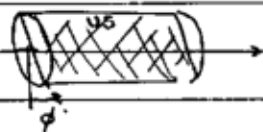
This can be complex!

⑧ Well, if $\Delta(s, s_1, s_2) > 0$, then $\tilde{E}_1$ real and $\tilde{p}_1^2 > 0$. So we learn $d\tilde{Q}_2 = 0$ if $\Delta < 0$. If $\Delta > 0$, then $\tilde{E}_1$ real $\forall z_1$. If $\Delta < 0$, then

⑨ What has been learned here?! If you try to evaluate dQ_2 in an arbitrary frame, it can be done. Even if $s \neq 0$, you can still do it if you drop the positive energy conditions. (Keeping these conditions forces $dQ_2 = 0$, that is for sure because you can go to a Breit frame and show it.) Most interesting is this: if $s \neq 0$, you can still define a z_1 as the physical angle of vector p_1 . However, it turns out that this variable is totally irrelevant. This is clearly indicated by the messy functional form of z_1 that appears in our $dz_1 dz_2$ integral in addition to the k -function. In fact, when you are doing Breit kinematics, it is the "angle" of the versor that is the significant variable, χ , not the angle of the vector. It is this versor angle which is related to a physical vector angle in the cms frame connected to this Breit frame by the operator o , which we recall was $(-i)$ times an imaginary z -boost.

The Planar Boost Theorem for Phase Space: dQ_2

- ① Consider the plane of paper to be at an arbitrary azimuth ϕ relative to a horizontal line on this paper.



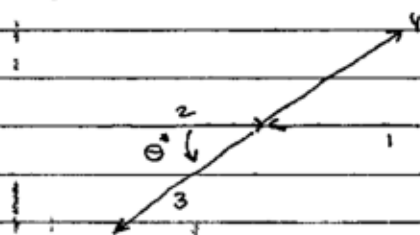
- ② Consider the reaction $1+2 \rightarrow 3+4$.

- ③ In the c.m.s. frame where p_1 and p_2 are along the horizontal [call this $\hat{z}$], we get:

Usually one writes:

$$dQ_2 = dp_3^+ dp_4^+ dp_3^- dp_4^- (\dots)$$

$$= \frac{p_3^+}{4\sqrt{s}} d(\cos\Theta^*) d\phi^*$$



This version of dQ_2 is special to the c.m.s. frame.

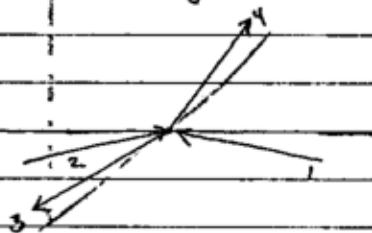
- ④ However, one can also write this as:

$$dQ_2 = [4\Delta'^2(s, m_1^2, m_2^2)]^{-1} dt d\phi^*$$

As Θ^* varies at fixed ϕ^* , t varies.

- ⑤ Now suppose we do an arbitrary boost in the $1+2 \rightarrow 3+4$ scattering plane [here, plane of this paper; $\phi = \phi^*$]. Then we might see:

Since all 4 particles are still in the plane $\phi = \phi^*$, ϕ^* is still the azimuth of $\vec{p}_3$.



- ⑥ Finally, it should make no difference where, in the scat plane, the $\hat{z}$ axis is chosen.

- ⑦ Theorem: In the reaction $1+2 \rightarrow 3+4$, the phase space of the final particles may be written:

$$dQ_{34} = \frac{dt d\phi}{4\Delta'^2(s, m_1^2, m_2^2)}$$

where ϕ is the azimuth of $\vec{p}_3$ (or $\vec{p}_4$), in any Lorentz frame with any $\hat{z}$ axis.

Cms two-body phase space

$$\textcircled{1} dQ_2 = d^4 p_1 \delta^4(p_1 - s_1) d^4 p_2 \delta^4(p_2 - s_2) \delta^4(p_1 + p_2 - P) \quad s_1 = m_1^2 \quad s_2 = m_2^2$$

$$= d^4 p_1 \delta^4(p_1 - s_1) \delta^4([P - p_1]^2 - s_2)$$

② If $s = P^2 > 0$, then a cms frame exists, and we can then go to that frame where $P = 0$. There,

$$dQ_2 = d^4 p_1 \delta^4(p_1 - s_1) \delta^4([P - p_1]^2 - s_2)$$

③ Since $P^2 = p_1^2 + p_2^2 + p_3^2 > 0$, we can always write $d^3 p_1 = dp_1^2 \pm |p_1| d\Omega_1$.
 I.e., we can do this variable change for any Euclidean vector. This is defined by a little jacobian: $p_i = |p| \sin \theta, \cos \phi$, etc, based on Euclidean geom.

④ So far we have:
$$dQ_2 = \int d^4 p_1 \int_0^\infty \frac{1}{2} |k| d|p_1|^2 \int d\Omega_1 \delta(p_1^2 - |p_1|^2 - s_1) \delta(\dots)$$

Do the $|p_1|^2$ integral into first δ . Then on: $|p_1|^2 = (p_1^0)^2 - s_1 = \tilde{|p_1|}^2 \quad \delta_0$

$$dQ_2 = \int dE_1 \cdot \frac{1}{2} \sqrt{|\tilde{p}_1|^2} \cdot \Theta(|\tilde{p}_1|) \cdot \int d\Omega \delta(s - 2\sqrt{s} E_1 + s_1 - s_2)$$

Do the E_1 :
$$dQ_2 = \frac{1}{2} |\tilde{p}_1| \Theta(|\tilde{p}_1|) \cdot \frac{1}{2\sqrt{s}} \cdot \Theta(\tilde{E}_1) d\Omega = \frac{p_1^*}{4\sqrt{s}} \Theta(\Delta) d\Omega$$

where now $|\tilde{p}_1| = \left[\frac{\Delta(s, s_1, s_2)}{2\sqrt{s}} \right] \quad \tilde{E}_1 = \frac{s + s_1 - s_2}{2\sqrt{s}}$ note
phases

⑤ Comments: There is no reason to believe any of this when $s < 0$, because then our hypothesis that $\exists$ a cms frame is wrong!
 In particular, our $\Theta(\tilde{E}_1)$ causes $dQ_2 = 0$ when $s < 0$, even when $\Delta > 0$, because $\sqrt{s} = \text{not real} \Rightarrow \tilde{E}_1 \text{ not real}$.

Two particle phase space, cms

$$\textcircled{1} dQ_2 = C_3 \chi_1 \chi_2 \cdot \cancel{\delta^4(\text{ext})} \cancel{d^4 p_1} \delta^+(p_1^2 - m_1^2) d^4 p_2 \delta^+(p_2^2 - m_2^2)$$

$$dQ_2 = C_3 \chi_1 \chi_2 \cdot \underbrace{d^4 p_2}_{\delta^+} \delta^+ \quad \text{where } \tilde{p}_1 = (P - p_2)^\mu$$

$$\begin{aligned} \textcircled{2} dQ_2 &= C_3 \chi_1 \chi_2 \cdot \int_{-\infty}^{\infty} dP_2^0 \int_0^{\infty} P_2^2 dp_2 \int d\Omega_2 \cdot d^4 p_2 \delta^+ \delta^+ \\ &= C_3 \chi_1 \chi_2 \cdot \int d\Omega_2 \cdot \underbrace{\left[\int dP_2^0 \int_0^{\infty} P_2^2 dp_2 \cdot \delta^+ \delta^+ \right]}_I \end{aligned}$$

③ Our problem is to compute I in the $\vec{p}_1 + \vec{p}_2 = 0$ cms frame.

$$\begin{aligned} I &= \int_{-\infty}^{\infty} dE_2 \int_0^{\infty} P_2^2 dp_2 \delta^+(E_1^2 - p_1^2 - m_1^2) \delta^+(E_2^2 - p_2^2 - m_2^2) \\ &= \int dE_2 \cdot \int_0^{\infty} \frac{1}{2} P_2 \cdot dP_2^2 \delta^+(E_2^2 - p_2^2 - m_2^2) \delta^+(E_1^2 - p_1^2 - m_1^2) \Theta(E_2) \\ &= \frac{1}{2} \int_0^{\infty} dE_2 \cdot \tilde{P}_2 \cdot \delta^+(\tilde{E}_1^2 - \tilde{P}_1^2 - m_1^2) \end{aligned}$$

So far, we have: a) $\tilde{E}_1 = E_1 - E_2$; $\tilde{p}_1 = p_2$

b) $\tilde{p}_2 = \sqrt{E_2^2 - m_2^2}$

So: $\Rightarrow \tilde{p}_1 = \sqrt{E_2^2 - m_2^2}$

$$\begin{aligned} I &= \frac{1}{2} \int_{E_2}^{\infty} dE_2 \cdot \tilde{P}_2 \delta^+((E_1 - E_2)^2 - (E_2^2 - m_2^2) - m_1^2) \Theta(E_1 - E_2) \\ &= \frac{1}{2} \int_0^{\infty} \tilde{P}_2 dE_2 \cdot \delta(E_1^2 - 2E_1 E_2 + m_2^2 - m_1^2) \\ &= \frac{1}{2} \cdot \tilde{P}_2 \cdot \frac{1}{|2E_1|} = \frac{P_2^*}{4\sqrt{s}} \end{aligned}$$

Conclusion:

$$dQ_2 = C_3 \chi_1 \chi_2 \cdot \frac{P_2^*}{4\sqrt{s}} \cdot d\Omega$$

④ Other forms:

$$dQ_2 = C_3 \chi_1 \chi_2 \cdot \left(\frac{1}{8\sqrt{s} p_a} \right) \cdot dt d\phi$$

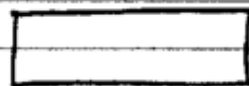
$$S = (E_a + E_b)^2$$

$$= C_3 \chi_1 \chi_2 \cdot \left(\frac{1}{4\Delta^{1/2}(s, m_a^2, m_b^2)} \right) \cdot dt d\phi$$

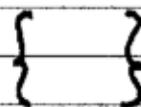
$$= C_3 \chi_1 \chi_2 \cdot \left(\frac{1}{8m_a m_b \sinh \eta} \right) \cdot dt d\phi$$

Legend Elastic Instability Forms.

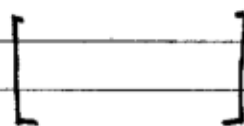
① All-norm, all-s results are boxed in red:



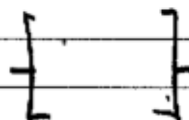
② Large s results are bracketed in pink:



③ Results in (Eden et al Ch 4 / Collins Squire / Horn Zach) norm ($\Gamma = +$) are bracketed in purple:



④ Red brackets are nothing:



⑤ A fiddle means "the delta form hit value:

$$\text{Eq, } \delta(x^2 - 4) = \frac{1}{2x} [\delta(x - \tilde{x}) + \delta(x + \tilde{x})] \quad \tilde{x} = 2.$$

⑥ Constants $C_1, C_3, \gamma_1, \gamma_2$ are defined in "dictionary"

⑦ In Collins + Squire, $D_S \equiv \frac{\text{disc } T}{2i}$ (= Im T if R.A.)

⑧ Always, nonidentical particles are assumed in the intermediate state. [except for $\pi^0\pi^0$]. If identicals there, add $\frac{1}{2}$ to RHS of elastic V. equations.

⑨ In delta integrations, I have left out the theta factors. Then I put them in at the end, eg, via $\Theta(-\Delta)$.

$$\text{Eq, } \int_{-1}^1 dz \delta(t-z) = 1, \text{ but really is } \overset{t \geq -1}{\Theta(t+1)} \overset{t \leq 1}{\Theta(1-t)}$$

Elastic Unitarity: conversion to angular

① In all norm, $dQ_2 = C_3 \delta^4(\text{ext}) \cdot \delta_1 d^4 p_1 \delta^+ \cdot \delta_2 d^4 p_2 \delta^+$
 $= (C_3 \delta_1 \delta_2) \cdot \delta^4(\text{ext}) d^4 p_1 \delta^+ d^4 p_2 \delta^+$

② Conversion to c.m.s. angle integration gives:

$$dQ_2^* = (C_3 \delta_1 \delta_2) \cdot \frac{P^*}{4\sqrt{s}} \cdot d\Omega^* = (\delta_1 \delta_2 C_3) \cdot \frac{\Delta^{1/2}(s, m_1^2, m_2^2)}{8s} \cdot d\Omega^*$$

③ Example: elastic unitarity with non identical m_1, m_2 intermediate says:

$$i[T_{fi}^* - T_{fi}] = C_1 \int dQ_2 T_{fz}^* T_{zi} \\ = (\delta_1 \delta_2 C_1 C_3) \cdot \frac{P^*}{4\sqrt{s}} \int d\Omega_1 T_{fz}^* T_{zi}$$

Note: for equal masses, $P^* = \Delta^{1/2}(s, m_1^2, m_1^2)/2\sqrt{s} = \frac{1}{2}\sqrt{s - 4m_1^2}$.
 For Eden, $C_1 = -1$ in ch. 4, $\delta_1 = \delta_2 = (2\pi)^{-3}$, $C_3 = (2\pi)^4$, so
 Eden would say:

$$i[T_{fi}^{(-)} - T_{fi}^{(+)}] = -(2\pi)^{-2} \cdot \frac{P^*}{4\sqrt{s}} \int d\Omega_1 T_{fz}^{(-)} T_{zi}^{(+)}$$

For equal masses, get

$$i[T_{fi}^{(-)} - T_{fi}^{(+)}] = -(2\pi)^{-2} \cdot \frac{1}{2} \cdot \sqrt{s - 4m_1^2} \cdot \frac{1}{4\sqrt{s}} \cdot \left\{\frac{1}{2}\right\} \int d\Omega_1 T_{fz}^{(-)} T_{zi}^{(+)}$$

This is Eden et al (4.7.2) I have added $\{1/2\}$ for identicals, since $m_1 = m_2$.

④ In general, assuming hermiticity, analyticity, we get

$$\left\{ \begin{aligned} -i \text{disc}(T_{fi}^{(+)}) &= (\delta_1 \delta_2 C_1 C_3) \cdot \frac{P^*}{4\sqrt{s}} \cdot \int d\Omega_1 T_{fz}^{(-)} T_{zi}^{(+)} \\ &= 2 \text{Im}[T_{fi}^{(+)}] \quad \text{if real analytic.} \end{aligned} \right.$$

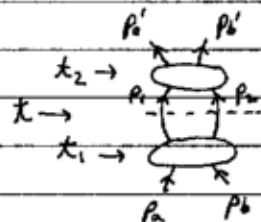
These are both for non-identical m_1, m_2 , + of course only one elastic channel, (m_1, m_2) . For large s , the HZ result should be

$$\left\{ \text{Im}[T_{fi}^{(+)}] = \frac{1}{64\pi^2} \int d\Omega T_{fz}^{(-)} T_{zi}^{(+)} \right\}$$

Conversion of dQ_2 to $dt_1 dt_2$ variables: reduction to $S(t_1, t_2, s)$

① Going one way, we can say:

$$\int dQ_2 \cdot f = (C_3 \chi_1 \chi_2) \cdot \int d^4 p_1 \delta^+ \delta_2^+ \cdot f$$



Now add:

$$1 = \int dt_1 dt_2 \delta(t_1 - [p_1 - p_2]^2) \delta(t_2 - [p_1' - p_2']^2)$$

Then:

$$\begin{aligned} \int dQ_2 \cdot f &= (C_3 \chi_1 \chi_2) \cdot \int dt_1 dt_2 f \cdot \int d^4 p_1 \delta^+ \delta_2^+ \delta(t_1 - \dots) \delta(t_2 - \dots) \\ &= (C_3 \chi_1 \chi_2) \int dt_1 dt_2 f \cdot Q(t_1, t_2, s) \end{aligned}$$

where $Q(t_1, t_2, s) = \int d^4 p_1 \delta^+(p_1^2 - m_1^2) \delta^+(p_2^2 - m_2^2) \delta[t_1 - (p_1 - p_2)^2] \delta[t_2 - (p_1' - p_2')^2]$

Thus: $\int dQ_2 = dt_1 dt_2 \cdot \chi_1 \chi_2 C_3 \cdot Q(t_1, t_2, s)$

② Alternatively, we can start with

$$\begin{aligned} \int dQ_2 \cdot f &= (\chi_1 \chi_2 C_3) \cdot \frac{P^*}{4\sqrt{s}} \cdot \int d\Omega \cdot f \\ &= \int dt_1 dt_2 \cdot \int d\Omega \cdot (\chi_1 \chi_2 C_3) \cdot \frac{P^*}{4\sqrt{s}} \cdot \delta[t_1 - \dots] \delta[t_2 - \dots] \cdot f \\ &= (\chi_1 \chi_2 C_3) \cdot \frac{P^*}{4\sqrt{s}} \cdot \int dt_1 dt_2 \cdot f \cdot S(t_1, t_2, s) \end{aligned}$$

where $S(t_1, t_2, s) = \int d\Omega \delta[t_1 - (p_1 - p_2)^2] \delta[t_2 - (p_1' - p_2')^2]$

Thus: $\int dQ_2 = dt_1 dt_2 \cdot \chi_1 \chi_2 C_3 \cdot \left(\frac{P^*}{4\sqrt{s}} \right) \cdot S(t_1, t_2, s)$

③ By comparison, it is clear that:

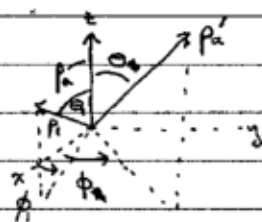
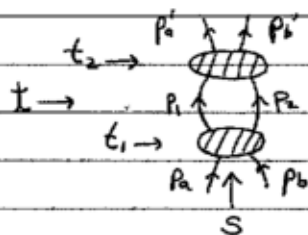
$$\int Q(t_1, t_2, s) = \frac{P^*}{4\sqrt{s}} \cdot S(t_1, t_2, s)$$

We have already reduced Q to S , so why do it again. Best to compute S at this point.

Compute $S(t, t_1, t_2, s)$

① $S(t, t_1, t_2, s) \equiv \int d\Omega_1 \delta[t_1 - (p_1 - p_a)^2] \delta[t_2 - (p_1 - p_a')^2]$

Picture is:



$$S = \int d\phi_1 \int dz_1 \delta[t_1 - m_1^2 - m_a^2 + 2E_1 E_a - 2p_1 p_a z_1] \delta[t_2 - (p_1 - p_a')^2]$$

All quantities are c.m.s. Do the z_1 integral:

$$S = \frac{1}{2p_1 p_a} \int d\phi_1 \delta[t_2 - m_1^2 - m_a^2 + 2E_1 E_a - 2p_1 p_a z_2]$$

Here, z_2 is $\cos\theta_2$, that messy angle between $\vec{p}_1$ and $\vec{p}_a'$.

$$z_2 = z \tilde{z}_1 + \sqrt{z^2 + 1} \sqrt{\tilde{z}_1^2 + 1} \cos(\phi_1 - \phi)$$

Thus:

$$S = \frac{1}{2p_1 p_a} \int d\phi_1 \delta[t_2 - m_1^2 - m_a^2 + 2E_1 E_a - 2p_1 p_a z \tilde{z}_1 - 2p_1 p_a \sqrt{z^2 + 1} \sqrt{\tilde{z}_1^2 + 1} \cos(\phi_1 - \phi)]$$

$$= \frac{1}{2p_1 p_a} \cdot \frac{1}{2p_1 p_a' \sqrt{z^2 + 1} \sqrt{\tilde{z}_1^2 + 1} \sin(\tilde{\phi}_1 - \phi)} \cdot 1 \times 2 \quad \text{since } \cos \text{ is double valued (see note)}$$

Here, $\tilde{z}_1$ and $\tilde{\phi}_1$ are the "hit values" of z_1 and ϕ_1 .

② Identity: $\sqrt{z^2 + 1} \sqrt{\tilde{z}_1^2 + 1} \sin(\tilde{\phi}_1 - \phi) = [(1 - z^2)(1 - \tilde{z}_1^2) - z^2 \tilde{z}_1^2 + 2z \tilde{z}_1 z_2 - z_2^2]^{1/2}$

$$= [1 - z^2 - \tilde{z}_1^2 - z_2^2 + 2z \tilde{z}_1 z_2]^{1/2} = [- (z^2 + \tilde{z}_1^2 + z_2^2 - 2z \tilde{z}_1 z_2)]^{1/2}$$

$$= \sqrt{-k(z, \tilde{z}_1, z_2)} \quad \text{defines } k. \quad \left[k(z, \tilde{z}_1, z_2) \equiv z^2 + \tilde{z}_1^2 + z_2^2 - 2z \tilde{z}_1 z_2 \right]$$

Thus: $S(t, t_1, t_2, s) = \frac{2 \cdot 1}{4p_1^2} \cdot \frac{1}{p_a' p_a} \cdot \frac{1}{\sqrt{-k(z, \tilde{z}_1, z_2)}} \cdot \Theta(-k)$

③ At large s , all three momenta become equal + we have:

$$\{k(z, \tilde{z}_1, z_2) \equiv 4\tilde{s}^2 \Delta(t, t_1, t_2)\} \Rightarrow \sqrt{-k} = 2\tilde{s} \sqrt{-\Delta}$$

Then: $S(t, t_1, t_2, s) = \frac{1}{4p^4} \cdot \frac{1}{2\tilde{s}^4} \cdot \frac{\Theta(-\Delta)}{\sqrt{-\Delta}} \quad p = \frac{1}{2}\sqrt{s}; p^4 = \frac{s}{16}$

$$= \frac{2 \Theta(-\Delta)}{s \sqrt{-\Delta(t, t_1, t_2)}}$$

H.E. Elastic Unitarity in $dt_1 dt_2$

① We know that $dQ_2 = dt_1 dt_2 \cdot \gamma_1 \gamma_2 C_3 \cdot \frac{p_1^*}{4\sqrt{s}} \cdot S(t, t_1, t_2, s)$

At any energy this becomes:

$$dQ_2 = dt_1 dt_2 \cdot \gamma_1 \gamma_2 C_3 \cdot \frac{1 \cdot 2}{16 p_1 p_2 \sqrt{s}} \cdot \frac{\Theta(-k)}{\sqrt{-k(z, z_1, z_2)}}$$

As $s \rightarrow \infty$, $p_i \rightarrow \sqrt{s}/2$ and $\sqrt{-k} \rightarrow 2\sqrt{-\Delta}/s$, so

$$\left\{ dQ_2 \rightarrow dt_1 dt_2 \cdot \gamma_1 \gamma_2 C_3 \cdot \frac{1 \cdot 2}{4s} \cdot \frac{\Theta(-\Delta)}{\sqrt{-\Delta(t, t_1, t_2)}} \right\}$$

② Thus, we expect

$$T_{fi}^{(+)} - T_{fi}^{(-)} = \text{disc } T_{fi}^{(+)} = i C_1 \int dQ_2 T_{f2}^{(-)} T_{2i}^{(+)}$$

$$\left\{ \text{disc } T_{fi}^{(+)}(s, t) = i \cdot C_1 \gamma_1 \gamma_2 C_3 \cdot \frac{2 \cdot 1}{4s} \cdot \int \frac{dt_1 dt_2}{\sqrt{-\Delta}} T_{f2}^{(-)}(s, t_2) T_{2i}^{(+)}(s, t_1) \right\}$$

This is for non-identical m_1, m_2 . For identical, add $\frac{1}{2}$ on RHS.

③ Example: in Eden/HZ/Jacks/CS, $\gamma_1 \gamma_2 C_3 = (2\pi)^{-2} = \frac{1}{4\pi^2}$

Also, $\text{disc } T_{fi}^{(+)} = 2i \text{Im } T_{fi}^{(+)}$ if real analytic.

$$\text{Then we get: } \left\{ \left[\text{Im } T_{fi}^{(+)} = \frac{1 \cdot 2}{32\pi^2 s} \int \frac{dt_1 dt_2}{\sqrt{-\Delta}} T_{f2}^{(-)}(s, t_2) T_{2i}^{(+)}(s, t_1) \cdot C_1 \right] \right\}$$

Identical particles would change this 32 to a 64.

General Elastic Unitarity

① We know that:

$$dQ_2 = dt_1 dt_2 \cdot \gamma_1 \gamma_2 C_3 \cdot \frac{1 \cdot 2}{16 p_1 p_2 p_3^2 \sqrt{s}} \cdot \frac{\Theta(-k)}{\sqrt{-k}}$$

Therefore:

$$\begin{aligned} T_{fi}^{(+)} - T_{fi}^{(-)} &= \text{disc } T_{fi}^{(+)} = i C_1 \int dQ_2 T_{fi}^{(+)} T_{fi}^{(+)} = 2i \text{Im } T_{fi}^{(+)} \quad \text{if P.A.} \\ &= i \cdot C_1 \gamma_1 \gamma_2 C_3 \cdot \frac{1 \cdot 2}{16 \sqrt{s} p_1 p_2 p_3^2} \cdot \int \frac{dt_1 dt_2}{\sqrt{-k}} T_{fi}^{(+)}(s, t_2) T_{fi}^{(+)}(s, t_1) \end{aligned}$$

② We might use z_1, z_2 instead of t_1, t_2 :

$$z = \frac{s^2 + s(2t - z) + \Delta^2}{4s q \cdot q} \Rightarrow dz = \frac{dt}{2qq}$$

$$\Rightarrow dz_1 dz_2 = dt_1 dt_2 \cdot \frac{1}{2 p_1 p_2} \cdot \frac{1}{2 p_1 p_2}$$

$$\Rightarrow dt_1 dt_2 = 4 p_1^2 p_2 p_2^2 \cdot dz_1 dz_2$$

Then:

$$dQ_2 = dz_1 dz_2 \cdot \gamma_1 \gamma_2 C_3 \cdot \frac{2 \cdot p_1}{4 \sqrt{s}} \cdot \frac{\Theta(-k)}{\sqrt{-k}}$$

And:

$$\text{disc } T_{fi}^{(+)}(s, z) = i \cdot C_1 \gamma_1 \gamma_2 C_3 \cdot \frac{2 \cdot p_1}{4 \sqrt{s}} \cdot \int \frac{dz_1 dz_2}{\sqrt{-k(z_1, z_2, z)}} \cdot T_{fi}^{(+)}(s, z_2) T_{fi}^{(+)}(s, z_1)$$

③ Example: in HD, $C_1 \gamma_1 \gamma_2 C_3 = +i(2\pi)^2$ and $p_1 = \sqrt{s}/2$, so

$$\left\{ \left[\text{Im } T_{fi}^{(+)}(s, z) = \frac{1 \cdot 2}{64 \pi^2} \int \frac{dz_1 dz_2}{\sqrt{-k}} T_{fi}^{(+)} T_{fi}^{(+)} \right] \right\}$$

④ At H.E., 1) $\{dz_1 dz_2 = dt_1 dt_2 \cdot \frac{1}{4 \cdot s^2 \cdot 2} = 4 \bar{s}^2 dt_1 dt_2\}$

$$2) \left\{ \frac{1}{\sqrt{-k}} = \frac{1}{2 \bar{s}} \frac{1}{\sqrt{-\Delta}} \right\}$$

Thus:

$$\left\{ \frac{dz_1 dz_2}{\sqrt{-k}} \rightarrow \frac{2}{s} \frac{dt_1 dt_2}{\sqrt{-\Delta}} \right\}$$

The double valuedness "subtlety"

- ① In computing $S(t, t_1, t_2, s)$, we encounter something like:

$$I = \int_0^{2\pi} d\phi \delta(\cos\phi - \frac{1}{\sqrt{2}});$$

But there are two hits, not one, in this integral:

$$\begin{aligned} \delta(\cos\phi - \frac{1}{\sqrt{2}}) &= \frac{1}{|\sin\phi|} \delta(\phi - 45^\circ) + \frac{1}{|\sin\phi|} \delta(\phi - 135^\circ) \\ &= \sqrt{2} [\delta(\phi - 45^\circ) + \delta(\phi - 135^\circ)]. \end{aligned}$$

Thus, $I = 2\sqrt{2}$, not $\sqrt{2}$.

Note that $|\sin\phi|$ is the same for both hits, because in both hits, $\cos\phi = 1/\sqrt{2}$. So in general, just add a $\times 2$.

Delta Practice:

Compute: $I \equiv \int d^2q \delta(x - [p+q]^2) \delta(y - [p-q]^2)$

This is all in two spatial dimensions.

① Brute force method:

$$I = \int_0^\infty q dq \int_0^{2\pi} d\theta \delta(x - p^2 - q^2 - 2pq \cos\theta) \delta(y - p^2 - q^2 + 2pq \cos\theta) \quad \checkmark$$

Write:

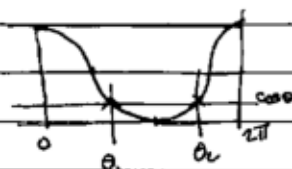
$$\delta(1^{\text{st}}) = \frac{1}{|2pq \sin\theta|} [\delta(\theta - \theta_1) + \delta(\theta - \theta_2)] \quad \checkmark$$

Since for any value of $\cos\theta$, $\exists$ two θ 's that give it.

Now:

$$\cos\theta = \frac{x - p^2 - q^2}{2pq} \quad \checkmark$$

$$\sin\theta = \frac{\sqrt{-\Delta(x, p^2, q^2)}}{2pq} \quad \checkmark$$



Thus, second δ is:

$$\delta(2^{\text{nd}}) = \delta(y - p^2 - q^2 + [x - p^2 - q^2]) = \delta(y + x - 2p^2 - 2q^2) \quad \checkmark$$

And now:

$$I = \int_0^\infty q dq \cdot \frac{2}{\sqrt{-\Delta}} \cdot \delta(2q^2 + 2p^2 - x - y) \quad \checkmark$$

since two hits in θ .

Here:

$$2q^2 = x + y - 2p^2 \quad \checkmark \Rightarrow \quad \tilde{q} = \sqrt{\frac{x+y}{2} - p^2} \quad \checkmark$$

$$\delta(2q^2 + 2p^2 - x - y) = \frac{1}{|4q|} [\delta(q + \tilde{q}) + \delta(q - \tilde{q})] \quad \checkmark$$

only this hits

So,

$$\frac{1}{2\sqrt{-\Delta}} = \frac{1}{2\sqrt{-\Delta/4}}$$

$$= \frac{1}{\sqrt{-\Delta}}$$

$$I = \frac{0(-\Delta)}{\sqrt{\Delta(x, y, 4p^2)}}$$

So:

$$I = \int_0^\infty q dq \cdot \frac{2}{\sqrt{-\Delta}} \cdot \frac{1}{|4q|} \cdot \delta(q - \tilde{q}) = \frac{1}{2\sqrt{-\Delta(x, p^2, q^2)}} \quad \checkmark$$

Write out this last result:

$$\Delta = x^2 + p^4 + \left(\frac{x+y}{2} - p^2\right)^2 - 2xp^2 - 2x\left(\frac{x+y}{2} - p^2\right) - 2p^2\left(\frac{x+y}{2} - p^2\right)$$

$$= x^2 + p^4 + \frac{1}{4}(x+y)^2 + p^4 - p^2(x+y) - 2xp^2 - 2x^2 - xy + 2xp^2 - xp^2 - yp^2 + 2p^4$$

$$= \left(\frac{1}{4}\right)x^2 + \frac{1}{4}y^2 + \frac{1}{4} \cdot 2xy + 4p^4 - 8xp^2 - 2yp^2$$

$$2 = \frac{8}{4} = \frac{1}{4} \cdot 2 \cdot 4$$

$$= \frac{1}{4} (x^2 + y^2 + (4p^2)^2 - 2xy - 2x(4p^2) + 2y(4p^2)) = \frac{1}{4} \Delta(x, y, 4p^2) \quad \checkmark$$

② Jacobian method:

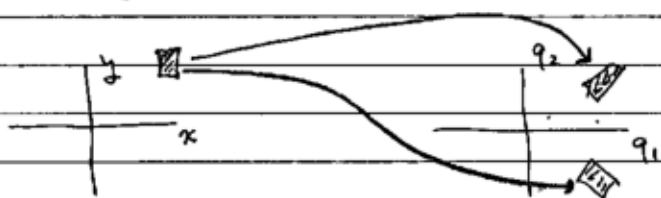
Consider: $\int dx dy f(x,y) \underbrace{\left[\int d^2 q \delta(x-\dots) \delta(y-\dots) \right]}_I = \int d^2 q \tilde{f}(q^2)$

It appears that:

$$I = 2 \cdot \frac{\partial(q_1, q_2)}{\partial(x, y)} \quad \frac{\partial(q_1, q_2)}{\partial(x, y)} dx dy = dq_1 dq_2$$

Locally, the jacobian relates volumes. But the mapping

from
(x,y) to
(q₁, q₂) is
1:2.



So, $\int dx dy f(x,y) \cdot \frac{\partial(q_1, q_2)}{\partial(x, y)} = \int dq_1 dq_2 f$

is wrong. The RHS covers q^2 plane once, but

$$\text{LHS} = \text{all } xy = \text{UHP}(q^2) = \text{LHP}(q^2) = \frac{1}{2}(U+L) = \frac{1}{2} \text{RHS}.$$

$$\Rightarrow 2 \cdot \frac{\partial(q_1, q_2)}{\partial(x, y)} = I$$

Why?

$$\begin{aligned} x &= (p+q)^2 = p^2 + q^2 + 2pq \cos \theta \\ y &= (p-q)^2 = p^2 + q^2 - 2pq \cos \theta \end{aligned}$$

$$\begin{aligned} x+y &= 2(p^2+q^2) \\ x-y &= 4pq \cos \theta \end{aligned}$$

← This sets q , then

← This gives 2 values of θ .

Now, $\frac{\partial(q_1, q_2)}{\partial(x, y)} = \text{volume relation} = \left[\frac{\partial(x, y)}{\partial(q_1, q_2)} \right]^{-1}$

But: $\frac{\partial(x, y)}{\partial(q_1, q_2)} = \begin{vmatrix} \frac{\partial x}{\partial q_1} & \frac{\partial x}{\partial q_2} \\ \frac{\partial y}{\partial q_1} & \frac{\partial y}{\partial q_2} \end{vmatrix} = \begin{vmatrix} 2(p_1+q_1) & 2(p_2+q_2) \\ -2(p_1-q_1) & -2(p_2-q_2) \end{vmatrix} = 4 \begin{vmatrix} q_1+p_1 & q_2+p_2 \\ q_1-p_1 & q_2-p_2 \end{vmatrix}$

since $\begin{aligned} x &= (p_1+q_1)^2 + (p_2+q_2)^2 \\ y &= (p_1-q_1)^2 + (p_2-q_2)^2 \end{aligned}$

Remark: $\det A = \det \tilde{A} = \sqrt{\det A \cdot \det \tilde{A}} = \sqrt{\det(A\tilde{A})}$

So: $\begin{vmatrix} q_1+p_1 & q_2+p_2 \\ q_1-p_1 & q_2-p_2 \end{vmatrix} = \sqrt{\begin{vmatrix} (q_1+p_1) & (q_2+p_2) & (q_1+p_1) & (q_1-p_1) \\ (q_1-p_1) & (q_2-p_2) & (q_2+p_2) & (q_2-p_2) \end{vmatrix}}$

$$= \sqrt{\begin{vmatrix} (q+p)^2 & (q^2-p^2) \\ (q-p)^2 & (q-p)^2 \end{vmatrix}}$$

$$q_1^2 - p_1^2 + q_2^2 - p_2^2 = q^2 - p^2$$

$$q^2 - p^2 = \left(\frac{x+y}{2} - p^2 \right) - p^2 = \frac{1}{2}(x+y-4p^2) = \frac{1}{2}(x+y-z)$$

$= \sqrt{\begin{vmatrix} x & \frac{1}{2}(x+y-z) \\ \frac{1}{2}(x+y-z) & y \end{vmatrix}} \cdot \text{Hence: } I = 2 \cdot \left\{ 4 \sqrt{\frac{1}{4}} \right\}^{-1} = \left\{ \sqrt{4} \right\}^{-1}$

$$I = \left\{ \sqrt{\begin{vmatrix} 2x & x+y-z \\ x+y-z & 2y \end{vmatrix}} \right\}^{-1} = \left\{ \sqrt{-\Delta(x, y, 4p^2)} \right\}^{-1}$$

$\Rightarrow$

$$I = \frac{\Theta(-\Delta)}{\sqrt{-\Delta(x, y, 4p^2)}}$$

so that

$$(2.13) \quad H(\xi, \xi_1, \xi_2) = \frac{1}{4}(-\xi^2 - \xi_1^2 - \xi_2^2 + 2\xi\xi_1 + 2\xi\xi_2 + 2\xi_1\xi_2),$$

$\frac{1}{2}\sqrt{H}$ represents the area of the plane triangle, whose sides are $\sqrt{\xi}$, $\sqrt{\xi_1}$, $\sqrt{\xi_2}$.

In the following we shall use the notation

$$(2.14) \quad \frac{\theta(H(\xi, \xi_1, \xi_2))}{\sqrt{H(\xi, \xi_1, \xi_2)}} = 2T(\xi, \xi_1, \xi_2).$$

In the variables ξ_1, ξ_2 eq. (2.11) becomes

$$(2.15) \quad (u_1 + \mu^2)(u_2 + \mu^2) B(s, t, u_1, u_2) = \frac{1}{4(2\pi)^4} \cdot$$

$$\int ds_0 A_0(s_0) \int_0^1 dx \int_0^\infty d\xi_1 d\xi_2 \cdot 2T(\xi, \xi_1, \xi_2) B\left(sx, t, \xi_1 + u_1x + \frac{s_0x}{1-x}, \xi_2 + u_2x + \frac{s_0x}{1-x}\right).$$

One can easily verify the following identity, which will often be useful

$$(2.16) \quad T(\xi, \xi_1, \xi_2) = 2 \int d^2q \delta((p+q)^2 - \xi_1) \delta((p-q)^2 - \xi_2),$$

where p is a two-dimensional vector such that $4p^2 = \xi = -t(1-x)$.

Using (2.16), eq. (2.15) can be expressed in the useful form

$$(2.17) \quad (u_1 + \mu^2)(u_2 + \mu^2) B(s, t, u_1, u_2) = \frac{1}{(2\pi)^4} \cdot$$

$$\int ds_0 A_0(s_0) \int_0^1 dx \int d^2q B\left(sx, t, (p+q)^2 + u_1x + \frac{s_0x}{1-x}, (p-q)^2 + u_2x + \frac{s_0x}{1-x}\right).$$

An interesting point concerning the asymptotic limit is to consider what happens in the forward direction where we know that the kinematical conditions impose that $V = u_1 - u_2 = 0$. If we consider the kinematics of the problem more closely, we see that $|V| < \sqrt{st}$. This means that if we set $s \rightarrow \infty$ before $t \rightarrow 0$ the integral equation we are considering no longer contains this limitation.

In fact if we take the limit for $t \rightarrow 0$ of our asymptotic equation (2.17)

$$I \equiv \int d^2q \delta(\xi_1 - \dots) \delta(\xi_2 - \dots) = \frac{\pi}{2} = \frac{\theta(H)}{4\sqrt{H}}$$

$$= \frac{\theta(H)}{2\sqrt{4H}} = \frac{\theta(-\Delta)}{2\sqrt{-\Delta}}$$

This paper claims: *of which this is*

H.E. elastic unitarity when $t=0$

① We found that:

$$A_{si}^{(+)} = \frac{\text{disc } T_{fi}^{(+)}}{2i} = \frac{C_1 \gamma_1 \gamma_2 C_3}{4s} \int \frac{dt_1 dt_2}{\sqrt{-\Delta(t, t_1, t_2)}} T_{f_2}^{(-)}(s, t_2) T_{i_1}^{(+)}(s, t_1)$$

② It is helpful to convert to

$$\begin{aligned} x &= t_1 + t_2 \\ y &= t_1 - t_2 \end{aligned} \quad dx dy = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} dt_1 dt_2 = 2 dt_1 dt_2$$

$$\text{Also: } \Delta(t, t_1, t_2) = (t_1 - t_2)^2 + t^2 - 2t(t_1 + t_2) \approx y^2 - 2tx \quad \text{for small } t \rightarrow 0.$$

So far then:

$$A^{(+)} = \frac{C_1 \gamma_1 \gamma_2 C_3}{4s} \cdot \frac{1}{2} \int_{-\infty}^0 dx \int_{-\infty}^{\infty} dy \cdot \frac{\Theta[-y^2 + 2tx]}{\sqrt{-y^2 + 2tx}} T^{(-)}(s, x, y) T^{(+)}(s, x, y)$$

$$\text{③ Consider: } \int_{-\infty}^{\infty} dy \frac{\Theta[-y^2 + \epsilon^2]}{\sqrt{-y^2 + \epsilon^2}} f(y) \cong \int_{-\epsilon}^{\epsilon} dy \frac{f(0)}{\sqrt{\epsilon^2 - y^2}} = \frac{\pi}{\epsilon} f(0) = \frac{\pi}{\lambda} f(0).$$

Thus:

$$A^{(+)} = \frac{\pi}{2} \frac{C_1 \gamma_1 \gamma_2 C_3}{4s} \int_{-\infty}^0 dx T^{(-)}(s, x, 0) T^{(+)}(s, x, 0)$$

But $y=0 \Rightarrow t_1 = t_2 \equiv t'$, so $x = 2t'$, and $dx = 2dt'$

$$\text{so: } \left\{ A^{(+)}(s, 0) = \frac{C_1 \gamma_1 \gamma_2 C_3}{4s} \pi \int_{-\infty}^0 dt' T^{(-)}(s, t') T^{(+)}(s, t') \right\}$$

④ Picture:

$$t=0 \quad \text{---} \bigcirc \text{---} \quad = \quad t' \rightarrow \begin{array}{c} \bigcirc \\ \bigcirc \end{array} \quad \left(= \quad \text{regge form} \quad \right)$$

Reduction of k for π -N + equal masses.

① We know that: $k = z^2 + z_1^2 + z_2^2 - 1 - 2z z_1 z_2$

In general,
$$z = \frac{s^2 + 2st - z_1^2 + B}{\Delta^{1/2} \dots \Delta^{1/2}}$$

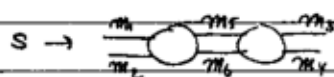
Or:

$$z = \frac{s + 2t - z_1 + B/s}{4q \cdot q'}$$

② In π -N mass situation, this simplifies to:

$$z = 1 + \frac{2st}{\Delta} \quad \text{where } \Delta = \Delta(s, m_\pi^2, m_N^2)$$

③ Let's take a slightly more general case?



Say:
$$\begin{cases} m_1 = m_3 \\ m_2 = m_4 \end{cases} = m$$

OK, π -N is most general "simple" case.

④ All Δ^k are the same, so: let: $2s/\Delta \equiv a$.

$$k = (1 + 2s \dots$$

$$\begin{aligned} k &= (1 + at)^2 + (1 + at_1)^2 + (1 + at_2)^2 - 1 - 2(1 + at)(1 + at_1)(1 + at_2) \\ &= \cancel{3} + 2a(t + t_1 + t_2) + a^2(t^2 + t_1^2 + t_2^2) - \cancel{1} - \cancel{2} - 2a(t + t_1 + t_2) \\ &\quad - 2a^2(tt_1 + tt_2 + t_1t_2) - 2a^3tt_1t_2 \end{aligned}$$

$$\begin{aligned} &= a^2 \Delta(t, t_1, t_2) - 2a^3 tt_1t_2 \\ &= a^2 [\Delta(t, t_1, t_2) - 2a tt_1t_2] \end{aligned}$$

Therefore:
$$k(z, z_1, z_2) = \left(\frac{2s}{\Delta}\right)^2 \left[\Delta(t, t_1, t_2) - \frac{4s}{\Delta} tt_1t_2 \right]$$

⑤ Now go to full equal mass by $\Delta \rightarrow \Delta(s, m^2, m^2) = s(s - 4m^2) = 4sp^2$

⑥ Note:

$$\boxed{\frac{1}{\Delta - k} = \frac{\Delta}{2s} \frac{1}{\sqrt{-\Delta + (4s/\Delta)tt_1t_2}}}$$

In equal mass,
$$\frac{4s}{\Delta} = \frac{1}{(s/4) - m^2} = \frac{1}{p^2}; \quad \frac{\Delta}{2s} = \frac{1}{2}(s - 4m^2) = 2p^2$$

and π -N case.
Equal mass, all-s elastic V.

① The angular stays the same, with $p^* = \frac{1}{2\sqrt{s}} [s(s-4m^2)]^{1/2} = \frac{1}{2}\sqrt{s-4m^2}$

② In " π -N threshold" mass case, we have:

$$dQ_2 = dt_1 dt_2 \cdot \chi_1 \chi_2 C_3 \cdot \frac{1}{8 p^* \sqrt{s}} \cdot \frac{\Theta(-k) \cdot (\Delta/2s)}{\sqrt{-\Delta(t_1, t_2) + \frac{(4s/\Delta) t_1 t_2}{1/p^2}}}$$

Rewrite:

$$dQ_2 = dt_1 dt_2 \cdot \chi_1 \chi_2 C_3 \cdot \frac{1}{4 p \sqrt{s}} \cdot \frac{\Theta(-k)}{\sqrt{-\Delta(t_1, t_2) + t_1 t_2 / p^2}}$$

where $p = \Delta^{1/2}(s, m_\pi^2, m_N^2) / 2\sqrt{s}$

So, once more: , use $2\sqrt{s} p = \Delta^{1/2} s_N$:

$$dQ_2 = dt_1 dt_2 \cdot \chi_1 \chi_2 C_3 \cdot \frac{1}{2 \Delta^{1/2}} \cdot \frac{\Theta}{\sqrt{-\Delta(t_1, t_2) + \frac{(4s)}{\Delta} t_1 t_2}}$$

In Eden et al, $\chi_1 \chi_2 C_3 = (2\pi)^{-2}$, so $\frac{1}{8\pi^2}$

③ In particular:

$$\begin{aligned} 2 \cdot \text{Im } T_{fi} &= (-i \text{disc } T_{fi}) = C_1 \int dQ_2 T^* T \\ &= (C_1 \chi_1 \chi_2 C_3) \cdot \frac{1}{2 \Delta^{1/2}(s; m_\pi^2, m_N^2)} \cdot \int \frac{dt_1 dt_2 T^{(*)} T^{(C)}}{\sqrt{-\Delta(t_1, t_2) + \frac{4s}{\Delta} t_1 t_2}} \end{aligned}$$

This gives $\frac{1}{16\pi^2 \Delta^{1/2}}$ for A.

See p. 160 of Abner et al.

Compute $k(z_1, z_2, z)$ to order $\dot{s}^2$

① $k = -1 + z_1^2 + z_2^2 + z^2 - 2z_1z_2z$

/we know.

② $z = 1 + (2t) \dot{s}^1 + (2t\varepsilon + 2G) \dot{s}^2 + \dots$

/see kinematics

$$z^2 = [1 + a\dot{x}^1 + b\dot{x}^2]^2 = 1 + a^2\dot{x}^2 + b^2\dot{x}^4 + 2a\dot{x}^1 + 2b\dot{x}^2 + 2ab\dot{x}^3$$

$$\approx 1 + (2a)\dot{x}^1 + (a^2 + 2b)\dot{x}^2$$

Thus, $z^2 = 1 + (4t) \dot{s}^1 + (4t^2 + 4t\varepsilon + 4G) \dot{s}^2$

$$z^2 = 1 + 4(t) \dot{s}^1 + 4(t^2 + t\varepsilon + G) \dot{s}^2 + \dots$$

③ Next, we need $z_1 z_2 z$

$$= (1 + a_1\dot{x}^1 + b_1\dot{x}^2)(1 + a_2\dot{x}^1 + b_2\dot{x}^2)(1 + a\dot{x}^1 + b\dot{x}^2)$$

$$= 1 + (a_1 + a_2 + a)\dot{x}^1 + (b_1 + b_2 + b + a_1a_2 + a_1a + a_2a)\dot{x}^2 + \dots$$

So, $z_1 z_2 z = 1 + (2t + 2t_1 + 2t_2) \dot{s}^1$

$$+ (2t\varepsilon + 2t_1\varepsilon_1 + 2t_2\varepsilon_2 + 2G_1 + 2G_2 + 2G + 4t_1t_2 + 4t_1t + 4tt_2) \dot{s}^2$$

$$= 1 + 2(t + t_1 + t_2) \dot{s}^1 + 2(t\varepsilon + t_1\varepsilon_1 + t_2\varepsilon_2 + G_1 + G_2 + G + 2t_1t + 2t_1t_2 + 2tt_2) \dot{s}^2$$

Thus, ~~$k = -1 + 2z_1z_2z$~~

④ Now add all to get k :

$$k = -1 + (1 + 4t\dot{s}^1 + (4t^2 + t\varepsilon + G)\dot{s}^2) + (1 + 4t_1\dot{s}^1 + 4(t_1^2 + t_1\varepsilon_1 + G_1)\dot{s}^2) + (1 + 4t_2\dot{s}^1 + 4(t_2^2 + t_2\varepsilon_2 + G_2)\dot{s}^2)$$

$$- 2 - 4(t + t_1 + t_2) \dot{s}^1 - 4(t\varepsilon + t_1\varepsilon_1 + t_2\varepsilon_2 + G_1 + G_2 + G + 2t_1t + 2t_1t_2 + 2tt_2) \dot{s}^2$$

$$\Rightarrow k = 0 \cdot \dot{s}^0 + 0 \cdot \dot{s}^1 + \boxed{4\dot{s}^2 \Delta(t, t_1, t_2)}$$

$$4\dot{s}^2 \times \left[\cancel{(t^2 + t\varepsilon + G)} + \cancel{(t_1^2 + t_1\varepsilon_1 + G_1)} + \cancel{(t_2^2 + t_2\varepsilon_2 + G_2)} - \cancel{t\varepsilon} - \cancel{t_1\varepsilon_1} - \cancel{t_2\varepsilon_2} - \cancel{G_1} - \cancel{G_2} - \cancel{G} - 2t_1t - 2t_1t_2 - 2tt_2 \right]$$

⑤ Discuss this result:

$$k \approx 4\bar{s}^2 \Delta(t, t_1, t_2) + O(\bar{s}^3)$$

This result is true for all mass configurations m_1, m_2, m_3, m_4 .
Thus, in this large s result, mass difference effects show up only in the next order, $O(s^{-3})$, at worst!

Thus, even for weird masses, $\Delta(t, t_1, t_2)$ still gives the correct boundary as $s \rightarrow \infty$.

This seems to say that you cannot have a positive t . I.e., if t were $+$, there would be no t_1, t_2 domain of integration for elastic unitarity.

So, the ^{2-body} ~~elastic~~ unitarity contribution^s to the absorptive ~~part~~ $2 \rightarrow 2$ amplitude vanishes as $s \rightarrow \infty$ for $t > 0$.

In fact, the larger s gets, the more the t_1, t_2 domain shrinks, if $t > 0$.

do...

NORMALIZATION DICTIONARY

$$S = 1 + i C_1 T, \quad C_1 = +, -$$

operator

$$\langle P|P' \rangle = 2E C_2 \delta^3(P-P') \quad (= 2E C_2 \cdot \frac{V}{(2\pi)^3} \cdot \delta_{P,P'}^{\text{box}}) \quad \text{single particle states}$$

$$T = 1 \otimes T \quad \text{little Hilbert space} \quad \langle P_f | P_i \rangle = C_3 \delta^4(P_f - P_i), \quad C_3 = 1, (2\pi)^4$$

$$|i\rangle = |P_i\rangle \otimes |q_i\rangle = C_3 \delta^4(\text{ext})$$

$$|\beta_i\rangle \equiv \prod_{\text{all}} C_4^{\dagger} |q_i\rangle$$

$$T_{fi} = \langle \beta_f | T | \beta_i \rangle$$

$$Y_i \equiv \left(\frac{C_4^2}{C_2} \right)$$

"matrix element"

$$S_{fi} = \delta_{fi} + i C_1 C_3 \left(\prod_{\text{all}} C_4 \right) T_{fi} \cdot \delta^4(\text{ext})$$

phase space

$$dQ_f \equiv C_3 \delta^4(\text{ext}) \cdot \prod_f \frac{d^3 p_f}{2E_f}$$

"flux"

$$= \sqrt{(P_a \cdot P_b)^2 - m_a^2 m_b^2} = v_a^L E_a^L E_b^L = m_b p_a^L = \sqrt{s} |\vec{p}_a^{\text{cm}}| = \frac{\lambda(s, m_a^2, m_b^2)}{2} \rightarrow s/2$$

cross section

$$d\sigma = \frac{(2\pi)^4 C_3}{4 \cdot \text{flux}} \cdot Y_a Y_b |T_{fi}|^2 dQ_f$$

decay rate

$$dW = \frac{C_3 Y_a}{4\pi E_a} |T_{fi}|^2 dQ_f$$

unitarity

$$i(T_{fi}^{\dagger} - T_{fi}) = C_1 \sum_n \int dQ_n T_{fn}^{\dagger} T_{ni}$$

$$= 2 \operatorname{Im} T_{fi} \quad \text{if } T_{fi} = T_{if}$$

optical theorem

$$\sigma_{\text{tot}} = \frac{C_1 C_3 (2\pi)^4}{2 \cdot \text{flux}} \cdot Y_a Y_b \operatorname{Im} T_{ii}$$

differential

$$\frac{d\sigma}{d\Omega} = \frac{(2\pi)^4 C_3^2 (\prod_{\text{all}} Y_i) (P_f/P_i)^* |T_{fi}|^2}{16s} \quad \frac{d\sigma}{dt} = \frac{(2\pi)^4 C_3^2 (\prod_{\text{all}} Y_i) \cdot \pi |T_{fi}|^2}{16s (P_i^*)^2}$$

Field Theory: $\phi = \int \frac{d^3 p}{2E} C_5 [a e^{ipx} + a^{\dagger} e^{-ipx}], \quad C_5 = \sqrt{\frac{1}{(2\pi)^3 C_2}}, \quad [\phi, a^{\dagger}] = \sqrt{\frac{C_2}{(2\pi)^3}} e^{ipx}$

Let C_6 in $\bar{u}u = -\bar{v}v = 2m/C_6, \quad C_6 = 1, 2m, 2E$
 $u^{\dagger}u = v^{\dagger}v = 2E/C_6$

external wavefunction

Then:

$$\psi = \int \frac{d^3 p}{2E} C_5 [b u e^{ipx} + d^{\dagger} v e^{-ipx}], \quad C_5 = \sqrt{\frac{C_6}{(2\pi)^3 C_2}}, \quad [\psi, b^{\dagger}] = \sqrt{\frac{C_2 C_6}{(2\pi)^3}} e^{ipx} u$$

Note: $\langle P|P' \rangle = \langle a | a^{\dagger} | 0 \rangle \Rightarrow [a, a^{\dagger}] = 2E C_2 \delta^3(P-P')$

Table:	author	$\langle \beta_f T \beta_i \rangle$	C_1	C_2	C_3	C_4	$Y = C_4^2/C_2$	C_6
	Commins	M_{fi}	+	$(2\pi)^4$	$\frac{(2\pi)^3}{2VE}$	$\frac{1}{\sqrt{2VE}}$	$\frac{1}{(2\pi)^3}$	1
	Sakurai	M_{fi}	-	$(2\pi)^4$	$(2\pi)^3/2VE$	$1/\sqrt{2VE f_i}$	$1/(2\pi)^3 f_i$	$2m$
	Kallen	$\langle f T i \rangle$	+	$(2\pi)^4$	$(2\pi)^3/2VE$	$1/\sqrt{2VE}$	$1/(2\pi)^3$	$2E$
	Bjorken & Drell	M_{fi}	-	$(2\pi)^4$	$1/2E$	$1/\sqrt{2E(2\pi)^3 f_i}$	$1/(2\pi)^3 f_i$	$2m$
	Jackson	M_{fi}	-	$(2\pi)^4$	$(2\pi)^3/2E$	$1/\sqrt{2E}$	$1/(2\pi)^3$	1
	Horn & Zach.	T_{fi}	+	$(2\pi)^4$	$(2\pi)^3/2E$	$1/\sqrt{2E}$	$1/(2\pi)^3$	
	Goldberger & Watson	β_{fi}	-	$(2\pi)^4$	$1/2E$	$1/\sqrt{E}$	2	
	Eden	T_{fi}, F	+	$(2\pi)^4$	$(2\pi)^3$	1	$1/(2\pi)^3$	
	Collins & Squire	$\langle f A^{\dagger} i \rangle$	+	$(2\pi)^4$	$(2\pi)^3$	1	$1/(2\pi)^3$	
	Williams	T_{fi}	+	$(2\pi)^4$	$(2\pi)^3$	1	$1/(2\pi)^3$	
	Gasiorowicz	T_{fi}	-	$(2\pi)^4$	f_i	1	$1/f_i$	
	Markus & Srednicki	$\langle f T i \rangle$	+	1	1	1	1	

$f_i = 1$ boson
 $= C_6^{-1}$ fermion

$V = \text{volume of box}$

Jackson Norm Formulas [but $\epsilon_i = +1$]

① All energy:

(a) total: $\sigma_{\text{tot}}(s) = \frac{1}{2\sqrt{s}} \sum_n dQ_n |T|^2$

(b) decay: $\Gamma = \frac{1}{\tau} = \frac{1}{2m_n} \sum_n dQ_n |T|^2$

(c) optical: $\text{Im} T = A = \sqrt{s} \cdot \sigma_{\text{tot}}$

(d) phase space: $dQ_n = (2\pi)^4 \delta^4(\dots) \prod_f \frac{d^3 p_f \delta(p_f^2 - m_f^2)}{(2\pi)^3}$

(e) unitarity: $2A = \sum_n dQ_n T^\dagger T$

(f) differentials: $\left. \frac{d\sigma}{d\Omega} \right|_{\text{cm}} = \frac{1}{64\pi^2 s} \sqrt{\frac{\Delta_f}{\Delta_i}} |T|^2$

$$\frac{d\sigma}{dt} = \frac{1}{16\pi s \Delta} |T|^2$$

② High energy: let $\sqrt{s} \rightarrow s$.

③ Two particle, cm phase space:

$$d^3 p_i \delta^3(p_i - m_i) d^3 p_f \delta^3(p_f - m_f) \delta^4(\text{ext}) = \frac{\pi}{2\sqrt{s}} dt$$

after $d\phi$ integration. Again, $\sqrt{s} \rightarrow s$.

P.W.A.

The infamous 1α "Jacobian" calculation

- ① The problem is to find out how the states $|\alpha\rangle$ are normalized. Think elastic, $p_1 + p_2 \rightarrow p_3 + p_4$.

We have: $\langle i | i' \rangle = \langle P | P' \rangle \cdot \langle \alpha | \alpha' \rangle$.

$$\text{or, } 2E_3 C_2^{(3)} \delta^3(\vec{p}_3 - \vec{p}_1) \cdot 2E_4 C_2^{(4)} \delta^3(\vec{p}_4 - \vec{p}_2) \quad (*)$$

$$= C_3 \delta^4(\text{ext}) \cdot \langle \alpha | \alpha' \rangle.$$

- ② One way to do this is compute the actual jacobian from $\vec{p}_3, \vec{p}_4$ to $\vec{p}_3 + \vec{p}_4, \vec{p}_3 - \vec{p}_4 \equiv \vec{P}, \vec{p}$.

$$\text{i.e. } \bar{J} = \frac{\partial(\vec{P}, \vec{p})}{\partial(\vec{p}_3, \vec{p}_4)} \Rightarrow d^3 \vec{p}_3 d^3 \vec{p}_4 = \bar{J} \cdot d^3 \vec{P} d^3 \vec{p}$$

$$\Rightarrow \delta^3(\vec{p}_3 - \vec{p}_1) \delta^3(\vec{p}_4 - \vec{p}_2) = \frac{1}{\bar{J}} \delta^3(\vec{P} - \vec{P}_1) \delta^3(\vec{p} - \vec{p}_1)$$

$$\text{Then we'd have: } 2E_3 C_2^{(3)} \cdot 2E_4 C_2^{(4)} \cdot \frac{1}{\bar{J}} \delta^3(\vec{P} - \vec{P}_1)$$

$$= C_3 \delta(\vec{P}_0 - \vec{P}_1) \langle \alpha | \alpha' \rangle \text{ etc. etc.}$$

- ③ Here is an easy way. We already know from elsewhere that:

$$\int_{(6)} d^4 p_1 \delta^4(p_1^2 - m_1^2) \cdot d^4 p_2 \delta^4(p_2^2 - m_2^2) \cdot \delta^4(\text{ext}) = \frac{P_1^*}{4\sqrt{s}} d\Omega_1^*$$

- ④ Now take (*) and mult both sides by $d^4 p_3 \delta^4(p_3^2 - m_3^2) d^4 p_4 \delta^4(p_4^2 - m_4^2)$. This gives:

$$2E_3 C_2 \cdot 2E_4 C_2 \cdot d^4 p_3 \delta^4(p_3^2 - m_3^2) \delta^3(\vec{p}_3 - \vec{p}_1) \cdot d^4 p_4 \delta^4(p_4^2 - m_4^2) \delta^3(\vec{p}_4 - \vec{p}_2)$$

$$= C_3 \delta^4(\text{ext}) \langle \alpha | \alpha' \rangle d^4 p_3 \delta^4(p_3^2 - m_3^2) d^4 p_4 \delta^4(p_4^2 - m_4^2).$$

Now do all 8 integrals on both sides to get

$$C_2^{(3)} \cdot C_2^{(4)} = C_3 \int \langle \alpha | \alpha' \rangle d^4 p_3 \delta^4(p_3^2 - m_3^2) d^4 p_4 \delta^4(p_4^2 - m_4^2) \delta^4(\text{ext})$$

$$= C_3 \int \langle \alpha | \alpha' \rangle \cdot \frac{P_0^*}{4\sqrt{s}} d\Omega_3^*$$

This really only makes sense when C_2 are independent of E . Suppose we pre-divide by the C_2 's: well, no matter because $E_3^* = f(s) = \text{constant}$ in our situation.

So far, then:
$$\underbrace{\left[\frac{C_2^{(3)} \cdot C_2^{(4)}}{C_3} \right]}_{\text{const. in } s} = \underbrace{\frac{P_3^*}{4\sqrt{s}}}_{\text{const. in } s} \int \langle \alpha | \alpha' \rangle d\Omega_3 \quad \checkmark$$

Therefore:
$$\langle \alpha | \alpha' \rangle = \frac{4\sqrt{s}}{P_3^*} \cdot \frac{C_2^{(3)} C_2^{(4)}}{C_3} \delta(\Omega_3 - \Omega_1)$$

⑤ Finally, go to the $|\beta\rangle = TC_Y |\alpha\rangle$ states to say. Not: $|\alpha\rangle = C_Y |\beta\rangle$.

$$\langle \beta | \beta' \rangle = C_Y^\dagger C_Y (C_Y^\dagger C_Y)^{-1} \cdot \langle \alpha | \alpha' \rangle = [C_Y^{(3)} C_Y^{(4)}]^{-2} \langle \alpha | \alpha' \rangle = C_Y^{-2} \cdot C_Y^2 \cdot \langle \alpha | \alpha' \rangle.$$

Then:

$$\langle \beta | \beta' \rangle = \frac{4\sqrt{s}}{P_3^*} \cdot \frac{\{[C_Y^{(3)}]^{-2} C_2^{(3)}\} \{[C_Y^{(4)}]^{-2} C_2^{(4)}\}}{C_3} \delta(\Omega_3 - \Omega_1).$$

a.
$$\langle \beta | \beta' \rangle = \frac{4\sqrt{s}}{P_3^*} \cdot \left[\frac{\cancel{\delta\Omega_3} \cdot \cancel{\delta\Omega_3}}{\delta\Omega_a \delta\Omega_b C_3} \right] \delta(\Omega_3 - \Omega_1)$$

⑥ In non-M.S. norms, this thing is not very simple. Eg, Jackson would have to say:

$$\langle \beta | \beta' \rangle = \frac{4\sqrt{s}}{P_3^*} \cdot \frac{(2\pi)^3}{4E_3^{*2}} \cdot \frac{(2\pi)^3}{4E_1^{*2}} \cdot (2\pi)^4 \cdot \delta(\Omega_3 - \Omega_1)$$

Whereas MS can say:

$$\langle \beta | \beta' \rangle = \frac{4\sqrt{s}}{P_3^*} \cdot \delta(\Omega_3 - \Omega_1)$$

⑦ General result:

$$\langle \beta | \beta' \rangle = \frac{4\sqrt{s}}{P^*} \cdot \frac{\delta(\Omega - \Omega')}{\delta\Omega_a \delta\Omega_b C_3}$$

↖ $\beta = 2$ particle state.

The all norm value of $\langle \beta | JM \rangle$

① We have shown that:

$$\langle \beta | \beta' \rangle = \frac{4\sqrt{s}}{p^*} \delta(\Omega - \Omega') \times \frac{\cancel{\delta_a \delta_b C_3}}{\delta_a \delta_b C_3} \quad \checkmark$$

$$\langle \alpha | \alpha' \rangle = \frac{4\sqrt{s}}{p^*} \delta(\Omega - \Omega') \times \frac{C_2 C_2}{C_3} \quad \checkmark$$

② We also know, however, that

$$\langle \theta \phi | \theta' \phi' \rangle = \delta(\Omega - \Omega')$$

$$|\theta \phi\rangle = \sum_{J,M} C_J \bar{D}_{M, \mu=\mu_0}^{(r)}(\phi, \theta, \phi) |JM\rangle$$

with $C_J = \sqrt{\frac{2J+1}{4\pi}}$

③ Comparison shows that, if

$$|\beta\rangle = k |\theta \phi\rangle, \text{ then}$$

$$\langle \beta | \beta' \rangle = \frac{4\sqrt{s}}{p^*} \frac{\cancel{\delta_a \delta_b C_3}}{\delta_a \delta_b C_3} \delta = k^2 \delta$$

$$\Rightarrow k = \sqrt{\frac{4\sqrt{s}}{p^*} \cdot \frac{\cancel{\delta_a \delta_b C_3}}{C_3 \delta_a \delta_b}} = \sqrt{\frac{4\sqrt{s}}{p^*}} \cdot \sqrt{\frac{1}{\delta_a \delta_b C_3}}$$

④ Therefore,

$$|\beta\rangle = k \sum_{J,M} C_J \bar{D} \cdot |JM\rangle$$

$$\langle JM | \beta \rangle = k \cdot C_J \cdot 1 \cdot 1 = k \cdot C_J$$

Result:

$$\langle \beta | JM \rangle = \left[\frac{4\sqrt{s}}{p^*} \right]^{1/2} \cdot \left[\frac{\cancel{\delta_a \delta_b C_3}}{C_3 \delta_a \delta_b} \right]^{1/2} \cdot \left[\frac{2J+1}{4\pi} \right]^{1/2} \cdot \left[\frac{1}{\delta_a \delta_b C_3} \right]^{1/2}$$

Result:

$$|\langle \beta | JM \rangle|^2 = \frac{4\sqrt{s}}{p^*} \cdot \frac{2J+1}{4\pi} \cdot \frac{1}{\delta_a \delta_b C_3}$$

Redefine the J-version of σ_{tot} using all-norm forms.

① Start with:

$$\sigma_{tot} = \sum_{n_f, \lambda_f} \frac{C_3 \pi^2}{\text{flux}} \cdot \overbrace{\left[\frac{1}{\text{flux}} \right]}^1 \cdot \delta_a \delta_b \cdot \int |T_{fi}|^2 dQ_f \quad \checkmark$$

where

$$T_{fi} = \langle \beta_f | T | \beta_i \rangle \quad \checkmark$$

as on "norm dictionary".

$$dQ_f = C_3 \delta^3(\text{ext}) \prod_f \frac{\delta_f d^3 p_f}{2 E_f} \quad \checkmark$$

② Now I could show that:
(completeness).

$$1 = \sum_{n, \lambda} \int dQ |\beta, n\rangle \langle \beta, n|$$

(The norm of state $|\beta\rangle$ is compensated for in dQ .)

③ Now examine

$$\begin{aligned} & \sum_{n_f, \lambda_f} \int dQ_f |T_{fi}|^2 \\ &= \sum_{n_f, \lambda_f} \int dQ_f \langle \beta_i | T^\dagger | \beta_f \rangle \langle \beta_f | T | \beta_i \rangle \\ &= \langle \beta_i | T^\dagger T | \beta_i \rangle \end{aligned}$$

Thus, of course, $\sigma_{tot} = \frac{C_3 \pi^2}{\text{flux}} \delta_a \delta_b \cdot \langle \beta_i | T^\dagger T | \beta_i \rangle$

④ Now inject $1 = \sum_{m, \lambda} \sum_{n, \lambda} |\alpha m\rangle \langle \alpha m|$ to get:

$$\begin{aligned} \sigma_{tot} &= \frac{C_3 \pi^2}{\text{flux}} \delta_a \delta_b \cdot \sum_{\alpha m} \sum_{n, \lambda} |\langle \alpha m n \lambda | T | \beta_i \rangle|^2 \\ &= \frac{C_3 \pi^2}{\text{flux}} \delta_a \delta_b \cdot \sum_{\alpha m, n, \lambda} |\langle \alpha m n \lambda | T | \beta_i \rangle|^2 \end{aligned}$$

⑤ Now we are stuck with the problem of expanding this plane wave state $|\beta\rangle = \Pi(C_i) |\alpha\rangle$ into αm states of the correct norm. We avoided this problem on the other side via completeness.

Now we have to face it! (see next page)

So now have:

$$\sigma_{\text{tot}} = \frac{G_3 \pi^2}{\text{flux}} \delta_{ab} \sum_J \sum_{n\lambda} |\langle JM n \lambda | T | \beta^0 \rangle|^2$$

Use result that

$$\begin{aligned} \langle JM n \lambda | T | \beta^0 \rangle &= \sum_{J' M' 2 \lambda_i} \langle JM n \lambda | X | J' M' 2 \lambda_i \rangle \langle J' M' 2 \lambda_i | \beta^0 \rangle \\ &= \langle JM n \lambda | 3 | J M 2 \lambda_i \rangle \langle J M | \beta^0 \rangle \delta_{M, n} \end{aligned}$$

Then,

$$\begin{aligned} |\langle JM n \lambda | T | \beta^0 \rangle|^2 &= |\langle JM n \lambda | T | J M 2 \lambda_i \rangle|^2 \cdot k^2 C_J^2 \delta_{M, n} \\ &= |T^J|^2 \cdot \frac{4\sqrt{s}}{p^*} \cdot \frac{\delta_{ab} G_3}{4\pi} \cdot \frac{2J+1}{4\pi} \delta_{M, n} \checkmark \end{aligned}$$

Inserting this gives:

$$\sigma_{\text{tot}} = \frac{\pi^2}{\text{flux}} \cdot \frac{4\sqrt{s}}{p^*} \cdot \frac{\delta_{ab} G_3}{4\pi} \cdot \sum_J \sum_{n\lambda} (2J+1) |T^J|^2$$

$$\sigma_{\text{tot}}^{sp} = \frac{\pi^2}{(p^*)^2} \cdot \frac{\delta_{ab} G_3}{4\pi} \cdot \sum_J \sum_{n\lambda} (2J+1) |T^J|^2$$

⑥ Thus, the all norm result is:

$$\sigma_{\text{tot}}(s) = \frac{\pi}{(p^*)^2} \sum_J (2J+1) \sum_{n\lambda} |\langle JM n \lambda | T | J M 2 \lambda_i \rangle|^2$$

Now we make the usual special cases:

$$a) |\langle JM 2 \lambda_i | T | J M 2 \lambda_i \rangle|^2 = |T_{el}^J|^2 = |(\eta_J e^{2i\delta_J} - 1)|^2$$

$$b) \sum_{n', \text{inel}} |\langle JM n' \lambda' | T | J M 2 \lambda_i \rangle|^2 = \sum_{n'} |S_{n'}^J|^2 = 1 - |S_{el}^J|^2 = 1 - \eta_J^2$$

These give the standard result.

About the state α

- ① Start with an n -particle state:

$$|p_1 \lambda_1; p_2 \lambda_2; \dots p_n \lambda_n\rangle$$

- ② Normalize this state (arbitrarily but invariantly) as

$$\begin{aligned} & \langle p'_1 \lambda'_1; p'_2 \lambda'_2; \dots p'_n \lambda'_n | p_1 \lambda_1; p_2 \lambda_2; \dots p_n \lambda_n \rangle \\ &= \delta^3(p'_1 - p_1) 2E_1 \delta_{\lambda'_1 \lambda_1} \otimes \delta^3(p'_2 - p_2) 2E_2 \delta_{\lambda'_2 \lambda_2} \otimes \dots \otimes \delta^3(p'_n - p_n) 2E_n \delta_{\lambda'_n \lambda_n} \end{aligned}$$

- ③ Now define a state $|\alpha\rangle$ by factoring off the "orbit"

$$|p_1 \lambda_1; p_2 \lambda_2; \dots p_n \lambda_n\rangle = |P_n\rangle \otimes |\alpha, n, \lambda\rangle$$

where now λ is really the whole set $\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_n$.

- ④ Since $\langle P' | P \rangle = \delta^3(P' - P)$, we have:

$$\begin{aligned} \delta_{n'n} \prod_{i=1}^n [\delta^3(p'_i - p_i) 2E_i \delta_{\lambda'_i \lambda_i}] &= \delta^3(P' - P) \cdot \langle \alpha', n, \lambda' | \alpha, n, \lambda \rangle \\ &= \delta_{\lambda' \lambda} \prod_{i=1}^n [\delta^3(p'_i - p_i) 2E_i] \delta_{n' n} \end{aligned}$$

where here $\delta_{\lambda' \lambda}$ means $\delta_{\lambda'_1 \lambda_1} \delta_{\lambda'_2 \lambda_2} \dots \delta_{\lambda'_n \lambda_n}$.

- ⑤ Now define $dQ_n = \delta^3(P - P') \prod_{i=1}^n \frac{d^3 p_i}{2E_i}$. Then mult both sides of ④ by $\prod_{i=1}^n \frac{d^3 p_i}{2E_i}$ to get:

$$dQ_n \langle \alpha', n, \lambda' | \alpha, n, \lambda \rangle = \delta_{\lambda' \lambda} \prod_{i=1}^n d^3 p_i \delta^3(p_i - p'_i) \delta_{n' n}$$

- ⑥ Now integrate over all of n -phase space to get

$$\int dQ_n \langle \alpha', n, \lambda' | \alpha, n, \lambda \rangle = \delta_{\lambda' \lambda} \delta_{n' n}$$

- ⑦ Now mult both sides by $|\alpha', n, \lambda'\rangle$ and sum on λ' and n' . Since $\langle \alpha | \alpha' \rangle dQ = \langle \alpha | \alpha' \rangle dQ'$, this leads to a completeness:

$$\sum_n \sum_{\lambda} \int dQ_n |\alpha, n, \lambda\rangle \langle \alpha, n, \lambda| = 1$$

(More on α)

- ⑧ Now we can generalize this result to $n' \neq n$. Back in ②, if $n' \neq n$, we simply add a $\delta_{n'n}$ to the right side. All changes are shown in red pencil. Thus: New results are:

$$\left[\begin{aligned} \int dQ_n \langle \alpha', n', \lambda' | \alpha, n, \lambda \rangle &= \delta_{\lambda'\lambda} \delta_{n'n} \\ \sum_{n, \lambda} \int dQ_n |\alpha n \lambda \rangle \langle \alpha n \lambda| &\equiv 1 \end{aligned} \right]$$

Clearly this is the only correct completeness relation.

- ⑨ We could include the index n with λ and call it λ . Then shorthand version would be:

$$\sum_{\lambda} \int dQ |\alpha \lambda \rangle \langle \alpha \lambda| = 1$$

See eg (4.166) of MS.

- ⑩ On cross sections. Now we have clearly normalized the states α . Next we normalize the operator T by saying

$$S = 1 + iT$$

$$\text{Then } \langle \alpha_f | S | \alpha_i \rangle = \langle \alpha_f | \alpha_i \rangle + i \langle \alpha_f | T | \alpha_i \rangle.$$

Then, one gets the usual result:

$$d\sigma = \frac{\pi^2}{\text{flux}} \left| \langle \alpha_f, n_f, \lambda_f | T | \alpha_i, n_i, \lambda_i \rangle \right|^2 dQ_{n_f}$$

$\uparrow$
 $n_i = 2 \text{ only}$

where, eg, $\text{flux} = p_i^* \sqrt{S}$. Better recall that $n_i = 2$ for $d\sigma$.

- ⑪ Then the total cross section is:

$$\left[\sigma_{\text{tot}} = \frac{\pi^2}{\text{flux}} \sum_{n_f, \lambda_f} \int dQ_{n_f} \left| \langle \alpha_f, n_f, \lambda_f | T | \alpha_i, 2, \lambda_i \rangle \right|^2 \right]$$

Shorthand:

$$\sigma_{\text{tot}} = \frac{\pi^2}{\text{flux}} \sum_{\lambda} \int dQ \left| \langle \alpha_f | T | \alpha_i \rangle \right|^2$$

(More on α_i)

(12) Now we have shown that:

$$\sigma_{\text{tot}} = \frac{\pi^2}{\text{flux}} \left(\sum_{n\lambda} d\Omega_n \right) \langle \alpha_i, 2, \lambda_i | T^\dagger | \alpha_i, n\lambda \rangle \langle \alpha_i, n\lambda | T | \alpha_i, 2, \lambda_i \rangle$$

This was result of (11). But, insert result of (8) to find:

$$\sigma_{\text{tot}} = \frac{\pi^2}{\text{flux}} \langle \alpha_i, 2, \lambda_i | T^\dagger T | \alpha_i, 2, \lambda_i \rangle$$

(13) If $S^\dagger S = S S^\dagger = 1$, and if $S = 1 + iT$, then you at once find that:

$$[T^\dagger T = T T^\dagger = -i(T - T^\dagger)]$$

$$\begin{aligned} \text{Then: } \langle \alpha_i | T^\dagger T | \alpha_i \rangle &= -i [\langle \alpha_i | T | \alpha_i \rangle - \langle \alpha_i | T^\dagger | \alpha_i \rangle] \\ &= -i [\langle \alpha_i | T | \alpha_i \rangle - \langle \alpha_i | T | \alpha_i \rangle^*] \\ &= -i [2i \operatorname{Im} \langle \alpha_i | T | \alpha_i \rangle] \\ &= +2 \operatorname{Im} \langle \alpha_i | T | \alpha_i \rangle. \end{aligned}$$

(14) Thus we find:

$$\left[\sigma_{\text{tot}} = \frac{2\pi^2}{\text{flux}} \operatorname{Im} \langle \alpha_i | T | \alpha_i \rangle \right] \quad / \text{standard optical}$$

(15) Including all n -manifolds, we have these completeness relations:

$$\left[\begin{aligned} \sum_{n\lambda} \int d\Omega_n | \alpha_i, n, \lambda \rangle \langle \alpha_i, n, \lambda | &= 1 \\ \sum_{n\lambda} \sum_{m\mu} | \alpha_i, n, \lambda \rangle \langle \alpha_i, m, \mu | &= 1 \end{aligned} \right]$$

Using these, we can quickly re-express (12) as:

$$\sigma_{\text{tot}} = \frac{\pi^2}{\text{flux}} \sum_{n, \lambda} \sum_{m, \mu} | \langle \alpha_i, m, \mu | T | \alpha_i, 2, \lambda_i \rangle |^2$$

(16) Since $|\alpha_i\rangle$ is a two particle state, we know explicitly how to project it:

$$|\alpha_i, 2, \lambda_i\rangle = \sqrt{\frac{4\sqrt{3}}{p_i}} |00\lambda_i\rangle = \sqrt{\frac{4\sqrt{3}}{p_i}} \times \sum_{\mu_1, \mu_2} C_{\mu_1 \mu_2}^{\lambda_i} | \alpha_i, \mu_1, \mu_2 \rangle$$

$\mu \rightarrow = \mu_1 \mu_2$

Where λ_i now means $\mu_1 \mu_2, \lambda_i$ and $C_{\lambda_i}^2 = (2\lambda_i + 1) / 4\pi$

(more on α_i)

(12) Now reexpress element in (15) as:

$$\begin{aligned}
 & \langle \sigma m n f | T_p | \alpha_i, 2, \lambda_i \rangle \\
 &= \sqrt{\frac{4\sqrt{5}}{p_i}} \cdot \sum_{\sigma'} C_{\sigma'} \langle \sigma m n f | T_p | \sigma' 1, 2, \lambda_i \rangle \\
 &= \sqrt{\frac{4\sqrt{5}}{p_i}} C_{\sigma} \langle \sigma m n f | T_p | \sigma 1, 2, \lambda_i \rangle \cdot \delta_{m\mu}
 \end{aligned}$$

Then, (15) becomes:

$$\begin{aligned}
 \sigma_{\text{tot}} &= \frac{\pi^2}{(\sqrt{5} p_i)} \cdot \sum_{n f} \frac{4\sqrt{5}}{p_i} \cdot \sum_{\sigma} \left(\frac{2\sigma+1}{4\pi} \right) |\langle \sigma m n f | T_p | \sigma 1, 2, \lambda_i \rangle \delta_{m\mu}|^2 \\
 \left[\sigma_{\text{tot}} &= \left[\frac{\pi}{(p_i)^2} \cdot \sum_{n f} \cdot \sum_{\sigma} (2\sigma+1) |\langle \sigma \mu; n f | T_p | \sigma \mu; 2, \lambda_i \rangle|^2 \right] \right]
 \end{aligned}$$

Notice that this result is correct with spin.

Unitarity Partial Wave Analysis

- ① We have shown that the total cross section may be projected as: (see notes under Martin/Spearman)

$$\sigma_{\text{tot}} = \frac{\pi}{k^2} \sum_{n_f, \lambda_f} \sum_{\sigma} (2J+1) |\langle J\mu; n_f \lambda_f | T_P | J\mu; 2\lambda_i \rangle|^2 \checkmark$$

Here, the state $|J\mu; n_f \lambda_f\rangle$ is just the projection of the state $|\alpha_f, n_f, \lambda_f\rangle$.

$$|J\mu; n_f \lambda_f\rangle = \int d\Omega_{\alpha_f} |\alpha_f n_f \lambda_f\rangle \langle \alpha_f n_f \lambda_f | J\mu n_f \lambda_f \rangle$$

$= 1$ within n_f manifold.

Notice that M is forced equal to $\mu = \mu_i = \mu_f = J_z$ for unitary state $\phi = \infty$.

- ② Now let's go back & examine unitarity of S_P in partial waves:

$$\left[\sum_{n_f, \lambda_f} |\langle J\mu n_f \lambda_f | S_P | J\mu, 2, \lambda_i \rangle|^2 = 1 \right] \checkmark$$

Thus:

$$\sum_{\substack{n_f, \lambda_f \\ \text{exc. elastic}}} |\langle \dots \rangle|^2 + \underbrace{|\langle J\mu 2 \lambda_i | S_P | J\mu 2 \lambda_i \rangle|^2}_{= \eta_0^2 \leq 1} = 1 \checkmark$$

This allows us to say:

$$\left[\begin{aligned} \langle J\mu 2 \lambda_i | S_P | J\mu 2 \lambda_i \rangle &= \eta_0 e^{2i\delta_J} \\ \Rightarrow \langle |T_P| \rangle &= -i [\eta_0 e^{2i\delta_J} - 1] \end{aligned} \right] \checkmark$$

- ③ Pulling out only the elastic piece of ① we can say:

$$\sigma_{\text{el}} = \frac{\pi}{k^2} \sum_{\sigma} (2J+1) |\eta_0 e^{2i\delta_J} - 1|^2 \checkmark$$

as shown in (4.172).

- ④ For inelastic, we have $S_i = iT$ and

$$\sum_{\substack{n_f, \lambda_f \\ \text{exc. el}}} |\langle J\mu; n_f \lambda_f | T_P | J\mu, 2, \lambda_i \rangle|^2 = 1 - \eta_0^2 \checkmark$$

Thus:

$$\sigma_{\text{in}} = \frac{\pi}{k^2} \sum_{\sigma} (2J+1) (1 - \eta_0^2)$$

(partial wave analysis)

⑤ So far then we have these very powerful results:

$$\left[\sigma_{el} = \frac{\pi}{k^2} \sum_j (2j+1) \left[\eta_j^2 - 2\eta_j \cos(2\delta_j) + 1 \right] \right]$$

$$\left[\sigma_{in} = \frac{\pi}{k^2} \sum_j (2j+1) \left[1 - \eta_j^2 \right] \right]$$

⑥ This implies directly:

$$\left[\sigma_{tot} = \frac{2\pi}{k^2} \sum_j (2j+1) \left[1 - \eta_j \cos(2\delta_j) \right] \right]$$

⑦ Recalling that $\langle \psi_{M;2,\lambda_i} | T_P | \psi_{M;2,\lambda_i} \rangle = -i \left[\eta_j e^{2i\delta_j} - 1 \right]$
elastic $\rightarrow$ We see that $\text{Im } T_P^{\text{elastic}} = 1 - \eta_j \cos(2\delta_j)$
 So:

$$\left[\sigma_{tot} = \frac{2\pi}{k^2} \sum_j (2j+1) \text{Im} \langle \psi_{M;2,\lambda_i} | T_P | \psi_{M;2,\lambda_i} \rangle \right]$$

A partial wave optical theorem.

⑧ Unitarity restrictions on partial wave amplitudes!
 We see that no controls the amount of inelastic,
 independent of δ_j . ie, $S_j = 0$, "total absorption"If $\eta_j = 0$, you have a maximal inelastic c.s. But
 this does not make the elastic go away:

$$\eta_j = 0 \Rightarrow \left. \begin{aligned} \sigma_{el} &= \frac{\pi}{k^2} \sum_j (2j+1) \\ \sigma_{in} &= \frac{\pi}{k^2} \sum_j (2j+1) \end{aligned} \right\} \begin{array}{l} \text{These are equal} \\ \text{in limit } \eta_j \rightarrow 0. \end{array}$$

$$\sigma_{tot} = \frac{2\pi}{k^2} \sum_j (2j+1)$$

(partial wave analysis).

⑨ Interpret this result.

$\eta_0 = 0$ means $\sigma_{\text{elastic}} = 0$ so $|T_{\ell}^0| = 1$, the "straight thru" component. This would be called diffraction and you get

$$\sigma_{\text{el}} = \frac{\pi}{k^2} \sum_{\ell} (2\ell+1) |T_{\ell}^0|^2 = \frac{\pi}{k^2} \sum_{\ell} (2\ell+1).$$

Notice that the total diffraction σ_{el} c.s. is infinite when $\eta_0 = 0$. So is the inelastic. So is the total in all waves.

But physically, what does $\eta_0 \neq 0$ mean? I think this is a black disc (at least within a certain partial wave.) (Maybe as ℓ increases, you need a larger + larger radius black disc, hence $\sigma_{\text{tot}} \rightarrow \infty$).

The black disc kills the elastic channel for sure if there is a hit. Thus:

$$\sigma_{\text{in}} = \frac{\pi}{k^2} (2\ell+1) = \pi R^2 \quad R = R(\ell, k).$$

But, unavoidably, this disc has a shadow. That shadow would not be there unless there were elastic scattering present to cause cancellation with the initial plane wave. Not surprising then that $\sigma_{\text{el}} = \pi R^2$ also.

⑩ The other extreme: $\eta_0 = 1$. Then $\sigma_{\text{el}} = \frac{\pi}{k^2} \sum_{\ell} (2\ell+1) \dots$

Then: $\sigma_{\text{in}} = 0$

$$\sigma_{\text{el}} = \frac{\pi}{k^2} \sum_{\ell} (2\ell+1) [1 - 2\cos(2\delta_{\ell}) + 1]$$

$$[\sigma_{\text{el}} = \frac{4\pi}{k^2} \sum_{\ell} (2\ell+1) \sin^2 \delta_{\ell} = \sigma_{\text{tot}}.]$$

All this stuff is due with spin. We have to restrict to a particular initial spin state λ_i , however.

(partial waves)

- ⑪ Now, back to unitarity restriction. Under any conditions, the most σ_{in} can be is (in Jⁿ pw) when $n_0 = 0$:

$$[\sigma_{in}^{(J)}(\max) = \frac{\pi}{k^2} (2J+1)]$$

- ⑫ For σ_{el} , the largest you can get is when $n_0 = 1$ and $\delta_0 = 90^\circ$, then:

$$[\sigma_{el}^{(J)}(\max) = \frac{4\pi}{k^2} (2J+1)]$$

- ⑬ For σ_{tot} , result occurs at same place as ⑫

$$[\sigma_{tot}^{(J)}(\max) = \frac{4\pi}{k^2} (2J+1)]$$

These both maximize when $\sigma_{in} = 0$.

- ⑭ Application. Consider $J=0$ pw (say $S=0, l=0$). Suppose $k = m_\pi$, a small value. Then:

$$\sigma_{in}^{(0)}(\max) = \frac{\pi}{m_\pi^2} = \underline{63.1 \text{ mb}} \quad (.139, .389)$$

$$\sigma_{el}^{(0)}(\max) = \frac{4\pi}{m_\pi^2} = 25.3 \text{ mb.}$$

Now, however, if σ_{in} is at its max, then $\sigma_{el} = \sigma_{in}$ ($n_0 = 0$).

M.S. unitarity, general and two-body

① As shown in "maxwell" notes, unitarity says

$$S^\dagger S = 1 \quad \Rightarrow \quad T^\dagger T = T T^\dagger = -i(T - T^\dagger)$$

all in operator form. T is normal, by $S = 1 + iT$. The states are normal so that

$$\sum_{n,\lambda} \int dQ_n |\alpha_{n\lambda}\rangle \langle \alpha_{n\lambda}| = 1 \quad d^4 p_i$$

where $dQ_n = \delta^4(\text{ext}) \cdot \prod_i \delta(p_i^2 - m_i^2)$. Note that there is no $(2\pi)^4$ in the norm of T , and no $(2\pi)^3$ in the state norms.

② Thus, general unitarity reads:

$$\sum_{n,\lambda} \int dQ_n \langle \alpha_f | T^\dagger | \alpha_{n\lambda} \rangle \langle \alpha_{n\lambda} | T | \alpha_i \rangle = -i \left[\langle \alpha_f | T | \alpha_i \rangle - \langle \alpha_f | T^\dagger | \alpha_i \rangle \right]$$

where λ here is a channel label to distinguish channels of the same n .

③ If we restrict $|\alpha_i\rangle$ and $|\alpha_f\rangle$ to be spinless, 2-body states, then time-reversal says

$$\text{RHS} = +2 \operatorname{Im} [\langle \alpha_f | T | \alpha_i \rangle]$$

as shown in "unitarity" notes.

④ Two-body-only restricted unitarity in the s-channel then reads:

$$\frac{i}{2} \sum_{\lambda} \int dQ_2 T(s, t') T(s, t'')^* = \operatorname{Im} T(s, t)$$

where t' and t'' are standard invariants.

⑤ We want to reduce this to the angular integral.

2-body reduction to angular integral

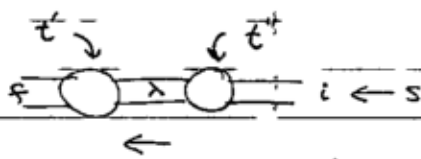
① So far we have:

$$\begin{aligned} \text{Im } T^{\text{fi}}(s, t) &= D_s(s, t) \quad dQ_2 \\ &= \frac{1}{2} \sum_{\lambda} \int \overbrace{\delta^4(\text{ext}) \delta^+(p_1^2 - m_1^2) \delta^+(p_2^2 - m_2^2)}^{dQ_2} T^{\text{fi}}(s, t'; \lambda) T^{\text{fi}}(s, t''; \lambda) \\ &\quad \times d^4 p_1 d^4 p_2 \end{aligned}$$

where $\delta^4(\text{ext}) = \delta^4(p_1 + p_2 - p_a - p_b)$. At once we reduce to:

$$D_s(s, t) = \frac{1}{2} \sum_{\lambda} \int d^4 p_1 \delta^+(p_1^2 - m_1^2) \delta^+(p_2^2 - m_2^2) T^{\text{fi}}(s, t'; \lambda) T^{\text{fi}}(s, t''; \lambda)$$

② Picture is:



③ To compute this phase space, go to cms. + say

$$\begin{aligned} &\int dE_1 \cdot \frac{1}{2} p_1 \cdot d p_1^2 \cdot d\Omega_1 \delta^+(E_1^2 - p_1^2 - m_1^2) \delta^+((\sqrt{s} - E_1)^2 - p_1^2 - m_2^2) \\ &= \int dE_1 \cdot \frac{1}{2} \tilde{p}_1 \cdot d\Omega_1 \cdot \delta^+[s - 2\sqrt{s}E_1 + E_1^2 - E_1^2 + m_1^2 - m_2^2] \end{aligned}$$

where $\tilde{p}_1 = \sqrt{E_1^2 - m_1^2}$. Next, get

$$= \frac{1}{2\sqrt{s}} \cdot \frac{1}{2} \tilde{p}_1 \cdot \int d\Omega_1$$

where now $\tilde{p}_1 = \frac{1}{2\sqrt{s}} [s - (m_1 + m_2)^2]^{1/2} [s - (m_1 - m_2)^2]^{1/2}$, the usual cms momentum. So,

$$D_s^{\text{fi}}(s, t) = \sum_{\lambda} \frac{q_{\lambda}}{8\sqrt{s}} \int d\Omega_1 [T^{\text{fi}}(s, t'; \lambda)]^* [T^{\text{fi}}(s, t''; \lambda)]$$

④ This differs from my C-S calc by the expected $(2\pi)^2$.

⑤ Pure elastic unitarity is thus:

$$\text{Im } T(s, t) = \frac{q_{\text{cms}}}{8\sqrt{s}} \int d\Omega T^*(s, t') T(s, t'') \quad \checkmark$$

2-body partial wave unitarity, M.S.

① We start off with:

$$\begin{aligned} \text{Im } T^{fi}(s, \theta) &= \sum_{\lambda} \frac{q_{\lambda}}{8\sqrt{s}} \int d\phi'' \int d(\cos\theta'') T^{\lambda f*}(s, \theta') T^{\lambda i}(s, \theta'') \\ &= \sum_{\lambda} \frac{\pi q_{\lambda}}{4\sqrt{s}} \int dz'' T^{\lambda f*}(s, z') T^{\lambda i}(s, z'') \end{aligned}$$

② Now do the spinless M.S. pw. expansion:

$$T^{\lambda i}(s, z'') = \left\{ \frac{\sqrt{s}}{\pi \sqrt{q_i q_{\lambda}}} \right\} \sum_{J} (2J+1) P_J(z'') T_J^{\lambda i}(s)$$

$$T^{\lambda f}(s, z') = \left\{ \frac{\sqrt{s}}{\pi \sqrt{q_{\lambda} q_f}} \right\} \sum_{J'} (2J'+1) P_{J'}(z') T_{J'}^{\lambda f}(s)$$

$$T^{fi}(s, z) = \left\{ \frac{\sqrt{s}}{\pi \sqrt{q_i q_f}} \right\} \sum_J (2J+1) P_J(z) T_J^{fi}(s)$$

③ Insert all this into ① to get

$$\text{Im } T_J^{fi}(s) = \frac{1}{2} \sum_{\lambda} T_J^{\lambda f*}(s) T_J^{\lambda i}(s)$$

where we have made use of:

$$\int dz'' P_{J'}(z') P_J(z'') = \frac{2}{(2J+1)} \delta_{J'J} P_J(z)$$

④ For pure elastic result we get

$$\text{Im } T_J(s) = \frac{1}{2} |T_J(s)|^2$$

$$\Rightarrow T_J(s) = i \left[1 - \eta_J e^{2i\delta_J} \right], \text{ unit circle}$$

⑤ Note that we could have obtained this result instantly from operator unitarity:

$$T^\dagger T = -i(T - T^\dagger) \Rightarrow |T_J|^2 = -i \cdot 2i \text{Im } T_J = 2 \text{Im } T_J$$

This works because the states in $T_J = \langle \Omega | T | \Omega \rangle$ are properly normalized, the big advantage of using such T_J 's.

M.S. partial wave projection

① From M.S. (4.107) or from -3- of MS notes, we know that:

$$I(s, t) = T_{ed, ab}(s, t) = \langle \langle \epsilon | T_P | \epsilon \rangle \rangle = \frac{4\sqrt{s}}{(q_i q_f)^{1/2}} \langle \langle \epsilon | T_P | \epsilon \rangle \rangle_{ab}$$

② From M.S. (4.110') or also from -3- we have this expansion:

$$\langle \langle \epsilon | T_P | \epsilon \rangle \rangle_{ab} = (4\pi)^{-1} e^{i(\mu - \mu')\phi} \sum_J (2J+1) d_{JJ'}^{\sigma}(\theta) \langle \langle \epsilon | T^{\sigma} | \epsilon \rangle \rangle_{ab}$$

③ Putting these together, we find

$$T_{ed, ab}(s, t) = \left\{ \frac{\sqrt{s}}{\pi \sqrt{q_i q_f}} \right\} e^{i(\mu - \mu')\phi} \sum_J (2J+1) d_{JJ'}^{\sigma}(\theta) T_{ed, ab}^{\sigma}(s)$$

④ For spinless this becomes

$$I(s, t) = \left\{ \frac{\sqrt{s}}{\pi \sqrt{q_i q_f}} \right\} \sum_J (2J+1) P_J(\cos\theta) T_J(s) \quad \checkmark$$

⑤ If we use $f_{ed, ab} = \frac{\pi}{2\sqrt{s}} T_{ed, ab}$ instead, then we get: (spinless)

$$f(s, \theta) = \frac{1}{2\sqrt{q_i q_f}} \sum_J (2J+1) P_J(\cos\theta) T_J(s) \quad (4.112)$$

This form shows the standard $\frac{1}{q}$ factor in the partial wave expansion of the f -amplitude. I like this expansion for two reasons.

a) uses f , so $ds/d\Omega = |f|^2 \cdot (q_f/q_i)$; f is a standardized amp.

b) uses T^{σ} , so $T_{ed, ab}^{\sigma}$ lies on a true unit circle.

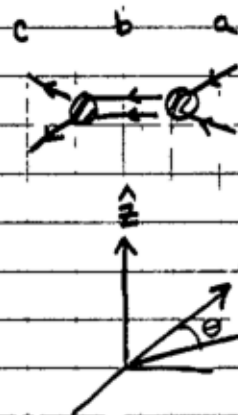
This unit circle is proved on the next few pages.

Unitarity, two particles, in partial waves

Start with three arbitrary channels:
Unitarity says:

$$\text{Im } A^{ca} = \frac{q}{32\pi^2 \sqrt{s}} \int d\Omega_1 A(s, \Omega_2) A^*(s, \Omega_1)$$

$\uparrow$ $\uparrow$ $\uparrow$ $\uparrow$
 θ, ϕ, Θ Θ_1, ϕ_1 Ω_2 Ω_1
 CS norm'd A.



where $q =$ cms momentum in channel b . The parameters of channels a and c are hidden within the Ω_2, Ω_1 relative to Θ_2, Θ_1 .

Define the (physical Θ) partial wave amplitude up to a constant C by:

$$[A = C \sum_l (2l+1) A_l(s) P_l(z)]$$

Use this on both sides to get: (now assume $c = a = b$)

$$C \sum_l (2l+1) \text{Im } A_l(s) P_l(z) =$$

$$= \frac{q}{32\pi^2 \sqrt{s}} \cdot C^2 \sum_{ll'} (2l+1)(2l'+1) A_l(s) A_{l'}^*(s) \int d\Omega_1 P_l(\Omega_2) P_{l'}(\Omega_1)$$

Note that for physical angles, $\text{Im } P_l(z) = 0$. The unitarity integral is only over physical states, after all.

Now, for any Ω_2 vector at (Θ_2, ϕ_2) you can show that

$$\int d\Omega_1 P_l(\Omega_1) P_{l'}(\Omega_2) = \delta_{ll'} \cdot \frac{4\pi}{2l+1} \cdot P_l(z)$$

where $z = \Omega_1 \cdot \Omega_2 = \cos \Theta_1 \cos \Theta_2 + \sin \Theta_1 \sin \Theta_2 \cos(\phi_1 - \phi_2)$. Thus everything cancels and you get:

$$[\text{Im } A_l(s) = \left(\frac{Cq}{8\pi\sqrt{s}} \right) |A_l(s)|^2]$$

or if you choose $C = 16\pi$, get:

$$\text{Im } A_l(s) = \frac{2q}{\sqrt{s}} |A_l(s)|^2$$

Why do C+S choose $C = 16\pi$? How should you set the scale of the partial waves?

(cont'd)

Reconsider the implications of the (2body) unitarity relation:

$$\text{Im} A_l = a |A_l|^2$$

2.

where $\{a = \left[\frac{Cq}{8\pi\sqrt{s}} \right]\}$. If you write $A_l = R e^{i\theta}$, then the relation is:

$$\sin\theta = aR \quad \text{so that} \quad -1 < aR < 1 \Rightarrow \boxed{R < \frac{1}{a}}$$

Unitarity restricts the magnitude of the partial wave amplitude. Namely,

$$|A_l| \leq \frac{8\pi\sqrt{s}}{Cq}$$

If you choose $C = 16\pi$, then at large s where $q = \sqrt{s}/2$, you find the "convenient" result:

$$|A_l(s)| \leq 1 \quad \text{large } s, \quad C = 16\pi$$

Notice that, if A were normalized differently, we would have to choose a different C , but we'd still have $|A_l| \leq 1$ for some C .

With $C = 16\pi$, at low s where q is small, $|A_l|$ has a larger disc.

You could choose $C = 8\pi\sqrt{s}/q$, then you'd have $|\tilde{A}_l(s)| \leq 1$ at all energies.

Argand plots: for elastic unitarity where $a = b = c$, we know that we get

$$\sin\theta = aR$$

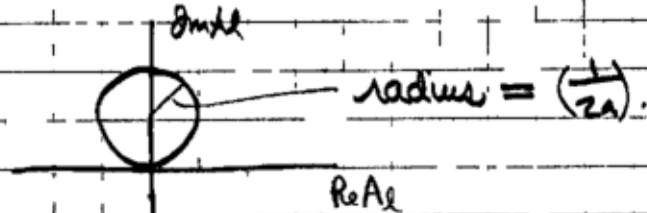
This says a lot more than just $R \leq \frac{1}{a}$. This actually tells you the trajectory of the partial wave amplitude!

Choose an arbitrary scale a for elastic amplitude. Then

$$\sin\theta = aR \Rightarrow (x - 0)^2 + (y - \frac{1}{2a})^2 = (\frac{1}{2a})^2$$

Thus, the amplitudes (one for each l) must — if elastic kinematics apply — lie on a circle.

The circle is:



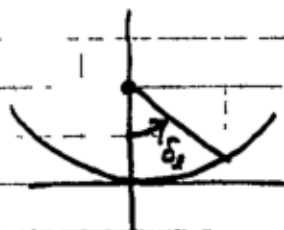
Of course unitarity does not tell us where on the circle you are at a given s . Notice that you could say:

$$A_L = i\left(\frac{1}{2a}\right) + \left(\frac{1}{2a}\right) e^{i2\delta_L(s)} \cdot e^{iK}$$

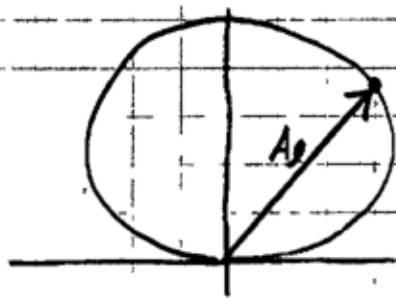
so that $\delta_L(s)$ tells you where you are on the circle. i.e., $\delta_L(s)$ has all the information that $A_L(s)$ has. Here K is an arbitrary angle tell us where $\delta_L = 0$ is on the picture. On (5.11), C+S choose $e^{iK} = -i$ so they get

$$2ia A_L(s) = -1 + e^{2i\delta_L(s)}$$

So $\delta = 0$ means $A = 0$; so angle is measured:



But keep in mind that δ_L contains the phase and magnitude of $A_L(s)$.



for further notes,
see "summary"

Hana '64

Analyticity Properties of Helicity Amplitudes and Construction of Kinematical Singularity-Free Amplitudes for Any Spin*

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(Received 11 June 1964)

The analyticity properties of helicity amplitudes for binary reactions of particles with arbitrary spins are studied using the following three properties: (i) the analyticity properties of scattering amplitudes, assuming these are correctly predicted by perturbation theory; (ii) the crossing relations of helicity amplitudes near $s=0$; (iii) the threshold behavior of partial-wave amplitudes. Making use of these properties, kinematical singularity-free amplitudes for any spin are constructed by modifying helicity amplitudes. MacDowell reciprocity is generalized to arbitrary spin. Helicity amplitudes are proved to satisfy the Froissart limit at the high-energy limit.

I. INTRODUCTION

THE S-matrix theory of strong interactions is based on the analyticity properties of scattering amplitudes.¹ Among various kinds of amplitudes, the helicity amplitude introduced by Jacob and Wick² is most convenient for practical applications. Until now, if one wanted to know the analyticity properties of helicity amplitudes, one had to look for the linearly-independent Lorentz-invariant scalars built up from the four-momenta and spin parameters of the external particles, the coefficients of which are free from kinematical singularities and satisfy the Mandelstam representation. (Here, the spin parameters include Dirac matrices, polarization vectors, and fermion spinors.) Then, one had to know the relation between helicity amplitudes and these coefficients. This has been done for $\pi\pi$, πN , and NN scatterings^{3,4} and their crossed reactions.^{4,5} For more complicated scattering problems, a prescription for finding kinematical singularity-free amplitudes has been given.⁶ However, it is not easy to follow the prescription. For example, it was difficult even for NN scattering. Therefore, this indirect method will not be used in the following. Instead, we will investigate the analyticity properties of helicity amplitudes from the beginning.

As will be shown in the following, the analyticity properties of helicity amplitudes are not so complicated for simple scattering problems.⁷

For πN scattering^{2,3}

$$\begin{aligned} f_{++} &= \cos(\theta/2) [2MA + (W^2 - M^2 - \mu^2)E], \\ f_{+-} &= \sin(\theta/2) [(W^2 + M^2 - \mu^2)A \\ &\quad + (W^2 - M^2 + \mu^2)MB]W^{-1}. \end{aligned} \quad (1.1)$$

For NN scattering⁴

$$\begin{aligned} \frac{1}{2}(f_{++} + f_{+-}) &= E^2 G_1 - z p^2 G_2 + m^2 G_3, \\ \frac{1}{2}(f_{++} - f_{+-}) &= (E^2 G_2 + m^2 G_4)z - p^2 G_5, \\ \frac{1}{2}[(1+z)^{-1}f_{+-} - (1-z)^{-1}f_{+-}] &= -p^2 G_3, \\ &\quad (z = \cos\theta), \quad (1.2) \\ \frac{1}{2}[(1+z)^{-1}f_{+-} + (1-z)^{-1}f_{+-}] &= m^2 G_2 + E^2 G_4, \\ (m/\gamma)f_{+-} &= -m^2 E(G_2 + G_4), \quad (\gamma = \sin\theta). \end{aligned}$$

For the $\pi\pi \rightarrow NN$ process⁵

$$\begin{aligned} f_{++} &= [-2pA + 2mqB \cos\theta], \\ f_{+-} &= 2EqB \sin\theta. \end{aligned} \quad (1.3)$$

Readers will see from Eqs. (1.1) to (1.3) that the modified helicity amplitude,

$$\begin{aligned} h_{\lambda_a \lambda_b, \lambda_c \lambda_d} &= [\cos(\theta/2)]^{-|\lambda_a + \mu|} [\sin(\theta/2)]^{-|\lambda_c - \mu|} \\ &\quad \times s^{-\xi/2} f_{\lambda_a \lambda_b, \lambda_c \lambda_d}, \\ [\lambda &= \lambda_a - \lambda_b, \quad \mu = \lambda_c - \lambda_d, \quad \xi = 0 \text{ (or } 1)] \\ &\quad \text{if } \lambda_a - \lambda_b - \lambda_c + \lambda_d \text{ is even (or odd)}, \end{aligned} \quad (1.4)$$

satisfies the Mandelstam representation if we neglect possible (kinematical) poles at $s=0$ and $p=0$. We can get rid of these by multiplying Eq. (1.4) by s^ξ and p^{2L} (ξ and L can be predicted).

The amplitude (1.4) does not satisfy the Mandelstam representation in general. However, it will be shown that we can modify helicity amplitudes for any spin in such a way that the modified amplitudes satisfy Mandelstam representations. A list of kinematical singularity-free amplitudes will be given in Sec. VII. Making use of this result, a generalized MacDowell reciprocity will be proved in Sec. V. Helicity amplitudes for any spin will be proved to satisfy the Froissart limit at high energies (Sec. VI).

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† On leave of absence from Physics Department, Tokyo University of Education, Tokyo, Japan.

¹ See, for example, G. P. Chew, *S-Matrix Theory of Strong Interactions* (W. A. Benjamin and Company, Inc., New York, 1961).

² M. Jacob and G. C. Wick, *Ann. Phys. (N. Y.)* **7**, 404 (1959).

³ G. F. Chew, M. L. Goldberger, F. E. Low, and Y. Nambu, *Phys. Rev.* **106**, 1337 (1957).

⁴ M. L. Goldberger, M. T. Grisaru, S. W. MacDowell, and Y. Wong, *Phys. Rev.* **120**, 2250 (1960).

⁵ W. R. Frazer and J. R. Fulco, *Phys. Rev.* **117**, 1603 (1960).

⁶ A. C. Hearn, *Nuovo Cimento* **21**, 333 (1961).

⁷ Note a difference by a factor of $(sp/p)^{1/2}$ in our definition of helicity amplitudes [see Eq. (2.4)]. In our definition, the helicity amplitude for $\pi\pi$ scattering satisfies the Mandelstam representation.

II. ANALYTICITY IN $\cos\theta$ AND IN t

In the following, we assume that the analyticity properties of scattering amplitudes are correctly predicted by perturbation theory. Then it has been shown⁸ that the amplitudes of binary reactions $a+b \rightarrow c+d$ can be written as⁸

$$T = \sum_j B_j(s, t, u) N_j(p_i, \beta_i), \quad (2.1)$$

where T is related to the corresponding S -matrix element through

$$\langle p_c p_d | S - 1 | p_a p_b \rangle = (2\pi)^4 i \delta^4(p_c + p_d - p_a - p_b) \times (p_a^0 p_b^0 p_c^0 p_d^0)^{-1/2} T, \quad (2.2)$$

$B_j(s, t, u)$ is an analytic function of s , t , and u and satisfies the Mandelstam representation. $N_j(p_i, \beta_i)$ is a polynomial in the four momenta of the external particles p_i and spin parameters β_i which include Dirac matrices, fermion spinors, and polarization vectors.

The expectation value of (2.1) between helicity states in the center-of-mass system of the s channel (helicity amplitudes) can be written as

$$T_{\lambda_a \lambda_d, \lambda_b \lambda_c} = \sum_j B_j(s, t, u) [\text{Polynomial in } p^0, p^2, p^2, p^2, \sin(\theta/2) \text{ and } \cos(\theta/2)] [p^0 \text{ or } p'^0] \times \prod_{i: \text{fermion}} [p_i^0 + m_i]^{-1/2} \quad (2.3)$$

In deriving (2.3) we have made use of the fact that the helicity states are linear combinations of direct products of $\epsilon_{\lambda}^{\mu}(p)$ and $u_{\lambda}(p)$ (λ stands for helicity and μ is a Lorentz index), where

$$\begin{aligned} \epsilon_1^{\mu}(p) &= (0; \cos\theta, i, -\sin\theta)/\sqrt{2}, \\ \epsilon_0^{\mu}(p) &= (p; p^0 \sin\theta, 0, p^0 \cos\theta)/m, \\ \epsilon_{-1}^{\mu}(p) &= (0; -\cos\theta, i, \sin\theta)/\sqrt{2}, \\ u_{1/2}(p) &= [(p^0 + m) \cos(\theta/2), (p^0 + m) \sin(\theta/2), p \cos(\theta/2), p \sin(\theta/2)] [2m(p^0 + m)]^{-1/2}, \\ u_{-1/2}(p) &= [- (p^0 + m) \sin(\theta/2), (p^0 + m) \cos(\theta/2), p \sin(\theta/2), -p \cos(\theta/2)] \times [2m(p^0 + m)]^{-1/2} \end{aligned}$$

for $p^{\mu} = (p^0, p \sin\theta, 0, p \cos\theta)$. For example, the helicity $\frac{3}{2}$ state of a spin $\frac{3}{2}$ particle is $\epsilon_{1/2} u_{1/2}$.

The helicity amplitude, $T_{\lambda_a \lambda_d, \lambda_b \lambda_c}$ in (2.3) is related to the conventional helicity amplitude of Jacob and Wick² through

$$T_{\lambda_a \lambda_d, \lambda_b \lambda_c} = 2\pi (s p/p')^{1/2} f_{\lambda_a \lambda_d, \lambda_b \lambda_c}(d\sigma/d\Omega = |f|^2) \quad (2.4)$$

In (2.3), p_i^0 is the energy of the particle i in the center-of-mass system of the s channel, $p(p')$ is the momentum of the initial (final) particles in the center-of-mass

$$s = (p_a + p_b)^2 = (p_c + p_d)^2, \quad t = (p_a - p_c)^2 = (p_b - p_d)^2, \quad \text{and} \quad u = (p_a - p_d)^2 = (p_b - p_c)^2.$$

system of the s channel, θ , is the scattering angle in the center-of-mass system of the s channel, and

$$p_a^0 = (s + m_a^2 - m_b^2)/2(s^{1/2}), \quad p^2 = [s - (m_a + m_b)^2][s - (m_a - m_b)^2]/(4s), \quad (2.5)$$

and

$$\cos\theta = [2st + s^2 - s \sum_i m_i^2 + (m_a^2 - m_b^2)(m_c^2 - m_d^2)] \times (4sp p')^{-1}.$$

In (2.3), $\eta = 0$ (or 1) if $\eta_a \eta_b \eta_c \eta_d = 1$ (or -1) due to the P invariance of the strong interactions.⁹

Next let us consider the following amplitude:

$$T_{\lambda_a \lambda_d, \lambda_b \lambda_c}(\epsilon) = [\cos(\theta/2)]^{-|\lambda_a + \mu|} [\sin(\theta/2)]^{-|\lambda_b - \mu|} T_{\lambda_a \lambda_d, \lambda_b \lambda_c}(\lambda = \lambda_a - \lambda_b \text{ and } \mu = \lambda_c - \lambda_d). \quad (2.6)$$

This amplitude has been shown^{10,11} to depend on $\cos\theta$ but not on $\cos(\theta/2)$ and $\sin(\theta/2)$. Then, we can write it as

$$T_{\lambda_a \lambda_d, \lambda_b \lambda_c} = \sum_j B_j(s, t, u) [\text{Polynomial in } p^0, p^2, p^2, p^2, \text{ and } \cos\theta] [p^0 \text{ or } p'^0] \times \prod_{i: \text{fermion}} [p_i^0 + m_i]^{-1/2}. \quad (2.7)$$

Hence, we have found that for fixed real s , T' is analytic in the $\cos\theta$ plane with cuts on the real axis. Since $\cos\theta$ is linear in t , we have also found that T' is analytic in the cut t plane (for fixed s) with cuts¹ $(t_{\min}, \infty)$ and¹ $(-\infty, \sum_i m_i^2 - s - u_{\min})$. Therefore, we have only to study the analyticity properties of helicity amplitudes as functions of s .

III. ANALYTICITY IN s ; GENERAL CASE

Thus far, we have only considered the helicity amplitude for the s reaction ($T_{\lambda_a \lambda_d, \lambda_b \lambda_c}$). As has been shown in the previous section, the helicity amplitudes for the t reaction ($T_{\lambda_a \lambda_d, \lambda_b \lambda_c}$) has no kinematical singularities in s if it is divided by

$$[\cos(\theta_t/2)]^{-|\lambda_a + \mu|} [\sin(\theta_t/2)]^{-|\lambda_b - \mu|}.$$

Crossing relations between helicity amplitudes in the s channel, $T_{\lambda_a \lambda_d, \lambda_b \lambda_c}$, and helicity amplitudes in the t channel, $T_{\lambda_a \lambda_d, \lambda_b \lambda_c}$, have been proposed by Trueman and Wick¹² and by Muzinich.¹³ According to Trueman and Wick,¹⁴

⁹ η_i is defined in Ref. 2.

¹⁰ M. Gell-Mann, M. L. Goldberger, F. E. Low, E. Marx, and F. Zachariasen, Phys. Rev. 133, B145 (1964).

¹¹ F. Calogero, J. Charap, and E. Squires (to be published); F. Calogero and J. Charap, Ann. Phys. (N. Y.) 26, 44 (1964).

¹² T. L. Trueman and G. C. Wick, Ann. Phys. (N. Y.) 26, 322 (1964).

¹³ I. Muzinich (to be published).

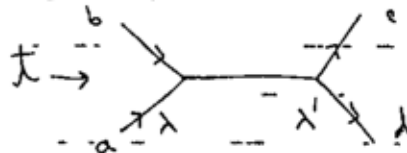
¹⁴ A proof based on Muzinich's relations can be given in the same way.

Wang '66

1. I am thinking in terms of Ling-Lie Wang at the moment. Normally, you write the t -channel helicity amplitude as a JW expansion in the t -channel, and you observe that it is nice to "pull" the half-angle factor. Thus:

$$f_{cd;ab}^{(t)} = \underbrace{\left[\left(\cos \frac{\theta_t}{2} \right)^{|\lambda+\lambda'|} \left(\sin \frac{\theta_t}{2} \right)^{|\lambda-\lambda'|} \right]}_{\equiv h(\theta_t)} \cdot \underbrace{\sum_j (z_j+1) f_{\lambda\lambda'}^j(t) \cdot \tilde{f}_{\lambda\lambda'}^j(\cos \theta_t)}_{\equiv \tilde{f}_{cd;ab}^{(t)}(t, z_t)}$$

$$\text{where } z_t = \cos \theta_t = \left(\frac{[2t + t^2 + \dots]}{2k p(t) p'(t)} \right)$$



The function f' defined in this way has only dynamical cuts in the z_t plane, the plane you use for going over to a FG projection. This same function, $f'^{(t)}$ is for the same reason analytic in s .

Here is an important point, however. In defining the function $f'^{(t)}(s, t)$, we have explicitly removed a factor which you can see is singular at the t -channel thresholds. I.e., at such thresholds, the p 's in the denominator go to zero so the factor which was removed blows up. BUT -- nevertheless -- the "cleansed" amplitude $f'^{(t)}(s, t)$ still contains ~~the~~ t -channel threshold singularities.

In other words, by exposing the half-angle factor as above, you have not exposed ^{all} the t -channel threshold singularities of the t -channel helicity amplitudes, although the half-angle factor pulled out does have ^{some} t -channel threshold singularities. The desired singularities are actually contained in the crossing matrices as well.

$$\tilde{f}^{(t)}(t, z_t) = \underbrace{h'(\theta_t) \cdot \sum d \cdot d \cdot d \cdot d}_{\text{all these factors needed to get } t\text{-channel thresh. sing.}} \cdot \underbrace{h(\theta_s) \cdot \tilde{f}^{(s)}(s, z_s)}_{\substack{\text{analytic in } t \\ \text{analytic in } t.}}$$

2. The fact that you have to include not only the $h(\theta_t)$ but also the $d \cdot d \cdot d \cdot d$ factors explains why my naïve Froissart-Gribov-Projection with spin Method did not give the correct answer!!! You start off with t -channel JW expansion; you pull off the half-angle factor thereby getting analyticity in z_t which allows the FGP. But, you never expose fully the t -channel threshold sings in the discontinuity you are integrating, so you do not "learn" the threshold behavior in this manner.

These results lead us to the following conclusions concerning the lowest order radiative corrections to muon-decay:

(1) There is no infinite renormalization of the direct term (proportional to D). There is no renormalization of C_1 .

(2) The coupling constant C_2 suffers an infinite renormalization.

When we compare the model proposed by Fronsdal and the conventional approach, we see that the situation in Fronsdal's model as far as μ decay is concerned is worse than in the conventional treatment, because of the occurrence of an infinite coupling-constant renormalization.

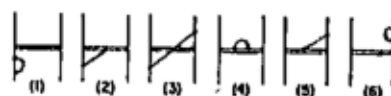
III. β DECAY OF NEUTRON

We consider the radiative corrections in lowest order to the process

$$n \rightarrow p + e^- + \bar{\nu}_e.$$

We proceed in exactly the same way as in the case of the μ decay. We state the results without going into the details of the calculation.

Fig. 5. Diagrams proportional to C_2^2 .



When we have only to deal with the direct four-fermion interaction, the lowest order radiative correction gives a divergent renormalization of the coupling constant. Also, after introduction of the charged scalar intermediate boson, we get in addition to an infinite renormalization of g an infinite renormalization of C_2 , contrary to the statement made by Fronsdal. So in this case Fronsdal's model does not give an improvement of the conventional approach either. This result contradicts Fronsdal's statement that the coupling constant renormalization is finite in this case.

ACKNOWLEDGMENTS

The author is especially indebted to Professor D. R. Yennie, who drew his attention to Fronsdal's model, for many stimulating and enlightening discussions. The assistance of Rae in the checking of some of the calculations is gratefully acknowledged.

General Method of Constructing Helicity Amplitudes Free from Kinematic Singularities and Zeros

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(Received 16 July 1965)

A simple and straightforward method to identify and remove the kinematic singularities of helicity amplitudes is constructed from the Trueman-Wick crossing relations. A set of amplitudes free of all kinematic singularities and zeros is obtained for two-particle $\rightarrow$ two-particle reactions of any spins and masses, except that for boson-fermion interactions with general mass assignments there is still a kinematic singularity left in the amplitude.

I. INTRODUCTION

IN dynamical calculations of scattering amplitudes, it is necessary to use kinematic-singularity-free amplitudes, which have only singularities of dynamical origin and satisfy the Mandelstam representation. D. N. Williams¹ has succeeded in constructing a complete set of invariant scalar amplitudes free of kinematic singularities and suitable for dynamical calculations. However, his amplitudes are not suitable for Reggeization. To Reggeize, first we have to remove all the kinematic singularities from the so-called parity-conserving helicity

amplitudes and then analytically continue their partial-wave helicity amplitudes with definite parity in the total angular momentum plane.² Therefore kinematic-singularity-free helicity amplitudes are not only suitable for dynamical calculation but also suitable for Reggeization. This is our motivation for investigating the kinematic singularities of helicity amplitudes.

Recently, Y. Hara³ has proposed a method to remove the kinematic singularities of helicity amplitudes by using perturbation field theory, with emphasis on threshold behavior of partial-wave amplitudes and crossing relations.⁴ In this paper, we develop a more

¹ D. N. Williams, Construction of Invariant Scalar Amplitudes Without Kinematical Singularities for Arbitrary-Spin Nonzero-Mass Two-Body Scattering Process, Lawrence Radiation Laboratory Report UCRL-1113 (unpublished).

² M. Gell-Mann, M. Goldberger, F. Low, E. Marx, and F. Zachariasen, Phys. Rev. 133, B145 (1964).

³ Y. Hara, Phys. Rev. 136, B507 (1964). For comparison of his results and ours, see Ref. 9.

Philosophy of the-Crossing Method of-doing Kinematic Singularities.

1. Perhaps this is best discussed by Wang. Consider $\tilde{T}^{(s)}(s,t)$, where the twiddle means that the famous factor of Θ_s has been removed. This function is analytic in t except for usual dynamical cuts. However, it does have threshold and other singularities in s .

A similar statement can then be made for $\tilde{T}^{(t)}(s,t)$.

The untwiddled amplitudes are connected by the Trueman Wick crossing relations, so that Wang's equation (III.4) really says it all. All the kinematic singularities of $\tilde{T}^{(s)}$ must be sitting in the crossing d-functions in equation (III.4).

2. Looking naively at these crossing functions, the desired factors seem to appear in the denominators if you think of $d(x) = (\cos x)^2$, so you immediately pick up things like $s_1 \pm s_2$. What you are developing here is Jackson Hite formula (26). Jackson's formula is much more efficient, I think. Obviously, since Wang does a lot of work, you cannot just think of the cosines.

3. Notice that Ling-Lie does not particularly mention any "partial wave threshold behavior" whereas Jackson and Hite use this as their physical starting point. Both methods have a certain elegance, but there is no denying: Jackson's is simpler.

4. I would note that Hara does use the threshold behavior as a starting point, but somehow the whole paper is obscure to me because there are no d-functions of Jacob and Wick. Wang claims that Hara is based somehow on perturbative field theory, though I didn't notice that when I perused Hara's paper.

5. Collins and Squires basically quotes Wang's results and thus endorses Wang's method. Jackson and Hite was not done, but does get mentioned in CS note added.

6. I find the Wang solution rather complicated, just hard to even read the results. I prefer Jackson and Hite presentation.

McK '68

Kinematic Singularities of Helicity and Transversity Amplitudes and Asymptotic Regge-Pole Contributions

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(Received 24 January 1968)

The crossing relation for transversity amplitudes is used in a simple proof of the kinematic-singularity structure of helicity amplitudes, both in the general-mass and in various particular-mass cases. A comparison with earlier results is made. The kinematic properties of helicity and transversity amplitudes are employed in the derivation of an asymptotic expression for the contribution of Regge poles to the amplitude.

1. INTRODUCTION

The kinematic-singularity structure of helicity amplitudes has been given by Hara¹ and by Wang.² Both the proofs and the final results are, however, rather complicated. At thresholds and pseudothresholds the behavior of helicity amplitudes may be related to that of partial-wave amplitudes, and Frautschi and Jones³ have shown how, at least in certain cases, this relationship may be used to obtain the powers of the singularities very simply. We show that this result is quite general.

Cohen-Tannoudji *et al.*⁴ also derived the kinematic-singularity structure of helicity amplitudes, this time from the properties of invariant amplitudes, without using crossing, and Stapp⁵ obtained similar results from basic analyticity properties. Reference 4 also employs the transversity amplitudes of Kotański⁶ in a derivation of kinematic constraints on the helicity amplitudes. Section 2 of the present paper is devoted largely to the presentation of a very simple proof of the kinematic-singularity structure of helicity amplitudes. The method employs the same physical assumptions as does Wang,² but by using transversity amplitudes⁶ and their kinematic properties⁴ great simplification is obtained.

In Sec. 3 some results on the high-energy limit of a Regge-pole contribution⁷⁻¹⁰ to the helicity amplitude are derived (for general mass), and questions about daughter contributions, conspiracy, and evasion are considered. In Sec. 4 it is shown how recourse to

transversity amplitudes may make for a simpler phenomenological analysis, in spite of the various helicity-dependent kinematic factors. Since this part of the paper is for application to inelastic processes, where data are at present rather inadequate, no apology is made for the somewhat crude approximations suggested.

The Appendix is devoted to showing the equivalence of the results of Sec. 2 with those of Wang^{2,3} and to the derivation of the kinematic-singularity structure of the helicity amplitudes in various particular-mass cases.

2. KINEMATIC-SINGULARITY STRUCTURE OF HELICITY AMPLITUDES

We review briefly the properties of helicity states in order to establish some notations and conventions. Single-particle rest states of a particle of mass m and spin s transform under rotations according to

$$R|\bar{p}\lambda\rangle = \sum_{\mu} D_{\mu\lambda}^s(R)|\bar{p}\mu\rangle, \quad \bar{p} = (m, 0). \quad (1)$$

We use the conventions of Edmonds¹¹ for the rotation matrices.

States of momentum p with $p^2 = m^2$ are defined by

$$|p\lambda\rangle = H(p)|\bar{p}\lambda\rangle, \quad (2)$$

where

$$H(p) = R(p)Z(p), \quad (3)$$

and, for¹

$$p = [m \cosh \zeta, m \sinh \zeta (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta)], \quad (4)$$

we have

$$Z(p) = \exp(-i\zeta K_z), \quad (5)$$

$$R(p) = \exp(-i\varphi J_z) \exp(-i\theta J_y) \exp(-i\varphi J_z). \quad (6)$$

In particular, for

$$q_1 = (m_1, 0, 0, q), \quad (7)$$

$$q_2 = (m_2, 0, 0, -q), \quad q \geq 0, \quad (8)$$

¹¹ A. R. Edmonds, *Angular Momentum in Quantum Mechanics* (Princeton University Press, Princeton, N.J., 1960), 2nd ed.

* Present address: Department of Applied Mathematics, The University, Liverpool 3, England.

¹ Y. Hara, Phys. Rev. 136, B507 (1964).

² L.-L. Wang, Phys. Rev. 142, 1187 (1966).

³ S. Frautschi and L. Jones, Phys. Rev. 164, 1918 (1967).

⁴ G. Cohen-Tannoudji, A. Morel, and H. Navelet, Ann. Phys. (N.Y.) 46, 239 (1968).

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McKerrell's approach to the kinematic sings.

McKerrel achieves a simplification over Wang by using the transversity amplitudes. He does not explain these objects, but it looks like it has something to do with quantizing spin transversely to the scattering plane, maybe. Cohen- etc. in their hugongous 80 page paper on this subject use such amplitudes, and they were invented by Kotanski in 1966.

The relation of transversity amplitudes to the usual JW helicity amplitudes is given in (21). Basically, the matrix u is $u(R_x(90^\circ))$. As you might guess, convoluting in these u -matrices as in (21) causes the helicity crossing matrices to go away and you get a simple phase relation. I gather that equation (27) really links the t -channel cms transversity amplitudes to the s -channel cms transversity angles. It looks like you get the usual angles in there, but they appear just as phases.

The general philosophy of identifying the kinematic sings is the same as that of Wang. You assume they are all given by the famous half-angle factor, and then you use crossing. Its just that here, crossing is so simple. For the transversity amplitudes, the 4 threshold conditions are given in (49) through (52). Then all you have to do is convert these back into the helicity amplitudes. The full answer is stated in (70), like Jackson Hite (26). The thetas indicate the parity step-ups which may be needed.

Section 3 of the paper deals with suggested forms for regge fitting, and Section 4 deals generally with phenomenology using transversity amplitudes. I note that he does show the "pole factors" in the fitting formulas. I am not sure whether McKerrell has fully considered the constraints that go along with the kine sings. Jackson and Hite showed that these in fact remove the poles.

In the appendix, results are shown to agree with those of Wang at Berkeley.

Jackson Hite is noted in a Noted Added in Proof.

Jack. Hite ('68)

Kinematic Singularities and Threshold Relations for Helicity Amplitudes*

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The kinematic singularities of two-body helicity amplitudes at thresholds and the concomitant relations among these amplitudes are discussed in a direct and elementary way, without recourse to the singularity structure of the crossing matrix. The tools are those of nonrelativistic quantum mechanics, as befits a situation where $p \rightarrow 0$, with spins combined into channel spins S and Russell-Saunders coupling of $L+S=J$. The kinematic singularities are shown to follow from a mismatch between J and L for each term in the partial-wave series. The method is applicable at pseudothresholds $(m_1-m_2)^2$ as well as normal thresholds $(m_1+m_2)^2$ with two formal changes involving an intrinsic parity and a helicity-dependent phase. The relations among the different helicity amplitudes at the thresholds are shown to result from the presence at threshold of fewer Russell-Saunders amplitudes than there are independent helicity amplitudes. The use of invariant amplitudes is shown to be an alternative which automatically yields the kinematic singularities and also the threshold relations among the helicity amplitudes. A discussion is given of dynamical exceptions to the threshold constraints, resulting from less singular than standard behavior at a threshold. The threshold relations are important constraints on the amplitudes, and must be satisfied by any realistic model. In the use of t -channel amplitudes for peripheral processes in the s -channel, the explicit imposition of all the relations at the t -channel thresholds is necessary in order to assure a differential cross section without spurious, polelike singularities in t whose variation could in some circumstances completely control the t dependence. The reactions $\pi N \rightarrow KY$ and $\pi N \rightarrow \pi' \Delta$ are used as illustrations. The latter process is especially illuminating because its t -channel amplitudes have a pole (rather than a simple inverse-square-root singularity) at the $\bar{N}\Delta$ pseudothreshold, $t=0.09$ (GeV/c)². The proximity of this point to the physical region of the s channel means that the threshold relations there are of crucial importance. The consequences of these constraints on the cross-section and decay density matrix of the Δ are discussed within the framework of the Regge-pole model. Comparison with experiment implies that the dynamics make the amplitudes for $\pi\pi' \rightarrow \bar{N}\Delta$ have less than the standard kinematic singularity at $\bar{N}\Delta$ pseudothreshold and so avoid almost all the threshold constraints. Examples are cited from the literature where use of Regge-pole formulas possessing the spurious kinematic factors has led to incorrect inferences concerning the dynamic behavior of Regge residues.

I. INTRODUCTION

THE question of kinematic singularities of S -matrix elements, that is, singularities associated with the threshold values of s , t , and u , and so depending on

the external masses, has received considerable attention in the past few years. Historically, the use of invariant

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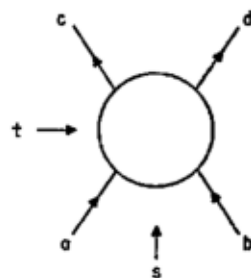
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amplitudes in combination with explicit kinematic factors made up from the momenta and the spin or Dirac operators automatically took into account the kinematic singularities of the problem. Classic examples are the A and B amplitudes in pion-nucleon scattering and the four invariant amplitudes $A_1, \dots, A_4$ in pion photoproduction.¹ The existence and construction of invariant amplitudes free of kinematic singularities for a general process has been discussed by Hepp,² Williams,³ and more recently by Fox.⁴ But with the consideration of processes involving particles of arbitrary spin, the use of helicity amplitudes⁵ became prevalent, chiefly because (a) the formalism is completely general, (b) the angular momentum and parity expansions are straightforward, and finally (c) the helicity amplitudes satisfy elegant crossing relations.^{6,7} The work of Hara⁸ and Wang⁹ solved, apart from a few details, the problem of determining the kinematic singularities of helicity amplitudes. Wang made extensive use of the crossing matrix, while Hara used partial-wave threshold behavior and the crossing matrix. Since then, other discussions of the kinematic-singularity structure of helicity amplitudes have been given from other points of view.^{10,11}

In peripheral reactions, the t -channel amplitudes often possess kinematic singularities that are sometimes close to the physical region of large s and small (negative) t . For example, in the process $ab \rightarrow cd$, illustrated in Fig. 1, the t -channel helicity amplitudes may have inverse-square-root (or worse) singularities at one or more of the points $t = (m_a + m_c)^2$, $(m_b + m_d)^2$ and $t = (m_a - m_c)^2$, $(m_b - m_d)^2$, the normal thresholds and pseudothresholds, respectively. The pseudothresholds can lie considerably closer to the physical s -channel region than dynamic singularities, such as t -channel poles. Consequently, it seems important to take proper account of such kinematic singularities in a theoretical model that is to be confronted with experiment. An attempt was made for the Regge-pole model to do this by exhibiting in the s -channel cross section all the t -channel kinematic singularities, leaving supposedly smoothly varying residue functions for phenomenological fitting.¹² This compendium of formulas for many different reactions was then to be viewed as the ultimate in Regge-pole phenomenology. Some analysis of data on the basis of these formulas has already been done.^{13,14}

FIG. 1. Diagram for the process $a+b \rightarrow c+d$.



But the structure of the formulas of Ref. 12 has been questioned, with special reference to the point $t=0$ by Lin¹⁵ and on general grounds by Stack.¹⁶

Another aspect of this general problem, recognized during the past year, is the existence of relationships between various helicity amplitudes at the kinematic thresholds. These threshold conditions or kinematic constraints are discussed by Jones¹⁷ in terms of partial-wave expansions and orbital angular momentum for the normal thresholds, by Diu and LeBellac¹⁸ in terms of the connection between invariant and helicity amplitudes, with special emphasis on $t=0$, and also by Cohen-Tannoudji, Morel and Navelet,¹⁹ and Fox.¹⁹ In Regge-pole theory with two particles of equal mass (e.g., $NN \rightarrow \pi\rho$), the appropriate pseudothreshold moves to $t=0$. There the problem of kinematic constraints is solved by "conspiracy" or "evasion,"²⁰ depending on whether or not a given trajectory needs the assistance of another trajectory in order to satisfy the conditions in a nontrivial fashion.

The main purposes of the present paper are (1) to present a unified and straightforward treatment of the kinematic singularities and threshold conditions for helicity amplitudes using orbital angular momentum, and (2) to show within the framework of the Regge-pole model how to incorporate properly the kinematic structure into the cross sections and density matrices. We show that the general results of Refs. 9 and 10 are obtainable by considerations of the thresholds alone, without reference to the crossing matrix. Our use of orbital angular momentum parallels the original work of Hara,⁸ but we are careful to distinguish between normal thresholds and pseudothresholds. Frautschi and Jones¹⁴ have also used orbital angular momentum arguments to verify and interpret the singularity structure in a number of specific examples.

The end results of the proper incorporation of the kinematic structure into the cross section and density matrices are phenomenological formulas very different from those of Wang¹² in that they conform to the requirements of Lin¹⁵ and Stack¹⁶ and possess no t -channel kinematic-singularity factors. The somewhat

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D. Regge Theory

1. Sing Structure and Residue Behavior

2. small t

3. choosing nonsense

note added in proof.

I. Introduction. Work of other people is reviewed briefly:

Hepp, Williams, Fox: construction of invariant amplitudes in special cases

Jacob, Wick: the general theory

Wang: used crossing matrices

Hara: partial wave threshold behavior and crossing matrices

Jones: threshold conditions = kinematic constraints (Note!)

Diu and LeBellac: k.c.

Cohen-T, Morel, Navelet, Fox: more on k.c.

McKerrell: has a paper on this whole subject, note added in proof.

Barmawi: regge theory in LS coupling.

Since pseudothresholds lie very close to physical scattering region, they should be taken into account! They will obtain previous results without going to crossing matrices.

II. Notation and Basic Concepts. They will base their arguments on non-rel QM concepts which are valid at threshold etc. They are the first to show that these simple ideas are enough to explain everything. Since regge theory in t-channel is of interest, that is their choice of channels here.

A. Notation. Figure 2 says it all. Particle helicities are not given. T_N etc are clearly defined. Certain nice kinematic relations are noted.

B. No spine. The angular momentum behavior is taken as a "law of nature". They say it can be shown from FGP, but they will later not show this in the spin case, but rather will again take it as law of nature. For no spins, they show how threshold behavior exactly compensates P_L function to give no kin sing in spinless scattering.

C. Outline of Method. They will do standard non-rel LS coupling. I.e., no need for the complicated relativistic exactness of McKerrell's LS states... Everything will boil down to sort of a counting problem.

D. Helicity Amplitudes and PW expansions. We open with the usual JW expansion and show the kin factor in the d-functions. Partial wave amplitudes of definite naturality are defined in the usual way (see ms) in the helicity formalism. Putting this back into the JW expansion, you get can define full amplitudes of definite naturality. These full amplitudes appearing in (15) have the kin sing factor cleaned out; i.e., these amplitudes are clean in z. Equation (16) shows the factor:

In order to express the full naturality amps in terms of partial wave naturality amps, you have to shuffle around the d-functions. This gives the e-functions of of GGLMZ. No big deal. Since these functions are shuffled d-functions with the kin factor removed, they have large z-dependence as shown in (18). Fine. No complaints.

E. Pseudothresholds are a little different. Consider the usual two-body helicity state. As you go down to the pseudothreshold, the state of the lighter particle m_1 has energy $E = -m$, which is certainly a bit strange. I.e., you need a complex LT to take your single particle helicity state to this point. This thing is just a phase. So, why not redefine amplitude to incorporate that phase as in (19). The $(-1)^S$ is just arbitrary, keeps exponent integral. Called a pseudoamplitude.

Second complication is in the parity relation of amplitudes when you are at pseudothreshold. I am not surprised to find that the lighter particle has an "effective" parity = parity times $(-1)^{2S}$. These, then, are the two important changes.

Implications: (1) different threshold power of T_p versus T_N for fermion amps. In general, this affects the constraints too for both bosons and fermions.

Threshold behavior of partial wave amps of definite naturality is then reduced to a "counting problem" in (20). Recall: law of nature.

III. Kinematic Singularities.

A. An example PI PI to N DELTA: Here is an example where you sit down and compute up the various minimum values of L allowed by angular momentum and parity considerations. For the fermion side of the system, the lighter fermion has its intrinsic parity effectively reversed at the pseudothreshold, so the entries in the table are not the same for N versus P.

I have done out this example fully on an attached sheet. Looks good.

B. Physical Boundary: powers of t. The "half-angle factor" given in (16) has certain singularities at $t=0$. These are shown above equation (24). For all masses different, the helicity amplitudes (full, left side of (10)) do not have any such singularities, since in physical region. Therefore, from (15), the definite naturality full amplitudes do have $t=0$ sings. These are shown in (24).

On the other hand, you can show that the definite naturality functions (15) have no singularities at Kibble $\phi = 0$, and therefore the regular helicity amplitudes 10, do have singularities at $\phi = 0$. Thus, to write a correct regular helicity amplitude, you should include this fact to make the thing smooth at $\phi=0$, as done in (25).

Both these effects have a common cause: the half-angle factor.

C. the general formula. For low values of J, as we have seen, there is "erratic behavior" of the minimum L values. When J is larger than all channel spins, a standard behavior sets in. For these low J values, the difference between the general form and the exact result is full zeros. Thus, the singularities are correctly given by the general formula. See footnote 30.

The general formula is given in (26). Clearly, the minimum value of L goes with the maximum value of the channel spin which is $s_1 + s_2$, for example. However, parity may increase the minimum L by one unit, as we have seen in the example. This fact is shown in equation (27) and the preceding equation.

Note: general formula (26) is not same thing as (20). Equation (20) is showing the threshold behavior of the definite-naturality partial wave amplitudes. I think the general form of (20) in the general (large J) case is this:

$$\begin{aligned} L_N &= J + s_1 - s_2 + \frac{1}{2} [1 - \eta_1 \eta_2 (-1)^{s_1+s_2} \eta] \quad \text{for } J \geq s_1+s_2 \\ L_P &= J - s_1 - s_2 + \frac{1}{2} [1 - \eta_1 \eta_2 (-1)^{s_1+s_2} \eta] \quad \text{for } J \geq s_1+s_2 \\ L'_N &= J - s_3 - s_4 + \frac{1}{2} [1 - \eta_3 \eta_4 (-1)^{s_3+s_4} \eta] \quad \text{for } J \geq s_3+s_4 \\ L'_P &= J - s_3 - s_4 + \frac{1}{2} [1 - \eta_3 \eta_4 (-1)^{s_3+s_4} \eta] \quad \text{for } J \geq s_3+s_4 \end{aligned}$$

These are the general minimum L values above the erratic region. Note that the pseudothreshold intrinsic effect is already built into these formulas. Notice that these things have a very intuitive form, aside from the parity factor which may step up $L_{\min}$ by one unit. The total J. minum the maximum spin. What could be more easy to remember!!!!

Equation (26) is showing not the partial wave threshold behavior, but rather the kin sings in the full (but definite naturality) amplitudes. As noted already, this is due to the "mismatch". As you can see from equation (17) and (18), the GGLM e-functions have as their leading power z^{j-m} where m is usual max of $|M_1|, |M_2|$. The "mismatch" is between this power $(j-m)$ and the minimum L listed above. Thus:

$$F_{\lambda_3 \lambda_4 \lambda_1 \lambda_2}(s, z) = (T_N)^{-\alpha_N} (T_P)^{-\alpha_P} (T_N')^{-\alpha_N'} (T_P')^{-\alpha_P'} (\sqrt{s})^{-m} \frac{1}{z} \frac{1}{1-z}$$

x (clean).

where:

$$\begin{aligned} \alpha_N &= (J - m - L_N) \\ \alpha_P &= (J - m - L_P) \\ \alpha_N' &= (J - m - L_N') \\ \alpha_P' &= (J - m - L_P') \end{aligned}$$

$m = \max(|\lambda_1|, |\lambda_2|)$

Comments: Notice that the threshold behavior of the partial wave amplitudes does not depend on the helicity of the particles involved, only on their spin. However, one could view this as $\alpha_{\min} = J - m$ + maximum helicities, modulo the parity factor which might step you up one unit, and modulo the low J erratic behavior.

In contrast, the "kinematic singularities" of the full amplitude do depend on the value $m = \max(|\lambda_1|, |\lambda_2|)$.

Comparison with crossing method results: put these comments somewhere else.

IV. Threshold Relations (also called Kinematic Constraints) At the t-channel thresholds, the t-channel helicity amplitudes have the kinematic singularities we have already noticed due to the mismatch of threshold behavior of the partial wave amplitudes, versus the behavior of the group functions.

The t-channel helicity amplitudes at the t-channel thresholds have also certain linear relations called kinematic constraints, an example of which is shown in (40). Often, both concepts must be dealt with together. Toller has shown how the two ideas are quite trivially connected as an inversion problem.

Notice that the t-channel helicity amplitudes do not have s-channel thresholds, etc and etc. I.e., the t-channel helicity amplitudes do not have s-channel kinematic singularities. How do I know this?

I think the reason is this: look at the t-channel JW expansion. All the s-dependence appears inside $\cos \theta = z$. But from (5), z is linear in s . Each term in the JW expansion is a polynomial in z , therefore in s . Thus, the t-channel helicity amplitudes are analytic in s . However, this little proof only works where the t-channel JW expansion converges, which is only negative s . However, I think the argument could be extended by replacing the JW with a SW representation. Then again, I'm not sure of this.

I don't know how one knows this.

I have skipped reading this section for now.

V. Cross Sections and Decays.

As we well know, in the large s physical s-channel, you can write the DCS in terms of the t-channel helicity amplitudes, due to the orthogonality of the helicity crossing matrices in the physical s-channel. But these t-channel helicity amplitudes have the kinematic singularities already noted. Thus, these kine sings should be taken into account when writing regge formulas, eg, as in equation (57). I.e., you will see the usual powers of t and "pole-like factors" in the denominator. I have seen this many times in regge fitting formulas. Fine.

But, $\rightarrow$ (over)

First, consider this fact: the t-channel helicity amplitudes have kinematic singularities at thresholds in the variable t (t-channel thresholds), because the arguments in the half-angle factor go infinite at these points. Going infinite takes you to the branch point in these functions.

However, a finite value of the variable s is never going to make one of the half-angle factor arguments infinite.

Therefore, the t-channel helicity amplitudes do not have the s-channel threshold singularities, and vice versa. The t-channel helicity amplitudes have only the t-channel threshold singularities.

Knowing this fact, how do we justify the equation (55)? The answer seems to be that this replacement is only valid in the physical s-channel, so you would never "hit" upon the t-channel threshold singularities which are present in the right side of (55) outside the physical s-channel.

If you try to make a more general version of (55) which is valid everywhere, it would have to show how the singularities work. The crossing matrices would have to exhibit singularities in the right way to make things work.

So this is why it is permissible to have "pole like" factors as in equation (57). You would never be allowed to continue this equation to positive t , because those poles do not exist in the t-channel. Equations like (57) are only applicable at negative t . Of course, that is where they are intended.

It turns out that the pole-like factors will be eliminated if you incorporate the kinematic constraints as well as the singularities. Stack thought this up.

Stop reading at this point, on to conclusions.

VI. Summary and Conclusions. - A good summary. Note on some dynamical exceptions, eg, if some channel spin is excluded by isospin or something like that.

They do not claim to do this for the first time. Claim is that the use of non-rel QM is a better way to go than via the crossing matrices.

A comment near the end about M-functions being nicer, but no mention of Toller at all. The obvious reason is that Toller's first mention of these functions is later in 1968 in the $t=0$ paper, and Cozensa papers.

Appendix D: Regge theory. - Considering all of the above, what should a regge pole term look like? They show the kinematic singularities factor called $K(t)$ here, and they assume "chooses sense" which I would always do. End result is (D.7).

A $t_{\min}$ effect factor is mentioned which does something at very small t .

Finally, the "chooses nonsense" mechanism (called Gell-Mann) is defined and they show what happens if you use it.

Section III.A: An example

① Since $\pi\pi$ system on left, communicating systems must be

$$J=L \quad P=(-1)^L \Rightarrow P=(-1)^J \Rightarrow P(-1)^J = + \Rightarrow \pi = +$$

So you can only couple to "natural systems". This defines the "natural parity sequence" $0^+, 1^-, 2^+, 3^-, 4^+, \dots$

J^P	S	$(\text{intrinsic } N)$	$(\text{intrinsic } P)$	$(\text{min } L^N)$	$(\text{min } L^P)$
0^+	0	+	+	0	0
1^-	0	+	+	1	1
2^+	0	+	+	2	2
3^-	0	+	+	3	3

② Now do the $N\Delta$ system

J^P	S	$(\text{intrinsic } N')$	$(\text{min } L^N)$	$(\text{intrinsic } P')$	$(\text{min } L^P)$
0^+	1	-	1	[+ $\emptyset, X, ? \text{ nada}$]	
	2	-	$\emptyset, X, 2, 3 \text{ nada}$	+	2
1^-	1	-	0	+	1
	2	-	$\emptyset, X, 2$	+	1
2^+	1	-	1	+	$\emptyset, X, 2$
	2	-	$\emptyset, 1$	+	0
3^-	1	-	$\emptyset, X, 3$	+	$\emptyset, X, 1, 2, 3$
	2	-	$\emptyset, X, 2$	+	$\emptyset, 1$

In the chart I have circled the "winners". You consider all possible S systems, and pick the lowest L you get throughout. Fine, I agree, all works good.

About "L-S coupling"

① First, combine your spins: $|s_1 s_2 m_1 m_2\rangle \equiv |s_1 m_1\rangle \otimes |s_2 m_2\rangle$

$$|SM_{s,s_2}\rangle = \sum_{m_1 m_2} \langle s_1 s_2 m_1 m_2 | SM_{s,s_2} \rangle |s_1 s_2 m_1 m_2\rangle \equiv |SM_s\rangle$$

② Second, mix in your L : $|LS m_L m_S\rangle \equiv |L m_L\rangle \otimes |S m_S\rangle$

$$|JM LS\rangle = \sum_{m_L m_S} \langle LS m_L m_S | JM \rangle |LS m_L m_S\rangle$$

③ Combine both steps together:

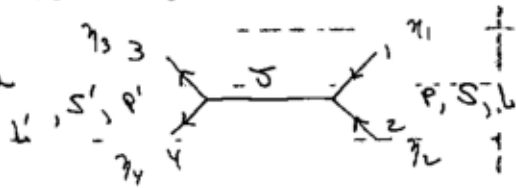
$$|JM LS\rangle = \sum_{m_L m_S} \sum_{m_1 m_2} \langle LS m_L m_S | JM \rangle \langle s_1 s_2 m_1 m_2 | SM_s \rangle |s_1 s_2 m_1 m_2\rangle$$

④ Ambiguity! s_1, s_2 are not at rest!

see McKersell - NC34, 1289 (1964).

pre-parity-regge theory with L-S coupling instead of helicity stuff.

① Picture is



What is a two-particle state in L-S?

$$|J M L S; n n_z\rangle$$

$$\langle J M L' S'; n_3 n_4 |$$

Jackson on kin. sing.

① He does his p.w. expansion in the t-channel! Good.

② Spinless analysis: $A(t, \theta_t) = \sum_L (2L+1) A_L(t) P_L(\cos \theta_t)$

But we know that $A_L(t) \sim (pp')^L \tilde{A}_L(t)$ — clean at PT, NT.
Then:

$$P_L(\cos \theta_t) = (\cos \theta_t)^L + \text{lower powers.}$$

$$\text{Then } A(t, \theta_t) = \sum_L (2L+1) \tilde{A}_L(t) (pp')^L (\cos \theta_t)^L [1 + O(\cos^2 \theta_t)].$$

Note "match". $(pp' \cos \theta_t)$ is analytic in s and t (neglect $t=0$).

Then claim: no "kin sing" in $A(t, \theta_t)$ in s or t .

③ Suppose we had $(pp')^L (\cos \theta_t)^J \sim (pp')^{L-J}$

Then $A(t, \theta_t) \sim (pp')^{L-J}$ and has singularities in t at $t_r, t_p, t=0$. This is the famous "mismatch". But we does not say sing at $\cos \theta_t = \pm 1$.

④ Now go to helicity case:

$$f_{\lambda_3 \lambda_4; \lambda_1 \lambda_2}(t, \theta_t) = \sum_J (2J+1) \langle \lambda_3 \lambda_4 | F^J(t) | \lambda_1 \lambda_2 \rangle d^J_{\lambda_3 \lambda_4}(\theta_t)$$

Go to natural parity: (big deal).

$$F^{\pi_2}_{\lambda_3 \lambda_4; \lambda_1 \lambda_2} = \langle \lambda_3 \lambda_4 | F^J(t) | \lambda_1 \lambda_2 \rangle + \pi \pi_1 \pi_2 \langle \lambda_3 \lambda_4 | F^J | -\lambda_1 -\lambda_2 \rangle$$

⑤ Then, JDT quotes reference "26" to get

$$F^{\pi_2}_{\lambda_3 \lambda_4; \lambda_1 \lambda_2}(t, \theta_t) = \xi(\theta_t) f_{\lambda_3 \lambda_4; \lambda_1 \lambda_2}(t, \theta_t) + \text{second term.}$$

↑
multiply by ξ to kill off "extra" $z = \pm 1$..

So, ref 26 and JDT extract these factors. OK, you can do that, but not clear that F is clean at $z = \pm 1$.

Partial Wave Analysis with LS-coupling.

1. I have never really done this before. Read about it in Werle. I think with the material of McKerrell I should be able to quickly run through the theory. I am interested in thresholds in this scheme.

2. How should we start off? Perhaps we should start with an S-matrix element in the two particle canonical states:

$$S^{-1} \langle q_3 v_3 q_4 v_4 | \hat{S} | q_1 v_1 q_2 v_2 \rangle_c$$

Lets say that initial and final are cms systems. Perhaps q_1 and q_2 are directed along the z axis, but the other system is rotated at some angle. So pull the rotation to get

$$|q_1 v_1 q_2 v_2\rangle_c = | \hat{q}_1 v_1 \hat{q}_2 v_2 \rangle_c$$

$$|q_3 v_3 q_4 v_4\rangle_c = R \cdot | \hat{q}_3 v_3 \hat{q}_4 v_4 \rangle_c$$

$$S_{\text{top}} = \langle \hat{q}_3 v_3 \hat{q}_4 v_4 | R^{-1} \cdot T \cdot | \hat{q}_1 v_1 \hat{q}_2 v_2 \rangle_c$$

Now the idea is simply to expand the two state systems into McKerrell's LS states.

③ From (4.13),

$$|\tilde{p}^m [w_j] l s\rangle = \sqrt{\frac{q}{2}} \sqrt{\frac{2l+1}{4\pi}} \sum_{l_3 s_3} C(l s j l_3 s_3 m) |\tilde{p} l l_3 s s_3\rangle$$

④ Set $L=1$ in (4.10):

$$|\tilde{p}^m [w_j] l s\rangle = \sqrt{\frac{q}{2}} \sqrt{\frac{2l+1}{4\pi}} \sum_{\substack{l_3 s_3 \\ v_1 v_2}} C(l s j l_3 s_3 m) D_{0 l_3}^l(0 0 \phi)$$

$$C(s_1 s_2 s v_1 v_2 s_3) R^L(-\phi, -0, 0) | \hat{q}_1 v_1 \hat{q}_2 v_2 \rangle_c$$

③ Try (4.14) instead: again, set $L(p) = 1$, $p = \tilde{p}$. Then:

$$| \hat{q}_1 v_1 \hat{q}_2 v_2 \rangle_c = \sum_{j l s} |\tilde{p}^m [w_j] l s\rangle \langle \tilde{p}^m [w_j] l s | \hat{q}_1 v_1 \hat{q}_2 v_2 \rangle_c$$

$$2\sqrt{w} \lambda^{-\frac{1}{2}} \sqrt{\frac{2l+1}{4\pi}} 2W \delta^{(ext)} \sum_{l_3 s_3} C(l s j l_3 s_3 m) D_{0 l_3}^l(0 0 \phi) C(s_1 s_2 s v_1 v_2 s_3)$$

$$\text{So: } |q_1, v_1, q_2, v_2\rangle_e = \sum_{j, l_3, s_3} \left[4\omega^{3/2} \lambda^{1/4} \sqrt{\frac{2l+1}{4\pi}} \right] \delta^4(\text{ext}) \times C(l, s, s_3, m) C(s_1, s_2, s_3, v_1, v_2, s_3) |\tilde{p}^m[\omega_j] l_s\rangle$$

Again:

$$|q_1, v_1, q_2, v_2\rangle_e = \sum_{j, l_3, s_3} \left[4\omega^{3/2} \lambda^{1/4} \sqrt{\frac{2l+1}{4\pi}} \right] \delta^4(\text{ext}) \times C(l, s, s_3, m) C(s_1, s_2, s_3, v_1, v_2, s_3) |\tilde{p}^m[\omega_j] l_s\rangle$$

note: $s_3 = v_1 + v_2$ removed s_3 sum.
 $m = l_3 + s_3$ is already "removed".

④ Now insert this twice:

$$\begin{aligned} \langle q_3, v_3, q_4, v_4 | R^\dagger T | q_1, v_1, q_2, v_2 \rangle_e &= \sum_{j, l_3, s_3} \left[4\omega^{3/2} \lambda^{1/4} \sqrt{\frac{2l+1}{4\pi}} \right] \delta^4(\text{ext}) C(l, s, s_3, m) C(s_1, s_2, s_3, v_1, v_2, s_3) \\ &\cdot \sum_{j', l'_3, s'_3} \left[4\omega'^{3/2} \lambda'^{1/4} \sqrt{\frac{2l'+1}{4\pi}} \right] \delta^4(\text{ext}) C(l', s', s'_3, m') C(s'_1, s'_2, s'_3, v'_1, v'_2, s'_3) \\ &\cdot \langle \tilde{p}^{m'}[\omega_{j'}] l'_{s'} | R^\dagger T | \tilde{p}^m[\omega_j] l_s \rangle \\ &\cdot \delta_{jj'} \delta_{ss'} \langle \tilde{p}^{m'}[\omega_{j'}] l'_{s'} | R^\dagger T | \tilde{p}^m[\omega_j] l_s \rangle \end{aligned}$$

But $m = l_3 + s_3$ $m' = l'_3 + s'_3$ so $\delta_{m, m'}$ kills $\sum_{l'_3}$.

Same here:

$$R^\dagger |\tilde{p}^m[\omega_j] l_s\rangle = \sum_{m''} D_{m''m}^j(R^\dagger) |\tilde{p}^{m''}[\omega_j] l_s\rangle$$

$$\begin{aligned} \text{So: } \text{---} &= \delta_{jj'} \sum_{m''} D_{m''m}^j(R^\dagger) \langle \tilde{p}^{m'}[\omega_{j'}] l'_{s'} | T | \tilde{p}^{m''}[\omega_j] l_s \rangle \\ &= D_{m''m}^j(R^\dagger) \delta_{jj'} \langle \tilde{p}^{m'}[\omega_{j'}] l'_{s'} | T | \tilde{p}^{m''}[\omega_j] l_s \rangle \end{aligned}$$

Add $\sum_{m', m''}$ if you like:

So:

$$\langle q_3 v_3 q_4 v_4 | T | q_1 v_1 q_2 v_2 \rangle_c$$

$$= \sum_{\substack{jls'l's' \\ m m'}} \left[4w^{j/2} \chi^{\frac{1}{2}} \sqrt{\frac{2l+1}{4\pi}} \right] \left[4w^{j/2} \chi^{\frac{1}{2}} \sqrt{\frac{2l'+1}{4\pi}} \right] \cdot \delta^y_{(ext)} \delta^y_{(ext)}$$

$$\times C(l s_j l s_3 m) \frac{C(s_1 s_2 s_1 v_1 s_3)}{C(l' s'_j l' s'_3 m')} \frac{C(s_3 s_4 s_3 v_3 v_4 s'_3)}{C(s_3 s_4 s_3 v_3 v_4 s'_3)}$$

$$\times \underline{D_{m'm}^{j-1}(\tilde{R})} \langle \tilde{p}'_{m'} [w_j] l' s' | T | \tilde{p}'_{m'} [w_j] l s \rangle$$

If $I=1$, get $\delta^y \cdot \delta^y \cdot \delta^y(0)$ on right & if you integrated $\int d\tilde{p}$ on both sides,

Here is the LS expansion. For each value of J , you must sum over all reasonable L and s .
I don't understand why there are δ delta functions on the right.

MCK (68)

On the Elimination of t Channel Threshold Singularities from the Differential Cross-Section.

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(ricevuto il 28 Maggio 1968)

It was pointed out by JACKSON and HITE⁽¹⁾ that the kinematic singularities⁽²⁻⁵⁾ of t -channel helicity amplitudes at thresholds and pseudo-thresholds in t do not lead to corresponding singularities in the differential cross-section. In ref. (1) certain reactions are studied in detail to demonstrate the cancellation of the pole, and new formulae are given for the differential cross-section to replace those of ref. (6).

In this letter the relation between helicity and transversity amplitudes⁽⁷⁾ is employed to prove, in the case of general mass and arbitrary spin, that the square of the modulus of a transversity amplitude, with the correct interpretation of its continuation to the neighbourhood of thresholds and pseudo-thresholds in t , has no singularity there. Explicit formulae are given which allow a parametrization truly independent of the kinematics.

We shall use much of the notation and some of the results of ref. (8), which, with its numbered equations, will be referred to in the following as M. Any explanations necessary for an understanding of the argument will, however, be given.

The t -channel process considered is $a+b \rightarrow c+d$, and it will be assumed that this channel has integral spin. It follows from the results of M that the contribution of a Regge pole of trajectory α to the t -channel helicity amplitude for negative t and s asymptotically large may be written

$$(1) \quad f_{cdab}^h(s, t) = K_{cdab}(t) \gamma_{cdab}(t) R(s, t).$$

(1) J. D. JACKSON and G. M. HITE: *Kinematic singularities and threshold relations for helicity amplitudes*, Berkeley preprint, UCRL-17959, November 1967; *Phys. Rev.* (to be published).

(2) Y. HARA: *Phys. Rev.*, **136**, B 507 (1964).

(3) L.-L. WANG: *Phys. Rev.*, **142**, 1187 (1966).

(4) G. COHEN-TANNOUDJI, A. MOREL and H. NAVELET: *Kinematical singularities, crossing matrix and kinematical constraints for two-body helicity amplitudes*, Saclay preprint, April 1967; *Ann. of Phys.* (to be published).

(5) A. MCKERRELL: *Kinematic singularities of helicity and transversity amplitudes and asymptotic Regge-pole contributions*, Cambridge preprint, DAMTP 68/1, November 1967; *Journ. Math. Phys.* (to be published).

(6) L.-L. WANG: *Phys. Rev.*, **153**, 1664 (1967).

(7) A. KOTANAWI: *Acta Phys. Polon.*, **29**, 699 (1966).

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question. The method of Wang makes essential use of an extra assumption. This assumption is that if certain singular kinematic functions with zeros are divided out of the helicity amplitude, then the resulting function has no kinematic singularities in certain variables. Any such singularities necessarily arise from a failure of a generalized Legendre expansion to converge, and it is asserted that this is a dynamical question. While this seems reasonable, it is not absolutely convincing; since we do not yet fully understand the dynamics of elementary-particle systems. Thus it is not absolutely inconceivable that a kinematic singularity could cause the series to diverge. In any case the question arises whether this assumption is a dynamical assumption that goes beyond the basic analyticity properties used by Hepp and Williams. We find that this extra assumption is not really needed.

An assumption essentially equivalent to the extra assumption of Wang is made also by Hara, who relies heavily on perturbation theory.

As in the work of Hara and Wang, only four-particle reactions are considered. It is further assumed that the two initial particles have unequal masses, and that the two final particles have unequal masses. The passage to equal-mass limits has been discussed by Wang.

II. SINGULARITIES AT $s=0$

The helicity amplitude is given by

$$H = RS, \quad (2.1)$$

where S is the S matrix and R is a product of rotation operators R_j , one for each final particle. The center-of-mass frame is used and the z axis is taken to lie along the direction of one of the incoming particles. The helicity λ_1 of this particle is just the z component of its spin. The other incoming particle has helicity λ_2 , which is minus the z component of its spin. The two final particles move in the x - z plane, the first moving in the direction θ , the second in the opposite direction.

The two rotations R_j act on the spin spaces of the two final particles, and each gives a rotation through angle θ . Specifically, for either final particle j , one has

$$R_j = \exp(i\theta J_{jy}), \quad (2.2)$$

where J_{jy} is the y component of the spin vector J_j that acts in the spin space of the final particle j .

The S matrix is related to the M function by⁵

$$S = BM, \quad (2.3)$$

where B is a product of boosts, one for each particle. We work in the representation where all indices are either lower dotted or lower undotted. For a particle associated with a lower undotted index the boosts for

particle j is expressed in terms of its covariant velocity $v_j = \mathbf{p}_j/m_j$ by

$$B_j = D^J[(v_j \cdot \sigma)^{1/2}] \quad (2.4)$$

which acts on M by multiplication from the left. For a particle associated with a lower dotted index the boost is given by the same function of its velocity acting on M by multiplication from the right. For any unitary 2-by-2 matrix A the matrix $D^J[A]$ is just the $(2J+1)$ -by- $(2J+1)$ matrix that represents the rotation specified by A in the $(2J+1)$ -dimensional irreducible representation of the rotation group.⁶ The matrix elements of the $D^J[A]$ are homogeneous polynomials in the matrix elements of A , and $D^J[A]$ for general A is defined by analytic continuation.

Consider first a system consisting of one spin- $\frac{1}{2}$ particle and one spin-zero particle. Then the boosts B_j take the form

$$\begin{aligned} B_j &= D^{1/2}[(v_j \cdot \sigma)^{1/2}] \\ &= (v_j \cdot \sigma)^{1/2} \\ &= (v_j^0 + 1 - \mathbf{v}_j \cdot \boldsymbol{\sigma}) / (2v_j^0 + 2)^{1/2} \\ &= \frac{1}{\sqrt{2}} [(v_j^0 + 1)^{1/2} - (v_j^0 - 1)^{1/2} \boldsymbol{\sigma} \cdot \mathbf{v}_j]. \end{aligned} \quad (2.5)$$

The rotation matrix R_j has elements

$$R_{\lambda\mu} = \pm (\cos \frac{1}{2}\theta)^{|\lambda+\mu|} (\sin \frac{1}{2}\theta)^{|\lambda-\mu|}, \quad (2.6)$$

where the sign is minus for $\lambda - \mu = -1$ and plus otherwise.

The basic analyticity assumption is that the M functions are analytic functions of the components of the momentum vectors, except at dynamical singularities.⁵ It follows from this, and Lorentz invariance, that M_i can be written in the form^{7,8}

$$M = a v_j \cdot \sigma + b v_i \cdot \sigma + c w \cdot \sigma + d v_j \cdot \sigma w \cdot \sigma, \quad (2.7)$$

where the coefficients a , b , c , and d are meromorphic functions of the scalar invariants with, at most, simple poles at $\phi=0$. Here ϕ is given by^{7,8}

$$\phi = (stu - sa^2 - tb^2 - uc^2 + 2abc), \quad (2.8)$$

where s , t , and u are the Mandelstam variables and

$$\begin{aligned} a &= \frac{1}{2}(m_1^2 + m_2^2 - m_3^2 - m_4^2), \\ b &= \frac{1}{2}(m_1^2 + m_3^2 - m_2^2 - m_4^2), \\ c &= \frac{1}{2}(m_1^2 + m_4^2 - m_2^2 - m_3^2). \end{aligned}$$

The surface $\phi=0$, which is the set of points where the rank of the gram determinant is less than three, includes the boundaries of the physical regions. The possibility of

⁵ Henry P. Stapp, Phys. Rev. 125, 2139 (1962); also in *Proceedings of the Seminar on High-Energy Physics and Elementary Particles, Trieste, 1965* (International Atomic Energy Agency, Vienna, 1965), pp. 3 to 40.

⁶ E. Wigner, *Group Theory and Its Application to the Quantum Mechanics of Atomic Spectra* (Academic Press Inc., New York, 1959).

⁷ A. O. Barut, I. Muzinich, and D. N. Williams, Phys. Rev. 130, 442 (1963).

⁸ Reference 5, p. 2156.

Analyticity Properties of Helicity Amplitudes*

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The analytic structure of helicity amplitudes is derived from basic analyticity properties. Previous derivations relied on crossing properties and extra assumptions.

I. INTRODUCTION

THE problem of expressing scattering amplitudes in terms of functions of scalar invariants without introducing extra singularities has been solved by Hepp¹ and Williams.² Their solution has a form that is not convenient, however, for many practical purposes. This is in part because it involves a reduction of the amplitude to its irreducible components. Though such a reduction is in principle straightforward, it is in practice cumbersome. Moreover, the irreducible components, though the natural mathematical quantities, are not nice physically. For example, the irreducible components mix different parity eigenstates. This means that the condition of invariance under space reflection does not lead to any simple reduction in the number of irreducible components. It leads rather to complicated relations between different irreducible components. For this reason, among others, the elegant results of Hepp and Williams have had little or no practical application.

For many purposes the most convenient form of the scattering amplitude is in terms of helicity amplitudes. The helicity amplitudes, like any others, become functions of scalar invariants when evaluated in the center-of-mass frame. This is because the components of the

momentum vectors become functions of scalar invariants. However, the functions that express these components in terms of the invariants have numerous kinematic singularities; which the amplitude itself is expected to inherit. Also, the various rotations and boosts needed to define the helicity amplitudes have kinematic singularities. Thus the analytic structure of the helicity amplitudes, considered as functions of the scalar invariants, would be expected to be very complicated. It turns out, however, that most of the singularities cancel, leaving the helicity amplitudes with reasonably simple analyticity properties. The purpose of the present paper is to show this.

The result is not new, having been obtained already by Hara³ and Wang.⁴ Their method is, however, circuitous. Rather than starting directly from the basic momentum-space analyticity properties, or equivalently from the analyticity properties deduced by Hepp and Williams, they base their conclusions on consistency with well-known crossing relations for helicity amplitudes. Since the crossing relations are themselves derived from the basic momentum-space analyticity properties, their procedure is evidently permissible. But it is roundabout. One would expect it to be simpler to work directly with the basic properties, and this is indeed the case.

There is a second reason for reconsidering the

* This work was done under the auspices of the U. S. Atomic Energy Commission.

¹ Klaus Hepp, *Helv. Phys. Acta* 37, 55 (1964).

² David N. Williams, Lawrence Radiation Laboratory Report No. UCRL-11113, 1963 (unpublished).

³ Yasuo Hara, *Phys. Rev.* 136, B507 (1964).

⁴ Ling-Lie Chau Wang, *Phys. Rev.* 142, 1187 (1966).

read July 29, 1977

Analytic Properties of Helicity Amplitudes

1. Henry observes that the fancy invariants scheme of Williams and everybody simply does not help much in practice, people are still going to use the helicity amplitudes. What singularities do they have? Hara and Wang answered this using crossing stuff. Since crossing follows from p-analyticity, Henry thinks there should be a faster way to show this. He seems to eliminate some assumption used by Wang and Hara.

2. The main content of the paper is to examine the various sing's by comparing helicity amplitudes to the M-functions. Henry only considers 4-particle amplitudes in the xz plane. Unequal masses all round are assumed. As usual, Henry treats general spin particle as combination of spin- $\frac{1}{2}$ with Clebsh's.

3. Sings come in the usual varieties:

- A. $s=0$. Turns out things are analytic as long as $\phi \neq 0$.
- B. threshold and pseudothreshold appear in this paper as $v_1^0 = \pm 1$ sings.
- C. $\phi = 0$. (Kiss)

4. I have not read this paper so much for the results as for studying a little Henry's M-function notation, since we are about to discuss this. So, for this purpose only the first section of the paper is relevant. We study the various equations:

(2.1). $H = RS$. I believe that Henry always uses S when he speaks of "canonical amplitudes" of McKerrell. Then the R here are the regular rotations which convert you to helicity amplitudes. So I would say: $S = T_C$ and $H = T_H$.

(2.3) $S = BM$. Here the B are just the $L(p)$ of McKerrell which convert T_C to M. I see no problem here.

(2.5) gave me a problem because I did algebra wrong, now OK. Point is the square root is gone.

(2.7) says that for the simple case of spin-0 and spin- $\frac{1}{2}$ you can write amplitude in terms of 4 scalar functions. This is explained in some other paper. I think probably parity would reduce this to 2 functions. Not important now.

So really there is not much here that is helpful to me in understanding Henry's notation. I think I do understand it already. File the paper and store for later use.

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only through the factors $\Lambda(v^0-1)^{1/2}$ and $\bar{\Lambda}(\bar{v}^0-1)^{1/2}$, respectively, whereas $R_{\bar{\Lambda}\Lambda}(\theta)$ is known.

If one chooses the frame where $\sigma_a = \sigma_3$ and $\sigma_b = -\sigma_3$, then the matrix elements of H are the "transversity amplitudes" of Kotanski.⁵ Then the rotation matrix $R(\theta)$ is diagonal, instead of $B(v^0, e)$ and $B(\bar{v}^0, \bar{e})$. This representation of H is denoted by H_T .

Note that if (v^0-1) [or $(\bar{v}^0-1)$] is zero then the boost factor $B(v^0, e)$ [or $B(\bar{v}^0, \bar{e})$] becomes unity. Then the dependence of H on Λ or $\bar{\Lambda}$ [or on τ or $\bar{\tau}$] is determined by the matrix elements of the known rotation operator $R(\theta)$. This immediately gives the kinematic constraints, as we shall see in the next section.

Processes with higher spins are dealt with by constructing their amplitudes from tensor products of spin- $\frac{1}{2}$ amplitudes. For the purpose of this (purely mathematical) construction one can consider a particle of spin J and velocity v to be a composite system (in a purely mathematical sense) of $n \geq 2J$ spin- $\frac{1}{2}$ particles of velocity v . Let the labels on the particles of $a+b \rightarrow c+d$ be chosen so that $J_a = J_c \pmod{1}$ and $J_b = J_d \pmod{1}$. Let $N_{ac} = \max(2J_a, 2J_c)$ and $N_{bd} = \max(2J_b, 2J_d)$. Then imagine a process with $N_{ac} + N_{bd} = N$ spin- $\frac{1}{2}$ particles, such that first N_{ac} particles come in with velocity v_a and leave with velocity v_c , and last N_{bd} particles come in with velocity v_b and leave with velocity v_d . Let $\mathcal{C}_a$ represent the set of Clebsch-Gordan operators that combine the last $2J_a$ of the N_{ac} particles constituting particle a into a particle with spin J_a . Let $\mathcal{C}_b$, $\mathcal{C}_c$, and $\mathcal{C}_d$ be similarly defined. Let $\mathcal{S}_a$ be the operator that projects each of the first $(\frac{1}{2}N_{ac} - J_a)$ pairs of particles from the set of N_{ac} particles constituting particle a onto a spin-0 system. That is, $\mathcal{S}_a$ is a tensor product of $(\frac{1}{2}N_{ac} - J_a)$ singlet projection operators, acting on these $(\frac{1}{2}N_{ac} - J_a)$ pairs of particles. And let $\mathcal{S}_b$, $\mathcal{S}_c$, and $\mathcal{S}_d$ be similarly defined. Then H for the composite system is written as

$$H = [(\mathcal{S}_c \otimes \mathcal{C}_c) \otimes (\mathcal{S}_d \otimes \mathcal{C}_d)] \prod_{i=1}^N \otimes H_i \\ \times [(\mathcal{S}_a \otimes \mathcal{C}_a) \otimes (\mathcal{S}_b \otimes \mathcal{C}_b)]. \quad (2.9a)$$

This equation is schematic, for it does not make explicit the particular way that the N variables for the operator in the center are separated into the four spaces of the outer operators. But this separation has already been explained. (See, also, the Appendix.) Also, (2.9a) does not convey the information that in forming the tensor product $\prod_{i=1}^N \otimes H_i$, each of the four terms corresponding to the four different possible values of $(\bar{e}_i, e_i)$ in H_i is to be combined independently with each of the four terms of each of the other H_i , to give altogether 4^N terms, which have independent coefficients $a(\bar{e}_1, \dots, \bar{e}_N; e_1, \dots, e_N, s, t)$. This fact is exhibited in

the explicit definition

$$\left[\prod_{i=1}^N \otimes H_i \right] = \sum_{e_{\alpha i} = \pm \frac{1}{2}} a(e_{c1}, \dots, e_{d1}, \dots; e_{a1}, \dots, e_{b1}, \dots; s, t) \\ \times \prod_{i=1}^{N_{ac}} \otimes B_i(v_c^0, e_{ci}) W^{|\epsilon_{ci} + e_{ai}|} R_i(\theta) B_i(v_a^0, e_{ai}) \\ \times \prod_{j=1}^{N_{bd}} \otimes B_j(v_d^0, e_{dj}) W^{|\epsilon_{dj} + e_{bj}|} R_j(\theta) B_j(v_b^0, e_{bj}). \quad (2.9b)$$

The summation on the right is over the 4^N combinations of signs of the various $e_{\alpha i}$, where $\alpha = a, b, c$, or d . It was shown in Ref. 4 that the coefficients $a(e_{\alpha i}; s, t)$ can be made functions of the invariants s and t that are free of kinematic singularities at $\phi \neq 0$.

One can write the equation analogous to (2.9a) for either the M function or the S matrix by simply replacing the H and H_i either by M and M_i or by S and S_i , respectively. The conversions between the three forms go through because both the boosts and the rotations are converted in passage through the $\mathcal{S}_a \otimes \mathcal{C}_a$ to the form appropriate to the space on the other side.⁶

Going to the helicity representation and regrouping factors, one obtains from (2.9) the analog of (2.8);

$$H_{\lambda_b \lambda_c \lambda_d \lambda_a} = \sum_{\gamma} F_{\lambda_b \lambda_c \lambda_d \lambda_a}^{\gamma} R_{\lambda_d \lambda_c \lambda_b \lambda_a}^{\gamma}(\theta), \quad (2.10a)$$

where the sum over γ is a sum arising from the linear combinations implied by the factors $\mathcal{S}_a \otimes \mathcal{C}_a$. The factor $R_{\lambda_b \lambda_c \lambda_d \lambda_a}^{\gamma}(\theta)$ is a linear combination of products of N elementary rotation operator matrix elements $R_{\lambda_d \lambda_b \lambda_c \lambda_a}(\theta)$ and $R_{\lambda_c \lambda_d \lambda_b \lambda_a}(\theta)$, which must satisfy

$$\sum \lambda_{\alpha i} = \lambda_{\alpha}. \quad (2.10b)$$

The function $F_{\lambda_b \lambda_c \lambda_d \lambda_a}^{\gamma}$ is a linear combination of products of N factors like $F_{\lambda_d \lambda_b \lambda_c \lambda_a}$ and $F_{\lambda_c \lambda_d \lambda_b \lambda_a}$ of (2.8b) which also must satisfy (2.10b). In particular, we have

$$F_{\lambda_b \lambda_c \lambda_d \lambda_a}^{\gamma} = F^{\gamma}[(v_a^0 + 1)^{1/2}, \\ \lambda_{\alpha i}(v_a^0 - 1)^{1/2}; s, t; W], \quad (2.10c)$$

where the $\lambda_{\alpha i}$ on the right-hand side satisfy (2.10b).

The representation (2.10) is discussed in detail in the Appendix. The main pertinent features are that the dependence of $F_{\lambda_b \lambda_c \lambda_d \lambda_a}^{\gamma}$ upon λ_{α} enters only through the factors $\lambda_{\alpha i}(v_a^0 - 1)^{1/2}$, and $R_{\lambda_b \lambda_c \lambda_d \lambda_a}^{\gamma}(\theta)$ is a linear combination of products of matrix elements of N elementary rotations (2.5) having the correct total helicities λ_{α} , as specified by (2.10b).

The arguments that follow hold for each term of the sum in (2.10a). Thus the index γ will be omitted.

* H. P. Stapp, in *Proceedings of the International Seminar in High-Energy Physics and Elementary Particles, Trieste, 1965* (International Atomic Energy Agency, Vienna, 1965), Eq. (2.29).

⁵ A. Kotanski, *Acta Phys. Polon.* 29, 699 (1966); 30, 629 (1966).

Read July 29, 1977

Kinematic Constraints on Helicity Amplitudes

1. The singularity of helicity amplitudes was a hot subject at the time 1968. Between the time of this and Henry's preceding paper, Jackson and Hite came out with their version based on partial wave analysis. Here, Henry adds to his earlier paper. There, he located the singularity branch points, but ignored the poles and zeros. Here he follows the others by computing the strengths of the various pole powers and zeros. As usual, he is forced into all kinds of messy stuff to explain the general spin case. Not very elegant, also the notation is messy. I see why Jackson and Hite caught on so well compared to this, say, although in principle this is a more basic derivation.

I have not read this paper, just a quick scan. Store for later use maybe.