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PAPERS I

Phil Lucht

Angular Functions with Complex Angular Momenta

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In the study of the partial amplitudes for many-particle processes, and also for processes involving particles with spin, use is made of matrix elements of the rotation group, $d_{\mu\nu}^j(z)$. In this paper the generalization of the functions $d_{\mu\nu}^j(z)$ to arbitrary arguments and indices is studied. At the same time the functions of the second kind, analogous to Legendre functions of the second kind, are investigated. The results obtained play an important part in the introduction of complex angular momenta in many-particle processes.

The study of the partial amplitudes for elastic scattering of two spinless particles for complex orbital angular momenta requires a knowledge of the properties of the Legendre functions $P_j(z)$ and $Q_j(z)$ for arbitrary values of the argument and index. In the case of the scattering of particles with spin or of many-particle processes the expansion of the total amplitude is carried out in terms of matrix elements of the rotation group, $d_{\mu\nu}^j(z)$,^[1,2] which satisfy the equation

$$\left[(1-z^2) \frac{d^2}{dz^2} - 2z \frac{d}{dz} + j(j+1) - \frac{\mu^2 - 2\mu\nu z + \nu^2}{1-z^2} \right] y(z) = 0. \quad (1)$$

Therefore in order to continue the partial amplitudes of such processes to complex values of the angular momentum it is first necessary to generalize the functions $d_{\mu\nu}^j(z)$ and the corresponding functions of the second kind to arbitrary values of the indices and the argument. In the treatment of the elastic scattering of particles with spin it is sufficient to have the functions $d_{\mu\nu}^j(z)$ for arbitrary j and z , but for "physical" values of the helicities μ and ν (i.e., for μ and ν either both integers or both half-integers). This case has been investigated, for example, in [3]. For many-particle amplitudes, however, even a preliminary study shows that they must obviously be continued not only with respect to the total angular momentum, but also with respect to the helicities.^[4] Owing to this we shall consider in the present paper the continuation of the functions $d_{\mu\nu}^j(z)$ to arbitrary values of z, j, μ, ν . Since, however, it turns out that the $d_{\mu\nu}^j(z)$ themselves have nonessential cuts in j, μ , and ν , we study instead of them functions which differ from $d_{\mu\nu}^j$ by a factor which does not depend on z . These functions are a generalization of the associated Legendre functions, so that it is natural to call them generalized Legendre functions.

The present paper contains a description of the fundamental properties of the generalized Legendre

functions, and is an aid in the study of many-particle amplitudes for complex angular momenta.

1. DEFINITION OF THE GENERALIZED LEGENDRE FUNCTIONS

We define the generalized Legendre functions of the first kind, $P_{\mu\nu}^j(z)$, and of the second kind, $Q_{\mu\nu}^j(z)$, for arbitrary values of j, μ, ν , and z , by the equations

$$P_{\mu\nu}^j(z) = \frac{1}{\Gamma(\nu - \mu + 1)} \left(\frac{z-1}{2} \right)^{(\nu-\mu)/2} \left(\frac{z+1}{2} \right)^{(\nu+\mu)/2} \times F\left(j + \nu + 1, -j + \nu; \nu - \mu + 1; \frac{1-z}{2}\right), \quad (2)$$

$$Q_{\mu\nu}^j(z) = e^{i\pi(\mu-\nu)} \frac{\Gamma(j + \mu + 1) \Gamma(j - \nu + 1)}{2\Gamma(2j + 2)} \times \left(\frac{z-1}{2} \right)^{-(j+1)} \left(\frac{z+1}{2} \right)^{(\nu+\mu)/2} \times F\left(j + \nu + 1; j + \mu + 1; 2j + 2; \frac{2}{1-z}\right). \quad (3)$$

Here $F(a, b; c; x)$ is the hypergeometric function, and $\arg(z-1) = \arg(z+1) = 0$ for real $z > 1$. It is not hard to verify that the functions $P_{\mu\nu}^j(z)$ and $Q_{\mu\nu}^j(z)$ defined by (2) and (3) satisfy Eq. (1).

It follows directly from (2) and (3) that for $\nu = 0$ the functions $P_{\mu\nu}^j$ and $Q_{\mu\nu}^j$ go over into the associated Legendre functions:

$$P_{\mu 0}^j(z) = P_j^\mu(z), \quad Q_{\mu 0}^j(z) = Q_j^\mu(z), \quad (4)$$

where $P_j^\mu(z)$ and $Q_j^\mu(z)$ are the associated Legendre functions.^[5,6] In the general case the generalized Legendre functions are connected in a simple way with the Jacobi functions; for example,

$$P_{\mu\nu}^j(z) = \frac{\Gamma(j - \nu + 1)}{\Gamma(j - \mu + 1)} \left(\frac{z-1}{2} \right)^{(\nu-\mu)/2} \left(\frac{z+1}{2} \right)^{(\nu+\mu)/2}$$

$$(\bar{z}-1) \cdot \left(\frac{d F_{\mu\nu}^j}{dz} \right) = \sqrt{\bar{z}^2-1} \cdot F_{\mu, \nu-1}^j - (\nu \bar{z} - \mu) F_{\mu\nu}^j$$

$$\Rightarrow (\bar{z}-1) \left[\frac{d}{dz} p_j^\mu \right] = \sqrt{\bar{z}^2-1} p_j^{\mu+1} + \mu \bar{z} p_j^\mu \text{ which checks with Bp160-161.}$$

$$e^{-i\mu\theta_1} Q_{\mu\nu}^j(z_0) e^{-i\nu\theta_1} = \vartheta(z_1 - z_2) \sum_{\lambda-\mu=n}^i \\ \times (-1)^{\lambda-\nu} Q_{\mu\lambda}^j(z_1) P_{\lambda\nu}^j(z_2) e^{i\lambda\theta} \\ + \vartheta(z_2 - z_1) \sum_{\lambda-\mu=n} (-1)^{\mu-\lambda} P_{\mu\lambda}^j(z_1) Q_{\lambda\nu}^j(z_2) e^{i\lambda\theta} \quad (47)$$

In (47) n runs through all integer values, and $\vartheta(x)$ is the usual discontinuous function: $\vartheta(x) = 1$ for $x > 0$ and $\vartheta(x) = 0$ for $x < 0$. It is easy to write down one further addition theorem: (where $\mu = \nu$?)

$$e^{-i\mu\theta_1} P_{\mu\nu}^j(z_0) e^{-i\nu\theta_1} \\ = \vartheta(z_1 - z_2) \sum_{\lambda-\mu=n=-\infty}^{+\infty} (-1)^n P_{\mu\lambda}^j(z_1) P_{\lambda\nu}^j(z_2) e^{i\lambda\theta} \\ + \vartheta(z_2 - z_1) \sum_{\lambda-\mu=n=-\infty}^{+\infty} (-1)^n P_{\mu\lambda}^j(z_1) P_{\lambda\nu}^j(z_2) e^{i\lambda\theta} \quad (48)$$

It is not hard to verify that (47) is equivalent to the analytic continuation of (42) to imaginary values of α . The addition theorem for $P_{\mu\nu}^j$ can also be written in integral form. It is, however, much more cumbersome than (42) and we shall not write it out here.

In conclusion we note the following. As can be seen from the definitions (2) and (3), the theory of generalized Legendre functions is equivalent to the theory of hypergeometric functions. Therefore for

each relation for generalized Legendre functions there are some corresponding relations for hypergeometric functions. In particular, there are some rather complicated nonlinear relations between hypergeometric functions which correspond to the addition and multiplication theorems. The author does not know whether these relations exist in the mathematical literature. At any rate they are not in the well-known reference books.^{15,16}

An important part in the accomplishment of this work has been played by discussions with A. A. Ansel'm, V. N. Gribov, G. S. Danilov, and I. T. Dyatlov, and some of the formulas relating to generalized Legendre functions were derived in collaboration with them.

¹M. Jacob and G. C. Wick, *Ann. Phys. (N. Y.)* **7**, 404 (1959).

²G. C. Wick, *Ann. Phys. (N. Y.)* **18**, 65 (1962).

³J. B. Hartle, *Phys. Rev.* **134**, B612 (1964). R. L. Omnès and V. A. Alessandrini, *Phys. Rev.* **136**, B1137 (1964).

⁴V. N. Gribov, I. Ya. Pomeranchuk, and K. A. Ter-Martirosyan, *Phys. Rev.* **139**, B184 (1965). Ya. I. Azimov, A. A. Ansel'm, V. N. Gribov, G. S. Danilov, and I. T. Dyatlov, *JETP* **48**, 1776 (1965), *Soviet Phys. JETP* **21**, 1189 (1965).

⁵H. Bateman (Bateman Manuscript Project). *Higher Transcendental Functions*, New York, 1953.

⁶I. S. Gradshteyn and I. M. Ryzhik, *Tablitsy integralov, summ, ryadov, i proizvedenii*, (Tables of Integrals, Sums, Series, and Products), 4th ed., Fizmatgiz, 1962.

Translated by W. H. Furry

Sequence for believing Azimov's Properties.

Sept 1, 1976

① First, I shall accept the fact that $P_{\mu\nu}^j$ and $Q_{\mu\nu}^j$ do satisfy (1).

② Next, do they really reduce for Bateman? Yes.

$$P_{j\mu}^{\mu} = P_{j\mu}^j = \frac{1}{\Gamma(1-\mu)} \left(\frac{z-1}{z}\right)^{-\frac{\mu}{2}} \left(\frac{z+1}{z}\right)^{\frac{\mu}{2}} \Gamma(j+1, -j; 1-\mu; \frac{1-z}{z})$$

Bateman (3.2) (1.4). ✓

$$e^{-i\pi\mu} Q_{j\mu}^{\mu} = e^{-i\pi\mu} Q_{j\mu}^j = \frac{\Gamma(j+1+\mu)\Gamma(j+1)}{2\Gamma(2j+2)} \left(\frac{z-1}{z}\right)^{-j-1} \left(\frac{z+1}{z-1}\right)^{\frac{\mu}{2}} \Gamma(j+1, j+\mu+1, 2j+2; \frac{z}{1-z})$$

Bateman 3.2 (37).

③ At (6): obvious.

④ At (7): $\phi_{\mu\nu}^j = \Gamma(j+\mu+1)\Gamma(j-\nu+1) f_2(\mu, \nu)$

$$\phi_{\nu\mu}^j = \Gamma(j+\nu+1)\Gamma(j-\mu+1) f_2(\nu, \mu)$$

Then: $\phi_{\mu\nu}^j = \frac{\Gamma(j+1+\mu)\Gamma(j+1-\nu)}{\Gamma(j+1-\mu)\Gamma(j+1+\nu)} \phi_{\nu\mu}^j = \phi_{\nu\mu}^j$

so ya ✓

⑤ Wronskian: $W(P_{\mu\nu}^j, P_{\nu\mu}^j) \sim$, try instead

$$W(Q_{\mu\nu}^j, Q_{\nu\mu}^{j-1}) = \frac{\Gamma(j+1+\mu)\Gamma(j+1-\nu)}{2\Gamma(2j+2)} 2^{j+1} \times \frac{\Gamma(j-\mu+1)\Gamma(j-\nu+1)}{2\Gamma(2j)} 2^{-j} \times W(\bar{z}^{j-1}, z^j)$$

$$= \frac{\Gamma(j+1+\mu)\Gamma(-j-\mu)\Gamma(j+1-\nu)\Gamma(-j+\nu)}{4\Gamma(2j+2)\Gamma(-2j)} \cdot 2 \cdot W(\bar{z}^{j-1}, z^j) \checkmark$$

$$= \frac{1}{2} \frac{1}{z^2} \frac{\pi}{\sin\pi(j-\mu)} \frac{\pi}{\sin\pi(-j+\nu)} \frac{(-\sin\pi(2j))}{\pi}$$

$$= \frac{1}{2} \frac{1}{z^2} S_{\mu\nu}^j \checkmark$$

$$\bar{z}^{j-1} j \bar{z}^{j-1} - z^j (-j-1) \bar{z}^{j-2} = z^{-2} [j+j+1] \checkmark$$

⑥ What about $A_z(\tau)$ $p_{\mu\nu}^j = p_{-\nu-\mu}^j$? Better check: ...

$$\begin{aligned}
 & F(j+v+1, -j+v, v-\mu+1; \frac{1-z}{z}) \\
 &= F(\cancel{j-\mu+1}, \cancel{-j+v}, \cancel{v-\mu+1+j}, v-\mu+1; \frac{1-z}{z}) \\
 &\quad \times \left[1 - \left(\frac{1-z}{z} \right) \right]^{\cancel{j-\mu+1}, \cancel{-j+v}, \cancel{v-\mu+1+j}} \\
 &= F(-\mu-j, -\mu+j+1, v-\mu+1; \frac{1-z}{z}) \left(\frac{z+1}{z} \right)^{-\mu-j} \\
 &= F(\underline{j+1-\mu}, \underline{-j-\mu}, (-\mu)-(-v)+1; \frac{1-z}{z}) \left(\frac{z+1}{z} \right)^{-\frac{2\mu-2v}{2}} \quad \checkmark
 \end{aligned}$$

Thus, $P_{\mu\nu}^j = P_{\nu-\mu}^j$ ✓

① Next: $Q_{j+1}^i = Q_{j-n}^i$

$$F(j+\nu+1, j+\mu+1; zj+2; \frac{z}{1-z})$$

$$= F(j+1-\nu, j+1-\mu; zj+1; \frac{z}{1-z}) \cdot \left(1 - \frac{z}{1-z}\right)^{-\nu-\mu}$$

$$\left(\frac{z+1}{z-1}\right)^{-\frac{2\nu-2\mu}{2}}$$

$$Q_{\mu\nu}^j = Q_{-\nu-\mu}^j$$

So $Q_{\mu\nu}^j = Q_{-\nu-\mu}^j$

② Axiom (18): $\not\exists \mu \neq \nu \text{ s.t. } \pi(\mu - \nu) = -p_{\mu\nu}^i - g_{\mu\nu}^i p_{\mu\nu}^i$.

Look at $B_{2.10}(2)$ which is really $(34) \checkmark$.

$$u_1 = \frac{P(c)P(b-a)}{P(c-a)P(b)} u_3 + \frac{P(e)P(a-b)}{P(c-b)P(a)} u_7$$

$$= \frac{\Gamma(v-\mu+1) \Gamma(-j+\cancel{j-1-\cancel{j}})}{\Gamma(\cancel{v-\mu+1-j-\cancel{j}}) \Gamma(-j+v)} \left(\frac{z-1}{z}\right)^{j+1-v} \cdot \Gamma(j+1+v, j+1+\mu, 2j+2; \frac{z}{1-z})$$

$$+ \frac{\Gamma(v-\mu+1) \Gamma(2j+1)}{\Gamma(v-\mu+1+j-\nu) \Gamma(j+1+v)}$$

START OFF

START OFF

Start again: $\Phi_{\mu\nu}^j \cdot \frac{z}{\pi} \sin \pi(\mu-\nu) = P_{\mu\nu}^j - G_{\mu\nu}^j P_{\nu\mu}^j$

$$u_1 = B_3 u_3 + B_4 u_4.$$

$$\begin{aligned} a+b &= -j+\nu \\ a+b &= \nu-\mu+1 \\ b+1-c &= -j+\mu \\ b+1-a &= \mu-\nu+1 \end{aligned}$$

$$u_1 = F(j+1+\nu, j+1+\mu; 2j+2; \frac{z}{1-z})$$

$$u_3 = \left(\frac{z-1}{z}\right)^{j+1+\nu} F(j+1+\nu, -j+\nu; \nu-\mu+1; \frac{1-z}{z})$$

$$u_4 = \left(\frac{z-1}{z}\right)^{j+1+\mu} F(-j+\mu, j+1+\mu; \mu-\nu+1; 1-\frac{z}{z})$$

$$B_3 = \frac{\Gamma(2j+2)\Gamma(\mu-\nu)}{\Gamma(j+1-\nu)\Gamma(j+1+\mu)}$$

$$B_4 = \frac{\Gamma(2j+2)\Gamma(\nu-\mu)}{\Gamma(j+1-\mu)\Gamma(j+1+\nu)}$$

Therefore:

$$\begin{aligned} \frac{F(\Phi_{\mu\nu}^j)}{\Gamma(2j+2)} &= \frac{\Gamma(\mu-\nu)}{\Gamma(j+1-\nu)\Gamma(j+1+\mu)} \left(\frac{z-1}{z}\right)^{j+1+\nu} F(P_{\mu\nu}^j) \\ &\quad + \frac{\Gamma(\nu-\mu)}{\Gamma(j+1-\mu)\Gamma(j+1+\nu)} \left(\frac{z-1}{z}\right)^{j+1+\mu} F(P_{\nu\mu}^j) \end{aligned}$$

Now apply $\left(\frac{z-1}{z}\right)^{j-1} \cdot \left(\frac{z-1}{z}\right)^{-\frac{\nu}{2}-\frac{\mu}{2}} \left(\frac{z+1}{z}\right)^{\frac{\nu+\mu}{2}}$. Also, set

$$\Gamma(\mu-\nu) = -\frac{1}{\Gamma(1+\nu-\mu)} \frac{\pi}{\sin \pi(\mu-\nu)}$$

$$\Gamma(\nu-\mu) = -\frac{1}{\Gamma(1+\mu-\nu)} \frac{\pi}{\sin \pi(\mu-\nu)}$$

Then mult by $-\frac{\sin \pi(\mu-\nu)}{\pi}$. Net result of all this is:

also do $\times \Gamma(j+1-\nu)\Gamma(j+1+\mu)$:

$$\frac{2}{2} \frac{F(\Phi_{\mu\nu}^j)}{\Gamma(2j+2)} \left(\frac{z-1}{z}\right)^{j-1} \left(\frac{z+1}{z}\right)^{\frac{\nu+\mu}{2}} \left(\frac{\sin \pi(\mu-\nu)}{\pi}\right) \Gamma(j+1-\nu)\Gamma(j+1+\mu)$$

$$= \frac{F(P_{\mu\nu}^j)}{\Gamma(1+\nu-\mu)} \left(\frac{z-1}{z}\right)^{\frac{\nu}{2}} \left(\frac{z+1}{z}\right)^{\frac{\nu+\mu}{2}} - \frac{F(P_{\nu\mu}^j)}{\Gamma(1+\mu-\nu)} G_{\mu\nu}^j \left(\frac{z-1}{z}\right)^{-\frac{\nu}{2}} \left(\frac{z+1}{z}\right)^{\frac{\nu+\mu}{2}}$$

$$\Rightarrow \boxed{\Phi_{\mu\nu}^j(z) \cdot \frac{z}{\pi} \sin \pi(\mu-\nu) = P_{\mu\nu}^j(z) - G_{\mu\nu}^j P_{\nu\mu}^j(z)}!$$



⑨ $A_2: (11)$ easy!

$$\begin{aligned} \frac{2}{\pi} \sin \pi(\mu-\nu) \Phi_{\mu\nu}^j &= P_{\mu\nu}^j - G_{\mu\nu}^j P_{\nu\mu}^j \\ \Rightarrow \frac{2}{\pi} \sin \pi(\mu-\nu) \Phi_{\mu\nu}^{j-1} &= P_{\mu\nu}^{j-1} - G_{\mu\nu}^{j-1} P_{\nu\mu}^{j-1} \\ &= P_{\mu\nu}^j - L_{\mu\nu}^j G_{\mu\nu}^j P_{\nu\mu}^j \end{aligned}$$

Subtract:

$$\begin{aligned} \frac{2}{\pi} \sin \pi(\mu-\nu) [\Phi_{\mu\nu}^j - \Phi_{\mu\nu}^{j-1}] &= -G_{\mu\nu}^j P_{\nu\mu}^j (1 - L_{\mu\nu}^j) \\ &= -G_{\mu\nu}^j P_{\nu\mu}^j (\sin \pi(\nu-\mu) S_{\mu\nu}^j) \\ &= + \sin \pi(\mu-\nu) G_{\mu\nu}^j S_{\mu\nu}^j P_{\nu\mu}^j \end{aligned}$$

$$\Rightarrow \boxed{\Phi_{\mu\nu}^j - \Phi_{\mu\nu}^{j-1} = \frac{\pi}{2} G_{\mu\nu}^j S_{\mu\nu}^j P_{\nu\mu}^j}$$

⑩ $A_3: R$

Try $F(a, b, c, z) = (1-z)^{-a} F(a, c-b, c; \frac{z}{1-z})$

$$\begin{aligned} \text{So } F(\Phi_{\mu\nu}^j) &= (1 - \frac{z}{1-z})^{-j-1-\nu} F(j+1+\nu, j+1-\mu; c; \frac{(\frac{z}{1-z})}{(\frac{z}{1-z})-1}) \\ &= (\frac{z+1}{z-1})^{-j-1-\nu} F(j+1+\nu, j+1-\mu; z_j+1; \frac{z}{1+z}) \end{aligned}$$

$$\begin{aligned} \text{So } F(\Phi_{\mu\nu}^j(z)) &= \frac{(\frac{z+1}{z-1})^{j+1-\nu}}{(\frac{z+1}{z-1})^{j-1-\nu}} \frac{F(\Phi_{\mu\nu}^j(-z))}{2\Gamma(z_j+2)} \frac{(\frac{z+1}{z-1})^{j+\mu}}{(\frac{z-1}{z})^{-j-1}} \\ &\quad \wedge \Gamma(j+1+\mu) \Gamma(j+1-\nu) \end{aligned}$$

$$\Rightarrow \Phi_{\mu\nu}^j(z) = \frac{F(\Phi_{\mu\nu}^j(-z))}{2\Gamma(z_j+2)} \cdot \Gamma(j+1-\mu) \Gamma(j+1-\nu) \cdot G_{\mu\nu}^j \cdot \left(\frac{z+1}{1-z}\right)^{j+1-\nu-\frac{\mu}{2}} \left(\frac{z-1}{z}\right)^{-j-1}$$

So consider:
$$\frac{\left(\frac{-z+1}{-z-1}\right)^{j+1} \left(\frac{z-1}{z}\right)^{-j-1} \left[\frac{\left(\frac{-z+1}{-z-1}\right)^{\frac{j+1}{2}}}{\left(\frac{-z-1}{z}\right)^{j+1}}\right]}{\left(\frac{-z-1}{z}\right)^{j+1}} = \left(\frac{-z-1}{z}\right)^{j+1}$$

$$\Rightarrow \Phi_{\mu\nu}^j(z) = G_{\mu}^j \Phi_{\mu\nu}^j(-z) \cdot \left\{ \left(\frac{-z+1}{-z-1}\right)^{j+1} \left(\frac{z-1}{z}\right)^{-j-1} \left(\frac{-z-1}{z}\right)^{j+1} \right\}$$

So:
$$\{ \} = \left(\frac{1-z}{z}\right)^{j+1} \left(\frac{z-1}{z}\right)^{-j-1} = \left(\frac{z+1}{z}\right)^{j+1} (e^{\mp i\pi})^{j+1} \left(\frac{z-1}{z}\right)^{j+1}$$

As I see these things,


$$\Rightarrow \boxed{\Phi_{\mu\nu}^j(z) = G_{\mu}^j \Phi_{\mu\nu}^j(-z) \cdot e^{\mp i\pi(j+1)} \quad \text{Im } z > 0.}$$

Then
$$G_{\mu}^j \Phi_{\mu\nu}^j = G_{\mu}^j \Phi_{\nu\mu}^j = G_{\mu}^j G_{\nu\mu} \Phi_{\mu\nu}^j = -G_{\nu\mu}^j \Phi_{\mu\nu}^j$$

① I want a version of (17),

A3.17 based on (17) + (18):

$$\Phi_{\mu\nu}^j(z) - \Phi_{\mu\nu}^{j-1}(z) = \frac{\pi}{2} S_{\mu\nu}^j G_{\mu\nu}^j P_{\mu\nu}^j(z)$$

$$\Rightarrow e^{\mp i\pi(j+1)} G_{\mu}^j \Phi_{\mu\nu}^j(-z) - e^{\mp i\pi(-j)} G_{\nu}^{j-1} \Phi_{\mu\nu}^{j-1}(-z) = \frac{\pi}{2} S_{\mu\nu}^j G_{\mu\nu}^j P_{\mu\nu}^j(z)$$

$$\Rightarrow \left\{ \begin{aligned} \Phi_{\mu\nu}^j(-z) &= G_{\nu}^j e^{\pm i\pi(j+1)} \left[e^{\mp i\pi(-j)} G_{\mu}^{j-1} \Phi_{\mu\nu}^{j-1}(-z) \right. \\ &\quad \left. + \frac{\pi}{2} S_{\mu\nu}^j G_{\mu\nu}^j P_{\mu\nu}^j(z) \right] \end{aligned} \right.$$

But directly:
$$\Phi_{\mu\nu}^j(-z) = \Phi_{\mu\nu}^{j-1}(-z) + \frac{\pi}{2} S_{\mu\nu}^j G_{\mu\nu}^j P_{\mu\nu}^j(z)$$

So subtract:

$$0 = \Phi_{\mu\nu}^{j-1}(-z) \left[e^{\pm i\pi(2j+1)} G_{\mu}^j G_{\nu}^{j-1} - 1 \right] + \frac{\pi}{2} \left[\Phi_{\mu\nu}^j e^{\pm i\pi(j+1)} S_{\mu\nu}^j G_{\mu\nu}^j P_{\mu\nu}^j(z) - S_{\mu\nu}^j G_{\mu\nu}^j P_{\mu\nu}^j(-z) \right]$$

$$\Rightarrow 0 = \Phi_{\mu\nu}^j(-z) \left[e^{\pm i\pi(-2j-1)} G_{\mu}^{j-1} G_{\nu}^j - 1 \right] - \frac{\pi}{2} \left[e^{\pm i\pi(-j)} S_{\mu\nu}^j G_{\mu\nu}^j P_{\mu\nu}^j(z) - S_{\mu\nu}^{j-1} G_{\mu\nu}^{j-1} P_{\mu\nu}^{j-1}(-z) \right]$$

$$\text{Now } G_{\nu}^{-j-1} G_{-\nu}^j = G_{\nu 0}^{-j-1} G_{-\nu 0}^j = L_{\nu 0}^j \cancel{G_{\nu 0}^j} \cancel{G_{-\nu 0}^j} = L_{\nu 0}^j$$

$$S_{\mu\nu}^{-j-1} = -S_{\nu\mu}^j$$

$$G_{\mu 0}^{-j-1} = L_{\mu 0}^j G_{\mu}^j$$

$$\Rightarrow (S_{\mu\nu}^{-j-1} G_{\mu}^{-j-1}) = -S_{\nu\mu}^j L_{\mu 0}^j G_{\mu}^j$$

$$S_{\mu-\nu}^{-j-1} = -S_{-\nu\mu}^j$$

$$G_{\mu-\nu}^{-j-1} = L_{\mu-\nu}^j G_{\mu-\nu}^j$$

$$\Rightarrow (S_{\mu-\nu}^{-j-1} G_{\mu-\nu}^{-j-1}) = -S_{-\nu\mu}^j L_{\mu-\nu}^j G_{\mu-\nu}^j$$

So:

$$Q_{\mu-\nu}^j(-z) [1 + L_{\nu 0}^j e^{\mp 2i\pi j}]$$

$$= \frac{\pi}{2} \left[e^{\mp i\pi j} (-1) S_{\nu\mu}^j L_{\mu 0}^j G_{\mu}^j P_{\nu\mu}^j(z) \right.$$

$$\left. + S_{-\nu\mu}^j L_{\mu-\nu}^j G_{\mu-\nu}^j P_{-\nu\mu}^j(-z) \right]$$

STOP.

GO do this on separate sheet since too messy.

The P(-z) Relation

① Use Oshiro's rule:
$$Q_{\mu\nu}^j(z) - Q_{\mu\nu}^{j-1}(z) = \frac{i}{2} S_{\mu\nu}^j G_{\mu\nu}^j P_{\mu\nu}^j(z)$$

This says:

$$\begin{aligned} \frac{i}{2} S_{\mu\nu}^j G_{\mu\nu}^j P_{\mu\nu}^j(-z) &= Q_{\mu\nu}^j(-z) - Q_{\mu\nu}^{j-1}(-z) \\ &= e^{\pm i\pi(j+1)} G_{\mu\nu}^j Q_{\mu\nu}^j(z) - e^{\pm i\pi(-j)} G_{\mu\nu}^{j-1} Q_{\mu\nu}^{j-1}(z) \end{aligned}$$

Apply $G_{\mu\nu}^j$ to both sides:

$$-\frac{i}{2} S_{\mu\nu}^j P_{\mu\nu}^j(-z) = e^{\pm i\pi j} G_{\mu\nu}^j Q_{\mu\nu}^j(z) + e^{\mp i\pi j} G_{\mu\nu}^{j-1} Q_{\mu\nu}^{j-1}(z)$$

Both sides is:

$$\boxed{-\frac{i}{2} G_{\mu\nu}^j S_{\mu\nu}^j P_{\mu\nu}^j(z) = e^{\pm i\pi j} Q_{\mu\nu}^j(z) + e^{\mp i\pi j} L_{\mu\nu}^{j-1} Q_{\mu\nu}^{j-1}(z)}$$

This one does not appear in Az. Rewrite:

$$-\frac{i}{2} G_{\mu\nu}^j S_{\mu\nu}^j P_{\mu\nu}^j(-z) = e^{\pm i\pi j} Q_{\mu\nu}^j(z) + e^{\mp i\pi j} L_{\mu\nu}^{j-1} Q_{\mu\nu}^{j-1}(z)$$

But obviously:

$$= \cancel{e^{\pm i\pi j}}$$

$$\underline{e^{\mp i\pi j} L_{\mu\nu}^{j-1} \frac{i}{2} S_{\mu\nu}^j G_{\mu\nu}^j P_{\mu\nu}^j(z)} = \underline{e^{\mp i\pi j} L_{\mu\nu}^{j-1} Q_{\mu\nu}^j(z)} - \underline{e^{\mp i\pi j} L_{\mu\nu}^{j-1} Q_{\mu\nu}^{j-1}(z)}$$

So add to get:

$$\left\{ \begin{aligned} \frac{i}{2} G_{\mu\nu}^j \left[e^{\mp i\pi j} L_{\mu\nu}^{j-1} S_{\mu\nu}^j G_{\mu\nu}^j P_{\mu\nu}^j(z) - S_{\mu\nu}^j P_{\mu\nu}^j(-z) \right] \\ = Q_{\mu\nu}^j(z) \left[e^{\pm i\pi j} + e^{\mp i\pi j} L_{\mu\nu}^{j-1} \right] \end{aligned} \right.$$

Now:

$$e^{\pm i\pi j} + e^{\mp i\pi j} L_{\mu\nu}^{j-1} = \cos \pi j \pm i \sin \pi j + (\cos \pi j \mp i \sin \pi j) \left[\frac{\sin \pi(j+1)}{\sin \pi(j-1)} \right]$$

$$= \frac{\cos \pi j \sin \pi(j-1) \pm i \sin \pi j (\sin \pi(j-1)) + \cos \pi j \sin \pi(j+1) \mp i \sin \pi j \sin \pi(j+1)}{\sin \pi(j-1)}$$

$$N = \cos \pi j 2 \sin \pi j \cos \pi j \mp i \sin \pi j 2 \sin \pi j \sin \pi j = \sin \pi(2j) e^{\mp i\pi j}$$

$$\Rightarrow \text{RHS} = Q_{\mu\nu}^j(z) \cdot \frac{\sin \pi(2j)}{\sin \pi(j-1)} e^{\mp i\pi j}$$

Meanwhile,

$$\left\{ \text{LHS} = \frac{\pi}{2} G_{\mu}^j \left[e^{\mp i\pi j} \frac{\sin\pi(j+\mu)}{\sin\pi(j-\mu)} \frac{\sin\pi(z_1)}{\sin\pi(j-\mu)} \frac{\sin\pi(j+\mu)}{\sin\pi(j-\mu)} G_{-\nu}^j P_{\nu\mu}^j(z) - \frac{\sin\pi(z)}{\sin\pi(j-\mu)} \frac{\sin\pi(j+\mu)}{\sin\pi(j-\mu)} P_{-\nu\mu}^j(-z) \right] \right.$$

Cancelling terms, I get:

$$\left\{ Q_{\mu\nu}^j(z) e^{\mp i\pi\nu} = \frac{\pi}{2} G_{\mu}^j \frac{e^{\mp i\pi j} G_{-\nu}^j P_{\nu\mu}^j(z) - P_{-\nu\mu}^j(-z)}{\sin\pi(j-\mu)} \right\}$$

Take $\mu \rightarrow -\mu, \nu \rightarrow -\nu$ then $Q_{\mu\nu}^j \rightarrow Q_{\mu\nu}^j = Q_{\nu\mu}^j$
 $P_{\nu\mu}^j \rightarrow P_{-\nu-\mu}^j = P_{\mu\nu}^j$

$$Q_{\nu\mu}^j(z) e^{\pm i\pi\nu} = \frac{\pi}{2} G_{-\mu}^j \frac{e^{\mp i\pi j} G_{\nu}^j P_{\mu\nu}^j(z) - P_{\nu-\mu}^j(-z)}{\sin\pi(j+\mu)}$$

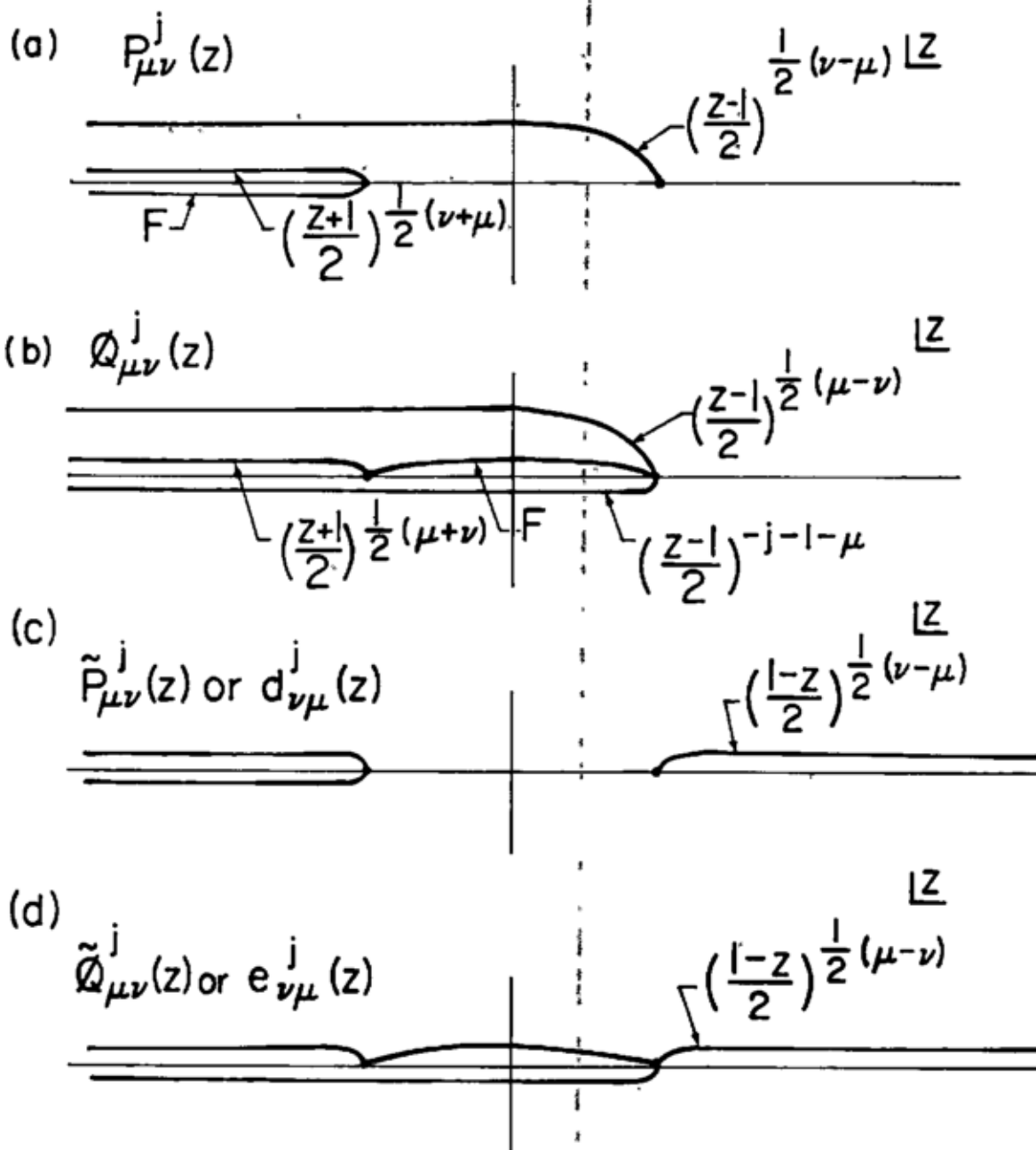
Apply $G_{\mu\nu}^j$:

- 1) $G_{\mu\nu}^j Q_{\nu\mu}^j = Q_{\mu\nu}^j(z)$
- 2) $G_{\mu\nu}^j G_{-\mu}^j G_{\nu}^j = 1$
- 3) $G_{\mu\nu}^j G_{-\mu}^j = G_{-\nu}^j$

Thus:

$$\boxed{\frac{\pi}{2} \sin\pi(j+\mu) Q_{\mu\nu}^j(z) e^{\pm i\pi\nu} = e^{\mp i\pi j} P_{\mu\nu}^j(z) - G_{\nu-\mu}^j P_{\mu-\nu}^j(-z)}$$

This proves the result I use, and therefore proves Az. 19.



XBL 7610 4295

Fig. 6

MISC paper notes

Notes on Cronstrom's Dec 6 1974 Diagonalization Preprint 19495

The question is to find out: what does he do, what does he not do?
I will take some typed notes then summarize at the end.

2. Integral equation for the scattering amplitude.

He starts here by writing a standard integral equation where d^4Q runs over all spacetime. He goes to versor notation, quoting old Serferio and Toller, and separates integral into timelike versor piece plus spacelike versor piece (does not use the word "versor"). His group variables are a boost and a rotation, so I's say he was definitely in the TT basis.

3. The generalized $O(1,2)$ transforms.

Quotes from his earlier paper. First, section #3A does the Q_j (chb) transform. He shows how you invert the Q_j projection, and shows there is a class of equivalent projections. Then section 3B repeats this with a so-called $Q_j(1 \text{ shb})$ transform. I.e. he writes projection then searches for inversion. Lots of crud about what classes of functions you can do this for.

4. The j-plane equations.

Holy cow! I think he does everything via the kinematic k-function method. He writes the abstracted integral equation and replaces the azimuth by a $\text{theat}(k)/\text{SQRT}(k)$ thing. Much yak about convergence etc. Then he gets all lost in a thing that looks like my Q_j -separation-integral, but he integrates only from 1 to inf thus causing lots of extra terms.

Next, he repeats all this for spacelike versor case, 4B. The diagonalized equation appears as 4.26 which has a j-integral in it and some kernel stuff. Then 4.91 gives his total result: not very enlightening to me, who has spent months at this game.

5. Reduction of the j-plane equation to algebra.

This is just a mess. Compares to Serferio Toller.

6. Summary and Discussion

OK, here he says what he does, good summary. Seeks an approach to reggeom calculus complimentary to White. (ie, $o(2,1)$ not $o(3)$.)

Appendix A. The P_j separation integral; the Q_j separation with same mistake that appears in AS, I think.

Comments after scanning: There is no mention of Haar invariance. Everything is done by brute force. Group representations are not mentioned, groups are not mentioned. There is no dj_{mn}^T . Not even P_j^m . Only P_j and Q_j . I.e. he is only analyzing spinless-four-point function (BSE). He speaks of AS as being "simpler" than his stuff.

In my humble opinion, C has simply failed to use the group machinery to his advantage. It sits idle in this paper. Lucky for me.

However, he does speak the question: does the modified transform still diagonalize integral equations?

Here is a quick summary of Cronstroems method of diagonalization. It starts on page 25. There he shows an integral equation with $dg = d\phi/d(chv)$ in my notation. Then he replaces the $d\phi$ integral with a dx integral and k -function square root. So far then

$$\int dg_2 f_z^{++}(-, \beta_2, -) K_z^{++}(-, \tilde{\beta}_1, -) g_1' g_2$$

$$= \int d\phi d(chv) \dots$$

$$= \int d(chv) f_z^{++}(chv) \int_{-\infty}^{\infty} dx K_z^{++}(x) \frac{\Theta(D)}{\sqrt{D}}$$

I show here only the right-hand-side. Next, he applies the Q_j to both sides and integrates so that:

$$f^L = \int dchv f_z(chv) \int_{-\infty}^{\infty} dx K_z^{++}(x) \int_{-\infty}^{\infty} dchv' \frac{\Theta(D)}{\sqrt{D}} Q_j^L(chv')$$

Here he uses the Q separation integral as in 4.12. Yes, he does realize that there are extra terms due to the endpoints being 1. Eg,

$$f^L = \int dy f_z(y) \int dy' \cdot \int dx K(x) \frac{\Theta(D)}{\sqrt{D}} Q_L(x) P_L(y)$$

but with O -factors.

It is because the Q did not cleanly separate that C now has to make some new moves. He next expands $K(x)$ onto Q 's, then he does the Q integral

$$K(x) = \int d\lambda (2\lambda+1) K(\lambda) Q_\lambda(x)$$

His final result is given in 4.26. It is:

$$f^L = \int d\lambda \frac{\lambda(2\lambda+1)}{(\lambda-2)(\lambda+2+1)} K_\lambda f_\lambda + \text{similar term.}$$

where all projections are standard Q -projections.

This is not a standard helicity contour. I am not sure this is even a group diagonalization! I suspect that the failure of C to get the simple result is due to his remaining in the TT basis for his Haar measure.

Barut + Raczka

"On non-compact groups." — 2 papers
PRAL, 287, 519 (1965)

Paper I: "Classification of non-compact Lie groups" 13 pages.

1. Preliminaries. Some interesting history here. Killing and Cartan in 1890, 1894 classified the complex Lie algebras. I.e., the famous $A_n \dots D_n + 5$ exceptional algebras. However, this does not really classify your real Lie $\sim$ eg, $SO(2,1)$ and $SO(3)$ are both the same in complex extension.

Rest: mentions Dynkin form, references etc. All math, but not too hard. In 2., they run through all the classical complex Lie, one at a time, + show what are the real lies. Only one of these is compact, like $O(3)$ of A_1 ; all rest are noncompact. They even treat the exceptionals.

Paper II: "representations of $SO(2,1)$ " 17 pages.

1. Introduction. OK, using the SL_2 monomials, and the differential rep of the generators, they construct the ranges of the various reps of $O(2,1)$ exactly as done in Wybourne's book, all is the same. Later, they deal with the Clebsch Gordon problem for $O(2,1)$. Then the last page constructs the madness! He calls them $S_{mm}^\phi(g)$. Ill copy this one page piece!

With these new ranges of the parameters (6.13) and (6.14) also continue to hold, except that the normalizers N_m of the rotation group must be replaced by the normalizers (4.8) appropriate to the 2+1 Lorentz group.

It is easy to extend the discussion of the invariants to the other products; e.g. to the product of two representations of the principal series. However, even when a formal invariant exists between, say, $\mathcal{D}_1$, $\mathcal{D}_2$ and $\mathcal{D}'$ it is not always true that it can be written in terms of normalizable vectors in the Hilbert space of the product representation $\mathcal{D}_1 \otimes \mathcal{D}_2$. We hope to complete our work on this problem in a future report.

7. MATRIX ELEMENTS FOR FINITE TRANSFORMATIONS

One advantage of the representation theory based on the multispinors (3.1) is that it allows us in a straightforward manner to evaluate the matrix elements of finite transformations. These are the quantities corresponding to the well known $\mathcal{D}'$ -functions in the case of the rotation group and are important quantities for applications.

Let us denote a representation of an element of the spinor group (2.7) by $S(g)$. Noting the inverse transformation

$$g^{-1} = \begin{pmatrix} \bar{\alpha} & -\beta \\ -\bar{\beta} & \alpha \end{pmatrix},$$

we have

$$S(g) |\Phi, m\rangle = N_m (\bar{\alpha} \xi_1 - \beta \xi_2)^a (\frac{1}{2} \bar{\beta} \xi_1 + \alpha \xi_2)^b. \quad (7.1)$$

Because the binomial expansions can be defined even for complex powers a and b we can expand the right hand of (7.1) and obtain

$$S(g) |\Phi, m\rangle = N_m \sum_x \sum_{x'} (-1)^x \frac{\bar{\alpha}^{a-x} \beta^x (-\bar{\beta})^{b-x'} \alpha^{x'}}{x! x'! (a-x)! (b-x')!} a! b! \xi_1^{a+b-x-x'} \xi_2^{x+x'}. \quad (7.2)$$

Let us now put in (7.2)

$$\begin{aligned} a+b-x-x' &= a', \\ -x'+x' &= b', \end{aligned}$$

and

$$\frac{1}{2}(a'-b') = \frac{1}{2}(a+b) - (x+x') = \Phi - (x+x') = m' + E_0.$$

We then obtain

$$S(g) |\Phi, m\rangle = \sum_{m'} S_{mm'}^\Phi(g) |\Phi, m'\rangle, \quad (7.3)$$

where the matrix elements are given by

$$\begin{aligned} S_{mm'}^\Phi(g) &= \frac{N_m}{N_{m'}} \frac{(\Phi+m+E_0)! (\Phi-m-E_0)!}{(m'-m)!} \bar{\alpha}^{\Phi+m+E_0} \alpha^{\Phi-m'-E_0} (-\bar{\beta})^{m'-m} \\ &\times \sum_x (-1)^x \frac{(\alpha \bar{\alpha})^{-x} (-\beta \bar{\beta})^x (m'-m)!}{(\Phi+m+E_0-x)! (\Phi-m'-E_0-x)! x! (m'-m+x)!} \end{aligned} \quad (7.4)$$

with

$$a = \Phi + m + E_0, \quad b = \Phi - m - E_0.$$

In the last sum we recognize the expansion of the hypergeometric function $F(a, b, c; -z)$ of argument

$$z = (\alpha \bar{\alpha})^{-1} (-\beta \bar{\beta}) = -\beta \bar{\beta} / (1 + \beta \bar{\beta})$$

1970-71

WINTERITZ

I. Introduction. It seems that W. will restrict to spin zero completely. Define $F(s,t) = M(p_1, p_2, p_3, p_4) = \mathcal{L}.I.$ W. is an good winter! The expansions will be designed to take advantage fully of group "assistance." This intro. ends with a long list of what one wants to know about each group. Lots of "explicit" knowledge will be required. .end p13

II. Little Group Expansions of Scattering Amplitudes

1. The Poincaré Group + its Little Groups The spacetime xform is $(1, a)$. Of the four pieces of $\mathcal{L}$, we care only about $\det A = 1$, $\Lambda_{00} \geq 1$, as usual. The Lie algebra is stated, $M_{\mu\nu}, P_\mu, K_i, K_i$ and the two Casimirs are noted. (Will check norms later). "Little group of vector k " is defined ["star of k ?"]. The four cases are timelike, spacelike, lightlike, and null.

① The little group of a lightlike vector is E_2 ! This will be explained later.

2. Little Group Expansions. Draw "all lines in". There are $3n-10$ scalar variables). There are basically two methods in doing these l.g. expansions:

(a) you consider S as a scalar operator. The initial state is $p_1 \otimes p_2 = |P\rangle \otimes |a\rangle$. Here, P is timelike, so the little group is $O(3)$. Thus, the "usual" partial wave expansion is "relativistically" correct. It is not clear exactly what the "Poincaré Clebsch's" are here, wait till later. The point is that you fix $p_1 + p_2 = P$ + then do your little group of P . This can be done in any channel.

(b) The second method is that of Toller '65. I.e., you write $M(P_A, P_B) = M(\Lambda_A P_A^\circ, \Lambda_B P_B^\circ)$ etc where $\Lambda_A \Lambda_B = g \in$ Little Group of $Q = \Sigma P_A = \Sigma P_B$.

① New touch: if P_A° has some little group itself, H_A , and P_B° has H_B , then you can say:

$$M(P_A^\circ, g P_B^\circ) = M(P_A^\circ, h_A g h_B)$$

Choice of $h_A \in H_A$ could lead to simplifications /p25
[OK, this just kills redundant variables.]

3. The $O(3)$ Expansions + Direct Channel Partial Waves

- ① Bang, he starts off with that kmelike-hyperboloid coordinate idea. At least it is not done "in generality" as in Banders RPM on Hydrogen. What you want is:

$$p_1 = (E_1, \vec{p}_1)$$

$$p_2 = (E_2, -\vec{p}_1)$$

$$Q = p_1 + p_2 = (E_1 + E_2, \vec{0})$$

The way to do this is:

$$p_1 = m_1 (\text{ch } a_1, \hat{n}_1 \text{sh } a_1)$$

$$E_1 = m_1 \text{ch } a_1$$

$$p_2 = m_2 (\text{ch } a_2, -\hat{n}_1 \text{sh } a_2)$$

$$|\vec{p}_1| = m_1 \text{sh } a_1$$

where $\hat{n} = (\sin \theta, \cos \phi, \sin \theta, \sin \phi, \cos \theta)$.

- ② Since "all lines in", we use

$$p_3 = -m_3 (\text{ch } a_3, \hat{n}_3 \text{sh } a_3)$$

↑ note.

- ③ Now of course $p_i^0 = m_i (\text{ch } a_i, 0, 0, \text{sh } a_i)$ and

$$p_i = \Lambda p_i^0 \quad \text{where} \quad \Lambda = R_z(\phi_1) R_y(\theta_1) R_z(\psi)$$

$$\Lambda_B = R_z(\phi_3) R_y(\theta_3) R_z(\chi)$$

- ④ Of the four a_i , only one is independent! See (2.25). Moreover, of all these angles, only $\theta_3 \equiv \theta$ counts. The ψ and χ are removed by the Λ trick. So, let $a_i = a$. Then,

$$M(p_i^0, g) = F(a, \theta)$$

where $\text{ch } a = E_1/m_1 = (s + m_1^2 - m_2^2)/2\sqrt{s}m_1$ of course.

- ⑤ Now comes that theorem I have never seen so plainly and cleanly stated:

$$f(\phi, \theta, \psi) = \sum_l \sum_{m,n} a_l^{mn} D_l^{mn}(g)$$

where $D_l^{mn}(g) = e^{-i(m\phi + n\psi)} P_l^{(mn)}(\cos \theta)$

Jacobi polynomials!

I never knew that!

But, $\frac{P_l^{mn}}{P_l^{mn}} \neq \text{dim}$ exactly.

⑥ The exact inversion is:

$$a_l^{mn} = \frac{(-1)^{m-n}}{8\pi^2} \int d(\cos\theta) d\phi d\psi f(\phi, \theta, \psi) Y_l^{mn}(g)$$

⑦ "Plancherel" says: (does this have a more common name?) ✓

$$\sum_{l,m,n} |a_l^{mn}|^2 = \frac{2l+1}{8\pi^2} \int |f|^2$$

Perhaps one could let: $\int d\sigma = (2l+1) \sum_{l,m,n}^{-1}$
and $dq = (8\pi^2)^{-1} \dots$
Then:

$$\int d\sigma |a_l|^2 = \int dq |f(q)|^2$$

end p 30

4. Crossed Channel Partial Waves: Now Q has the four possibilities previously stated, and we consider them one at a time.

[a] spacelike Q, $O(2,1)$, and regge poles.

① How now do you set up the hyperbolic coordinates? The vector p_1 is of course still timelike. What he does is this:

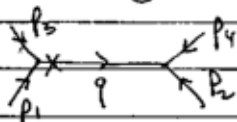
$$p_1 = m_1 (\text{ch} \alpha_1, \check{p}_1, \text{sh} \alpha_1)$$

where $\check{p}$ is a timelike $O(2,1)$ "versor":

$$\check{p}_1 = (\text{ch} \beta_1, \text{sh} \beta_1 \cos \phi_1, \text{sh} \beta_1 \sin \phi_1)$$

restart:

② OK, he is writing p_1^0 and p_3^0 in the standard Toller frame:



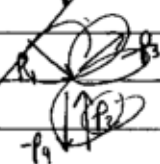
$$\text{Thus: } \begin{aligned} p_3^0 &= -m_3 (\text{ch} \alpha_3, 0, 0, \text{sh} \alpha_3) \\ p_1^0 &= m_1 (\text{ch} \alpha_1, 0, 0, \text{sh} \alpha_1) \end{aligned} \Rightarrow p_1^0 + p_3^0 = (0, 0, 0, \dots)$$

Then we do an arbitrary $O(2,1)$ on these 7 like gammas to right side 3, picking up β_1 and ϕ_1 ; it does nothing. Repeat on p_1^0, p_2^0 pair.

(9.3) Having parametrized our p_i , we now know that

$$M(p_1, p_2, p_3, p_4) = f(\beta_1, \beta_2, \phi_1, \phi_2)$$

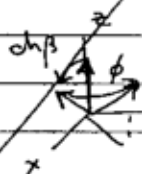
Of course, if we choose "as lab" the right Toller frame $\rightarrow x$, then $\beta_2 = 0$ and ϕ_2 is irrelevant. ~~The picture of the process is:~~



~~If we choose xyz as the axis:~~

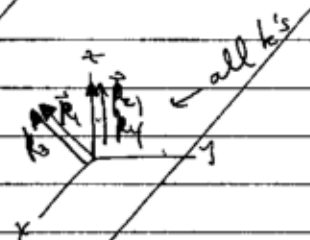
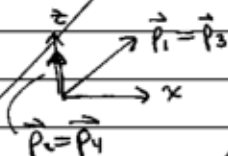
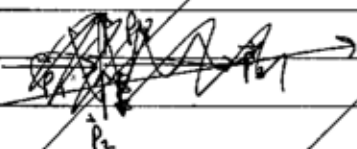
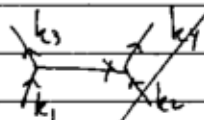


(9.4) Warning: be careful about the meaning of "azimuth" ϕ_1 . Start with a vector $\vec{p}_1$ in standard form:



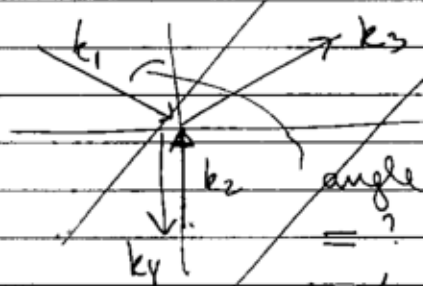
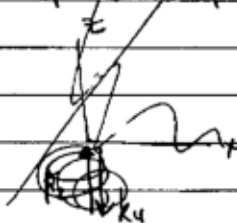
Then add some β to get some x ;
Then β to get some ϕ . So ϕ is
The usual azimuth,

Thus, to get $\phi_1 = 0$ we have to say:



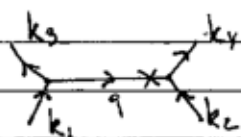
So setting $\phi_1 = 0$ means choosing $\hat{x} = \hat{p}_1$. Note that the right Toller frame $\rightarrow x$ is ~~not~~ a brick wall [need $-\vec{p}_4$ to see it]. The $\phi = 0$ merely makes $\vec{p}_1$ lie in the xz plane.

This is the analog of cms. Here, you can see θ as angle, whereas above the single variable is β . In fact, using k_1, k_2, k_3, k_4 all F.T.:



$$\tan \delta = \frac{\sin \phi_1}{f(\phi)}$$

(a.3, contd.) So, in the right Toller frame, here is what happens:



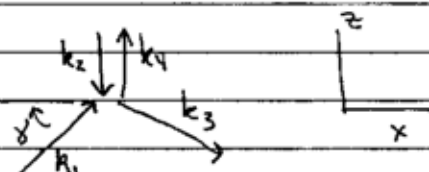
In particular,

$$(k_4)_z = (k_2)_z + \sqrt{-q^2}$$

$$E_4 = E_2$$

If $m_2 = m_4$, then $|(k_4)_z| = |(k_2)_z| \Rightarrow (k_2)_z = -\frac{1}{2}\sqrt{-q^2}$
 $(k_4)_z = +\frac{1}{2}\sqrt{-q^2}$

Then:



This is the brick wall. If $m_2 \neq m_4$, k_2 still $\parallel$ to k_4 but along the change. In this case, Toller frame is not the brick wall. I.e., $E_2 = E_4$, $m_2 \neq m_4 \Rightarrow |k_2| \neq k_4$.

(a.4) choose $\phi_1 = 0$ means k_1 is in xz plane. Where is β ?

$$\tan \delta = (k_1)_z / (k_1)_x = \text{sh} \alpha_1 / \text{ch} \alpha_1 \cdot \text{sh} \beta_1 = \text{th} \alpha_1 / \text{sh} \beta_1$$

~~If $m_2 \neq m_4$, $E_2 \neq E_4$ is not by values of S :~~

~~$$m_2 \neq m_4 \Rightarrow (k_2)_z \neq (k_4)_z \Rightarrow \text{th} \alpha_1 \neq \text{th} \alpha_2 \Rightarrow \alpha_1 \neq \alpha_2 \Rightarrow S \neq S'$$~~

So angle δ is a measure of β , the one parameter left.

(a.5) This Toller frame is the analog of the cms frame for $O(3)$.

(a.6) From standard frame we know that:

$$\text{ch} \alpha_1 = (t + m_1^2 - m_3^2) / (2m_1 \sqrt{-E}) = \text{ch} \alpha$$

At fixed t , β is a measure of S .

Finally: $F = F(\alpha, \beta) \quad \leftarrow$

⑦ Now we are ready to project this thing onto $o(2,1)$. Explicitly W shows all:

$$f(\phi, \beta, \psi) = \frac{1}{4\pi^2} \sum_{m,n} \sum_{\epsilon} \left[\int_0^{2\pi} dq \cdot a_{mn}^{\epsilon}(q) q \cdot \ln(\pi(q+i\epsilon)) d_{-\frac{1}{2}+iq, \epsilon}^{mn}(g) + \sum_{l=1-\epsilon}^M (l-\frac{1}{2}) b_{mn}^{\epsilon}(l) d_{l, \epsilon}^{mn}(g) \right]$$

Here, we are projecting onto ~~four~~ ^{four} separate classes:

$$\{\epsilon, q, m, n\}; a_{mn}^{\epsilon}(q) \quad \{\epsilon, l, m, n\}; b_{mn}^{\epsilon}(l).$$

These are the representation labels. The functions are:

$$d_{l, \epsilon}^{mn}(g) = e^{-i(m'\phi + n'\psi)} P_l^{m'n'}(\cosh \beta)$$

where

(a) m and n are integers!

(b) $\epsilon = 0, \frac{1}{2}$

(c) $m' = m + \epsilon$ $n' = n + \epsilon$

(d) $M = \min\{|m|, |n|\}$ if $mn > 0$; $M = 0$ if $mn \leq 0$.

It looks to me that all four reps are involved here!

reference: Vilenkin "Special Functions and Theory of Group Reps" (1968)

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T7
vol. 22

Note: This guy is a colleague of Gelfand et al in USSR. LBL does not have this book listed under V.

⑧ So OK, how do you invert?

$$a_{mn}^{\epsilon}(q) = \int_0^{2\pi} d\phi \int_0^{2\pi} d\psi \int_0^{\infty} \sinh \beta \cdot d\beta \cdot f(g) d_{-\frac{1}{2}+iq, \epsilon}^{mn*}(g)$$

$$b_{mn}^{\epsilon}(l) = (-1)^{m-n} \frac{\Gamma(l+m+\epsilon+1) \Gamma(l-m-\epsilon+1)}{\Gamma(l+n+\epsilon+1) \Gamma(l-n-\epsilon+1)} \cdot \int dg \cdot f(g) \cdot d_{l, \epsilon}^{m, n*}(g)$$

where dg is $d\phi d\psi \sinh \beta d\beta$, I presume.

⑨ Plancherel Formula:

$$\int |f(g)|^2 dg = \frac{1}{4\pi^2} \sum_{m, n, \epsilon} \left[\int_0^{\infty} dq \cdot q \cdot \ln[q+i\epsilon] \pi \right] |a_{mn}^{\epsilon}(q)|^2 + \sum_{l=1-\epsilon}^M X \cdot (l-\frac{1}{2}) |b_{mn}^{\epsilon}(l)|^2$$

B. The case $Q^\mu = \text{lightlike}$; E_2

① Why fancy coordinates. In standard frame, choose:

$$p_1^\mu = m_1 (\cosh \chi_1, 0, 0, \sinh \chi_1)$$

Now, consider this new 4×4 matrix:

$$\begin{bmatrix} 1 + \eta_1^2/2 & \eta_1 & 0 & -\eta_1^2/2 \\ \eta_1 & 1 & 0 & -\eta_1 \\ 0 & 0 & 1 & 0 \\ \eta_1/2 & \eta_1 & 0 & 1 - \eta_1^2/2 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = \begin{bmatrix} A + A\eta_1^2/2 + B\eta_1 - \eta_1^2/2 D \\ \eta_1 A + B - \eta_1 D \\ C \\ \eta_1/2 A + \eta_1 B + D - \eta_1^2/2 D \end{bmatrix} = \begin{bmatrix} A' \\ B' \\ C' \\ D' \end{bmatrix}$$

In general, this makes a mess. Notice however that:

$$A' - D' = A - D \quad / \text{is left invariant}$$

This matrix must be that famous so-called "lightlike boost".
When it acts on a lightlike vector, you get: *this particular*

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} A \\ 0 \\ 0 \\ A \end{bmatrix} = \begin{bmatrix} A \\ 0 \\ 0 \\ A \end{bmatrix}$$

/this thing leaves lightlike
vectors untouched!

② Why is the group E_2 ?

Define a map: $F: p^\mu \rightarrow \xi \quad F: \mathbb{R}^4 \rightarrow \mathbb{R}^2$

namely: $\xi_1 = \frac{p_1}{p_0 - p_3} \quad \xi_2 = \frac{p_2}{p_0 - p_3}$

Now also map the 4×4 matrix to a 2×2 :

$$\xi_1' = \frac{p_1'}{p_0' - p_3'} = \frac{p_1'}{p_0 - p_3} \quad \text{since invariant } p_0 - p_3 = a p_0 + p_1 - a p_3$$

$$\xi_2' = \frac{p_2'}{p_0 - p_3} = \frac{p_2}{p_0 - p_3} = \frac{p_1}{p_0 - p_3} + a$$

Thus: $T_1(a) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} \rightarrow \begin{bmatrix} \xi_1 \\ \xi_2 + a \end{bmatrix}$

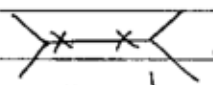
Clearly, $\begin{pmatrix} \xi_1' \\ \xi_2' \end{pmatrix} = \begin{pmatrix} \xi_1 + a \\ \xi_2 \end{pmatrix}$, so a translation.

Similarly, $T_2(\xi_1) = (\xi_1 + b)$ etc. So the little group of a lightlike vector is isomorphic to E_2 .

③ What the hell are these creatures? It must be a "rotation" about a lightlike vector?

$$T_1(a) \begin{bmatrix} E \\ x \\ y \end{bmatrix} = \begin{bmatrix} 1 + \frac{a^2}{2} & a & -a/c \\ a & 1 & -a \\ a^2/2 & a & 1 - a^2/2 \end{bmatrix} \begin{bmatrix} E \\ x \\ y \end{bmatrix}$$

Yes. The particular lightvector you rotate around is $p^\mu = (A, 0, 0, A)$. Other lightlike vectors do get changed, just as $O(2,1)$ in general changes a spacelike vector.

④ So as in $O(2,1)$, you can draw a picture  with some standard frames. In right frame, $\alpha_2 = 0$, $\phi_2 = \text{irrelevant}$. Choose $\alpha_1 = 1$, $\phi_1 = 0$ as before. So, again the amp. boils down to

$$f(\alpha).$$

⑤ Now expand this thing in E_2 reps [again, Vilenkin, NT, is the reference]. The rule is:

$$a_{nm}^k = \frac{1}{4\pi^2} \int d\alpha d\phi \int da f(\alpha, \phi) a_{nm}^k(g)$$

$$f(g) = \sum_{nm} \int_0^\infty k dk a_{nm}^k a_{nm}^k(g)$$

$$\text{where } a_{nm}^k(g) = i^{n-m} e^{-i[n\alpha + (m-n)\phi]} J_{n-m}(kr)$$

In our case, only J_0 enters, looks like "Fourier Bessel".

⑥ References are given here

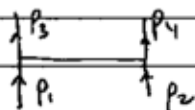
- (a) modification for regge pole convergence
- (b) relation to integrals

C. The case $Q^\mu = \text{timelike}$ (unequal masses only)

- ① OK, as usual write $p_i = m_i (\cosh a_i, \hat{n}_i \sinh a_i)$ etc. For spinless, expand $f(\theta)$ in $P(\cos \theta)$. "Little use has been made of this".

D. The case $Q^\mu = (0, 0, 0, 0)$; the $O(3, 1)$ expansion.

- ① Elastic forward scattering with π - N mass structure



$$p_i = m (\cosh a_i, \hat{n}_i \sinh a_i)$$

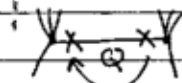
$$p_z = M (\cosh b, \hat{n}_z \sinh b)$$

Standards are $p_i^0 = p_z^0 = m(1, 0, 0, 0)$. Now how does he parametrize the general $O(3, 1)$?

$$\Lambda = \underbrace{R_z(\phi) R_y(\theta) R_z(\alpha)}_{\text{final orient}} \underbrace{B_z(a)}_{\text{boost}} \underbrace{R_y(\beta) R_z(\gamma)}_{\text{boost plane pre-set}}$$

This is to be applied to a F.T. vector, so clear.

- ② Comment: suppose you have $\equiv \bigcirc \equiv \leftarrow Q^\mu \equiv 0^\mu$. Still, you set up standard frames:

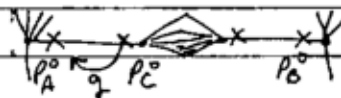


And you connect via Λ above.

No need to worry about designing a $\Lambda \in O(3, 1)$ for all those spacelike lines ... But what about

$$d^4 q_1, d^4 q_2, \dots, d^4 q_n, \delta(Q - \sum_{i=1}^n q_i) = d^4 q_1, \dots, d^4 q_{n-1}, d^4 Q$$

Perhaps picture should be:



Then go to frame P_A^0 +

write all your p_i 's in

terms of P_C^0, g . But, again you will have to know how to write your P_C^0 standard vectors! Wait on this.

- ③ In our case, $\vdash \times$ choose frame shown $\Rightarrow a=0, \theta=0, \phi=0$. Or, choose cms, so

$$p_i = m (\cosh a, \hat{n}_i \sinh a)$$

$$p_z = M (\cosh b, -\hat{n}_z \sinh b)$$

④ So far, we are down to $\Lambda_A = \Lambda_A(\alpha, \beta/2, \tilde{\beta}, \gamma)$
 $\Lambda_B = \Lambda_B(\pi, -\pi, 0, \beta/2, \mu, \nu)$

Each ends with an $O(3)$ transform, but $O(3)$ does nothing to your standard vectors, so get rid of dependence on $(\alpha, \tilde{\beta}, \gamma)$ (π, μ, ν) . Thus, the end result is:

$F(\beta)$ $\beta = \text{overall boost.}$

⑤ How do we project square integrables? Write

$$g = u_1(\phi, \theta, \psi) a_{03}(\beta) u_2(0, \pi, \chi) \quad N1, V4 \text{ are refs.}$$

where now $u, a = 2 \times 2$ complex unitary matrices, since

$$SO(3,1) = SL(2, \mathbb{C})$$

Now the "rotation matrix" will carry 6 indices:

$$\rightarrow D_{\sigma\sigma', mm'}^{\nu\sigma}(g) = \sum_{\lambda=-x}^x D_{\sigma}^{m\lambda}(u_1) d_{\sigma\sigma'\lambda}^{\nu\sigma}(\beta) D_{\sigma'}^{\lambda m'}(u_2)$$

$x = \min(\sigma, \sigma')$

Here, $D_{\sigma}^{m\lambda}(u_1)$ is the $O(3)$ rotation matrix recorded earlier, but

$$d_{\lambda\lambda\sigma}^{\nu\sigma}(\beta) = \text{messy hypergeometric } {}_2F_1.$$

The labels $\sigma\sigma' mm' \nu$ are all integral, but g is continuous.

Namely:

$$f(g) = (2\pi)^{-4} \sum_{\nu=-\infty}^{+\infty} \int d\rho (4\nu^2 + \rho^2) \sum_{\sigma, \sigma'=|\nu|}^{\infty} \sum_{mm'} A_{\sigma\sigma' mm'}^{\nu\sigma} D_{\sigma\sigma' mm'}^{\nu\sigma}(g)$$

Inverse:

$$A_{\sigma\sigma' mm'}^{\nu\sigma} = \int_0^2 \sinh \beta d\beta \cdot \sin \theta d\theta \cdot \sin \chi d\chi \cdot d\phi d\psi f \cdot D^*$$

$\uparrow$ This is the kind of result I got too!

⑥ Special case:

$$D_{0000}^{\sigma\sigma}(g) = d_{000}^{\sigma\sigma}(\beta) = \frac{2}{\beta} \frac{\sin(\beta/2)}{\sinh \beta}$$

$$\text{Then } f(\beta) = \int_{-1-i\infty}^{-1+i\infty} d\sigma A(\sigma) P_{\frac{\sigma}{2}+\sigma}^{\frac{\sigma}{2}}(\cosh \beta) (\sigma+1)^{-\frac{\sigma}{2}}, \text{ etc.}$$

- ⑦ An outcropping of $O(3,1)$ work: For unequal masses, $t=0$ cuts through the physical region. The $O(3,1)$ analysis [Freedman + Wang] shows that to have both regge behavior and analyticity in t , you need daughters! Some $O(4)$ RSE analysis also gives this result.

The way this works is easy to see. We get something like:

$$f(\beta) = \int d\sigma \cdot A(\sigma) \cdot \int dl \cdot P_l(ch\beta) \cdot \Gamma(\sigma + l + 2)$$

Thus, if $A(\sigma)$ has a pole at $\sigma = \sigma_0$, the σ -integrand will also have poles at $\sigma + l + 2 = \text{neg. integer}$. These are the daughters, + each has an effect on $f(\beta)$.

Conclusion of this Chapter: There now exists a covariant formalism based on $W^\mu = \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} P_\nu M_{\rho\sigma}$ [Pauli-Lubanski spin operator] which gives all the cases just discussed as special cases [see PR 168, 1587-1593 (1968)]. i.e., FF3, FF4.

- ⑧ Note: W. never clearly discusses how you go to "second kind" functions to get better convergence.

Exercise: reduction of $\alpha(2,1)$ projection for $F(\beta)$

① $m=n=0$, so choosing the $\epsilon=0$ principle series we have:

$$F(\beta) = \frac{1}{4\pi^2} \int_0^\infty a(q) P_l(\cosh \beta) q \ln[\pi(q)] dq.$$

where $l = -\frac{1}{2} + iq$.

② Change variables after using $P_l = P_{-l}$ for conicals, so:

$$F(\beta) = \frac{1}{8\pi^2} \int_{-\infty}^\infty q dq \ln(\pi q) a(q) P_l(\cosh \beta).$$

Well,

$$l = -\frac{1}{2} + iq \quad dl = i dq \quad q = -\infty \Rightarrow l = -\frac{1}{2} - i\infty.$$

$$\ln\left[\pi\left[\frac{l+\frac{1}{2}}{i}\right]\right] = -\ln[\pi i(l+\frac{1}{2})] = -i \tan\left[\pi l + \frac{\pi}{2}\right] = +i \cot(\pi l)$$

Thus: $\frac{1}{8\pi^2} q dq \ln(\pi q) = \frac{1}{8\pi^2} \cdot \frac{l+\frac{1}{2}}{iq} \cdot \frac{dl}{i} \cdot i \cot(\pi l)$

$$= \frac{1}{16\pi^2 i} (2l+1) dl \frac{\cos(\pi l)}{\sin(\pi l)} = \frac{1}{16\pi^2 i} (2l+1) dl \cot(\pi l)$$

③ Thus: $F(\beta) = \frac{1}{2i} \cdot \frac{1}{8\pi^2} \cdot \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} dl \frac{(2l+1)}{\sin(\pi l)} \cdot a(l) \cos \pi l \cdot P_l(\cosh \beta)$

Now group: $\frac{a(l) \cos \pi l}{8\pi^2} \equiv A(l)$

Then:

$$F(\beta) = \frac{1}{2i} \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} dl \frac{(2l+1)}{\sin(\pi l)} A(l) P_l(\cosh \beta)$$

This is (2.57) of Winternitz.

book says (-1) here.

④ The inverse is:

$$a(l) = \int d\phi \int d\psi \cdot F(\beta) \cdot \int_0^\infty \sinh \beta d\beta \cdot P_l(\cosh \beta) = (2\pi)^2 \cdot \int_0^\infty F(\beta) \sinh \beta d\beta P_l(\cosh \beta)$$

$$\Rightarrow A(l) = \frac{\cos(\pi l)}{8\pi^2} \cdot (2\pi)^2 \cdot \int_0^\infty F(\beta) \sinh \beta d\beta P_l(\cosh \beta)$$

May 11, 1976

Boyce et al. The big PWA thing

1. U-reps of Poincaré
definitions, algebra, little groups.
2. Classification of IR's.
 - (i) timelike
 - (ii) spacelike
 - (iii) lightlike
 - (iv) nullparametrization, relations between parameters
3. Reduction of Direct Product.
lots of cases, many explicit results.
4. Analytic Properties
The Casimir DE's
expansion theorems
first kind functions
second kind functions
all subgroups!
5. Improper.
6. PWA.

Comments: no continuous or mixed basis of $O(3,1)$
no NSI, no diagonalization.
never published.
no part II even as preprint!
no references!

What information is missing? Mixed basis matrix elements? Maybe stick to continuous basis. Expansion & Addition Theorems in the continuous basis.

Relation ~ Why do we not use the actual ME's themselves? Why no bi-valued index? Need Hilbert Space + other ideas.

!) derivation of continuous basis ME's for principle & discrete series. ~ must set up scalar products in order to have V.I.R's. What is the relevant scalar product?? See

Munkunda:-

What is a representation? $U(g) f(x) = f(\tilde{g}x)$ a what ever - point transformation. Also, c

Maybe I should revert to Valentini's forms:

$$T_x(g) f(e^{i\theta}) = |\beta e^{i\theta} + \bar{\alpha}|^{-2} f\left(\frac{\alpha e^{i\theta} + \bar{\beta}}{\beta e^{i\theta} + \bar{\alpha}}\right)$$

This is a "representation of $SU(1,1)$ " ~ no particular series or anything. Derived in Valentini. Still on unit circle.

circle $\theta \in 0, 2\pi$

function on circle

=

M-Space

=

R-space

function = measure.

How do you make $f(e^{i\theta}) \in \mathcal{H}$ a Hilbert Space? (f, g) needs definition. OK.

$$(f, g) = \int_0^{2\pi} d\theta f(\theta) g(\theta)$$

Why does this go with principle series??

$$(U(g)f_1, U(g)f_2) = (f_1, f_2)$$

invariant scalar product?

$$\text{Why need this? } \langle Uf_1 | Uf_2 \rangle = \langle f_1 | U^\dagger U | f_2 \rangle \quad U^\dagger U = 1$$

$$\text{OK, } U^\dagger U = 1 \text{ unitary rep}$$

If a rep is to be unitary, then $U(g)$ as an operator must ~ unitary means w.r.t a scalar product. So

So take your value of l + search for a unitary scalar product.

But ~ who cares if unitary? Also, why not use all of G as your M -space

$$(f_1, f_2) = \int dg \, f_1(g) f_2(g)$$

$$\text{Then, } (T(g_1) f_1, T(g_2) f_2) = \int dg \, [T(g_1) f_1(g)]^* [T(g_2) f_2(g)]$$

But ok, This is an inner product, but the rep is not an IR. Where does circle come from? Subspace?

From where do you drag up your M -space??

With each VIR goes a special HS. Why? - VIR. $R = \text{rep}$ by operators in a space... $V \Rightarrow$ operators and unitary.
 $\Rightarrow$ HS: $D_k^\dagger \rightarrow \mathcal{H}_k$

✓ Wang
 "Clebsch ... for $O(2,1)$ "
 JMP 11, 2077 (1970) (19 pages)

Another Wang paper on "Diag. of V .
 Role..." PRD4, 489 (1971)

I. Introduction. Says AG have already done the $D_K^+ \otimes D_K^+ \otimes D_K^+$ Clebsch problem in their same 1964 paper. HB did something, but not "continuable". Wang will do $Q \times Q$ à la AG

II. Decomposes $Q \times Q$ (restricted to $v_i > \mu_i > 0$ where $d_{\nu_i \mu_i}^{j_i}$). $Q = -J^2$. Wang's j is offset by $\frac{1}{2}$ unit from usual j . Uses AG-like d -function. Second-kind function is called $a_{\nu \mu}^j(z)$.

Out of the blue, it is claimed that $C(j_1, j_2, j_3; \nu_1, \nu_2)$ should satisfy

- (a) Clebsch Series relation in terms of d -functions
- (b) a certain recursion relation
- (c) orthogonality.

Next, the rule that $F(\dots) \times F(\dots) = \sum_n \text{coeff} \cdot F(\dots)$ is used to write:

$$a_{\nu_1 \mu_1}^{j_1}(z) a_{\nu_2 \mu_2}^{j_2}(z) = \sum_{n=0}^{\infty} (\text{coeff}) \cdot a_{\nu_3 \mu_3}^{j_3}(z)$$

$(j_3 \equiv j_1 + j_2 + n + \frac{1}{2})$

and $\nu_3 = \nu_1 + \nu_2$ $\mu_3 = \mu_1 + \mu_2$. The (coeff.) involves two $3F_2$ or "Morse-Whipple $F_2(0.45)$ " functions. Each of these two guys is then called $G(j_1, j_2, j_3; \nu_1, \nu_2)$.

The sum over n is then SOT + the total result is:

$$a_{\nu_1 \mu_1}^{j_1}(z) a_{\nu_2 \mu_2}^{j_2}(z) = \sum_{\substack{\text{my } j \\ \text{from } (j_1, j_2)}} \frac{d_{j_3}(z_{j_3+1})}{\text{factor}} [G \cdot G + G \cdot G] d_{\nu_3 \mu_3}^{j_3}(z) + \sum_{j_3=1}^{\infty} (z_{j_3+1}) [G \cdot G + G \cdot G] (11)$$

Compare to the $\alpha(3)$ result.

$$D_{m_1 \mu_1}^{j_1} D_{m_2 \mu_2}^{j_2} = \sum_j C(\dots) C(\dots) D_{m_1 + m_2, \mu_1 + \mu_2}^j$$

So it's roughly the same, only Plancherel differs, + there is multiplicity of 2.

Next, a pair of Clebschs C_1 and C_2 are carefully defined to solve the principal series problem.

Next, the case $j_1 = Q$ $j_2 = D_K^+$ is done, also vice versa. Then only one CGC. Finally the D_K^+, D_K^+ case is done.

Outline of this paper:

- II. define: 1) "d", "a" functions
2) CG series + coeff. properties
3) derive the rules:

$$a \cdot a = \sum \text{res}[a \cdot a] a$$

$$d \cdot d = \sum_{i,j} d_{ij} [aa + \bar{a}\bar{a}] a^{ij}$$

← fancy contour zigzag.

~~II.1~~

$$= \sum_{\text{all labels}} d_{ij} [aa + \bar{a}\bar{a}] \cdot d$$

These things are found by brute force. This ↑ is the result for $C_9 \times C_9$.

- III. math properties of core functions
from which you make your CG's.
[G, $3F_2$, Whipple + Thomae, etc].

- IV. Explicit CG's calculated for $\nu > \mu > 0$ in cases
- | | | | |
|----|------------------|-------------------|--------------------------|
| A. | $C_9 \times C_9$ | ← C_1 and C_2 | } only one in each case. |
| B. | $C_9 \times D_k$ | | |
| C. | $D_k \times C_9$ | | |
| D. | $D_k \times D_k$ | | |

V. Extend results to all helicity values

VI. $O(2,2)$ group = $o(2,1) \otimes o(2,1)$.

VII. Conclusions:

✓ Ferretti + Verde

"On the Wigner coeff's for $\alpha(2,1)$ "
NC 55A, 110 (1968)

[14 pages].

I. Intro. Will not do the continuous basis problem, until future paper. $[j]$ = notation. Bailey book ref'd (1935) on Gen. HG series.Will only do $Q \times C_q$, refs BH, P and AG for others.1. VIR's of $\alpha(2,1)$ [called (3)] $d_{j\mu}^j(z) \approx B_{j\mu}^j(z)$. Writes Casimirc DE! Martine Green Fock Method!Orthog + Completeness stated.
← Peter Weyl. $A_{j\mu}^j \approx A_{j\mu}^j = \text{second-kind} \approx a_{j\mu}^j$ 2. Assumes CGC's (to be found below) + gets properties of same.

$$F_{\nu}(0,14) \sim {}_3F_2(\dots; 1)$$

3. Evaluation of 3-j Coeffs

$$F(\dots) F(\dots) = \sum_n {}_3F_2 {}_3F_2 \cdot F(\dots)$$

$$\Rightarrow a \cdot a = \sum_n C_n \cdot a$$

$$\Rightarrow a \cdot a = \int d_{j3} \text{ fancy } [c \cdot c + c \cdot c] \cdot a$$

$$\Rightarrow d \cdot d = \int d_{j3} [cc + \bar{c}\bar{c}] a$$

Finally:

$$d^{j_1 j_2}_{j_3} d^{j_1 j_2}_{j_3} = \int d_{j_3} [cc + \bar{c}\bar{c}] d_{j_3} \left. \vphantom{\int} \right\} \text{results.}$$

$\alpha(2,1)$ product

$$\text{and } \int d_{j_3} d \cdot d \cdot d \cdot d = cc + \bar{c}\bar{c}.$$

Comments:

Contains a reference

Giovannini + Verde NC 34, 1306 (1964) 8 pages.
which does my Green Fctn method
 $\delta(z-z')$ for $O(2,1)$, $O(3)$ and $E(2)$!

The $E(2)$ equation is different from
the two, so different fctns, etc.

reference	YEAR of writing	basis: 1=0(2), 2=0(1,1) 3=mixed												all series?	the transport problem?	SOC at same time?	matrix elements?	application?	Clebsch problem?
		finite x forms?	function H space?	invariant H.S.?	boson calculus?	expansion theorem?	D, C, ...												
Bargmann	1947	1	✓	✓			✓	✓			✓								
Bayes	1966	1					✓	✓		✓	✓								
Cronstrom	1970	∞							✓										
Abarlo+Sandera	1970	∞															✓		
Goddard White IA	1970	1,3					✓	G,D			✓	✓							
KMS	1967	2	no	no	no	no	no	Y	n	Y	n	n							
Andrews+Ginsman	1964	1	no	no	no	no	Y	C,D	no	Y	Y	no.							
CDM	1969	2,3	Y	Y	no	no	no	no!	no	no	Y	Y							
MM	1969	2,3	Y	Y	-	-	Y	only	-	-	Y	Y							
Mukunda	1967	2	Y	Y	-	-	NO	P,D	no	no	no	no.							
JLY, PRD	1971	3	-	-	-	-	Y	Poly	-	Y	no	Y							
JLY-70	1970	1	Y	Y	-	-	Y!	P,D	Y!	-	Y	-							
JLY-71	1971	3	Y	no	-	-	Y	P,D	Y!	-	Y	-							
MOK-S1	1968	2	Y	no	-	-	no	D!	no	no	Y!	-							
MUK-S2	1968	2	Y	no	-	-	no	C!	no	no	Y!	-							
MUK-S3	1972	1,2	Y	no	-	-	Y!	P,D	no	no	-	-							
MR-71	1971	~2	Y	Y!	-	Y!	NO	P,D	NO	no	na	-							
PR-73	1973	2	-	-	-	-	Y!	P,D	Y!	no	Y!	-							
Barut-Friedal	1965	1	✓	*	Y	no	NO	all	no.	Y!	Y!	yes.	Y	* relation to Bargmann!					
Holman, Bietenham	1966	1	-	-	no	Y!	NO	all	no	Y!	Y	-	Y						
Barut Phillips	1968	1,2*	Y	no.	Y	Y	NO	all	no	Y!	Y.	-	-				* multiphoton.		
MR, S1	1973	2	-	-	-	Y!	no	-	-	-	-	-	Y!						
Wybourne book	1973	1	Y	no	Y	Y	no	all	no	Y	Y	Y	Y						
Vilenkin book	1965	1	Y	Y!	-	-	Y!	all	no	no!	Y!	Y*					* special Schur.		
Gelfand et al.	1962	1		Y!			?	✓											
Ours+Klink	1971	1	Y	no	-	-	Y	C,D	Y!	no	Y	Y	no						

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того уравнения, выведенного в первой статье этой серии. Исследование проводится путем замены решетки точек для четырехмерного пространства и времени континуумом. Затем уравнение сводится к системе уравнений в частных производных с непостоянными коэффициентами. Структура этой системы подробно анализируется, и мы приходим к выводу, что только подмножество этих уравнений должно рассматриваться как уравнения в частных производных, а оставшиеся имеют лишь алгебраическое содержание. Затем мы переходим к непосредственному конструированию решений, удовлетворяющих дополнительным ограничениям симметрии и трансляционной инвариантности. Для этого класса решений система коллапсирует в систему обыкновенных дифференциальных уравнений. Это существенное упрощение позволяет сконструировать точные решения, которые обнаруживают существенную особенность для исчезающе малой величины константы связи. Расходящееся пертурбационное решение для этой системы может быть получено из точного решения, используя непосредственную процедуру, которая заменяет точное выражение для остатка ряда. Из этого видно, что при соответствующих ограничениях на матрицу распространения пертурбационное разложение может быть асимптотикой для точных решений. Допускается, чтобы пропагаторы при ограничениях не представляли с необходимостью физически значащие пропагаторы. Однако, имеется другое разложение для этих решений, которое сходится для величины g^2 , отличной от нуля, при слабых ограничениях на матрицу распространения. Вопрос о том, какие из полученных заключений справедливы для решетки, которая может быть распространена на четырехмерный пространственно-временной континуум, не исследуется в этой статье.

Harmonic Analysis in an $SU_{1,1}$ Basis.

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(ricevuto il 26 Febbraio 1969)

Summary. — The theory of harmonic analysis on $SL_{2,0}$ is developed using an $SU_{1,1}$ basis. It is found that regarded as an integral kernel, the Fourier transform (FT) of a function defined over the group acquires two additional labels τ representing the two disjoint subspaces needed in the analysis. The representation functions are used to discuss $SU_{1,1}$ -bivariant distributions: we find that the FT cannot be carried out using them alone, because of parametrization difficulties. The problem of equivalence is discussed; conditions upon the transforms obtained, and functions of the second kind, shown to be just the representation functions themselves. The work is applied to analysis of the two-particle elastic-scattering amplitude at and away from $t_\mu = 0$, and nonunitary representations used to give a natural description of daughter trajectories and their properties.

1. — Introduction.

In a previous paper ⁽¹⁾ we have examined the decomposition of $SL_{2,0}$ under the subgroup $SU_{1,1}$ and evaluated the matrix elements in the corresponding basis of finite transformations of the group. That paper ended with an indication of the applicability of the work to the description of daughter trajectories: in the present article we intend to give a more detailed and rigorous demonstration of this.

The motivation of our discussion is the need to parametrize the scattering amplitude in such a way that the «daughters» of a Lorentz pole appear naturally. We have been struck by the apparent inconsistency in spirit of

⁽¹⁾ N. W. MACFADYEN: Cambridge preprint DAMTP 68/28, to be published. We refer to this as I.

equation. In fact, this phenomena occurs for all values of A closed to the square of an integer, but for $A > 36$ the unstable zones are so small that they cannot be shown on Fig. 1. If Q/A is small enough the width of each zone is of the order of $A^{-1/2}(Q/A)^{1/4}$ and in the adiabatic limit such an effect is totally ignored.

On the μ curve of Fig. 2 these instable zones appear around $A = 25$ and $A = 16$. We notice that μ_2 just bridges the discontinuities of the exact μ curve and comes back on this last curve while the μ_1 approximation is definitively too large. Finally, for the value $A < 10$ the adiabatic limit has no meaning.

We notice the large improvement of the μ_1 approximation upon the lowest-order WKB method.

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¹ H. R. Lewis, Phys. Rev. Letters 18, 510 (1967).

² H. R. Lewis, Phys. Rev. 172, 1313 (1968).

³ H. R. Lewis, J. Math. Phys. 9, 1976 (1968).

⁴ S. Chandrasekhar, *The Plasma in a Magnetic Field*, R. K. M. Landshoff, Ed. (Stanford U.P., Palo Alto, Calif., 1958).

⁵ E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis* (Cambridge U.P., Cambridge, 1966).

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The Reduction $O(3, 1) \supset O(2, 1) \supset O(1, 1)$

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We set up a representation of the principal series of the group $SL(2, C)$ in terms of the sequence of noncompact subgroups $SL(2, C) \supset SU(1, 1) \supset O(1, 1)$. The basis functions are just the cross-basis matrix elements of $SU(1, 1)$ between $O(2)$ and $O(1, 1)$ bases, and we derive much material on these before using them to calculate the representation functions. The latter have two pairs of discrete labels distinguishing between equivalent representations occurring in the reduction, and these are related to two discrete reflection operators that are introduced in order to obtain a maximal Abelian set.

INTRODUCTION

In a previous paper¹ (to be referred to as I) we examined the reductions $SL(2, C) \supset SU(1, 1) \supset O(2)$ and $SL(2, R) \supset O(1, 1)$ and gave a complete account of the global properties of the irreducible representations in the corresponding bases. We did not, however, consider the reduction in the sequence of noncompact subgroups $SL(2, C) \supset SU(1, 1) \supset O(1, 1)$; while this has no very new features, it is of interest to see how our previous results combine, and so here we present the theory of this reduction for the principal continuous series of representations.

The interesting part of the problem is concerned with the basis functions we must use. There are now no fewer than four sets of these, distinguished by a pair of degeneracy labels, because each group representation contains those of its subgroup twice over; and they are just the "cross-basis" matrix elements of $SU(1, 1)$ that we can label schematically

$$\langle O(2) | \exp(iK_2\xi) | O(1, 1) \rangle.$$

Much of the paper is devoted to defining and examining these functions; indeed the first two sections and Appendix A are devoted entirely to these preliminaries, and only in Sec. III do we at last discuss the eponymous problem.

We achieve a formulation of the reduction that in principle is complete, but in practice the only coset class whose representation functions we can calculate is that one which we were unable to treat in I. As expected, the asymptotic behavior of these functions in the complex plane of the Casimir operator σ is just that of I; we can also show that with this basis we do not obtain matrix elements of an element $a \notin SU(1, 1)$ that are simultaneously of simple behavior in both σ and j , thus disposing of any hopes of a "completely second-kind" set of representation functions. Hence, the main interest (or usefulness!) of the work lies in the cross-basis matrix elements; there are some indications that these may be of use in the multi-Regge theory.

This paper is essentially a sequel to I, to which we refer constantly; hence all the results of that paper

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10. V. A. Fock, CR (Doklady) Acad. Sci. U.R.S.S. (N.S.) 39, 253 (1943).
11. Bateman Manuscript Project, Vol. 1, Higher Transcendental Functions, edited by A. Erdelyi et al (McGraw-Hill, New York, 1953).
12. A. O. Barut, The Theory of the Scattering Matrix (MacMillan, New York, 1967), Section 9.1.
13. T. K. Gaisser and C. E. Jones, Phys. Rev. 184, 1602 (1969).
14. E. T. Whittaker and G. N. Watson, A Course of Modern Analysis (Cambridge, 1927), Chap. 11.

EXPANSION THEORY AND THE LORENTZ GROUPS IN NON-CANONICAL BASES†

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I. Introduction

There are several reasons for studying the reduction $O(3,1) \supset O(2,1)$: the intrinsic interest of the topic; the elucidation of the role of "second-kind" functions;¹⁾ the possible application to Regge theory; and, most pertinent to this symposium, the study of how things can go wrong if we choose an unusual basis. Since there are so many reductions of $SO(4,2)$, it may be of interest to examine the unexpected behavior of such a well known group as $SL(2,C)$ in an $SU(1,1)$ basis.

The results I shall present are by no means all new: the reductions $SL(2,C) \supset SU(1,1)^{2)}$ and $SL(2,R) \supset O(1,1)^{3)}$ have been treated by many people, and I cannot mention them all here. As far as I am aware, the actual use of the matrix elements of finite transformations in this basis to expand functions defined over the group is new, as is also the brief summary I shall give of the chain $SL(2,C) \supset SU(1,1) \supset O(1,1)^{4)}$. This is not the place to give explicit proofs or detailed arguments, and so these will be almost completely absent. They can be found in the references.

II. The Reduction $SL(2,R) \supset O(1,1)$

Because this problem displays so many features of the higher dimensionality without also having its complexities, I shall go into most detail here and simply give an umbrella assurance that proofs for that more interesting case follow the same lines.¹⁾ I apologize to all those to whom this is quite familiar.

Consider then a representation j of $SL(2,R)$. Instead of letting the operators act on special functions defined over a homogeneous space of the group--i.e.,

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$$T_r: Y_j^m(\Omega) = Y_j^m(\Omega_r), \quad (1)$$

it is much more convenient to carry the j -dependence in the action of the operators T_r , and let the basis functions be simple. Following the Russian school, we set up our representation on a space of functions defined over a hyperbola we shall parametrize by β : the τ -label specifies the sheet, and $\beta \in (-\infty, \infty)$ the position thereon, and the operators T_r^j are specified by

$$[T_r^j: f]^\tau(\beta) = |\lambda|^{2j} (\text{sign } \lambda)^\nu f^{\tau'}(\beta') \quad (2)$$

where the parameters are defined uniquely by

$$\begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \text{ch } \beta/2 & \text{sh } \beta/2 \\ \text{sh } \beta/2 & \text{ch } \beta/2 \end{pmatrix} \begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix} = \begin{pmatrix} \lambda^{-1} & \mu \\ -\lambda & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \text{ch } \beta'/2 & \text{sh } \beta'/2 \\ \text{sh } \beta'/2 & \text{ch } \beta'/2 \end{pmatrix} \quad (3)$$

or in matrix notation, $\epsilon^\rho \beta r = k \epsilon^{\rho'} \beta'$. The parameters τ and ρ are discrete; $\tau = \pm 1$ only and $\rho = \frac{1}{2}(1 - \tau)$. The label ν is either 0 or 1, depending on whether the eigenvalues of J_3 in the given representation are integer or half-integer.

It is easy to check that this is indeed a representation of $SL(2, R)$, and clearly we have diagonalized the generator K_1 , conjugate to the boost β : that is,

$$[T_{\beta_1}^j: f]^\tau(\beta) = f^\tau(\beta + \beta') \quad (4)$$

The new feature is the appearance of the label τ . We can look at this in two ways, of which the simpler is algebraic: the operators $\Delta = J^2 - K_1^2 - K_2^2$ and K_1 do not form a maximal Abelian set but can be augmented by a reflection R which in the standard homomorphism of $SL(2, R)$ and $O(2, 1)$ has the significance of a change of sign of the 2-axis:

$$R: (x_0, x_1, x_2) = (x_0, x_1, -x_2). \quad (5)$$

The other way of regarding this is as a corollary of Gel'fand's horospheric method--it is exactly analogous to the two terms required^{(4), (5)} to expand a function defined over a hyperboloid with a hyperbolic coordinate system. In either case, (2) tells us that provided the representation j is irreducible, each representation μ of $SO(1, 1)$ is doubly degenerate therein. We shall treat principal series representations: if $\text{Re } j = -\frac{1}{2}$, we can introduce an inner product

$$(f, g) = \frac{1}{2\pi} \sum_{\tau} \int_{-\infty}^{\infty} \overline{f^{\tau}(\beta)} g^{\tau}(\beta) d\beta \quad \checkmark \quad (6)$$

and then provided $f(\beta)$ belongs to a certain space of functions $\mathfrak{H}_j$, the representation is both unitary and irreducible.

Having thus set up a representation, we must choose a set of basis functions. Since the τ are discrete labels, these are obviously also 2-vectors, and a convenient choice is just

$$\varphi_{\mu}^{+}(\beta) = \begin{pmatrix} e^{i\mu\beta} \\ 0 \end{pmatrix} \quad \varphi_{\mu}^{-}(\beta) = \begin{pmatrix} 0 \\ e^{i\mu\beta} \end{pmatrix}. \quad (7)$$

Obviously many other choices are possible: a different one with some advantages (particularly for the supplementary series of representations) is to take the sum and difference of these, corresponding to eigenvalues of the reflection R . Clearly, $(\varphi_{\mu}^{+}, \varphi_{\mu'}^{+}) = \delta_{\tau\tau'} \delta(\mu - \mu')$.

The vectors actually do not belong to $\mathfrak{H}_j$ because they are not square-integrable, but we can regard them as members of the dual space in the usual way.

III. Matrix Elements

Let us first consider what we have to calculate. It is found that the parametrization $r = \beta b \beta'$ covers $SL(2, R)$ only with three choices of b : convenient ones are:

$$b_1 = \xi = \begin{pmatrix} e^{\xi/2} & \\ & e^{-\xi/2} \end{pmatrix} \quad b_2 = \xi c = \begin{pmatrix} & e^{\xi/2} \\ -e^{-\xi/2} & \end{pmatrix} \quad b_3 = \theta = \begin{pmatrix} \cos \theta/2 & \sin \theta/2 \\ -\sin \theta/2 & \cos \theta/2 \end{pmatrix} \quad (8)$$

with $-\infty < \xi < \infty$, $-2\pi \leq \theta < 2\pi$. The first double coset class is exactly the analogue of the single one in the canonical parametrization $r = \theta \xi \theta'$, and we calculate its matrix elements in a straightforward manner. Clearly these will have a pair of τ -labels attached; indeed, by (2) we find

$$\begin{aligned} [T_{\xi}^j : \varphi^+]^+(\beta) &= (e^{-\xi} \operatorname{ch}^2 \beta/2 - e^{\xi} \operatorname{sh}^2 \beta/2)^j \varphi^+(\beta') \\ [T_{\xi}^j : \varphi^+]^-(\beta) &= 0 \\ [T_{\xi}^j : \varphi^-]^{-}(\beta) &= (e^{\xi} \operatorname{ch}^2 \beta/2 - e^{-\xi} \operatorname{sh}^2 \beta/2)^j \varphi^-(\beta'') \\ [T_{\xi}^j : \varphi^-]^+(\beta) &= (-\operatorname{sign} \beta)^j (e^{\xi} \operatorname{sh}^2 \beta/2 - e^{-\xi} \operatorname{ch}^2 \beta/2)^j \varphi^-(\beta''') \end{aligned} \quad (9)$$

where

$$\begin{aligned} \operatorname{th} \beta'/2 &= e^{\xi} \operatorname{th} \beta/2 \\ \operatorname{th} \beta''/2 &= e^{-\xi} \operatorname{th} \beta/2 \\ \operatorname{th} \beta'''/2 &= e^{-\xi} \operatorname{coth} \beta/2. \end{aligned} \quad (10)$$

Notice that although the subspace $\tau = +$ is invariant, its complement $\tau = -$ is not so; for $\xi < 0$ the situation is reversed. Inserting the basis functions we find

$$\begin{aligned} d_{\mu, \mu'}^{j--}(\xi) &= \frac{\Gamma(-j-i\mu') \Gamma(-j+i\mu')}{\Gamma(-2j)} \cdot \frac{1}{2\pi} (\operatorname{sh} \xi/2)^{2j} (\operatorname{th} \xi/2)^{-1(\mu+\mu')} \times \\ &\times F(-j+i\mu, -j+i\mu'; -2j; -1/\operatorname{sh}^2 \xi/2) \end{aligned} \quad (11)$$

$$d_{\mu, \mu'}^{j++}(\xi) = \frac{\Gamma(j+i\mu+1) \Gamma(j-i\mu+1)}{\Gamma(j+i\mu'+1) \Gamma(j-i\mu'+1)} d_{\mu, \mu'}^{j-1--}(\xi) \quad (12)$$

$$d_{\mu, \mu'}^{j+-}(\xi) = 0$$

$$\cos \pi(j+\nu/2) d_{\mu, \mu'}^{j+-}(\xi) = \cos \pi(i\mu' - \nu/2) d_{\mu, \mu'}^{j--}(\xi) - \cos \pi(i\mu - \nu/2) d_{\mu, \mu'}^{j++}(\xi). \quad (13)$$

All this is valid for $\xi > 0$; if $\xi < 0$ we find

$$d_{\mu, \mu'}^{j\tau\tau'}(-|\xi|) = d_{\mu, \mu'}^{j-\tau-\tau'}(|\xi|) = d_{\mu, \mu}^{-j-1-\tau-\tau'}(|\xi|). \quad (14)$$

Now notice what effect this decomposition has had upon the matrix elements. Consider in particular the integral defining (11):

$$d_{\mu, \mu}^{j--}(\xi) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\mu'\beta} (e^{\xi} \operatorname{ch}^2 \beta/2 - e^{-\xi} \operatorname{sh}^2 \beta/2)^j e^{i\mu\beta} d\beta. \quad (15)$$

The term in parentheses is always positive; and since it has a minimum value of unity, the asymptotic behavior in the complex j -plane (or the ξ -plane) comes entirely from the ends of the integration, at both of which it has behavior $e^{j\xi}$. This agrees with (11). Similarly, (12) has specific and simple behavior of the opposite kind: only (13) is of mixed type. Closer examination shows that (12) and (11) are exactly the classical "second kind" functions for the group $SL(2, R)$, analytically continued to imaginary values of $m = i\mu$ (that is, modulo irrelevant phase factors). The significance of (13) we shall see shortly. Notice that for $\xi = 0$ the first two matrix elements degenerate into $\delta(\mu - \mu')$.

We just mention that the matrix elements of the other double-coset classes can be calculated similarly. We find

$$d_{\mu, \mu'}^{j\tau\tau'}(\xi) = (-1)^{\nu(\rho'+1)} d_{\mu, \mu'}^{j-\tau-\tau'}(\xi), \quad (16)$$

and $d_{\mu, \mu'}^{j++}(\theta)$ turns out¹⁾ to be just a representation function of $SU(2)$ after analytic continuation in j, μ, μ' . The discrete series k^t , $t = \pm$, of representations of $SL(2, R)$ behave similarly; but $O(1, 1)$ is not degenerate in any of these representations and consequently all the matrix elements vanish if $\operatorname{sign}(\tau\mu) \neq t$. Those remaining are exactly the continuations of the principal continuous series matrix elements.

Now let us return to (12) and (13). Since these relate expressions involving j and $-j-1$, which label equivalent representations, we expect that a study of the intertwining operator will be useful. Recall that this is an isometric operator $A: \mathcal{H}_j \rightarrow \mathcal{H}_{-j-1}$, which satisfies

$$A T_r^j = T_r^{-j-1} A \quad \forall r \in SL(2, R). \quad (17)$$

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Crossing-Symmetric Expansions of Scattering Amplitudes, Threshold Behavior, and Asymptotics*

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A two-variable, explicitly crossing-symmetric expansion of the scattering amplitude is discussed for the two-body scattering of spinless particles (with arbitrary masses). It converges simultaneously in the physical regions of at least two channels, has the correct threshold behavior, and allows for amplitudes growing asymptotically as arbitrary powers of s and t . The expansion is based on the representation theory of the group $O(2,1)$ in a basis not corresponding to any subgroup reduction and making use of Lamé functions.

I. INTRODUCTION

THE purpose of this paper is to discuss a new expansion formula representing a general class of scattering amplitudes and having the following properties.

- (i) It is a two-variable expansion, simultaneous in terms of both the Mandelstam variables s and t .
- (ii) The expansion is explicitly crossing-symmetric term by term in two channels and converges simultaneously for amplitudes defined in the physical regions of both channels. Both the direct and inverse expansion formulas involve only amplitudes defined in physical regions.
- (iii) It automatically incorporates the correct threshold behavior.
- (iv) The simplest assumptions about the analytic continuation of the expansion leads to amplitudes growing asymptotically as arbitrary powers of s and t .
- (v) The expansion is based on the representation theory of the group $O(2,1)$, using a basis that does not correspond to the reduction of $O(2,1)$ to any subgroups. The basis functions turn out to be Lamé functions.

Our approach is part of a general program,¹⁻⁴ the aim of which is to develop a scattering theory based on

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¹ N. Vilenkin and J. Smorodinsky, Zh. Eksperim. i Teor. Fiz. 46, 1793 (1964) [Soviet Phys. JETP 19, 1209 (1964)].

² P. Winternitz, J. Smorodinsky, and M. Sheftel, Yadern. Fiz. 8, 833 (1968) [Soviet J. Nucl. Phys. 8, 489 (1969)].

³ P. Pajás and P. Winternitz, Phys. Rev. D 1, 1105 (1970).

⁴ P. Winternitz, in *Lectures in Theoretical Physics*, edited by W. E. Brittin et al. (Gordon and Breach, New York, 1971), Vol. XIII. This paper and Ref. 3 contain numerous references to previous work.

two-variable (or more generally, multivariable) expansions of scattering amplitudes. The dependence on all kinematic parameters (energies, angles, etc.) is made explicit, the dynamics are contained in the expansion coefficients, so that a greater separation of "kinematics" and "dynamics" is achieved, thus giving a tool for implementing general principles, making dynamical assumptions, and performing phenomenological fits to larger bodies of data.

The method of this approach (so far for two-body scattering) is to consider the scattering amplitude as a function of the momenta p_i (or rather the relativistic velocities $v_i = p_i/m_i$) of the particles involved ($i=1, \dots, 4$) and then show that, making use of the conservation laws and relativistic invariance, it is possible to express the components of three of the momenta in terms of the fourth. The scattering amplitude can thus be considered to be the function of one four-momentum only, i.e., the function of a point on a three-dimensional hyperboloid $v^2 = v_1^2 - v_2^2 - v_3^2 - v_4^2 = 1$ (or on the mass shell of one of the particles). The group of motions of this space is the homogeneous Lorentz group $O(3,1)$ and the natural thing to do is to expand the amplitude $f(v)$ in terms of the basis functions of this group.

Crucial in the obtaining of such expansions are three related features—the choice of a Lorentz frame of reference, choice of coordinates on the hyperboloid, and choice of a basis for group representations.

In this paper our principal aim is to discuss crossing-symmetric expansions, so we must choose such a frame of reference and such coordinates that we obtain a symmetric mapping of the Mandelstam variables s and t onto some curvilinear coordinates α, β on the hyperboloid. In a previous publication,² we have considered this problem for the two-body scattering of spinless particles with equal masses. Here we shall consider a more general case.

* N. W. Macfadyen and P. Winternitz, J. Math. Phys. 12, 281 (1971), hereafter referred to as I.

II. MAPPING OF MANDELSTAM VARIABLES ONTO VELOCITY SPACE

As in I, we simplify the mathematics by restricting ourselves to spinless particles and choosing the scattering plane to be such that the third space components of all the momenta vanish. This makes it possible to consider the momenta as three-dimensional objects and to consider an $O(2,1)$ hyperboloid, instead of an $O(3,1)$ one.

To make the choice of the curvilinear coordinates unique, we demand that the Laplace operator on the hyperboloid allow the separation of variables in these coordinates and that the corresponding eigenfunctions—the basis functions for our expansions—can be written as the product of two identical functions. An inspection of separable coordinates^{6,7} leads us directly to elliptic coordinates (see I), in which the nonzero components of the momenta are

$$p_r = m_r (-\operatorname{cn} \alpha_r \operatorname{cn} \beta_r, i \operatorname{sn} \alpha_r \operatorname{dn} \beta_r, i \operatorname{dn} \alpha_r \operatorname{sn} \beta_r, 0), \quad r = 1, \dots, 4. \quad (1)$$

Here $\operatorname{cn} \alpha$, $\operatorname{sn} \alpha$, and $\operatorname{dn} \alpha$ are the Jacobian elliptic functions⁸ of modulus $k = 1/\sqrt{2}$, with arguments in the

regions $\alpha_r \in (iK, iK + 2K)$, $\beta_r \in (iK, iK + 2K)$, where $K = [F(1/2)]^2 / \sqrt{\pi}$ is the quarter-period of the elliptic functions.

To eliminate the redundant variables we must choose a convenient frame of reference. We shall use the coordinates of momentum p_2 as our basic variables.

In order to obtain simple properties under the crossing transformation, we let $\alpha_2 \equiv \alpha$, $\beta_2 \equiv \beta$ and demand that the transformation $\alpha \rightarrow \beta$, $\beta \rightarrow 2K - \alpha$, $m_2 \rightarrow m_3$, $m_3 \rightarrow m_2$, $m_1 \rightarrow m_1$, $m_4 \rightarrow m_4$ should correspond to $p_2 \rightarrow -p_2$, $p_3 \rightarrow -p_3$, $p_1 \rightarrow p_1$, $p_4 \rightarrow p_4$. Further we specify our frame of reference (a generalization of the brick-wall system) by putting $p_1 \parallel p_4$ (see Fig. 1). It is easy to check that in such a frame of reference we have

$$\begin{aligned} p_1 &= (-m_1 \operatorname{cn} \alpha_1 \operatorname{cn} \beta_1, -i(a+1)A, -i(a+1)B, 0), \\ p_2 &= m_2 (-\operatorname{cn} \alpha \operatorname{cn} \beta, i \operatorname{sn} \alpha \operatorname{dn} \beta, i \operatorname{dn} \alpha \operatorname{sn} \beta, 0), \\ p_3 &= m_3 (-\operatorname{cn} \alpha \operatorname{cn} \beta, -i \operatorname{sn} \alpha \operatorname{dn} \beta, -i \operatorname{sn} \alpha \operatorname{dn} \beta, 0), \\ p_4 &= (-m_4 \operatorname{cn} \alpha_1 \operatorname{cn} \beta_1 + (m_3 - m_2) \operatorname{cn} \alpha \operatorname{cn} \beta, \\ &\quad -iaA, -iaB, 0), \end{aligned} \quad (2)$$

where

$$\begin{aligned} A &= m_2 \operatorname{sn} \alpha \operatorname{dn} \beta + m_3 \operatorname{dn} \alpha \operatorname{sn} \beta, \\ B &= m_2 \operatorname{dn} \alpha \operatorname{sn} \beta + m_3 \operatorname{sn} \alpha \operatorname{dn} \beta, \end{aligned} \quad (3)$$

$$\frac{\{\operatorname{cn}^2 \alpha_1\}}{\{\operatorname{cn}^2 \beta_1\}} = \frac{\mp(a+1)^2(A^2 - B^2) - \{[2m_1^2 - (a+1)^2(A+B)^2][2m_1^2 - (a+1)^2(A-B)^2]\}^{1/2}}{2m_1^2}, \quad (4)$$

and

$$a = [\mp Y + (Y^2 - 4XZ)^{1/2}] / 2X, \quad (5)$$

with

$$\begin{aligned} X &= 4(A^2 + B^2)(m_2 + m_3)^2[2m_2m_3(A+B)^2 + (m_2^2 - m_3^2)^2], \\ Y &= 4(A^2 + B^2)(m_2 + m_3)^2\{2m_2m_3(A+B)^2 \\ &\quad + (m_2 + m_3)^2[(m_2 - m_3)^2 - m_1^2 + m_4^2]\}, \\ Z &= \{2m_2m_3(A+B)^2 + (m_2 + m_3)^2[(m_2 - m_3)^2 - m_1^2 \\ &\quad - m_4^2]\}^2 + 4m_4^2(m_2 + m_3)^4(A^2 + B^2 - m_1^2). \end{aligned} \quad (6)$$

Such a frame of reference exists if a is real, the condition for which is

$$(m_2 - m_3)^2 \leq (m_1 - m_4)^2, \quad (7)$$

In the chosen frame of reference, all momenta obviously depend on α and β only, which in turn can be related to the Mandelstam variables s and t , so that the scattering amplitude $f(s, t) \equiv f(\alpha, \beta)$ is given as a function of a point on the hyperboloid $p_2^2 = m_2^2$.

The properties under the crossing transformation are of interest mainly when the two exchanged particles

are identical. Let us further assume that $m_2 = m_3$. In such a case the formulas (2)–(6) simplify considerably.

Indeed for $m_2 = m_3$ we can give the relation between s , t , u and α , β explicitly:

$$\begin{aligned} \begin{cases} s \\ t \end{cases} &= m_1^2 + m_2^2 + 2\{\mp m_1 m_2 y \\ &\quad \times [1 - (1/8 m_1^2 m_2^2 x^2)(2m_2^2 x^2 + m_1^2 - m_4^2)^2]^{1/2} \\ &\quad - \frac{1}{4}(2m_2^2 x^2 + m_1^2 - m_4^2)\}, \\ u &= 2m_2^2 x^2, \end{aligned} \quad (8)$$

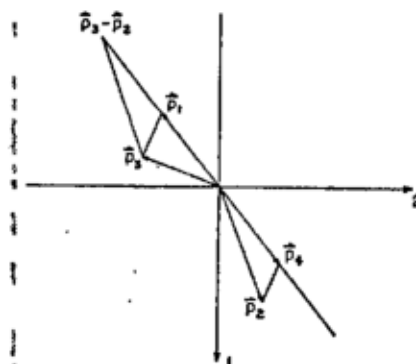


FIG. 1. Symmetric frame of reference.

⁶ M. N. Olevsky, *Mat. Sbornik* 27(69), 379 (1950).

⁷ P. Winternitz, J. Lukač, and J. Smorodinsky, *Yadern. Fiz.* 7, 192 (1968) [*Soviet J. Nucl. Phys.* 7, 139 (1968)].

⁸ Bateman Manuscript Project, *Higher Transcendental Functions*, edited by A. Erdélyi et al. (McGraw-Hill, New York, 1953), Vol. II, Chap. 13.

MacFadyen (Carnegie-Mellon) and Winternitz (same):

1/23/77

"Crossing-Symmetric Expansions..."

PR.D3, 1874 (1971)

example of a "two-variable expansion"

This paper strikes me as rather a curiosity. In the case of 2 to 2 scattering (which is the only case they treat), the authors write the spinless amplitude in terms of (α, β) in place of usual (s, t) . These new variables are just coordinates of a point on a velocity hyperboloid. See, eg, equation (2) line 2. The hyperboloid is an $O(2,1)$ one here because z-component is dropped (spinless); $v^2 = 1$ so has two sheets. Upper sheet point are the s channel, lower sheet points are the t-channel.

Once you have your scattering amplitude defined on an $O(2,1)$ surface, it is clear that some kind of $O(2,1)$ expansion will work out. It was shown in an earlier paper that if you want some $O(2,1)$ functions that separate in the α (sense (ie, separate in elliptic coordinates), then you must diagonalize the usual $O(2,1)$ casimir as well as $(K_1^2 - J_3^2)$. Also, certain reflections X and Y will be diagonalized by the basis functions. This peculiar choice of the second diagonal object leads to the Lamé differential equation instead of the Legendre, and the expansion functions are then products of Lamé functions. Labels ℓ and h are "rep labels".

The expansion theorem is then stated in (19) and (20). $A^{pq}(\ell, h)$ is the partial wave amplitude for the expansion of the s-channel amplitude. The t-channel expansion is then shown in (24) and (25). Crossing is then conveyed term by term via (27). This simple crossing relation is one of their claims to success.

Somehow, power asymptotic behavior and correct threshold behavior come out naturally. ~~Well, one could say that the little group analysis applied to the 2x2 case replaces with a group variable but leaves the other is, Why is this method better than the usual $O(2,1)$ little group method?~~

Answer: for spinless 2-2 case, the little group method would replace s with group variable z, but would leave t fixed, ie, a one-variable expansion. The two-variable expansion gets both variables into a group theoretic significance.

I might counter that to me, the strength of the little group method is its ability to handle spin (helicity carried by the azimuthals). Oh, but I guess you could do this analysis starting with M-functions and write $\tilde{M}_{m_1 m_2 m_3 m_4}(\alpha, \beta)$.
(over)

A stronger critique of the two-variable expansion might be this: you want to keep the variable t because it is the regge mass variable and you want to see Poincare group entities "exchanged". These exchanges will be totally unclear in the two-variable formalism. I think the fact that such exchanges are important experimentally will kill the usefulness of the 2-variable approach. But I may be wrong.

The authors admit that they are still working on the connection to Regge theory. Also they have to add spin. Their expansion does have some niceties though, I do not mean to dump on it.

CMP '72

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A Laplace Transform on the Lorentz Groups

I. Quasiregular Representations

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Abstract. As a first step in the generalisation of the Laplace transform to a non abelian group, we examine the representations of the groups $SO(n, 1)$ by means of transformations of (not necessarily integrable) functions defined over the hyperboloids $O(n, 1)/O(n)$. We define a regularised version of the Gelfand-Graev transformation from the n -dimensional hyperboloid to its associated cone, which is valid (under certain restrictions) for polynomially bounded functions. Upon the cone we then carry out a pair of classical Laplace transforms parallel to a generator. We give inversion formulas for both these procedures, and express the Laplace transform/inversion pair directly in terms of the function on the hyperboloid.

For integrable functions our results reduce to those already known; in the non-integrable case they are new. New features include the divergence of the transform for certain discrete asymptotic behaviours; the existence of a finite dimensional kernel subspace which is annihilated; good asymptotic behaviour of both Laplace projection and inversion formulas; and the existence of discrete terms contributing to the inversion formula for even dimension. Our results are valid for all dimensions and are completely independent of the usual "Laplace transforms" involving projection by means of "second-kind representation functions"; in a final section of the paper we examine briefly the significance of that approach in the light of our own.

I. Introduction

There has been recently [1, 2] considerable interest in possible generalisations of the Fourier transform on locally compact non-Abelian groups, with the hope of deriving expansion theorems valid for non-square-integrable functions over a non-compact group — typically one of the Lorentz groups $SO(n, 1)$. Two approaches to the problem can be distinguished: the distribution-valued-transform methods, as exemplified by (for instance) the work of Rühl [2]; and the special-function approach [1] which is often called the Laplace transform but which we shall call the Legendre transform. The former is the direct analogue of the classical one-dimensional theory; but it has the disadvantage that the function $f(g)$ with which we are concerned has to be regarded as a distribution, and although the Fourier transform is then certainly defined, there is no

general way of evaluating it without a regularisation procedure involving $f(g)$ itself. We shall not consider this method further.

The second approach is more interesting, although the theory behind it is somewhat ad hoc. Despite any claims made by its protagonists, it still rests essentially upon the integral relation between Legendre functions

$$\int_1^\infty P_j(x) Q_l(x) dx = [(l-j)(l+j+1)]^{-1} \quad (1)$$

[valid for $\text{Re}(l-j) > 0$ and $\text{Re}(l+j) > -1$], and its generalisations whereby the P_l become the representation functions (matrix elements) of the group and the Q_l the so-called "second-kind" functions [3]. There are three limitations to this approach: one (the asymptotic behaviour) we shall discuss shortly; a second is that the entire theory is cast in very strongly basis-dependent language; and the third (and most significant) is that it is restricted to $SO(2, 1)$ and is incapable of generalisation [4] to higher groups like $SO(3, 1)$. The last point tells us at once that if there exists a generalisation of the Laplace transform to non-Abelian groups, then this is not it.

Nonetheless, because no other approach to this important problem has so far been found, there is a considerable literature on the subject. Several authors [5] have noticed indeed that in a non-compact basis the representation functions themselves are of "second-kind" behaviour, but unfortunately this tantalising discovery merely exchanges one divergent integral for another, and we are no further advanced. Otherwise, the complete reliance upon special-function theory in a particular basis is in striking contrast to the basis-independent formalism [4, 6] developed for Fourier transforms.

In this paper we make a start on the problem of defining a true Laplace transform over a group, using the methods of group theory and integral geometry. For the sake of clarity and intuitiveness, we discuss first not the regular representation itself (that is, the "non-integrable regular representation") but rather the quasi-regular representation of $SO(n, 1)$ by means of transformations of functions defined over one sheet of the hyperboloid $O(n, 1)/O(n)$; we shall extend our results to the regular representations themselves in a later paper. Let us summarise the Legendre theory for $SO(2, 1)$, in the especially simple case of a function $f(u)$ over the 2-dimensional hyperboloid $u \cdot u = 1$ which in a spherical coordinate system has no angular dependence: $f(u) = f(ch\zeta)$. We have the transform pair

$$\tilde{f}(l) = \int_1^\infty f(ch\zeta) Q_l(ch\zeta) d(ch\zeta), \quad (2a)$$

$$f(ch\zeta) = \frac{1}{2\pi i} \int_C (2l+1) \tilde{f}(l) P_l(ch\zeta) dl, \quad (2b)$$

where C is a contour running from $C - i\infty$ to $C + i\infty$; $C = \text{Re} l$ is chosen to the right of the least value for which (2a) converges. If indeed f is integrable we can take instead

$$\tilde{f}(l) = \int_1^\infty f(ch\zeta) P_l(ch\zeta) d(ch\zeta), \quad (3a)$$

$$f(ch\zeta) = \frac{1}{2\pi i} \int_C (2l+1) \tilde{f}(l) Q_{-l-1}(ch\zeta) dl; \quad (3b)$$

but (3a) diverges for general f , whereas (2a) may converge. The pair (2) is generally called the Laplace transform; only the pair (3) can be extended [4] to higher groups.

This theory suffers however from an important defect. The classical Laplace transform on the real line converges for all exponentially bounded $f(x)$; and it gains a large measure of its usefulness from the fact that when the contour of integration in the inversion formula is pulled back to the left half-plane, we can ultimately ignore its contribution to the integral when compared with those of the singularities it has crossed. Such is not the case in (2b): the Legendre P_l -functions are ill-behaved as $\text{Re} l \rightarrow \pm\infty$. Certainly it is true that (3b) is satisfactory as $\text{Re} l \rightarrow -\infty$; but then (3a) does not usually converge. Thus the Legendre transform pair for $SO(2, 1)$ can either be arranged to have a convergent projection formula or to have an attractive inversion formula – but not both. We shall present a theory of a Laplace transform where both these desirable features are present.

To see how our results arise, consider the classical Laplace transform. This is usually (incorrectly) stated to be an expansion in non-unitary representations of the additive group of the reals; in fact, it derives its entire usefulness from the fact that it is no such thing. Instead, it is a transform over representations of a *semigroup* (the transform integral is only from zero to infinity); the group property is possessed only by the two-sided Laplace transform, whose domain of definition is small indeed. If we want to recover the entire function $f(x)$, $-\infty < x < \infty$, we need a pair of Laplace transforms – one for each half of the real axis.

This idea tells us how to proceed: the (generalised) Laplace transform on a group G is to be identified with a projection over representations of some semigroup contained in G . We shall elaborate these ideas in a later paper; here we concern ourselves only with the "quasi-regular" representations of $SO(n, 1)$, which we know how to treat. The general procedure for decomposing an (integrable) function on the hyperboloid into its irreducible components has been given by Gel'fand and Graev [7]; see also Ref. [6], Chapters V, VI, and Refs. [8–10]. Its cardinal feature is the

Macfadyeh "A Laplace Transform on the Lorentz Groups"
CMP 28,87 (1972) --- also received 1972.

1. Introduction. He notes that there is an "interest" in this matter, due to Abarbanel&Saunders, Cronstrom&Klinck, Jones&Young; separately Ruhl. There are two separate methods: distribution, and special functions; we will only consider the second one:

Claims that the entire theory rests on a simple integral relation between the P and Q functions, (I agree). Talks about you have a choice: either take a divergent recovery, or a no good projection. The problem is not really solved.

To solve this problem, Mac will use the hyperboloid/cone technique of GG. He will do this for the general case $SO(n,1)$, then at the end will consider our special case of $O(2,1)$. Should be interesting. Mentions some "later paper".

2. The Gél'fand-Graev Transform. x Define the following junk:

H = upper hyperboloid with points u and measure du
K = upper light-cone with points k and measure dk

Suppose you integrate some function $f(u)$ over the part of the hyperb which satisfies the condition $u.k=1$ for some fixed k on the light cone K. This surface is I think a surface of one dimension less than the hyperb and is called a horosphere. You can do this for each k on K, and thereby you construct a function $F(k)$ on the light-cone!

GGV book shows that if $f(u)$ is very nice (compact support), then $F(k)$ is infinitely differentiable and vanishes at infinity and at the origin of the light cone.

Creating $F(k)$ from $f(u)$ in this way (integrating over a horosphere) is called the Gel'fand-Graev transform.

P = space of functions on H $f(u)$ which are polynomially bounded.

F_1 = subspace of P which have compact support, called $f_1(u)$

F_∞ = subspace of P of functions which grow faster than $1/u^{1-n}$, where n is of course the $O(n,1)$ dimension.

Q = space of functions which are linear combinations of the appropriate Gegenbauer functions for this value of n. These functions are polynomials, like the Legendre polynomials for $O(2,1)$ case. Variable ξ is hyperbolic variable as usual.

Lemma 1 says: if your function $f(u)$ grows faster than a certain powerxxx no. If your function has certain integer fixed powers in it, then the GGT screws up and does not exist. The exact type of "disallowed power" depends on whether n is even or odd integer. Also, the "kernel" of the GGT consists of functions in the space Q defined above.

So the conclusion is: as long as you avoid certain singular cases, the GGT generalized to any power bounded function is valid. We have successfully converted our representation from functions on the hyperb into a representation with functions moving on a cone. Reminds me of Kalnins.

3. The Inversion Formula. If someone gives you $F(k)$, how do you reconstruct $f(u)$? The answer is given here as Theorem 1. Note that you recover your starting function modulo arbitrary functions in the space Q. This does not seem helpful...

~~xxxRaphae~~

4. Laplace Transform. Recall that k was a point on the cone. Through such a point there is a line passing through the origin of the cone. Mac says to do a fourier analysis projection by integrating the function over this semi-infinite line. Or, to get convergence, only go half way and you have your "honest" Laplace transform. The projection is now a function of some " Q " parameter, but it is also a function of I think the azimuth of the cone defining your line. Thus, we see $f(k, Q)$.

If we combine this true Laplace projection with the GGT generalization given in (8), we get the full "Laplace projection:" (29) with kernel (32). Not too messy at that!

5. Laplace Inversion. Now mess around by combining the usual Laplace inversion with the GGT inversion.

Full result is stated as Theorem 2.

Comments: it is pointed out that now in the recovery formula you can move the contour to the left and have your background integration disappear. Thus, he has solved a problem which earlier had caused trouble. Secondly, it is noted that in the even case only, n -even, you get some sums in the recovery. Very reminiscent of the "discrete series" of usual theory. But he notes that a baloid really cannot do discrete series. Not fully understood.

Earlier restriction on the transform is removed in Lemma 5.

6. The $O(2,1)$ case. He shows that if you start with the P and Q functions and compute the GGT, you get very simple first and second kind functions on the cone, very nice indeed.

The P -projection as shown in (50) is exactly the same as doing a fourier projection on the cone.

But the Q -projection is not like doing a Laplace transformation on the cone because, I gather, there is an extra tangent factor floating around. Not very clear why the Q fails the criterion.

Recall that he was saying in the very beginning that the Q -projection idea was non-generalizable to higher groups.

Anyway, he does not say what the $O(2,1)$ true Laplace looks like in u -space.

Final note: Cronstrom has done my Q - Q thing, it is claimed! IC/72/41.

Bargmann '46

IRREDUCIBLE UNITARY REPRESENTATIONS
OF THE LORENTZ GROUP

By V. BARGMANN

(Received September 18, 1946)

PART I

Introduction

It is the purpose of this paper to construct and to analyze the irreducible unitary representations of the Lorentz group which satisfy certain regularity conditions stated below. More specifically, we deal with the proper Lorentz group, i.e., the group of all homogeneous linear transformations in four variables x^0, x^1, x^2, x^3 which leave the quadratic form $(x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2$ invariant, have the determinant 1, and do not reverse the direction of time (the variable x^0). This group will be denoted by $\mathcal{L}_+$. It is known that $\mathcal{L}_+$ (as well as the group $\mathcal{L}_-$ defined below) has only *infinite-dimensional* unitary representations (by operators in Hilbert space) except for the trivial one-dimensional case, where every group element is represented by 1 [Wigner, p. 165].¹ In addition to $\mathcal{L}_+$ we shall also investigate the corresponding group $\mathcal{L}_-$ of all homogeneous linear transformations in the three variables x^0, x^1, x^2 which leave the form $(x_0)^2 - (x^1)^2 - (x^2)^2$ invariant—with the same restrictions as above.

Apart from possible applications in Mathematical Physics [cf. Dirac 2] this investigation has an intrinsic mathematical interest as a detailed analysis of the unitary representations of a non-compact group. This holds in particular in the case of $\mathcal{L}_-$ where the results are fairly explicit and complete. Moreover, the representations of both $\mathcal{L}_+$ and $\mathcal{L}_-$ play a part in Wigner's classification of the unitary representations of the *inhomogeneous* Lorentz group [Wigner, p. 192], $\mathcal{L}_+$ being the "little group" in the case $P < 0$, and $\mathcal{L}_-$ being the little group in the case 0_0 (the representations of these two groups are not classified in Wigner's paper). It should be noted however that the conditions which we impose on the representations are more stringent than Wigner's.

PLAN OF THE INVESTIGATION. We shall discuss both single- and double-valued representations and hence deal with the corresponding spinor groups $\mathcal{S}_+$ and $\mathcal{S}_-$ rather than with $\mathcal{L}_+$ and $\mathcal{L}_-$. If $\mathcal{S}$ (which stands here for either $\mathcal{S}_+$ or $\mathcal{S}_-$) is represented on a Hilbert space $\mathcal{H}$ by unitary operators $U(a)$ which are continuous in a (a is a group element of $\mathcal{S}$), every one parameter subgroup may be expressed, by Stone's theorem, in the form $U_t = \exp(-itH)$ where H is a self-adjoint operator on $\mathcal{H}$. Since on the other hand, every one parameter subgroup is generated by an infinitesimal transformation (an element of the Lie algebra $\mathfrak{g}$) of the group $\mathcal{S}$, there is a correspondence between the operators H and the elements of $\mathfrak{g}$. Our main assumption will be that the operators H define a representation of $\mathfrak{g}$ which will be called an *infinitesimal representation* of

$\mathcal{S}$. It is sufficient to require that the sum of two elements of $\mathfrak{g}$ is mapped into the sum of the corresponding operators H . The H being unbounded, their sum can only be properly defined if the common part of the domains of the different H is large enough. We need, therefore, a condition about these domains, which will be stated in §5. If these requirements are satisfied the possible irreducible infinitesimal representations of $\mathcal{S}$ may be classified. By an explicit construction it is shown that to every infinitesimal representation corresponds an irreducible unitary representation of $\mathcal{S}$ itself. In each case the Hilbert space $\mathcal{H}$ is defined as a function space over a properly chosen manifold $\mathcal{M}$ on which $\mathcal{S}$ acts as a transformation group (to every element a of $\mathcal{S}$ corresponds a homeomorphism of $\mathcal{M}$ into itself denoted by $y = ax$, where x, y are points on $\mathcal{M}$). The group property requires that $a(bx) = (ab)x$. The operators $U(a)$ are obtained as follows. If $f(x)$ is an element of $\mathcal{H}$, then $U(a)f(x) = \mu(a, a^{-1}x) \cdot f(a^{-1}x)$, where $\mu(a, x)$ is a fixed non-vanishing function of the group element a and the point x which satisfies the condition $\mu(ab, x) = \mu(a, bx) \cdot \mu(b, x)$, and is called a *multiplier* of the transformation group. It is easily seen that $U(ab) = U(a)U(b)$, and with a suitable definition of the inner product the operators $U(a)$ are unitary. We turn now to a brief summary of the results obtained.

I. The group $\mathcal{S}_+$. The infinitesimal representation contains three linearly independent elements H_{12}, H_{20} , and H_{01} , where H_{12} corresponds to a transformation of the $(k-1)$ plane into itself. For any irreducible representation the operator

$$-Q = (H_{01})^2 + (H_{20})^2 - (H_{12})^2$$

is a scalar, i.e., it has the form $Q = q \cdot 1$, q being a real number. Moreover, H_{12} , whose spectrum is always discrete, has simple proper values m which are either all integral or all half-integral and which characterize the representations of the rotations in the $(1-2)$ plane. In the half-integral case we obtain a double-valued representation of $\mathcal{S}_+$. The representations may be classified according to the value of q and the values of m . The following possibilities are found: (1) C_0^q : q may be any positive number while m assumes all integral values $0, \pm 1, \dots$. (2) $C_{\frac{1}{2}}^q$: q may be any number in the interval $\frac{1}{4} < q < \infty$, while m assumes all half integral values $\pm \frac{1}{2}, \pm \frac{3}{2}, \dots$. (3) D_k^+ : k may be one of the numbers $\frac{1}{2}, 1, \frac{3}{2}, \dots$; q has the value $k(1-k)$, and m assumes all values $k, k+1, k+2, \dots$. (4) D_k^- : k may be one of the numbers $\frac{1}{2}, 1, \frac{3}{2}, \dots$; q is equal to $k(1-k)$, and m assumes all values $-k, -(k+1), -(k+2), \dots$. The two classes C_0^q and $C_{\frac{1}{2}}^q$ are termed *continuous* because in each case the possible values of q fill an interval. By contrast, the two classes $D_k^\pm$ are termed *discrete*, because q may only assume the values $k(1-k)$.

The unitary representations of $\mathcal{S}_+$ corresponding to the two continuous classes C_0^q and $C_{\frac{1}{2}}^q$ may be realized on a function space over the unit circle, which is the manifold $\mathcal{M}$ mentioned above, with suitably chosen multipliers. The group $\mathcal{S}_+$ —and hence also the group $\mathcal{S}_-$ —acts on $\mathcal{M}$ as the group of the projective transformations of $\mathcal{M}$ into itself. As long as $q \geq \frac{1}{4}$, the Hilbert space $\mathcal{H}$ consists of all square integrable functions over $\mathcal{M}$, while the inner product in $\mathcal{H}$ is defined

¹ See bibliography at the end of the paper.

Bargmann Symbols

$$\mathcal{L} = \mathcal{L}_3 = \text{SU}(1,1) = \text{"spinor group of } \mathcal{L}_3\text{"}$$

$$\mathcal{L} = \mathcal{L}_3 = \text{SO}(2,1)$$

$$W = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix}$$

$$\Omega = (\chi_0)^2 - (\chi_1)^2 - (\chi_2)^2$$

$$Q = (L_0)^2 - (L_1)^2 - (L_2)^2$$

$$Q = (H_1)^2 + (H_2)^2 - (H_0)^2$$

$$[L_0, L_1] = L_2$$

$$[L_1, L_2] = -L_0$$

$$[L_2, L_0] = L_1$$

$$q = \text{eigenvalue of } Q$$

$$H_i = iL_i, i = 0, 1, 2$$

$$q = -\lambda(\lambda+1)$$

$$q = k(1-k) \text{ for both discrete series}$$

$$[H_0, H_1] = iH_2$$

$$[H_0, H_2] = -iH_1$$

$$[H_1, H_2] = -iH_0$$

$$q = \frac{1}{4} - \sigma^2 = \frac{1}{4} + s^2$$

$$\sigma = is$$

$$q = \frac{j}{2}(1 - \frac{1}{2})$$

$$k = \frac{1}{2}l$$

$$\sigma = k - \frac{1}{2}$$

$$q = -j(j+1) \text{ for finite reps}$$

Note: Bargmann always uses 1) q, σ, s for continuous + excep. series
 All above equations are direct quotes
 2) q, l, k for discrete series
 3) j for finite reps

The object I call " $l=j$ " is what Bargmann calls " $j=\lambda$ ".

Because:

$$l^{me} = -\frac{1}{2} + is = -\frac{1}{2} + \sigma \Rightarrow \sigma = l^{me} + \frac{1}{2} \Rightarrow q = -l^{me}(l^{me}+1)$$

Because that Barg. has separate usage of " l " for discrete series:

$$\sigma = k - \frac{1}{2} \Rightarrow k = l^{me} + \frac{1}{2}$$

$$k = \frac{1}{2}l \Rightarrow l/2 = l^{me} + \frac{1}{2}$$

$$q = \frac{j}{2}(1 - \frac{1}{2}) \Rightarrow \uparrow \text{ also.}$$

Bargmann

me

q

$-j(j+1)$

σ

is

k

$j+1$

$j = -\frac{1}{2} + is$

Matrix elements:

$$V_{mn}(\xi) = \langle jm | e^{\xi X_2} | jn \rangle$$

But $X_2 = L_2 = -i H_2$.

$$= \langle jm | e^{-i \xi H_2} | jn \rangle$$

$$= \langle jm | e^{+i \xi K_1} | jn \rangle$$

<u>me</u>	<u>Bay</u>	<u>Bay</u>	<u>Bay</u>
K_2	H_1	$i L_1$	$i X_1$
$-K_1$	H_2	$i L_2$	$i X_2$
J_3	H_3	$i L_3$	$i X_3$

$$X_1 = -i K_2$$

$$X_2 = i K_1$$

$$X_3 = -i J_3$$

The same goes for $\phi(0,y)=y$. And

$$\phi(x,y) = x + y + axy + bx^2y + cxy^2 + dx^3y + ex^2y^2 + fxy^3 + \dots$$

In specific form, using Pontryagin's labels.

$$\phi^{\alpha}(x,y) = x^{\alpha} + y^{\alpha} + a_{\beta\gamma}^{\alpha} x^{\beta} y^{\gamma} + g_{\beta\gamma\delta}^{\alpha} x^{\beta} x^{\gamma} y^{\delta} + h_{\beta\gamma\delta}^{\alpha} x^{\beta} y^{\gamma} y^{\delta} + \dots$$

Result 3 $Q_{\beta}^{\alpha}(x,y) \equiv \frac{\partial \phi^{\alpha}}{\partial x^{\beta}}(x,y)$

$$= \delta_{\beta}^{\alpha} + 0 + a_{\beta\gamma}^{\alpha} \delta_{\gamma}^{\beta} y^{\gamma} + g_{\beta\gamma\delta}^{\alpha} y^{\delta} [x^{\beta} \delta_{\gamma}^{\beta} + \delta_{\gamma}^{\beta} x^{\beta}] + h_{\beta\gamma\delta}^{\alpha} \delta_{\gamma}^{\beta} y^{\gamma} y^{\delta}$$

Now reset $i = \beta$:

$$Q_{\beta}^{\alpha}(x,y) = \delta_{\beta}^{\alpha} + a_{\beta\gamma}^{\alpha} y^{\gamma} + g_{\beta\gamma\delta}^{\alpha} x^{\beta} y^{\delta} + g_{\beta\gamma\delta}^{\alpha} x^{\gamma} y^{\delta} + h_{\beta\gamma\delta}^{\alpha} y^{\gamma} y^{\delta} + \dots$$

So:

$$Q_{\beta}^{\alpha}(x,y) = \delta_{\beta}^{\alpha} + a_{\beta\gamma}^{\alpha} y^{\gamma} + [g_{\beta\gamma\delta}^{\alpha} + g_{\gamma\beta\delta}^{\alpha}] x^{\beta} y^{\delta} + h_{\beta\gamma\delta}^{\alpha} y^{\gamma} y^{\delta}$$

Result 4: for small x,y , $\phi(x,y) \approx x+y \Rightarrow x^{\gamma} \approx -x + O(x^2)$

Therefore

$$Q_{\beta}^{\alpha}(x,x) \approx \delta_{\beta}^{\alpha} - a_{\beta\gamma}^{\alpha} x^{\gamma} - [g_{\beta\gamma\delta}^{\alpha} + g_{\gamma\beta\delta}^{\alpha}] x^{\beta} x^{\gamma} + h_{\beta\gamma\delta}^{\alpha} x^{\beta} x^{\gamma}$$

↳ + unknown order x^2 .

So

$$\psi_{\beta}^{\alpha}(x) = \delta_{\beta}^{\alpha} - a_{\beta\gamma}^{\alpha} x^{\gamma} + O(x^2)$$

②

Result 5. Define:

$$\chi_{\beta}^{\alpha}(x) = \frac{\partial \phi^{\alpha}(y, x)}{\partial y^{\beta}} \Big|_{y=0}$$

define χ .

Now evaluate result 1 at $x=0$, using result 4 which says

$$\psi_{\beta}^{\alpha}(0) = \delta_{\beta}^{\alpha}$$

Get $\delta_{\beta}^{\alpha} = \psi_{\sigma}^{\alpha}(y) \cdot \chi_{\beta}^{\sigma}(y)$

Assume $(\chi^{-1})^{\alpha}_{\beta}(y)$ exists such that: $(\chi)_{\beta}^{\alpha} (\chi^{-1})^{\beta}_{\gamma} = \delta_{\gamma}^{\alpha}$

Therefore

$$\delta_{\beta}^{\alpha} (\chi^{-1})^{\beta}_{\gamma} = \psi_{\sigma}^{\alpha} \cdot \underbrace{\chi_{\beta}^{\sigma} (\chi^{-1})^{\beta}_{\gamma}}_{\delta_{\gamma}^{\sigma}}$$

$$\Rightarrow (\chi^{-1})^{\alpha}_{\gamma} = \psi_{\gamma}^{\alpha}$$

(χ^{-1}) is presumably a two-sided inverse matrix, since finite,
so

$$\delta_{\beta}^{\alpha} = \chi_{\sigma}^{\alpha} \psi_{\beta}^{\sigma} = \psi_{\sigma}^{\alpha} \chi_{\beta}^{\sigma}$$

So:

$$\delta_{\beta}^{\alpha} = \chi_{\lambda}^{\alpha}(x) \psi_{\beta}^{\lambda}(x) = \psi_{\lambda}^{\alpha}(x) \chi_{\beta}^{\lambda}(x)$$

$$1 = \chi \cdot \psi = \psi \cdot \chi$$

where we use $A \sim A^{\alpha}_{\beta}$

Result 6. Use result 5 on result 1 to get:

$$\chi_{\lambda}^{\alpha}(\phi) \psi_{\beta}^{\lambda}(x) = \underbrace{\chi_{\sigma}^{\alpha}(\phi) \psi_{\sigma}^{\lambda}(\phi)}_{\delta_{\sigma}^{\alpha}} \frac{\partial \phi^{\sigma}}{\partial x^{\beta}}$$

$$\Rightarrow \frac{\partial \phi^{\alpha}(x, y)}{\partial x^{\beta}} = \chi_{\lambda}^{\alpha}(\phi(x, y)) \psi_{\beta}^{\lambda}(x)$$

This is the famous Bargmann 1.1.

Result 7:

$$\chi_{\beta}^{\alpha}(\bar{x}) = \frac{\partial \bar{\phi}^{\alpha}(y, x)}{\partial y^{\beta}} \Big|_{y=0} = Q_{\beta}^{\alpha}(y, x) \Big|_{y=0}$$

$$\boxed{\chi_{\beta}^{\alpha}(x) = Q_{\beta}^{\alpha}(0, x)}$$

$$= \delta_{\beta}^{\alpha} + a_{\beta\gamma}^{\alpha} x^{\gamma} + \frac{1}{2} h_{\beta\gamma\lambda}^{\alpha} x^{\gamma} x^{\lambda} + \dots$$

So certainly:

$$\boxed{\chi_{\beta}^{\alpha}(x) = \delta_{\beta}^{\alpha} + a_{\beta\gamma}^{\alpha} x^{\gamma} + \frac{1}{2} h_{\beta\gamma\lambda}^{\alpha} x^{\gamma} x^{\lambda} + \dots}$$

Result 8: Let us simply assume [or, maybe it follows from our assumed analyticity of ϕ] that

$$\boxed{\frac{\partial \phi^{\alpha}(x, y)}{\partial x^i \partial x^j} = \frac{\partial \phi^{\alpha}(x, y)}{\partial x^j \partial x^i}}$$

Insert result 6 to get

$$\frac{\partial}{\partial x^i} [\chi_{\lambda}^{\alpha}(\phi) \psi_{\cdot j}^{\lambda}(x)] = (i \leftrightarrow j)$$

Expand:

$$\chi_{\lambda}^{\alpha}(\phi) \frac{\partial \psi_{\cdot j}^{\lambda}(x)}{\partial x^i} + \frac{\partial \chi_{\lambda}^{\alpha}(\phi)}{\partial \phi^{\alpha}} \left[\frac{\partial \phi^{\alpha}(x, y)}{\partial x^i} \right] \psi_{\cdot j}^{\lambda}(x) = (i \leftrightarrow j)$$

Reinsert result 6 again to get

$$\chi_{\lambda}^{\alpha}(\phi) \frac{\partial \psi_{\cdot j}^{\lambda}(x)}{\partial x^i} + \frac{\partial \chi_{\lambda}^{\alpha}(\phi)}{\partial \phi^{\alpha}} [\chi_{\cdot b}^{\alpha}(\phi) \psi_{\cdot i}^b(x)] \psi_{\cdot j}^{\lambda}(x) = (i \leftrightarrow j)$$

Left multiply by $\psi_{\cdot a}^{\lambda}(\phi)$ to get

$$\boxed{\frac{\partial \psi_{\cdot j}^{\lambda}(x)}{\partial x^i} + \psi_{\cdot a}^{\lambda}(\phi) \frac{\partial \chi_{\lambda}^{\alpha}(\phi)}{\partial \phi^{\alpha}} \chi_{\cdot b}^{\alpha}(\phi) \psi_{\cdot i}^b(x) \psi_{\cdot j}^{\lambda}(x) = (i \leftrightarrow j)}$$

③

Result 9. Rewrite result 8 as:

$$\left[\frac{\partial \psi_j^l(x)}{\partial x^i} - \frac{\partial \psi_i^l(x)}{\partial x^j} \right] = \psi_a^l(\phi) \frac{\partial X_a^l(\phi)}{\partial \phi^a} X_b^a(\phi) [\psi_j^b(x) \psi_i^a(x) - \psi_i^b(x) \psi_j^a(x)]$$

Apply to the right $X_J^j(x) X_I^i(x)$ to get

$$\begin{aligned} \left[\frac{\partial \psi_j^l(x)}{\partial x^i} - \frac{\partial \psi_i^l(x)}{\partial x^j} \right] X_J^j(x) X_I^i(x) &= \psi_a^l(\phi) \frac{\partial X_a^l(\phi)}{\partial \phi^a} X_b^a(\phi) [\delta_{Jj}^b \delta_{Ii}^a - \delta_{Ii}^b \delta_{Jj}^a] \\ &= \psi_a^l(\phi) \left[\frac{\partial X_{Ii}^a(\phi)}{\partial \phi^a} X_J^a(\phi) - \frac{\partial X_{Jj}^a(\phi)}{\partial \phi^a} X_I^a(\phi) \right] \end{aligned}$$

Therefore: (9.1)

$$\boxed{\left[\frac{\partial \psi_j^l(x)}{\partial x^i} - \frac{\partial \psi_i^l(x)}{\partial x^j} \right] X_J^j(x) X_I^i(x) = C_{IJ}^l = \psi_a^l(\phi) \left[\frac{\partial X_{Ii}^a(\phi)}{\partial \phi^a} X_J^a(\phi) - \frac{\partial X_{Jj}^a(\phi)}{\partial \phi^a} X_I^a(\phi) \right]}$$

Now set $y = x'$ on right side. It depends only on ϕ which then takes the value 0 since $0 = \phi(x, x')$. Both sides must equal some constant

$$A_{IJ}^l = C_{IJ}^l = \text{structure constant}$$

This yields two equations: first, apply $\psi_j^j(x) \psi_i^i(x)$ to top equation:

$$(9.2) \quad \boxed{\left[\frac{\partial \psi_j^l(x)}{\partial x^i} - \frac{\partial \psi_i^l(x)}{\partial x^j} \right] = C_{ba}^l \psi_j^a(x) \psi_i^b(x)} \quad \leftarrow \text{equation for the } \psi$$

second, apply $X_a^a(\phi)$ to lower eqn; set $I=a$ $J=b$.

$$\begin{aligned} \left[\frac{\partial X_a^a(x)}{\partial x^a} X_b^a(x) - \frac{\partial X_b^a(x)}{\partial x^a} X_a^a(x) \right] &= X_{Ii}^a(x) C_{ab}^l \\ \Rightarrow (9.3) \quad \boxed{\left[\left(\frac{\partial X_b^a}{\partial x^i} \right) X_a^i(x) - \left(\frac{\partial X_a^a}{\partial x^i} \right) X_b^i(x) \right] = X_{Ii}^a(x) C_{ba}^l} &\quad \leftarrow \text{equation for the } X \end{aligned}$$

Result 10: in order that (9.2) have a solution, you must have:

$$\begin{cases} C_{ab}^l = -C_{ba}^l \\ C_{ab}^l C_{cd}^l + \dots + \dots = 0 \end{cases}$$

If you pick some constants that do this, then with gauge freedom, you can find a solution ψ of 9.1, and you can compute χ from ψ . Then you integrate result 6 to get $\phi(x, y)$. For detail, see P.

Result 11: differential generators on the group!

Define:

$$X_r = -X_r^i(x) \frac{\partial}{\partial x^i}$$

Then use usual commutator:

$$\begin{aligned} [X_r, X_s] f(x) &= \left[X_r^i(x) \frac{\partial}{\partial x^i}, X_s^j(x) \frac{\partial}{\partial x^j} \right] f(x) \\ &= X_r^i(x) \frac{\partial}{\partial x^i} \left[X_s^j(x) \frac{\partial}{\partial x^j} f(x) \right] \\ &= X_s^j(x) \frac{\partial}{\partial x^j} \left[X_r^i(x) \frac{\partial}{\partial x^i} f(x) \right] \\ &= X_r^i(x) \frac{\partial X_s^j(x)}{\partial x^i} \frac{\partial f(x)}{\partial x^j} + X_r^i(x) X_s^j(x) \frac{\partial^2 f(x)}{\partial x^i \partial x^j} \\ &\quad - X_s^j(x) \frac{\partial X_r^i(x)}{\partial x^j} \frac{\partial f(x)}{\partial x^i} - X_s^j(x) X_r^i(x) \frac{\partial^2 f(x)}{\partial x^j \partial x^i} \end{aligned}$$

But $\frac{\partial^2 f}{\partial x^j \partial x^i} = \frac{\partial^2 f}{\partial x^i \partial x^j}$ we presume, so

$$= \left[X_r^i(x) \frac{\partial X_s^j(x)}{\partial x^i} - X_s^j(x) \frac{\partial X_r^i(x)}{\partial x^j} \right] \frac{\partial f(x)}{\partial x^j}$$

$$= X_{sr}^l(x) C_{sr}^l \frac{\partial f}{\partial x^l} = C_{sr}^l \cdot \underbrace{X_s^j(x) \frac{\partial}{\partial x^j}}_{X_s} f$$

↑ using (9.3).

④

Thus we have shown that:

$$X_r = X_r^i(x) \frac{\partial}{\partial x^i} \rightarrow$$

$$[X_r, X_s] = -C_{rs}^l X_l$$

Not minus sign!

We have realized the Lie algebra as differential generators on the group manifold.

Result 12 Recall that $X_\rho^a(x) \approx \delta_\rho^a + a_\rho^b X^b + \mathcal{O}(x^2)$

$$\text{Thus, } \frac{\partial X_b^a(x)}{\partial x^i} = a_{bi}^a + \mathcal{O}(x)$$

Insert into result (9.3) and take zeroth order on both sides:

$$a_{bi}^a \delta_a^i - a_{ai}^b \delta_b^i = \delta_l^a C_{ba}^l \Rightarrow$$

$$a_{ba}^a - a_{ab}^a = C_{ba}^a$$

The structure constants determine ?

$$a_{ba}^a = s_{ba}^a + A_{ba}^a$$

$$\Rightarrow s_{ba}^a + A_{ba}^a - s_{ab}^a - A_{ab}^a = C_{ba}^a$$

$$\Rightarrow 2 A_{ba}^a = C_{ba}^a$$

 $\Rightarrow$

$$A_{ab}^a = \frac{1}{2} C_{ab}^a$$

The structure constants determine the Antisym part of a only.

structure constants of $SO(2,1)$.

① Let

$$G_1 = K_1$$

Use:

$$[K_1, K_2] = -i J_3$$

a)

$$G_2 = K_2$$

$$[K_1, J_3] = -i K_2$$

b)

$$G_3 = J_3$$

$$[K_2, J_3] = +i K_1$$

c)

Then, since $[G_i, G_j] = C_{ij}^k G_k$ we find:

$$a) \quad C_{12}^1 = 0 \quad C_{12}^2 = 0 \quad C_{12}^3 = -i$$

$$b) \quad C_{13}^1 = 0 \quad C_{13}^2 = -i \quad C_{13}^3 = 0$$

$$c) \quad C_{23}^1 = +i \quad C_{23}^2 = 0 \quad C_{23}^3 = 0$$

② How many independent entries are there? Let $n = \text{order}$ (here $n=3$).

Then for each upper index $\exists \frac{n(n-1)}{2}$ elements (3 in our case).

Thus, total # is

$$N = \frac{n^2(n-1)}{2} = \frac{3 \cdot 3 \cdot 2}{2} = 9.$$

All nine are shown here.

Questions: 1) why are there diff ops on the group which satisfy the Lie ?

2) why are the "matrix elements" eigenfunctions of the Casimir?

1. In answer to the first question, we know that there is such a thing as the χ matrix because this is a certain derivative of the ϕ function. Also, we know that the χ functions satisfy a certain equation involving the structure constants. It is easy to show that the differential generators (which are defined in terms of the χ functions) satisfy the Lie Algebra, ^{except for a sign} using this equation. I proved this as Result 11. As sort of a corollary, we know that the λ functions satisfy the same structure constants equation as the χ 's. Thus, the M-space generators called M by Bargmann must also satisfy the Lie Algebra.

2. As for the second question, I quote Bargmann: ^{for} every irreducible representation of G , the operator Q has the constant value q . Now, if you go inside an irreducible rep, you will have some little basis vectors and $Q/a = q/a$ for all the basis vectors $/a$. If you act on $/a$ with a group element operator $U(g)/a$, the result is a linear combination of the basis elements, i.e., lies in the same space and is therefore still an eigenfunction of the operator Q . In specific coordinates, if you use the r, ϕ for the light cone thing, you will find that $Q = d/dr$ roughly and basis functions in a given IR all have the same r dependence, hence the above will be true.

This leads at once to the idea that matrix elements are eigenfunctions of the Casimir operator. Recall that you apply the group-diffop to the matrix element and move that operator inside to act directly upon $T(a)/m$. More specifically, the group-operator acts on $T(a) \cdot f_m(x)$ where x are some kind of coordinates. Since this is a simple multiplier representation, you can replace the group-diff-op with an M-space diffop. But, once you are in the M-space, you realize that the thing is an eigenfunction because you have an IR. OK, so this is really no big thing.

Calculation of the Invariant Measure as a Determinant. (Bargmann)

1. Here is what we want a right-invariant measure to do:-

$$\int_G da_1 \dots da_n \nu_r(a) f(a) = \int_G da_1 \dots da_n \nu_r(a) f(ab)$$

The game is to find a right-invariant measure $\nu_r(a)$ that makes this equation true.

Define $c = ab$. Then we have a jacobian,

$$da_1 \dots da_n = \frac{\partial(a_1, \dots, a_n)}{\partial(c_1, \dots, c_n)} dc_1 \dots dc_n = J^{-1} da_1 \dots da_n$$

where

$$J = \frac{\partial(c_1, \dots, c_n)}{\partial(a_1, \dots, a_n)} = \left| \frac{\partial(ab)^i}{\partial a^j} \right| = \left| \chi_s^i(c) \right| \cdot \left| \psi_k^s(a) \right| = \frac{|\chi(c)|}{|\chi(a)|}$$

Here we have made use of Bargmann's equation 1.1 which tells how the new parameters c vary as the parameters a are varied. We also use fact that ψ is the inverse matrix of χ . So far, our equation becomes:

$$\int_G da_i \nu_r(a) f(a) = \int_{G_b} dc_i \frac{|\chi(a)|}{|\chi(c)|} \nu_r(a) f(c)$$

2. There is a slight question as to the range of integration in the second integral.

On the left side, you integrate over a however is necessary to cover the group.

As the a range over G , it is clear that the c are also on the group, but it is not exactly clear that the c exhaust the group just because the a do. But clearly we want this to be the case so we can run the second integral also over G . I will skip this point for now and assume true.

3. The obvious choice for the invariant measure is:

$$\nu_r(a) = \text{const} \times |\chi(a)|^{-1}$$

$$\Rightarrow \int_G da_i |\chi(a)|^{-1} f(a) = \int_G dc_i |\chi(c)|^{-1} f(c)$$

4. We can repeat for a left-invariant measure by using $f(ba)$. We end up with

$$\nu_l(a) = |\chi(a)|^{-1} \quad \text{as from B (1.3).}$$

proof that $\psi_i^j(a) \frac{da^j}{dt} = \text{constant}$:

$$a(s), a(t) = a(s+t) \text{ means}$$

$$\phi^i(a(s), a(t)) = a^i(s+t)$$

diff wrt s

$$\left[\frac{\partial \phi^i(a(s), a(t))}{\partial a^j(s)} \right] \frac{\partial a^j(s)}{\partial s} = \frac{\partial a^i(s+t)}{\partial (s+t)}$$

replace with $\psi_i^j(a(s+t))$

$$\psi_i^j(a(s+t)) \psi_j^s(a(s)) \frac{\partial a^j(s)}{\partial s} = \frac{\partial a^i(s+t)}{\partial (s+t)}$$

Apply $\psi_i^j(a(s+t))$:

$$\psi_j^s(a(s)) \frac{\partial a^j(s)}{\partial s} = \psi_i^j(a(s+t)) \frac{\partial a^i(s+t)}{\partial (s+t)}$$

Set $t = -s$ to conclude

$$\psi_j^s(a(s)) \frac{\partial a^j(s)}{\partial s} = \text{const} = k^s \quad \text{QED.}$$

prove that:
$$\frac{\partial g(a, x)}{\partial a^k} + \Psi_k^r(a) \lambda_r^i(x) \frac{\partial g(a, x)}{\partial x^i} = 0 \quad (1.24)$$

① Since $g(a, ay) = f(y)$, we know $\frac{dg(a, ay)}{da^k} = 0$,
total derivative.

② Thus,

$$0 = \frac{\partial g(a, ay)}{\partial a^k} + \frac{\partial g(a, ay)}{\partial (ay)^i} \underbrace{\frac{\partial (ay)^i}{\partial a^k}}_{= \lambda_r^i(ay) \Psi_k^r(a)}$$

③ Now set $ay \equiv x$ to get

$$0 = \frac{\partial g(a, x)}{\partial a^k} + \frac{\partial g(a, x)}{\partial x^i} \lambda_r^i(x) \Psi_k^r(a) \quad \text{QED.}$$

prove that
$$\chi(T(a)f(x)) = \Lambda_\chi(T(a)f(x))$$

① I.e., show $\chi g(a, x) = \Lambda_\chi g(a, x)$

Well,

$$\begin{aligned} \chi g(a, x) &= \chi^k(a) \frac{\partial g(a, x)}{\partial a^k} \\ &= \chi^k(a) \left[- \frac{\partial g(a, x)}{\partial x^i} \lambda_r^i(x) \Psi_k^r(a) \right] \\ &= - \underbrace{\chi^k(a) \Psi_k^r(a)}_{\delta_{kr} = 1} \cdot \left[- \lambda_r^i(x) \frac{\partial g(a, x)}{\partial x^i} \right] \\ &= - \chi^r \lambda_r^i(x) \frac{\partial g(a, x)}{\partial x^i} = - \lambda_\chi^i(x) \frac{\partial}{\partial x^i} g = \Lambda_\chi g \quad \text{QED.} \end{aligned}$$

The Basic Facts about Multiplier Representations.

1. First, we must review some of the characteristic functions which are involved. When you consider simple shift representations on some M-space, you run into some functions :

$$c = \phi(a, b) = "ab" \quad \chi_\beta^a(b) = \left. \frac{\partial \phi(a, b)}{\partial a^\beta} \right|_{a=0} \quad a \in G \quad b \in G$$

$$y = ax = \sum (a, x) \quad \lambda_\beta^i(x) = \left. \frac{\partial \sum^i(a, x)}{\partial a^\beta} \right|_{a=0} \quad a \in G \quad x \in \mathcal{M}$$

In terms of these very special functions, we may write down some differential operators which satisfy the lie algebra of the group:

$$-X_r = -\chi_r^i(a) \frac{\partial}{\partial a^i}$$

$$\Lambda_{r,i}^0 = -\lambda_r^i(x) \frac{\partial}{\partial x^i} \leftarrow \text{generates of the left shift rep } T^0$$

If $T^0(a)$ is a simple left-shift representation (no multiplier), then we know we get this magic equation:

$$\chi_r [T^0(a) f(x)] = \Lambda_r^0 [T^0(a) f(x)].$$

2. Now we move to the context of multiplier representations. A multiplier is analytic in both variables and satisfies two equations. The "multiplier representation" then indeed is a group representation:

$$T(a) f(x) = \mu(a, \bar{a}x) f(\bar{a}x)$$

$$\mu(e, x) = 1 \quad \mu(ab, x) = \mu(a, bx) \mu(b, x) :$$

Given some multiplier, you can then show that certain new functions can be defined namely:

$$\tau_r(x) = \left. \chi_r \mu(a, x) \right|_{a=0}$$

$$\text{Then: } \chi_r \mu(a, x) = \tau_r(ax) \cdot \mu(a, x)$$

One then defines some differential operators Λ on the M-space with no superscript:

$$\Lambda_r = \tau_r + \Lambda_r^0$$

You then show that these new operators satisfy the "magic equation" for the multiplier representation:

$$\Lambda_r [T(a) f(x)] = \chi_r [T(a) f(x)]$$

and then it follows at once that these Λ operators are the differential generators of the multiplier representation! They satisfy the Lie algebra, ie.

One other interesting fact is that the product of two multipliers is again a multiplier, and the new τ function is the sum of the two τ functions.

3. The scene is now set to investigate some of the dynamite aspects of multiplier representations. I will do these on separate sheets because I know they will take some work.
4. One other detail. Corresponding to the diff gens Λ of the mult rep on the group are certain diff gens of the mult rep on the M-space. These are given in an obvious way:

Related Lie Algebras.

① I claim to have derived these facts:

$$[X_r, X_s] = -C_{rs}^t X_t$$

$$[\Lambda_r, \Lambda_s] = +C_{rs}^t \Lambda_t$$

② To check consistency, use "magic equation":

$$\Lambda_r [T(a) f(x)] = X_r [T(a) f(x)]$$

OR $\Lambda_r g(a, x) = X_r g(a, x).$

Well: $[X_r, X_s] g = -C_{rs}^t X_t g = -C_{rs}^t \Lambda_t g.$

Alternatively: $[X_r, X_s] g = (X_r X_s - X_s X_r) g$

$$= (X_r \Lambda_s - X_s \Lambda_r) g$$

$$= (\Lambda_s X_r - \Lambda_r X_s) g$$

$$= (\Lambda_s \Lambda_r - \Lambda_r \Lambda_s) g$$

$$= [\Lambda_s, \Lambda_r] g = -[\Lambda_r, \Lambda_s] g = -C_{rs}^t \Lambda_t g.$$

/since $[\Lambda_r, X_s] = 0$

Here, you can consider T to be either a regular left shift rep, or a multiplier rep, with Λ the corresponding generators.

A comment on invariant densities.

1. Having considered Bargmann pages 610-612, it seems to me that you could have a local scalar product for real j if you could solve this equation:

$$|\alpha + \beta e^{i\phi}|^{4\sigma} W(\phi) = W(\phi') = W\left(\phi - i \ln \left[\frac{\bar{\alpha} + \beta e^{-i\phi}}{\alpha + \beta e^{i\phi}} \right]\right)$$

Ie, if you could solve this for W , then W would be the invariant density and you would have a nice local scalar product for the D_k^+ scalar product, say.

2. The nature of the above equation is this:

$$a(x) f(x) = f[b(x)].$$

You are given a and b and are told to find f . I have never thought about this type of equation before. I am not even sure what the conditions are that this equation have a solution. Does this equation have some name??

3. I have asked Hassam about this, didn't know. Halpern mentioned it once. I could probably drag it up somewhere, but will not bother. The point is clear: in some cases this thing will not have a solution. This translates into the statement: in some cases, you will not get a local ~~scalar~~ scalar product in your M -space.
4. Comments on D_k . In the z -rep on this disc, you do get a local scalar product, but in the ϕ -rep, you get a non-local scalar product. Bargmann gives an example of a non-local scalar product for his exceptional interval. In the m -space representation of D_k , the scalar product again becomes local with a simple weight function.

1. The idea that you associate a one-parameter subgroup unitary operator as an exponentiated hermitian operator is called Stone's Theorem.
2. Lots of notation is set up. $\mathcal{L}$ stands for a group which we would call $SO(2,1)$. I think, the group of Lob motions. Whereas, $\mathcal{D}$ stands for the related "spinor group" wherein you often need $\frac{1}{2}$ on your rotations to return to unity. The carrier space, or rep-space, of a rep will be called $\mathcal{H}$, a Hilbert-Space. The $\mathcal{M}$ -space on which functions of $\mathcal{H}$ are defined is called $\mathcal{M}$. The generators are taken as H_{01} , H_{20} (the boosts), and H_{12} . The spectrum of this last generator is discreet.
3. Bargmann summarizes first the $O(2,1)$ results. He lists off the UIR's, including the exceptional as part of C^q_0 . Except for the trivial, then, there are really four kinds of series. Note that group elements are called a not g . The operators are called $U(a)$. Basis functions are f_m . Matix elements of the xform operators which define a rep are called u_{mn} .

A new idea for me presented here is this: you know what dg is, and you can look at the asymptotic behavior of the matrix elements in the various classes of reps. It turns out, eg, that $D^{+\frac{1}{2}}$ blows up so fast that you can not integrate it on dg . This is connected with the fact that $D^{+\frac{1}{2}}$ does not show up in the expansion theorem. Also, concerning expansion theorem, the idea that certain UIR functions (not all) of them are dense in the space of functions of g means, basically, that arb function can be expanded in them.
4. Next, he summarizes some results for the group $SO(3,1)$. Again, the series are listed off, /Casimirs are stated, it is remarked that there are no discreet series, etc. However, B will discuss this group $SO(3,1)$ which he calls $\mathcal{L}_4$ in another paper called "Part II". The paper I have is Part I, and will stick to $SO(2,1)$ except that certain parts of I are general to both papers.

1a. introduction. When you do $ab=c$ in a group, this means that two vectors in parameter space are mapped into a third, called $\phi^i(a,b) = c^i$. If you differentiate this function wrt a^i or b^i , you get equations like 1.1 and 1.3, special cases of which were discussed in Wybourne. These derivatives are described by a set of entities called the $\chi_k^i(g)$. Certain combinations of these objects with their own first derivatives gives certain primitive commutation relations, as in 1.2. In fact, these appear in Wybourne page 31.

1b. one parameter subgroups. Let a^i be a point in parameter space. Let t be the parameter of a OPS (one parameter subgroup). Then as t changes, a^i change for all i . In fact 1.6a shows exactly how. A certain linear comb of those χ_k^i gives you an object χ^i which $= da^i/dt$. Of course, i runs from 1 to n where n parameter Lie group, so χ is the vector whose components are χ^i . Basically, these objects $\chi^i(g)$ act as a set of generators. They are really the infinitesimal transformations when you take the group itself as an M-space. However, these χ^i are not the same as the usual X^i because (I suspect) they are more general, not taken to be near the identity. Thus, the comrels 1.10 are not yet AB-BA, but appear as 1.9 instead. Adjoint group is mentioned here.

1c. group invariant integration. It turns out that the inverse of the determinant of the objects $\chi_k^i(g)$ is the invariant measure, up to a norm factor. I bet this ties in with Vilenkin page 159. The existence of an equal right and left invariant measure is tied to a condition on the structure constants, 1.18, new to me. We shall stick to such cases.

1d realization of G on the M-space. Here, things are repeated as before. Instead of using G itself as the M-space, use some arbitrary M-space whose coords are x^i . G will act on this "transitively" such that x goes into $Z^i(a,x)$, which plays the role of $\phi^i(a,b)$ earlier. Now the derivatives of these Z_i are given in terms of some old objects τ_k^i (inverse of χ_k^i) which are functions on G, and in terms of some new objects $\lambda_a^i(x)$ which are functions on m . These new objects λ_a^i then satisfy the primitive com rels 1.21, still not AB-BA.

Now, out of the blue sky comes : the standard (ie, shift) representation. This is : $T(a)f(y) = f(a^{-1}y)$ which he also calls $g(y;a)$. Look at 1.25 and 1.26. These define $\mathcal{X}_i$ are the differential generators of the Lie algebra acting on G, whereas Λ_i are the diff generators which act on the M-space! Presumably these are in 1-to-1 correspondence, they just act on different spaces. Thus, in 1.27 χ is

some arbitrary generator acting on the G space. Corresponding to it, is Λ_X given in 1.28 as the diff generator which acts on x . Now consider $g(y;a) = f(a^{-1}y)$. The action of generators $X(a)$ on $g(y;a)$ is the same as the action of $\Lambda_X(x)$! See 1.29. Now these new generators which act on x , these $\Lambda_X(x)$, these are the guys which satisfy the AB-BA com rels!

1e. remarks on the linear reps of G . Look back at 1.19 and 1.20. If $\mathcal{G}$ our transformations on the space M are linear (like $SL(n, \mathbb{C})$), then the objects $\Lambda_X(x)$ appearing in 1.20 must be linear in the x 's. Thus, the objects L_x appear, as $n \times n$ matrices. These are the usual generator matrices (like Pauli matrices) which you associate with a Lie algebra. Of course they satisfy the Lie algebra. When you exponentiate these matrices, you get finite dimensional linear group representations. (of course for $so(2,1)$ all such reps will be non-unitary).

Bargman makes a good point here. In the compact theory, you always deal with generator matrices L , and matrix reps $U(a)$ acting on vectors in the finite dim vector space. In the noncompact theory, ~~xxx~~ our generators are Λ_X and our rep operators are $T(a)$. These act on an infinite dim Hilbert space of functions of x . By the way, notice that the shift representation is characterized by a linear operator $T(a)$, even though the action of ~~xxxx~~ group element a on x may be highly non linear. Proof:

want to show that $T(a)\{f_1 + f_2\} = T(a)f_1 + T(a)f_2$
 then $T(a)$ is linear, more or less.

Use basis x :

$$\langle x | T(a) | f_3 \rangle = \langle a'x | f_3 \rangle = f_3(a'x)$$

But $f_3(y) = f_1(y) + f_2(y) \quad \forall y \in M$. Thus $\rightarrow = f_2(a'x) + f_1(a'x)$.

$$\text{So } \langle x | T(a) | f_3 \rangle = \langle x | T(a) \{f_1 + f_2\} \rangle = f_3(a'x) = \text{this} \uparrow \\ = \langle a'x | f_2 \rangle + \langle a'x | f_1 \rangle = \langle x | T(a) | f_2 \rangle + \langle x | T(a) | f_1 \rangle, \quad \forall x.$$

$$\text{Therefore: } T(a)\{f_1 + f_2\} = T(a)f_1 + T(a)f_2. \quad \checkmark$$

By the way, the symbol X will always denote an arbitrary generator of the Lie Algebra, ie, a linear combination of some chosen basis generators X_i .

1f. multipliers. A multiplier is defined on both G and M , hence $u(a;x)$. If it satisfies two simple properties: (1) $u(e;x) = 1$ for all x ; and (2) $u(ab;x) = u(a;bx) u(b;x)$ for all a, b, x ; then it really is a "multiplier" and the generalized shift operators $T(a)f(x) = u(a;a^{-1}x) f(a^{-1}x)$ define a linear rep of your group. It is noted that the set of multipliers forms a group under

multiplication. Also, it is noted that if you can find a function $g(x)$ that is invariant under your multiplier rep $T(a)$, then the multiplier must be none other than $u(a;x) = g(ax)/g(x)$.

lg. infinitesimal multipliers. Since $u(a;x)$ is a function of x , we can obviously try to apply our M -space diff generators χ to the multiplier. Here is what you find:
 $\chi u(a;x) = [\chi u(a;x)]_{a=e} \cdot u(a;x)$... In other words, the multiplier is an eigenvector of the operator χ with eigenvalue as shown. This eigenvalue is given a new notation: $\tau_\chi(ax)$. The object $\tau_\chi(a) \equiv \chi u(a;x)|_{a=e}$. This thing is called the "infinitesimal multiplier". Actually, it is a particular derivative of the multiplier near the identity. (at the identity). See 1.45.

Now, let $T(a)$ be some multiplier rep, denote by $T^0(a)$ the bare shift rep. What is the action of χ on $[T(a)f(x)]$? Back in 1.29 we found the action of χ on $[T^0(a)f(x)]$. There we learned that $\chi T^0(a)f(x) = \Lambda_\chi^0$ (on same). (Let us now use Λ_χ^0 for the differential generators of the bare shift rep.) Then, the new result we find is this:

$$\chi[T(a)f(x)] = (\tau_\chi(x) + \Lambda_\chi^0) [T(a)f(x)]$$

Thus, define new diff oprs as:

$$\Lambda_\chi(x) \equiv \Lambda_\chi^0(x) + \tau_\chi(x)$$

Here is the point: if you know the diff oprs of the bare shift rep, and if you know the "infinitesimal multiplier" of your multiplier rep, then you get the new diff oprs of the multiplier rep as shown by this equation. Of course these new diff oprs "realize" the com rels just as the Λ_χ^0 did.

This section closes with various comments about how multipliers multiplied give an infinitesimal multiplier as a sum, etc. Things are also stated in terms of $g(x)$ which we recall as a way to reparametrize the multiplier.

lh. Method of constructing Multipliers. This section will, I feel, be fundamental for Bargmann's purposes later on. It involves fiber bundles and, I think, induced representations, but these things are never really mentioned. Here is the idea: start with a space M^* having $m-1$ coordinates x^i . (Treat as base of bundle). The group acts on M^* in some known way $Z^i(a;x)$. Now, for each point in M^* define a line over that point (a fiber) whose parameter is x^m , a new coordinate. Thus, a point in the fiber has m coordinates. The action of the group on this fiber requires extra knowledge about what happens to this extra coordinate. Suppose we try: $U_X^*(g) x^m = u(g;x^*) x^m$. I.e., this is the action of the group G on the fiber over point x^* . Then the rule for the consistency of a homogeneous fiber

bundle becomes exactly the rule for $u(a;x)$ to be a multiplier!!!!

Now suppose we have spaces M^* and M (ie, a bundle) with the multiplier $u(a;x^*)$. Because u is a multiplier, we have ourselves a multiplier rep on M^* (not on M). Ie, $T(a)f(x^*) = u(a;a^{-1}x^*)f(a^{-1}x^*)$. What Bargmann now wants to do is construct multiplier reps which have powers of this multiplier as the new multiplier. So, let $T^h(a)$ be this new rep on M^* . Now, he is going to construct a new rep, the bare shift rep on the space M , called $T^0(a)$. If you restrict your functions (on which T^0 shall act) to functions of the form $F(x) = (x^m)^{-h} f(x^*)$, then you can show this: $T^0(a) F(x) = (x^m)^{-h} T^h(a) f(x^*)$. Ie, the action of T^0 on its space of F 's is equal to a multiple of the action of the h^{th} multiplier rep on M^* on functions in its domain. Similarly, the action of the generator X on the "outcome" of $T^0 F$ is given in terms of the action of X on the outcome of $T^h f$, see under 1.57.

My own comments: Possibly the space M^* contains a point x^*_0 which has an invariance group H , a subgroup of G . If so, then the fiber bundle characterized by Bargmann's multiplier could also be characterized by some representation of the subgroup; H . Perhaps then some correspondence could be made between induced reps and multiplier reps.

In the last section here B shows how you might actually construct a multiplier to go from M^* to M . Suppose you start with some space M and some linear transformations of M onto M (like $SL(m,C)$!). Denote the transformation matrices by w^i_j . Then you can define the space M^* by taking as coordinates the ratios $x^i = \frac{w^i}{w^m}$ where w^m is the last coordinate of space M . Let $x^m = \xi^m$. Then the x^i up to $m-1$ are coords of M^* , and coords up to m are coords of M , and the multiplier connecting them is given in 1.59 in terms of the matrices w^i_j ! Therefore, from the m -dimensional (nonunitary) rep of $GL(m,C)$ (ie, those matrices) apparently you can explicitly construct a multiplier rep on the space M^* of $m-1$ coordinates. Moreover, as discussed above, the various "multiples" of this multiplier rep on M^* can be associated with bare shift reps in the larger m -coordinate space M , provided that the functions under these bare shift reps are suitable restricted as in 1.56! I think this is very significant. For example, the reps of $GL(2,C)$ should be realizable either as multiplier reps on a 2-dim space M^* , or as a bare shift rep on a restriction of the space of functions of 2 variables. I bet the space M^* is the circle, and the 2 dim space is (z_1, z_2) where the restriction is homogeneous polys, or something like that.

1i. (last) invariant densities. Define $I(a), \omega(x), \eta(a,x)$. Save a reading of this section until it is needed.

2a. linear transformations leaving a quadratic form invariant. Very nice section. Here all at once B treats the standard analysis of $SO(p,q)$ where $p+q = m$, fixed. There is a certain diagonal metric tensor, g_{ij} . The requirement that any generator X acting on $x \cdot x$ do nothing (invariant) forces the $m \times m$ generator matrices to be antisymmetric, as in 2.5. Thus, there are only $m(m-1)/2$ indep generators, and that is the order of the Lie group. The standard generators are defined in terms of the metric tensor as in 2.6, the way I like. Then the structure constants are implicitly as shown in 2.7 and it is noted that these yields a right/left invariant measure on the group. These generators so far act on m -component vectors and we can talk about the finite dim reps. The scene now shifts to the standard "bare" shift representation on functions of m -variables x^i . The generators of this rep are given in 2.8, along with the reminding notation Λ_X^0 where $X = (kl)$ labels the generator of interest.

I think this is new to me. The differential operators that I have always used for angular momentum, eg, are the generators of the shift representation where the M -space is E^3 . The orbits in E^3 are of course spheres. Since $SO(3)$ preserves the length of vector x^i , the unit sphere alone can be used for the carrier space of this shift representation. Notice that the action of group elements on the points in the M -space are given by the m -dimensional group rep, whereas the action of group operators on the space of functions is given by a shift. Thus, we have here a case of a linear rep being defined in terms of another linear rep. I suspect that the general shift rep so described is highly reducible in general.

2b. the adjoint group. Not clear what this is all about. If you similarity transform all your generator by some fixed operator $U(b)$, you get a new set of generators. This similarity xform that generates a new generator $X'_r = U(b) X_r U(b)^{-1}$ can of course be written as $= S_{ri}(b) X_i$ because the original generators span the lie algebra. The set of matrices S forms a new finite dim rep of the group, of size m . This is called the "adjoint" rep. Equation 2.10a shows how you can construct this rep from the usual finite dim rep, w^i_j .

2c. the operators Q and R . Q is the usual Casimir $= \frac{1}{2} L_{ij} L^{ij}$. In three dimensions (ie, $m=3$) it is the only one. Ie, there is no other way you can combine the tensors L_{ij} to get a scalar. All you have to use is g_{ij} and ϵ_{ijk} . But when you go to $m=4$, you can also use $R = \frac{1}{4} \epsilon_{jkl} L^{ij} L^{kl}$. These scalars Q and R correspond to the

labels (P, Q) used in Liubarski's book. Recall that Lorentz group reps there were characterized by $\mathcal{P}_Q$. A "vector" particle belonged to the rep $\mathcal{C}_{11}$, whereas a Dirac spinor particle was something like $\mathcal{C}_{0\frac{1}{2}} \oplus \mathcal{C}_{\frac{1}{2}0}$. Etc.

But now we are not really interested in $SO(3,1)$, so drop this stuff for now.

2e. the case $m=3$. first, the generators are recast into the three L_i . These will be the same generators for $SO(3)$ as for $SO(2,1)$. Remember that the metric tensor appears in the com rels!! So, 2.18 shows the com rels for $SO(3)$ and 2.19 gives Q . Similarly, for $SO(2,1)$, equation 2.22 gives the algebra. It is observed that for the first case, the "definite metric" means that $k.k$ is always negative, so only one case to consider. In the latter case, the metric is "indefinite" so you have to consider three kinds of "orbits" depending on nature of $k.k$. These are called elliptic, parabolic, hyperbolic. Parabolic means "on the light cone". On a parabolic orbit, you can find a point (say, $(1,1,0)$) whose little group is a one-parameter group whose single generator is the lightlike boost. On an elliptic orbit, you can find a point, say $(0,0,1)$, whose one-parameter little group is $SO(1,1)$. Finally, there is always a point on a hyperbolic orbit, say $(1,0,0)$ with little group $SO(2)$. The point is that depending on your type of orbit, there are three kinds of little groups to think about here, whereas for the rotations group, only one little group: $SO(2)$.

3a. The group G_4 . By plopping the 4 components of vector x^i into a 2×2 matrix in the usual way, see 3.1, we construct an element of group $SL(2, \mathbb{C})$. This matrix realization of the 4-vector is Hermitian and has determinant equal to the invariant length of the vector. If you act on this on both sides by some unimodular matrix, say W , the determinant is preserved. Thus, any real Lorentz transformation can be cast into some $SL(2, \mathbb{C})$ element W . Of course $(-W)$ also works. The map from group $G_4 = SL(2, \mathbb{C})$ to $L_4 = SO(3, 1)$ is therefore 2-to-1. These two groups are "locally isomorphic" and therefore have the same Lie algebra.

3b. The group G_3 . There are basically two ways to do this. First, take a subgroup of $SL(2, \mathbb{C})$ with the usual $\begin{pmatrix} z & 1 \\ 1 & \bar{z} \end{pmatrix}$. Set $x^3 = 0$ to get new X as in 3.4+. Now, if you consider the Mobius transformation $z' = (az + b)/(cz + d)$ as a map from the plane $\mathbb{C}$ to itself, you see that actually you are mapping the interior of the unit circle onto itself.

On the other hand, there is another way to do G_3 using 2×2 matrices with real elements. This is $SL(2, \mathbb{R})$. Given any $SU(1, 1)$ matrix, you can make the isomorphic $SL(2, \mathbb{R})$ matrix as shown in 3.6. If you consider the same Mobius transform with this new real 1×1 , you find that your group G_3 can be realized as transformations of the upper half z -plane into itself, instead of the interior of the unit circle. Just two ways to do it.

By the way, these $SU(1, 1)$ matrices have a property that looks like unitarity, but not quite. Vilenkin refers to $SU(1, 1)$ as $QU(2)$ for "quasiunitary". If you restrict to that part of $SL(2, \mathbb{C})$ which gives you $su(2)$, your 2×2 matrices come out really being unitary. See 3.7.

3c. multi-valued reps? Take the "elementary rep" of $su(1, 1)$, namely, those 2×2 matrices themselves. Obviously, in this rep, the "rep" of W is different from the rep of $-W$ because $\text{rep}(W) = W$ and $\text{rep}(-W) = -W$. But both W and $-W$ are the same group element from the point of view of $SO(2, 1)$. Thus, the "elem rep" of $su(1, 1)$, when considered as a rep for $SO(2, 1)$, is double-valued. Each point in $SO(2, 1)$ has two matrices associated with it. In general, if you examine some rep of $SU(1, 1)$ and you find that $\text{rep}(W) \neq \text{rep}(-W)$, then as a rep of $SO(2, 1)$, that rep is double valued. If $\text{rep}(W) = \text{rep}(-W)$, then it is single valued. The usual example is that $R_z(0) = \text{rep}(I)$ and $R_z(2\pi) = \text{rep}(-I)$. (You can show this). Thus, in a double-valued rep of $SO(2, 1)$, these two things are not the same!

However, there is another sense of "multi valued" that I want to get straight. The topological space associated with $SU(2)$ (I think a 3-sphere) is simply connected. This means that $SU(2)$ is its own universal covering group, and it also means that the representations of $SU(2)$ are always single-valued representations of $\widetilde{SU(2)}$ (notation for the covering group). On the other hand, the group $SU(1)$ has a topology which is infinitely connected (ie, the circle). The representations of the covering group $\widetilde{SU(1)}$ are $e^{im\phi}$ where m is any real number. For example, if $m = \frac{1}{4}$, then such a rep would be a four-valued rep of $SU(1)$. (I think) This single valued rep of $\widetilde{su(1)}$, $e^{i\phi/4}$, is called a "projective rep" of $su(1)$.

Similarly, it turns out that the top space associated with $su(1,1)$ is circle $\times$ plane and is therefore infinitely connected, just like $su(1)$. Thus, there will be reps of $\widetilde{su(1,1)}$ which will be many-valued reps of $su(1,1)$. As just stated, these are called projective reps of $\widetilde{su(1,1)}$. See page 193-4 Wybourne.

3f. infinitesimal xforms for spinor group. The point here is to find the action of the generators χ on some $\chi \chi X$. These turn out to be 2×2 -matrices given in 3.10. For G_3 these are not quite the Pauli matrices. Of course for $su(2)$ that is exactly what they are.

3g. remarks on spinors. We know that we can parametrize a 4-vector or three vector by the 2×2 matrix called X , eg, 3.4+. Call elements of this matrix x^{ij} . Thus, 3.12 shows group action in this notation.

Now for some reason I do not yet see, he wants to imagine that our 4- or 3-vector is lightlike ("null"). Then you can define two 2-spinors as in 3.14 such that $x^{ij} = s^i s^j$. Notice that this must give $\det(X) = 0$.

Next, you can also parametrize these x^{ij} by three polar variables as shown. Then 3.16a shows how the spinor and polar params compare. (Note that the first spinor is real, so in fact only three params in the spinor param). Now, if you insist that $x_3 = 0$ (ie, restrict to G_3), then $x^{11} = x^{22}$ and you get 3.17.

Now comes the point. Suppose you look at a G_3 transformation 3.12_{on a null vector} where the initial and final 3-vectors are param'd by polars, ie, r and ϕ for initial X and r' and ϕ' for resultant Y . Then of course you can compute $r'(r, \phi, \psi)$ and also ϕ' (same). These are given in 3.18 and 3.19. Remember that this only applies to transformations "on the light cone" of $SO(2,1)$. These little formulas look extremely familiar in another context. (ie, Vilenkin's $QU(2)$ reps).

4a. introduction of parameters. The first set of parameters to be introduced are the complex number γ which really is β/α , and the real parameter ω which is the phase of α . In terms of these new variables, α and β are given in 4.2. From the unimodularity condition, we see that $|\alpha| \geq |\beta|$, and that therefore the variable γ is restricted to lie inside the unit circle somewhere. For the group $G (=G_3$ from now on) variable ω lies in $(-\pi, \pi)$. For the covering group, ω lies anywhere on the real line, as already discussed.

4b. infinitesimals and invariant integration. Note the usage of symbols $\kappa^0, \kappa^1, \kappa^2$. These define a particular linear combination of the standard generators χ_r . I.e., you write an arbitrary generator as $\chi = \kappa^r \chi_r$. Recall that these generators χ act as diff ops right on the group G parametrized by some a^i . Now we already found the 2×2 generators for any generator of $SL(2, \mathbb{C})$ back in 3.10. Now, 4.7 merely specializes to any arbitrary lincomb of the base generators of G . Now, the idea is to insert these known generators M_r and to solve the first equation in 4.7 for the numbers $\chi_r^i(a)$. These numbers $\chi_r^i(a)$ recall are very fundamental to the group, and were introduced back on page 1 of this paper! So, once you calculate these χ_r^i for this special parametrization, you can at once do two things: (1) you can write out the explicit G -diff generators χ_r as in 4.8; and (2) you can compute up the determinant of the $\chi_r^i(a)$ because you know its inverse gives the invariant measure, hence 4.9.

4c. the parameters μ, ξ, ψ . Very familiar terrain. Under the standard ranges shown in 4.15, we are reminded that even so, the group G is covered twice. These new parameters give back the original ones γ, ω as shown in 4.16. Again, you can solve up equation 4.7 for the various 2×2 generators to compute the diff ops χ_r as shown in 4.17. And the invariant measure shown in 4.20. In fact, Bargmann even goes so far as to compute the diff op for Q , which he calls Ω , see 4.19. I skip section 4d which discusses ds^2 and geometry

4e. conformal xforms of unit circle to itself. Here we have an example of that M and M^* business encountered earlier, in the case $m = 2$. Start with a two dim space M with w^i_j given just by a β matrix. At once from 1.59 we can calculate the multiplier $u(a; z)$ and we have therefore constructed a multiplier rep on the space M^* which has just one (complex) variable; i.e., z . The multiplier is just $(1 + \beta z)$. He then computes the three infinitesimal multipliers, see 4.26. Recall that these are needed to compute up the M -space diff gens.

5a. introductory remarks. Sets up notation for Hilbert Space scalar product and norm, follows QM instead of math notation. Then he carefully defines what a UIR is: unitary operators on a Hilbert Space $\mathcal{H}$, operators are $U(a)$. Must satisfy the "rep property" which I call the multiplication rule. Also, certain continuity condition which I will always skip over. Finally, no invariant subspaces.

5b. criterion for irreducibility. Here we have Lemma 1 which is a modification of Schur's Lemma. Schur says that in a UIR, any all commuting operator must be a multiple of the identity. Here he claims the same, and more: if a rep is reducible, then there does exist some other all commuting operator. This operator can be explicitly constructed, it is not just a multiple of 1, and is called E . Basically, it is the operator which projects functions in $\mathcal{H}$ onto the invariant subspace $\mathcal{U}$.

5c. one-parameter subgroups. Here again we have Stone's Theorem which says the ~~invariant~~ one parameter subgroups can be set up as exponentiated Hermitian generators. Thus, we have generators H_r , or, if you still insist on arbitrary lin combs, H_ψ

5d. assumptions about infinitesimal reps. Warning: do not confuse the notation $\mathcal{D}_r$ with the same symbol used in Vilenkin. Think of r as labelling one of your gens, $r = 1, 2, 3$ lets say. I really don't care much for this subsection. The end result is that each generator H_r on our Hilbert Space has its own domain, called $\mathcal{D}_r$ and the intersection of these domains is called $\mathcal{U}$. As far as I am concerned, this $\mathcal{U}$ is the same as $\mathcal{H}$.

5e. the operator Q . Of course the operator Q acting on $\mathcal{H}$ is given in terms of the generators H_r acting on same. Page 602 is a long harangue that concludes that Q is a hermitian operator which in a UIR is a multiple of the identity. I.e., see equation 5.14. The "multiple" of the identity is called q . Thus, q is real. Also, he concludes in 5.17 that $\mathcal{D}_0$, the domain of the one compact generator, is equal to $\mathcal{U}$, the intersection of the domains of all three generators.

5f. playing with the generators. We know that the compact subgroup acting on our Hilbert Space looks like so, generatorwise: $H_0 g_n = \lambda_n g_n$. If we impose at worst double valuedness here, we get the usual integral/half integral spectrum. So,

let g be some eigenfunction of H_0 , and let λ be its eigenvalue. Define: $H_+ = F$ and $H_- = G$. Then 5.21 is standard. So is 5.22. Note that F^s means F applies s times. Now, notice that $GF+FG$ is a diagonal operator and has eigenvalue $2(q+\lambda^2)$ when applied to function g . See 5.23+. Next define ϑ_1 and σ_1 as what you get when you apply GF and then FG , respectively, to g . See 5.24 to get these in terms of q and λ . It does not take much then to get 5.25. If you raise s steps, then lower one, you get ϑ_s times something. Also, 5.26 follows at once. Therefore, you are forced to have σ_s and ϑ_s be positive!

5g. classification of the reps.

First, assume that λ is integral and there is no truncation so we go all the way in both directions. Choose $\lambda = 0$. The requirement that all those little ϑ_s and σ_s be positive ~~xx~~ forces you to keep to $q > 0$ only! Then 5.28 gives you your basis f_m functions. The whole show is summed in 5.29. Notice that he allows each basis function to have an arbitrary phase.

Next, assume no truncation again, but half-integral spectrum on H_0 . Then choose $\lambda = \frac{1}{2}$. Turns out that you need $q > \frac{1}{4}$ now. All rest is same. These two classes are called C_q^0 and $C_q^{\frac{1}{2}}$.

Next, assume truncation on the right. Choose g and its λ as the rightmost eigenvalue. I.e, $H_+g = 0$. Thus, $\vartheta_1 = 0$, and q locks onto λ : $q = -\lambda(\lambda+1)$. Easy to show that $\lambda \leq 0$. If $\lambda = 0$, then $\sigma_1 = 0$ also and you "die" on one step in either direction. This is the trivial representation. If $\lambda < 0$ (of course must be half or integral), then I have shown that $\sigma_s = (s+k)^2$ where $k = -\lambda$. Thus, σ_s is always positive def and thing does not truncate on the left. NB: there is no such thing as a finite dim-UIR here!! Or with any noncompact group. Repeat arguments to get the other type of discrete series. So far, there is nothing exceptional about the exceptional range of the continuous integral series.

5h. discussion. Although we have only set ~~of~~ the infinitesimal generators (ie, we know the action of F, H, H_0 on any basis function f_m in the Hilbert space spanned by such functions for a given ~~representation~~ UIR, we claim that the finite group transforms so generated are the UIR's

6a. transforms on the light cone. Here at last is the section where all our work begins to pay off, especially that stuff about multipliers. By simply considering the transformations of vectors on the future light cone, we are going to end up with specific operators which represent the finite group transformations on a carrier space. This rep will be valid for the $q > \frac{1}{4}$ part of C^0_q , the so-called nonexceptional interval.

As we know well by now, euclidean vectors on the future light cone can be paramed by r, ϕ . With no trouble at all, we can write down the diff generators Λ_r which give the bare shift rep in the Euclidean space. Thus, we just copy results from 2.8a and change to these r, ϕ coordinates. Hence, you have 6.4. Keep in mind that functions $f(r, \phi)$ form the Hilbert Space upon which these Λ_r generators act, and this bare shift rep is not an irreducible rep by any means. Ie, it contains lots of invariant subspaces, and we are about to identify one of these subspaces! To this end, we note that any function of the form 6.6b is an eigenvector of diff-op Q . Thus, if you stick within the subspace of functions of this form, you might be staying within a UIR because Q is the same for all function in this subspace. Notation ϕ is for later use.

6b. transformations of the unit circle. The above section was motivation for this section. We shall use the result of the previous section to choose a proper power of a certain multiplier, ie, $h = \frac{1}{2} + i\sigma$ where $\sigma = \sqrt{\frac{1}{4} - q}$.

So, here we go. Let the space M have variables r and ϕ . Let this r be the "added coordinate. Notice from 6.7, how you get r' from r . This is exactly what you want such that $|\alpha + \beta e^{i\phi}|^2$ is a multiplier for a representation in the space M^* whose only variable is ϕ . (M^* was the base of fiber bundle M). Now, take a power h of this multiplier. We know that is also a multiplier. Hence we have constructed the multiplier representation 6.11; there is no question now about what is going on. Equation 6.10 shows how this multiplier rep is related to a simple shift rep on a certain subspace of the larger space M . In fact, we see that that subspace is exactly functions of the form 6.6b which we found above!

Next, we utilize the "sandwich rule" 1.58 to construct the generators of our multiplier rep. Ergo, equation 6.12. Again, no muss, no fuss. By comparing the Λ_r to the Λ_ϕ we know we can get the "infinitesimal multipliers" by just taking the difference, so 6.13. Finally, 6.14 gives the operator Q in our new multiplier rep, again via the sandwich rule.

6c. discussion of this new multiplier rep. Out of the blue, once again, Bargmann guess the scalar-product 6.15. Looks simple enough. Craftily, he shows that if you choose q so that ω is imaginary, then the operators $T_\alpha(a)$ will be unitary (with respect to the chosen scalar product). In 6.18, elements u_{mn} are defined for later calculation.

6d. reps of the nonexceptional C_q^∞ series. Note that $\bar{H}_\alpha = i\Lambda_\alpha$. We have just found a set of Λ_α . He claims that they do in fact satisfy the equations 5.29, provided the phases are chosen in a special way. (Not trivial). Remember that 5.29 follows just from the Lie Algebra, really, - so any "infinitesimal rep" must satisfy them. Without further ado, we state that these generators Λ_α acting on the circle with scalar product as shown and with $q > \frac{1}{2}$ so that $\omega = \text{imag} = i$ with s real, ... these generators are the inf rep for the UIR's so claimed. It follows that the finite transform ops defined by our multiplier rep or by the associated shift rep on M , these finite transforms are the rep operators.

In 6.25 he simply rephases the basis vectors. Then matrix elements will be called v_{mn} instead of u_{mn} .

Connection between the theory of multiplier representations and
the theory of induced representations.

1. I think this is the way it works: start with a group G and take as your subgroup the trivial subgroup E . This subgroup has one representation, namely the real number 1. As the base of your fiber bundle choose some homogeneous space of $m-1$ dimensions; this will be Bargmann's space M^* . This will then be Mackey's space S . Any point in this space can serve as the invariance point of the base, s_0 . That is because the trivial subgroup E is an invariant-subgroup (recall this detail). So pick some s_0 . For our fibers, over each point in the base put a "complex line". I.e., the vector space which is each fiber is just a complex plane. Thus, the elements of these fibers are merely complex numbers. In general, the "fiber operator" is supposed to act on a vector in your Hilbert space (fiber), but here the vector space is so simple that the fiber operator is just itself a complex number. I.e., that is a possible fiber operator. Let us now identify the fiber operator $U_s(g)$ with the multiplier $u(x^*; g)$, where, of course x^* is a point on the base M^* and plays the role of s . Notice that the rules of multipliers are two. One of them we identify with the fiber operator consistency condition for a homogeneous fiber bundle. The other we realize says that for h in H (i.e., for $g = e$), the fiber operator is supposed to form a rep of subgroup H . But the multiplier rule says that this particular fiber operator ~~ix~~ reduces to unity for $g = e$, i.e., for h in H . But unity is a rep of our subgroup H (i.e., the trivial subgroup; it is the only rep.) Thus, everything is hanging together.

However, I thought that the trivial rep of the trivial subgroup always induced just the regular representation! Ah, but I think that is only true if you use the coset space G/H for your base S . Here, that coset space would be G itself, but this is not what we are using for S here. Ergo, the bundle is not completely determined by specifying G , $H=E$, and $L(h) = 1$. What you really need to completely specify your bundle is the multiplier. For each multiplier you get a unique bundle whose fiber operator is defined to be that multiplier. Then that bundle together with $G, H=E$ and $L(h) = 1$ specifies a unique induced representation.

Think back. The induced representation is just a shift operator which acts on things we called $f(g)$ or $\tilde{f}(g)$ as in Hermann. These are points in the reference fiber which you get via some "reference" cross-section. In our present case, the reference cross section is a complex function on M^* , i.e., $f(x^*)$. To get the objects $f(g)$, you act on the cross section with $U_s(g^{-1})$. But this is

just $u(x^*, g^{-1})$. Thus, we identify the elements of the reference fiber $\mathcal{F}(g)$ with $u(x^*, g^{-1}) f(x^*)$. This is very close to Bargmann's 1.56, considering his 1.54. Details aside, we conclude that Bargmann's "multiplier rep" in either of its forms (ie, as multiplier rep on M^* , or as shift rep on a piece of M) is to be identified with the induced representation of the group G by the subgroup E , rep 1, on the fiber bundle whose base is M^* and which is effectively labelled by the multiplier itself.

7a. reps of the half-integral principle series. The multiplier we used for the integers was pos def, no good here because we want $u(-e; x^*) = -1$. It turns out that only a small change is needed. This new rep called $T_0(a)$ is obtained on the same space by the rep $T_0(a)$ multiplied by the phase of the quantity $w(a, \phi)$, see 7.4. Hence, 7.5 will be the new rep: In general, it turns out that in order to get operators in this new rep from the integer rep, you use a modified sandwich rule, as in 7.7 for the differential generators, and as in 7.8 for Q .

7b. reps of the class $C_{\frac{1}{2}}^q$. Now define your basis functions as f_m' shown in 7.9. You can then show that your new Λ_x on these f_m' give you the desired Lie algebra stuff, except again you have to choose the phases in a certain way, viz. 7.10. Again, you use $\omega = -i\delta$, so $q = \frac{1}{4} + \delta^2$ and you again get $q > \frac{1}{4}$. This is the same range covered in the integral case. Finally, they note that again you can rephase your basis states to get g_m' , and again there will be two kinds of matrix elements u_{mn} and v_{mn} . Section closes with a comment on these reps for the covering group. Basically there is a new continuous label called h (I see this as E_0) which specifies the arbitrary real starting point for the spectrum of H_0 , which is still discreet.

8a. the kernel $L(b, \psi)$ We cannot just carry over previous stuff because in order to make it work you need a complex, not just imaginary (maybe real), but then your operators are no longer ~~hermitian~~ unitary. This will be cured simply by doctoring up the scalar product. Hopefully, we shall find a nonlocal scalar product of the form 8.1 that will work. Notice that this scalar product carries a rep label!

It is important to understand that, although we are about to change the scalar product, this only has an influence on the matrix elements. The basis functions will be the same as in the nonexceptional interval (up to norm) because the generators are the same. The rep is the same also, of course. The only place that the scalar product is felt is when you go to compute matrix elements. So this entire section is just a search for a scalar product relative to which ~~our~~ our already established rep is unitary when q lies in $(0, \frac{1}{4})$. The solution is given in equation 8.4, but the constant is not yet fixed. But in 8.7 the kernel is

stated again with a constant such that the unit function on the unit circle has unit norm. Next, B shows that if a function on the unit circle has finite norm via the old scalar product, then the norm is also finite under this new scalar product. This proof is lengthy.

8c. the Hilbert Space $\mathcal{H}$. Here is just remarked that you can define new basis functions that are normalized by using g_m instead of f_m . The notation $\mathcal{H}$ just reminds us to use this new "scalar product". Only the type v_{mn} matrix elements will be used. However, these are not involved in the expansion theorem to come, so not important to me.

Bargmann Section 9 The Discrete Classes

9a. the multiplier rep T_λ . Look back for a moment at page 597. There we used for the space M (the fiber) little complex tw-vectors $\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$. We allowed these to be acted upon by the elementary rep of $SL(2, \mathbb{C})$. Then we took the top variable z_1 as our "add-on variable" and the ratio $z = z_2/z_1$ as the variable of the space M^* . We found that if you shuffle the points in the space M^* by the usual fractional linear transformation, then, that corresponded exactly to shuffling the little two-spinors by the usual 2×2 $SL(2, \mathbb{C})$ matrices. Also, the variable z_1 ends up just getting multiplied by a certain quantity, which we identify with the multiplier. Thus, we cleanly defined a multiplier representation on the space M^* (notice that M^* is the space of z , which in turn is just the inside of the unit circle for $SU(1,1)$.) Obviously, you can use the multiplier raised to any power and you still have a multiplier rep. Going back to the space of $\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$, we know that the rep appears as a bare shift rep on a certain class of function of these two variables. From page 583 we see that that class of functions is precisely: functions of the form $(z_1)^h f(z_2/z_1)$, where h is the power you chose. This certainly looks like Vilenkin's equation (5), page 109. (Of course there he is talking about $SU(2)$ piece of $SL(2, \mathbb{C})$.)

So, to get on with it, Bargmann is going to use $M^* = \text{unit disc}$ to deal with the discrete UIR's. It seems that Vilenkin got away with doing everything on the circle itself, but remember that at his "integral points" the reps he got were actually reducible and extra work was needed to get the discrete UIR's. He is very sloppy on this business, I thought. Bargmann uses the circle for the continuous UIR's, but he uses the disc for the discreets.

So, having said all this, we accept M^* = disc in z -plane and we have the multiplier rep 9.4. The next problem is to define a scalar product over this disc such that the rep operators are unitary. This is simply achieved by identifying an invariant density via the method described back in section 1, leading us to equation 1.61. The appropriate density is stated in 9.8, normed so that the norm of a unit function over the disc has norm = 1. Therefore our scalar product on M^* is given by 9.9 (it is local). So far, l can be any positive number.

9b. the Hilbert space H_l . It is functions of z on the disc with scalar product given by 9.9. As such, our rep operators are unitary for any pos real l . It is noted that the scalar product of two powers of z is very simple, see 9.11. It turns out that $h_m(z)$ given by 9.14, (just normalized powers z^m) form a complete and orthonormal basis in M^* .

9c. reps of the class D_k^+ . Now given the rep 9.4 we can always find the infinitesimal representation. I.e, we can always go from finite to infinitesimal. For example, stick some OPS into 9.4, differentiate both sides wrt the parameter, then set parameter = zero. This will give you a diff op for the generator. But we have no need to do that because we have already recorded the answer back in equation 4.26 (ie, this gives the infinitesimal multipliers and you just add these to the "bare shift" generators Λ^0 which are determined by the $\lambda(y)$ factors given in equation 4.24.). So, boom: we have generators as in 9.16.

Notice that the lowering operator is d/dz . Thus, if you have a basis which is powers of z , eventually H_- will kill off your series. Thus, what we are doing really is appropriate for the discrete series. In fact the basis is powers of z , as given now in 9.18 in terms of 9.14. These functions do just the right thing under action of $H_0 = \Lambda_0$. Notice, however, that in 9.18 you must have $XXXXX (m-k)$ be an integer, otherwise the thing will not truncate under H_- action. This is the thing that quantizes l to be an integer. Note that 9.17 for Q is just right. Matrix elements will be of v_{mn} type on these g_m basis.

9d. Reps of the class D_k^- . Basically you take your fractional xform and complex conjugate it. Then you CC all your generators. This negates the spectrum of Λ_0 , which is just what you want. However, we do not CC the scalar product (this could introduce a minus sign) so the basis is chosen a little differently, see 9.22.

10a. differential relations. All along, Bargmann has made lots of use of these "a-diff" generators, the χ . Contrast these generators with the "x-diff" generators, the Λ_x . For example, all those generators used by Mukunda were of the x-diff type. In equation 4.8, Barg gives some examples of the a-diff's. Wybourne only mentions these guys in exercise 4.5 page 27. So I still have a lot to learn along these lines. Now, a very important result we learned way back concerning these two kinds of diff generators was a certain relation between the two: namely, that $\chi[T(a)f(x)] = \Lambda_x[T(a)f(x)]$. Of course χ only acts on an a-variable, whereas Λ_x only acts on an x-variable.

This little idea is applied many times here. Look first at 10.3 which defines the matrix element. It is a function of a, hence we can apply χ as in 10.4. Application of χ to 10.1 gives 10.5+, for example. Similarly for 10.5. Ultimately, we can show that the matrix elements of any ops must solve an infinite set of coupled differential equations, as in 10.7. Notice that the quantities called λ_{mn} are completely known, given some label χ , ie, some particular generator. The parameter is then called t. Now, because these λ_{mn} only "couple" indices not more than one apart, it turns out that you can solve this infinite set of equations for small t. That solution is shown in 10.8, all fine. However, we want to know our matrix elements for large t as well!

10b. form of the matrix elements. We now go directly to the $\mu\eta$ group param, so that you can "do" the ends at once, and the entire problem boils down to working out the elements of the OPS whose generator is χ_2 . By the way, this χ_2 should go with L_2 of the lie algebra shown in 2.22. This it turns out is a y-boost. So we now have to work out these OPS elements $V_{mn}(\xi)$. For small ξ we can use 10.11. Notice that the functions $\Theta_{mn}(q)$ are defined here. But again, we want large ξ results.

10c. discussion of the functions $V_{mn}(\xi)$. First we change from ξ to y. So now we are after the $W_{mn}(y)$. We now apply equation 10.5. The [a-diff operator for Q in these params $\mu\eta$ was given earlier, it is called Ω . But we have simple μ and η dependence, so replace Ω with $-\Omega_{mn}$ when acting on this simple class of functions. Hence, 10.16. Recast one more time to get functions $Y_{mn}(y)$. Then the "Casimir differential equation" for Y is the hypergeometric equation. The solution of 10.18 is any multiplex of 10.20. How do we know it is the particular multiplex shown? OK, this follows directly from our small ξ forms found back in 10.11. Fine.

10d. remarks on hypergeometrics. The various translation formulas are stated, big deal, but this paper was written pre-Bateman so I guess that's why.

10e. reps of the continuous class. Now we just put the pieces together and out comes the net results for $v_{mn}(a)$. Notice that he has gone back to the original group variables α and β and as usual you have to watch the various regions of m and n . We now come to an important point: Bargmann derived these matrix elements without ever having to use an explicit scalar product, or an explicit form for one of the operators $T(a)$. Basically, the whole show was done via the diff op for the casimir Q as an a -diff generator combination. A few other moves were needed to get the normalization of the result, but the whole derivation is beautifully independent of the class of UIR !

10f. reps for the discrete series. The formulas 10.27 as just stated are valid for all the classes, so just replace $k = k - \frac{1}{2}$ and you're all set to fly with the discrete series. It turns out that these are polynomials (not unlike the P_1 that you catch in SWT stuff when you catch the nonsense poles moving your contour back to $\text{Re } \lambda = -\frac{1}{2}$). Notice fancy notation introduced in 10.29 for these matrix elements.

10g. remarks on the finite dim reps. In deriving the above matrix elements, it was crucial that we had unitary operators, eg, 10.6 made use of 10.1, and so on. To get these non-unitary finite dim reps, he uses the monomials method. I wonder if Bargmann realized that he could have gotten the $SU(1,1)$ matrix elements also from the monomials method. This is what most modern authors do, see page 191 of Wybourne. Ie, just use the space M and don't bother with going down to M^* and getting all involved with multipliers. I really should look more into this comparison.

Weylman: "The UIR's of the Lorentz group"

Annals of Math., Vol. 48, (1947), 568

Overall Comments: This is the one that everybody references. Now is the time to learn how to manipulate $SO(2,1)$ with confidence. I have dabbled too long, still don't know what the reps are, etc. The paper is 70 pages long, here is an outline:

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Introduction

$\mathcal{H}$ = Hilbert space of the unitary operators $U(a)$.

H = element of $\mathcal{H}$, a "generator", an "infinitesimal rep."

$\mathcal{M}$ = a manifold, a function space. $a \in \mathcal{A}$, $y = ax$ so $\beta: \mathcal{M} \rightarrow \mathcal{M}$.

The space $\mathcal{H}$ has, as elements, functions defined on some variable, say x . Here, $x \in$ the space called $\mathcal{M}$. So, $f(x) \in \mathcal{H}$, $x \in \mathcal{M}$ makes sense. Our operators $U(a)$ shall act on "vectors" in $\mathcal{H}$, so: $U(a)f(x) = f(\bar{a}'x) \times \mu(a, \bar{a}'x)$. I.e., our operators shuffle around these functions. The function $\mu(a, x)$ is called a multiplier. Notice that a can act directly on x inside $\mathcal{M}$, whereas $U(a)$ acts on $f(x)$.

Now comes a "result summary". $Q = (H_{01})^2 + (H_{20})^2 - (H_{12})^2 = q \cdot 1$ is a scalar; H_{12} has usual m spectrum, discreet, like J_3 . You can label all reps by (q, m) . The reps are listed:

C_q^0	$q \geq 0$	$m = 0, \pm 1, \dots$
$C_q^{\frac{1}{2}}$	$q > \frac{1}{4}$	$m = \pm \frac{1}{2}, \pm \frac{3}{2}, \dots$
D_k^+	$q = k(1-k)$	where $k = \frac{1}{2}, 1, \frac{3}{2}, \dots$; $m = k, k+1, \dots$
D_k^-	$q = k(1-k)$	where $k = \frac{1}{2}, 1, \frac{3}{2}, \dots$; $m = -k, -k-1, \dots$

For the continuous classes, $\mathcal{M}$ = the unit circle. So, a group element will be represented as a "projective transformation" of $\mathcal{M} \rightarrow \mathcal{M}$. $\mathcal{H} = \{ \text{square integrable fctns. on } \mathcal{M} \}$. Similarly, the two discrete classes are "realized" as "conformal x-forms" on $\mathcal{M}$ = open unit circle. More, of course, later.

Let $\{f_m\}$ be a set of basis vectors (i.e., functions) in $\mathcal{H}$, e.g., f_m are eigenfunctions of H_{12} . Then we can certainly speak

The matrix elements in $\mathcal{K}$: $(f_m, U(a)f_n) \equiv u_{mn}(a)$. Here, we can think of either a function space inner product $\int dx f_m(x) \dots$, or, the abstract matrix element of an arbitrary group element in some basis $\{f_m\}$. In any class continuous, u_{mn} is analytic in a and in q . Later we shall find what these u_{mn} look like for various "large a " limits. These asymptotic limits shall have some bearing on whether an "orthogonality relation" is possible, for a particular rep.

All the above results were for $\mathcal{G}_3 (\mathcal{L}_3)$. We shall also have results for the 6 parameter group $\mathcal{G}_4 (\mathcal{L}_4) \equiv SO(3,1)$. There are two Casimirs Q and R . There are reps C_q^0 and $C_{k,r}$, but no discrete reps now. $M_2 =$ unit sphere. An exceptional interval is now $0 < q < 1$. The basis can be f_{jm} , as obvious, so $(f_{jm}, U(a)f_{jn}) =$ matrix. Again, asymp. limits are taken. As with $\mathcal{G}_3$, in $\mathcal{G}_4$ certain classes of reps have trouble with square integrability.

Part II of this work must be published elsewhere. It is to concentrate on $\mathcal{G}_4$, to generalize all our $\mathcal{G}_3$ results.

1. Onlie groups + multipliers

This section is much of the "Theory" of this paper, much of what follows will be application of this Theory.

1a. introduction. $\mathcal{G} =$ a lie group. $a =$ an element, described by n real parameters $a^1, \dots, a^n$. $(ab)^i \equiv$ parameters of the new element $(ab) = \phi^i(a, b) =$ functions of parameters of a, b . Also, $(a^{-1})^i = \Theta^i(a^i)$.

Now, what is $\partial(ab)^i / \partial a^k$? Even this trivial thing I have

...er worked through, have to see a reference. The reason is that such facts are not mentioned in modern works! Perhaps we can identify:

$$\hat{X}_s^i(a) = V_s^i(a) = \left(\frac{\partial \phi^i(a, b)}{\partial b^s} \right) \Big|_{b=0}$$

Actually, consider:

$$\left. \frac{\partial (ab)^i}{\partial b^k} \right|_{b=0} = \hat{X}_k^i(a) = V_k^i(a) \quad \checkmark$$

Similarly:

$$X_k^i(b) = \left. \frac{\partial (ab)^i}{\partial a^k} \right|_{a=0}$$

So, we know how to identify X and $\hat{X}$ as derivatives of $\phi(a, b)$ w.r.t. the identity of the derived parameter. Not at all clear where (1.1) comes from. I find it strange that such things do not appear in Wybourne or Hammermesh.

Reading on, it seems that $\vec{X} = \vec{K} \cdot \vec{X}$ is like $\vec{O} \cdot \vec{J}$, so $\{X^i\}$ are a set of generators which you make out of the $n \times n$ matrix of functions X_s^i . I.e., these X^i are the "inf. x forms".

$$X^i = \sum_s X_s^i K^s$$

The "algebra" is: $X'' = [X, X']$.

We read this as: $[X_s X_k] = C_{sk}^i X_i$

So, we now depart this subsection totally mystified.

1. Group invariant integration.

"The right invariant group integration" $= \nu_h(a) \Rightarrow \int_G da^i \nu_h(a) f(a) = \int_G da^i \nu_h(a) f(ab)$

● They show that $V_1(a) = \text{const} / \det |X^i_j|$.

Similarly, $\exists V_2(a) = \text{const} / \det \hat{X}^i_j$. If $c^i_{ir} = 0 \forall r$, then it turns out that both $V_1(a)$, $V_2(a)$ are equal, and $\equiv V(a)$. So, from here on, $\int da^i f(a) V(a)$ is just written $\int da f(a)$.

Id. Transformations. As yet, no mention of X^i coordinates + all that. Most texts introduce these right at the start. Here, we set up $x^1, x^2, \dots, x^m \equiv \vec{x}$ as a point in space M . When we say $y = ax$, we think $a: M \rightarrow M$. Not necessarily a linear x form:

$$(x')^i = (ax)^i = Z^i(a^1, \dots, a^m; x^1, \dots, x^m)$$

● Now: $\frac{\partial Z^i(a, x)}{\partial a^k} = \lambda^i_k(ax) \psi^i_k(a)$, it turns out.

Now what about starting with $f(x)$ + generating new function $g(x)$ via $g_a(x) = f(\vec{a}'x)$. I.e., g eval at x , is f eval at some weird point $\vec{a}'x$, where $\vec{a}'$ is a general x form. Well, then $g_a(x) = f(\vec{a}'x) = T(a)f(x)$. We can think of $T(a)$ as giving us a new function $g, \in \mathcal{F}$. Since $T(a)T(b) = T(ab)$, these operators represent G , called "standard rep."

As usual, my attitude remains: "huh?"

I.e. linear reps of G . Another big "huh?"

● multipliers. Now we come to something significant. Consider:

$$g(a, x) = T(a)f(x) = \mu(a, \vec{a}'x) \cdot f(\vec{a}'x)$$

Here, $a \in G$, $x \in M$. From function $f(x)$, we generate a new function in this way. Question: when is $T(a)$, so defined, a group rep? If $\mu = 1$, it is the standard rep, $T^0(a)$. Answer: if the function $\mu(a, x)$ satisfies:

$$\mu(e, x) = 1 \quad \mu(ab, x) = \mu(a, bx) \cdot \mu(b, x)$$

Then yes, T is a rep.

So, $\mu(a, x)$ is called a multiplier. The $T(a)$ are called "the multiplier rep". For each $a \in G$, $T(a)$ is some transformation on $f(x)$. Hence, $a \sim P \Rightarrow$ a group of transformations.

Properties of multipliers? They form a group under mult.

I.e., $\mu_1(a, x), \mu_2(a, x) = \text{multiplier} \Rightarrow \mu_1 \times \mu_2 = \mu_3 = \text{also a mult.}$

Here is a little theorem: if $T(a)f(x) = f(x)$ for all a ,

Then the multiplier must be $\mu(a, x) = f(ax)/f(x)$!

Discussion of infinitesimal multipliers follows, skipped. Pause, this stuff is clear very far afield ~ and may actually never be relevant. Let us now jump to the next section + see how it goes there.

2. The inf. xforms of the Lorentz & Rotation Groups.

$w_j^i =$ the L.T. which takes x to y , i.e., a linear x-form:
 $= [U(a)]^i_j$

$L, l_{ij}, b_{ij}, \dots$ lots of new symbols.

$Q, R, L_n, K, \dots$ etc etc.

Nothing relevant yet, but next section looks promising.

● The spinor groups. Let x^0, x^1, x^2, x^3 be your spacetime point. Crunch into 2×2 matrix $X = \begin{bmatrix} x^0 + x^3 & \\ & \end{bmatrix}$, herm. Now, transform via $W = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ of $\det 1$. So, this is a proper L.T. There are 6 parameters in here, hence $\mathcal{L}_4$.

Note that W and $-W$ are the same in $\mathcal{L}_4$, but differ in $\mathcal{L}_4$. So $\mathcal{L}_4 \sim \mathcal{L}_4/\mathbb{Z}_2$.

Now, let $\mathcal{L}_3 =$ S.G. of $\mathcal{L}_4$ where x^3 held constant. Hence, $W = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$, $\det = 1$. This is $SU(1,1)$. Fine, all is well so far. If we like, we can associate $f(w): \mathbb{C} \rightarrow \mathbb{C}$ where $z' = (\delta z + \gamma)/(\beta z + \alpha)$. A group of linear xforms is thus isomorphic to the group $\mathcal{L}_4$. Now, for $W \in \mathcal{L}_3$, this transform maps interior of unit circle in $\mathbb{C}$ onto itself, conformally.

● A "null vector" is one on the light cone, so

$$x^\mu = (r, r \sin \theta \sin \phi, r \sin \theta \cos \phi, r \cos \theta)$$

From this, compute the x^i . Thence, compute S' and S'' . Now, set $\theta = \pi/2$ to make $x^3 = 0$. Then, (3.1) reduces to (3.4)[†].

We are still looking at $Y = W X W^*$, so W has $\alpha \beta$.

Notice that only r, ϕ are needed to describe the special $x^3 = 0$ null vectn. Y will also be such a null vectn, (r', ϕ') . The claim is that $Y = W X W^*$ can be written in this special case as:

$$r' = r |\alpha + \beta e^{i\phi}|^2$$

$$e^{i\phi'} = e^{i\phi} \frac{\alpha}{\beta} \frac{1 + (\beta/\alpha) e^{-i\phi}}{1 + (\beta/\alpha) e^{i\phi}}$$

● But who cares about null vectns with $z^3 = 0$? He does not say.

● Detailed study of $\mathcal{H}_3$. Define $\gamma \equiv \beta/\alpha = \gamma_1 + i\gamma_2$, so that two of our three parameters are now γ_1, γ_2 . The third will be ω defined by (4.2). Now, let $a'' = a' a$, where $a = (\gamma, \omega)$ and $a' = (\gamma', \omega')$. What are the parameters of a'' ? Do by brute force to find:

$$\gamma'' = \frac{\gamma + \gamma' e^{-2i\omega}}{1 + \bar{\gamma}\gamma' e^{-2i\omega}}$$

$$\omega'' = \omega + \omega' + \frac{1}{2i} \log [\dots]$$

Next, using these params, we get the "differential operator realization" where $X_0 = -\frac{1}{2} \partial/\partial \omega$, etc. Also, we can compute the X'_s , hence the $\Delta = |X'_s(a)| = \frac{1}{8}(1 - \gamma\bar{\gamma})^2$. Hence, since measures are equal, we find:

$$da = \frac{d\gamma_1 d\gamma_2 d\omega}{2\pi^2 (1 - \gamma\bar{\gamma})^2}$$

Next, we come at last to the famous μ, ν, ρ parameters.

$$\alpha = \cosh e^{-i(\mu+\nu)}$$

$$\beta = \sinh e^{i(\nu-\mu)}$$

$$a = e^{2\mu X_0} e^{2\nu X_2} e^{2\rho X_0}$$

How do we get γ, ω ?

$$\gamma = \tanh e^{2i\omega}$$

$$\omega = -(\mu+\nu), \text{ mod } 2\pi$$

Again, from the workings of the general theory which I still do not get, we can construct differential realization for X_0, X_1, X_2 in terms of these new parameters, so in the end, I'll have to go back to (4.7) and beyond.

$$da = (2\pi)^{-2} \sinh^2 d\mu d\nu d\rho$$

now skip to section 10 to look at the explicit matrix elements.

$$V_{mn}(a) = (g_m, T(a) g_n) = \bar{e}^{-2im\eta} \bar{e}^{-2in\eta} V_{mn}(\xi)$$

Define: $y = \sinh^2 \xi$ $da = (2\pi)^2 dy d\eta d\phi$

$$V_{mn}(\xi) = W_{mn}(y) = y^{\frac{1}{2}(m-n)} (1+y)^{-\frac{1}{2}(m+n)} Y_{mn}(y)$$

$$Y_{mn}(y) = \Theta_{mn}(q) \cdot F(c_1, c_2, c_3; y)$$

$\uparrow$ some constants. $\uparrow$ $q = \text{casimir label}$

I now doubt that this paper has what I am really looking for. I wonder if the Russian book is back yet, Vilenkov...?

Andrews + '64
Gunson

Complex Angular Momenta and Many-Particle States. I. Properties of Local Representations of the Rotation Group

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(Received 20 March 1964)

From (14.1):

$$d_{j,m}^{m,m'}(z) = \langle jm | e^{+i\sigma J_2} | jm' \rangle$$

The properties of the "local representations" of the rotation group corresponding to complex angular momentum are further developed. Completeness and bi-orthogonality relations are derived and a reduction of products is carried out, giving a generalization of the Clebsch-Gordan reduction. The connection with the representation theory of the group $SL(2, R)$ is considered and a generalization of Regge's use of the Sommerfeld-Watson transform is made to the case where three momentum transfer variables occur in the description of scattering amplitudes.

1: INTRODUCTION

RECENT investigations¹⁻³ into the connection between stable particles and Regge poles have suggested examination of the following problems: Given a certain set of particles all of which are Regge particles, i.e., they are represented by poles in scattering amplitudes, which lie on Regge trajectories, how can one define scattering amplitudes for these particles taking nonphysical values of the spin variable? Are these amplitudes uniquely defined, and how are the analytic and unitary properties related to those for physical scattering amplitudes?

In this series of papers, we develop first some of the basic techniques required in attacking these problems and use them to provide partial answers in certain important cases. The definition of an "off-spin-shell" amplitude is based here on the factorization of residues at Regge poles, in complete analogy with the corresponding definition of unstable particle amplitudes. (See Gunson's "Analyticity and Unitarity",⁴ hereafter referred to as A).

In one sense, these amplitudes may be regarded as "off-mass-shell" continuations, but differing from the more usual versions in that the spin is continued simultaneously with the mass, so as to follow a Regge trajectory. The other important difference is that of uniqueness. We expect the Regge particle amplitudes to be unique, being defined entirely in terms of on-mass-shell amplitudes without any extra external parameters (see Sec. 6.5 of A).

Before we can investigate these amplitudes in detail, we have to develop suitable mathematical

tools for handling the transforms of many-particle amplitudes to and from the complex angular momentum planes. The two main obstacles met are (i) the extension in full of the representations of the rotation group to complex angular momenta and (ii) the problem of "complex singularities" in the momentum transfer variables. Our solutions to these problems are given in the first and second papers of this series. In the third paper, we treat unitarity and analyticity.

For a physical interpretation of states involving particles with nonphysical spins, we refer the reader to papers by Regge⁵ and Ford and Wheeler.⁶ In many senses, these states have the same status as unstable or virtual particle states. Both are somewhat ephemeral and wavefunctions representing them cannot be constructed without violating the usual boundary conditions. In the case of particles with nonphysical spins, the relevant condition is that of single (or double) valuedness over the unit sphere. However, it often occurs that the physical effect of the violation of the boundary conditions is small. The linking of mass and spin variables on a Regge trajectory demonstrates the complementary nature of these two forms of ephemeral state. In the boson case, the transformation properties under rotations of the states with well-defined complex spin j in the rest frame are those of the $Y_{l,m}(\theta, \varphi)$. These and related properties have been studied from a mathematical standpoint in terms of "local representations" of the rotation group in three dimensions in an appendix to A. In particular, definitions were given for functions $D_{j,m}^{m,m'}(R)$, of rotations R , which gave a natural extension of the well-known unitary representations to complex values of j —

¹ S. C. Frautschi, M. Gell-Mann, and F. Zachariasen, Phys. Rev. 126, 2204 (1962).

² M. Gell-Mann, and M. L. Goldberger, Phys. Rev. Letters 9, 275, (1962).

³ M. Gell-Mann, et al., Phys. Rev. 133, B145, B161 (1964).

⁴ J. Gunson "Unitarity and On-mass-shell Analyticity as a Basis for S-Matrix Theories," Birmingham Preprint (1962) (to be published).

⁵ T. Regge, Nuovo Cimento 18, 947, (1960).

⁶ K. W. Ford, and J. A. Wheeler, Ann. Phys. (N. Y.) 7, 272 (1959).

JMP 6: 821-44 (1965)¹³⁹¹ = part I
845-51 = part II
852-8 = part III.

Note: $d_j^{m,m'}(\theta=\pi \Rightarrow z=-1)$ does not exist because $z=-1 \Rightarrow \xi=+1$, a singular point; unless, reduces to a polynomial. Thus, in general you cannot relate $d(z)$ to $d(-z)$ as in MS.

Unitarity and On-Mass-Shell Analyticity as a Basis for S-Matrix Theories. III

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(Received 20 January 1964; final manuscript 30 June 1964)

The singularity structure of many-particle scattering amplitudes in momentum transfer variables is investigated in terms of the many-particle unitarity conditions. Cauchy-type kernels for analytic functions defined on complex rotation group in three dimensions are constructed and related to a theory of local representations of the rotation group, corresponding to complex angular momenta.

IN Parts I and II of this series, we have studied singularities of scattering amplitudes whose location depends only on the energy invariants for one particular channel, in particular the total energy W . By generalizing the treatment given in a previous paper,¹ we investigate below the implications that the many-particle unitarity relations hold for the structure of singularities whose locations depend on one or more of the momentum transfer variables for the chosen channel. For this purpose, we have found it convenient to construct a special Cauchy-type reproducing kernel for holomorphic functions defined on the manifold of the group $O(3, C)$, regarded as a complex space of three (complex) dimensions. In the original derivation of this kernel, there appeared incidentally certain formulas which had, at least locally, some of the structural properties of representations of the rotation group in three dimensions. These formulas have subsequently proved to be of importance in the complex angular momentum analysis of many-particle amplitudes and so a detailed mathematical treatment of certain basic properties is given in Sec. 2. Applications to scattering theory will appear in later papers.

1. CONTINUATION IN MOMENTUM TRANSFER VARIABLES

As an example we consider a three-particle unitarity term

$$[\mathbb{A}] = i [\mathbb{Z}][\mathbb{X}] \quad (1)$$

occurring in (3.36) of I, whose partial wave projection appears in (3.38). The total amplitude

$$A(W^2, \dots, \rho) = \sum_{i=0}^{\infty} \sum_{m=-i}^i \sum_{m'=-i}^i A_i^{m,m'}(W^2, \dots) \mathcal{D}_i^{m,m'}(\rho) \quad (2)$$

is, according to our postulates, analytic in ρ on the

group manifold of the complex orthogonal group $O(3, C)$ (or rather the component of this group which is connected with the identity element) except for certain sets of singularities. These must include at least the ones which intersect the physical region itself, i.e., the submanifold which forms the real rotation group. These comprise the single-particle pole and anomalous threshold terms introduced in Sec. (3.3) of I and further branch points at higher energies.

In Sec. 2, there is constructed an invariant kernel function associated with a special type of closed hypercontour M in the manifold of $O(3, C)$ and possessing the reproducing property

$$f(R) = \oint_M dS K(R, S) f(S) \quad (3)$$

for any function f holomorphic in a neighborhood of the physical region. In order to apply this theorem, we must reintroduce the $(-i\epsilon)$ addenda to the masses of the intermediate particles as in Sec. 2 of I, in order to produce a neighborhood of the physical region in which the functions are holomorphic. Let us now write the amplitudes occurring in the unitarity integral¹(1) in this form, i.e.,

$$A(R_2) = \int_G dR_1 \int_G dR_2 \delta(R_2, R_2 R_1) Z(R_2) X(R_1) \quad (4)$$

after omitting irrelevant variables and integrations over the intermediate phase space. G is the manifold of $SO(3)$. Using (3) on Z alone, we get

$$A(R_2) = \int_{R_1} dR_1 \oint_M dS_2 K(R_2 R_1^{-1}, S_2) Z(S_2) X(R_1). \quad (5)$$

Applying to both Z and X , we get

$$A(R_2) = \oint_M dS_1 \oint_M dS_2 H(R_2, S_1, S_2) Z(S_2) X(S_1), \quad (6)$$

where

$$H(R_2, S_2,$$

These for formula for many-particle that if $Z(S_i)$ sets $\{\beta_i\}$ are elements we find the points by Lardner Ref. (1) in contrast with the not intersect to a multiphysical region; real rotation.

The natural terminated part (7). An integral obtained by for the K kernel. We have an expression of contours of function and R_2 , we obtain on the three $R_2 R_1$, square $z_1^2 - 1 - 2z_1$ on $z_1 = \pm$ singularity a two-particle

For many the invariant between the the initial and the Euler invariants, the singularities again selection of integral

The impossibility of momentum space

¹R. Lardner (1962).

²S. Mandel-

¹J. Gunson and J. G. Taylor, Phys. Rev. 121, 343 (1961).

Andrews-Garsen : cuts in j .

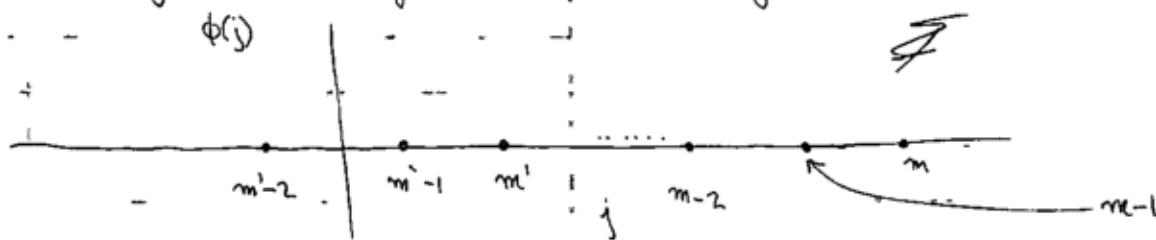
① Examine the normalizer.

$$\phi_j^{m, m'} = \left[\begin{matrix} m & -m' \\ -m & m' \end{matrix} \right]_j = \left[\frac{(j+m)!(j-m')!}{(j-m')!(j+m)!} \right]^{\frac{1}{2}}$$

In region A, $m \geq m'$ and $m \geq -m'$, ie, $|m'| \leq m$ and $m \geq 0$.
Since $m \geq m'$, let $m = m' + h$, $h = \text{pos. int.}$

$$\begin{aligned} &= \left[\frac{(j+m'+h)(j-m')!}{(j-m'-h)(j+m')!} \right]^{\frac{1}{2}} = \left[\frac{(j+m'+h)(j+m'+h-1)\dots(j+m'+1)}{(j-m'-h)(j-m'-h-1)\dots(j-m'-h+1)} \right]^{\frac{1}{2}} \\ &= [j+m)(j+m-1)\dots(j+m'+1) \times (j-m')(j-m'-1)\dots(j-m'+1)]^{\frac{1}{2}} \end{aligned}$$

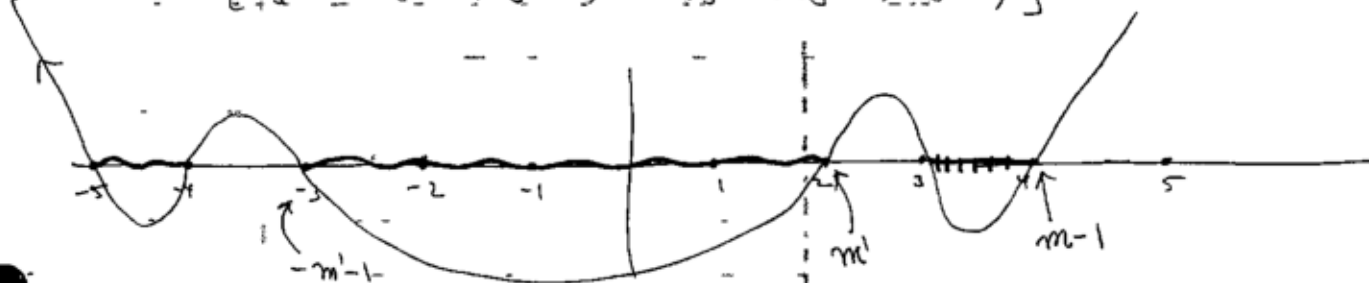
② Let m, m' be fixed in region A. Plot this function:



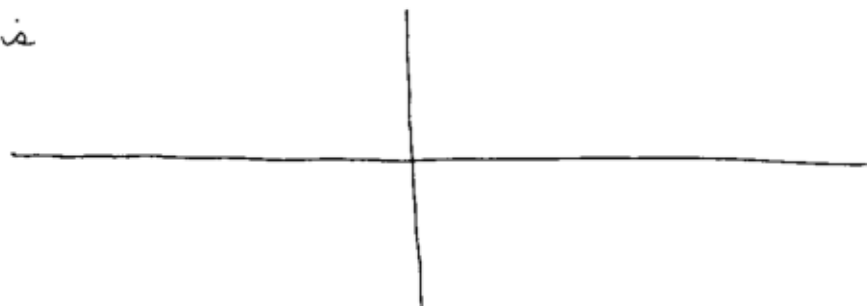
The first (.) to vanish as j comes down is that with largest negative idem: So this is $m-1$.

Example: $m = 5$ $m' = 2$

$$[(j+5)(j+4)(j+3) \times (j-2)(j-3)(j-4)]^{\frac{1}{2}}$$



The general case is

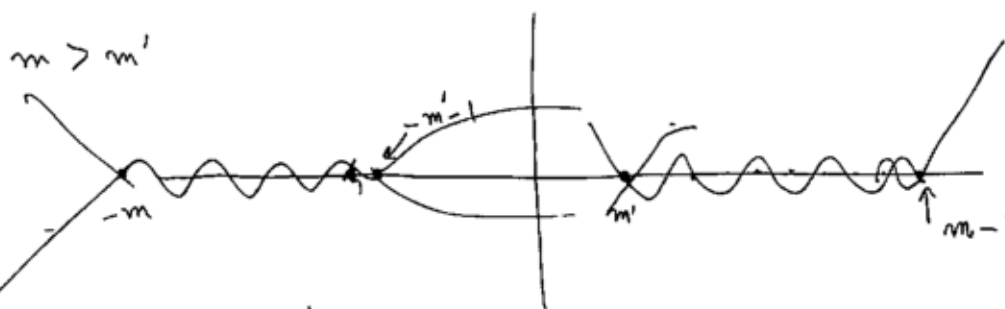


Try $m=8, m'=2$

$$(j+8)(j+7) \dots (j+3)$$

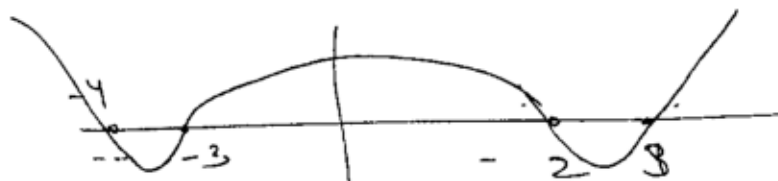
Generally: $(j+m)(\dots)(j+m'+1) \times (j-m') \dots (j-[m-1])$
 $= (j-[m]) \dots (j-[m'+1]) \times (j-[m']) \dots (j-[m-1])$

where $m > m'$



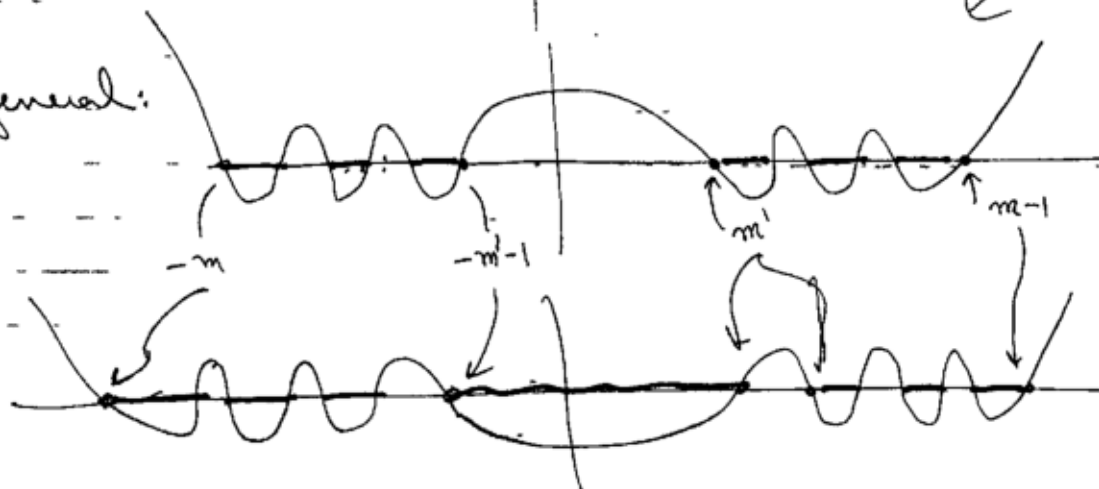
Example: $m=4, m'=2$

$$(j+4)(j+3) \times (j-2)(j-3)$$



see Fig 2.

In general:



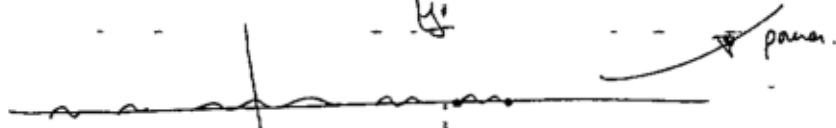
$m-m' = \text{even}$

$m'-m = \text{odd}$

③ Phase of normalizer:

$$\Phi_j^{m,m'} = \begin{bmatrix} m & -m' \\ -m & m' \end{bmatrix}^{\frac{1}{2}} \xrightarrow{j \rightarrow +\infty} [(j^{m-m'})^2]^{\frac{1}{2}} = j^{m-m'}$$

--- i.e., this rises as a + power, $m \geq m'$.



$$\Phi_{-j-1}^{m,m'} \rightarrow (-j-1)^{m-m'} \approx (-j)^{m-m'}$$

If $a(j)^2 = -a(-j-1)^2$, then $a(j) = \underset{\pm 1}{\text{phase}} \times a(-j-1)$.
What is this phase:

$$a(j) = \text{phase} \cdot a(-j-1)$$

Take $j \rightarrow +\infty$ on both sides: $(j)^{m-m'} = \text{ph} \times (-j)^{m-m'}$

$$\Rightarrow \text{phase} = (-1)^{m-m'}$$

Final conclusions for Weds Jan 28 on the AG d and e function.

1. Here are the objects which I have calculated:

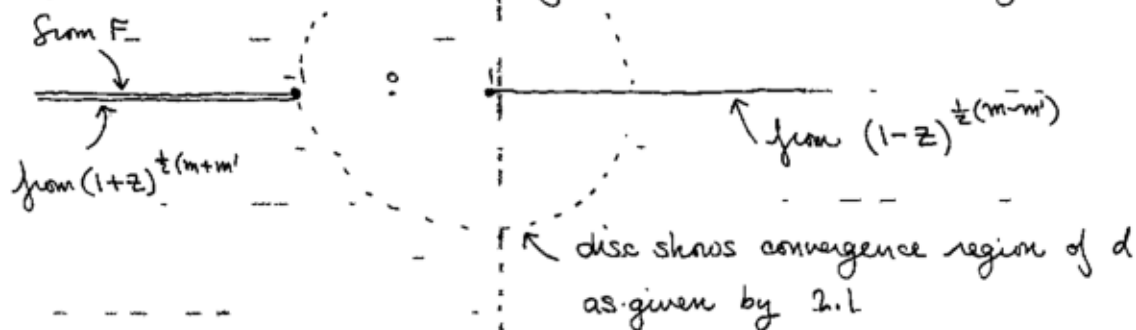
- a) expansion of d into two terms well behaved at $z = \infty$; see page 4
- b) explicit expression for $d_{m, -m}^j(-z)$ made to look very close to regular d.
For comparison of these two, see page 4 backside.
- c) on pages 5 and 6 I showed that the definition (3.1) does indeed give formulas (3.2) and (3.3) when the "extra cut" in d runs to the left. This cut is shown in page 4 backside, second box.
- d) page 7 shows disc(d) on left related to disc(e) on right, with other stuff.
- e) page 8 shows disc(d) on left calculated directly. On back of page 8 I write this disc in terms of $d(z)$ and $d(-z)$. This is my first disagreement with (14.3) except when $m-m' = \text{even integer}$.
- f) on page 9 we have disc(e) on the right. See backside for alternate result
- g) when results (d) and (f) are combined, we get result on page 10 for disc-d on the left. This still disagrees with (14.3) in the same way.
- h) on page 11 I consider an alternative cutting scheme to rescue the results so that (14.3) is OK. My rescue messes up other formulas, however, and conflicts with the scheme AG explicitly choose.

2. The reason I am worried about the result (14.3) is that AG use it directly to obtain (14.4), the $O(2,1)$ completeness relation. If (14.3) is really wrong, as I suspect it is, then so is (14.4) unless there are other compensating errors.

i) derived (3.5)

Properties of AG d-functions.

- ① For $(m, n) \in A$, d is shown in (2.1). They select cuts in this way:



- ② If $-j+m, j+m, m-m'$ etc. were all real, d could be real, but depends on gammas.

- ③ It is easy to calculate the disc of d across the right cut.

Let

$$f(z) = (1-z)^\alpha \quad \text{with } \alpha = \frac{1}{2}(m-m').$$

Let $z-1 = |z-1| e^{i\theta}$. Choose $1-z = |z-1| e^{i\theta} e^{-i\pi}$

so that

$$f(z) = |z-1|^\alpha e^{i(\theta-\pi)\alpha}$$

Then $f(z)$ is real when $\theta = \pi$, i.e. when $-1 < z < 1$. This is AG's choice.

Thus:

$$\text{disc } f(z) = |z-1|^\alpha \{ e^{-i\pi\alpha} - e^{+i\pi\alpha} \} = -2i \sin \pi\alpha |z-1|^\alpha$$

$$= -2i \sin \pi\alpha e^{+i\pi\alpha} f(z+i\epsilon)$$

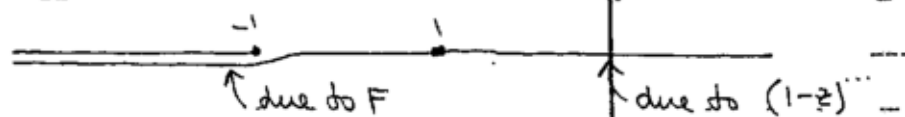
④ Therefore:

$$\text{disc } d_{mn}^j(z) = -2i \sin \frac{\pi}{2}(m-m') e^{+i\frac{\pi}{2}(m-m')} d_{mn}^j(z+i\epsilon)$$

- ⑤ Cannot get disc on the left because that cut lies outside convergence region of F . You must continue via the Bateman formulas.

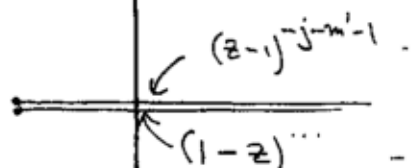
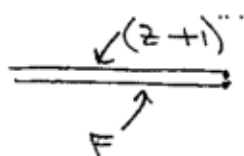
- ⑥ I suspect this is most easily done via the e -functions.

- ⑦ In fact, for the ϵ function, F converges outside exact same circle.



But picture cannot be right. The F has bp at $\frac{z}{1-z} = 1$, i.e. $z = -1$. And $(z-1)^{m-1}$ at 1.

So true picture is



- ⑧ Perhaps what he means is this:



Then $e_j^{m,n}(z) = \text{real positive for } z > 1$.

He could also mean

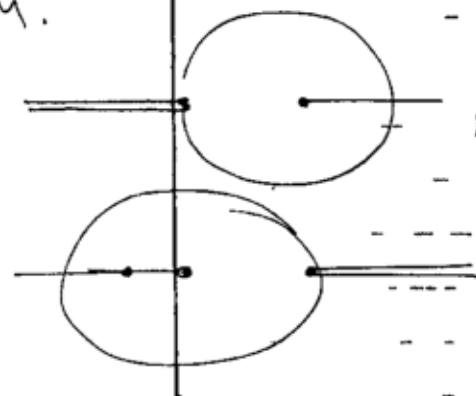


But this would be dumb.

- ⑨ Assume $\mathcal{E}$ is correct. Then $\text{disc}_{z>0} \mathcal{E}_z = 0$. I assume $\mathcal{E}^{j-1}$ is out same way.
Ergo: $\text{disc } d(z) = 0$ on right.

- ⑩ Idea. Given that $d(z) =$
we can continue to entire z -plane.

What is $d'(z) \equiv d(-z) =$



Analytic continuation of d-function.

① Define: $a = -j+m$ then: $c-a = j-m'+1$
 $b = j+m+1$ $c-b = -j-m'$
 $c = 1+m-m'$ $c-a-b = -m'-m$

then: $m-m' = c-1$ $j+m'+1 = b=c+1$ $b-a = 2j+1$
 $m+m' = a+b=c$ $j-m+1 = 1-a$ $b+a = 2m+1$

So: $d_j^{m,m'}(z) = \left[\frac{\Gamma(b)\Gamma(c-a)}{\Gamma(b-c+1)\Gamma(1-a)} \right]^{\frac{1}{2}} (1-\xi)^{\frac{1}{2}(a+b-c)} (\xi)^{\frac{1}{2}(c-1)} \underbrace{F(a,b,c;\xi)}_{u_1} / \Gamma(c)$

where $\xi = \left(\frac{1-z}{2}\right)$ and $1-\xi = \left(\frac{1+z}{2}\right)$ and $z = 1-2\xi$.

② Now want to use

$$u_1 = \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(c-a)\Gamma(b)} u_3 + \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(c-b)\Gamma(a)} u_4$$

where

$$u_3 = (-\xi)^{-a} F(a, a+1-c, a+1-b; \xi')$$

$$u_4 = (-\xi)^{-b} F(b, b+1-c, b+1-a; \xi')$$

③ Therefore:

$$d_j^{m,m'}(z) = \left\{ \Gamma(b-c+1)\Gamma(1-a)\Gamma(c-a)\Gamma(b) \right\}^{\frac{1}{2}} (1-\xi)^{\frac{1}{2}(a+b-c)} (\xi)^{\frac{1}{2}(c-1)} (-\xi)^{-a} F(a, a+1-c, a+1-b; \xi')$$

$$+ \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(c-b)\Gamma(a)} \left\{ \frac{\Gamma(b)\Gamma(c-a)}{\Gamma(b-c+1)\Gamma(1-a)} \right\}^{\frac{1}{2}} \text{ stop.}$$

④ Therefore:

$$d_j^{m,m'}(z) = \frac{\Gamma(b-a)}{\Gamma(c-a)\Gamma(b)} \left\{ \right\}^{\frac{1}{2}} (1-\xi)^{(a+b-c)/2} (\xi)^{(c-1)/2} (-\xi)^{-a} F(a, a+1-c, a+1-b; \xi')$$

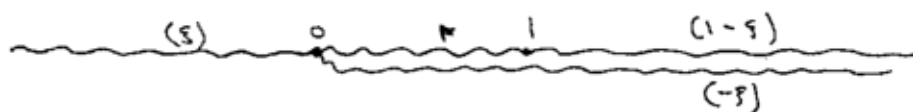
$$+ \frac{\Gamma(a-b)}{\Gamma(c-b)\Gamma(a)} \left\{ \right\}^{\frac{1}{2}} (1-\xi)^{(a+b-c)/2} (\xi)^{(c-1)/2} (-\xi)^{-b} F(b, b+1-c, b+1-a; \xi')$$

How do you draw the ends?

$$\textcircled{5} d_j^{m,m'}(z) = \left\{ \frac{\Gamma(b)\Gamma(c-a)}{\Gamma(b-c+1)\Gamma(1-a)} \right\}^{\frac{1}{2}} (1-\xi)^{(a+b-c)/2} (\xi)^{(c-1)/2} \dots$$

$$\times \left[\frac{\Gamma(b-a)}{\Gamma(c-a)\Gamma(b)} (-\xi)^{-a} F(a, a+1-c, a+1-b; \xi^{-1}) + (a \leftrightarrow b) \right].$$

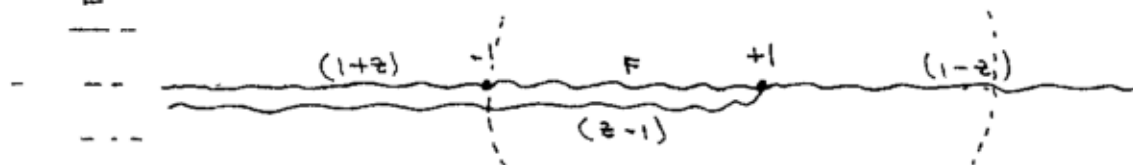
Each term has the same cut structure, namely



⑥ In the z -plane we have

$$d_j^{m,m'}(z) = \left\{ \right\}^{\frac{1}{2}} \left(\frac{1+z}{2} \right)^{(a+b-c)/2} \left(\frac{1-z}{2} \right)^{(c-1)/2}$$

$$\times \left[\frac{\Gamma}{\Gamma \Gamma} \left(\frac{z-1}{2} \right)^{-a} F(a, a+1-c, a+1-b; \frac{z}{1-z}) + (a \leftrightarrow b) \right]$$



④ Now, when $z > 1$, the cut disc comes entirely from $\left(\frac{1-z}{2} \right)$ which is common to both terms. We get

$$\left(\frac{1-z}{2} \right)^{(c-1)/2} \rightarrow -2i \sin \pi \left[\frac{c-1}{2} \right] \left| \frac{1-z}{2} \right|^{(c-1)/2}$$

When $z > 1$, $z = z + i\epsilon$, $\left(\frac{1-z}{2} \right) = ?$ $z-1 = |z-1| e^{i\theta}$

$$1-z = |z-1| e^{i(\theta-\pi)}$$

$$1-z^* = |z-1| e^{-i\pi}$$

so $\left(\frac{1-z}{2} \right)^a = \left(\left| \frac{1-z}{2} \right|^a \right) e^{-i\pi a}$

So $\left| \frac{1-z}{2} \right|^a = \left(\frac{1-z}{2} \right)^a e^{+i\pi a} \rightarrow -2i \sin \pi \left[\frac{c-1}{2} \right] e^{i\pi \left[\frac{c-1}{2} \right]} \left(\frac{1-z}{2} \right)^{(c-1)/2}$

This yields same result we got earlier for this disc.

⑤ The real problem is when $z < -1$ because there we have disc due to two separate cuts.

Compute disc $d(z)$ for $z < -1$.

① The first step has already been done - we have:

$$d_j^{mm'}(z) = \left\{ \frac{\Gamma(\alpha)}{\Gamma(\beta)} \right\}^{\frac{1}{2}} \left(\frac{1+z}{z} \right)^{\alpha+\beta-c/2} \left(\frac{1-z}{z} \right)^{(c-1)/2} \times$$

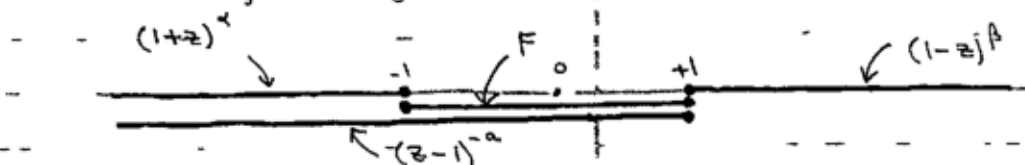
$$\left\{ \frac{\Gamma(b-a)}{\Gamma(c-a)\Gamma(b)} \left(\frac{z-1}{z} \right)^{-a} F(a, a+1-c, a+1-b, z/1-z) + (a \leftrightarrow b) \right\}$$

$$= \left\{ \right\}^{\frac{1}{2}} \cdot 2^q (1+z)^\alpha (1-z)^\beta \left\{ \frac{\Gamma(b-a)}{\Gamma(c-a)\Gamma(b)} (z-1)^{-a} F(a, a+1-c, a+1-b, z/1-z) + (a \leftrightarrow b) \right\}$$

where $q = (-a-b+c)/2 + (-c+1)/2 = (-a-b+1)/2 = -m$

$\alpha = (a+b-c)/2$ $\beta = (c-1)/2$

② The cut structure of each of the two terms is:



③ For $z < -1$, we are dealing with $g(z) = (1+z)^\alpha (z-1)^{-a}$ in first term.

④ We know that $\text{disc } (z-a)^\alpha = 2i \sin \pi \alpha (a-z)^\alpha$ when $z < a$.

⑤ So, why not just add discs: $\text{disc } g(z) = (1+z)^\alpha$

⑥ Easier to do directly: from math notes I get

$$\text{disc}_{z < -1} g(z) = 2i \sin \pi (\alpha-a) (-1-z)^\alpha (1+z)^{-a}$$

Therefore:

$$\text{disc}_{z < -1} d_j^{mm'}(z) = \left\{ \right\}^{\frac{1}{2}} 2^q \underbrace{(-1-z)^\alpha}_{2i \sin \pi (\alpha-a)} (1-z)^\beta$$

$$\times \left\{ \frac{\Gamma(b-a)}{\Gamma(c-a)\Gamma(b)} \left(\frac{1-z}{z} \right)^{-a} F(a, a+1-c, a+1-b, z/1-z) + (a \leftrightarrow b) \right\}$$

Compare $d(z)$ to $d(-z)$.

4

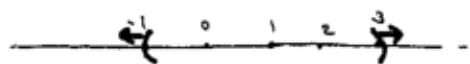
① First, write out our original result:

$$d_{j,m,m'}(z) = \left\{ \frac{\Gamma(j+m+1)\Gamma(j-m'+1)}{\Gamma(j-m+1)\Gamma(j+m'+1)} \right\}^{\frac{1}{2}} 2^{-m} (1+z)^{\frac{1}{2}(m+m')} (1-z)^{\frac{1}{2}(m-m')} \\ \times \left\{ \frac{\Gamma(2j+1)}{\Gamma(j-m'+1)\Gamma(j+m+1)} 2^{-j+m} (z-1)^{-j-m} F(-j+m, j+m'; -2j; \frac{z}{1-z}) + (j \leftrightarrow -j+1) \right\}$$

② Rewrite second term first. Also, cancel $2^{-m} \cdot 2^m$. Then: a ... ✓

$$d_{j,m,m'}(z) = \left\{ \frac{\Gamma(j+m+1)\Gamma(j-m'+1)}{\Gamma(j-m+1)\Gamma(j+m'+1)} \right\}^{\frac{1}{2}} \left(\frac{1+z}{2} \right)^{\frac{1}{2}(m+m')} \left(\frac{1-z}{2} \right)^{\frac{1}{2}(m-m')} \\ \times \left\{ \frac{\Gamma(-2j-1)}{\Gamma(-j-m)\Gamma(-j+m)} \left(\frac{z-1}{2} \right)^{-j-1-m} F(j+1+m, j+1+m'; 2j+2; \frac{z}{1-z}) + (j \leftrightarrow -j-1) \right\}$$

③ This formula is valid when $|z-1| > 2$,
in particular for $z > 3$ and $z < -1$.



④ Imagine that $x = \frac{z}{1-z}$. Then $\frac{z}{1+z} = x/(x-1)$ and $(1-x) = \left(\frac{z+1}{z-1} \right)$

$(1-x) = \left(\frac{z+1}{z} \right)^{-1} \cdot \left(\frac{z-1}{z} \right)^{-1}$. We can use a little shift then

to say:

$$\left[\begin{aligned} & F(j+1+m, j+1+m'; 2j+2; \frac{z}{1-z}) \\ &= \left(\frac{z+1}{z} \right)^{-(j+1+m)} \left(\frac{z-1}{z} \right)^{j+1+m} F(j+1+m, j+1-m'; 2j+2; \frac{z}{1+z}) \end{aligned} \right]$$

$\nwarrow$ new factor. $\nwarrow$ This exactly cancels extant factor.

Thus:

$$d_{j,m,m'}(z) = \left\{ \frac{\Gamma(j+m+1)\Gamma(j-m'+1)}{\Gamma(j-m+1)\Gamma(j+m'+1)} \right\}^{\frac{1}{2}} \left(\frac{1+z}{2} \right)^{\frac{1}{2}(m+m')} \left(\frac{1-z}{2} \right)^{\frac{1}{2}(m-m')} \\ \times \left\{ \frac{\Gamma(-2j-1)}{\Gamma(-j-m)\Gamma(-j+m)} \left(\frac{z+1}{z} \right)^{-(j+1+m)} F(j+1+m, j+1-m'; 2j+2; \frac{z}{1+z}) + (j \leftrightarrow -j-1) \right\}$$

⑤ From this result we find that:

$$d_j^{m,m'}(-z) = \left\{ \frac{\Gamma(j+m+1)\Gamma(j-m'+1)}{\Gamma(j-m+1)\Gamma(j+m'+1)} \right\}^{\frac{1}{2}} \left(\frac{1+z}{2} \right)^{\frac{1}{2}(m+m')} \left(\frac{1-z}{2} \right)^{\frac{1}{2}(m-m')} \\ \times \left\{ \frac{\Gamma(-2j-1)}{\Gamma(-j+m')\Gamma(-j+m)} \left(\frac{1-z}{2} \right)^{-(j+1+m)} F(j+1+m, j+1+m', 2j+2, \frac{z}{1-z}) + (j-j-1) \right\}$$

This is very similar to $d_j^{m,m'}(z)$ which I copy here:

$$d_j^{m,m'}(z) = \left\{ \frac{\Gamma(j+m+1)\Gamma(j-m'+1)}{\Gamma(j-m+1)\Gamma(j+m'+1)} \right\}^{\frac{1}{2}} \left(\frac{1+z}{2} \right)^{\frac{1}{2}(m+m')} \left(\frac{1-z}{2} \right)^{\frac{1}{2}(m-m')} \\ \times \left\{ \frac{\Gamma(-2j-1)}{\Gamma(-j-m')\Gamma(-j+m)} \left(\frac{z-1}{2} \right)^{-(j+1+m)} F(j+1+m, j+1+m', 2j+2, \frac{z}{1-z}) + (j-j-1) \right\}$$

⑥ The trick now is to add these in a clever way $\Rightarrow$ the second terms cancel. First trick is:

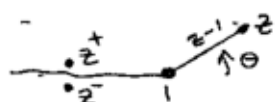
$$\frac{1}{\Gamma(-j+m)} = -\Gamma(j-m+1) \cdot \frac{\sin \pi(j-m)}{\pi}$$

$$\frac{1}{\Gamma(-j+m')} = -\Gamma(j-m'+1) \cdot \frac{\sin \pi(j-m')}{\pi}$$

These tricks alone cause:

$$d_j^{m,m'}(-z) = \left\{ \Gamma(j+m+1)\Gamma(j-m+1)\Gamma(j+m'+1)\Gamma(j-m'+1) \dots \text{STOP} \right\}$$

⑦ Look at this: $z-1 = |z-1| e^{i\theta} \Rightarrow z-1 = (1-z) e^{\pm i\pi}$ when z above & below.



$$z-1 = |z-1| e^{i\theta} \Rightarrow \begin{aligned} z^+-1 &= |z-1| e^{+i\pi} \\ z^--1 &= |z-1| e^{-i\pi} \end{aligned}$$

$$\text{So } z < 1 \Rightarrow (z^+-1) = (1-z) \cdot e^{+i\pi}$$

⑧ Look at

$$\left(\frac{1-z}{z}\right)^{-(j+1+m)} = (1-z)^{-j-1-m} \cdot \frac{z^+}{z^-}$$

Now claim: $1-z^\pm = |z-1| e^{\mp i\pi}$ so let's be thinking of $z > 4$, say. Then

$$1-z^\pm = (z-1) e^{\mp i\pi}$$

$$\Rightarrow \left(\frac{1-z}{z}\right)^{-j-1-m} = \left(\frac{z-1}{z}\right)^{-j-1-m} \times e^{\mp i\pi(-j-1-m)}$$

$$z = |z| \pm i\epsilon \\ z > 4.$$

⑨ Alternatively, — no, shake it ↑. Insert this into the $d_j^{m,m'}$ term. Now compute:

$$[e^{\mp i\pi(j-m)} d_j^{m,m'}(z) - d_j^{m,m'}(-z)]$$

Result:

$$\begin{aligned} & \left\{ \frac{\Gamma(j+m+1)\Gamma(j-m'+1)}{\Gamma(j-m+1)\Gamma(j+m'+1)} \right\}^{\frac{1}{2}} \left(\frac{1+z}{z}\right)^{\frac{1}{2}(m+m')} \left(\frac{1-z}{z}\right)^{\frac{1}{2}(m-m')} \\ & \times \left\{ \frac{\Gamma(-2j-1)}{\Gamma(-j-m')\Gamma(-j+m)} \left(\frac{z-1}{z}\right)^{-(j+1+m)} e^{\mp i\pi(j-m)} F(j+1+m, j+1+m', 2j+2, z/1-z) \right. \\ & + \frac{\Gamma(2j+1)}{\Gamma(j+1-m')\Gamma(j+1+m)} \left(\frac{z-1}{z}\right)^{j-m} e^{\mp i\pi(j-m)} F(-j+m, -j+m', -2j, z/1-z) \left. \right\} \\ & - \left\{ \frac{\Gamma(j+m+1)\Gamma(j+m'+1)}{\Gamma(j-m+1)\Gamma(j-m'+1)} \right\}^{\frac{1}{2}} \left(\frac{1+z}{z}\right)^{\frac{1}{2}(m+m')} \left(\frac{1-z}{z}\right)^{\frac{1}{2}(m-m')} \\ & \times \left\{ \frac{\Gamma(-2j-1)}{\Gamma(-j+m')\Gamma(-j+m)} \left(\frac{z-1}{z}\right)^{-(j+1+m)} e^{\mp i\pi(-j-1-m)} F(j+1+m, j+1+m', 2j+2, z/1-z) \right. \\ & + \frac{\Gamma(2j+1)}{\Gamma(j+1-m')\Gamma(j+1+m)} \left(\frac{z-1}{z}\right)^{j-m} e^{\mp i\pi(j-m)} F(-j+m, -j+m', -2j, z/1-z) \left. \right\} \end{aligned}$$

⑩ Now want to show that second terms cancel. This happens if:

$$\left\{ \frac{\Gamma(j-m'+1)}{\Gamma(j+m'+1)} \right\}^{\frac{1}{2}} \frac{1}{\Gamma(j+1-m')} = \left\{ \frac{\Gamma(j+m'+1)}{\Gamma(j-m'+1)} \right\}^{\frac{1}{2}} \frac{1}{\Gamma(j+1+m')} \quad \checkmark$$

⑩ Comment: One phase used by AG in (3.1) — and shown in $\square$ — is exactly what you need to make the two "second terms" cancel.

⑪ It now remains to add the two "first terms". Easy to do. The tricky stuff is:

$$\left\{ \frac{\Gamma(j+m+1)\Gamma(j-m'+1)}{\Gamma(j-m+1)\Gamma(j+m'+1)} \right\}^{\frac{1}{2}} \frac{1}{\Gamma(-j-m')\Gamma(-j+m)} e^{\mp i\pi(j-m)}$$

$$+ \left\{ \frac{\Gamma(j+m+1)\Gamma(j+m'+1)}{\Gamma(j-m+1)\Gamma(j-m'+1)} \right\}^{\frac{1}{2}} \frac{1}{\Gamma(-j+m')\Gamma(-j+m)} e^{\mp i\pi(j-1-m)}$$

All other terms are the same. From P. ⑥ we get a common factor of

$$\left\{ \Gamma(j+m+1)\Gamma(j-m+1)\Gamma(j-m'+1)\Gamma(j+m'+1) \right\}^{\frac{1}{2}} \cdot \frac{(-1)^{\frac{1}{2}} \sin \pi(j-m)}{\pi}$$

Residuals are:

$$e^{\mp i\pi(j-m)} (-1)^{\frac{1}{2}} \frac{\sin \pi(j+m')}{\pi}$$

$$e^{\mp i\pi(j-1-m)} (-1)^{\frac{1}{2}} \frac{\sin \pi(j-m')}{\pi}$$

$$= \frac{1}{\pi} \left[\sin \pi(j-m') e^{\mp i\pi(j-1-m)} - \sin \pi(j+m') e^{\mp i\pi(j-m)} \right]$$

$$= \frac{e^{\mp i\pi(-m)}}{\pi} \left[\sin \pi(j-m') e^{\mp i\pi(-j-1)} - \sin \pi(j+m') e^{\mp i\pi(j)} \right]$$

But $e^{\mp i\pi(-1)} = (-1)$: so

$$= -\frac{e^{\mp i\pi(-m)}}{\pi} \left[\sin \pi(j-m') e^{\pm i\pi j} + \sin \pi(j+m') e^{\mp i\pi j} \right]$$

Now mult by $e^{\mp i\pi m'}$ to get

$$= -e^{\mp i\pi(-m)} e^{\mp i\pi m'} \left[\sin \pi(j-m') e^{\pm i\pi(j+m')} + \sin \pi(j+m') e^{\mp i\pi(j-m')} \right]$$

$$= -e^{\pm i\pi(m+m')} [\text{mess}], \text{ poor choice Change in red.}$$

$$= -e^{\pm i\pi(m-m')} [\sin(2j)]$$

⑬ Next, here's what we have so far:

$$[] = \frac{\sin \pi(j-m)}{\pi} \cdot e^{\pm i\pi(m-m')} \cdot \frac{\sin \pi(2j)}{\pi} \cdot \left\{ \prod \prod \prod \right\}^{\frac{1}{2}} \cdot \Gamma(-2j-1) \\ \times \left(\frac{1+z}{2} \right)^{\frac{1}{2}(m+m')} \left(\frac{1-z}{2} \right)^{\frac{1}{2}(m-m')} \cdot \left(\frac{z-1}{2} \right)^{-j-1-m} \\ \times F(j+1+m, j+1+m', 2j+2; z/1-z).$$

⑭ Now use $\Gamma(2j+2) \Gamma(-2j-1) \frac{\sin \pi(2j+2)}{\pi} = 1$.

$$\text{Thus, } \Gamma(-2j-1) \frac{\sin \pi(2j)}{\pi} = \frac{1}{\Gamma(2j+2)} \checkmark$$

⑮ Now the z -factors: we have so far:

$$\left(\frac{1+z}{2} \right)^{\frac{1}{2}(m+m')} \left(\frac{1-z}{2} \right)^{\frac{1}{2}(m-m')} \left(\frac{z-1}{2} \right)^{-j-1-m} e^{\pm i\pi(m-m')}$$

We were thinking of $z = 4 \pm i\epsilon$, so phase of $(\frac{1-z}{2})$ factor is chosen in usual way. Rewrite as

$$\left(\frac{z+1}{2} \right)^{\frac{1}{2}(m+m')} \left(\frac{z-1}{2} \right)^{-j-1-m'} \left[\left(\frac{z-1}{2} \right)^{m'-m} \left(\frac{1-z}{2} \right)^{\frac{1}{2}(m-m')} e^{\pm i\pi(m-m')} \right] \textcircled{A}$$

Convert $\uparrow$ term to other form:

$$\text{Then: } \left(\frac{z-1}{2} \right)^{m'-m} = \left(\frac{1-z}{2} \right)^{m'-m} \cdot e^{\pm i\pi(m'-m)} \quad \leftarrow \text{cancels.}$$

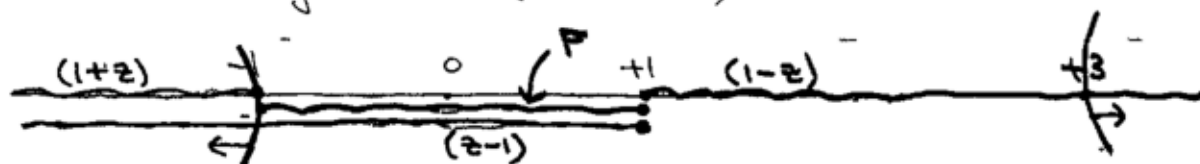
$$\text{Thus get } \left(\frac{1-z}{2} \right)^{\frac{1}{2}(m-m')} \left(\frac{1-z}{2} \right)^{m'-m} = \left(\frac{1-z}{2} \right)^{-\frac{1}{2}(m-m')}$$

Thus, from (3.1) I have at last obtained (3.2)

⑯ Thus, put all factors together to get:

$$\left\{ \begin{aligned} & \left[e^{\pm i\pi(j-m)} d_{j,m,m'}(z) - d_{j,m,-m'}(-z) \right] \\ & = \frac{2 \sin \pi(j-m)}{\pi} \cdot \left\{ (j-m)! (j+m)! (j-m')! (j+m')! \right\}^{\frac{1}{2}} \frac{1}{2} \\ & \times \left(\frac{z+1}{2} \right)^{\frac{1}{2}(m+m')} \left(\frac{z-1}{2} \right)^{-j-m'-1} \left(\frac{1-z}{2} \right)^{-\frac{1}{2}(m-m')} \frac{F(j+1+m, j+1+m', 2j+2; \frac{z}{1-z})}{\Gamma(2j+2)} \end{aligned} \right.$$

⑪ Where are the cuts of C ? (see (3.2))



This is the same picture as we got for each of the d -terms, of course.

⑫ Finally, let's get (3.3).

$$x = \frac{z}{1-z} \Rightarrow (1-x) = \left(\frac{z+1}{2}\right)^1 \left(\frac{z-1}{2}\right)^{-1}$$

$$\text{so we pick up: } c-a-b = 2j+2 - (j+m+1) - (j+m'+1) = -(m+m')$$

$$\left(\frac{z+1}{2}\right)^{-(m+m')} \left(\frac{z-1}{2}\right)^{+(m+m')}$$

$$\text{Thus, new } z\text{-factors are: } \left(\frac{z+1}{2}\right)^{-\frac{1}{2}(m+m')} \left(\frac{z-1}{2}\right)^{-j-\cancel{m}-1+m+\cancel{m}'}$$

$$\text{And: } c-a = j+1-m \checkmark$$

$$c-b = j+1-m' \checkmark$$

try to calculate disc.

$$\begin{aligned}
 & [e_j^{m,m'}(z+i\epsilon) - e_j^{\bar{m},\bar{m}'}(z-i\epsilon)] \frac{2}{\pi} \sin \pi(j-m) \\
 &= \left[e^{-i\pi(j-m)} d_j^{m,m'}(z+i\epsilon) - e^{+i\pi(j-m)} d_j^{m,m'}(z-i\epsilon) \right] \\
 &+ \left[d_j^{m,-m'}(-z+i\epsilon) - d_j^{m,-m'}(-z-i\epsilon) \right] \\
 &= \cos \pi(j-m) \left[d_j^{m,m'}(z+i\epsilon) - d_j^{m,m'}(z-i\epsilon) \right] \\
 &- i \sin \pi(j-m) \left[d_j^{m,m'}(z+i\epsilon) + d_j^{m,m'}(z-i\epsilon) \right] + \left[-d_j^{m,-m'}(-z+i\epsilon) - d_j^{m,-m'}(-z-i\epsilon) \right].
 \end{aligned}$$

Summarize: $\text{disc}_z e_j^{m,m'}(z) = \cos \times$

read "disc at $z = -z$."

$$\frac{2}{\pi} \sin \pi(j-m) \text{disc}_z e_j^{m,m'}(z) = -\cos \pi(j-m) \text{disc}_z d_j^{m,m'}(z) - i \sin \pi(j-m) \sum_z d_j^{m,m'}(z) + \text{disc}_{(-z)} d_j^{m,-m'}(z)$$

Again: $\frac{2}{\pi} \sin \cdot \text{disc}_z e_j^{m,m'} = \cos \text{disc}_z d_j^{m,m'} + \text{disc}_{-z} d_j^{m,-m'} - i \sin \sum_z d_j^{m,m'}$

- ② This relates:
- A) $\text{disc}(z, d_j^{m,m'})$
 - B) $\text{disc}(-z, d_j^{m,-m'})$
 - C) $\sum (z, d_j^{m,m'})$
 - D) $\text{disc}(z, e_j^{m,m'})$.

③ Easy to see that

$$\sum(z > 1, d_j^{m,m'}) = +2 \cos \frac{\pi}{2}(m-m') e^{+i\frac{\pi}{2}(m-m')} d_{mm'}^j(z+i\epsilon)$$

$$\text{disc}(z > 1, d_j^{m,m'}) = -2i \sin \frac{\pi}{2}(m-m') e^{+i\frac{\pi}{2}(m-m')} d_{mm'}^j(z+i\epsilon)$$

④ Thus, these two terms yield:

$$-2i \sin(m-m') \cdot \pi \cdot e^{+i\frac{\pi}{2}(m-m')} d_{mm'}^j(z+i\epsilon) \rightarrow$$

$$-2i \sin \frac{\pi}{2}(m-m') e^{+i\frac{\pi}{2}(m-m')} d_{mm'}^j(z+i\epsilon) \cos \pi(j-m) - 2i \cos \frac{\pi}{2}(m-m') e^{+i\frac{\pi}{2}(m-m')} d_{mm'}^j(z+i\epsilon) \sin \pi(j-m)$$

⑤. In order to use, we must learn how to get $d(-z)$. OK, give it a go: -----

$$d_{j,m,m'}^{m,m'}(z) = \left(\frac{1+z}{2}\right)^{\frac{1}{2}(m+m')} \left(\frac{1-z}{2}\right)^{\frac{1}{2}(m-m')} F(-j+m, j+m+1; 1+m-m'; \left(\frac{1-z}{2}\right))$$

Let $\xi = \left(\frac{1-z}{2}\right) \Rightarrow \left(\frac{1+z}{2}\right) = 1-\xi$. Now

$$d_{j,m,m'}^{m,m'}(\xi) = (1-\xi)^{\frac{1}{2}(m+m')} (\xi)^{\frac{1}{2}(m-m')} \underbrace{F(-j+m, j+m+1; 1+m-m'; \xi)}_{= u_1}$$

⑥. Now, $u_1 = A u_2 + B u_3$

$$= A \cdot F(-j+m, j+m+1; \quad ; 1-\xi)$$

$$\pm B \cdot (1-\xi)^{c-a-b} F(c-a, c-b, c+1-a-b; 1-\xi)$$

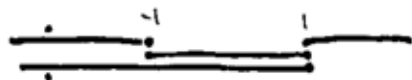
Specifically: ----- This is done.

Calculate disc d when $z < -1$.

① From our formula for d , we see that we are really computing

$$\text{disc}_z \left[\left(\frac{1+z}{2} \right)^{\frac{1}{2}(m+m')} \left(\frac{z-1}{2} \right)^{-j-1-m} \right]$$

since these are the cuts of interest



② Recast as:

a) remove the 2 's temp

b) let $a = 1$ $\alpha = -j-1-m$

$$\alpha + \beta = -j-1 + \frac{1}{2}(m'-m)$$

c) let $b = -1$ $\beta = \frac{1}{2}(m+m')$

Then use: $\text{disc}_z (z-a)^\alpha (z-b)^\beta = 2i \sin \pi(\alpha+\beta) (a-z)^\alpha (b-z)^\beta$

Answer to ① is:

$$= 2i \sin \pi \left[-j-1 + \frac{1}{2}(m'-m) \right] \left(\frac{-1-z}{2} \right)^{\frac{1}{2}(m+m')} \left(\frac{1-z}{2} \right)^{-j-1-m}$$

③ Now pull the $2i$ and one z factor outside. I get

$$\begin{aligned} \text{disc}_{z < -1} d_{j, m'}^{m, m'}(z) &= 2i \left\{ \frac{\Gamma(j+m+1) \Gamma(j-m'+1)}{\Gamma(j-m+1) \Gamma(j+m'+1)} \right\}^{\frac{1}{2}} \left(\frac{-1-z}{2} \right)^{\frac{1}{2}(m+m')} \left(\frac{1-z}{2} \right)^{\frac{1}{2}(m-m')} \\ &\times \left\{ \frac{\Gamma(-2j-1) \sin \pi \left[-j-1 + \frac{1}{2}(m'-m) \right]}{\Gamma(-j-m') \Gamma(-j+m)} \left(\frac{1-z}{2} \right)^{-j-1-m} F(j+1+m, j+1+m', 2j+2; \frac{1-z}{2}) \right. \\ &\quad \left. + (j \rightarrow -j-1) \right\} \end{aligned}$$

④ Now do the standard shift on HGF shown on page 4 — no.

⑤ Instead, look at forms on back of p 4. Since,

$$\begin{aligned} \text{disc}_{z > 1} \left[\left(\frac{1-z}{2} \right)^{\frac{1}{2}(m-m')} \left(\frac{1-z}{2} \right)^{-(j+1+m)} \right] &= \text{disc} \left(\frac{1-z}{2} \right)^{-j-1-\frac{1}{2}(m+m')} \\ &= 2i \sin \pi \left[-j-1-\frac{1}{2}(m+m') \right] \left(\frac{z-1}{2} \right)^{\frac{1}{2}(m-m')} \left(\frac{z-1}{2} \right)^{-j-1-m} \\ &\quad \rightarrow = e^{\pm i \frac{\pi}{2}(m-m')} \left(\frac{1-z}{2} \right)^{\frac{1}{2}(m-m')} \end{aligned}$$

⑥ What is

$$\sin \pi \left[-j-1-\frac{1}{2}(m+m') \right] e^{\pm i \frac{\pi}{2}(m-m')} = \sin \pi \left[j+\frac{1}{2}(m+m') \right] e^{\pm i \frac{\pi}{2}(m-m')}$$

① Try other method; use second term:

$$\text{disc} \cdot \left(\frac{1-z}{z}\right)^{\frac{1}{2}(m-m')} \left(\frac{1-z}{z}\right)^{j-m} \\ = \left(\frac{1-z}{z}\right)^{\frac{1}{2}(m-m')} 2i \sin \pi(j-m) \left(\frac{z-1}{z}\right)^{j-m} + \text{second term.}$$

This is just what we want for 14.3. But, it is not the full result, as I see it: There are 3 other terms.

disc

② We really have:

$$\text{disc} \cdot \left(\frac{1-z}{z}\right)^{\frac{1}{2}(m-m')} \left[\left(\frac{1-z}{z}\right)^{j-1-m} + \left(\frac{1-z}{z}\right)^{j-m} \right] \\ = \left(\frac{1-z}{z}\right)^{\frac{1}{2}(m-m')} \left[2i \sin \pi(j+m) \left(\frac{z-1}{z}\right)^{j-1-m} + 2i \sin \pi(j-m) \left(\frac{z-1}{z}\right)^{j-m} \right] \\ + 2i \sin \pi\left(\frac{1}{2}(m-m')\right) \left(\frac{z-1}{z}\right)^{\frac{1}{2}(m-m')} \left[\left(\frac{1-z}{z}\right)^{j-1-m} + \left(\frac{1-z}{z}\right)^{j-m} \right]$$

Use $\sin \pi(j+m) = \sin \pi(j-m) \times (-1)^{2m}$

$$\text{Then: } = \left(\frac{1-z}{z}\right)^{\frac{1}{2}(m-m')} 2i \sin \pi(j-m) \left[(-1)^{2m} \left(\frac{z-1}{z}\right)^{j-1-m} + \left(\frac{z-1}{z}\right)^{j-m} \right] \\ = \left(\frac{1-z}{z}\right)^{\frac{1}{2}(m-m')} \cdot e^{\pm \frac{\pi}{2}(m-m')}.$$

③ Rephase

Then if $m = \text{integer}$, then I get

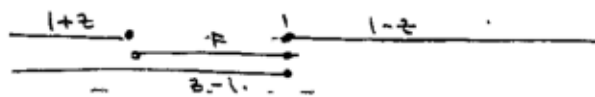
$$\text{disc } d_j^{m,m'}(-z) = 2i \sin \pi(j-m) d_j^{m,m'}(z) \\ + 2i \sin \frac{\pi}{2}(m-m') e^{\pm \frac{\pi}{2}(m-m')} d_j^{m,m'}(-z)$$

This disagrees with (14.3).

Calculate disc $\in$ when $z > 1$.

① Use (3.2). Then

$$\text{disc}_z \left[\left(\frac{1-z}{z} \right)^{-\frac{1}{2}(m-m')} \left(\frac{z-1}{z} \right)^{-j-m-1} \right]$$



Oh yes, very easy. Only one cut, so

$$\text{disc}_z \left(\frac{1-z}{z} \right)^{-\frac{1}{2}(m-m')} = -2i \sin \left[-\frac{1}{2}(m-m') \right] \left(\frac{z-1}{z} \right)^{-\frac{1}{2}(m-m')}$$

② But now change back to other form:

$$z^{\pm} - 1 = |z-1| e^{i\left(\frac{0}{2\pi}\right)} = |z-1| e^{+i\pi} e^{-i\pi}$$

$$z-1 = |z-1| e^{i0}$$

$$1-z = |z-1| e^{i(\pi-0)}$$

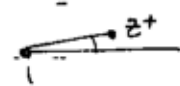
$$1-z^+ = |z-1| e^{-i\pi}$$

$$1-z^- = |z-1| e^{+i\pi}$$

$$\Rightarrow (1-z^{\pm}) = (z-1) e^{\mp i\pi}$$

Thus:

$$\boxed{\left(\frac{1-z^{\pm}}{z} \right)^{-\frac{1}{2}(m-m')} = \left(\frac{z-1}{z} \right)^{-\frac{1}{2}(m-m')} e^{\pm i\frac{\pi}{2}(m-m')}$$



③ Thus, replace:

$$\begin{aligned} \text{disc} &= -2i \sin \left[-\frac{1}{2}(m-m') \right] e^{\mp i\frac{\pi}{2}(m-m')} \left(\frac{1-z^{\pm}}{z} \right)^{-\frac{1}{2}(m-m')} \\ &= 2i \sin \frac{\pi}{2}(m-m') e^{\mp i\frac{\pi}{2}(m-m')} e. \end{aligned}$$

④ Conclusion:

$$\left[\text{disc}_{z>1} e_j^{m,m'}(z) = 2i \sin \frac{\pi}{2}(m-m') e^{\mp i\frac{\pi}{2}(m-m')} e_{-m,m'}^j(z^{\pm}) \right]$$

⑤ If $m-m' = \text{even} = 2N$, this disc vanishes due to sine. $N = m'_j$
 If $m-m' = \text{odd} = 2N+1$, then

$$\begin{aligned} \text{factn} &= 2i \sin \frac{\pi}{2}(2N+1) e^{\mp i\frac{\pi}{2}(2N+1)} = 2i (-1)^N e^{\mp i\pi} (-1)^N = 2i e^{\mp i\pi} \\ &= 2i (\mp i) = \pm 2 \end{aligned}$$

⑥ Thus, when $m-m' = \text{integer}$, I get

$$\text{disc}_{z>1} d_j^{m,m'}(z) = \begin{cases} 0 & \text{if } m-m' = \text{even} \\ \pm 2 d_j^{m,m'}(z \pm i\epsilon) & \text{if } m-m' = \text{odd} \end{cases}$$

The 0 is rather obvious, since $\left(\frac{1-z}{z}\right)^{-\frac{1}{2}(m-m')}$ gives the entire out.

⑦ Implication: $+ d_j^{m,m'}(z+i\epsilon) = - d_j^{m,m'}(z-i\epsilon)$

$$\Rightarrow d(z+i\epsilon) + d(z-i\epsilon) = 0$$

But, since d is cut everywhere, may not be RA.
If it is RA, then:

$$d_j^{m,m'}(z+i\epsilon) + d_j^{m,m'}(z-i\epsilon) = \dots \quad \text{so what.}$$

⑧ Now go to page 7:

$$\frac{z}{\pi} \sin \pi(j-m) \cdot \left[2 d_j^{m,m'}(z) \right] = -2i \sin(m-m')\pi e^{i\frac{\pi}{2}(m-m')} d_j^{m,m'}(z+i\epsilon) \\ + \text{disc}_{z>1} d_j^{m,m'}(-z)$$

Quick.

⑨ So get: $\frac{z}{\pi} \sin \pi(j-m) \left[2i \sin \frac{\pi}{2}(m-m') e^{i\frac{\pi}{2}(m-m')} d_j^{m,m'}(z \pm i\epsilon) \right]$
 $= \text{disc}_{z>1} d_j^{m,m'}(-z)$
 $- 2i d_j^{m,m'}(z+i\epsilon) e^{i\frac{\pi}{2}(m-m')} \left[\sin \frac{\pi}{2}(m-m') \cos \pi(j-m) + \cos \frac{\pi}{2}(m-m') \sin \pi(j-m) \right]$

Thus:

$$\text{disc}_{z>1} d_j^{m,m'}(-z) = + 2i \sin \pi(j-m) d_j^{m,m'}(z+i\epsilon)$$

but only if $m-m' = \text{even}$

Restate:

$$\sin \frac{\pi}{2}(m-m') = \begin{cases} 0 & \text{even} & m-m' = 2N \\ (-1)^N & \text{odd} & m-m' = 2N+1 \end{cases}$$

$$\cos \frac{\pi}{2}(m-m') = \begin{cases} (-1)^N & \text{even} & m-m' = 2N \\ 0 & \text{odd} & m-m' = 2N+1 \end{cases}$$

$$e^{\mp i \frac{\pi}{2}(m-m')} = \begin{cases} (-i)^N & \text{even} \\ (-1)^N (\mp i) & \text{odd} \end{cases}$$

So: $\frac{2}{\pi} \sin \pi(j-m) \cdot 2i \cdot \begin{bmatrix} 0 \\ (-i) \end{bmatrix} e_j^{m,m'}(z+i\epsilon)$

$$+ 2i d_{mm'}^j(z+i\epsilon) \cdot \begin{bmatrix} 1 \\ (+i) \end{bmatrix} (-1)^N \left\{ \cos \pi(j-m) \begin{bmatrix} 0 \\ (-i)^N \end{bmatrix} + \sin \pi(j-m) \begin{bmatrix} (-1)^N \\ 0 \end{bmatrix} \right\}$$

$$= \begin{bmatrix} 2i d_{mm'}^j(z+i\epsilon) \sin \pi(j-m) \\ -2 d_{mm'}^j(z+i\epsilon) \cos \pi(j-m) \end{bmatrix} + \frac{4}{\pi} \sin \pi(j-m) e_j^{m,m'}(z+i\epsilon) \begin{matrix} \leftarrow \text{even} \\ \leftarrow \text{odd} \end{matrix}$$

$$= \underset{z>1}{\text{disc}} d_{m,-m'}^j(-z)$$

Again, (14.3) is just no damn good!

Discussion of the d and e functions.

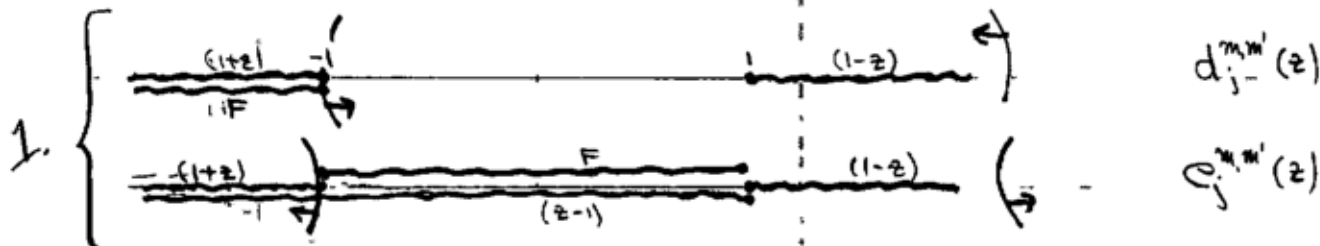
1. Question: when I showed that AG definition (3.1) gave result (3.2) for the e function, what assumptions about cuts were made?
2. Answer: examine the procedure. My first move was to expand $d(z)$ into two terms as shown on page 4 in orange box. Going from AG equation (2.1) defining d to this equation, no cut choices need be made. The expansion into two terms is independent of cut choices.

Next, on the back of page 4 I wrote two formulas: one is for $d_j^{m,m'}(-z)$ and the other is the usual $d_j^{m,m'}(+z)$. We are still cut independent.

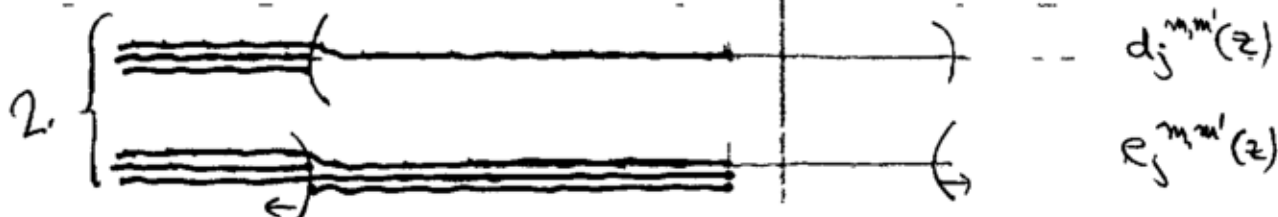
Finally, on top of page 5 we make a definite choice: the $(1-z)$ cut in $d(-z)$ runs from $z=1$ off to the right. This is the only assumption needed in order to have the two second-terms cancel. I believe that this corresponds to saying that the "extra cut" in $d(+z)$ runs from $z=1$ off the the left. I.e., see lower box on page 4 backside.

No new assumptions were made until page 6. But at line A, no new assumptions have yet been made. In fact, I think none are made for the rest of the say.

3. Conjecture: in formula (3.2) for e, you must choose the $(z-1)$ cut to run off to the left, but the other two cuts can be chosen however you decided to choose the cuts in d. Lets consider some of the possibilities:



This is the one that AG start out using. They like it because it makes $d(z)$ clean in the -1 to 1 region. Notice that P_j and Q_j never have the $(1+z)$ and $(1-z)$ cuts. However, I would prefer in some sense to have the $z=1$ cuts all run off the the left so $e(z)$ is clean when $z.GT. 1$. (These cuts are not really as fundamental as the other two cuts). Another choice is:



However, with this choice the rule (3.7) does not work because d has an internal discontinuity.

Try getting disc $d(-z)$ from (3.5) + (3.6).

$$① \quad \pi \cot \pi(j-m) d_j^{m,m'}(-z) = e_j^{m,m'}(-z) - e_{-j-1}^{-m,m'}(-z)$$

$$\text{from (3.4)} \quad | = -e^{\pm i\pi(j-m)} e_j^{m,m'}(z) + (-1)^{m-m'} e^{\pm i\pi(j-m)} e_{-j-1}^{m,m'}(z)$$

(3.5)

(3.6)

$$= -e^{\pm i\pi(j-m)} e_j^{m,m'}(z) + (-1)^{m-m'} e^{\mp i\pi(j-1-m)} e_{-j-1}^{m,m'}(z)$$

$$② \text{ Now use: } \text{disc}_{z>1} e_j^{m,m'}(z) = 2i \sin \frac{\pi}{2}(m-m') e^{\mp \frac{\pi}{2}(m-m')} e_j^{m,m'}(z \pm i\epsilon)$$

$\uparrow$
 $e_j^{m,m'}(z)$

This factor is very strange:

$$\begin{aligned} \pi \cot \pi(j-m) d_j^{m,m'}(-z-i\epsilon) &= -e^{+i\pi(j-m)} e_j^{m,m'}(z+i\epsilon) + (-1)^{m-m'} e^{+i\pi(j-1-m)} e_{-j-1}^{m,m'}(z+i\epsilon) \\ \pi \cot \pi(j-m) d_j^{m,m'}(-z+i\epsilon) &= -e^{-i\pi(j-m)} e_j^{m,m'}(z-i\epsilon) + (-1)^{m-m'} e^{-i\pi(j-1-m)} e_{-j-1}^{m,m'}(z-i\epsilon) \end{aligned}$$

$$\text{So: } \pi \cot \pi(j-m) [d_j^{m,m'}(-z+i\epsilon) - d_j^{m,m'}(-z-i\epsilon)]$$

$$= (-1)^{m-m'} \left[e^{-i\pi(j-1-m)} e_{-j-1}^{m,m'}(z-i\epsilon) - e^{+i\pi(j-1-m)} e_{-j-1}^{m,m'}(z+i\epsilon) \right]$$

$$= \left[e^{-i\pi(j-m)} e_j^{m,m'}(z-i\epsilon) - e^{+i\pi(j-m)} e_j^{m,m'}(z+i\epsilon) \right]$$

$$= (-1)^{m-m'} \left[\right]$$

Derive AG (3.5)

$$\begin{aligned} \textcircled{1} \quad e_j^{m,m'}(z) - e_{j-1}^{m,m'}(z) &= e_j^{m,m'}(z) - (-1)^{m-m'} e_{j-1}^{m,m'}(z) \\ &= \frac{\pi}{2 \sin \pi(j-m)} \left\{ e^{i\pi(j-m)} d_j^{m,m'}(z) - d_j^{m,m'}(-z) \right\} \\ &\quad - \frac{(-1)^{m-m'} \pi}{2 \sin \pi(-j-1-m)} \left\{ e^{i\pi(-j-1-m)} d_{j-1}^{m,m'}(z) - d_{j-1}^{m,m'}(-z) \right\} \end{aligned}$$

This uses (3.1), (3.4). Now use (2.2) to get:

$$\begin{aligned} &= \frac{\pi}{2 \sin \pi(j-m)} \left\{ e^{i\pi(j-m)} d_j^{m,m'}(z) - d_j^{m,m'}(-z) \right\} \\ &\quad - \frac{(-1)^{m-m'} \pi}{2 \sin \pi(-j-1-m)} \left\{ e^{i\pi(-j-1-m)} (-1)^{m-m'} d_j^{m,m'}(z) - (-1)^{m+m'} d_{j+1}^{m,m'}(-z) \right\} \end{aligned}$$

② Look at second pair. $(-1)^{m-m'} (-1)^{m+m'} = (-1)^{2m}$ so

$$\text{pair} = \frac{\pi}{2} d_j^{m,m'}(-z) \left[\frac{1}{\sin \pi(j-m)} + \frac{(-1)^{2m}}{\sin \pi(-j-1-m)} \right]$$

Now, $\sin(-j-1-m)\pi = \sin \pi(j+m) = \sin \pi(j-m+2m) = (-1)^{2m} \sin \pi(j-m)$.
Thus: pair = 0

③ Now look at first pair:

$$\begin{aligned} &= \frac{\pi}{2} d_j^{m,m'}(z) \left[\frac{e^{i\pi(j-m)}}{\sin \pi(j-m)} - (-1)^{2(m-m')} \frac{e^{i\pi(-j-1-m)}}{\sin \pi(-j-1-m)} \right] \\ &= \frac{\pi}{2} d_j^{m,m'}(z) \left[\cot \pi(j-m) - \cot \pi(-j-1-m) \right] \\ &= \frac{\pi}{2} d_j^{m,m'}(z) \left[\cot \pi(j-m) - \cot \pi(j-m) \right] \end{aligned}$$

Now $\cot \pi(-j-m) = -\cot \pi(j+m) = -\cot \pi(j-m+2m) = -\cot \pi(j-m)$
so $[] = 2 \cot \pi(j-m)$

$$= \pi d_j^{m,m'}(z) \cot \pi(j-m) = e_j^{m,m'}(z) - e_{j-1}^{m,m'}(z) \quad \checkmark$$

Understanding AG integral representation.

① Set $\alpha = \beta = 0$ so $X = \bar{X} = 1$.

②
$$\begin{bmatrix} \omega' \\ \tilde{\omega}' \end{bmatrix} = \begin{bmatrix} \cos \beta/2 & \sin \beta/2 \\ -\sin \beta/2 & \cos \beta/2 \end{bmatrix} \begin{bmatrix} \omega \\ \tilde{\omega} \end{bmatrix} = \begin{bmatrix} \cos \beta/2 \omega + \sin \beta/2 \tilde{\omega} \\ -\sin \beta/2 \omega + \cos \beta/2 \tilde{\omega} \end{bmatrix}$$

③ $|j, m\rangle \rightarrow N_m^j \omega^{j+m} \tilde{\omega}^{j-m}$

④
$$\begin{aligned} D(R)|j, m\rangle &\rightarrow N_m^j (\omega')^{j+m} (\tilde{\omega}')^{j-m} \\ &= N_m^j (C\omega + S\tilde{\omega})^{j+m} (-S\omega + C\tilde{\omega})^{j-m} \end{aligned}$$

⑤ Assume this can be expanded to read:

$$= \sum_{m'} D(R)_{mm'} \left[N_{m'}^j (\omega)^{j+m'} (\tilde{\omega})^{j-m'} \right]$$

⑥ Note wrong order here $\uparrow$; maybe designed to counter use of left shift versus right shift.

⑦ Set $N_m^j = 1$ even though AG do not. Easy to replace later.

Then:

$$(C\omega + S\tilde{\omega})^{j+m} (-S\omega + C\tilde{\omega})^{j-m}$$

$$= \sum_{m'} D(R)_{mm'} (\omega)^{j+m'} (\tilde{\omega})^{j-m'}$$

⑧ Now, use $(\omega)^{j+m} (\omega)^{j-m} = (\omega)^{2j}$ on left.

$$(\omega)^{j+m'} (\tilde{\omega})^{j-m'} = (\omega)^{2j} (\tilde{\omega}/\omega)^{j-m'} \quad \text{on right.}$$

Cancel $(\omega)^{2j}$ on both sides + define $s = \tilde{\omega}/\omega$. Then:

$$(C + Ss)^{j+m} (-S + Cs)^{j-m} = \sum_{m'} D_{mm'}^j \cdot (s)^{j-m'}$$

Mult both sides by $(s)^j$ to have:

$$(s)^j (C + Ss)^{j+m} (-S + Cs)^{j-m} = \sum_{m'} D_{mm'}^j \cdot (s)^{j-m'}$$

So far we have:

$$* \quad s^j (C + Ss)^{j+m} (-S + Cs)^{j-m} = \sum_{m'=0}^j D_{mm'}^j \cdot s^{-m'}$$

① Now mult both sides by s^{n-1} . Then execute $\oint \frac{1}{2\pi i} ds$ on both sides where $\oint$ is some closed contour around origin:

note: $\frac{1}{2\pi i} \oint ds \cdot s^{-m'+n-1} = \delta_{m',n}$

Thus, we find:

$$D_{mn}^j = \frac{1}{2\pi i} \oint ds \cdot s^{-j+n-1} (C + Ss)^{j+m} (-S + Cs)^{j-m}$$

Comments: so far, the contour in this integration is any contour looping the origin in the positive sense. For example, a circle of any radius.

Note that we simply assumed that an expansion existed at an early step.

② Singularities of the integrand are:

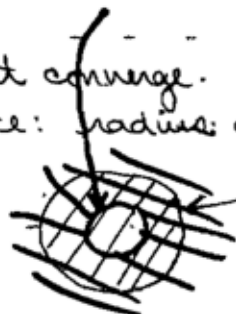
$$\begin{array}{ll} s = 0 & \text{exp} = -j+n-1 \\ s = -C/S & \text{exp} = j+m \\ s = -S/C & \text{exp} = j-m \\ s = \infty & \text{exp} = j+n-1 \end{array}$$

Now here is something strange. Look at equation (*) above. If the right side converges, then presumably it converges to the function shown on the left. But, the function on the left has cuts as shown here. In order to apply a contour integral to both sides of equation (*) and then interchange order, you must apply this contour so that the contour lies entirely within a portion of the s -plane in which the series converges uniformly to the function on the left. Therefore, any old contour around the origin will not do!!

③ Consider RHS of (*) as two power series.

$$\sum_{m'=0}^{\infty} D_{mm'}^j \left(\frac{1}{s}\right)^{m'} + \sum_{m''=1}^{\infty} D_{m-m''}^j (s)^{m''}$$

Both must converge. Each will have some radius of convergence: radius of second must be larger.



Then you will have annular convergence + analyticity.



12. So we have found that the only chance we have of doing what we did is to assume there is some convergence annulus. Given the function on the right, and assuming that tangent is smaller, we have to cut the function in this way:

$$e^{-s} (C + Ss)^{j+m} (-S + Cs)^{j-m} = \sum D \cdot s^{-m}$$

1. $A = 0$

2. $A = -\cot \beta/2$

3. $A = \tan \beta/2$

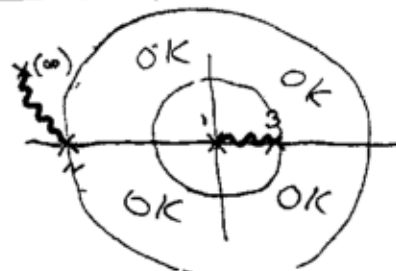
$A = \infty$

exp = $-j$

exp = $j+m$

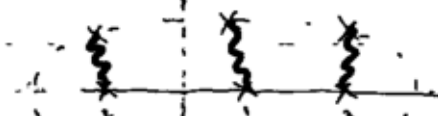
exp = $j-m$

exp = $-j, j+m, j-m$

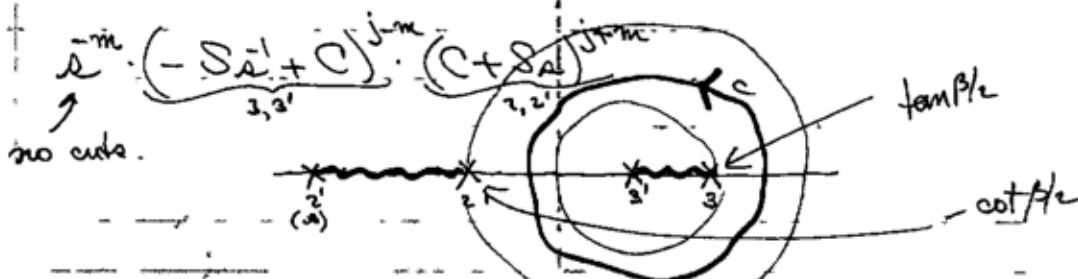


Notice that this method of cutting yields nonsense if you let $j-m = \text{integer}$ but $j \neq \text{integer}$. In such a case, you have to join the cuts differently.

13. Lets clean this cut business up a little. With each factor in the product of functions you need a cut. The way I have written the function above implies this cut structure:



This is definitely NO GOOD. Suppose we may assume that $m = \text{integer}$. Then much better way to write the function is this:



Must remember that the cut structure of a function depends on how you write the factors !!!!!

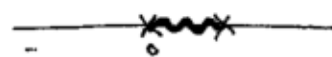
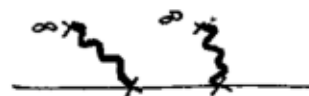
14. Therefore, when $0 < \beta < \pi$, we obtain the result of P 9, but the contour must lie inside the annulus, eg, see graph.

$$D_{mn}(\cos \beta) = \frac{1}{2\pi i} \oint_C ds \cdot e^{-m+n-1} (-Ss^{-1} + C)^{j-m} (C + Ss)^{j+m}$$

15. Further clarification. We are considering that there is a distinction in how you write these two functions:

$$A) s^{-j+n-1} (-s + C_s)^{j-m}$$

$$B) s^{-m+n-1} (-s^{-1} + C)^{j-m}$$



Look at form (A):

$$s = 0$$

$$s = \tan \beta/2$$

$$s = \infty$$

$$-j+n-1$$

$$j-m$$

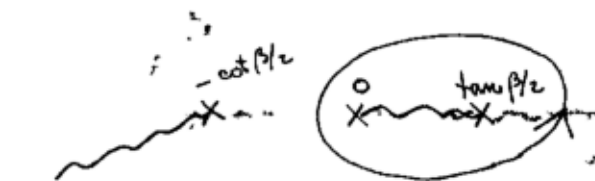
$$n-m-1$$

Suppose you really know that $n-m = \text{integer}$. Then looking at (A) we see that although we can pretend there are two-branch points at infinity, in fact in the total function they "cancel" so really the cut is just passing through infinity when you draw cuts as in top picture. But a cut passing through infinity can just be deformed to give the simpler picture shown in the lower sketch. (Credit Vilenkin for showing this.) (See his page 307 comment.)

Thus, with this in mind, we can go back to the form given in P9 for the matrix elements.

16. I have now studied carefully the technology needed to do this contour integral for the matrix elements. We have:

$$D_{mn}^j = \frac{1}{2\pi i} \oint ds s^{-j+n-1} (C + s)^{j+m} (-s + C_s)^{j-m}$$



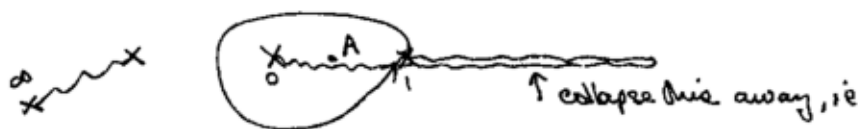
Originally this contour was in the convergence annulus mentioned above, but now we are free to deform the thing.

$$D_{mn}^j = (C)^{j+m} (-s)^{j-m} (2\pi i)^{-1} \oint ds s^{-j+n-1} (1 + \tan \beta/2 s)^{j+m} \times (1 - \cot \beta/2 s)^{j-m}$$

Let $t = \cot \beta/2 s$

$$D_{mn}^j = (\cos \beta/2)^{j+m} (-\sin \beta/2)^{j-m} (\tan \beta/2)^{j+n} \oint dt t^{-j+n-1} (1-t)^{j-m} (1 + \tan^2 \beta/2 t)^{j+m}$$

17. Now be careful. Notice that the cut for the first factor is chosen in non-standard fashion. Normally, $(t)^a$ means the cut runs off to the left not to the right as drawn. But I can just assume that we are drawing the cut strangely. The first factor is still positive at point A.



$$b = -j+n$$

$$c = 1+n-m$$

$$c-b = j+1-m$$

$$-a = j+m$$

18. I think I have a way to write the integral with less ambiguity. Use:

$$s = \tan \beta/2(t) \quad ds = \frac{1}{2} (-\tan \beta/2) dt$$

$$(s)^{-j+n-1} = (-\tan \beta/2)^{-j+n-1} (-t)^{-j+n-1}$$

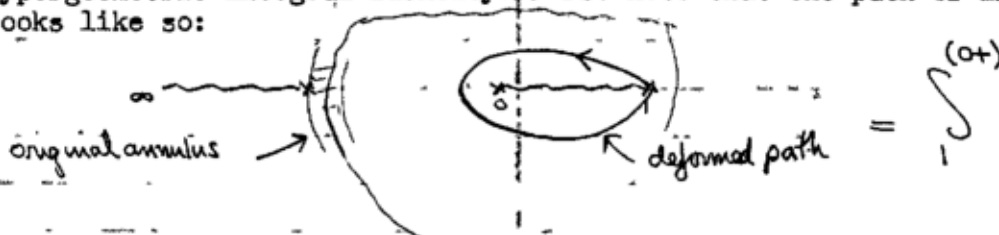
$$ds (s)^{-j+n-1} = -(-\tan \beta/2)^{-j+n} dt (-t)^{-j+n-1}$$

Then we can write the matrix element so that the hypergeometric integral assumes standard form. Also, the sine factor works out OKM:

$$(\cos \beta/2)^{j+m} (-\sin \beta/2)^{j-m} (-\tan \beta/2)^{-j+n} = (\cos \beta/2)^{2j+m-n} (-\sin \beta/2)^{n-m}$$

$$D_{mn}^j = [\cos \beta/2]^{2j+m-n} [-\sin \beta/2]^{n-m} \cdot \frac{1}{2\pi i} \oint dt (-t)^{-j+n-1} (1-t)^{j-m} (1+\tan^2 \beta/2 t)^{j+m}$$

19. Now we apply our second form of the hypergeometric integral identity. First note that the path of integration looks like so:



$$\text{Use: } \int_1^{\infty} dt (-t)^{b-1} (1-t)^{c-b-1} (1-tz)^{-a}$$

$$= -2\pi i \sin \pi b \Gamma(b) \Gamma(c-b) F(a, b; c; z) / \Gamma(c)$$

Identify: $b = -j+n$
 $c-b-1 = j-m \Rightarrow c = 1+n-m$
 $c-b = j+1-m \quad 1-b = j+1+n$
 $a = -j-m$

$$\sin \pi b \Gamma(b) = \frac{\pi}{\Gamma(1-b)}$$

$$\text{So: } \oint = -2\pi i \frac{\Gamma(b) \Gamma(c-b)}{\Gamma(1-b) \Gamma(c)} F(a, b; c; z)$$

So, the $2\pi i$'s cancel, so do the two minus signs. Also, $z = -\tan^2 \beta/2$. Thus:

$$D_{mn}^j = [\cos \beta/2]^{2j+m-n} [-\sin \beta/2]^{n-m} \cdot \frac{\Gamma(j+1-m)}{\Gamma(j+1-n)} \frac{F(-j-m, -j+n, 1+n-m, -\tan^2 \beta/2)}{\Gamma(1+n-m)}$$

Now use: $z = \cos \beta$

$$\sin^2 \beta/2 = \left[\frac{1-z}{2} \right] \quad (-\sin \beta/2)^{n-m} = (-1)^{n-m} \left(\frac{1-z}{2} \right)^{\frac{n-m}{2}}$$

$$(\cos \beta/2)^{2j+m-n} = \left(\frac{1+z}{2} \right)^{\frac{2j+m-n}{2}}$$

$$\tan^2 \beta/2 = (1-z)/(1+z)$$

Putting in all these relations, we arrive at the following result for D. This is not standard form, but I record it anyway for possible later use:

$$D_{mn}^j(z = \cos \beta) = \left(\frac{1+z}{2} \right)^{\frac{2j+m-n}{2}} (-1)^{n-m} \left(\frac{1-z}{2} \right)^{\frac{n-m}{2}}$$

$$\frac{1}{\Gamma(1+n-m)} \frac{\Gamma(j+1-m)}{\Gamma(j+1-n)} F(-j-m, -j+n; 1+n-m; \frac{z-1}{z+1})$$

20. Now we may apply the usual shift formulas to get standard form. Define various z 's and write both results:

$$\xi \equiv \frac{z-1}{z+1}, \quad \xi' \equiv \left(\frac{1-z}{2} \right) \Rightarrow \xi' = \xi/(\xi-1)$$

$$\begin{aligned} a &= -j-m, & c-b &= j+1-m, & 1-\xi &= \left(\frac{z+1}{2} \right)^{-1} \\ b &= -j+n, & c-a &= j+1+n. \\ c &= 1+n-m \end{aligned}$$

$$(1-\xi)^{-a} = \left(\frac{z+1}{2} \right)^a = \left(\frac{z+1}{2} \right)^{-j-m}$$

$$(1-\xi)^{-b} = \left(\frac{z+1}{2} \right)^b = \left(\frac{z+1}{2} \right)^{-j+n}$$

With these facts, we now can use Bateman 2.9 (3) and (4) to replace the F with:

$$\begin{aligned} F(\text{above}) &= \left(\frac{z+1}{2} \right)^{-j-m} F(-j-m, j+1-m, 1+n-m, \left(\frac{1-z}{2} \right)) \\ &= \left(\frac{z+1}{2} \right)^{-j+n} F(j+1+n, -j+n, 1+n-m, \left(\frac{1-z}{2} \right)) \end{aligned}$$

However, now each of these can be shifted according to Bateman 2.9 (2) to give:

$$\xi' \equiv \left(\frac{1-z}{2} \right), \quad (1-\xi') = \left(\frac{1+z}{2} \right)$$

$$c-a-b = \begin{cases} n+m & \text{top} \\ -(n+m) & \text{bottom} \end{cases}$$

$$c-a = \begin{cases} j+1+n \\ -j-m \end{cases}$$

$$c-b = \begin{cases} -j+n \\ j+1-m \end{cases}$$

Therefore, we actually have four different ways to replace the original F. They are:

$$F(\text{above}) = \left(\frac{z+1}{z}\right)^{-j-m} F(-j-m, j+1-m, 1+n-m, \frac{1}{z} - \frac{1}{z} z)$$

$$\left(\frac{z+1}{z}\right)^{-j+n} F(j+1+n, -j+n, 1+n-m, \frac{1}{z} - \frac{1}{z} z)$$

$$\text{top} \rightarrow \left(\frac{z+1}{z}\right)^{-j+n} F(j+1+n, -j+n, 1+n-m, \frac{1}{z} - \frac{1}{z} z)$$

$$\text{bot} \rightarrow \left(\frac{z+1}{z}\right)^{-j-m} F(-j-m, j+1-m, 1+n-m, \frac{1}{z} - \frac{1}{z} z)$$

Well that was stupid because I know there are only two ways to do this. Thus they are equal in pairs. Inserting these into D we get standard form results:

$$D_{mn}^j(\cos\beta) = \left(\frac{1+z}{z}\right)^{-\frac{(m+n)}{2}} \left(\frac{1-z}{z}\right)^{-\frac{(n-m)}{2}} (-1)^{n-m}$$

$$\frac{1}{\Gamma(1+n-m)} \frac{\Gamma(j+1-m)}{\Gamma(j+1-n)} F(-j-m, j+1-m, 1+n-m, \frac{1}{z} - \frac{1}{z} z)$$

$$D_{mn}^j(\cos\beta) = \left(\frac{1+z}{z}\right)^{-\frac{(m+n)}{2}} \left(\frac{1-z}{z}\right)^{-\frac{(n-m)}{2}} (-1)^{n-m}$$

$$\frac{1}{\Gamma(1+n-m)} \frac{\Gamma(j+1-m)}{\Gamma(j+1-n)} F(j+1+n, -j+n, 1+n-m, \frac{1}{z} - \frac{1}{z} z)$$

21. At this point, it appears that Ag correct for their reversal of indices in their equation (18). So in fact, the above equations should really be valid with D_{nm}^j on the left. Also, they drop the inessential phase. Thus, we get:

$$D_{nm}^j(\cos\beta) = \left(\frac{1+z}{z}\right)^{-\frac{(m+n)}{2}} \left(\frac{1-z}{z}\right)^{-\frac{(n-m)}{2}} \frac{(j-n)!}{(j-m)!} \frac{1}{(m-n)!} \\ \times F(j+1+m, -j+m, 1+m-n, \frac{1}{z} - \frac{1}{z} z)$$

This agrees with equation (22) of Ag II, except of course that I have been using a different normalizer. Obviously this form is only valid when $m-n =$ positive integer or 0 due to the usual business. I.e., you have to make surface corrections to get the other sign, do not bother with that here. Also, I have assumed that BETA is positive and in 0 to PI in order to have done what I did.

22. So, being careful to switch the indices, the upshot of all this work is basically to verify the integral representation of D:

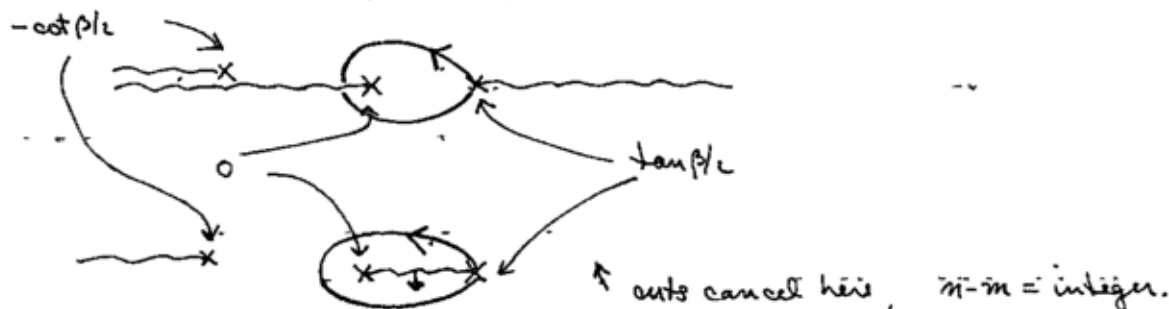
no! see
over

23. Actually, I may be wrong in saying that AG are correcting for their index reversal and dropping the $(-1)^{m-n}$. In fact, I know that these two operations exactly compensate, so the result comes out exactly!!

Thus, the exact integral representation is:

$\downarrow N_m = 1$

$$d_{mn}^{(j)}(\cos\beta) = \frac{1}{2\pi i} \oint ds (s)^{-j+n-1} \left(\cos\frac{\beta}{2} + \sin\frac{\beta}{2} s\right)^{j+m} \left(-\sin\frac{\beta}{2} + \cos\frac{\beta}{2} s\right)^{j-m}$$



24. Comments: notice that if $n-m$ is not an integer, you cannot even go through with the above 23-step procedure. The reason is that you cannot run your contour through an annulus because there is no convergence annulus. In other words, the double-infinite power-series we encountered does not have a convergence annulus so you are not justified in interchanging the order of summation and integration around the pole at $s=0$. In other words, you only get a group representation if $m-n = \text{integer}$.

25. We may rewrite the integral above in this way:

$$\oint = \int_{\tan\beta/2}^{(0+)} = \oint_{|s|=1}$$



26. Significance of the integral representation:

- the dependence on variable n is ultrasimple, compared to the hypergeometric function representation where n is tangled up in a very complicated way.
 - of course by using the symmetry properties, you can make the m dependence similarly simple.
 - the integral rep is really the direct thing you get using the operator representation. (See page 307 of V).
27. So far I have been thinking $SU(2)$, but notice that $\tan\beta/2$ goes over into $\tanh(x/2)$ and this is still less than 1 in absolute value for all x . Thus, integral rep is also correct for $SU(1,1)$ matrix elements.

Barut + '65
Fronsdal

On non-compact groups

II. Representations of the 2+1 Lorentz group

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A simple algebraic method based on multispinors with a complex number of indices is used to obtain the linear (and unitary) representations of non-compact groups. The method is illustrated in the case of the 2+1 Lorentz group. All linear representations of this group, their various realizations in Hilbert space as well as the matrix elements of finite transformations have been found. The problem of reduction of the direct product is also briefly discussed.

1. INTRODUCTION

Recently physicists have become more interested in the theory of non-compact groups. The early work on this subject was pioneered by Wigner (1939) in his study of the unitary representations of the inhomogeneous Lorentz group and its various subgroups. Later a systematic study of the representations of simple non-compact Lie groups was initiated by Mackey (1955) and by the Russian school.†

Instances where the unitary representations of non-compact groups have been worked out completely are remarkably few. They include the 2+1 Lorentz group (Bargmann 1947; Gel'fand, Graev & Vilenkin 1962), the homogeneous Lorentz group (Naimark 1964), some work on the 3+2 and 4+1 de Sitter groups (Thomas 1941; Newton 1949, 1950; Ehrman 1957; Phillips 1962) and on some non-compact forms of unitary groups (Graev 1958). Considering the large number of semi-simple non compact Lie groups‡ the study of non-compact groups seems to be only beginning, at least for physicists. Essentially the difficulties arise from the fact that the interesting (unitary) representations of non-compact groups are all infinite dimensional and seem to require a formidable mathematical apparatus. The purpose of this paper is to show that simpler algebraic techniques are sufficient to obtain the representations of non-compact groups. These methods seem to us to be sufficiently general that we hope this paper may serve as an introduction to the theory of representations of non-compact groups, although we shall be concerned with a simple example.

The example is the 2+1 Lorentz group which is not only the smallest of all non-trivial simple real non-compact Lie groups, but is also fundamental in that it appears as a subgroup of all others. It is also interesting in its own right for physical applications. In other reports we show that these methods are by no means confined to the 2+1 Lorentz group.

In §2 we list some properties of the 2+1 Lorentz group, the essential spinor group, and the covering group. These were all given by Bargmann (1947), but are included here for completeness.

In §3 we write down the spinor representation of the associated Lie algebra, and introduce the principal technique: the construction of multi-spinors with non-integral, in general complex, number of indices. In this way all linear representations of the algebra are found. In §4 those representations that are equivalent to unitary representations are selected and their unitary forms are given.

In §5 we show how a number of equivalent realizations of the underlying Hilbert space may be obtained. In particular, some light is thrown on the relation between the three main series of representations. In particular we study the contraction of some of the representations into unitary, irreducible representations of the two-dimensional Poincaré group.

In §6 some problems of the reduction of product representations into sums of irreducible representations are discussed. This problem was solved by Pukánszky (1961) in a very special case.

In §7, finally, we rederive the matrix elements of finite transformations found already by Bargmann, using algebraic rather than analytic methods.

2. THE GROUP AND THE LIE ALGEBRA

The 2+1 Lorentz group is the group ‡ of transformations in a real three-dimensional vector space that leaves invariant the indefinite form

$$x^2 = x_1^2 + x_2^2 - x_3^2 = g^{\mu\nu} x_\mu x_\nu. \quad (2.1)$$

This group has three real parameters and is generated by a Lie algebra spanned by three independent generators. The generators and their commutation relations are

$$\left. \begin{aligned} L_{12}, \quad L_{13}, \quad L_{23}; \quad L_{\mu\nu} + L_{\nu\mu} &= 0, \\ [L_{\mu\nu}, L_{\rho\lambda}] &= -ig_{\mu\rho} L_{\nu\lambda}, \\ g_{11} = g_{22} &= 1, \quad g_{33} = -1. \end{aligned} \right\} \quad (2.2)$$

Another basis for the Lie algebra is particularly useful, namely

$$\left. \begin{aligned} M^\pm &= \sqrt{\frac{1}{2}}(iL_{13} \pm L_{23}), \quad L_{12}, \\ [L_{12}, M^\pm] &= \pm M^\pm, \quad [M^+, M^-] = L_{12}. \end{aligned} \right\} \quad (2.3)$$

We have defined $M^\pm$ and L_{12} in such a way that their commutation relations are exactly the same for the 2+1 group as for the three-dimensional rotation group. The difference between the two groups lies in the range of the parameters. Thus, if

$$L = \epsilon_{12} L_{12} + \epsilon_+ M^+ + \epsilon_- M^- \quad (2.4)$$

is an element of the algebra, then $\bar{\epsilon}_+ = \epsilon_-$ for the rotation group and $\bar{\epsilon}_+ = -\epsilon_-$ for the 2+1 Lorentz group; ϵ_{12} is real in either case. This means that in a unitary representation

$$L_{12}^\dagger = L_{12}, \quad (M^+)^\dagger = M^- \quad \text{so } (3) \quad (2.5)$$

‡ Like the 3+1 Lorentz group, this group consists of four disconnected parts. Here we study only that which is continuously connected to the identity.

§ In deference to the custom of mathematicians we shall write $\bar{a}$ for the complex conjugate

† See for example the review by Berezin, Gel'fand, Graev & Naimark (1960).

‡ See for example the § 1.1 in the book by Gel'fand, Graev & Vilenkin (1962).

1. General comments: this is the paper which really forms the basis for the entire Chapter 17 of Wybourne's book. This is the first paper I have read which bypasses the messy Hilbert Space function realizations of the representations and deals with the "multispinor method". They even make a comment to this effect, page 538. I.e., instead of using a messy space of functions on a circle and all that, they use a Hilbert space of functions of two variables, and their "representation" is given by their equation (7.1), see also equations (17.67) through (17.69) of Wybourne. Using this rep, it is completely trivial to compute the rep functions d_{mn}^1 which other people get only via hairy integral representations, etc. Here, all you need is the binomial theorem.

2. There are several other points to be noted in this paper. First, notice that it is a very early paper, 1965. Secondly, the phrase "spinor group" is clarified a little. Recall that the parameter space is a topological space. Now, you can parametrize the $su(1,1)$ matrices by the variables shown in 2.8 which suggests that the parameter space is the topological product of a complex plane and a circle. However, you can imagine the covering space of that circle to be a real line which contains the circle in some sense infinitely many times. Somehow the covering group associated with this covering space will contain $su(1,1)$ many times; somehow this has to do with multivalued representations. The group $su(1,1)$ itself is called the "spinor group" of the Lorentz group $2+1$ (?) because $su(1,1)$ is a two-valued representation of $SO(2,1)$. In other words, $su(1,1)$ contains $so(2,1)$ two times.

Now this all has a connection with the values taken by parameter E_0 . If you let E_0 have its full continuum of values, then your representations so obtained are reps for the entire covering group or line X plane. I.e., there is no necessity that going around the circle once or twice or three times should give you the same "value" for a function defined on the circle. If you then restrict down the acceptable values of E_0 to integers and half integers, then you are guaranteeing that a 4π rotation gives same value, so on the circle there are only two values. Then you are talking about reps of the "spinor group" $su(1,1)$. Finally, if you further restrict E_0 to just the integers, then 2π is identity element, and rep is "single valued". Then you are talking $so(2,1)$ reps. One can say, of course, that: "the two-valued reps of $so(2,1)$ are the single valued reps of $su(1,1)$ " . . Remember that a spin- $\frac{1}{2}$ particle is associated with a two valued rep of $SO(3)$, which is a single valued rep of $SU(2)$.

I have always wondered about this business of values of E_0 . This paper should help to straighten that out.

3. Section 2. The Lie algebra is stated in such a way that it is the same for $SU(1,1)$ as for $SU(2)$, see 2.3. The difference between the two groups lies in the acceptable values of the parameters which "push" these generators (i.e., real or imaginary or what). Remember that the parameters must be such that the vector magnitude, or vector magnitude, is preserved by a group transformation.

It is noted that the unitarity conditions on the generators are, therefore, different for these two groups. The non-unitary 2-dim rep is stated in 2.9, this is I think the "elementary rep" which you associate with the Dynkin diagram.

Section 3: Here he sets up the linear space of monomials. I.e., the space is defined to be the functions of two variables. If you restrict this space to functions which have the "monomial form", then you can show that such functions form an invariant subspace and will therefore tell you about the irreducible representations.

- Now, on this space of functions of two variables, the generators can be realized as in 3.2, you just show this by brute force. Thus, 3.3 follows, and we notice that in this particular differential operator representation, J_3 is diagonal. Later I suspect I will learn about other possible forms for the differential operators which allow other generators to be diagonal.

If we restrict our linear space to monomial form functions such that $a+b$ is a fixed constant (homogeneous polynomials, recall from Vilenkin), you can show that such a subspace of the Hilbert space is invariant. I.e., just look at the action of the generators shown in 3.3. Thus, you expect $a+b$ to be connected to the Casimir Q which labels the reps. Then in 3.6 he shows that there is another number E_0 which is part of the rep label. I.e., E_0 tells you where to start the spectrum of J_3 .

Next, the various possible reps are enumerated. The unitarity conditions have not yet been enforced.

Section 4: The unitarity conditions force the various rep labels to have special values. These are all shown in the picture. Also, see (4.9) for the form $\sum x_i$ of a general vector in $\mathbb{R}^n$ a IR subspace. I.e., this is just a sum over monomials such that $a+b = \text{fixed}$. By the way, we want our rep space to be a Hilbert space which means we need a scalar product. For vectors inside the subspaces, this is given by 4.10 and 4.3. The requirement that such a vector have finite norm is stated in 4.11.

Section 5: the first half of this section discusses the connection between their nice simple reps and those messy reps of Bargmann. I will probe this later, glad to see that they dealt with this question. The second part of this section deals with sort of an application: what would particle physics look like if Lorentz group were replaced by deSitter group, $SO(3,2)$. I'll have to give this section a special reading to find out what the hell it is all about and why it is relevant.

Section 6: here are some comments on the Clebsh Gordon problem for $SO(2,1)$. IK-skip over this but note that: this same stuff is also mentioned a little in Wybourne. Also, Mukunda is involved in a huge series of 4 papers which attack exactly this problem. I think Mukunda does a complete solution of the situation, whereas these guys are looking at a special case.

Section 7: Here, finally, the action of the Group Operator on our freshly defined monomial hilbert space is defined, also see Wybourne as already noted. Now we have a complete representation. In fact we know we are dealing with irreducible reps because we are in the invariant subspaces of the total Hilbert-space. Finally, if we keep to the particular values of the rep labels, we shall have unitary irreducible reps. Thus, bang, with no trouble at all you get the matrix elements for g parametrized in terms of α , β , and g_{33} . So we have the reps of $so(3)$ and $so(2,1)$ both at the same time in the J_3 basis. This is truly dynamite to handle these things at the same time. They even take the rep down to hypergeometric function, which Wybourne does in equation 17.82.

By the way, notice that this gives the reps of all series. However, notice that the "normalizers" appear in 7.4 and in Wybourne, and that these take different forms for the different types of reps. That explains why JLY-factored out that normalizer factor to get rep functions valid for all series (I hold off on details; the notion is hereby noted).

Conclusions: this paper has added to my understnading in several different ways. This reinforces my plan to keep reading as many papers as possible, picking up bits and pieces here and there. Nobody in a review or a book puts in all these little facts that you really ought to know. Original papers are best to get a nice description of these "facts".

Holman f
Biedenharn

'66

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Complex Angular Momenta and the Groups $SU(1, 1)$ and $SU(2)^*$

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The Wigner coefficients coupling any two discrete representations of the noncompact group $SU(1, 1)$ are derived, and it is shown that they can be expressed as analytic continuations of one another and of the Wigner coefficients of $SU(2)$ in the planes of complex angular momenta. It is further shown that these Wigner coefficients, in certain well-defined limits, approach the form of the representation functions: The Wigner coefficients coupling two discrete representations of $SU(1, 1)$ approach the representation functions of $SU(1, 1)$ while those of $SU(2)$ approach the $SU(2)$ functions. The Wigner coefficients coupling a positive and a negative discrete representation of $SU(1, 1)$ are used to define states of the continuous series of its unitary representations. It is shown that a definition of the norm of these states exists under which the states of all the unitary representations of $SU(1, 1)$ —positive and negative discrete and both the principal and exceptional series of the continuous representations—are mutually orthogonal in the same Hilbert space.

I. INTRODUCTION

The following paper is concerned with the interrelation and generalization of several important concepts from the quantum theory of angular momentum; while many of the specific results are not new, we believe that the connections which we shall establish are both new and interesting. We shall define a generalized Wigner coefficient (GWC) which has the property that it agrees precisely with the $SU(2)$ coefficient and at the same time possesses a unique analytic continuation everywhere in the complex planes of the three angular momenta which it couples. This GWC in turn agrees precisely with the Wigner coefficients which couple three members of the discrete series of unitary representations of the group $SU(1, 1)$, where $SU(1, 1)$ is defined as the group of transformations which leaves the Hermitian form $|x_1|^2 - |x_2|^2$ invariant. We further demonstrate that the analytic continuation of the GWC also serves to couple representations no more than two of which belong to the continuous series, and in fact can be used to define these representations in terms of a coupling of discrete ones. (We shall not consider the case of the coupling of three continuous representations.)

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Holman and Biedenharn (published 1966)

"Complex J and groups $su(2)$ and $su(1,1)$ "

1. Overall comments: This is the first paper to use the "boson calculus" method. Some ref back to 1935 however. E_{12} and E_{21} are raising and lowering operators, the e_i are the cartesian generators. However, most of the effort of this paper goes into handling Wigner coefficients for $su(1,1)$, continued from $su(2)$ somehow. These notes will only concern what the authors say about things other than Wigner coefficients.

Introduction. Summary, mention of Pchammer double contour integral.

II. Review of $su(2)$ and $su(1,1)$. THE CREATION realisations of the generators is set up. It is claimed that this creation method very quickly gives you the $su(2)$ CGC's, which are quoted in 2.11, along with the group reps quoted in 2.12.

Then on page 5 they review $su(1,1)$. Generators are called K_i . The fact that m gets quantized is observed to have nothing to do with the Lie algebra, but rather with global group properties. (ie, spinor group). The possible reps are investigated in usual Wybourne form, terminating chains etc. In particular, the explicit HS forms 2.43 and 2.44 provide rep~~s~~ space for the discrete series only (respectively D^+ and D^-). The continuous series are postponed to section IV which I have not copied. I think the reason for this is that the discrete series are involved in the simple Wigner coupling analysis in $su(1,1)$, whereas dealing with the continuous series is more involved.

Next, the various rep matrices are quoted. First, 2.53 is for the (nonunitary) finite dim reps. Then 2.59 and 2.61 are for the two types of discrete series. Then 2.64 gives the rep functions for all types of continuous series. Then follows the various orthogonality properties for all these reps. See 2.69 eg. I note that this notation looks very much like Bargmann's. Finally, this section closes with remarks about the asymptotic properties of all these rep functions.

From the point of view of CGC's, the nonunitary finite dim reps of $su(1,1)$ will couple in the same way as the unitary reps of $su(2)$; that is why these nonunitary reps are mentioned. (Usually they are never mentioned).

III. The coupling of two discrete series to produce a third. Here I stop. But I note that this is also the subject of the first paper in Mukunda's four paper CGC series of 1973. Apparently this is the simplest case.

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Matrix Elements of Representations of Non-Compact Groups in a Continuous Basis*

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Abstract. Explicit formulas are obtained by a simple algebraic method for the representations of the finite group transformations of $O(2,1)$ in a continuous basis when a non-compact generator is diagonalized. Compact and non-compact cases are treated in a unified form and the nature of analytic continuation is determined. The transformation function between the discrete and the continuous bases is also given. These explicit formulas have not been obtained in the literature before.

I. Introduction

The use of the unitary infinite-dimensional representations of non-compact groups to describe the properties of bound states of quantum-mechanical systems is by now well understood. However, not much has been done to deal with the scattering states. There is a need for explicit forms of unitary representations when a continuous spectrum is used to label the states (i.e. diagonalized). There arise here some peculiar and unfamiliar (at least to physicists) problems that must be solved.

There has been a number of recent discussions on the unified representation theory of compact and non-compact groups having the same complex extension [1-3]. A number of recent papers deal with the specific cases of the representations of $O(3,1)$ with respect to the non-compact group $O(2,1)$ and the analytic continuation problem between $O(2,1)$ and $O(3)$ [1-3, 8-10]. In none of the previous work there appears the explicit form of the representation in a continuous basis and the relation of the continuous basis to the discrete one. The purpose of this work is to fill this gap. The simple algebraic method that we give in this paper as an extension of the previous work [3] not only determines the representations in a continuous basis $|\lambda\rangle$, but also gives explicit formulas for the transformation function $\langle m|\lambda\rangle$ between the discrete basis $|m\rangle$ and the continuous basis $|\lambda\rangle$, and the matrix elements $\langle\lambda|U|\lambda'\rangle$, where

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U is any group element. These are the quantities of prime interest in physical applications. We confine ourselves to the simple prototype case, namely the group $O(2,1)$, but the results can easily be generalized to higher groups.

In Section II we derive as a preliminary the unitary representations of $O(2,1)$ from those of $O(3)$ by analytic continuation. This is a correspondence between two unitary representations, and of course, is different than the usual correspondence of finite dimensional unitary representation of $O(3)$ and the finite dimensional non-unitary representation of $O(2,1)$. In Section III, the main part of the paper, we introduce the continuous basis by a suitable redefinition of the creation and annihilation operators and derive the above mentioned quantities $\langle m|\lambda\rangle$ and $\langle\lambda|U|\lambda'\rangle$. The nature of analytic continuation from the discrete basis to the continuous basis is of course not unique and straightforward as one might think, and special attention is given to this problem.

II. Collection of Formulas. Derivation of Unitary $O(2,1)$ - Representations by Analytic Continuation

(1) Lie Algebras

The elements of the Lie algebras of $O(3) - O(2,1)$ system satisfy [3]

$$\begin{aligned} [L_{12}, L_{13}] &= iL_{12} \\ [L_{23}, L_{13}] &= ig_{33}L_{12} \\ [L_{23}, L_{12}] &= iL_{13} \end{aligned} \quad (2.1)$$

$$g_{33} = -1 \text{ for } O(3), \quad g_{33} = +1 \text{ for } O(2,1),$$

or, in canonical form, with $L^\pm \equiv \frac{1}{\sqrt{2}}(L_{13} \pm iL_{23})$,

$$\begin{aligned} [L_{12}, L^\pm] &= \pm L^\pm \\ [L^+, L^-] &= -g_{33}L_{12} \end{aligned} \quad (2.2)$$

Thus, starting with the $O(3)$ algebra, L_{12}, L_{13}, L_{23} , the $O(2,1)$ algebra can be obtained by the new elements

$$L_{12} = L_{12}, \quad L_{13} = iL_{13}, \quad L_{23} = iL_{23}. \quad (2.3)$$

(2) Fundamental Representation of the Groups

The "spinor representations" of the group $O(3)$ and $O(2,1)$ (not algebras!) are given by [3]

$$W = \begin{pmatrix} \alpha & \beta \\ g_{33}\bar{\beta} & \bar{\alpha} \end{pmatrix}, \quad \det W = 1; \quad \begin{cases} g_{33} = -1 & \text{for } O(3) \\ g_{33} = +1 & \text{for } O(2,1) \end{cases} \quad (2.4)$$

note!

$$\begin{aligned} L_{12} &= J_0 = J_z \\ L_{23} &= J_2 = K_y \\ L_{13} &= J_1 = K_x \end{aligned}$$

"Matrix Elements of Representations of Non-compact Groups in Continuous Basis"

1. General: in this paper the "monomial" Hilbert space is used, nothing Bargmannic. The creation algebra is also used. Finally, the g_{33} notation is used to both $su(2)$ and $su(1,1)$ are handled at the same time. Also, after doing everything in the Bargmann basis, they go to the "space-space" basis. The transformation connecting these two bases is written as a hypergeometric function. The matrix elements for both groups are stated in both basis.
2. BUT, the expansion theorems are not stated in either basis. Also, that discreet index for the continuous basis seems to be MISSING! This question must be resolved.
1. Introduction. Brief summary of how this work builds on a previous Barut article in a 1967 book.
2. Summary of Usual Basis Results: a well organized section!!!
 1. Lie algebras. Stated in g_{33} notation. The hermitian $su(2)$ gens are called L , the hermitian $su(1,1)$ gens are called $\pm$.
 2. Fund. Reps. The two dim "basis rep" is stated, just the usual 2×2 matrix. Note that unitary for $su(2)$, not so for $su(1,1)$, but irreducible in both cases. Then 2.6 shows the usual param method. Param $\xi = \frac{1}{2}$ is what appears for $su(1,1)$. The usual pauli 2×2 generator basis is stated.
 3. General linear rep. The monomial HS is set up, the group rep for finite elements is stated as in Wy. Via usual binomial theorem method, matrix element is given as an F in 2.11 for certain range of m 's, then 2.12 for another range. Notice several things: (a) the "normalizers A_m " appear, hence these forms are good for all series; (b) E_0 appears as in 2.12, thus we can consider these as reps for entire covering group. The forms 2.13 and 2.13' are just F translations of earlier forms.
 4. Irreducible reps. So far ϕ and m are general complex. What values restrict HS to invariant SS's? This question is rephrased: when does D_{mm} vanish? 3 cases are identified. If a and b neither integers, etc etc.
 5. Unitary Reps. An unusual treatment of this question, but gets the usual result for conditions on the "normalizers".
3. Continuous Basis. Starts by writing the same generators in usual creation form with J_3 diagonal. In this "realization" the basis vectors are given as creations on the "vacuum". Then 3.2 defines some new creators c_i such that they are just lincombs of the a_i . In terms of these c_i , the gens are given in 3.3. The point is that now $L_{23} = K_y$ is diagonal!!
So, in terms of these new c 's, the monomial HS is shown in 3.6, whereas the old monomial HS is shown top of page 59. Using a scalar product defined by usual QM creation operators, its easy via binomial to compute the basis xform, see 3.10 for answer!

(over)

Now that we have a new basis in terms of creations, it is easy to transform the old "REP" into this new basis. I.e., we know how D acts on old basis, and we know connection to new basis. After a very painful calculation requiring an integral form of the binomial theorem, the resultant noncompact matrix elements are given in 3.18; actually this must be good for noncompact too since e appears.

Finally, in the last section the "normalizers" are computed in each basis by means of the unitarity conditions. They seem to use the same ones for all series.

General comments on this paper: the monomial method seems to give all reps of $so(2,1)$ without much distinction between the different classes of reps. The authors never mention the rep classes when they are talking about matrix elements.

The big question remains: how did they miss on the doubling of the continuous spectrum? I would guess that they make a comment on this in some subsequent letter. Surprisingly, they do reference Mukunda's first article in preprint form, yet they make absolutely no comment whatsoever about the degeneracy of the spectrum. It makes me wonder whether these guys bothered to read Mukunda's paper, or Bargmann, for that matter!