

Parity, Naturality, T, C, TCP

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$$D_{m'm'}(2Y) = (-1)^{s-m'} D_{m',m'}(a) \checkmark$$

Time Reversal.

1. On single particle rest state:

$$T |\tilde{p}, m\rangle = \epsilon \gamma |\tilde{p}, m\rangle = \epsilon \cdot (-1)^{s-m} |\tilde{p}, -m\rangle \checkmark$$

$$T |\tilde{p}, -m\rangle = \epsilon \cdot (-1)^{s+m} |\tilde{p}, m\rangle \Rightarrow T^2 |\tilde{p}, m\rangle = T \{ \epsilon \cdot (-1)^{s-m} |\tilde{p}, -m\rangle \}$$

$$\Rightarrow T^2 = (-1)^{2s} \quad \forall \epsilon \quad = \bar{\epsilon} \cdot (-1)^{s-m} \epsilon \cdot (-1)^{s+m} |\tilde{p}, m\rangle$$

As MS show, the phase ϵ can depend only on s . Notice that no matter what phase you take, you always get $T^2 = (-1)^{2s}$ on the rest, single-particle state. You cannot arrange things to get $T^2 = 1$.

2. Properties of the operator T :

- a) anti-linear so that $Ta = a^*T$ and $(a/T/b) = (a/\bar{T}/b)^*$
- b) anti-unitary so that $T^{-1} = T^\dagger$.
- c) $TRT^{-1} = R$ for any rotation, but $TBT^{-1} = B^{-1}$ for any boost.
- d) of course $TcT^{-1} = c^*$ for any c -number.

3. On canonical spin states:

$$L(p)^{-1} = L(sp) \checkmark$$

$$|p, m\rangle_c = L(p) |\tilde{p}, m\rangle \checkmark$$

$L(p) \doteq$ direct boost, as in McK.

check: $\checkmark$

$$T |p, m\rangle_c = T L(p) T^{-1} T |\tilde{p}, m\rangle = L(p)^{\dagger} \epsilon \cdot (-1)^{s-m} |\tilde{p}, -m\rangle \checkmark$$

$$= \epsilon \cdot (-1)^{s+m} L(sp) |\tilde{p}, -m\rangle \checkmark = \epsilon \cdot (-1)^{s-m} |sp, -m\rangle_c \checkmark$$

where $sp = p$ ie $s\vec{p} = -\vec{p}$ but $sE = E$. $\checkmark$

$$T |p, m\rangle_c = \epsilon \cdot (-1)^{s-m} |sp, -m\rangle_c \checkmark \rightarrow \text{go to A}$$

4. On spinor states:

$$|p, m\rangle = |p, m\rangle_c D_{m'm'}(L(p)^{-1}) \checkmark$$

$$|p, m\rangle_p = |p, m\rangle \bar{D}_{m'm'}(\gamma L(p)^{-1})$$

$$T |p, m\rangle = \epsilon \cdot (-1)^{s-m'} |sp, -m'\rangle_c \bar{D}_{m'm'}(L(p)^{-1}) \checkmark$$

$$= \epsilon \cdot (-1)^{s-m'} |sp, m''\rangle \bar{D}_{m'm'}(\gamma L(p)^{-1}) \bar{D}_{m'm'}(L(p)^{-1}) = \epsilon |sp, m''\rangle \bar{D}_{m'm'}[L(sp) \gamma L(p)^{-1}]$$

$$\Rightarrow T |p, m\rangle = \epsilon \cdot |sp, m''\rangle \bar{D}_{m'm'}[L(p)^{-1} \gamma L(p)^{-1}] \checkmark$$

→ A:

4. On spinor states:

$$T|p^m\rangle = (T|p^m\rangle) \cdot D_{m'm} (L(p)^{-1})^* \checkmark$$

$$= \{ \epsilon (-1)^{s-m'} |sp, -m'\rangle_c \} D_{m'm} (L(p)^{-1})^* \checkmark \quad \text{cc } T.$$

But,

$$|p, m\rangle_c = |p^{m''}\rangle D_{m''m} [\bar{Y}^T, L(p)^{-1}] \checkmark$$

$$\Rightarrow |sp, -m'\rangle_c = |p^{m''}\rangle D_{m''-m'} [\bar{Y}^T, L(sp)^{-1}] \checkmark$$

But, $\bar{Y}^T L(sp)^{-1} = \bar{Y}^T L(p) = \bar{Y}^T L(p) Y Y^T = L(p)^{-T} \cdot \bar{Y}^T \checkmark$

So $D_{m''-m'} [\bar{Y}^T, L(sp)^{-1}] = \sqrt{D_{m''-m'} [L(p)^{-T} \cdot \bar{Y}^T]} = D_{m''-m'} [L(p)^{-T} \cdot Y] (-1)^{2s} \checkmark$

So $|sp, -m'\rangle_c = |p^{m''}\rangle D_{m''m'} (L(p)^{-1})^* \cdot (-1)^{s-m'}$

$$= (-1)^{s+m'} D_{m''m'} (L(p)^{-1})^* (-1)^{2s} \checkmark$$

$$= (-1)^{s-m'} D_{m''m'} (L(p)^{-1})^* \checkmark$$

Thus:

$$T|p^m\rangle = \epsilon (-1)^{s-m'} |sp^{m''}\rangle D_{m''m'} (L(p)^{-1})^* \cdot (-1)^{s-m'} D_{m'm} (L(p)^{-1})^*$$

$$= \epsilon |sp^{m''}\rangle D_{m'm} (L(p)^{-2})^* \checkmark$$

So

$$T|p^m\rangle = \epsilon \cdot |sp^m\rangle D(L(p)^{-2})^* \checkmark$$

Next:

$$T|p^m\rangle = T|p^m\rangle D(L^2(p)Y)^* = \{ \epsilon |sp^m\rangle D(L(p)^{-2})^* \} D(L^2(p)Y)^* \checkmark$$

$$= \epsilon |sp^m\rangle D(Y)^* = \epsilon |sp^m\rangle D(Y) \checkmark$$

$$= \epsilon (-1)^{2s} |sp^m\rangle D(Y) = \epsilon (-1)^{2s} |sp^m\rangle$$

$$= \epsilon |sp^m\rangle D[L^2(sp)Y] D(Y) \checkmark$$

$$T|p^m\rangle = \epsilon \cdot (-1)^{2s} \cdot |sp^m\rangle D[L(p)^{-2}] \checkmark$$

$$T^2|p^m\rangle = \epsilon (-1)^{2s} T|sp^m\rangle D[L(p)^{-2}]^* = (-1)^{2s} |p^m\rangle \checkmark$$

$$= |p^m\rangle D(L(p)^{-2})^*$$

Therefore we have shown:

$$T|p^m\rangle = \epsilon |sp^{m'}\rangle D_{m'm} (L^2(p))^*$$

$$\langle p^m | T^\dagger = \bar{\epsilon} (-1)^{2s} \langle sp^{m'} | D_{m'm} (L(p)^2)^*$$

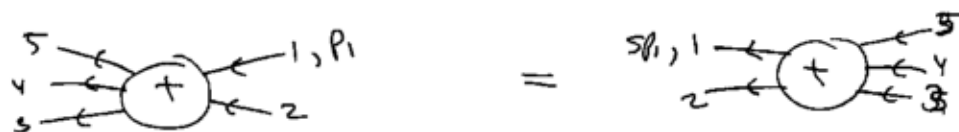
5. Now use Taylor M-fun:

$$\begin{aligned} \langle p_1^{m_1} | \dots S_c \dots | p_2^{m_2} \rangle &= \langle p_1^{m_1} | \dots T^\dagger S_c^\dagger T \dots | p_2^{m_2} \rangle^* \\ &= \left\{ (\bar{\epsilon}_1 \dots) (\epsilon_2 \dots) (-1)^{2(s_1 + \dots)} \langle sp_1^{m_1'} | \dots S_c^\dagger | \dots sp_2^{m_2'} \rangle \right\}^* \\ &\quad D_{m_1' m_1} (L^2(p))^* \dots \end{aligned}$$

$$\text{So } \langle p_3^{m_3} | \otimes \langle p_4^{m_4} | \otimes \langle p_5^{m_5} | S_c | p_1^{m_1} \rangle \otimes | p_2^{m_2} \rangle$$

$$= \bar{\epsilon}_3 \bar{\epsilon}_4 \bar{\epsilon}_5 (-1)^{2(s_3 + s_4 + s_5)} \bar{\epsilon}_1 \bar{\epsilon}_2$$

$$\langle p_1^{m_1'} | \otimes \langle p_2^{m_2'} | S_c \langle p_3^{m_3'} | \otimes \langle p_4^{m_4'} | \otimes \langle p_5^{m_5'} | \prod_{i=1}^5 D_{m_i' m_i} (L^2(p_i))$$



$$M^{m_1 m_2 m_3 m_4 m_5} (1, p_1; 2, p_2; 3, p_3; 4, p_4; 5, p_5) = \left(\prod_{i=1}^2 \bar{\epsilon}_i \right) \left(\prod_{i=3}^5 \epsilon_i (-1)^{2s_i} \right)$$

$$\times M^{m_5' m_4' m_3' m_2' m_1'} (5, sp_5; 4, sp_4; 3, sp_3; 2, sp_2; 1, sp_1)$$

$$\times \prod_{i=1}^5 D_{m_i' m_i}^{(0, s_i)} (L^2(p_i))$$

6: Restate

(new version)

π

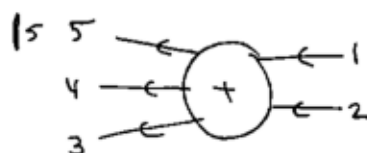
$$M^{m_1 m_2 m_3 m_4 m_5} (1, p_1; 2, p_2; 3, p_3; 4, p_4; 5, p_5)$$

$$= \bar{\epsilon}_1 \bar{\epsilon}_2 \epsilon_3 \epsilon_4 \epsilon_5 \cdot \underbrace{(-1)}_{\sigma}^{2(s_3 + s_4 + s_5)}$$

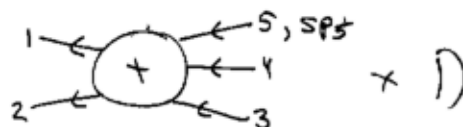
$$M^{m'_5 m'_4 m'_3 m'_2 m'_1} (5, s p_5; 4, s p_4; 3, s p_3; 2, s p_2; 1, s p_1)$$

Graphical

$$\times \prod_{i=1}^5 D_{m'_i m_i}^{(0, s_i)} (L^{-2}(p_i))$$



$$= \bar{\epsilon}_1 \bar{\epsilon}_2 \epsilon_3 \epsilon_4 \epsilon_5 \cdot \sigma$$



$$\sigma = \prod_{\text{all}} (-1)^{2s_i} = (-1)^{2(s_3 + s_4 + s_5)} = (-1)^{2(s_1 + s_2)}$$

Comments: in time reversal invariance, you pick up an obvious product of ϵ phases which appeared at the start. Also, a characteristic spin phase called σ . Top to bottom order is unaffected, but cyclic order is reversed. What came in, now goes out with negated 3-momentum. Spins are mixed.

Parity.

1. On single particle rest states:

$$P|\tilde{p}, m\rangle = \eta |\tilde{p}, m\rangle \quad P^2 = \eta^2 = \begin{cases} 1 & \text{MS} \\ (-1)^{2s} & \text{Toller} \end{cases} \checkmark$$

We could insist that $P^2 = 1$, but to be general we will regard η as a complex phasor.

2. The operator P is a standard unitary operator, linear.

3. On canonical spin states:

$$P P^\dagger = R \quad P B P^\dagger = B^{-1}$$

$$|p, m\rangle_c = L(p) |\tilde{p}, m\rangle \checkmark$$

$$P|p, m\rangle_c = L(p)^{-1} \eta |\tilde{p}, m\rangle \checkmark = \eta L(sp) |\tilde{p}, m\rangle \checkmark = \eta |sp, m\rangle_c \checkmark$$

So $\boxed{P|p, m\rangle_c = \eta |sp, m\rangle_c} \checkmark$

4. On spinor states:

$$|p, m\rangle = |p, m\rangle_c D_{m'm}(L(p)^{-1}) \checkmark$$

$$|p, m\rangle_c = |p, m\rangle D_{m'm}(L(p)) \checkmark$$

$$P|p, m\rangle = \eta |sp, m'\rangle_c D_{m'm'}(L(p)^{-1}) \checkmark$$

$$= \eta |sp, m'\rangle D_{m'm'}(L(sp)) D_{m'm'}(L(p)^{-1}) \checkmark = \eta |sp, m'\rangle D_{m'm}(L(p)^{-2}) \checkmark$$

So $\boxed{P|p, m\rangle = \eta |sp, m'\rangle D_{m'm}(L(p)^{-2})} \checkmark$

Similarly:

$$P|p, m\rangle = \eta |sp, m'] \bar{D}_{m'm}(L(p)^{-2}) \checkmark$$

$$[p, m| P^\dagger = \bar{\eta} [sp, m'| D_{m'm}(L(p)^{-2}) \checkmark$$

5. Parity invariance for Taylor M-functions

$$\begin{aligned} P|p, m\rangle &= \{P|p, m\rangle\} D(L^2(p)Y) = \{\eta |sp, m'\rangle D(L(p)^{-2})\} D(L^2(p)Y) = [\eta |sp, m'] D(Y) \\ &\stackrel{\text{recom}}{\text{put}} = [\eta |sp, m'] \cdot D(Y^{-1} L^{-2}(p)) D(Y) = \eta |sp, m'] D(Y^{-1} L^2(p)Y) \\ &= \eta |sp, m'] D(L^2(p)^T) = \eta |sp, m'] D(L^2(p))^* \end{aligned}$$

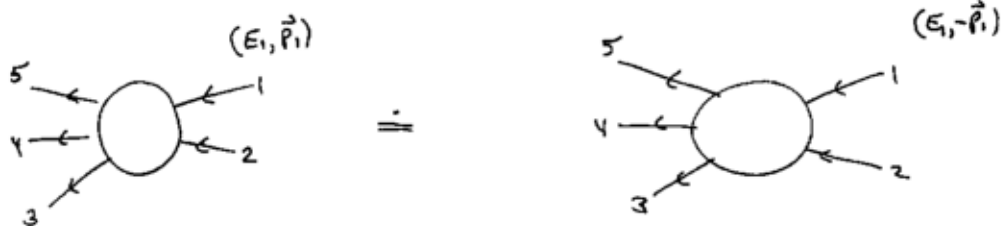
$$M^{m_1 m_2 m_3 m_4 m_5} (1, p_1; 2, p_2; 3, p_3; 4, p_4; 5, p_5)$$

P ✓

$$= \eta_1 \eta_2 \bar{\eta}_3 \bar{\eta}_4 \bar{\eta}_5 \cdot \prod_{i=1}^5 D_{m'_i m_i}^{(\alpha_i s_i)} (L(p_i)^{-2})$$

$$M^{m'_1 m'_2 m'_3 m'_4 m'_5} (1, s p_1; 2, s p_2; 3, s p_3; 4, s p_4; 5, s p_5)$$

Graphical :



Particle in the initial state of course stays there. The 3-momenta are negated. The helicities are mixed in a certain way.

Charge Conjugation

1. This unitary operator C is the simplest since it moves through rotations and boosts. However, it changes particle type, so there might be two phases going with this operator:

$$C|\tilde{p}, m, t\rangle = \alpha_t |\tilde{p}, m, \bar{t}\rangle$$

$$C|\tilde{p}, m, \bar{t}\rangle = \alpha_{\bar{t}} |\tilde{p}, m, t\rangle$$

$$C^2 = \text{any phase you want} = \epsilon_c$$

$$\Rightarrow \alpha_t \alpha_{\bar{t}} = \epsilon_c \quad \forall t$$

2. Since C does not affect Lorentz operators, the action of C is the same for all types of states:

$$C|\tilde{p}, m, t\rangle = \alpha_t |\tilde{p}, m, \bar{t}\rangle$$

$$C|p, m, t\rangle_c \doteq \alpha_t |p, m, \bar{t}\rangle_c$$

$$C|p, m, t\rangle = \alpha_t |p, m, \bar{t}\rangle \quad \text{etc.}$$

Because C commutes with all Poincare generators, it cannot change momentum or spin.

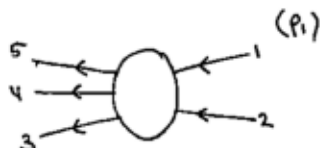
3. Charge conjugation invariance of M-functions of Taylor:

$$M^{m_1 m_2 m_3 m_4 m_5}(1, p_1; 2, p_2; 3, p_3; 4, p_4; 5, p_5)$$

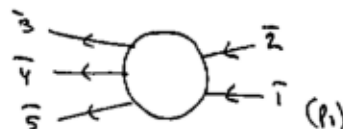
$$= \alpha_1 \alpha_2 \bar{\alpha}_3 \bar{\alpha}_4 \bar{\alpha}_5$$

$$\times M^{m_2 m_1 m_5 m_4 m_3}(\bar{2}, p_2; \bar{1}, p_1; \bar{5}, p_5; \bar{4}, p_4; \bar{3}, p_3)$$

graphical:



$\doteq$



For purposes of Ring amplitudes, I have assumed that C reorders each channel, hence reverses overall order.

PT and CPT invariance as products of separate operators.

① First do T to get

$$M^{m_1 m_2 m_3 m_4 m_5} (1, p_1; 2, p_2; 3, p_3; 4, p_4; 5, p_5) = \bar{E}_1 \bar{E}_2 E_3 E_4 E_5 \times \sigma$$

$$\times M^{m'_5 m'_4 m'_3 m'_2 m'_1} (5, p_5; 4, p_4; 3, p_3; 2, p_2; 1, p_1) \prod_{m_i, m'_i}^{(0, s_i)} D_{m_i m'_i} (L(p_i)^{\dagger} L(p_i)^{-1})$$

② Now do P on this amplitude:

$$\eta_5 \eta_4 \eta_3 \bar{\eta}_2 \bar{\eta}_1 \cdot M^{m''_5 m''_4 m''_3 m''_2 m''_1} (5, p_5; 4, p_4; 3, p_3; 2, p_2; 1, p_1) \prod_{m_i, m'_i}^{(0, s_i)} D_{m_i m'_i} (L(p_i)^{\dagger 2})$$

③ Now use group property to conclude

$$L(p_i)^{\dagger 2} = L(p_i)^{\dagger 2}$$

NOTE!

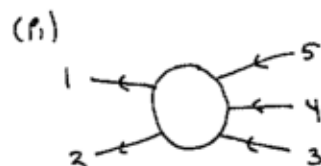
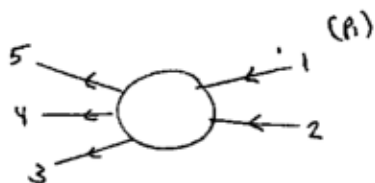
$$M^{m_1 m_2 m_3 m_4 m_5} (1, p_1; 2, p_2; 3, p_3; 4, p_4; 5, p_5)$$

PT ✓

$$= \bar{E}_1 \bar{\eta}_1 \cdot \bar{E}_2 \bar{\eta}_2 \cdot E_3 \eta_3 \cdot E_4 \eta_4 \cdot E_5 \eta_5 \times \sigma$$

$$M^{m'_5 m'_4 m'_3 m'_2 m'_1} (5, p_5; 4, p_4; 3, p_3; 2, p_2; 1, p_1)$$

graphical:



④ Now do C on last amp.

$$= d_5 d_4 d_3 \bar{d}_2 \bar{d}_1 M^{m'_3 m'_4 m'_5 m'_1 m'_2} (\bar{3}, p_3; \bar{4}, p_4; \bar{5}, p_5; \bar{1}, p_1; \bar{2}, p_2)$$

So here is CPT:

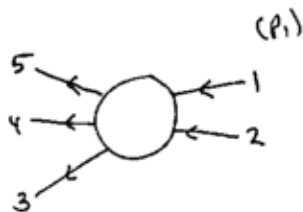
CPT

$$\mathcal{M}^{m_1 m_2 m_3 m_4 m_5} (1, p_1; 2, p_2; 3, p_3; 4, p_4; 5, p_5)$$

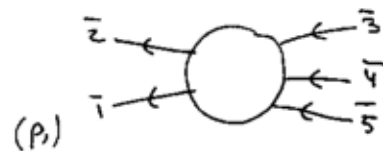
$$= \overline{d_1 n_1 \epsilon_1} \cdot \overline{d_2 n_2 \epsilon_2} \cdot d_3 n_3 \epsilon_3 \cdot d_4 n_4 \epsilon_4 \cdot d_5 n_5 \epsilon_5 \quad \times \quad \delta$$

$$\times \mathcal{M}^{m'_2 m'_4 m'_5 m'_1 m'_3} (\bar{3}, p_3; \bar{4}, p_4; \bar{5}, p_5; \bar{1}, p_1; \bar{2}, p_2).$$

graph:



=



On the Question of CPT Theorem and Reversing Order of Particles.

1. My first approach to considering reversal of order was to consider just a state. Assuming S-matrix theory has the "relative statistics property", here is what you get for reversing a state:

$$|1,2\rangle = |2,1\rangle \epsilon_{12}$$

$$|1,2,3\rangle = |3,2,1\rangle \epsilon_{12} \epsilon_{13} \epsilon_{23}$$

$$|1,2,3,4\rangle = |4,3,2,1\rangle \epsilon_{12} \epsilon_{13} \epsilon_{14} \epsilon_{23} \epsilon_{24} \epsilon_{34}$$

$$\epsilon_{ij} = (-1)^{4s_i s_j}$$

In the general case, the factors you get can be considered to lie on one side of a certain matrix:

$$\begin{pmatrix} \epsilon_{11} & \epsilon_{12} & \epsilon_{13} & \dots & \epsilon_{1n} \\ & \epsilon_{22} & \epsilon_{23} & \dots & \epsilon_{2n} \\ & & \epsilon_{33} & \dots & \epsilon_{3n} \\ & & & \ddots & \\ & & & & \epsilon_{nn} \end{pmatrix}$$

Thus, when you reverse an n -particle state, there are N factors:

$$N = (n^2 - n)/2.$$

Actually, I now find that this is a clumsy way to write the phase factor.

2. Consider again a general state containing both fermions and bosons. Because of usual relative statistics, you are free to shift all the boson fields to one side:

$$|F_1, B_2, F_3, F_4, B_5, B_6\rangle = |F_1, F_3, F_4, B_2, B_5, B_6\rangle$$

$$\text{and } |B_6, B_5, F_4, F_3, B_2, F_1\rangle = |F_4, F_3, F_1, B_6, B_5, B_2\rangle$$

Now we see that to reverse the order of a state, you can consider the bosons and fermions separately. In reversing the bosons, you get no signs at all. Thus, the only signs you get come from reversing the order of the fermions. Assume there are F fermions. Then each of the $(F^2 - F)/2$ exchange factors is a minus sign, so that the overall phase due to state reversal is:

$$\text{reversal phase} = (-1)^I \text{ where } I = \frac{1}{2}(F^2 - F)$$

By just fiddling, it is easy to show that this becomes:

$$\text{reversal phase} = \begin{cases} (1)^F & \text{when } F = \text{even} \\ (1)^{F-1} & \text{when } F = \text{odd} \end{cases}$$

3. Thus, when you reverse order for an S-matrix element or Wightman function, you get this factor i^F because F must be even, otherwise the function vanishes.

$$\langle 0 | \phi(x_1) \phi(x_2) \dots | 0 \rangle = i^F \langle 0 | \dots \phi_2(x_2) \phi_1(x_1) | 0 \rangle$$

4. Now we can write F and B in terms of the spins as follows:

$$\boxed{F = \sum_{i=1}^n \left[\frac{1 - (-1)^{2s_i}}{2} \right]} \quad \text{ie, a zero for boson, a one for fermion}$$

$$\boxed{B = \sum_{i=1}^n \left[\frac{1 + (-1)^{2s_i}}{2} \right]} \quad \text{//}$$

5. Therefore, we can say for reversal of order on a ~~full S-matrix element~~ ^{Wightman function}:

$$\underline{\text{reversal phase}} = (i)^{\sum \left[\frac{1 - (-1)^{2s_i}}{2} \right]} = \prod_{j=1}^n \left((i)^{\left[\frac{1 - (-1)^{2s_j}}{2} \right]} \right)$$

Example:

$$|1, 2, 3\rangle = \text{phase} \times |3, 2, 1\rangle$$

$$\text{phase} = (i)^{\left[\frac{1 - (-1)^{2s_1}}{2} \right]} \times (i)^{\left[\frac{1 - (-1)^{2s_2}}{2} \right]} \times (i)^{\left[\frac{1 - (-1)^{2s_3}}{2} \right]}$$

Thus, when the total number of fermions is even -- as it is for ^{a W.F.} ~~an S-matrix element~~ -- the reversal phase can be written as a product of factors where each factor depends only on one particle type. I.e., the phase does factorize. This is not at all obvious in the other notation:

$$\text{phase} = (-1)^{4[S_1 S_2 + S_1 S_3 + S_2 S_3]} \quad \text{with} \quad (-1)^{2[S_1 + S_2 + S_3]} = +1$$

I have been unable to show directly the following theorem, but I have proven it indirectly so no need to show it directly:

$$\text{If } (-1)^{2[S_1 + S_2 + \dots + S_n]} = 1, \text{ then}$$

$$(-1)^{4[S_1 S_2 + S_1 S_3 + S_1 S_4 + \dots + S_1 S_n] + (S_2 S_3 + S_2 S_4 + \dots + S_2 S_n) + \dots + (S_{n-1} S_n]}$$

$$= \prod_{i=1}^n \left[(i)^{\left[\frac{1 - (-1)^{2s_i}}{2} \right]} \right]$$

Comments about this CPT result:

1. Using the complex Lorentz group we have already derived this thing and we have done so in such a way that the only phase is δ . In deriving the TCP relation through the complex Lorentz group, we used these ingredients:

- a) Taylors crossing rule for all five particles
- b) the complex lorentz transformation

However, in Taylor's crossing rule we have set a certain phase Iagolnitzer calls $\lambda_c = 1$.

So, in order to make our present result agree with that earlier phase convention, we shall agree to choose the time reversal phase to compensate the parity and charge conjugation phase.

$$\epsilon = (\eta \cdot \alpha)^{-1}$$

Recall that there can never be any significance to the time-reversal phase, but there is significance to the intrinsic parity and charge conjugation phases.

Misc

6. We now make the corresponding statements for S-matrix elements. Suppose we reverse the order of both initial and final channels of an S-matrix element. We get:

$$\left(\begin{array}{c} a \\ b \\ c \end{array} \left| \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \right. \right) = (-1)^{\frac{F_i(F_i-1)}{2}} (-1)^{\frac{F_f(F_f-1)}{2}} \left(\begin{array}{c} c \\ b \\ a \end{array} \left| \begin{array}{c} 3 \\ 2 \\ 1 \end{array} \right. \right)$$

However, in order that this amplitude not vanish, we know that F_1 and F_2 are either both even or both odd. The phases are then:

$$\text{phase} = \begin{cases} (-1)^{F/2} & F_i, F_f \text{ even} \\ -(-1)^{F/2} & F_i, F_f \text{ odd} \end{cases}$$

Let us define this symbol sigma:

$$\sigma = \prod_{\text{chan}} (-1)^{2s_i} = (-1)^{F_i} = (-1)^{F_f} = \begin{cases} +1 & \text{even} \\ -1 & \text{odd} \end{cases}$$

Then we can write

$$\text{phase} = \sigma (-1)^{F/2}$$

Result:

$$\left(\begin{array}{c} 6 \\ 5 \\ 4 \end{array} \left| \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \right. \right) = \sigma (-1)^{F/2} \left(\begin{array}{c} 4 \\ 5 \\ 6 \end{array} \left| \begin{array}{c} 3 \\ 2 \\ 1 \end{array} \right. \right)$$

7. Lemma:

$$\underset{\substack{\uparrow \\ A}}{(-1)^{2s}} \cdot \underset{\substack{\uparrow \\ B}}{(i)} \left[\frac{1 - (-1)^{2s}}{2} \right] = \underset{\substack{\uparrow \\ C}}{(-i)} \left[\frac{1 - (-1)^{2s}}{2} \right]$$

Proof:

$\frac{3}{B}$	$\frac{A}{+1}$	$\frac{B}{1}$	$\frac{C}{1}$	$\frac{A \times B}{1}$	
				$\leftrightarrow$	
F	-1	i	$-i$	$\leftrightarrow$	$-i$

QED.

8. Theorem:

$$\sigma (-1)^{F/2} = \prod_{\text{initial}} \left((i) \left[\frac{1 - (-1)^{2s_i}}{2} \right] \right) \times \prod_{\text{final}} \left((i) \left[\frac{1 - (-1)^{2s_i}}{2} \right] \right)$$

Proof: $(-1)^{F/2} = \prod_i (i)^{[]} \cdot \prod_f (i)^{[]} \cdot \prod_f (-1)^{2s_i} = \prod_i (i)^{[]} \cdot \prod_f (-i)^{[]} \checkmark$

What is the Wigner rotation of a rotation?

1. First, let's do the Helicity Wigner rotation. WE have:

$$R_H(R, p) = H(Rp)^{-1} \cdot R \cdot H(p)$$

$$H(p) = R(p) Z(p).$$

$$\Rightarrow R_H = Z(Rp)^{-1} R(Rp)^{-1} R R(p) Z(p)$$

Let $R(p) = (\phi, \theta, \phi')$ and $R = (\alpha, \beta, \gamma)$. Then $R \cdot R(p) = (\alpha', \beta', \gamma')$, some new rotation. Then:

$$R(Rp) = (\alpha', \beta', \alpha')$$

I.e., the 3-vector $\vec{R}p$ is at (α', β') . So

$$\begin{aligned} R_H &= Z(p)^{-1} \cdot (\alpha', \beta', \alpha') \cdot \underbrace{(\alpha, \beta, \gamma) \cdot (\phi, \theta, \phi')}_{= (\alpha', \beta', \gamma')} \cdot Z(p) = Z(p)^{-1} \cdot (\alpha', 0, \gamma') Z(p) \\ &= R_Z(\alpha' + \gamma') \end{aligned}$$

Then:

$$\boxed{\begin{aligned} R_H^W(R, p) &= R_Z(\alpha' + \gamma') \quad \text{where } R \cdot R(p) = (\alpha', \beta', \gamma') \\ R(p) &= (\phi, \theta, \phi') \quad R = (\alpha, \beta, \gamma) \end{aligned}}$$

Thus:

$$\boxed{R|p, m\rangle_H = e^{-im(\alpha' + \gamma')} |R p, m\rangle_H}$$

$$R(Rp) \cdot R_Z(\alpha' + \gamma') = R \cdot R(p)$$

Thus, when you rotate a helicity state, you do not change the helicity of the state, but you do pick up an unpleasant phase as shown here.

2. Now the Canonical case:

$$\begin{aligned} R_C^W(R, p) &= L(Rp)^{-1} \cdot R \cdot L(p) = R(Rp) H(Rp)^{-1} R \cdot H(p) \cdot R(p)^{-1} \\ &= R(Rp) R_H(R, p) R(p)^{-1} = R(Rp) \cdot R_Z(\alpha' + \gamma') \cdot R(p)^{-1} \\ &= R \cdot R(p) \cdot R(p)^{-1} = R. \end{aligned}$$

So

$$\boxed{R_C^W(R, p) = R} \quad / (3.23) \text{ McKenell.}$$

So the Canonical Wigner rotation is the rotation itself, very simple.

Some comments about parity.

1. Consider this equation:

$$s a s^{-1} = (a^+)^{-1}$$

Here I am imagining that a is some matrix which represents the group $SL(2, \mathbb{C})$ with some number of dimensions, and s is also a matrix representing parity in that same number of dimensions. Why is this what parity should do? Start out in 4-dimensions where you know about things. Then you know that for $a = \text{rotation}$, $a = \text{unitary matrix}$ and thus you have $s a = a s$? To see why this must be true, apply both sides to a general 4-vector:

$$s = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix} \quad a = e^{-i\vec{\alpha} \cdot \vec{J}} = \text{unitary}$$

$$ka = at \text{ of course.}$$

$$s \cdot R \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} = s \cdot \begin{pmatrix} t \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} t \\ -x' \\ -y' \\ -z' \end{pmatrix}$$

$$k = -1 = \begin{pmatrix} -1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

$$R \cdot s \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} = R \begin{pmatrix} t \\ -x' \\ -y' \\ -z' \end{pmatrix} = R \cdot k \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} = k R \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} = t \cdot \begin{pmatrix} -x' \\ -y' \\ -z' \end{pmatrix} = \begin{pmatrix} t \\ -x' \\ -y' \\ -z' \end{pmatrix}$$

You want parity to commute with rotations. But you want parity to negate boosts. Note that in 4-vector space, boost matrix is hermitian because the boost generators are anti-hermitian. Actually, the boost matrix is real and symmetric since we are only here talking about real Lorentz stuff. Consider:

$$s \cdot B_x \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} = s \cdot \begin{pmatrix} t \cosh \alpha + x \sinh \alpha \\ t \sinh \alpha + x \cosh \alpha \\ y \\ z \end{pmatrix} = \begin{pmatrix} t \cosh \alpha + x \sinh \alpha \\ -t \sinh \alpha - x \cosh \alpha \\ y \\ -z \end{pmatrix}$$
$$B_x s \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} = B_x \begin{pmatrix} t \\ -x' \\ -y' \\ -z' \end{pmatrix} = \begin{pmatrix} t \cosh \alpha + x' \sinh \alpha \\ -t \sinh \alpha - x' \cosh \alpha \\ -y' \\ -z' \end{pmatrix} \checkmark$$

2. So the above equation which tells what parity does is motivated in this way:

a) assume rotation generators are hermitian and boost generators are anti hermitian. This means that rotations are unitary and boosts are hermitian.

b) then the equation says: $s R s^{-1} = R$ and $s B s^{-1} = B^{-1}$. We want these facts to be true for the 4x4 space. In the 4x4 space we can find an explicit 4x4 matrix called s which does the job.

3. Now read the equation in the 2x2 space of actual $SL(2, \mathbb{C})$ itself. Assumption (a) is still correct, but the fact is that there exists no 2x2 matrix s which does the required job. The reason is this:

$$s a s^{-1} = (a^+)^{-1} \quad \forall a \in SL(2, \mathbb{C}) \Rightarrow \text{a equivalent to } (a^+)^{-1} \text{ which we know is not true!}$$

OK, $s A^\mu \sigma_\mu s^{-1} = \bar{A}_\mu \sigma_\mu$ obviously has no solution valid $\forall$ complex A_μ with $A \cdot A = 1$.

So, when we come to the 2×2 $SL(2, \mathbb{C})$ world, we know what we want the parity operator s to do, but we cannot find a 2×2 matrix that will work. Note that we cannot find a matrix of any determinant that will work. So we have a peculiar situation that the operator s somehow lies "outside" of $SL(2, \mathbb{C})$ and must be added on. The fact that no matrix 2×2 works is connected to the fact that the representation a is not equivalent to the representation $(a^+)^{-1}$. Somehow, the parity operator s is faking a connection between these two different representations of $SL(2, \mathbb{C})$.

4. Now what are we going to do? Since we cannot "represent" the parity operator s in the 2×2 space, we probably cannot represent s in any of the $D^{(s,0)}$ spaces. Thus, we will never be able to find a statement like this:

$$M_{m_1, \dots} (p^\lambda \dots) = \dots D_{m_1, m_1}^{(s,0)}(s) M_{m_1, \dots} (p_\mu \dots)$$

This is in line with the fact that parity simply does not belong to the complex Lorentz group $L_+(C)$, or not to mention the real proper Lorentz group. So how do you make the statement of parity invariance of your theory?

5. Looking at Toller, it seems that the only time you can represent parity is if you are talking about one of the little groups.

Why does an anti-fermion have opposite parity from a fermion?

1. When you start looking this up in the standard texts, you are referred to the usual field theoretical argument which is somewhere in BD. But I want a general S-matrix argument for why this should be true. I checked Willimas (refers to BD), Gas. (field theory), etc etc. Not even Baryt seems to do this. But, Martin and Spearman gave a hint, which I now follow up, and it works.

2. Consider a Taylor M-function. I will show that the above rule may be proved from the combination of crossing and parity invariance. For reference, see Taylor M-function notes elsewhere. The basic idea is this:

$$\begin{aligned}
 \text{Diagram: circle with 3 incoming lines on the left, 2 outgoing lines on the right labeled } 2, p_2 \leftarrow &= M^{m_1 \dots m_2 \dots} (\dots; 2, p_2; \dots) \\
 &= M^{m'_1 \dots m'_2 \dots} (\dots; 2, -p_2; \dots) \eta_2 D_{m'_2 m_2}^{(0, s_2)} (L(p_2)^{-2}) \dots
 \end{aligned}$$

This is the statement of parity invariance for the Taylor M-function, no ordering needed but OK to use ordering if you want. I only show relevant factors. Similarly:

$$\text{Diagram: circle with 3 incoming lines on the left, 2 outgoing lines on the right labeled } 2, p_2 \leftarrow = + \text{Diagram: circle with 3 incoming lines on the right, 2 outgoing lines on the left labeled } \bar{2}, -p_2 \leftarrow \text{ by crossing.}$$

Then:

$$\begin{aligned}
 &= M^{m_1 \dots m_2 \dots} (\dots; \bar{2}, -p_2; \dots) \\
 &= M^{m'_1 \dots m'_2 \dots} (\dots; \bar{2}, -p_2; \dots) \bar{\eta}_2 D_{m'_2 m_2}^{(0, s_2)} (L(-p_2)^{-2}) \dots
 \end{aligned}$$

but these are also equal by crossing.

$$\Rightarrow \eta_2 = \bar{\eta}_2 \cdot (-1)^{2s_2}$$

provided that $D_{m'_2 m_2}^{(0, s_2)} (L(-p_2)^{-2}) = (-1)^{2s_2} D_{m'_2 m_2}^{(0, s_2)} (L(+p_2)^{-2}) \checkmark$

Note: $L^2(-p) = (-1)^{2s_z} L^2(p) \rightarrow D(L^2(-p)) = D(-1) D(L^2(p))$

↓

$1 \cdot (-1)^{2s_z} \checkmark$

Thus: $\boxed{\eta_{\bar{2}} = \overline{\eta_2} \cdot (-1)^{2s_z}}$

So, in the usual system of conventions where $P^2 = +$ we have real parity for bosons and fermions. Then you see that for fermions, the intrinsic parities must be opposite for F versus $\bar{F}$. For bosons the Boson-bar has same parity as a boson. But, for mesons, all anti-mesons are also mesons, so good thing answer comes out plus!

What does parity do to a multiparticle helicity amplitude?

① Each helicity is negated, each Breit is negated, pick up huge phase:

$$M_{m_1 m_2 m_3 \dots m_n}(\vec{p}_1, \vec{p}_2, \dots, \vec{p}_n) \rightarrow \prod \eta_i \prod (-1)^{S_i - m_i} M_{-m_1 -m_2 -m_3 \dots -m_n}(-\vec{p}_1, -\vec{p}_2, \dots, -\vec{p}_n)$$

② In 4 particle case, the phase is:

$$\eta_1 \eta_2 \eta_3 \eta_4 (-1)^{S_1 - m_1} (-1)^{S_2 - m_2} (-1)^{S_3 - m_3} (-1)^{S_4 - m_4}$$



$$= \eta_1 \eta_2 \eta_3 \eta_4 (-1)^{m_3 - S_3} (-1)^{S_1 - m_1} (-1)^{S_4 - m_4} (-1)^{m_2 - S_2}$$

$$\mu = -m_3 - m_1, \mu' = -m_4 - m_2 \Rightarrow \eta_1 \eta_2 \eta_3 \eta_4 (-1)^{S_1 + S_4 - S_2 - S_3} (-1)^{\mu - \mu'}$$

This agrees more or less with my symmetry notes in the TK. There is some ambiguity as to the meaning of the helicity of each particle. Also, recall that the two-particle helicity state has an extra phase thrown in to it which I do not want to use. (See page 124 MS). This phase will I think change the phase in the above relations.

3. My strategy is this: consider a multiparticle amplitude to have an initial and final state which is simply the direct product of single particle helicity states. Thus, the action of parity is completely known if you know what parity does to a single particle helicity state. There is no need to make cms states, etc etc, .

What does TCP say for a reaction with spin?

1. First, it is important to understand what P and T do to helicity states.

2. First, parity:

Let $\underline{p} = +p_z \hat{z}$.

$$P |p, m\rangle_H = (P Z(p) P^\dagger) P |\tilde{p}, m\rangle_H = (R_y(\pi) Z(p) R_y(\pi)) \cdot \chi |\tilde{p}, m\rangle$$

$$= (-1)^{s-m} \chi R_y(\pi) |p, m\rangle_H$$

Here, parity just pulls an intrinsic phase off the rest state: the thing is at rest, so you imagine leaving the spin put (parity no change magnetic field) but reversing its "motion". However, parity on the moving state of course will convert Z_p into Z_{-p} . If you are going to only have helicity states defined by positive boosts, you have to flip the thing back to Z_p somehow. The way is a y -rotation. Of course such a rotation acting on the rest state flips the helicity. The final y -rotation then flips everything over, but helicity stays flipped.

On the more physical level, parity changes the momentum but not the spin component, therefore the helicity must flip.

3. Secondly, time-reversal:

$$T |p, m\rangle_H = (T Z_p T^\dagger) T |\tilde{p}, m\rangle_H = (R_y(\pi) Z(p) R_y(\pi)) |\tilde{p}, -m\rangle$$

$$= R_y(\pi) |\tilde{p}, m\rangle_H$$

This goes exactly as above, except that in the rest frame, T does flip the helicity. So it gets flipped back again by the same y -rotation, and the end result is that time-reversal does not flip helicity.

On the more physical level, once again, you gather that time reversal should flip momentum and spin, and that therefore helicity should remain unflipped.

4. Thirdly, lets do the combination of PT on a state. The result is easily:

$$PT |p, m\rangle_H = (-1)^{s-m} |\tilde{p}, -m\rangle_H$$

Under this combination, the momentum ends up where it was, but helicity is flipped. Pick up a phase in obvious way. Note that 2π rotation is $(-1)^{2s}$.

5. Now consider $TCP = T$ invariance. Recalling that T is an antiunitary operator, recall that you get the switching of initial and final states as part of the invariance rule. Thus:

$$\langle f | S | i \rangle = \langle T i | S | T f \rangle \quad \leftarrow \text{no stars or phases.}$$

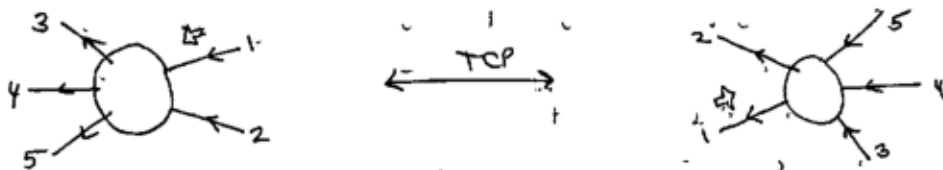
We have already shown what PT does to each state, except I only discussed states along the z -axis. Recall that the operator PT moves through all Lorentz transformations, so we can consider the T operators to be acting directly on the states along the z -axis, then we can use information just presented. Action of C is to charge conjugate all particles, maybe with a conventioned phase. So TCP invariance therefore would say:

$$\begin{aligned} & \langle p_3, m_3; p_1, m_1; p_5, m_5 | S | p_1, m_1; p_2, m_2 \rangle \\ &= (\text{phase}) \cdot \langle p_1, -m_1; p_2, -m_2 | S | p_3, -m_3; p_4, -m_4; p_5, -m_5 \rangle \end{aligned}$$

where $(\text{phase}) = \pi \eta_i \pi \eta_c \pi (-1)^{m_i}$

today, no η , but result still clear.

6. Therefore, TCP relates amplitudes with all helicities flipped, but otherwise only change is initial to final state. Here is a picture. I do not show helicities, but arrow directions are supposed to represent actual momenta:



These are two reactions related by TCP . The magnitude of these two amplitudes must be precisely equal. Note that same 4-momenta throughout!

7. The phase is of course not observable. Perhaps some bode detail if you like.

More detail on phase in TCP:

① Assume some phase η_i under $\mathcal{P}$

$$\langle f | S | i \rangle = \langle \tilde{f} | S | \tilde{f} \rangle \text{ where } \tilde{f} = \text{CPT}$$

$$\text{Now, } \mathcal{PT} \rightarrow \eta_i \in (-1)^{S_i - M_i} \text{ and } C \rightarrow \eta_i$$

so

$$\langle f | S | i \rangle = (\text{phase}) \times \langle \tilde{f} | S | \tilde{f} \rangle$$

as already noted
(relative flipped)

The phase is:

$$\left(\prod_i [\eta_i \in \eta_i (-1)^{S_i - M_i}]^* \prod_{\tilde{f}} [\eta_i \in \eta_i (-1)^{S_i - M_i}] \right)$$

= TCP phase

Reconsider Action of P, T on $|p, m\rangle_H$ states.

Toller = Lee-Wick convention

Sources: JW, ms. Let $|p, m\rangle_H = 2(\tilde{p}) |\tilde{p}, m\rangle = \gamma_{pm}$ of JW.

① Action on rest states:

$$P |\tilde{p}, m\rangle = \eta |\tilde{p}, m\rangle$$

$$T |\tilde{p}, m\rangle = \epsilon (-1)^{s-m} |\tilde{p}, -m\rangle$$

$$PT |\tilde{p}, m\rangle = \eta \epsilon (-1)^{s-m} |\tilde{p}, -m\rangle$$

$$TP |\tilde{p}, m\rangle = \eta^* \epsilon (-1)^{s-m} |\tilde{p}, -m\rangle, \text{ same as } PPT$$

② Action on $\underline{z}$ -states:

$$P |p, m\rangle_H = \eta (-1)^{s-m} e^{+i\pi J_y} |p, -m\rangle_H \quad /ms,$$

$$T |p, m\rangle_H = \epsilon e^{-i\pi J_y} |p, m\rangle_H = \epsilon (-1)^{2s} e^{+i\pi J_y} |p, m\rangle_H \quad /ms, \epsilon = 1$$

$$PT |p, m\rangle_H = \eta \epsilon (-1)^{s-m} |p, -m\rangle_H$$

$$TP |p, m\rangle_H = \eta^* \epsilon (-1)^{s-m} |p, -m\rangle_H, \text{ same as } PPT$$

③ Squared-actions:

$$P^2 |p, m\rangle_H = \eta^2 |p, m\rangle_H$$

$$T^2 |p, m\rangle_H = |\epsilon|^2 (-1)^{2s} |p, m\rangle_H$$

$$(PT)^2 |p, m\rangle_H = \eta^2 |\epsilon|^2 (-1)^{2s} |p, m\rangle_H$$

I see no way to "duplicate" the Lee-Wick field theory results here. For example, acting on single particle states of even or odd $2J_z$ operators P and T always commute. You cannot change this fact by any phase convention that I can see. Let's try to make a chart of conventions:

Note that T is not constant

④ So far we have:

$$P^2 = \eta^2$$

$$T^2 = (-1)^{2s} \text{ since } |\epsilon|^2 = 1$$

$$(PT)^2 = (-1)^{2s}$$

Now everything hinges on the choice you make for η .

Try: $\eta^2 = (-1)^{2s}$ - just a convention. regardless of choice of ϵ

Then $\eta = \pm (i)^{2s}$ and $\eta^* = (-1)^{2s} \eta$

$$s = \text{integer} \Rightarrow (-1)^{2s} = 1 \Rightarrow \eta^2 = 1 \Rightarrow \eta = \pm 1 \Rightarrow \eta^* = \eta$$

$$s = \text{half-int} \Rightarrow (-1)^{2s} = -1 \Rightarrow \eta^2 = -1 \Rightarrow \eta = \pm i \Rightarrow \eta^* = -\eta.$$

Then:

$$\begin{aligned} P \cdot T &= T \cdot P \cdot (-1)^{2s} \\ P^2 &= T^2 = (-1)^{2s} \end{aligned}$$

Toller
Lee-Wick

Assume that C plays no role, so that $t = PT$. Ans

$$t^2 = (-1)^{2s}$$

Now: $t \cdot P = PT \cdot P = 1$

$$P \cdot t = P \cdot PT = (-1)^{2s}$$

$$\Rightarrow \boxed{t \cdot P = P \cdot t \cdot (-1)^{2s}}$$

So hurray! We presume that C is part of the internal symmetry group for hadrons and therefore will commute with P and T . Thus, anywhere C appears, it will slip through and have no effect"

$$P(TCP) = (-1)^{2s} (TCP) P, \quad (TCP)^2 = (-1)^{2s}.$$

$$\textcircled{C^2 = 1}$$

Thus we now see why Toller makes that peculiar choice for the parity: fermions have imaginary parity. This allows Toller to duplicate the Lee Wick relations. This is of course just convention. Another convention would be this:

$$\begin{aligned} \eta = +1 &\Rightarrow \begin{aligned} P^2 &= +1 \\ T^2 &= (-1)^{2s} \\ (PT)^2 &= (-1)^{2s} \end{aligned} & \begin{aligned} PT &= TP \\ (PT)P &= P(PT) \end{aligned} \end{aligned}$$

MS-

File: Time Reversal.

Antiunitary, antilinear operators.

1. An operator A is anti-linear if it has this property:

$$A(c_1|1\rangle + c_2|2\rangle) = c_1^* A|1\rangle + c_2^* A|2\rangle$$

2. Easy to show that product of two antilinear ops is a linear op, etc.

3. Constants get starred as they move through an antilinear operator.

4. The action of an antilinear operator on a bra is defined by this:

$$\langle x|\vec{A}|y\rangle = \langle x|\vec{A}|y\rangle^* \quad / \text{ change of direction accompanied by star for a.l.o.}$$

Notice that the second expression (the RHS of last equation) is a linear function of the ket y , because of the star. The name of this function is $\langle x/\vec{A}$, and this is the definition of the action of A to the left.

5. If A is antilinear, how do you define the adjoint $A^\dagger$? Answer:

$$\langle x|\vec{A}^\dagger|y\rangle = \langle y|\vec{A}|x\rangle^* = \langle y|\vec{A}|x\rangle$$

6. The rules for $(AB)^{-1}$ or $(AB)^\dagger$ are the same rules as usual for linear ops.

7. What is an anti-unitary operator? Answer: one such that $A^{-1} = A^\dagger$ but $A =$ anti-linear.

8. Facts about anti-unitary operators. Let A be antiunitary, $B =$ linear operator. Then one can easily show that:

$$\langle \hat{x}|\hat{B}|\hat{y}\rangle = \langle x|B|y\rangle^* \quad \text{and} \quad A c A^\dagger = c^* \cdot I$$

$$\text{where } |\hat{x}\rangle \equiv A|x\rangle, \quad \langle \hat{y}| \equiv \langle y|A^\dagger, \quad \hat{B} \equiv A B A^\dagger$$

9. Time-reversal as an example of anti-unitary operator. In general, let $A = T$, the time-reversal "operator" which is anti-unitary. Then we know from the above that ($S = S$ matrix = a linear operator):

$$\langle f|S|i\rangle = \langle \hat{f}|\hat{S}|\hat{i}\rangle^* \quad \text{where} \quad |\hat{i}\rangle = \vec{T}|i\rangle, \quad \langle \hat{f}| = \langle f|\vec{T}^\dagger$$

and $\hat{S} \equiv T S T^\dagger$. So far we have not said anything. Now suppose it happens that $T S T^\dagger = S^\dagger$. Then you have this:

$$\langle f|S|i\rangle = \langle \hat{f}|\hat{S}^\dagger|\hat{i}\rangle^* = \langle \hat{i}|S|\hat{f}\rangle$$

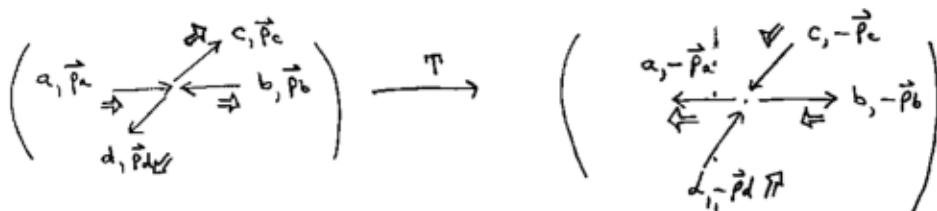
$$\text{or: } \langle f|S|i\rangle = \{\langle \hat{i}|\vec{T}^\dagger\} S \{\vec{T}|\hat{f}\}$$

$$\text{or: } \langle f|S|i\rangle = \langle \hat{i}|S|\hat{f}\rangle \iff \vec{T} S \vec{T}^\dagger = S^\dagger$$

This last line written in operator and matrix element form is the statement of time reversal invariance.

A naive discussion of time-reversal and parity for S-matrix theory.

1. This discussion begins with the stuff mentioned in Williams page 105 to 110.
2. What is the time-reversed reaction? Just think classically. The 3-momenta should reverse, the canonical spin projections should reverse. First, lets do a picture:



$$|\langle \vec{p}_c, m_c; \vec{p}_d, m_d | S | \vec{p}_a, m_a; \vec{p}_b, m_b \rangle| = |\langle -\vec{p}_c, -m_c; -\vec{p}_d, -m_d | S | -\vec{p}_a, -m_a; -\vec{p}_b, -m_b \rangle|$$

$$\left| \begin{array}{c} c, E_c, \vec{p}_c, m_c \\ d, E_d, \vec{p}_d, m_d \end{array} \leftarrow \begin{array}{c} \text{S} \\ \text{---} \end{array} \begin{array}{c} a, E_a, \vec{p}_a, m_a \\ b, E_b, \vec{p}_b, m_b \end{array} \right| = \left| \begin{array}{c} a, E_a, -\vec{p}_a, -m_a \\ b, E_b, -\vec{p}_b, -m_b \end{array} \leftarrow \begin{array}{c} \text{S} \\ \text{---} \end{array} \begin{array}{c} c, E_c, -\vec{p}_c, -m_c \\ d, E_d, -\vec{p}_d, -m_d \end{array} \right| \quad T_c$$

Several comments are appropriate here:

- a) Williams sometimes referse to time-reversal as "reciprocity".
 - b) notice that the phase has not been specified here.
 - c) we use canonical spin, not helicity. Helicity projections would not get negated.
 - d) energies and particle types do not get changed, only 3-momenta get changed.
3. What is the parity-reversed reaction? Obviously this is going to involve reversed 3-momenta, but what about the spin. The spin 3-vectors like s , which we usually do not use in S-matrix theory, stay the same because they are axial vectors. If you view the canonical spin projection as spin vector s dotted into some fixed z-axis which does not move under parity, then canonical spin projections do not change. Anyway, this is the result williams gives. Again, in helicity the helicities get negated; jus t the opposite result as above. Fro parity we then have:

$$|\langle \vec{p}_c, m_c; \vec{p}_d, m_d | S | \vec{p}_a, m_a; \vec{p}_b, m_b \rangle| = |\langle -\vec{p}_c, m_c; -\vec{p}_d, m_d | S | -\vec{p}_a, m_a; -\vec{p}_b, m_b \rangle|$$

$$\left| \begin{array}{c} c, E_c, \vec{p}_c, m_c \\ d, E_d, \vec{p}_d, m_d \end{array} \leftarrow \begin{array}{c} \text{S} \\ \text{---} \end{array} \begin{array}{c} a, E_a, \vec{p}_a, m_a \\ b, E_b, \vec{p}_b, m_b \end{array} \right| = \left| \begin{array}{c} c, E_c, -\vec{p}_c, m_c \\ d, E_d, -\vec{p}_d, m_d \end{array} \leftarrow \begin{array}{c} \text{S} \\ \text{---} \end{array} \begin{array}{c} a, E_a, -\vec{p}_a, m_a \\ b, E_b, -\vec{p}_b, m_b \end{array} \right| \quad P_c$$

Comments on this:

- a) particles stay in same side of reaction, no type changes.
- b) the only change is the three momenta are negated.
- c) again, phases not specified.

A list of conflicting concepts regarding TCP, crossing, etc.

1. Conflict #1: according to Taylor or anyone else, say MS, when you "cross" a particle, you negate its 4-momentum. Therefore, if you simulate TCP by crossing all four particles of a 4-body reaction, you would get a relation relating an amplitude where a is incoming with positive energy, to an amplitude where $\bar{a}$ is outgoing with negative energy. I.e., the two amplitudes are continuations of each other and are not physical at the same time. This conflicts with the idea that TCP à la Williams connects two reactions both in physical regions, i.e., $\bar{a}$ has positive energy in the second reaction of Williams. This Conflict #1 does not need spin.
2. Conflict #2. In Williams or me, the combination PT negates helicity or canonical spin projection, and therefore so does TCP. This means that TCP relates a with momentum p_a and helicity m_a to reaction with particle $\bar{a}$ having p_a but $-m_a$. This seems to conflict with both Taylor and MS who say that when you cross a particle, the helicity does not get negated but the momentum does. Of course there are two complications here:
 - a) Taylor uses spinorial states rather than helicity.
 - b) this conflict #2 is complicated by existence of conflict #1.

Resolution #1: I now regard TCP as the combination of crossing all particles and the complex I.T. which negates all momenta.

Stapp ('62)

read 7/21/77.

Stapp: Intrinsic Parity Problem.

1. Definition: parity invariance for a process means reverse all 3-momenta, do not reverse spin vectors. Magnitude of amplitude is then the same for both. (2.1). There will then be some phase called $\epsilon(T)$ which you can show depends only on particle types, not on their momenta. This thing is the "total process phase" due to parity inversion.
2. Theorem: the total phase you get from parity is either +1 or -1. This is what (3.1) says. I do not really know why this is, but postpone this question till later. Seems that maybe two applications of P could give you an arbitrary phase. I suspect this question is a larger question than I care to analyze right now. Assume correct.
3. One way to think of parity inversion is to go to the multiple-L representation. Then parity gives a phase which is minus one to the sum of all L_i . In this case, you see that your phase must be plus or minus one.
4. In section 5 Henry shows that product of process parities for process and another process with one crossed particle must be $(-1)^{2s}$ where s = spin of crossed particle. I have not followed this because I do not yet understand Stapp notation, eg, (5.9) or its generalization (5.4). *But I have my own proof of this.*
5. In section 6 Henry asks: can you write the total process parity as a product of particle intrinsic parities? If so, is this assignment unique?
Uniqueness is considered first. It is easy to show that, if your theory happens to have something else conserved, like charge, this fact enables you to generate an equivalently useful set of intrinsic parities from the original set, as in (6.2+). This thing is referred to as a gauge transformation. In the rest of this section, he shows the existence of a set of intrinsics.
Should you always take them +1 or -1? You certainly can do this. Then any additive conservation law would allow you to gauge these intrinsics into a set of complex numbers.
Since self-conjugate fermions have not yet been observed, you can forget them. But if they did exist, you would have to give them purely imaginary intrinsic parity, so that $\epsilon = -1$. Interesting point. IF the world had no conserved quantities except integrality (then fermions would have to be self-conjugate), you would be forced to give all fermions in such a theory imaginary parity! Perhaps this is really the best way to make the assignment, since integrality may be the only perfectly conserved quantum number! Then you are OK if self-conjugate fermions are discovered some day. This must be why Toller opts for this version!

STAPP
(please return)

Reprinted from THE PHYSICAL REVIEW, Vol. 128, No. 4, 1963-1969, November 15, 1962
Printed in U. S. A.

Intrinsic Parity from the S-Matrix Viewpoint

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(Received July 2, 1962)

The concept of intrinsic parity is developed as a construct of observables within the S-matrix framework, and ambiguities arising in more conventional approaches are avoided. It is shown that invariance of transition probabilities under spatial inversion implies the existence of a set of particle intrinsic parities that is real and unique to within a gauge transformation, provided all processes not forbidden by additive conservation laws do occur in nature. If certain of the conservation laws are multiplicative, then nonreal intrinsic parities are possible. The intrinsic parity of particle-antiparticle pairs is fixed by the general S-matrix postulates to be negative for fermions and positive for bosons, a result that parallels the one of field theory.

I. INTRODUCTION

THE question whether the sign of the intrinsic parity of particle-antiparticle pairs follows directly from the S-matrix postulates has served to focus attention on the more general question of the meaning of intrinsic parity in the S-matrix framework. In this paper the concept of intrinsic parity is systematically developed from observable S-matrix quantities. The approach is more general than the usual one, as the existence of a parity operation in the abstract space of field operators is not assumed; the only assumed invariance is that of the physically observed transition probabilities. The notion of an intrinsic parity for individual particles is not a manifest part of this assumption, but the concept can be established by construction, and the questions of existence and uniqueness completely answered. The problem of superselection rules, which clouds the conventional approach, causes no difficulty in this one. A more detailed comparison with the conventional approach is given in the final section.

* This work was done under the auspices of the U. S. Atomic Energy Commission.

II. INVARIANCE OF A PROCESS UNDER SPATIAL INVERSION

A process is taken to mean a reaction having specified initial and final particles. A process will be said to be invariant under spatial inversion (reflection) if and only if the transition probabilities associated with the process are invariant under an inversion through the origin of all polar three-vectors. The space parts of momentum-energy vectors are defined to be polar three-vectors, whereas spin vectors are not polar vectors. Thus, in the S-matrix framework, invariance under spatial reflection implies the equation^{1,2}

$$|R(K_p)| = |R(K)|, \quad (2.1)$$

where K_p is the set of variables obtained from the set K by reversing the space parts of all the momentum-energy vectors k_i contained in K . This equation can also

¹ Henry P. Stapp, Phys. Rev. 125, 2139 (1962) and *Lectures on S-Matrix Theory* (W. A. Benjamin Inc., New York, 1963). These references will be referred to as SI and SII, respectively.

² See SI, Appendix E.

where $\epsilon' = \epsilon(-1)^{N_f}$, and N_f is the number of spinor indices associated with initial particles.

The expression for $M(K)$ in Eq. (5.7) is the analytic function that describes the processes in all channels. Since ϵ' is independent of K over the physical region, it is an absolute constant. The intrinsic parities ϵ defined in (5.1) are given, therefore, by

$$\epsilon = \epsilon'(-1)^{N_f}. \quad (5.8)$$

The immediate consequence of this equation is that the intrinsic parities of two processes are opposite if one is obtained from the other, changing a fermion in the initial configuration to its antiparticle in the final configuration. That is, if the sum over both initial and final particles of the angular momentum quantum numbers is even for some process, it must be odd for the process obtained by switching an initial fermion to a final antifermion, or vice versa. For bosons the intrinsic parity of the process is not changed by such a switch.

To understand the essential idea involved here one may consider the simple case of the scattering of a spin $-\frac{1}{2}$ particle by a spin-zero particle. If the process is invariant under spatial inversion the M function takes the form

$$M(K) = v(K) \cdot \sigma - \epsilon(k^a \cdot \sigma) [v(K_p) \cdot \sigma] (k^b \cdot \sigma), \quad (5.9)$$

where the k are *mathematical* momentum-energy vectors, and the mass of the fermion has been taken to be unity; v is a combination of the four-vectors of the problem. Going to the nonrelativistic limit, $k \cdot \sigma$ becomes plus one for final particles and minus one for initial particles. Thus in the scattering process the surviving contribution is

$$[v(K) + \epsilon V(K_p)] \cdot \sigma, \quad (5.10)$$

and only the part even or odd under $k \rightarrow k_p$ contributes for ϵ plus one or minus one, respectively. But if one analytically continues the momentum-energy vector, for, say, the initial fermion from the physical region where $k_t^0 < 0$ to the physical region where $k_t^0 > 0$, in order to obtain the function representing pair production, the nonrelativistic limit of $k \cdot \sigma$ is changed from minus one to plus one; the nonrelativistic limit for pair production is

$$[v(K) - \epsilon v(K_p)] \cdot \sigma. \quad (5.11)$$

Thus if the even part of $v(K)$ contributes in the scattering, the odd part contributes in the pair production, and vice versa.

For a scattering process, in which the incoming and outgoing particles are the same, the intrinsic parity is plus one, since it is plus one for the no-scattering part of this process. That is, if the sum of the initial orbital angular momenta is even (odd), then the sum of the final ones is also even (odd). By virtue of the result shown above, this connection will be reversed in the

production of a fermion particle-antiparticle pair by a boson particle-antiparticle pair; even orbital states will go to odd orbital states, and vice versa. Also, a fermion-antifermion pair must be in an odd orbital state if it is to be emitted in a reaction without changing the orbital states of the remaining particles. These relations are just what is meant by the statement that the intrinsic parity of a fermion-antifermion pair is -1 .

VI. INTRINSIC PARITY OF A PARTICLE

Suppose all processes are invariant under spatial reflection. The intrinsic parities $\epsilon(T)$ of all processes T are then well defined. If there is a set of numbers ω_i , one for each particle type i , such that the intrinsic parity of every process is the product of these ω_i 's, extended over the particles participating in the process, the set of ω_i will be called a (possible) set of particle intrinsic parities. Specifically, we require of a set of particle intrinsic parities ω_i , for all physically occurring processes T , that

$$\epsilon(T) = \prod_i \omega_i^{N_i(T)} = \prod_{[T]} \omega_i, \quad (6.1)$$

where $N_i(T)$ is the number of particles of type i occurring in the process T , and $[T]$ is the set of particles in the process T .

For generality we may consider, in these formulas, that the intrinsic parities of initial and final particles are represented by relative reciprocals (inverses), since this condition is already imposed by the no-scattering case in which $\epsilon(T) = 1$. Specifically, in (6.1) the ω_i will be considered to be replaced by ω_i^{-1} for initial particles. This added generality becomes effective only if the ω_i are different from plus or minus unity, a possibility we mean to include.

It is not obvious that a set of particle intrinsic parities can be found. And if their existence can be demonstrated the question of their uniqueness arises.

If there are conservation laws, such as charge conservation, that forbid the occurrence of certain processes, then the particle intrinsic parities, if they exist, are certainly not unique. An additive conservation law requires, for occurring processes, that

$$\sum_{[T]} q_i = \sum q_i = 0, \quad (6.2)$$

where q_i is the number of units of the conserved quantity carried by particles of type i . The contributions associated with initial particles are to appear in (6.2) with reversed signs. But then all M functions are invariant under multiplication by $\exp(2\pi i \sum q_i \alpha)$, for arbitrary α . This implies that if $\omega = (\omega_i)$ is one set of particle intrinsic parities then $\omega' = (\omega'_i)$, given by

$$\omega'_i = \omega_i \exp(2\pi i q_i \alpha), \quad \checkmark$$

example
↓

is another set, since

$$\prod_{(T)} \omega_t = \prod_{(T')} \omega'_t \quad (6.3)$$

for any occurring process T .

For the case in which there are several independent additive conservation laws restricting the class of occurring processes, let A be the matrix whose integer elements A_{tg} are the units of the conserved quantity labeled by g and carried by particles of type t . It will always be possible to use integers, since noncommensurate units give independent conserved quantities and any common factor can—and will—be factored out. The requirement that the conservation laws be independent means that the N_g columns of A_{tg} are linearly independent. The number of particle types N_t will be assumed finite, and N_t must be at least as large as N_g . Generally, A will have more rows than columns.

The direct extension of the above argument shows that if $\omega = \{\omega_t\}$ is a set of particle intrinsic parities, then $\omega' = \{\omega'_t\}$, given by

$$\omega'_t = \omega_t \exp(2\pi i \sum_g A_{tg} \alpha_g), \quad (6.4)$$

is another set.

There is an N_g -fold ambiguity in the assignment of particle intrinsic parities associated with the gauge transformation (6.4). The question now posed is whether a set of particle intrinsic parities exists and, if it does exist, whether it is unique to within this gauge transformation. It will be assumed, for the moment, that all processes not forbidden by the additive conservation laws do occur. Multiplicative conservation laws will be discussed later.

It is clear that if the process intrinsic parities are completely arbitrary one can never find a set of particle intrinsic parities. Specifically, if the particles occurring in some process are the sum of the particles occurring in a set of other processes, then the intrinsic parity of the first process must be the product of the intrinsic parities of the processes of the set. Otherwise one would obtain from (6.1) an immediate contradiction. However, it is a basic assumption of S -matrix theory (the decomposition law ansatz⁴) that the R functions contain terms that are products of the R functions for the various separate processes that can occur. This implies the following composition requirement for process intrinsic parities: The intrinsic parity of a process must equal the product of the intrinsic parities of any set of processes that, combined, have the same set of initial and final particles, since the R function for the former contains a term that is a product of the R functions of the latter. This composition requirement also guarantees the consistency of the assignments of process intrinsic parities with the unitarity relations.

The mathematical problem may now be framed. The processes allowed by the conservation laws can be

characterized by vectors C_p with integer components C_{pt} giving the number of particles of type t occurring in the process. Initial particles will be represented by negative numbers. The conservation law requirement is, then,

$$\sum_t C_{pt} A_{tg} = 0 \quad \text{for all } p, g. \quad (6.5)$$

For every C_p there is a given intrinsic parity ϵ_p . These are subject to the composition requirement: If there is a set of integers b_p such that

$$\sum_p b_p C_{pt} = 0 \quad \text{for all } t, \quad (6.6a)$$

then

$$\prod_p \epsilon_p^{b_p} = 1. \quad (6.6b)$$

The positive b_p represent the number of times the corresponding processes p occurs in one set of processes, and the negative b_p represent the number of times processes p occur in another set. If the two sets combine to give the same set of particles, then the products of the two sets of intrinsic parities must be equal. In constructing this formulation of the composition requirement the fact is used that the process intrinsic parity must be unchanged if the same particle is added both initially and finally. This is a part of the composition requirement that follows from the positiveness of the intrinsic parity of the no-scattering parts of scattering functions. A single C_p actually represents the whole class of processes generated from one by adding the same sets of initial and final particles. The problem is to show the existence of a set of ω_t satisfying

$$\prod_t \omega_t^{C_{pt}} = \epsilon_p \quad \text{for all } p, \quad (6.7)$$

and the uniqueness of this solution to within the gauge transformation

$$\omega'_t = \omega_t \exp(2\pi i \sum_g A_{tg} \alpha_g). \quad (6.8)$$

This algebraic problem may be solved as follows. Note first that it is not necessary to stick to the original conserved quantities. Linear combinations of conserved quantities are also conserved, and one can choose any set of N_g linearly independent ones. It is possible to choose the conserved quantities in such a way that for each conserved quantity g there is a particular linear combination of particles, specified by integer coefficients M_{gt} , such that this combination carries one unit of this conserved quantity and zero units of all the others. (The proof of this statement will be deferred until the end of the argument.) The intrinsic parity of this group of particles, defined by

$$\epsilon_g = \prod_t \omega_t^{M_{gt}}, \quad (6.9)$$

is transformed by the gauge transformation into

$$\begin{aligned} \epsilon'_g &= \prod_t (\omega'_t)^{M_{gt}} = \epsilon_g \exp(2\pi i \sum_t M_{gt} A_{tg} \alpha_g) \\ &= \epsilon_g \exp(2\pi i \alpha_g), \end{aligned} \quad (6.10)$$

⁴ See SI, Appendix I.

Marginal uniqueness

where the last line follows from the property of M ,

$$\sum_t M_{gt} A_{tg} = \delta_{gg}. \quad (6.11)$$

According to (6.10) the ϵ_g can be fixed arbitrarily by means of the gauge transformations. Conversely, the specification of the ϵ_g completely removes the ambiguity in the ω_t associated with the gauge transformation, since the α_g and the A_{tg} in (6.4) are both integers.

For every particle there is an allowed process involving only this particle and the multiples of the groups of particles constructed above. In particular, for a particle of type s the process represented by

$$C_{st} = \delta_{st} - \sum_g A_{sg} M_{gt} \quad (6.12)$$

will be an allowed process, since, by virtue of (6.11),

$$\sum_t C_{st} A_{tg} = A_{sg} - A_{sg} = 0. \quad (6.13)$$

(A and M are generally nonsquare.) Thus, there will be an equation from the set (6.7) that reads

$$\begin{aligned} \epsilon_s &= \prod_t \omega_t^{C_{st}} \\ &= \omega_s \prod_t \omega_t^{-\sum_g A_{sg} M_{gt}} \\ &= \omega_s \prod_g \epsilon_g^{-A_{sg}}, \end{aligned} \quad (6.14)$$

which can be solved to give

$$\omega_s = \epsilon_s \prod_g \epsilon_g^{A_{sg}}. \quad (6.15)$$

This shows that if a solution exists it is unique, aside from the ambiguity given by the gauge transformation.

To show that the solution exists one must confirm that the solution of the particular equations from (6.7) used above will also satisfy the remaining infinite number of equations in (6.7). Also, one must show that the solution (6.15) for the ω_s is consistent with the values of the ϵ_g , which are products of ω_t .

To verify the consistency of (6.15) with (6.9) one must show that

$$\prod_g [\epsilon_s \prod_{g'} \epsilon_{g'}^{A_{sg'}}]^{M_{gs}} = \epsilon_g. \quad (6.16)$$

In virtue of (6.11) this is equivalent to the condition

$$\prod_g \epsilon_s^{M_{gs}} = 1, \quad (6.17)$$

which is a special case of (6.6), with the C_{st} given by (6.12). The self-consistency is thus demonstrated.

To show that this solution will satisfy all the equations (6.7), consider an arbitrary process, represented by C_{pt} . The left-hand side of (6.7) is

$$\prod_t \omega_t^{C_{pt}} = \prod_s [\epsilon_s \prod_{g'} \epsilon_{g'}^{A_{sg'}}]^{C_{ps}}, \quad (6.18)$$

which in virtue of (6.5) becomes

$$\prod_t \omega_t^{C_{pt}} = \prod_s \epsilon_s^{C_{ps}}. \quad (6.19)$$

The C_{ps} can be written as

$$C_{ps} = \sum_{s'} C_{ps'} C_{s's}, \quad (6.20)$$

where $C_{s's}$ is the particular set of C 's given by (6.12), and (6.5) is used again. Thus Eq. (6.6) can be invoked to give

$$\epsilon_p = \prod_s \epsilon_s^{C_{ps}}, \quad (6.21)$$

which, when combined with (6.19), gives the desired (6.7). Thus, the solution of the system of Eqs. (6.5) through (6.7) exists and is unique aside from gauge transformations (6.8). If the ϵ_g are chosen to be real, the particle intrinsic parities will be real.

It remains to be demonstrated that one can choose the conserved quantities in such a way that for each conserved quantity, labeled by g , there is a linear combination of particles, with integer coefficients M_{gt} , that carries one unit of the conserved quantity g and zero units of the other conserved quantities.

The original conservation laws are represented by the matrix A . Factors common to all elements of a column can be divided out. Consider the first column. It is possible to find a linear combination, with integral coefficients, of the particle types such that this combination bears one unit of the first conserved quantity. Starting with any two elements of the first column of A one can find, as is well known,⁶ integral coefficients such that the corresponding linear combination of these two elements is the greatest factor common to them. This combination may then be combined with any other element, using integral coefficients, to give the greatest factor common to these. And one can continue. Since the greatest common factor of all the elements of the first column is unity, the procedure must produce after a finite number of steps a linear combination of rows corresponding to a single unit of the first conserved quantity. The remaining conserved quantities can now be changed by subtracting off appropriate integer multiples of the first conserved quantity so that this particular combination bears zero units of all other conserved quantities. Considering these operations as matrix operations on A , this matrix is now transformed to a form with one in, say, the first row of the first column, and zeros in the rest of the first row. Common factors may again be divided out and the procedure applied to the second column. The greatest common factor, unity, can be moved to the second row and appropriate multiples of the second column subtracted from all other columns. Repeated application of the procedure gives the desired result.

⁶ See, for instance, A. A. Albert, *Introduction to Algebraic Theories* (University of Chicago Press, Chicago, Illinois, 1941), pp. 9, 118.

The linear independence of the original columns of A ensures that these operations never lead to a column of zeros. $\downarrow$

It is possible to take intrinsic parities of all particles either plus or minus unity, and there is no reason to do otherwise. Starting from the originally real set, a gauge transformation would generally take the particle intrinsic parities to complex values. One could contemplate adding a certain gauge transformation for each of the conservation laws. If the various α_p were taken to be incommensurate with each other and with unity, then the absolute physical requirement that the intrinsic parity of all processes be real would, by itself, entail the existence of all the conservation laws. One might try, therefore, to claim that the conservation laws were a consequence of invariance under spatial inversion. Such a terminology must be regarded as illogical. The question of whether invariance under spatial reflection is maintained or not is simply the question of whether $|R(K_p)|$ equals $|R(K)|$. Any restriction more stringent than this is not an expression of invariance under spatial inversion alone.

To get a simple picture of the freedom available in the assignment of intrinsic parities, it is helpful to use the notion of a null group. A set of particles will be called a null group if it carries zero units of all constants of the motion. According to the above analysis, the product of the intrinsic parities of the particles of a null group is specified uniquely, since it is invariant with respect to the gauge transformations. This product will be called the intrinsic parity of the null group. Since all processes not forbidden by the conservation laws are assumed to occur, one can consider for every occurring process A a related occurring process A_n that differs from A by the presence of the null group n of final particles. The intrinsic parity of the null group is then

$$\epsilon_n = \epsilon(A_n) / \epsilon(A), \quad (6.22)$$

the quotient of the two observable process intrinsic parities. That this quotient is independent of A is ensured by the above analysis. It can also be seen directly from the composition requirement, which implies that

$$\epsilon(A_n)\epsilon(B) = \epsilon(B_n)\epsilon(A), \quad (6.23)$$

since the sum of the initial and final particles of the set of A_n and B and of B_n and A are the same. [Recall that $\epsilon^2(T)$ is unity.] Null groups of four or more particles are associated with occurring processes; one can switch certain of the final particles to initial antiparticles. The relationship between the intrinsic parities of the null group and the various associated processes obtained in this way is fixed by the results of Sec. V. Particle-antiparticle pairs are simple null groups whose intrinsic parities are plus or minus one for bosons or fermions, respectively. Self-conjugate particles are the simplest type of null group.

The discussion, so far, has been based on the simplifying assumption that all processes not forbidden by additive conservation laws actually occur. Of course, selection rules may also be expressed by multiplicative conservation laws. However, these generally do not affect the analysis. As mentioned before, the processes represented by any C_p constitute a whole class of processes that differ from one another by arbitrary numbers of particles occurring both initially and finally. Multiplicative conservation laws generally forbid only certain of these. But the occurrence of any one is sufficient for the analysis, since the corresponding ϵ_p is then physically determined.

There is one type of multiplicative law that does forbid the occurrence of all processes of classes represented by certain C_p 's. These are multiplicative conservation laws that can be represented by an additive conservation law that is valid to within multiples of some modulus.

The selection rule forbidding processes involving an odd number of fermions is a multiplicative law of this type. In actual practice this particular law is evidently not needed, since the selection rule is already guaranteed by the additive conservation laws of baryon and lepton number, but it would be necessary if, for instance, there were self-conjugate fermions.

To extend the analysis to include additive "quasi-conservation" laws, that are valid, modulo m , a type of fictitious momentumless particle carrying m units of the quasi-conserved quantity, can be introduced. These fictitious particles can be considered to supply the missing units of the quasi-conserved quantity, which can then be considered exactly conserved. The previous analysis is then applicable. The intrinsic parity of the fictitious particle can be fixed at plus one, by choice of gauge, and simply ignored. The results are, therefore, the same as before except that the gauge transformation will introduce multiples of $(\omega_f)^{1/m}$ into the intrinsic parities of the physical particles, ω_f being the original real intrinsic parity of the fictitious particle. If ω_f is minus one the new intrinsic parities of the real particles may no longer be real. An alternative procedure would be to keep the intrinsic parities real, but simply add the negative unit of intrinsic parity for each missing m units of the quasi-conserved quantity. This second procedure departs somewhat from the original program, but it is just as useful a method for cataloging the angular momentum selection rules.

If one uses the original procedure, in which the process intrinsic parities are factorized into contributions from the participating particles, the intrinsic parity of a self-conjugate fermion is forced to be purely imaginary. This follows immediately from the result of Sec. V, since the intrinsic parity of a pair of these particles is minus one. Indeed, if the only selection rule were the one requiring the total number of fermions—initial plus final—to be even, then the intrinsic parities

of bosons would necessarily be purely real and those of fermions purely imaginary. As this selection rule follows from the basic requirement of Lorentz invariance, whereas the baryonic and leptonic and the other conservation laws are not required by our general postulates, it may be sensible to choose the intrinsic parity assignments consistent with the possibility that the former is the only absolute conservation law—to take the intrinsic parities of bosons either plus or minus unity and the intrinsic parities of fermions either plus or minus the imaginary unit. The intrinsic parities of each particle would then become the same as that of its antiparticle, and self-conjugate combinations would have well-defined intrinsic parities. This condition removes most of the ambiguity associated with the gauge transformations, and is perhaps to be preferred over the arbitrary convention that the intrinsic parities be chosen real, since the latter precludes the possibility of self-conjugate fermions. That it is possible to choose the particle intrinsic parities in this way follows from the above analysis, provided the restriction to even numbers of fermions is the only selection rule represented by a nonadditive conservation law that forbids the occurrence of all the processes represented by any C_p .

VII. COMPARISON WITH OTHER TREATMENTS

The point of view regarding intrinsic parities developed here grows naturally out of an S-matrix philosophy, but is presumably not restricted to this approach. It is, in fact, essentially the view that has always been favored by the author. It differs somewhat, however, from what seems to be the prevailing view.

The standard approach⁶ is to start with the idea that there may be a certain transformation on the field operators that can be called the parity operation. This operation must express the field operators at each point in space-time in terms of the field operators at the point obtained by inversion through the origin in space, with time unchanged. If this operator takes all observable quantities into observable quantities, and if the transformation of these observables is consistent with the classical physical meaning of spatial inversion, then the operation may be called a possible parity operation. It is specified that particles are to be con-

sidered as transforming into themselves, not their antiparticles, under spatial inversion, this being a convention that distinguishes a "parity" operation from its product with antiparticle conjugation.

Divergences between this conventional attitude and the one adopted in the present paper may be noted:

First, the existence of particle intrinsic parities is assumed right at the start of the usual approach. The transformation on each field will contain a possible phase factor and the states constructed using these operators will, under the transformation, be multiplied by these factors, which are, then, essentially the particle intrinsic parities. These quantities are therefore introduced at the outset; their existence is never in question. In the present approach the starting point is a possible invariance of transition probabilities under spatial inversion. The notion of a particle intrinsic parity emerges only after analysis, by a construction based on observable quantities; the discussion is not predicated on the supposition that the physically observed invariance is a manifestation of some corresponding symmetry operation, of a particular form, on the field operators of an abstract Hilbert Space.

A second difference revolves about the question of what quantities are observables. For Wick, Wightman, and Wigner⁶ this rather abstruse question is the key to the entire situation. Since the parity operation is required, from their point of view, to give well-defined and physically permissible values to all observable quantities, the question of what quantities are observable becomes an essential one. This leads to the question whether one can measure the relative phases between states corresponding to different values of good quantum numbers. For z components of angular momentum the relative phases of the various eigenstates are clearly observable. But for different charge states the question is unresolved. Yet questions regarding intrinsic parities devolve to this perhaps insoluble problem of whether such phases are, or are not, measurable.

In the S-matrix approach developed here particle intrinsic parities are nothing more or less than a factorization of the observable "angular momentum parity defect" $(-1)^{2J}$. Questions like whether or not all observables are properly transformed under some formal operation do not enter; particle intrinsic parities are merely a convenient device for cataloging the angular momentum selection rules implied by an invariance of transition probabilities under spatial inversion.

⁶ G. C. Wick, A. S. Wightman, and E. Wigner, Phys. Rev. 88, 101 (1952). This reference will be taken as the standard in the field. Some elaboration of what these authors say has been made, however, so the reader should go to the reference for an exact statement of their position.

dumps
m.H.S.

Dear Phil

I thought you might be
interested in this paper.

See last paragraph of Sec VI.

✓
Hm

Lee + Wick

Space Inversion, Time Reversal, and Other Discrete Symmetries in Local Field Theories*

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(Received 28 February 1966)

The general algebraic relations between space inversion, time reversal, and the internal symmetry group are analyzed within the framework of a Lorentz-invariant local field theory. The problem of unitary representations of the full Poincaré group including the space and time reflection operators has been studied by Wigner, and the representations are classified into 4 cases. It is shown that, with the added assumption of the local field theory, Wigner's cases 2, 3, and 4 either do not occur or can be reduced to his case 1. The concept of minimal group extension is introduced and the related mathematical analysis is given. The symmetry properties under space inversion, time reversal, and other discrete operators such as charge conjugation are analyzed separately for each of the three known interactions: strong, electromagnetic, and weak.

I. INTRODUCTION

OUR views of discrete symmetries such as space inversion P , time reversal T , and charge conjugation C have undergone great changes in recent years. Since 1957, it has been well established¹ that both P and C symmetries are only approximately valid. Rather naturally, this discovery led immediately to speculations as to the possible need of a profound revision of some of our concepts. Until the recent discovery² of the $\pi^+\pi^-$ decay mode of the long-lived K_2^0 meson, however, it was possible to retain intact the structure of the full Poincaré group and in particular to believe in the essential symmetry between left and right by using CP , instead of P . From a fundamental point of view, therefore, the strong evidence now existing of a CP violation in K_2^0 decay, and the concurrent indirect conclusion that T is also not an exact symmetry, are a more decisive blow to our notions on geometric symmetry principles than earlier results.

Strictly speaking, since the very existence of symmetry operators such as T and P , or CP , is deduced from the alleged equivalence of certain reference systems, the presence of phenomena violating that equivalence implies that the operators themselves cannot be exactly defined. The language employed most often in describing the situation is in fact somewhat inconsistent. One says: there is a P or CP , etc., operator and in specific theories one often proceeds to give exact definitions of these operators; subsequently one assumes

a Hamiltonian containing terms which do not commute with P , CP or T , etc. This is the same as saying, however, that these operators do not satisfy exactly the multiplication laws of the coordinate transformations they are alleged to represent within the full Poincaré group. More explicitly, the transformation

$$r \rightarrow -r, \quad t \rightarrow +t,$$

corresponding to P or CP , should commute with a time translation, represented infinitesimally by H , and the transformation

$$r \rightarrow +r, \quad t \rightarrow -t,$$

corresponding to T , transforms the time translation $t \rightarrow t+\tau$ into $t \rightarrow t-\tau$ which, because of the anti-unitary nature³ of T , again implies commutativity of the corresponding operators. Thus the "exact" definition of the symmetry operators is a purely formal convention, that in fact does not satisfy the basic geometrical requirements. One can similarly think of other possibilities of "exact" definitions. For example, one might start with the physical single-particle state $|k, \lambda\rangle$, and define P and T to be, respectively, the unitary and anti-unitary operators that satisfy

$$P|k, \lambda\rangle = \eta_P | -k, -\lambda \rangle, \quad (1.1)$$

and

$$T|k, \lambda\rangle = \eta_T | -k, +\lambda \rangle, \quad (1.2)$$

where k and λ are, respectively, the momentum and the helicity of the particle, and η_T and η_P are phase factors which may depend on λ . (Helicity is defined to be the spin component along the direction of k .)

In equations such as (1.1) and (1.2), the states $|k, \lambda\rangle$ and $| -k, \pm\lambda \rangle$ refer to the same physical particle. The equations, however, have an unambiguous meaning only for an exactly stable particle. The state vector of an unstable particle must sooner or later develop components corresponding to the decay products. No satisfactory generalization of (1.1) or (1.2) exists for such

* This research was supported in part by the U. S. Atomic Energy Commission.

¹ The first experiment that conclusively established the approximate nature of P and C symmetries was made on β decay by C. S. Wu, E. Ambler, R. W. Hayward, D. D. Hoppes, and R. P. Hudson, Phys. Rev. 105, 1413 (1957). This was immediately followed by the observation of the same noninvariance properties in π , μ decays by R. L. Garwin, L. M. Lederman, and M. Weinrich, Phys. Rev. 105, 1413 (1957) and by J. J. Friedman and V. L. Telegdi, Phys. Rev. 105, 1681 (1957). The possibility that P , C , and T are only approximate symmetries was suggested theoretically by T. D. Lee and C. N. Yang, Phys. Rev. 104, 254 (1956) and further discussed by T. D. Lee, R. Oehme, and C. N. Yang, Phys. Rev. 106, 340 (1957).

² J. H. Christenson, J. W. Cronin, V. L. Fitch, and R. Turlay, Phys. Rev. Letters 13, 138 (1964). See also A. Abashian *et al.*, Phys. Rev. Letters 13, 243 (1964).

³ E. P. Wigner, Gött. Nach. Math. Naturw. Kl., p. 546 (1932). For a general discussion of anti-unitary operators see, e.g., E. P. Wigner, *Group Theory* (Academic Press Inc., New York, 1959), Chap. 26.

1. My motivation for reading this : Toller uses certain mysterious phases in his paper on signature and TCP. These phases involve the operators P and TCP which Toller calls s and t. Eg, on page 316 Toller notes that $t^2 = s^2 = (-1)^{2j}$ and $st = ts(-1)^{2j}$. I want to know: is this just some convention (which MS do not use) or is there some significance to these phases.

1. Introduction. With the discovery of T violation = CP violation, you find that you can no longer "rescue" parity by combining it with C. Therefore, the very existence of operators P and T is brought into question. Maybe they only exist for some unperturbed Hamiltonian. Maybe different interactions have their own P, C, T operators? Notation: $\bar{G}$ = internal symmetry group like isospin, elements are S. Operator T and $\bar{S}T$ are equivalent "time-reversal operators" so general T operator called $\bar{T}$, a representative. I think the idea here is this: if you turn off your weaks and E&M, then can't tell p and n apart, so you can't tell whether there is an isospin mixer in your chosen T operator.

This paper will operate in a field theory context. I am not yet sure what $\bar{T}$ is going to be.

conclusions: after playing with the single-particle states of Jacob and Wick and ignoring field theory altogether, I am able to make a phase choice so as to completely duplicate the equations of Lee Wick. However, there is nothing at all magic about this business. The only definite convention-free results I get are these: $T^2 = (-1)^{2s}$ and $(PT)^2 = (-1)^{2s}$. It seems that you cannot change these results by changing conventions. In the special case of field theory with spinor particles, these two equations are in fact both out. But I have shown them generally. The value of P^2 however is a convention; eg, Martin and Spearman differ from Toller. This convention determines whether or not P and T will commute. By allowing fermions to have imaginary parity, you start on a path which leads to the Lee-Wick conditions.

The Lee-Wick Dirac Transformation Algebra

In my notes in QED binder, last section, I have an old page which shows the effects of P, C, T, and TCP on field operators. I suspect that Lee-Wick algebra just derives from these rules. I will try some examples:

$$\Theta \psi(x) \Theta^\dagger = i \gamma^5 \psi^\dagger(-x), \quad \Theta = PCT.$$

$$\Theta \psi^\dagger(x) \Theta^\dagger = i \gamma^5 \psi(-x)$$

Note: only applying $\dagger$ to operators: $[i \gamma^5 \psi^\dagger(-x)]^\dagger$

$$\text{Then } \Theta^2 \psi(x) \Theta^2 = \Theta [\Theta \psi(x) \Theta^\dagger] \Theta^\dagger = i \gamma^5 \Theta \psi^\dagger(-x) \Theta^\dagger = i \gamma^5 i \gamma^5 \psi(x) = -\psi(x)$$

$$\text{Whereas: } \Theta^2 \phi(x) \Theta^2 = \Theta [\Theta \phi(x) \Theta^\dagger] \Theta^\dagger = \Theta [\phi(-x)] \Theta^\dagger = \phi(x)$$

So:

$$\boxed{\begin{aligned} \Theta^2 \phi(x) \Theta^2 &= + \phi(x) \\ \Theta^2 \psi(x) \Theta^2 &= - \psi(x) \end{aligned}}$$

$$\begin{aligned} \text{Next: } (\Theta P) \psi(x) (\Theta P)^\dagger &= \Theta [P \psi(x) P^\dagger] \Theta^\dagger = \gamma_0 \Theta \psi(-x, t) \Theta^\dagger \\ &= \gamma_0 i \gamma^5 \psi^\dagger(x, -t) \end{aligned}$$

$$(P \Theta) \psi(x) (P \Theta)^\dagger = P [\Theta \psi(x) \Theta^\dagger] \dots = i \gamma^5 \gamma_0 \psi^\dagger(x, t)$$

Reverse order of ops.

$$\text{So: } P \Theta: b_s^\dagger(p) \Theta = \dots$$

$$\boxed{\begin{aligned} (\Theta P) &= (P \Theta) \cdot (-1)^{2s} \\ (\Theta T) &= (T \Theta) \cdot (-1)^{2s} \end{aligned}}$$

I write this for general spin field, but have only shown it in one case. The reason you get a minus sign for spinor field is that doing these operations in reverse order causes gamma matrices to appear in reverse order and they anticommute.

So, I can derive all these little equations from field theory, but I don't know their content, as Halpern used to like to say. Why should you expect to get minus signs for fermions? Why don't Lee and Wick give a simple reason for this, if there is such a simple reason. Otherwise it is just convention.

What is the evidence pro and con for LeeWick phase having relevant significance?

1) Nowhere in their paper do I find any interpretation of the interesting facts they claim. I.e., you find certain minus sign for spinor field, and this appears in all half-integer spin fields because you can make these out of spinor fields. Nowhere do they associate this phase $(-1)^{2s}$ with any particular meaning/.

2) For my standard hadronic single particle states, I have:

$$CPT |p, m\rangle_H = \eta (-1)^{2s} |p, -m\rangle_H^*$$

$$\Rightarrow (CPT)^2 |p, m\rangle_H = +1 |p, m\rangle_H \quad \text{so} \quad (CPT)^2 = +1 \text{ regardless of spin.}$$

Also:

$$P^2 |p, m\rangle_H = (-1)^{s+m} \eta R_y(\pi) \cdot (-1)^{s+m} \eta R_y(\pi) |p, m\rangle_H$$

$$\text{so} \quad P^2 = +1$$

$$\text{Finally:} \quad T^2 |p, m\rangle_H = (-1)^{s+m} T R_y(\pi) |p, m\rangle_H$$

$$= (-1)^{s+m} \{(-1)^{s+m} R_y(2\pi) |p, m\rangle_H$$

$$= (-1)^{2s+2m-2s} \dots \Rightarrow T^2 = (-1)^{2s}$$

Now that is strange. It seems that this magic spin-dependent phase may have to pop up somewhere or other. With my choice of phases I seem to get this:

$$\begin{array}{lll} P^2 = 1 & PT = TP (-1)^{2s} & C^2 = 1 \\ T^2 = (-1)^{2s} & (PT)^2 = 1 & (PT)T = T(PT) \cdot (-1)^{2s} \\ & & (PT)P = P(PT) \cdot (-1)^{2s} \end{array}$$

This is very much what Lee Wick claims. How did this happen? I was not expecting it? Is it just accident?

Goldstein Owens

(46) then would require C_{yy} to vanish as $\sin^2 \frac{1}{2} \theta$. Thus, the suppression of unnatural parity at the $p\bar{p}$ vertex predicts $A_a \sim \sin \frac{1}{2} \theta$, $A_b \sim \sin \frac{1}{2} \theta$, $C_{yy} \sim \sin^2 \frac{1}{2} \theta$ and all other polarization observables will be suppressed further.

6.4. $p\bar{p} \rightarrow \Lambda^+ X$ with beam polarized

For this reaction in general, there are 20 amplitudes. However, with the target unpolarized, only six combinations of amplitudes can be measured. In the beam fragmentation region, the amplitudes combinations that are measurable are analogous to the amplitudes in $p\bar{p} \rightarrow \Lambda^+ X$, (57). With particle b the unpolarized target, the relevant amplitude combinations are

$$\begin{aligned} \sum_b D_{g+b+, \pm b \pm} &= \sum_b (D_{g+b+, \pm b+}^{(+, +, +)} \pm D_{g+b+, \pm b+}^{(-, -, +)}), \\ \sum_b D_{g+b-, \pm b \pm} &= \sum_b (\pm D_{g+b-, \pm b-}^{(+, +, +)} + D_{g+b-, \pm b-}^{(-, -, +)}), \\ \sum_b D_{g+b+, \pm b \mp} &= \sum_b (\pm D_{g+b+, \pm b-}^{(+, +, +)} + D_{g+b+, \pm b-}^{(-, -, +)}), \end{aligned} \quad (62)$$

which is analogous to (57). The relationships between these definite naturality amplitude combinations and the single and double polarization observables are of the same form as in (58), except for the sum over b. Hence the predictions based on natural parity dominance in (59) should be the same for this reaction.

In the target fragmentation region the Λ is associated with the unpolarized target proton and a different decomposition into definite naturality amplitudes is appropriate. This is done by noting that the beam particle (a) now appears at the lower vertex and the target particle (b) appears at the upper vertices in fig. 2. Then the observables can be written in terms of pairs of definite naturality amplitudes appropriate for the target fragmentation region:

$$s \frac{d^2 \sigma}{dt dM_x^2} = \frac{1}{64\pi^2 s} F, \quad F = 4 \sum_b [D_{g+b+, \pm b+}^{(+, +, +)} + D_{g+b+, \pm b+}^{(-, -, +)}],$$

$$FA = 4 \operatorname{Im} \sum_b [D_{g+b+, \pm b+}^{(+, +, +)} + D_{g+b+, \pm b+}^{(-, -, +)}],$$

$$FP = 4 \operatorname{Im} \sum_b [D_{g+b+, \pm b-}^{(+, +, +)} + D_{g+b+, \pm b-}^{(-, -, +)}],$$

$$FD_{xx} = 4 \sum_b [D_{g+b+, \pm b-}^{(+, +, -)} + D_{g+b+, \pm b-}^{(-, -, -)}],$$

$$FD_{xz} = 4 \sum_b [D_{g+b+, \pm b-}^{(+, +, -)} + D_{g+b+, \pm b-}^{(-, -, -)}],$$

$$\begin{aligned} FD_{yy} &= 4 \sum_b [D_{g+b+, \pm b-}^{(+, +, +)} + D_{g+b+, \pm b-}^{(-, -, +)}], \\ FD_{zx} &= 4 \sum_b [D_{g+b+, \pm b-}^{(+, +, -)} + D_{g+b+, \pm b-}^{(-, -, -)}], \\ FD_{zz} &= 4 \sum_b [D_{g+b+, \pm b+}^{(+, +, -)} + D_{g+b+, \pm b+}^{(-, -, -)}], \end{aligned} \quad (63)$$

where the superscripts now refer to the naturality coupling at the unpolarized $p \rightarrow \Lambda(c)$ vertex, the $\Lambda(c') \rightarrow$ unpolarized p vertex, and the polarized $p(a) \rightarrow$ polarized $p(a')$ vertex, respectively. For natural parity dominance at the polarized proton vertex, D_{xx} , D_{xz} , D_{zx} , and D_{zz} will all be small. The angular constraints (51) on the observables in the target fragmentation region ($\cos \frac{1}{2} \theta$ small) along with this natural parity dominance, then require that $D_{yy} \sim \cos^2 \frac{1}{2} \theta$ ($D_{g+b-, \pm b+}$ must be evasive). Hence, for this hypothesis, all the polarization observables will be small in the target fragmentation region.

7. Conclusion

The formalism presented herein provides a convenient framework, both from the experimental and theoretical point of view, in which the spin dependence of inclusive reactions may be studied. The constraints due to conservation of parity and angular momentum conservation have been discussed and their effect on the various inclusive observables has been shown to be of importance. In particular, the collinear constraints provide a substantial suppression of these observables over a significant portion of the relevant kinematic regions. The three-body amplitude naturality decomposition relevant for triple-Regge model analyses has been presented. This, together with the above constraints, allows a number of interesting predictions to be made.

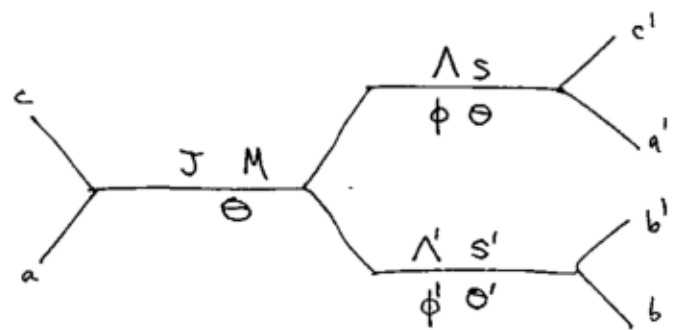
Appendix

NP B103 (1976)

Definite naturality three-body amplitudes

Extending the procedure used to write the angular momentum decomposition of the three-body amplitudes, we can form linear combinations of amplitudes to which reggeon exchanges of definite naturality contribute in the fragmentation and triple-Regge kinematic regions of the Regge-Mueller model. This procedure is a generalization of the work of Salin and Soffer [6].

Consider the amplitude for $abc \rightarrow abc$ continued into the physical region for $a + \bar{c} \rightarrow (\bar{b}b') + (a'\bar{c}')$ (see fig. 2). We will form definite angular momentum states for the $(a\bar{c})$, $(\bar{b}b')$, $(a'\bar{c}')$ systems, combine these to obtain definite parity combina-



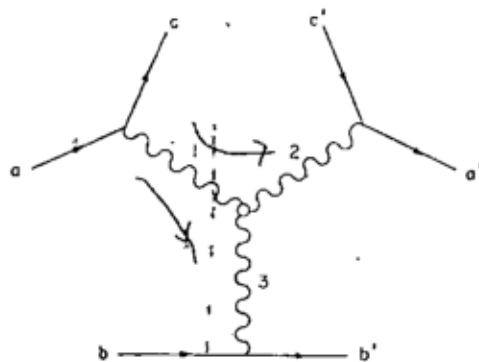


Fig. 2. Triple-Regge diagram showing the notation used for the naturality decomposition.

tions of the $(a\bar{c})$ system to leading order in the cosine of the relevant scattering angle, and then cross back to the three-body physical region.

Begin in the $(a\bar{c})$ center-of-mass frame in which $p_a = -p_{\bar{c}} = p$ are the incoming momenta and $Q = q_{\bar{b}} + q_{b'} = -q_{a'} - q_{\bar{c}'}$ are the outgoing momenta. The two body helicity state $|b\bar{b}'\rangle$ can be decomposed into states of angular momentum S

$$\langle \bar{b}b' | = \sum_{S\Lambda} D_{\Lambda, \bar{b}-b'}^{S*}(\theta_q, \phi_q) \langle S\Lambda \bar{b}b' |,$$

where θ_q and ϕ_q orient the single particle momentum $q_{\bar{b}}$ with respect to Q which is taken along the polar axis. Similarly

$$\langle a'\bar{c}' | = \sum_{S'\Lambda'} D_{\Lambda', a'-\bar{c}'}^{S'*}(\theta'_q, \phi'_q) \langle S'\Lambda' a'\bar{c}' |,$$

where θ'_q and ϕ'_q orient $q_{a'}$ with respect to Q , and

$$|a\bar{c}\rangle = \sum_{J,M} d_{M, a-\bar{c}}^J(\theta) |JM a\bar{c}\rangle,$$

where θ is the scattering angle $\cos \theta = p \cdot Q / |p||Q|$. Then the $a + \bar{c} \rightarrow (\bar{b}b') + (a'\bar{c}')$ amplitude can be written

$$F_{(\bar{b}b')(a'\bar{c}'), a\bar{c}} = \sum_{J S \Lambda S' \Lambda'} d_{\Lambda-\Lambda', a-\bar{c}}^J(\theta) D_{\Lambda, \bar{b}-b'}^{S*}(\theta_q, \phi_q) \times D_{\Lambda', a'-\bar{c}'}^{S'*}(\theta'_q, \phi'_q) \langle S\Lambda \bar{b}b'; S'\Lambda' a'\bar{c}' | T^J | JM a\bar{c} \rangle. \quad (A.1)$$

Now form the states

$$|a\bar{c}\rangle \pm = \sqrt{\frac{1}{2}} [|JM a\bar{c}\rangle \pm \eta_a \eta_{\bar{c}} (-1)^{S_a+S_{\bar{c}}-J} |J - a - \bar{c}\rangle], \quad (A.2)$$

where η_a and $\eta_{\bar{c}}$ are the intrinsic parities of a and $\bar{c}$, S_a and $S_{\bar{c}}$ are their spins, and $v=0$ for J an integer but $\frac{1}{2}$ for J a half-integer. These states have definite naturality $\pm(-1)^{J-v}$. Inserting these states into (A.1), with

$$|a\bar{c}\rangle = \sqrt{\frac{1}{2}} [|JM a\bar{c}\rangle_+ + |JM a\bar{c}\rangle_-],$$

and observing that

$$d_{\Lambda-\Lambda', a-\bar{c}}^J(\theta) \approx (-1)^{\bar{c}-a} d_{\Lambda-\Lambda', -a-\bar{c}}^J(\theta)$$

to leading order in $\cos \theta$ (which is proportional to s/M_X^2), we obtain the amplitudes for naturality $\pm(-1)^{J-v}$ of the $a\bar{c}$ system

$$F_{(\bar{b}b')(a'\bar{c}'), a\bar{c}}^{\pm} = F_{(\bar{b}b')(a'\bar{c}'), a\bar{c}} \pm \left[\uparrow \eta_a \eta_{\bar{c}} \right] (-1)^{S_a+S_{\bar{c}}-J-v-a-\bar{c}} \times F_{(\bar{b}b')(a'\bar{c}'), -a-\bar{c}}. \quad (A.3)$$

to leading order in $\cos \theta$.

We next cross into the three-body forward scattering region,

$$g_{\lambda_a \lambda_b \lambda_{\bar{c}}, \lambda_{a'} \lambda_{b'} \lambda_{\bar{c}'}} = \sum_{\substack{a a' \\ b b' \\ \bar{c} \bar{c}'}} d_{\lambda_a a'}^{S_a}(\chi_a') d_{\lambda_b b'}^{S_b}(\chi_b') d_{\lambda_{\bar{c}} \bar{c}'}^{S_{\bar{c}}}(\chi_{\bar{c}}') \times d_{\lambda_a \lambda_{a'}}^{S_a}(\chi_a) d_{\lambda_b \lambda_{b'}}^{S_b}(\chi_b) d_{\lambda_{\bar{c}} \lambda_{\bar{c}'}}^{S_{\bar{c}}}(\chi_{\bar{c}}) F_{(\bar{b}b')(a'\bar{c}'), a\bar{c}}, \quad (A.4)$$

where the χ 's are the relevant crossing angles. Using the properties of the d functions, it follows that

$$g_{a'b'\bar{c}', ab\bar{c}}^{\pm(a, \bar{c})} = g_{a'b'\bar{c}', ab\bar{c}} \pm \eta_a \eta_{\bar{c}} (-1)^{S_a+S_{\bar{c}}-a+\bar{c}-v} g_{a'b'\bar{c}', -ab-\bar{c}}, \quad (A.5)$$

has $(a\bar{c})$ coupling only to states of definite naturality $\pm(-1)^{J-v}$, to leading order in s/M_X^2 .

The same procedure could have been applied to the amplitude for $\bar{a}' + c' \rightarrow (\bar{b}, b') + (\bar{a}, c)$, so that an analogous conclusion is that

$$g_{a'b'\bar{c}', ab\bar{c}}^{\pm(\bar{a}', c')} = g_{a'b'\bar{c}', ab\bar{c}} \pm \eta_{\bar{a}'} \eta_c (-1)^{S_{\bar{a}'}+S_c-a'+\bar{c}'-v} g_{-a'b'-\bar{c}', ab\bar{c}}, \quad (A.6)$$

has $(\bar{a}'c')$ coupling only to states of definite naturality $\pm(-1)^{J-v}$ to leading order in s/M_X^2 . Finally, consideration of the reaction $b + \bar{b}' \rightarrow (c\bar{a}) + (a'\bar{c}')$ leads to the conclusion that

$$g_{a'b'\bar{c}', ab\bar{c}}^{\pm(b, \bar{b}')} = g_{a'b'\bar{c}', ab\bar{c}} \pm (-1)^{b-b'} g_{a'-b'\bar{c}', a-b\bar{c}}, \quad (A.7)$$

has $(b, \bar{b}')$ coupling only to states of definite naturality $\pm(-1)^J$ to leading order in M_X^2 .

Given the amplitudes for definite naturality of any of the pairs that couple to Regge poles in the triple-Regge limit, it is then possible to form combinations with definite naturality at all three vertices (see fig. 2) by applying relations (A.5), (A.6), and (A.7) simultaneously. Let ξ_1, ξ_2, ξ_3 be the naturality (+1 or -1) that couples to the $(a\bar{c}), (ac'), (b\bar{b}')$ vertices, respectively. Then definite triple naturality amplitudes (to leading order in M_X^2 and s/M_X^2) will be

$$g_{a'b'\bar{c}', ab\bar{c}}^{(\xi_1, \xi_2, \xi_3)} = \frac{1}{8}(1 + \xi_1 \xi_2 \xi_3) [g_{a'b'\bar{c}', ab\bar{c}}^{S_a + S_c - u - a' + \bar{c}} + \xi_1 \eta_a \eta_{\bar{c}} (-1)^{S_a + S_c - u - a' + \bar{c}} \\ \times g_{a'b'\bar{c}', -ab - \bar{c}}^{S_a + S_c - u - a' + \bar{c}} + \xi_2 \eta_a \eta_{\bar{c}} (-1)^{S_a + S_c - u - a' + \bar{c}} g_{-a'b' - \bar{c}', ab\bar{c}}^{S_a + S_c - u - a' + \bar{c}} \\ + \xi_3 (-1)^{a - a' - \bar{c} + \bar{c}'} g_{-a'b' - \bar{c}', -ab - \bar{c}}^{S_a + S_c - u - a' + \bar{c}}], \quad (A.8)$$

where parity conservation relations (17) have been used to simplify the form. Note that (due to parity conservation) $\xi_1 \xi_2 \xi_3$ must be +1 in order for this expression to be non-zero. Hence, naturality combinations that are odd will not occur at all.

The inverse relations will be of use in the decomposition of observables into definite triple naturality amplitudes in the text.

$$g_{a'b'\bar{c}', ab\bar{c}}^{(+, +, +)} = g_{\Lambda}^{(+, +, +)} + g_{\Lambda}^{(+, -, -)} + g_{\Lambda}^{(-, +, -)} + g_{\Lambda}^{(-, -, +)}, \\ g_{a'b'\bar{c}', -ab - \bar{c}}^{(+, +, +)} = \eta_a \eta_{\bar{c}} (-1)^{S_a + S_c - u + a - \bar{c}} [g_{\Lambda}^{(+, +, +)} + g_{\Lambda}^{(+, -, -)} - g_{\Lambda}^{(-, +, -)} - g_{\Lambda}^{(-, -, +)}], \\ g_{-a'b' - \bar{c}', ab\bar{c}}^{(+, +, +)} = \eta_a \eta_{\bar{c}} (-1)^{S_a + S_c - u + a' - \bar{c}'} [g_{\Lambda}^{(+, +, +)} - g_{\Lambda}^{(+, -, -)} + g_{\Lambda}^{(-, +, -)} - g_{\Lambda}^{(-, -, +)}], \\ g_{-a'b' - \bar{c}', -ab - \bar{c}}^{(+, +, +)} = (-1)^{a - a' - \bar{c} + \bar{c}'} [g_{\Lambda}^{(+, +, +)} - g_{\Lambda}^{(+, -, -)} - g_{\Lambda}^{(-, +, -)} + g_{\Lambda}^{(-, -, +)}], \quad (A.9)$$

where $\Lambda \equiv a'b'\bar{c}', ab\bar{c}$. Note that if only one particular set of naturalities, (ξ_1, ξ_2, ξ_3) , contributes, the collinearity constraints (21) in the $\theta \rightarrow 0$ limit will force some of the amplitudes to have additional powers of $\sin \frac{1}{2}\theta$ (i.e. some amplitudes will evade.) If several sets of naturalities contribute, there will be the possibility of conspiracy.

Note also, that for single factorizable reggeon contributions to definite naturality amplitudes, the resulting amplitude will be proportional to the product of residue functions

$$g_{a'b'\bar{c}', ab\bar{c}}^{(\xi_1, \xi_2, \xi_3)} \propto \beta_{ac}^{(1)}(t) \beta_{a'c'}^{(2)}(t) \beta_{bb'}^{(3)}(0). \quad (A.10)$$

Then the collinear constraints (21) for $t \rightarrow 0$ ($\sin \frac{1}{2}\theta \rightarrow 0$) will be satisfied (evasively) since $\beta_{\lambda\mu}(t) \sim (\sin \frac{1}{2}\theta)^{|\lambda - \mu|}$. For factorizable reggeon amplitudes this implies

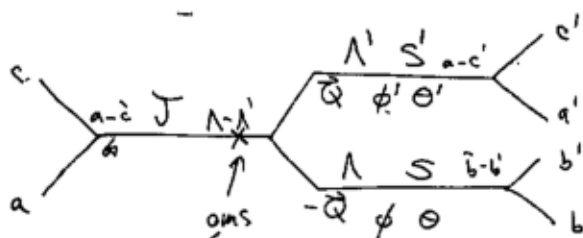
$$g_{a'b'\bar{c}', ab\bar{c}}^{(\xi_1, \xi_2, \xi_3)} \propto \delta_{bb'} (\sin \frac{1}{2}\theta)^{0 - c' + a' - c}, \quad (A.11)$$

which immediately requires that the (bb') vertex cannot involve helicity flip. That is just the result first shown by Abarbanel and Gross [9] for factorizable poles.

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5/7/77.



→ think physical process
in this direction

(A.1) is then usual type $O(3)$ pw analysis.

The idea of this little appendix is to construct 6-point functions of definite triple-naturality. We know of course how to do this for a 4-point function. The thing begins by doing a White-like partial wave analysis in a physical 2 to 4 channel. You recombine all your two-body states so that each system has a definite parity. Then you fiddle around as in (A.3) to get the entire amplitude to have a definite parity with respect to the initial channel only. Then using the helicity crossing d-functions you convert to some useful Mueller region from your chosen 2-to-4 channel.

Having done all this wrt a particular 2-body initial channel, you repeat for the other two channels to get (A.6) and (A.7). Each amplitude involves naturality in a particular channel. Finally, you combine everything to get amplitudes of definite naturality in all channels, see (A.8). It is claimed that the product of the naturalities must be 1 or else the thing vanishes due to parity. I'm not sure this is exact or asymptotic claim. Asymptotics were used in step top p 169.

For more details, see their ref's 6 and 9.

By the way, when I do this stuff I have been using Toller so have not had to do crossing. Something maybe to wonder about, item # 53,674.

By the way, this analysis is for $A_0 = \text{XO}$, not restricted to triple regge "vertex" etc.

(weak)

P, C, T - DIRAC

PCT and all that. (mainly for QED and weaks)

I. This first section is a discussion of how time reversed, parity inverted, and charge conjugated wavefunctions must be related to the unmanipulated wavefunction such that the Dirac Equation either remains invariant (P and T) or stays the same except for a charge change. (C) In each subsection below we investigate several aspects:

- 1) must the transformation operator contain complex conjugation?
- 2) how must the Dirac Hamiltonian transform under the transformation operator (which acts in the Dirac 4×4 space)?
- 3) what happens to Dirac's ALPHA and BETA matrices under xform?
- 4) what is the energy of the transformed wavefunction if the untransformed one is in an energy eigenstate?
- 5) How should the external electromagnetic potential be treated under the various transformations?
- 6) Do all analyses using both ALPHA and GAMMA forms of the Dirac equation.
- 7) Special section on PCT all at once.

Here is a list of contents:

1. TRI and the Dirac Equation	1
2. External Potentials	3
3. CCI and the Dirac Equation	4
4. PI and the Dirac Equation	6
5. PCT invariance	8
6. typed comments	11

II. A miscellany selection of working papers.

1. Behavior of the spinors $u(p,s)$ etc under P, C and T	12
2. What is time reversal?	15
3. Verify that Mott is TRI.	16
4. Mott and the "Commins Table"	17
5. Reverse spins and momenta but <u>not</u> i and f .	18
6. The TR contradiction and resolution	19
a. Commins M, M^* and M'	
b. Hermiticity has two senses	
c. photon absorption as ex of TRI	
7. Typed comments	22
a. when do you know a current is Hermitian?	
b. why are em proton form factors real?	
c. "Hermiticity" in weaks: $T = T^\dagger$	

III. Invariances of Bilinear forms.

1. Recap sheets for P, C, T, H	24
a. specific PCTH transformed bilinears	
b. table of transformations	
c. direct effect of PC, T on spinors	
d. two statements of invariances	
2. When is a Bilinear TRI? When is it PI? When is it CCI?	26
3. Hermiticity Invariance	29
4. Example of how Hermiticity forces real form factors.	30

Not covered here:

1. anti linear operators.
2. S-matrix and PCT.
3. field operators and PCT.

Time Reversal Invariance of the Dirac Equation

① Start with $i \frac{\partial \psi(x,t)}{\partial t} = H(x,t) \psi(x,t)$.

Let $\bar{\psi}(t') = \bar{S} \bar{\psi}(t)$ $\bar{S}$ as yet unknown.

$\psi'(t')$ is the Dirac wavefunction as "seen" by observer S' . Assume that $\bar{S}$ (real number) $\bar{S}^\dagger = (\text{real number})$, no effect. Then we transform by "inserting" $\bar{S}$ everywhere:

$$(\bar{S} i \bar{S}^\dagger) (\bar{S} \frac{\partial}{\partial t} \bar{S}^\dagger) \bar{S} \psi(t) = \bar{S} H(t) \bar{S}^\dagger \bar{S} \psi(t)$$

$$\Rightarrow -(\bar{S} i \bar{S}^\dagger) \frac{\partial}{\partial t'} \psi'(t') = (\bar{S} H(t) \bar{S}^\dagger) \psi'(t')$$

② Why not have $\bar{S} i \bar{S}^\dagger = i$ and $\bar{S} H(t) \bar{S}^\dagger = -H'(t')$? This does not work, because no 4×4 Dirac $\bar{S}$ can take $+e\phi(t) = +e\phi'(t')$ to $-e\phi'(t')$.

③ Thus, try $\bar{S} i \bar{S}^\dagger = -i$. This means $\boxed{\bar{S} = TK}$ so that

$$\boxed{c = \bar{S} c \bar{S}^\dagger} = TK \cdot c \cdot K^\dagger T^\dagger = T c^* T^\dagger = c^* \quad \text{for any } c\text{-number } c.$$

So, in this case we want: $\bar{S} H(t) \bar{S}^\dagger = H'(t')$. I.e.,

$$\boxed{T H^*(t) T^\dagger = H'(t') = \bar{S} H(t) \bar{S}^\dagger}$$

④ Now,
$$\begin{aligned} H(t) &= \vec{\alpha} \cdot (-i \vec{\nabla} - e \vec{A}(t)) + \beta m + e \phi(t) \\ H^*(t) &= \vec{\alpha}^* \cdot (+i \vec{\nabla} - e \vec{A}(t)) + \beta^* m + e \phi(t) \\ H'(t') &= \vec{\alpha} \cdot (-i \vec{\nabla} - e \vec{A}'(t')) + \beta m + e \phi'(t') \end{aligned}$$
 $\vec{A}, \phi, e, m, \vec{\nabla}$ all real.

But, we know that $\begin{aligned} \vec{A}'(t') &= -\vec{A}(t) \\ \phi'(t') &= +\phi(t) \end{aligned} \quad \left. \begin{array}{l} \text{appended.} \\ \text{/ see previous sheet.} \end{array} \right\}$

So: $H'(t') = \vec{\alpha} \cdot (-i \vec{\nabla} + e \vec{A}(t)) + \beta m + e \phi(t)$

And:
$$\begin{aligned} T H^*(t) T^\dagger &= (T \vec{\alpha}^* T^\dagger) \cdot (+i \vec{\nabla} - e \vec{A}(t)) + T \beta^* T^\dagger m + e \phi(t) \\ &= \underline{(T \vec{\alpha}^* T^\dagger)} \cdot (-i \vec{\nabla} + e \vec{A}(t)) + \underline{(T \beta^* T^\dagger)} m + e \phi(t) \end{aligned}$$

⑤ Now, if we choose a T such that: --

$$\begin{aligned} \vec{\alpha} &= -T \vec{\alpha}^* T^{-1} \\ \beta &= +T \beta^* T^{-1} \end{aligned}$$

Then, observer O' will see: $i \frac{\partial}{\partial t'} \psi'(t') = H'(t') \psi'(t')$

I.e., he will see the same Dirac equation. This is the meaning of TRI.

⑥ The matrices $\vec{\alpha}$ and β are fixed, standard, so the equations above are true in all γ systems.

⑦ In the BD system we have:

$$\left\{ \begin{aligned} T &= i \gamma^1 \gamma^3 & T \gamma_\mu T^{-1} &= \gamma_\mu^T = \gamma_\mu^* \\ T &= T^{-1} = T^\dagger = -T^* = -T^T & T \gamma_\mu T^{-1} &= \gamma_\mu \end{aligned} \right.$$

⑧ Now, suppose $\psi(t)$ is an energy eigenstate of energy E .
Then:

$$i \frac{\partial \psi(t)}{\partial t} = E \psi(t)$$

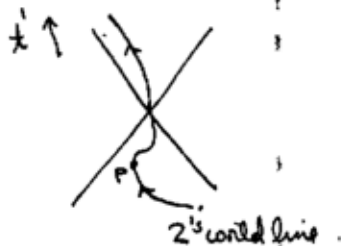
$$\Rightarrow i \frac{\partial \psi(t')}{\partial t'} = E \psi(t') \quad \text{by applying 3.}$$

Thus, $E = E'$. [Both observers O and O' see this electron having the same (positive, say) energy. Also, same charge]

What is the meaning of $\vec{A}'(t') = -\vec{A}(t)$?

① This is the same as asking about $\vec{B}'(t') = -\vec{B}(t)$

② Imagine two observers, one travels forward in time, or two travel backward in time. Each has his own lightcone.

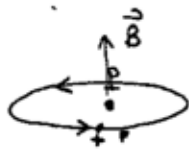


At $t=t'=0$ they "collide".
They have the same world line,
but travel it in opposite
directions.

③ Let P be a point at some $\vec{x}$, and $t = 5 \text{ sec}$, $t' = -5 \text{ sec}$.
Shift "start", or time origin, so observers collide at P .

④ At P , observer O measures $\vec{B}(t = 5 \text{ sec})$
observer O' measures $\vec{B}'(t' = -5 \text{ sec})$.

How do the results of their measurements compare? Near point P ,
 O sees a $+$ charge going in a circle CCW , and he sees
that it is causing his $\vec{B}(t = 5)$ to be in $+\hat{z}$.



But O' sees $+$ charge going CW ! Thus, he
measures $\vec{B}'(t' = -5)$ to be in $-\hat{z}$.

Therefore:

$$\vec{B}'(t') = -\vec{B}(t)$$

⑤ Note that, although $t' = -t$, t' and t do label the same point P
in the t -spacetime.

⑥ By similar argument, $\phi'(t') = +\phi(t)$

⑦ Therefore:

$$A'_\mu(x') = A^\mu(x)$$

under Time reversal, BD .

Charge Conjugation and the Dirac Equation

- ① Use BD γ forms:

$$(i\cancel{\partial} - eA - m)\psi = 0$$

- ② Now, we seek a wavefunction $\psi_c = G\psi$ such that:

$$(i\cancel{\partial} + eA - m)\psi_c = 0$$

Then ψ_c will describe a positron if ψ describes an electron.

- ③ In order that G change the relative sign between $i\cancel{\partial}$ and eA , we shall require that $G = C_1 K$ then, from ①:

$$[(i\partial_\mu - eA_\mu)\gamma^\mu - m]\psi = 0$$

$$\Rightarrow [(-i\partial_\mu - eA_\mu)(C_1 \gamma^\mu C_1^{-1}) - m]\psi_c = 0$$

$$\rightarrow ((i\partial_\mu + eA_\mu) \underline{[-C_1 \gamma^\mu C_1^{-1}]} - m)\psi_c = 0$$

Clearly, we need:

$$\underline{C_1 \gamma^\mu C_1^{-1} = -\gamma^{\mu*}}$$

From our Dirac notes, $\underline{C_1 = i\gamma^2}$ will do the trick.

- ④ Relate to BD: $C_1 = C\gamma^0 \Rightarrow C = C_1\gamma^0 = i\gamma^2\gamma^0 \checkmark$

$$\text{Also: } \psi^* = (\psi^\dagger)^T \Rightarrow \psi^* = (\psi^\dagger)^T$$

$$\gamma_0 \bar{\psi}^* = [(\gamma_0 \psi^*)^T]^T = (\psi^{*T} \gamma_0^T)^T = (\bar{\psi})^T$$

$$\text{So } \psi_c = C_1 \psi^* = C(\gamma^0 \psi^*) = C(\bar{\psi})^T$$

- ⑤ Note: The operator G did nothing to the number e , and did nothing to A_μ . I.e., we consider the same potential in either case.

- ⑥ Note: $\underline{C_1 = C_1^{-1} = C_1^T = C_1^* = C_1^\dagger}$

⑥ Now repeat using:

$$i \frac{\partial \Psi}{\partial t} = H \Psi$$

$$= H(e) \Psi$$

$$\Psi_c = G \Psi$$

What we want is:

$$i \frac{\partial \Psi_c}{\partial t} = H(-e) \Psi_c$$

This requires that:

$$G H(e) G^{-1} = -H(-e) = C_1 H(e) C_1^{-1}$$

↑ because $G i G^{-1} = -i$

Let $G = C_1 K$. Hence:

$$H(e) = [\vec{\alpha} \cdot (-i \vec{\nabla} - e \vec{A}) + \beta m + e \phi]$$

$$H^*(e) = [\vec{\alpha}^* \cdot (+i \vec{\nabla} - e \vec{A}) + \beta^* m + e \phi]$$

$$+ H(-e) = [\vec{\alpha} \cdot (-i \vec{\nabla} + e \vec{A}) + \beta m - e \phi]$$

$$C_1 H^*(e) C_1^{-1} = [(C_1 \vec{\alpha}^* C_1^{-1}) \cdot (i \vec{\nabla} - e \vec{A}) + (C_1 \beta^* C_1^{-1}) m + e \phi]$$

$$= [(-C_1 \vec{\alpha}^* C_1^{-1}) \cdot (-i \vec{\nabla} + e \vec{A}) + (C_1 \beta^* C_1^{-1}) m + e \phi] =$$

$$-H(-e) = [\vec{\alpha} \cdot (+i \vec{\nabla} - e \vec{A}) + \beta m + e \phi]$$

⑦ Comparison requires:

$$\begin{cases} C_1 \vec{\alpha}^* C_1^{-1} = \vec{\alpha} \\ C_1 \beta^* C_1^{-1} = -\beta \end{cases}$$

$$G H(+e) G^{-1} = -H(-e)$$

⑧ Now, suppose Ψ = energy eigenstate of E .

$$i \frac{\partial \Psi}{\partial t} = E \Psi \rightarrow -i \frac{\partial \Psi_c}{\partial t} = E \Psi_c$$

$$\Rightarrow i \frac{\partial \Psi_c}{\partial t} = E_c \Psi_c \quad \text{where} \quad \boxed{E_c = -E} !$$

Thus, if Ψ = ~~positive~~ negative energy, negative charge,
 then Ψ_c = positive energy, positive charge.

⑨ spin S_z is same for Ψ and Ψ_c , see p68.

Parity and the Dirac Equation

$$\textcircled{1} \quad i \frac{\partial \psi(x)}{\partial t} = H(x) \psi(x)$$

Let $\psi'(x') = P \psi(x)$. We want invariance under parity, so:

$$i \frac{\partial \psi'(x')}{\partial t} = H'(x') \psi'(x'), \quad \text{desired result.}$$

Thus, want: $\boxed{P H(x) P^{-1} = H'(x')}$ / assume $P P^{-1} = 1$.

Now write $P = P \cdot 1$, so $P H(x) P^{-1} = H'(x')$

$$H(x) = \vec{\alpha} \cdot (-i \vec{\nabla} - e \vec{A}(x)) + \beta m + e \phi(x)$$

$$H'(x') = \vec{\alpha} \cdot (-i \vec{\nabla}' - e \vec{A}'(x')) + \beta m + e \phi'(x')$$

$\textcircled{2}$ Parity reflection does not change pseudovector $\vec{B}$, but it does change vector $\vec{A}$. So

$$\vec{A}'(x') = -\vec{A}(x) \quad \phi'(x') = \phi(x)$$

Thus: $\rightarrow H'(x') = \vec{\alpha} \cdot (+i \vec{\nabla} + e \vec{A}(x)) + \beta m + e \phi(x)$

$$\rightarrow P H(x) P^{-1} = \underline{(-P \vec{\alpha} P^{-1})} \cdot (i \vec{\nabla} + e \vec{A}(x)) + \underline{(P \beta P^{-1})} m + e \phi(x).$$

$\textcircled{3}$ So, P must obey:

$$\boxed{\begin{aligned} P \vec{\alpha} P^{-1} &= -\vec{\alpha} \\ P \beta P^{-1} &= \beta \end{aligned}}$$

$$P = \gamma^0 \quad \text{works.}$$

$$\vec{\alpha} = \gamma^0 \vec{\gamma}$$

$$\gamma^0 (\gamma^0 \gamma^i) \gamma^0 = \gamma^i \gamma^0 = -\gamma^0 \gamma^i \quad \checkmark$$

$\textcircled{4}$ Energy unaffected, $E' = E$.

$\textcircled{5} \quad P = P^{-1} = P^* = P^T = P^\dagger$

⑤ Now let's "do parity" again, this time in γ notation:

$$(i\gamma - m)\psi(x,t) = 0 \quad (i\gamma^0 \partial_t - i\gamma^i \partial_i - m)\psi(x,t) = 0$$

$$[i\gamma^0 \frac{\partial}{\partial t} + i\gamma \cdot \nabla - m]\psi(x,t) = 0.$$

Again, let $\psi'(x',t) = P\psi(x,t)$, Use $\bar{P} = P$ to say:

$$P[i\gamma^0 \frac{\partial}{\partial t} + i\gamma \cdot \nabla - m]P^{-1}\psi'(x',t) = 0$$

Replace $\nabla = -\nabla'$ and use $P\gamma^0 P^{-1} = \gamma^0$ $P\gamma P^{-1} = -\gamma$ so

$$[i\gamma^0 \frac{\partial}{\partial t} + i\gamma \cdot \nabla' - m]\psi'(x',t) = 0$$

Then: $(i\gamma' - m)\psi'(x',t) = 0$ Equation is invariant.

PC3 invariance

① We now know what each does separately. So, what do they do all together? Define

$$\Psi_{\text{PCT}}(x') = P \tilde{G} T \Psi(x)$$

$$\Psi_{\text{PCT}}(x') = P(C, K)(TK) \Psi(x) = P C, T^* \Psi(x) = -P C, T \Psi(x)$$

Note that the * does not survive. Also:

$$P C \tilde{G} = P(C, K)(TK) = P C, T^* = -P C, T, \text{ period.}$$

So, PC3 has no effect on a c-number. So

$$\boxed{i \frac{\partial \Psi(x)}{\partial t} = H(e, x) \Psi(x)}$$

because $t = -t'$, in derivative.

$$\Rightarrow i \frac{\partial \Psi_{\text{PCT}}(x')}{\partial t'} = -(P C \tilde{G}) H(e, x) (P C \tilde{G})^{-1} \Psi_{\text{PCT}}(x')$$

Now compute:

$$\begin{aligned} & P C \tilde{G} H(e, x) \tilde{G}^{-1} \tilde{P}^{-1} \\ &= P C [\tilde{G} H(e, x, t) \tilde{G}^{-1}] \tilde{G}^{-1} \tilde{P}^{-1} = P C H'(e, x, t) \tilde{G}^{-1} \tilde{P}^{-1} \\ &= P [C H'(e, x, t) \tilde{C}^{-1}] \tilde{P}^{-1} = \tilde{P} [-H'(-e, x, t')] \tilde{P}^{-1} \\ &= -[\tilde{P} H'(-e, x, t') \tilde{P}^{-1}] = -H'(-e, x', t') = -H'(-e, x') \end{aligned}$$

② Therefore,

$$\boxed{i \frac{\partial \Psi_{\text{PCT}}(x')}{\partial t'} = H'(-e, x') \Psi_{\text{PCT}}(x')}$$

③ If $i \frac{\partial \Psi(x)}{\partial t} = E \Psi(x)$, then $i \frac{\partial \Psi_{\text{PCT}}(x')}{\partial t'} = -E \Psi_{\text{PCT}}(x')$

$$\boxed{(P C \tilde{G}) H(e, x) (P C \tilde{G})^{-1} = -H'(-e, x')}$$

② Now repeat P3 all at once:

(5.19.80)

$$\rightarrow [\vec{\alpha} \cdot (-i \vec{\nabla} - e \vec{A}(x)) + \beta m + e \phi(x)] \psi(x) = E \psi(x)$$

Well, "do" P3 together now. Combined, $A'_\mu(x') = \bar{A}_\mu(x)$, so the equation is invariant, hence

$$[\vec{\alpha} \cdot (-i \vec{\nabla}' - e \vec{A}'(x')) + \beta m + e \phi'(x')] \psi_{P3}(x') = E \psi_{P3}(x')$$

③ Now do G. We want only e to change, so we know result is:

(5.19.85)

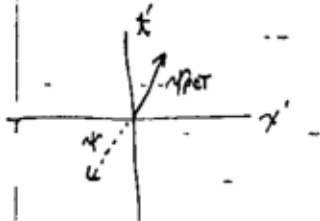
$$\rightarrow [\vec{\alpha} \cdot (-i \vec{\nabla}' + e \vec{A}'(x')) + \beta m - e \phi'(x')] \psi_{P3}(x') = -E \psi_{P3}(x')$$

Recall that G causes energy to change sign.

④ These are exactly those BD equations that have puzzled me for so long...

- 1) spacetime "reversed"
- 2) charge reversed
- 3) energy reversed.
- 4) velocity not reversed, $v = \frac{dx}{dt} = \frac{dx'}{dt'}$.

σ' sees positron moving at $\vec{v}$, $+$ energy, forward in time t' .
 σ sees electron moving at $\vec{v}$, $-$ energy, forward in time t .



← "A positron is $(i\gamma_5) \times$ a negative energy electron running backwards in space and time"

$$\rightarrow \psi_{PET}(x') = i\gamma_5 \psi(-x')$$

⑤ Note: $\psi_{PET}(x') = P3 \psi(x) = -i\gamma_5 PCT \psi(x) = -\gamma^0 i\gamma^2 i\gamma^1 \gamma^3 \psi(x)$
 $= +i [i\gamma^0 \gamma^1 \gamma^2 \gamma^3] \psi(x)$
 $= i\gamma^5 \psi(x).$

$$\boxed{\psi_{PET}(x') = i\gamma^5 \psi(x)}$$

⑥

⑦ Repeat once more:

$$i \frac{\partial \psi(x)}{\partial t} = H(e, x) \psi(x)$$

Do P3 at the same time:

$$\begin{aligned} P3 H(e, x) (P3)^{-1} &= H'(e, x') \\ P3(i) (P3)^{-1} &= -i \end{aligned}$$

$$\Rightarrow i \frac{\partial \psi_{P3}(x')}{\partial t'} = H'(e, x') \psi_{P3}(x')$$

⑧ N.B.: nothing happens. $\uparrow$ The equation is invariant under P3, since it's invariant under P3 separately.

⑨ Now do C:
$$\begin{aligned} C H'(e, x') C^{-1} &= -H'(-e, x') \\ C i C^{-1} &= -i \end{aligned}$$

Thus,

$$i \frac{\partial \psi_{PCT}(x')}{\partial t'} = H'(-e, x') \psi_{PCT}(x')$$

⑩ Thus:
$$i \frac{\partial \psi_{PCT}(x)}{\partial t} = H(-e, x) \psi_{PCT}(x)$$

So, ψ_{PCT} is exactly like an electron, but opposite charge.

Comments on P, C, and T.

1. There is an essential difference in our handling of T and P as compared with C. T and P involve spacetime and observers O and O'. We spoke about the relations between the external electromagnetic potentials "as seen from different frames, the primed and the unprimed." There, we were observing the same electron from two different reference frames. In both frames, the electron seen had the same energy, the same charge
2. Charge conjugation is very different. We never "go to a charge conjugated reference frame." There is only one frame in our consideration of C. But, there are two different electrons. Ψ is an electron, Ψ_c is something else. If Ψ were a negative energy electron, then Ψ_c would be a regular positron of positive energy. So, the two objects involved here, and observed in the same referential, have equal and opposite charges and energies. Since there is only one referential, we never ask questions like: "What happens to A under charge conjugation?" In fact, in treating C, it is best to just suppress the spacetime indices.
3. We have now shown that the Dirac equation is separately "invariant" under P, C and T. For P and T (or PT together) this means that in either frame, the Dirac equation looks the same. For C, this means that Ψ_c satisfies the same equation as does Ψ , except charge is reversed. That is, C invariance says that the anti-electron acts exactly as an electron with reversed charge.
4. This, we are not surprised to find that the Dirac theory also has combined PCT invariance. The interesting facts are:
 - a) the "PCT wavefunction" satisfies almost the same Dirac equation as the "original wavefunction". However, energy, spacetime, and charge are all reversed in the PCT equation, relative to the original equation.
 - b) the fact that a C wavefunction has reversed energy is only clear if you look at an energy eigenstate equation.
 - c) fact a) motivates the idea used in propagator theory which says that a positron is a negative energy electron moving backwards in space time. By "positron" we mean the PCT wf.
5. Also interesting is the fact that outside the realm of Dirac theory and QED, the product PCT is still supposed to be an invariance of the exact theory, even though the separate pieces are known to be violated. The PCT theorem has been proven in axiomatic FT and SMT.
6. Note that in this whole business we have given up on the idea of giving a meaning to the action of some operator P, C or T on the external potential. We simply do not consider one of these operators as acting on a function. They only act in the Dirac space.

Behavior of the plane wave spinors under $P, \mathcal{C}, T$ (all OK)

① Up to now we have made no mention of $u(p, s)$, but have instead dealt with general wavefunctions. Eg,

$$(i\not{\partial} - m)\psi(x) = 0$$

② However, a plane wave solution is: $\psi(x) = u(p, s) e^{-ip \cdot x}$
(for example), where

$$u(p, s) = \sqrt{E+m} \begin{bmatrix} u(s) \\ \frac{\vec{\sigma} \cdot \vec{p}}{E+m} u(s) \end{bmatrix}$$

We replace: $i\not{\partial} \rightarrow i(-i\not{\partial}) = \not{p}$, so

$$(\not{p} - m)u(p, s) = 0.$$

③ Alternatively, we have the two negative energy solutions,

eg $\psi(x) = v(p, s) e^{+ip \cdot x}$ and $(\not{p} + m)v(p, s) = 0$

and $v(p, s) = \sqrt{E+m} \begin{bmatrix} \frac{\vec{\sigma} \cdot \vec{p}}{E+m} u(s) \\ u(s) \end{bmatrix}$ is the solution.

④ What does parity have to say? $\psi'(x) = P\psi(x)$. If we use plane wave $\psi(x)$, then

$$P[u(p, s) e^{-ip \cdot x}] = u(p, s) e^{-ip' \cdot x} = \psi'(x')$$

However, $Pu(p, s) = \gamma^0 u(p, s) = u(-\vec{p}, s)$ / still + energy.

Meanwhile, $p \cdot x = p' \cdot x'$ plays no role, so

$$u(p', s') = Pu(p, s) = u(-p, s), \text{ or } P: u(p, s) \rightarrow u(-p, s).$$

The spin is unchanged.

⑤ Similarly: $v(p', s') = Pv(p, s) = \gamma^0 v(p, s) = -v(-p, s)$

$$\text{or } P: v(p, s) \rightarrow -v(-p, s)$$

⑥ Of course, $u'(p', s')$ satisfies

$$(\not{p}' - m)u'(p', s') = 0 \quad p' = (E, -\vec{p})$$

⑧ Similarly we may conclude that, under $\bar{J}$:

$$\bar{J} u(p, s) = T u^*(p, s) = i \gamma^3 u^*(p, s)$$

$$\text{or: } \bar{J}: u(p, s) \rightarrow i \gamma^3 u^*(p, s)$$

$$\text{But, } u^*(p, s) = [u^\dagger(p, s)]^T = [u^\dagger(p, s) \gamma_0 \gamma_0]^T = [\bar{u}(p, s) \gamma_0]^T$$

$$\text{⑨ Next, } G u(p, s) = C_1 u^*(p, s) = C \gamma_0 u^*(p, s) = C [\bar{u}(p, s)]^T$$

⑩ Now, suppose we have a matrix element, eg:

$$\bar{u}(p_1, s_1) \gamma_\mu u(p_2, s_2) = M$$

Observed from a parity inverted frame, this is

$$\begin{aligned} \bar{u}(p'_1, s'_1) \gamma_\mu u(p'_2, s'_2) &= [u(p'_1, s'_1)]^\dagger \gamma_0 \gamma_\mu [u(p'_2, s'_2)] \\ &= [P u(p_1, s_1)]^\dagger \gamma_0 \gamma_\mu [P u(p_2, s_2)] = u^\dagger(p_1, s_1) P^\dagger \gamma_0 \gamma_\mu P u(p_2, s_2) \\ &= \bar{u}(p_1, s_1) (\gamma_0 \gamma_\mu \gamma_0) u(p_2, s_2) = \bar{u}(p_1, s_1) \gamma^\mu u(p_2, s_2) \end{aligned}$$

$$\text{Thus: } \bar{u}(p'_1, s'_1) \gamma_\mu u(p'_2, s'_2) = - \bar{u}(p_1, s_1) \gamma_\mu u(p_2, s_2)$$

N.B. As this is the vector current, we would expect that

$$\bar{J}' = -\bar{J}$$

⑪ Now we see The meaning of Commies table:

$$\bar{u}_b F u_a \xrightarrow{P} \bar{u}_b \gamma_0 F \gamma_0 u_a = \bar{u}'_b F u'_a$$

← The element in inverted frame:

⑫ Application: Let $A^\mu(x)$ be an external potential. Under Parity, we know that $A^\mu(x') = A_\mu(x)$, i.e., the vector changes sign. Thus, consider:

$$\begin{aligned} M &= \bar{J}_\mu A^\mu = \bar{u}(p'_1) \gamma_\mu u(p'_2) A^\mu(x') = \bar{u}(p_1) \gamma_0 \gamma_\mu \gamma_0 u(p_2) A_\mu(x) \\ &= \bar{u}(p_1) \gamma^\mu u(p_2) A_\mu(x) = \bar{u}(p_1) \gamma_\mu u(p_2) A^\mu(x) \\ &= \bar{J}_\mu A^\mu \end{aligned}$$

⑫ The C column in the Commie Table.

Start with $\bar{u}(2) \gamma_\mu u(1)$.

Consider a charge conjugated reaction: (ie, a different process):

$$\begin{aligned}
 [\bar{u}(2)] \gamma_\mu [u(1)] &= \bar{u}_c(2) \gamma_\mu u_c(1) \\
 &= [C, u^*(2)]^\dagger \gamma_0 \gamma_\mu C, u^*(1) \\
 &= \{ (C^* u(2))^\dagger \gamma_0 \gamma_\mu^* C^\dagger u(1) \}^\dagger = u^\dagger(1) C, \gamma_\mu^* \gamma_0 C, u(2) \\
 &= \bar{u}(1) \gamma_0 C, \gamma_\mu^* \gamma_0 C, \bar{u}(2) = \bar{u}(1) C, \gamma_0 \gamma_\mu^* C, \gamma_0 u(2) \\
 &= \bar{u}(1) i \gamma^2 \gamma^0 \gamma_\mu i \gamma^2 \gamma^0 u(2)
 \end{aligned}$$

Commie uses $M = i \gamma^2 \gamma^0$, BP use $C = i \gamma^2 \gamma^0$

⑬ Result: $\bar{u}_c(2) \gamma_\mu u_c(1) = \bar{u}(1) C \gamma_\mu C u(2)$

this is what GC lists

⑭ Again, there is no frame of CC.

⑮ Keep parity result:

$$\bar{u}(2) \gamma_\mu u(1) \rightarrow \bar{u}'(2) \gamma_\mu u'(1) = \bar{u}(2) \gamma_0 \gamma_\mu \gamma_0 u(1) \quad \text{parity}$$

OK on list of GC.

Only T and H are confused in his table:

What is time reversal?

① For spin- $\frac{1}{2}$ particles, each plane wave state can be described by a momentum vector + a spin pole vector, so let

$$\mathcal{M} = \langle \vec{p}_3, \vec{s}_3; \vec{p}_1, \vec{s}_1 | T_P | \vec{p}_2, \vec{s}_2; \vec{p}_1, \vec{s}_1 \rangle$$

Then:

$$\mathcal{M}_{TR} = \langle -\vec{p}_3, -\vec{s}_3; -\vec{p}_1, -\vec{s}_1 | T_P | -\vec{p}_2, -\vec{s}_2; -\vec{p}_1, -\vec{s}_1 \rangle$$

* Time reversal invariance requires that $|\mathcal{M}| = |\mathcal{M}_{TR}|$.

② For Mott scattering, we might say:

$$\mathcal{M} = \langle \vec{p}_3, \vec{s}_3 | T_P | \vec{p}_1, \vec{s}_1 \rangle \quad T_P = \gamma_\mu \bar{A}^\mu$$

$$\text{Then } \mathcal{M}_{TR} = \langle -\vec{p}_3, -\vec{s}_3 | T_P | -\vec{p}_1, -\vec{s}_1 \rangle$$

③ Another way to imagine this is: observer O' in time reversed frame S' sees:

$$\mathcal{M}' = \langle \vec{p}'_3, \vec{s}'_3 | T_P | \vec{p}'_1, \vec{s}'_1 \rangle \quad \vec{p}'_3 = -\vec{p}_3, \text{ etc.}$$

Then, TRI means: $|\mathcal{M}| = |\mathcal{M}'|$ ($\mathcal{M}' \equiv \mathcal{M}_{TR}$)

④ Another way to say this is $T_P = \bar{T}^\dagger T_P^\dagger \bar{T}$ ie TRI, where $\bar{T}$ is antiunitary and, eg, $\bar{T} | \vec{p}, \vec{s} \rangle = | -\vec{p}, -\vec{s} \rangle$.

$$\begin{aligned} \text{Then: } \langle \vec{p}_3, \vec{s}_3 | T_P | \vec{p}_1, \vec{s}_1 \rangle &= \langle \vec{p}_3, \vec{s}_3 | (\bar{T}^\dagger T_P^\dagger \bar{T} | \vec{p}_1, \vec{s}_1 \rangle) \\ &= (\langle \vec{p}_3, \vec{s}_3 | \bar{T}^\dagger) (T_P^\dagger \bar{T} | \vec{p}_1, \vec{s}_1 \rangle)^* \\ &= \langle -\vec{p}_3, -\vec{s}_3 | T_P^\dagger | -\vec{p}_1, -\vec{s}_1 \rangle^* \\ &= \langle \vec{p}_1, \vec{s}_1 | T_P | -\vec{p}_3, -\vec{s}_3 \rangle = \mathcal{M}_{TR}. \end{aligned}$$

But, we do not have to get all involved in anti-linear operators to see TR.

here, $M' \equiv M^{TR}$

16

Verify that Mott is TRI

$$\textcircled{1} M = \int \langle \vec{p}_3, \vec{s}_3 | J_\mu A^\mu | \vec{p}_1, \vec{s}_1 \rangle d^4x = (2\pi)^4 \delta^4(\dots) \langle \vec{p}_3, \vec{s}_3 | J_\mu(0) \epsilon^\mu | \vec{p}_1, \vec{s}_1 \rangle$$

$$= (2\pi)^4 \delta^4(\dots) \bar{u}(p_3, s_3) \gamma_\mu \epsilon^\mu u(p_1, s_1)$$

$$\textcircled{2} M' = (2\pi)^4 \delta^4(\dots) \bar{u}'(p'_1, s'_1) \gamma_\mu \epsilon'^\mu u'(p'_3, s'_3)$$

$$\textcircled{3} \text{ We know: } \epsilon'^\mu = \epsilon_\mu \quad u'(p'_3, s'_3) = T u^*(p_3, s_3)$$

$$p'_3{}^\mu = p_{3\mu} \text{ etc.}$$

$\textcircled{4}$ Insert these facts into M' to get:

$$M' = (2\pi)^4 \delta^4(\dots) [\overline{T u^*(p_1, s_1)}] \gamma_\mu \epsilon_\mu T u^*(p_3, s_3)$$

Extract the star: $M' = (2\pi)^4 \delta^4(\dots) [\overline{T^* u(p_1, s_1)}] \gamma_\mu^* \epsilon_\mu^* T^* u(p_3, s_3)]^*$

Unload $-T = T^*$

$$M' = (2\pi)^4 \delta^4(\dots) \{ [T u(p_1, s_1)]^\dagger \gamma_0 \gamma_\mu^* \epsilon_\mu^* T u(p_3, s_3) \}^*$$

Now replace overall * with † and execute:

$$M' = (2\pi)^4 \delta^4(\dots) [T u(p_3, s_3)]^\dagger \epsilon_\mu^* \gamma_\mu^* \gamma_0 T u(p_1, s_1)$$

$$= (2\pi)^4 \delta^4(\dots) u^\dagger(p_3, s_3) T^\dagger \epsilon_\mu^* \gamma_\mu^* \gamma_0 T u(p_1, s_1)$$

$$= (2\pi)^4 \delta^4(\dots) \epsilon_\mu^* \bar{u}(p_3, s_3) \gamma_0 T^\dagger \gamma_\mu^* T u(p_1, s_1)$$

But $\gamma_0 T = T \gamma_0$, so

$$\bar{u}(p_3, s_3) T \gamma_\mu^* T u(p_1, s_1)$$

They are equal!

So: $\begin{cases} M' = (2\pi)^4 \delta^4(\dots) \epsilon_\mu^* \bar{u}(p_3, s_3) \gamma_\mu u(p_1, s_1) \\ M = (2\pi)^4 \delta^4(\dots) \epsilon_\mu \bar{u}(p_3, s_3) \gamma_\mu u(p_1, s_1) \end{cases}$

Well, actually phase of ϵ does not matter, $M \propto M'$.

$\textcircled{5}$ Conclusion: ~~Assume ϵ is real [why not]. Then~~ time reversal invariance holds for Mott ~~when $\epsilon = \epsilon^*$, ϵ is real.~~

~~Example of how TRI makes a coupling real.~~

$\textcircled{6}$ It's also real because Sunitarity $\Rightarrow \mathcal{L} = \text{hermi.}$, lowest order.

Relate TRI in Mott. to ^{my} Commune Table.

$$\begin{aligned} \textcircled{1} \quad \mathcal{M} &\sim \bar{u}(3) \gamma_\mu e \epsilon^\mu u(1) = e \epsilon^\mu [\bar{u}(3) \gamma_\mu u(1)] \\ \mathcal{M}' &\sim \bar{u}'(1') \gamma_\mu e \epsilon'^\mu u'(3') = e \epsilon'^\mu [\bar{u}'(1') \gamma_\mu u'(3')] \end{aligned}$$

② But $u'(3') = T u^*(3)$ etc, so

$$\begin{aligned} \bar{u}'(1') \Gamma u'(3') &= [T u^*(1)]^\dagger \gamma_0 \Gamma T u^*(3) \\ &= \{ u(1)^\dagger T \gamma_0 \Gamma^* T u(3) \}^* = \{ u^\dagger(1) T \gamma_0 \Gamma^* T u(3) \}^T \\ &= u^\dagger(3) T \Gamma^T \gamma_0 T u(1) = \bar{u}(3) \gamma_0 T \tilde{\Gamma} \gamma_0 T u(1) \\ &= \bar{u}(3) \gamma_0 T \tilde{\Gamma} T \gamma_0 u(1) \end{aligned}$$

③ Thus, we could say:

$$\bar{u}(3) \Gamma u(1) \xrightarrow{T.R.} \bar{u}'(1') \Gamma u'(3') = \bar{u}(3) \gamma_0 T \tilde{\Gamma} T \gamma_0 u(1)$$

④ In the case of Mott, $\Gamma = \gamma_\mu$, and

$$\gamma_0 T \tilde{\gamma}_\mu T \gamma_0 = \gamma_0 \gamma_\mu \gamma_0 = \gamma_\mu^\dagger = \gamma_\mu$$

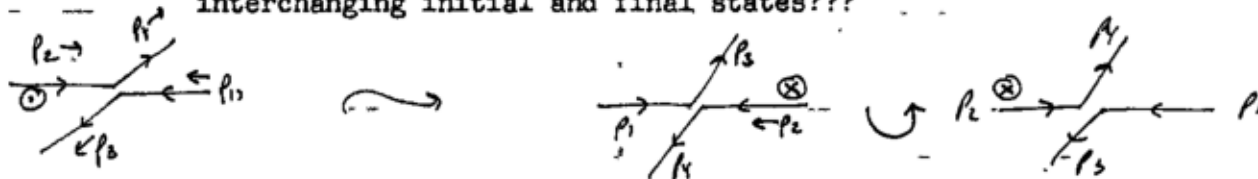
$$\text{So } \bar{u}(3) \gamma_\mu u(1) \xrightarrow{\epsilon^\mu} \bar{u}(3) \gamma_\mu u(1) \quad \text{so}$$

$$\left[\text{Or: } \underbrace{\bar{u}(3) \gamma_\mu u(1)}_{\epsilon^\mu} \right]_{\mathcal{M}} \rightarrow \underbrace{\bar{u}(3) \gamma_\mu u(1)}_{\epsilon^\mu} \Big|_{\mathcal{M}'} \times$$

$$\cancel{\left[\bar{u}(3) \gamma_\mu u(1) \right]_{\epsilon^\mu}} \quad \left[\bar{u}'(1') \gamma_\mu u'(3') \right]_{\epsilon'^\mu} = \left[\bar{u}(3) \gamma_\mu u(1) \right]_{\epsilon^\mu}$$

$= \mathcal{M}' \qquad \qquad \qquad = \mathcal{M}$

Question: What is the effect of reversing all spins and momenta, but not interchanging initial and final states???



It would give you the same amplitude you started with, except for the possibility of polarizations perpendicular to the reaction plane.

By the way, this is not TR!

Towards a Resolution.

I have read page *86 of Commins and learn, much to my surprise, that M' is not the TR of process M . Instead, it is the TR of process M^* . So, when he says to compare $M^* \stackrel{?}{=} M'$, he means to compare a process directly with its TR, no stars involved. This is supposed to line up the spinors for comparison.

Well, for me it does line up the spinors. So now it seems that everything is in order again.

However, now we see that it is not time reversal invariance which requires that the coupling constants be real. Instead, they are real because ~~as discussed earlier, the bilinear forms must be such that the matrix element is real... well, what I mean is this:~~

$$M_{fi} = \langle f | T | i \rangle = \langle i | T^\dagger | f \rangle^* = \langle i | T | f \rangle^* = M_{if}^*$$

$\downarrow$ $\qquad\qquad\qquad \downarrow$
 $\bar{u}(b) F u(a) \times \bar{u}(d) G u(c)$ $\qquad\qquad\qquad \{ \bar{u}(a) F u(b) \times \bar{u}(c) G u(d) \}^*$

the T-op is Herm.

$\rightarrow |\bar{u}(b) F u(a)| = |\bar{u}(b) \gamma_0 F^\dagger \gamma_0 u(a)|$ etc, at least.

So, the hermiticity of the T-operator (see other sheet why) implies that these two objects must have the same magnitude:

$$\left. \begin{array}{l} \bar{u}(b) F u(a) \\ \bar{u}(b) \gamma_0 F^\dagger \gamma_0 u(a) \end{array} \right\} \text{ and that in turn will } \rightarrow \text{ real complex.}$$

Notice that this involves the complex conjugate of F , whereas the time reversal relation, to follow, only involves the transpose. Time reversal invariance says these two must be equal in magnitude:

$$\begin{array}{l} \bar{u}(b) F u(a) \\ \bar{u}(b) \gamma_0 T \tilde{F} T \gamma_0 u(a) \end{array}$$

Detail on Hermitian conjugating: Let M be a bilinear form. This thing is just a number, so in the 4×4 matrix sense, taking a hermitian conjugate is exactly the same as complex conjugating. Ie, if you transpose a number as a 1×1 matrix nothing happens.

HOWEVER, when you think of M as the matrix element of the T-operator in the usual quantum mechanical Hilbert space, then $M_{ba} = \langle b | T | a \rangle$. With respect to this "states Hilbert space" there is a different meaning to the Hermitian conjugate. Ie, although M_{ba} is a 1×1 matrix in the Dirac space, it is a $N \times N$ matrix in the QM Hilbert space (eg, a is a matrix subscript in this sense). So, when you demand that M be Hermitian in the QMHS sense, you get:

$$M = M^\dagger \Rightarrow M_{ba} = M_{ab}^* \leftarrow \text{"hermiticity invariance"}$$

$\dagger = \dagger^{(QMHS)}$

Time Reversal Contradiction.

Suppose we have an amplitude or bilinear $M = \bar{b} \dots a$. What is the amplitude M^{TR} ? I seem to have two contradictory answers to this question.

First, it seems reasonable to say:

$$M = \bar{u}(\vec{p}_b, \hat{s}_b) F u(\vec{p}_a, \hat{s}_a) \Rightarrow M_{TR} = \bar{u}(-\vec{p}_a, -\hat{s}_a) F u(-\vec{p}_b, -\hat{s}_b)$$

Now, if we use Gene's page 366 claim, then

$$M_{TR} = [\bar{u}(p_a, s_a)]^\dagger \gamma_0 F \bar{u}(p_b, s_b) = u^\dagger(p_a, s_a) K T \gamma_0 F T K u(p_b, s_b)$$

Now if we reverse the direction of action of the first K operator, we have to star the whole thing,

$$M_{TR} = [u^\dagger(p_a) K T \gamma_0 F T K u(p_b)]^* = [\bar{u}(p_a) T F^* T u(p_b)]^*$$

Thus, with this definition of M^{TR} I find that:

$$M = \bar{u}(b) F u(a) \quad M_{TR}^* = \bar{u}(a) T F^* T u(b)$$

$$\Rightarrow M_{TR} = \bar{u}(b) \gamma_0 T \tilde{F} T \gamma_0 u(a) \quad \text{wrong.}$$

Now these equations disagree violently with Commins. (He always has to use M_{TR}^* in order to compare with M) so that the spinors agree, whereas my spinors already agree without the *. Lets go over Commins method for TR. It is as follows:

$$M = \bar{u}(b) F u(a) \Rightarrow M_{TR} = \bar{u}_{TR}(b) F u_{TR}(a)$$

The difference is that Commins has the roles

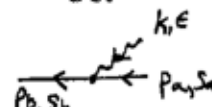
of a and b reversed from the above analysis. Thus, in the end he gets:

$$M = \bar{u}(b) F u(a)$$

$$M_{TR}^* = \bar{u}(b) T F^* T u(a)$$

$$M_{TR} = \bar{u}(a) \gamma_0 T \tilde{F} T \gamma_0 u(b)$$

To clarify this TR contradiction

① Let's consider the process $M =$ 

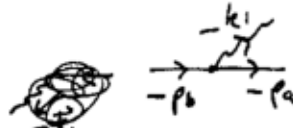
$$M = \int d^4x \langle p_b, s_b | : \bar{\psi}(x) F_{\mu} \psi(x) : A^{\mu}(x) | p_a, s_a; k, \epsilon \rangle$$

$\underbrace{\quad}_{b+d^+} \quad \quad \quad \underbrace{\quad}_{b^+} \quad \quad \underbrace{\quad}_{a^+}$

$$M = (2\pi)^4 \delta^4(\text{ext}) \bar{u}(b) F_{\mu} u(a) \epsilon^{\mu}(k)$$

Here, F_{μ} is some combination of momenta + $\gamma^5, \sigma^{\mu\nu}, \dots$. M is really T of $T = 1 \otimes 3p$. Thus: (used $C_4 = 1$, dagger norm).

$$\bar{T}_R = \bar{u}(b) F_{\mu} u(a) \epsilon^{\mu}(k)$$

② Now consider the time reversed reaction: $M_{TR} =$ 

$$M_{TR} = \int d^4x \langle -p_a, -k | : \bar{\psi}(x) F_{\mu} \psi(x) : A^{\mu}(x) | -p_b \rangle$$

$$M_{TR} = (2\pi)^4 \delta^4(\text{ext}) \bar{u}(-p_a) F_{\mu} u(-p_b) \epsilon^{\mu}(-k)$$

$$\bar{T}_R^{TR} = \bar{u}(-a) F_{\mu} u(-b) \epsilon^{\mu}(-k)$$

→ This supports my point of view on this matter ✓

③

Comments on Time Reversal, Weak Interactions, and Reality of form factors.

1. First, a comment on the hermiticity of currents. The electromagnetic current is, at zero momentum transfer, an observable thing. The current operator corresponds to a physical observable, so the expectation value must be real:

$$[\bar{u}(1) \gamma_\mu u(1)]^* = \bar{u}(1) \gamma_\mu u(1)$$

If we were to concoct some kind of electromagnetic current operator, say using Dirac fields, then we might expect that operator to be Hermitian (at least the normal ordered current ought to be Hermitian.) It is Hermitian operators which have real expectation values, always.

Similarly, the V-A current of weak interactions is real at zero momentum transfer. To check this, we just look at the last column in Commins' table. In general, we would expect the weak leptonic current operator to be Hermitian.

2. In QED field theory, the Lagrangian is an operator $L = J \cdot A$. The vector potential A may or may not be an operator. If it is not, then it is real. If it is, then it is Hermitian because photons are neutral. The electron current J_μ is also hermitian, and normal ordered. Thus, the lagrangian density operator is also Hermitian.

However, in field theory the Lagrangian is always Hermitian because the S operator is unitary. Thus, the Hermiticity of L is really a starting point. We should not think of it as arising from the hermiticity of some current. In fact, in weak interactions there are charged currents, and these are not Hermitian. (You always have to add the piece of opposite charge to get a Hermitian current.)

3. How do we know that the form factors of the proton for deep inelastic are real? At zero transfer we know they are both real because they are both observables (charge and anomalous moment), . But why couldn't they go complex as you leave the forward direction? Well, there is this idea that even at $q^2 \neq 0$ this current couples to A^μ . Again, $L = J \cdot A$, and the Hermiticity of L , together with the reality of A , implies the Hermiticity of J , so the form factors must maintain zero relative phase.

4.. In the study of weak interactions, however, we often encounter currents with various form factors, which currents are not Hermitian. Of course these currents are combined by hand in a way which makes the phenomenological L Hermitian. The question now is: how to we show that these form factors are real?

Well, the answer is as follows: you know that these form factors are going to show up in matrix elements of the T -operator. Now, if you could show that the T -operator were Hermitian, ~~and if you assume time reversal invariance~~ then you can show that the form factors are real.:

$$M = \langle 2 | T | 1 \rangle$$

$$M_{TR} = \langle 1' | T | 2' \rangle = \langle 1' | T^\dagger | 2' \rangle$$

$$= \langle 1 | T | 2 \rangle = \langle 2 | T^\dagger | 1 \rangle^*$$

$$= \langle 2 | T | 1 \rangle^*$$

$$= M^*$$

$$\Rightarrow M_{TR} = M^*$$

/ time reversal assumption

/ $T = T^\dagger$ assumption

$$|T|_{ba} = |M|_{ab}^*$$

see invariance sheets

Now if you take a specific example for M , you have to go look up in the last two columns of Commins table to see how things transform. Eg,

$$M = [\bar{u}(2) \gamma_\mu u(1) g_V + \bar{u}(2) \gamma_\mu \gamma^5 u(1) g_A] \epsilon^\mu$$

$$M^* = ([\bar{u}(2) \gamma_\mu u(1)]^\dagger g_V^\dagger + [\bar{u}(2) \gamma_\mu \gamma^5 u(1)]^\dagger g_A^\dagger) \epsilon^\mu$$

(a)

$$= (\bar{u}(2) \gamma_\mu u(2) g_V^* + \bar{u}(1) \gamma_\mu \gamma^5 u(2) g_A^*) \epsilon^\mu$$

/ last columns.

$$M_{TR} = [\bar{u}(1') \gamma_\mu u(2') g_V + \bar{u}(1') \gamma_\mu \gamma^5 u(2') g_A] \epsilon'^\mu$$

$$= \{ [\bar{u}(2) \gamma_\mu u(1)]^{TR} g_V + [\bar{u}(2) \gamma_\mu \gamma^5 u(1)]^{TR} g_A \} \epsilon'^\mu$$

ie, look up in TR column

/ lowered, $\epsilon'^\mu = \epsilon_\mu$

(b)

$$= \{ \bar{u}(1) \gamma^\mu u(2) g_V + \bar{u}(1) \gamma^\mu \gamma^5 u(2) g_A \} \epsilon_\mu$$

Comparing lines (a) and (b) shows that g_A and g_V must have real relative phase.

5. It remains only to justify the idea that the T -operator is Hermitian. Usually it is not so, because the difference is $TT^\dagger$, according to unitarity. But, in weak interactions it is fair to set all higher order graphs = 0 since G is so small. This means that $TT^\dagger = 0$ and $T = T^\dagger$. Another way to look at it is this: If you consider only first order, then $T = \text{integral } d^4x \text{ Lag}$. Since the Lagrangian is Hermitian, so is T .

Recap on PCT invariance (and H)

① Parity $M_{ba} = \bar{u}(b) F u(a)$ $u_P(a) = \gamma_0 u(a)$

$$(M_{ba})^P = \bar{u}_P(b) F u_P(a) = \bar{u}(b) \gamma_0 F \gamma_0 u(a)$$

② Time Reversal:

$$M_{ba} = \bar{u}(b) F u(a)$$

$$(M_{ba})^T = \bar{u}_T(a) F u_T(b) \quad u_T(b) = T u^*(b)$$

$$= \bar{u}(b) T \gamma_0 \tilde{F} \gamma_0 T u(a)$$

③ Charge Conjugation. $M_{ba} = \bar{u}(b) F u(a)$

$$(M_{ba})^C = \bar{u}_c(b) F u_c(a) = \bar{u}(a) C \tilde{F} C u(b)$$

$$(M_{ba}^*)^C = \bar{u}(b) \gamma^2 F^* \gamma^2 u(a) \quad u_c(a) = C u^*(a)$$

④ Hermiticity "Invariance": (for $T = T^\dagger$ in F.O.P.T.)

$$M_{ba} = \bar{u}(b) F u(a) \quad M_{ba}^* = M_{ab}^* = \bar{u}(b) \gamma_0 F^\dagger \gamma_0 u(a)$$

I.e., $M_{ba}^* = M_{ab}^* = M_{ab}^\dagger = [\bar{u}(a) F u(b)]^\dagger = \bar{u}(b) \gamma_0 F^\dagger \gamma_0 u(a)$

⑤ All together:

$$M_{ba}^* = M_{ab}^* \equiv (M_{ba})^H$$

M_{ba}	$=$	$\bar{u}(b) F u(a)$	$GC \checkmark$
$(M_{ba})^P$	$=$	$\bar{u}(b) \gamma_0 F \gamma_0 u(a)$	$GC \checkmark$
$(M_{ba})^T$	$=$	$\bar{u}(b) T \gamma_0 \tilde{F} \gamma_0 T u(a)$	$\times$
$(M_{ba})^C$	$=$	$\bar{u}(a) C \tilde{F} C u(b)$	$GC \checkmark$
$(M_{ba})^H$	$=$	$\bar{u}(b) \gamma_0 F^\dagger \gamma_0 u(a)$	$\times$

$$T = i \gamma^1 \gamma^3$$

$$C = i \gamma^2 \gamma^0$$

$$(M_{ba}^*)^C = \bar{u}(b) \gamma^2 F^* \gamma^2 u(a)$$

⑥ Invariances:

P	$ M_{ba} = (M_{ba})^P $
T	$ M_{ba} = (M_{ba})^T $
C	$ M_{ba} = (M_{ba}^*)^C $
H	$ M_{ba} = (M_{ba})^H $

② Supertable: [$\bar{u}(b) \dots u(a)$ elided]

M_{ba}	$(M_{ba})^T$	$(M_{ba})^P$	$(M_{ba})^C$	$(M_{ba})^H$
c	c	c	$-c^*$	c^*
γ^5	$-\gamma^5$	$-\gamma^5$	γ^5	$-\gamma^5$
γ^μ	γ_μ	γ_μ	γ^μ	γ^μ
$\gamma^\mu \gamma^5$	$\gamma_\mu \gamma^5$	$-\gamma_\mu \gamma^5$	$-\gamma^\mu \gamma^5$	$\gamma^\mu \gamma^5$
$\sigma^{\mu\nu}$	$-\sigma_{\mu\nu}$	$\sigma_{\mu\nu}$	$-\sigma^{\mu\nu}$	$\sigma^{\mu\nu}$
$\sigma^{\mu\nu} \gamma^5$	$\sigma_{\mu\nu} \gamma^5$	$-\sigma_{\mu\nu} \gamma^5$	$\sigma^{\mu\nu} \gamma^5$	$-\sigma^{\mu\nu} \gamma^5$

N.B. This is not the same table as 'Common'!

③ Direct effect on spinors:

<u>Spinor</u>	<u>$\mathcal{Z} = T, K$</u>	<u>$\mathcal{P} = \gamma^0$</u>	<u>$\mathcal{C} = C, K$</u>
$u(p, s)$	$(-i) u(-\vec{p}, -\hat{s})$	$u(-\vec{p}, \hat{s})$	$\bar{v}(p, s)$
$v(p, s)$	$(+i) v(-\vec{p}, -\hat{s})$	$-v(-\vec{p}, \hat{s})$	$u(p, s)$

④ Direct statement of invariances: (all in magnitudes only)

$$\mathcal{P} \quad \bar{u}(\vec{p}_b, \hat{s}_b) \not{F} u(\vec{p}_a, \hat{s}_a) = \bar{u}(-\vec{p}_b, \hat{s}_b) \not{F} u(-\vec{p}_a, \hat{s}_a)$$

$$\mathcal{T} \quad \bar{u}(p_b, s_b) \not{F} u(p_a, s_a) = \bar{u}(-\vec{p}_a, -\hat{s}_a) \not{F} u(-\vec{p}_b, -\hat{s}_b)$$

$$\mathcal{C} \quad \bar{u}(p_b, s_b) \not{F} u(p_a, s_a) = \bar{v}(p_b, s_b) \not{F} v(p_a, s_a)$$

Other results are obvious (ie, other than $\bar{u} \dots u$). Note that all these things make sense!

① When is a Dirac Matrix Element TRI?

note that b is now put on the right.
 $u_{TR}(\vec{p}, \vec{s}) = u^*(\vec{p}, s) \neq$ initial state
 in TR frame.

$$\text{Let } \mathcal{M} = \bar{u}(b) F u(a)$$

$$\begin{aligned} \text{Then: } \mathcal{M}_{TR} &= \bar{u}_{TR}(a) F u_{TR}(b) = [\bar{u}(a)] F [\bar{u}(b)] \\ &= [\bar{T} u^*(a)] F [T u^*(b)] = [T u^*(a)]^\dagger \gamma_0 F T u^*(b) \\ &= u^{*\dagger}(a) T \gamma_0 F T u^*(b) \end{aligned}$$

$$\text{But, } \mathcal{M} = \mathcal{M}^{**} = \mathcal{M}^{\dagger*} = u^\dagger(a) \tilde{F} \gamma_0 u^*(b)$$

$$\text{Or, } \mathcal{M}_{TR} = \mathcal{M}_{TR}^{\dagger*} = u^\dagger(b) T \tilde{F} \gamma_0 T u(a) = \bar{u}(b) T \gamma_0 \tilde{F} \gamma_0 T u(a).$$

So, we have:

$$\left. \begin{aligned} \mathcal{M} &= \bar{u}(b) F u(a) \\ \mathcal{M}_{TR} &= \bar{u}(b) \gamma_0 T \tilde{F} T \gamma_0 u(a) \end{aligned} \right\} \begin{array}{l} \text{Commutator calls this } \mathcal{M}^* \\ \text{Com. calls this } \mathcal{M}' \end{array}$$

$$\mathcal{M}_{TR}^* = \bar{u}(a) T \tilde{F} T u(b) \quad \leftarrow \text{so what.}$$

$$\text{Thus, TRI} \Rightarrow F = \gamma_0 T \tilde{F} T \gamma_0$$

yes

Table	F	$\gamma_0 T \tilde{F} T \gamma_0$	(TRI?) $\leftarrow$ but, see ⑤
	1	1	Yes
	γ_5	$-\gamma_5$	No
	γ_μ	γ_μ	No
	$\gamma_\mu \gamma_5$	$\gamma_\mu \gamma_5$	No
	$\sigma_{\mu\nu}$	$-\sigma_{\mu\nu}$	No
	$\sigma_{\mu\nu} \gamma_5$	$\sigma_{\mu\nu} \gamma_5$	No

② When is a DME Parity Invariant?

$$\mathcal{M} = \bar{u}(b) F u(a)$$

$$\begin{aligned} \mathcal{M}_P &= \bar{u}_P(b) F u_P(a) = [\bar{P} u(b)] F P u(a) = (P u(b))^\dagger \gamma_0 F P u(a) \\ &= u^\dagger(b) P \gamma_0 F P u(a) = \bar{u}(b) P F P u(a) \end{aligned}$$

$$\begin{aligned} \text{So: } \mathcal{M} &= \bar{u}(b) F u(a) \\ \mathcal{M}_P &= \bar{u}(b) \gamma_0 F \gamma_0 u(a) \end{aligned}$$

$$\text{Thus, PI} \Rightarrow F = \gamma_0 F \gamma_0$$

Table	F	$\gamma_0 F \gamma_0$	PI?
	1	1	yes
	γ_5	$-\gamma_5$	no
	γ^μ	$-\gamma^\mu$	no
	$\gamma^\mu \gamma_5$	$-\gamma^\mu \gamma_5$	no
	$\sigma^{\mu\nu}$	$\sigma^{\mu\nu}$	no
	$\sigma^{\mu\nu} \gamma_5$	$-\sigma^{\mu\nu} \gamma_5$	no

③ When is a DME Charge Conj. invariant?

$$= \bar{u}(a) C \gamma_0 \tilde{F} C \gamma_0 u(b) \\ = \bar{u}(a) C \tilde{F} C u(b)$$

$$\mathcal{M} = \bar{u}(b) F u(a)$$

$$\mathcal{M}_c = \bar{u}_c(b) F u_c(a)$$

$$= [C u(b)]^\dagger \gamma_0 F C u(a) = u^\dagger(b) C \gamma_0 F C u(a) \\ = ? = u^\dagger(a) C \tilde{F} \gamma_0 C u(b) = \bar{u}(a) \gamma_0 C \tilde{F} \gamma_0 C u(b)$$

$$\mathcal{M}_c^* = u^\dagger(b) C \gamma_0 F^* C u(a) = \bar{u}(b) \gamma_0 C \gamma_0 F^* C u(a) \\ = -\bar{u}(b) C F^* C u(a) = \bar{u}(b) \gamma^2 F^* \gamma^2 u(a)$$

So:

$$\mathcal{M} = \bar{u}(b) F u(a)$$

$$\mathcal{M}_c^* = \bar{u}(b) \gamma^2 F^* \gamma^2 u(a)$$

$$\mathcal{M}_c = \bar{u}(a) C \tilde{F} C u(b)$$

CCI $\Rightarrow F = \gamma^2 F^* \gamma^2$

Table:	F	$\gamma^2 F^* \gamma^2$	CCI?
	1	1	no
	γ_5	$+\gamma_5$	yes
	γ^μ	γ^μ	yes
	$\gamma^\mu \gamma_5$	$-\gamma^\mu \gamma_5$	no
	$\sigma^{\mu\nu}$	$-\sigma^{\mu\nu}$	no
	$\sigma^{\mu\nu} \gamma_5$	$\sigma^{\mu\nu} \gamma_5$	yes

④ Example on TR. Gene shows that

Thus:

$$\mathcal{M} = \bar{u}(\vec{p}_b, \hat{s}_b) F u(\vec{p}_a, \hat{s}_a)$$

$$\mathcal{M}_{TR} = \bar{u}(-\vec{p}_a, -\hat{s}_a) F u(-\vec{p}_b, -\hat{s}_b)$$

$$= \bar{u}(\vec{p}_b, \hat{s}_b) \gamma_0 T \tilde{F} T \gamma_0 u(\vec{p}_a, \hat{s}_a)$$

$$3 u(\vec{p}, \hat{s}) = T u^*(\vec{p}, \hat{s})$$

$$= (-i) u(-\vec{p}, -\hat{s})$$

phase, not important.

/makes sense to me.

/pure math.

⑤ Remark: by just looking at a bilinear form, you cannot say whether the entire amplitude, of which it is only a part, has a certain symmetry. You need to know all parts of the amplitude, eg, $e_p^\mu = e^\mu = e_n$, etc. In weak interactions there are usually two bilinear forms. So this table is a tool to help analyze the whole amplitude:

$\mathcal{M}$	$\mathcal{M}^* = \mathcal{M}^\dagger$	$\mathcal{M}_{TR}$
$\bar{u}(b) F u(a)$	$\bar{u}(a) \gamma_0 F^\dagger \gamma_0 u(b)$	$\bar{u}(b) \gamma_0 T \tilde{F} T \gamma_0 u(a)$
$\mathcal{M}_P$	$\mathcal{M}_C$	$\mathcal{M}_C^*$
$\bar{u}(b) \gamma_0 F \gamma_0 u(a)$	$\bar{u}(a) C \tilde{F} C u(b)$	$\bar{u}(b) \gamma^2 F^* \gamma^2 u(a)$

So here is a table:

$\mathcal{M}$	$\bar{u}(b) F u(a)$
$\mathcal{M}^*$	$\bar{u}(a) \gamma_0 F^\dagger \gamma_0 u(b)$
$\mathcal{M}_{TR}$	$\bar{u}(b) \gamma_0 T \tilde{F} T \gamma_0 u(a)$
$\mathcal{M}_{TR}^*$	$\bar{u}(a) T F^* T u(b)$
$\mathcal{M}_P$	$\bar{u}(b) \gamma_0 F \gamma_0 u(a)$
$\mathcal{M}_P^*$	$\bar{u}(a) F^\dagger u(b)$
$\mathcal{M}_C$	$\bar{u}(a) C \tilde{F} C u(b)$
$\mathcal{M}_C^*$	$\bar{u}(b) \gamma^2 F^* \gamma^2 u(a)$
H.I. $\rightarrow \mathcal{M}_{ab}^*$	$\bar{u}(b) \gamma_0 F^\dagger \gamma_0 u(b)$

$$T = i \gamma^1 \gamma^3$$

$$C = i \gamma^2 \gamma^0$$

And another table: (all $\bar{u}(b) \dots u(a)$)

$\mathcal{M}$	$\mathcal{M}_{TR}$	$\mathcal{M}_P$	$\mathcal{M}_C^*$	$\mathcal{M}_{TR}^*$
F	$\gamma_0 T \tilde{F} T \gamma_0$	$\gamma_0 F \gamma_0$	$\gamma^2 F^* \gamma^2$	$\gamma_0 F^\dagger \gamma_0$
1	1	1 — S	-1	1
γ^5	$-\gamma^5$	$-\gamma^5$ — P	γ^5	$-\gamma^5$
γ^μ	γ^μ	γ^μ — V	γ^μ	γ^μ
$\gamma^\mu \gamma^5$	$\gamma^\mu \gamma^5$	$-\gamma^\mu \gamma^5$ — A	$-\gamma^\mu \gamma^5$	$\gamma^\mu \gamma^5$
$\sigma^{\mu\nu}$	$-\sigma^{\mu\nu}$	$\sigma^{\mu\nu}$ — T	$-\sigma^{\mu\nu}$	$\sigma^{\mu\nu}$
$(\sigma^{\mu\nu} \gamma^5)$	$\sigma^{\mu\nu} \gamma^5$	$-\sigma^{\mu\nu} \gamma^5$ — AT	$\sigma^{\mu\nu} \gamma^5$	$-\sigma^{\mu\nu} \gamma^5$

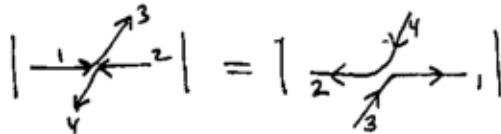
Applications

X ⑥ Suppose one of your kinematic frames is. Wait.

⑥ "Hermiticity Invariance" ~ We should certainly have:

$$M_{fi} = \langle f | T | i \rangle = \langle i | T^\dagger | f \rangle^* = \langle i | T | f \rangle^* = T_{if}^* \quad \text{if } T = T^\dagger$$

Thus: $|\bar{u}(b) F u(a)| = |\bar{u}(a) F u(b)|$. This is not time reversal. This is that thing about reverse initial with final but do not change momenta.



We have:

$$M_{ba} = \bar{u}(b) F u(a) \quad ; \quad M_{ba}^* = \bar{u}(a) \gamma_0 F^\dagger \gamma_0 u(b), \text{ no relations to us here.}$$

But $M_{ab}^* = \bar{u}(b) \gamma_0 F^\dagger \gamma_0 u(a)$
 $M_{ba} = \bar{u}(b) F u(a)$ $\leftarrow$ These must be equal in mag.

This is just like parity, except: F is $F^\dagger$!

⑦ Repeat

$$M_{ab}^* = M_{ba}$$

Compare M_{ba} with $M_{ab} \rightarrow M_{ab}^*$.

$$M_{ba}^* = M_{ab} \quad \bar{u}(a) \gamma_0 F^\dagger \gamma_0 u(b) = \bar{u}(a) F u(b)$$

So $M_{ab} = \left[M_{ba} \rightarrow M_{ba}^* \right]$ shown in table.

so $M_{ba}^* = (M_{ab})^\dagger$
 $\uparrow$
 $\bar{u}(a) F u(b)$

Table:

F	$\gamma_0 F^\dagger \gamma_0$
1	1
γ^5	$-\gamma^5$
γ^μ	γ^μ
$\gamma^\mu \gamma^5$	$\gamma^\mu \gamma^5$
$\sigma^{\mu\nu}$	$\sigma^{\mu\nu}$
$\sigma^{\mu\nu} \gamma^5$	$-\sigma^{\mu\nu} \gamma^5$
c	c^*

So define

$$(M_{ij})^\dagger = M_{ji}^*$$

Then HI says:

$$|M_{ij}| = |(M_{ij})^\dagger|$$

⑧ Now let's do some testing. First:

$$M = \bar{u}(b) \gamma^\mu (1 + \gamma_5) u(a)$$

$$M_P = \bar{u}(b) \gamma_\mu (1 - \gamma_5) u(a)$$

$$M_C^* = \bar{u}(b) \gamma^\mu (1 - \gamma_5) u(a)$$

$$M_{TR} = \bar{u}(b) \gamma_\mu (1 + \gamma_5) u(a)$$

$$M_{\text{bar}}^* = \bar{u}(b) \gamma^\mu (1 + \gamma_5) u(a)$$

P violated!

C violated!

T OK.

H OK.

⑨ Next, try

$$M = \bar{u}(b) [f_1 \gamma^\mu + g_1 \gamma^\mu \gamma_5] u(a)$$

$$M_P = \bar{u}(b) [f_1 \gamma^\mu - g_1 \gamma^\mu \gamma_5] u(a)$$

$$M_C^* \quad \text{same concl.}$$

$$M_{TR} = \bar{u}(b) [f_1 \gamma^\mu + g_1 \gamma^\mu \gamma_5] u(a)$$

$$M_H = \bar{u}(b) [f_1^* \gamma^\mu + g_1^* \gamma^\mu \gamma_5] u(a)$$

P violated unless f_1 or $g_1 = 0$.

T OK, even if f_1, g_1 complex.

H OK if relative phase real!!

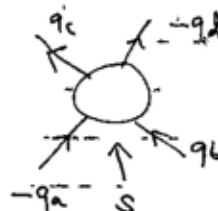
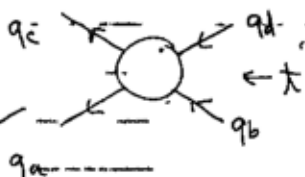
Here is our first crispy example of how it is that Hermiticity is what forces coupling constants to be real, not TRI.

crossing

Compare Crossing as Stated in Helicity Amplitudes versus as Stated in Toller M-functions.

1. First, I quote the helicity amplitude result from MS. See self-notes in S-matrix binder for one page review. Crossing says:

$$\langle \hat{q}_c \mu_c; \hat{q}_a \mu_a | I^{(t)} | q_d \mu_d; q_b \mu_b \rangle = \langle \hat{q}_c \mu_c; -q_d \mu_d | I^{(s)} | -\hat{q}_a \mu_a; q_b \mu_b \rangle$$



I would like to know how this equality appears in Toller M-function notation.

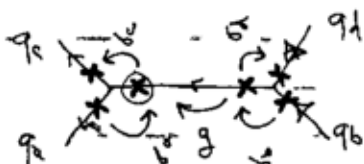
$$M_{\mu_c \mu_a \mu_d \mu_b}^{(t)}(a_c^{(t)}, a_b^{(t)}, a_d^{(t)}, a_a^{(t)})$$

where $a_c^{(t)}$ and $a_a^{(t)}$ contain τ , others do not.

$$M_{\mu_c \mu_a \mu_d \mu_b}^{(s)}(a_c^{(s)}, a_b^{(s)}, a_d^{(s)}, a_a^{(s)})$$

where $a_d^{(s)}$ and $a_c^{(s)}$ contain τ , others do not.

But I would not say these general objects are equal unless you put in the right transformations a_i . These are supposed to be helicity amplitudes, so construct your a_i accordingly:



$$\begin{aligned} a_a^{(t)} &= \sigma_a \cdot \tau \\ a_b^{(t)} &= g \cdot \sigma_b \end{aligned}$$

$$\begin{aligned} a_c^{(t)} &= \sigma_c \cdot \tau \\ a_d^{(t)} &= g \cdot \sigma_d \end{aligned}$$



$$\begin{aligned} a_a^{(s)} &= v_a \\ a_b^{(s)} &= v_b^{-1} \end{aligned}$$

$$\begin{aligned} a_c^{(s)} &= g' \cdot v_c \cdot \tau \\ a_d^{(s)} &= g' \cdot v_d \cdot \tau \end{aligned}$$

The group rotations g and g' are presumably determined by the above equations.

2. Comment: with these values of the a_i we really want to say that the two Toller M-functions are equal. They are equal when evaluated at two different point in the huge space $\hat{A}^4$. However, these two points are not connected by any $a \in \hat{A}$. To see this, you just realize that such an "a" would either have to include TCP or not include it, in either case you cannot generate the right TCP factors. Thus:

The Toller M-functions include TCP, but they do not include crossing.!

3. Let me try to restate this in other words.

We know that there are many amplitudes related to the same analytic function. If you neglect Dalitz decays, then usually there are 6 amplitudes to be considered.

The Toller formalism including TCP "incorporates" TCP invariance. Ie, two reactions which are related by TCP are simply related because the TCP operator is just an element of the invariance group. In other words, the comparison of a reaction to its TCP reaction is as trivial as comparing a reaction in one Lorentz frame to the same reaction in another Lorentz frame.

However, there is no such simple connection between the s-channel amplitude and the t-channel amplitude. In other words, the Toller formalism does not include analyticity as one of its assumptions.

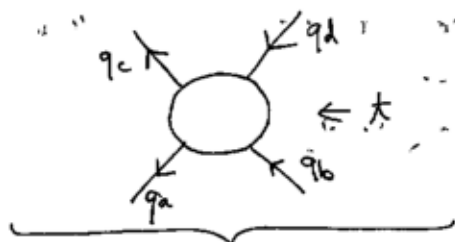
Toller nevertheless does want to include all 6 reactions into his one function, recall equation (4.7) in signature paper, where he allows for all reactions to be "present". However, this is of course a doctoring up of the Hilbert Spaces by adding their conjugate spaces, and then letting the doubled Moussa Stora wavefunction select out the reaction of interest.

Still, there is no analytic connection between the various reactions, except the TCP connection as already noted.

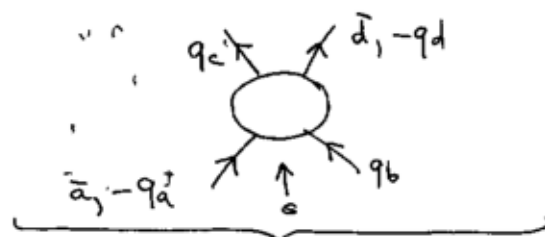
1. I had trouble reconstructing the argument, so this time I will record the argumtn in more detail. Open MS to page 329. Examine the equations on that page. We are going to prove that equation (7/14) is true, ie, we will prove this:

(*)

$$\langle q_c q_a; \mu_c \mu_a | T^{(t)} | q_d q_b; \mu_d \mu_b \rangle = \langle q_c, -q_d; \mu_c \mu_d | T^{(s)} | -q_a, q_b; \mu_a \mu_b \rangle$$



q_i all futuretimelike, this amp is evaluated in region (s,t) where process really occurs [t-ch phys region]



s-channel amplitude continued to where it cannot really happen, eg, $(-q_a)$ is past timelike.

$$s = (q_b - q_a)^2 < 0 \quad t = (q_b + q_d)^2 > 0$$

We are claiming that if you take the normal s-channel helicity amplitude and analytically continue it to a point (s,t) which is in the t-channel physical region, then you will end up with something which is numerically equal to the t-channel helicity amplitude. Look at the right thing above. This cannot of course be a physical s-channel process because t is positive and s is negative. Another way to say this is: some of the momenta which should be future are past timelike. Thus, this s-channel amplitude is not being "evaluated" in the s-channel physical region.

2. Now, let Λ be a real Lorentz transformation (not a complex one) which connects two frames in the t-channel physical region. Then the t-channel amplitude shown above on the left must transform exactly as shown in equation (7.17) where $\bar{R}$ is the proper Wigner rotation, ie, $\bar{R} = \bar{R}(\Lambda, q_i)$ for each particle. Note of course that at both ends of this Lorentz transformation, the t-channel amplitude is physical, ie, is evaluated in the t-channel physical region.

Now, apply this same Lorentz transformation to the object on the right, above, ie, on the continuation of the s-channel amplitude to the t-channel physical region. You will get equation (7.21), although MS use the p_i as shown in (7.16). The Wigner angles here are $\bar{R}(\Lambda, q_c)$ for particle c, the same Wigner rotation appearing in (7.17). BUT, what is $\bar{R}(\Lambda, -q_d)$? Ie, you must face up to this minus sign.

If the vector $-q_d$ is defined in terms of q_d according to (7.23), then you can calculate the rotation $R(\Lambda, -q_d)$ and you get the result shown on top of page 332, ie,

$$D_{\mu'\mu} [R(\Lambda, -q_d)] = (-1)^{\mu'-\mu} D_{\mu\mu'}^* [R(\Lambda, +q_d)]$$

This allows you to write the transformation of the (continued) s-channel amplitude as in (7.26). Comparing (7.26) to (7.17) then shows that the two objects given on top of last page (back of this page) transform in the same way under real Lorentz transformations.

If they transform the same way, it is reasonable to conclude that they are equal. This is the gist of the MS proof.

3. The story almost ends here but not quite. In equality (*) we could choose a t-channel cms frame if we liked. We would then be equating (a t-channel amplitude evaluated in ~~the~~ a t-channel cms frame) with (a s-channel amplitude continued to the t-channel cms frame).

Now a careful distinction is needed. A t-channel amplitude evaluated in a special t-channel cms frame may be identified with the "t-channel cms amplitude" $T^{(t)}(s, t)$. You can continue this "t-channel cms amplitude" into the s-channel physical region, eg, to the s-channel cms frame, but it is still the "t-channel cms amplitude".

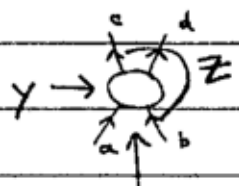
Now consider equation (7.31). This shows a relation between the s-channel amplitude evaluated in the s-channel cms frame, and some other frame. The s-channel amplitude evaluated in the s-channel cms frame is identified with the "s-channel cms amplitude", which as noted above you could if you liked continue anywhere you like in s and t, but it is still the "s-channel cms amplitude."

Now so far, equation (7.31) links the s-channel amplitude at two points in the s-channel physical region. (here specialized to case where the LT is a z boost). The trick is now to continue the entire equation, including the wigner rotations, to the t-channel region, in particular to the t-channel cms frame. Then the object on the right of (7.31) is the s-channel amplitude continued to the t-channel cms frame, and is therefore exactly the object on the right of (*) and via (*) may be identified with the ~~unknown~~ t-channel cms amplitude, as noted above. Thus we get (7.34).

Equation (7.34) then relates the t-channel cms amplitude in the t-channel cms frame, to the s-channel cms amplitude continued to the t-channel cms frame. More generally, (7.34) relates the (t-channel cms amplitude) to the (s-channel cms amplitude) in any frame. The frame is denoted roughly by values of s and t.

CROSSING

① Let this be our reference process:



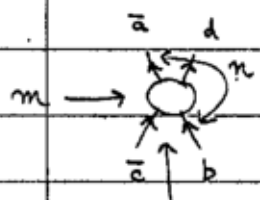
$$Y \rightarrow \text{circle} \rightarrow Z \equiv T(ab \rightarrow cd; p_a, p_b, p_c, p_d) \quad \text{all } p_i \text{ are f.t.}$$

$$X = A(ab \rightarrow cd; -(p_a + p_b)^2, (p_c - p_a)^2, (p_d - p_a)^2)$$

Now arbitrarily define invariants as shown. Then

$$= A(ab \rightarrow cd; X, Y, Z) \equiv f(X, Y, Z)$$

② We now wish to express all crossing-related amplitudes in terms of the function f . Example:



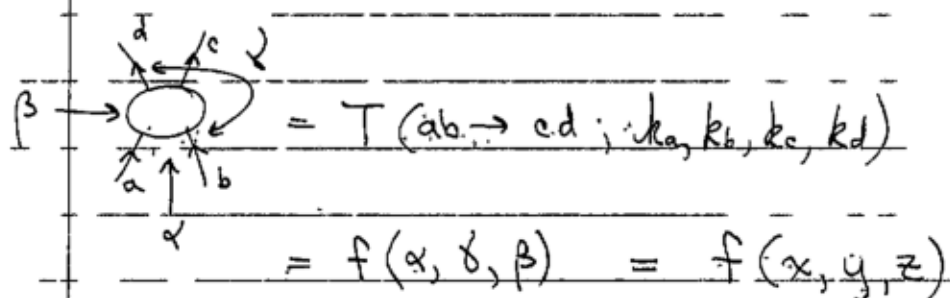
$$m \rightarrow \text{circle} \rightarrow n \equiv T(\bar{c}b \rightarrow \bar{a}d; q_{\bar{c}}, q_b, q_{\bar{a}}, q_d)$$

$$l = f(n, m, l)$$

How did we know this to be true? My rule is this: The first argument refers to a and b . It was named X above. Below the variable relating to a and b is called n . Hence, replace X with n .

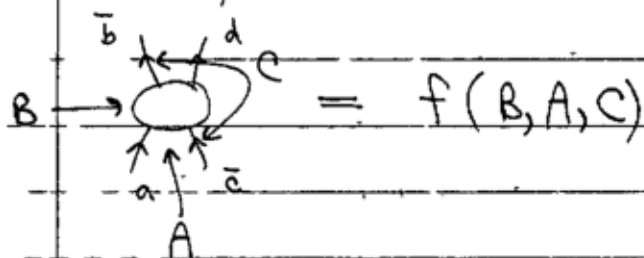
N.B: in the above case, $X = -(p_a + p_b)^2 = \text{positive only}$; in the lower case, $n = (q_{\bar{c}} - q_b)^2 = \text{usually negative}$.

③ Here is a "crossing related" process, ~~that is the~~ same as the reference process.

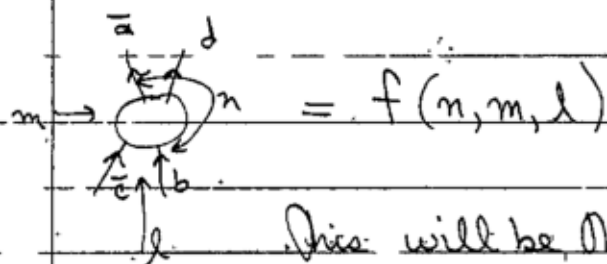


Well, here $\alpha = x$, $y = \delta$, $z = \beta$. Trivial, but could cause confusion.

④ Next, consider:



Since the second argument is energy, we could call this the "2-channel" of the reference process which is the 1-channel. Then



This will be the "3-channel" of the reference process.

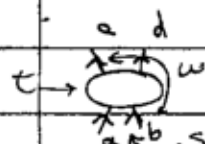
③ Once we have arbitrarily named the invariants for the reference process [here, xyz], then we could if we liked call

$$1\text{-channel} = "x\text{-channel}" = \text{reference process}$$

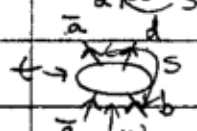
$$2\text{-channel} = "y\text{-channel}" =$$

$$3\text{-channel} = "z\text{-channel}"$$

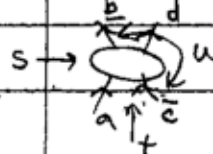
④ I have been careful to give the invariants of different processes different names. [eg, (xyz) , ABC , nml]. Some authors like to "reuse" the same names, like so:



$$= f(s, t, u) \quad "s\text{-channel}"$$



$$= f(s, t, u) \quad "u\text{-channel}"$$



$$= f(s, t, u) \quad "t\text{-channel}"$$

As usual, the process is named by its energy invariant.

In this system, all three processes "look" the same (as f.)

But, we must keep in mind the $s \neq s \neq s$ just as

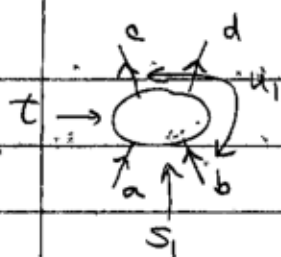
$$x \neq y \neq z.$$

$$S = (fa + pb)^2$$

$$S = (qa - qb)^2$$

no relation

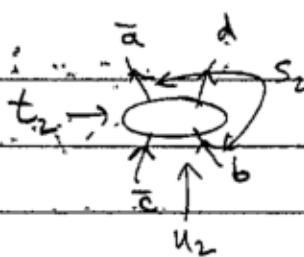
⑤ Here is a notation which perhaps bridges the two notations above:



||

$$f(s_1, t_1, u_1)$$

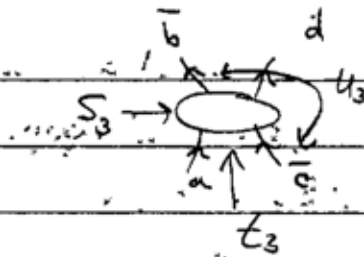
s-channel



||

$$f(s_2, t_2, u_2)$$

u-channel

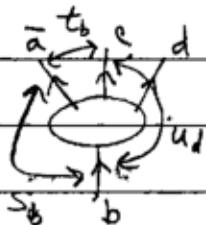


||

$$f(s_3, t_3, u_3)$$

t-channel

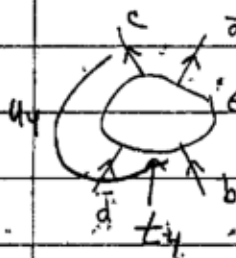
⑧ We can also do decays, for example:



$$= f(s_4, t_4, u_4)$$

decay of 'b'

⑨ All other $2 \rightarrow 2$ processes can be encompassed:



$$= f(s_4, t_4, u_4)$$

= the 4-channel

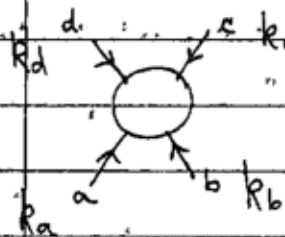
Note: This is not the "time-reversed" of the 3-process. It is the "all particles crossed" version of '3'. One might claim that 3 and 4 are equal by TC or TCP or something, but here we claim they are equal by the General Crossing Theorem, which we are now illustrating by various examples, for spinless particles only.

⑩ Thus far we have given the statement of crossing in terms of the invariants; for spinless particles.

The basic idea is to set up a "reference" process $ab \rightarrow cd$, and to label s, t, u , somehow, + then to call that amplitude $T(s, t, u)$. Then, all "crossing-related" processes can be written in terms of $T(z_1, z_2, z_3)$ by the method just discussed.

I find an alternative scheme to be clearer and more easily generalized beyond $n=4$. In this alternative scheme, one states the General Crossing Theorem (for spinless particles) in terms of momenta rather than invariants.

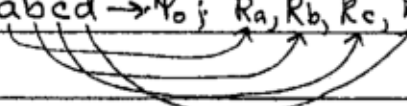
⑪ Let us now develop this scheme. We define, for reference, a Master Amplitude as follows:



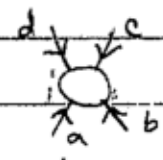
$$= M(a, k_a; b, k_b; c, k_c; d, k_d)$$

This is the amplitude for a, b, c and d (all incoming) to collide and annihilate into the vacuum. With (conventional) particles, such a process is always kinematically forbidden; nevertheless, we use this as a reference amplitude because it is so symmetric. You can draw the incoming particles in any order you like, it is all the same.

So, we are saying:

$$M(a, k_a; b, k_b; c, k_c; d, k_d) = T(abcd \rightarrow \emptyset; k_a, k_b, k_c, k_d)$$


⑫ Now suppose we wish to "get to" some interesting "crossing related" process. Here is the rule which is the General Crossing Theorem:

"In reference to the Reference Process , if you reverse a given line but leave the particle as is, then in the Master Amplitude, the physical momentum appears negated, and the particle label is barred."

Example:

$$R \quad \text{Diagram: Circle with incoming lines } a, b, c, d \text{ and momenta } k_a, k_b, k_c, k_d. \quad = \mathcal{M}(a, k_a; b, k_b; c, k_c; d, k_d)$$

/note: $k_a + k_b + k_c + k_d = 0$
/is kinematic condition which
/blocks the "reference process"

Then,

$$D_b \quad \text{Diagram: Circle with incoming lines } a, b, c, d \text{ and momenta } q_a, q_b, q_c, q_d. \text{ Line } a \text{ is crossed out.} \quad = \mathcal{M}(\bar{a}, -q_a; b, q_b; c, q_c; d, q_d)$$

Here, q_a is the F.T. "physical momentum" of the (outgoing) a particle. Note that it appears negated. Also, we have the label $\bar{a}$ instead of a .

So, we are claiming:

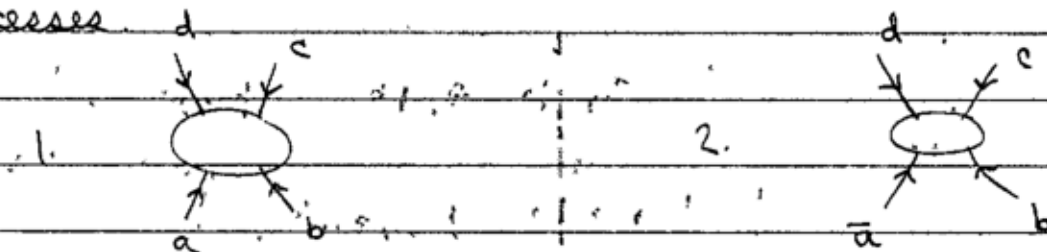
$$T(b \rightarrow adc; q_b, q_a, q_d, q_c) = \mathcal{M}(\bar{a}, -q_a; b, q_b; c, q_c; d, q_d)$$

The processes $abcd \rightarrow \gamma_0$ and $b \rightarrow adc$ are not crossing related. That is, the Crossing Theorem provides no equality or other relation between these amplitudes:

$$M(a, k_a; b, k_b; c, k_c; d, k_d) \quad \dots 1.$$

$$M(\bar{a}, k_a; b, k_b; c, k_c; d, k_d) \quad \dots 2.$$

These are the Mandel Amplitudes for two different processes.



- ⑬ Notice that the arrow on a leg merely indicates whether the particle is incoming or outgoing. The sense of this arrow has nothing to do with particle or antiparticle, unlike the situation of electrons + positrons in Feynman diagrams. What the particle is (in our notation) is indicated by the particle label.

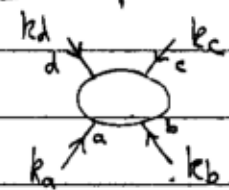
Eg. in 1. above the label is an incoming a.

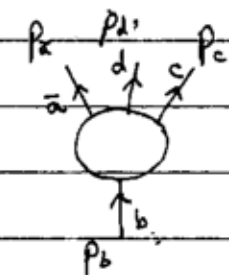
In 2. " " " an incoming $\bar{a}$.

- ⑭ In order to go from a master reference process to a crossing related process, each time you "reverse a line" [or, "cross" a particle; or "change a particle from incoming to outgoing"], you must also bar [charge conjugate] the particle label. When you do this, the physical momentum

still appears negated in the master amplitude, but the particle label in the M.A. is not changed.

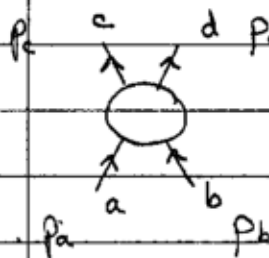
Example:

R  = $M(a, k_a; b, k_b; c, k_c; d, k_d)$

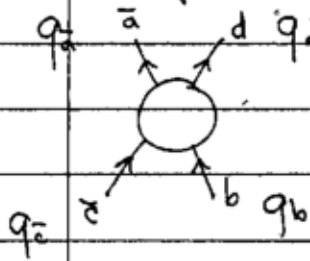
D_b  = $M(a, \bar{p}_a; b, p_b; c, p_c; d, p_d)$
 = $T(b \rightarrow \bar{a}dc; p_b, \bar{p}_a, p_d, p_c)$

no bar.
negated

⑮ Now (at last) we can attempt to relate the various 2→2 scattering amplitudes to the master amplitude. Eg,

 = $M(a, p_a; b, p_b; \bar{c}, -p_c; \bar{d}, -p_d)$
 = $T(ab \rightarrow cd; p_a, p_b, p_c, p_d)$

Compare this to:

 = $M(a, -q_a; b, q_b; \bar{c}, q_c; \bar{d}, -q_d)$
 = $T(\bar{c}b \rightarrow \bar{a}d; q_c, q_b, q_a, q_d)$

⑥ Now, via the reference amplitude, we can make a relation between these two $2 \rightarrow 2$ scattering amplitudes. This relation is:

$$T(\bar{c}b \rightarrow \bar{a}d; q_{\bar{c}}, q_b, q_{\bar{a}}, q_d) \\ = T(ab \rightarrow cd; -q_{\bar{a}}, q_b, -q_{\bar{c}}, q_d)$$

⑦ Perhaps it is more convenient if we use this as the reference amplitude:

$$T(ab \rightarrow cd; p_a, p_b, p_c, p_d)$$

We may relate any "crossing related" amplitude to this one, by the rule:

"If a particle a is "crossed" relative to the reference amplitude, then it must appear as $\bar{a}$ in the new amplitude, and its momentum q_a will appear negated in the reference amplitude."

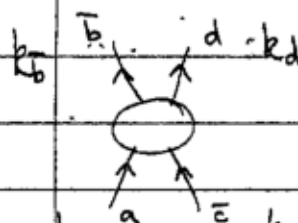
examples:

$$T(\bar{c}b \rightarrow \bar{a}d; k_{\bar{c}}, k_b, k_{\bar{a}}, k_d) = T(ab \rightarrow cd; -k_{\bar{a}}, k_b, -k_{\bar{c}}, k_d)$$

Diagram illustrating the crossing of particle a (labeled $\bar{a}$) from the initial state to the final state, resulting in the crossing of particle c (labeled $\bar{c}$) from the initial state to the final state. The diagram shows a central vertex with four external lines: $\bar{a}$ (top), d (right), b (bottom), and $\bar{c}$ (left). Arrows indicate the flow of particles. The diagram is labeled with momenta: $k_{\bar{a}}$, k_b , $k_{\bar{c}}$, and k_d .

Annotations: "appears as $\bar{a}$ in new amplitude" (pointing to $\bar{a}$) and "momentum is negated in ref. amp." (pointing to $-k_{\bar{a}}$ and $-k_{\bar{c}}$).

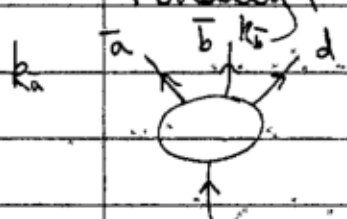
Another example 2



$$= T(a\bar{c} \rightarrow \bar{b}d; k_a, k_{\bar{c}}, k_{\bar{b}}, k_d)$$

$$= T(ab \rightarrow cd; k_a, -k_{\bar{b}}, -k_{\bar{c}}, k_d)$$

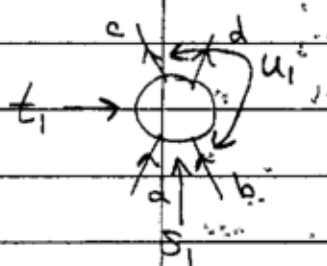
Finally,



$$= T(\bar{c} \leftrightarrow \bar{a}\bar{b}d; k_{\bar{c}}, k_{\bar{a}}, k_{\bar{b}}, k_d)$$

$$= T(ab \rightarrow cd; -k_{\bar{c}}, -k_{\bar{b}}, -k_{\bar{a}}, k_d)$$

- ⑮ We can now show that the "invariants method" and the "momenta method" are in fact the same. Let's define

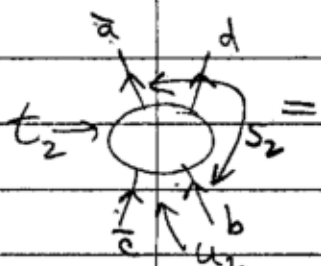


$$= T(ab \rightarrow cd; p_a, p_b, p_c, p_d)$$

$$\equiv F(ab \rightarrow cd; (p_a + p_b), (p_c - p_a), (p_d - p_a)^2)$$

$$= F(ab \rightarrow cd; s_1, t_1, u_1)$$

We wish to compare this with



$$= T(\bar{c}b \rightarrow \bar{a}d; q_{\bar{c}}, q_b, q_{\bar{a}}, q_d)$$

$$= T(ab \rightarrow cd; -q_{\bar{a}}, q_b, -q_{\bar{c}}, q_d)$$

$$= F(ab \rightarrow cd; (-q_a + q_b)^2, (-q_c + q_d)^2, (q_d + q_a)^2)$$

$$= F(ab \rightarrow cd; s_2, t_2, u_2)$$

So, we again get: $f(s_1, t_1, u_1) = f(s_2, t_2, u_2)$.

(19) I like the idea of exhibiting both the process and the momenta inside the parentheses, eg,

$$T(ab \rightarrow cd; p_1, p_2, p_3, p_4)$$

It is understood that p_1 is the momentum of the incoming a , and so on. The momenta are ordered in the same way, left to right, as the particle labels. Another notation might be:

$$T^{(ab \rightarrow cd)}(p_1, p_2, p_3, p_4)$$

(20) For particles with spin, we might use:

$$T(ab \rightarrow cd; p_a, p_b, p_c, p_d; \mu_a, \mu_b, \mu_c, \mu_d)$$

← helicities

Then the crossing relation is something like:

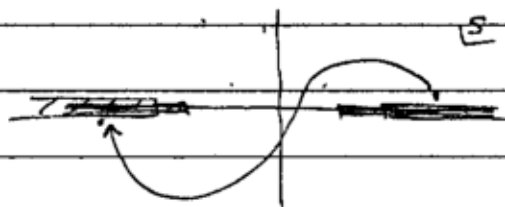
$$T(a\bar{c} \rightarrow \bar{b}d; k_a, k_c, k_{\bar{b}}, k_d; \mu_a, \mu_c, \mu_{\bar{b}}, \mu_d)$$

$$= T(ab \rightarrow cd; k_a, -k_{\bar{b}}, -k_c, k_d; \mu_a, \mu_{\bar{b}}, \mu_c, \mu_d)$$

I think under "crossing" each particle "keeps" its helicity.

The fancy linear combinations on helicity only arise when you try to compare the center-of-mass amplitudes of these two processes. To relate such amplitudes, you have to do a fancy complex Lorentz transformation which brings in all these d 's.

(2) Crossing is the statement that all "crossing-related" amplitudes can be obtained from the same function $f(s, t, u)$. We have discussed specifically what this means. Somehow, it seems to me that the crossing "assumption" necessarily follows from the "analyticity assumption". If $T(s, t)$ is analytic in s , then you can "get to" the u -ch from the s -ch. by analytic continuation:



Since such continuation is unique, crossing is implicit.

As we learned from Eden et al., however, this continuation should more correctly be considered in a four-dimensional space: $[E_s, \Delta s, E_t, \Delta t]$.

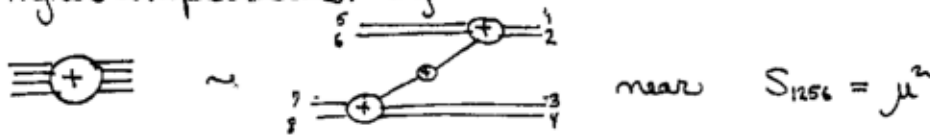
The path shown above is $\sim$ is really a curve in this $\mathbb{R}^4 = \mathbb{C}^2$. We imagine that $\Delta t = 0$, $\Delta s = 0$ during this cont, and we claim that no strange cuts are encountered [this is "analyticity of the physical sheet" of LS].

Eden et al. section 4.8 "Anti-Particles, TCP, crossing"

Note: in my last Eden et al reading, I stopped short of this section, but read some of section 4.9

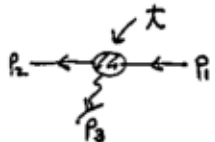
- ① In earlier sections it was shown that there exist stable particle poles in the higher amplitudes. Eg

claim:
P pole
↓
P pole.



near $S_{1256} = \mu^2$

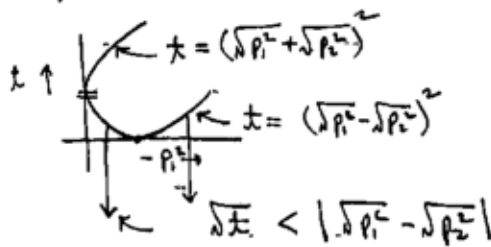
- ② What does Gram say about S_{1256}, S_{12}, S_{56} ?



$$G\left(\begin{smallmatrix} 12 \\ 12 \end{smallmatrix}\right) < 0 \Rightarrow \Delta(P_1^2, P_2^2, (\alpha P_1 + \beta P_2)^2) > 0 \quad \alpha, \beta = \pm 1.$$

Thus $\Delta(P_1^2, P_2^2, (P_2 - P_1)^2) = \Delta(P_1^2, P_2^2, t) > 0$

Thus, there is a restriction on values t may take:

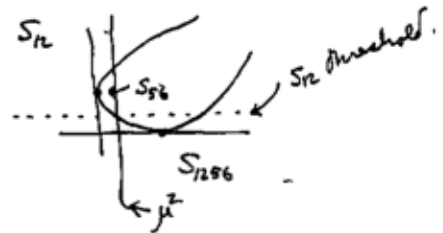


$t = (\sqrt{P_1^2} + \sqrt{P_2^2})^2$
 $t = (\sqrt{P_1^2} - \sqrt{P_2^2})^2$
 $\sqrt{t} < |\sqrt{P_1^2} - \sqrt{P_2^2}|$

$$S_{1256} \leq (\sqrt{S_{12}} - \sqrt{S_{56}})^2$$

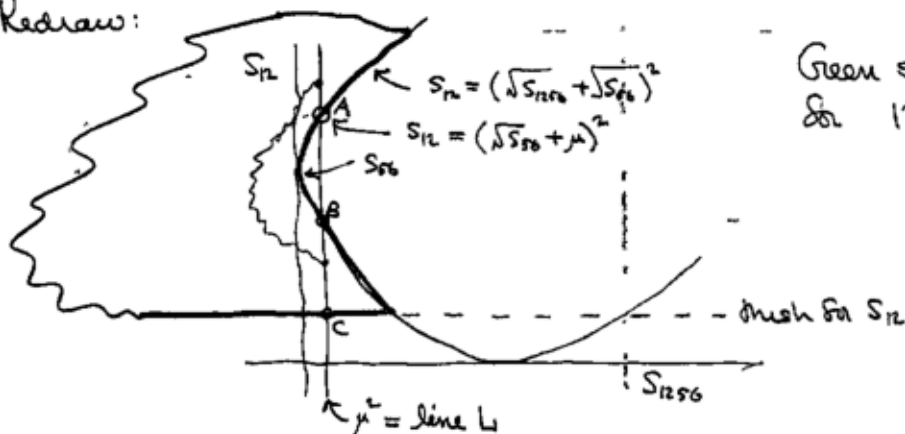
In our case, $P_1^2 \rightarrow S_{12}$ $P_2^2 \rightarrow S_{56}$ $t \rightarrow S_{1256}$, so

- ③ Flip the graph.



Choose an S_{56} above its threshold.

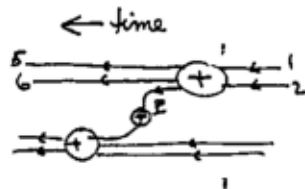
Redraw:



Green shows phys. region
 for $1234 \rightarrow 5678$.

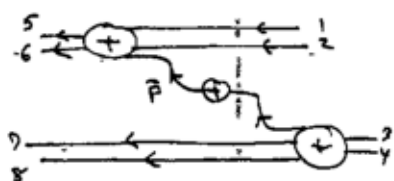
④ Now consider this picture. We are claiming that our $4 \rightarrow 4$ amplitude has a pole at $S_{1256} = \mu^2$; we claim this in general, but the pole appears in a physical region above A ~ ie., we interpret it as:

ie, P has the quantum numbers apropos for $12 \rightarrow 56P$



Two "successive" connected parts, everything on shell.

How then do we interpret the re-entrance into the physical region of this same pole at BC? We see that a possible interp is:



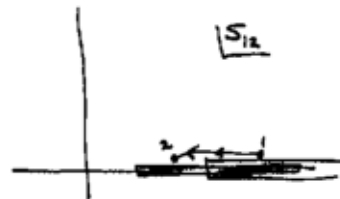
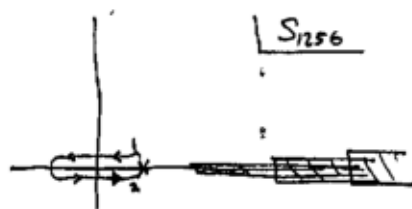
$$\sqrt{S_{76}} \geq (\mu + \sqrt{S_{12}})$$

Assuming this interp is correct, then there must exist a pole $\bar{P}$ for every pole P, but with all additive quantum numbers reversed.

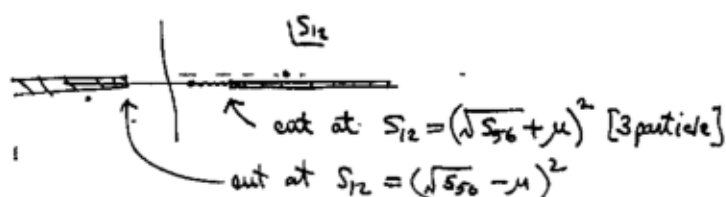
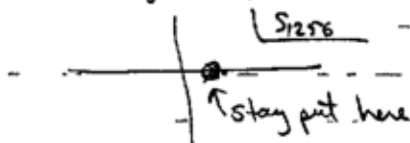
catch 22

⑤ In order to prove that this pole is on the physical sheet, we have to find a path of continuation in the multivariable complex space Consider:

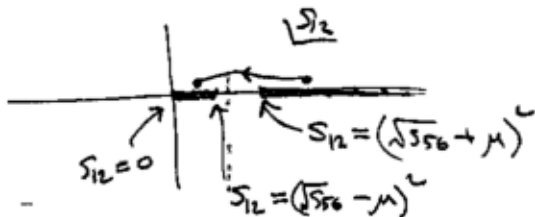
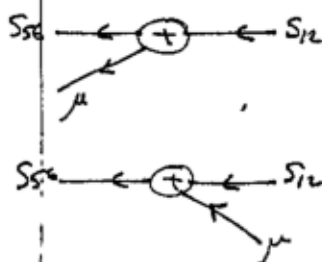
Suppose we were to wander from above A to the BC area as shown pl. Why? Why not ride the pole right down L? In that case we'd have:



Why? Why not ride the pole right down L? In that case we'd have:



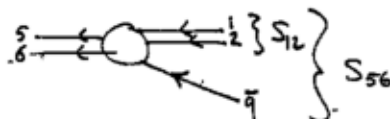
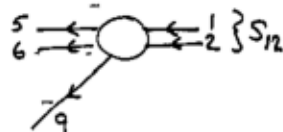
⑥ Simplify:



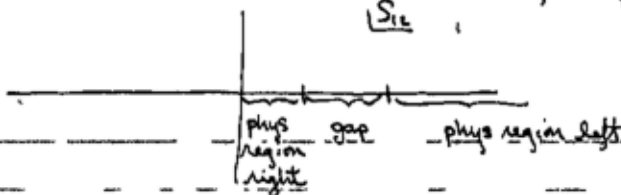
We have to leap across an unphysical gap here.

The second continuation requires continuation in S_{12} through a place we know nothing about. However, we may avoid the branch points by deforming.

① Better example

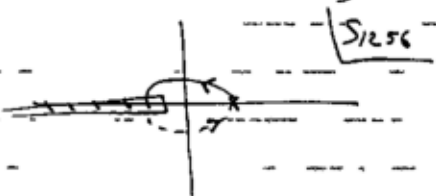


In the second case, S_{12} is like a mass variable. At given S_{56} , this amp vanishes if S_{12} is too large. But, I do not think the amplitude has a singularity at this point in S_{12} . There is a ring at $S_{12} = (\sqrt{S_{56}} + \mu)^2$, via unitarity? Well, no, not even there. So



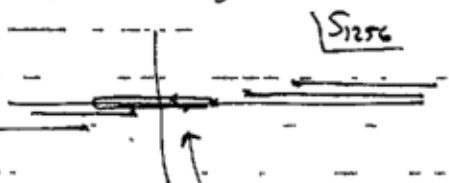
If we could continue also in μ^2 , we could get rid of this gap, but that does not make sense to me. $\mu^2 = \text{mass}^2$ - you do not usually continue in an external mass? But μ^2 is an internal pole, so you can do this.

So, in the $S_{12}S_{56}$ plane, back at ⑤, one must be careful that one does not accidentally slide through a cut:



If you did, you would be on the unphysical sheet of the $S_{12}S_{56}$ plane, & a pole there is not a pole in the amplitude.

③ Well, Eden argues that even if there were cuts, we use $+i\epsilon$, so if we stay on $S_{12}S_{56} = \text{real}$, there cannot be caught:



However, there could be some (not normal threshold) singularity sitting in here that has real b.p. so that would invalidate the proof.

⑨ Conclusion: as long as the SISO-plane is "clean", then the "pole above A" will "persist" and will be present also in the "BC-physical region." Thus, every particle has an antiparticle of exactly the same mass.

⑩ Review hermitian analyticity: ✓

⑪ Now consider

Claim:

$$\begin{cases} A = T_{fi} \\ A' = T_{if} \\ \text{h.a.} \Rightarrow A = A' \end{cases}$$

$$\text{Diagram with incoming lines from left and outgoing lines to right} = A$$

$$A \text{ cont'd to other region} = \alpha \times \bar{A}$$

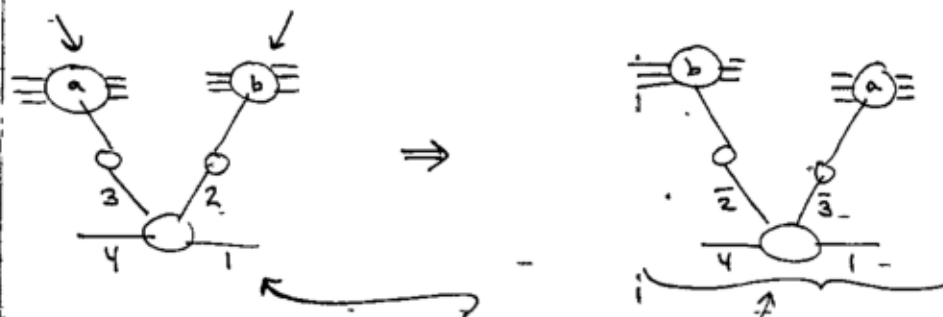
$$\text{Diagram with incoming lines from right and outgoing lines to left} = \bar{A}$$

one can show that $\alpha =$ just a phase. Often, one sets the phase from other considerations (like isospin, etc).

By playing with h.a.,

⑫ Thus, we have established the principle of crossing for one particle. For two particles at once, use

Crossing (proof of)



ie, simultaneous poles here at approx subenergies $\Rightarrow$ same poles occur in this other amp. Interpret these new poles as antiparticles. Continue all residues to conclude:

$$(12 \rightarrow 34) \text{ continues to } (1\bar{3} \rightarrow 24)$$

TCP (proof of)

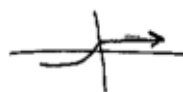
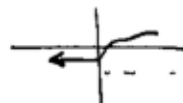
This is CROSSING!

⑬ Always be above cut in channel of interest (+iε prescription).

⑭ Consider 4.8.10, 11, 12, and figure 4.8.2. Read p243-4. Claim is that we have in fact proven that $10 = 12$. This is TCP:

$$\text{amp } (ab \rightarrow cd) = (\bar{c}\bar{d} \rightarrow \bar{a}\bar{b})$$

The idea is that you cross once to get to $10 \rightarrow 11$.
 Then you cross the other pair to go $11 \rightarrow 2$.
 This takes you back to the start!
 So, amps are equal.



⑤ Why called TCP? Cannot answer because I do not know what T does to an amp. Perhaps:

$$\langle 3, \vec{p}_3, m_3; 4, \vec{p}_4, m_4 | T_P | 1, \vec{p}_1, m_1; 2, \vec{p}_2, m_2 \rangle \rightarrow \text{spin amp.}$$

Recall that:

$$\begin{aligned} \langle a | U | b \rangle &= \langle a | U | b \rangle \quad \text{for unitary} \\ \langle b | A | b \rangle &= [\langle b | A | b \rangle]^* \quad \text{for antiunitary} \end{aligned}$$

So perhaps we say:

□ Time reversal is "on the list."

$$\langle 34 | T_P | 12 \rangle = \langle 34 | (T^\dagger T_P T) | 12 \rangle$$

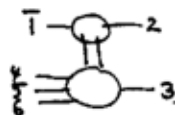
$$\begin{aligned} &= [\langle 34 | T^\dagger T_P T | 12 \rangle]^* \\ &= \langle 34 | T_P | 12 \rangle^* \dots \text{unclear.} \end{aligned}$$

poles

⑩ Implications of crossing. First, an example:



cross

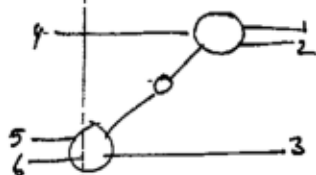


Thus, the process:

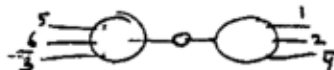


has a 2 particle normal threshold in the t -variable because, if you crossed, you would see it.

⑪ Important example:



cross



our first direct channel pole.

⑫ Eden claims that using this fancy disc cross too.

it can be shown that even has poles.

⑬ Cross on disc. of the "fancy disc. formulae". I just read that fact.