

Correspondence
with
Reuben Freeman

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Feb 10, 1978

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Mar 28, 1978

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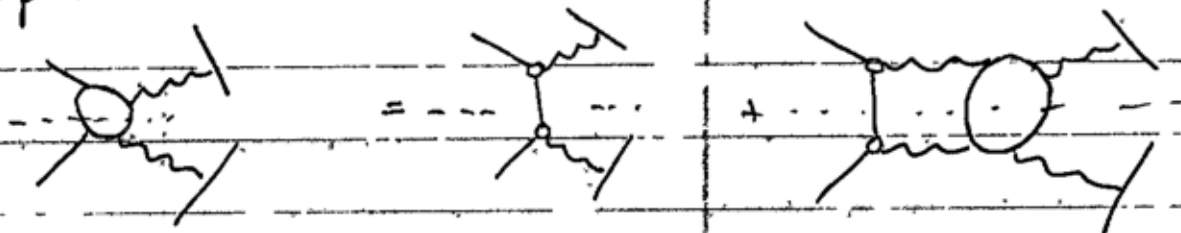
Phil :

Feb. 8/78

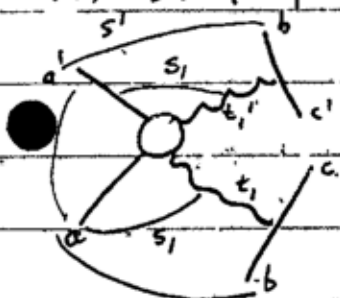
Thanks for "the" copy of your thesis. I confess that I have not waded through every detail (nor am I likely to do so) but I do have some comments to which you may want to respond.

I'm worried whether the analytic structure of Regge theory has not somehow been glossed over, your footnote on p. 6. is perhaps overly optimistic that sweeping this problem under the rug is O.K. Certainly using group-theoretic variables affords a significant simplification in kinematics but isn't the analytic structure obscured. In particular I'm concerned about the requirement that the only allowed multiple discontinuities are from cuts in non-overlapping channels. (Steinmann conditions). Using invariant variables (as in the review by Brower, DeTar and Lewis) allows one to handle Steinmann-ism simply. I don't think it can be ignored in ordered unitarity, cylinder building etc. because the analytic structure specifies what variables are to be held fixed while others are integrated over in unitarity sums.

At the risk of boring you, let me outline a typical unitarity relation in invariant variables then try to convince me why it should be substantially different in group-theoretic variables. Consider



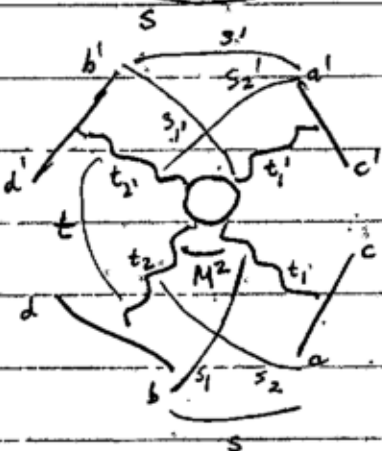
FIRST STEP: Define Reggeon amplitudes



$$A_b = \gamma_{bc}(t_i) \gamma_{b'c'}(t_{i'}) \Gamma(-\alpha(t_i)) \Gamma(-\alpha(t_{i'}))$$

$$(-s)^{\alpha(t_i)} (-s')^{\alpha(t_{i'})} A(s_i, t_i, t_{i'}, t_{i'})$$

$R, d \rightarrow R, a'$



A_s (helicity pole limit)

$$= \gamma_{ac}(t_1) \gamma_{bd}(t_2) \gamma_{a'c'}(t_{i'}) \gamma_{b'd'}(t_{i'})$$

$$\Gamma(-\alpha(t_1)) \Gamma(-\alpha(t_{i'})) \Gamma(-\alpha(t_2)) \Gamma(-\alpha(t_{i'}))$$

$s' = s^*$

$$(-s_1)^{\alpha(t_2)} (-s_2)^{\alpha(t_1)} (-s_{i'})^{\alpha(t_{i'})} (-s_{i'})^{\alpha(t_{i'})}$$

$$A(M^2, t, t_i, t_{i'}, t_2, t_{i'}, \frac{s_1 s_2}{s})$$

$R_2 R_1 \rightarrow R_2' R_1'$

(3)

Note $\frac{s_1 s_2}{s} = \frac{s_1' s_2'}{s} = K = M^2 + E(t, t_1, t_1', t_2, t_2')$ in H.P. limit

Now the four Reggeon amplitude is an extremely complicated object. It has, for example, several Steinmann pieces.

NEXT write down the unitarity sum
say for $t=0$ to simplify life.



$$\gamma_{ac}(t_1) \gamma_{a'c'}(t_1') P(-\alpha(t_1)) P(-\alpha(t_1'))$$

set $t_2 = t_2' = t_1'$

$$\int \frac{dt_1'}{16\pi^3} \sum_d \gamma_{bd}(t_2) \gamma_{b'd'}(t_2') P(-\alpha(t_2)) P(-\alpha(t_2'))$$

$$\int \frac{dM^2}{s_1} s_1 f(t_1') (-s_1)^{\alpha(t_2) + \alpha(t_2')} (-s_2)^{\alpha(t_1) + \alpha(t_1')}$$

threshold

$$\cos \pi (\alpha(t_2) - \alpha(t_2'))$$

$$\frac{1}{2i} \bigtriangleup_{M^2} A(M^2, K, t=0, t_1, t_1', t_2, t_2')$$

$k_1, k_2 \rightarrow k_1, k_2$

Now I must express s_2 in terms of K and s_1 and s to do the integration as the fixed variables

(4)

the output are s and s_1 . The underlying sum is then

$$\int dt' \dots \int_{\text{thresh}} \frac{dM^2}{s_1} \left(-\frac{s K}{s_1} \right)^{\alpha(t_1) + \alpha(t_1')} (s_1)^{\alpha(t_2) + \alpha(t_2')} \frac{1}{2i} \Delta A_{M^2}$$

$K = M^2 + \epsilon$

This should integrate into something that behaves like

$$(-s)^{\alpha(t_1) + \alpha(t_1')} \frac{1}{2i} \Delta A_{K, a \rightarrow K, a}(s_1, \dots)$$

which for large s goes like

$$(-s)^{\alpha(t_1) + \alpha(t_1')} s_1^{\alpha(t=0) - \alpha(t_1) - \alpha(t_1') + 1}$$

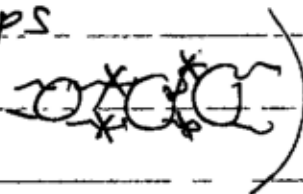
the point is: the 4-Reggeon amplitude is weighted in a particular way in the unitarity integrals. No one suggests, as you seem to think on p.60 that the 4 Reggeon amplitude is asymmetric. It is symmetric but the weighting of the unitarity integrals is not. I don't see how going to different variables is going to alter this.

(5)

Stated another way: having defined a four Reggeon amplitude A_{RR} by amputating the appropriate external legs of A_8

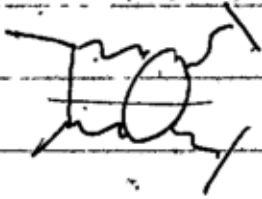


one must be careful just how the 4-Reggeon amplitude appears in loops (in unitarity or cylinder building



One can check the structure of a loop sum in various models: just to be sure.

For example the structure of the unitarity type loop



can be checked

in the Dual Resonance Model and agree with what I wrote down.

So, my suspicion [perhaps I'm wrong] is that somehow analytic structure is washed out in your approach. If not, what's the analog of the asymmetric integral that I've described?

(6)

This letter is getting far too long, but let me continue.

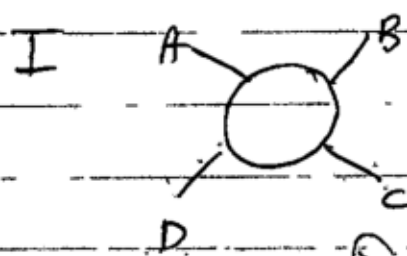
Your statement following eq. (11.12) that $C^{(1)}$ of (11.6) may be trivially Mellin diagonalized is not entirely correct. It is true provided the functions $A(t_1^\pm, x, t_2^\pm)$ etc. (i.e. the integrand of 11.6) is not pathological — very singular. The reason is that interchange of variables of integration becomes rather tricky if the integrand has singularities near the integration boundaries. Now your particular choice (11.9) of amplitudes when inverse - Mellin transformed back to the energy (or rapidity) plane is very singular (worse than the derivative of a δ -function), (11.9) does not correspond simply to (11.8)

$$s^\alpha \longrightarrow \overline{j-\alpha}$$

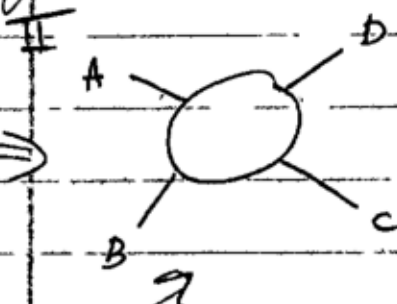
So if you want non-trivial A (as in 11.9) you'd better be careful about "instant diagonalization." (See the paper by Zarmi and me.)

(7)

Finally, perhaps you could enlighten me as to why ordered amplitudes necessarily have an orientation. Could not one consistently leave out amplitudes that are just flips.



flip about AC $\Rightarrow$



different order but same channel structure

Why not everywhere leave out II. Isn't this something like what Stapp was suggesting last summer?

Keep well!

Reuben

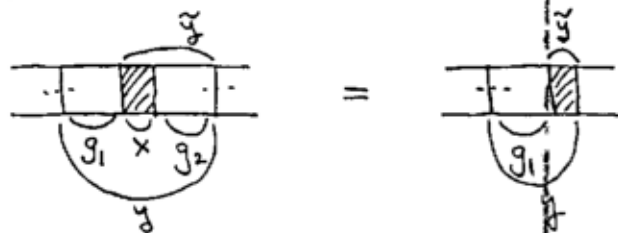
Feb 28, '78

Dear Reuben,

Thanks for your comments, and for looking at my Thesis. I rather regret that, while I was a grad student at Berkeley, there was no one else there who was interested in the technical side of these bootstrap and cylinder calculations. Geoff and his students Weissman, Millan and Sursock were more interested in such things as "proving" the existence of quarks + "generalizing" to baryons. As a result, I read all the Rutherford papers + Veneziano's + yours + did the best I could to assimilate their contents, without the benefit of "discussions". You have no doubt located some errors in my perception of the whole business, but I will try to make a defense anyway (below). Ironically, now that I have invested so heavily in the TE business, my present course is directed to infrared QCD calculations (integral equations, however!) since this is Tim Ball's interest.

① Concerning the famous "weight factor" in the FMSR:

Let's start with a little rapidity bootstrap equation that is sort of halfway in your variables + halfway in mine:



Roughly, the connection to your variables is

$$s_1 = e^{(g_1+x)}$$

$$s_2 = e^{(g_2+x)}$$

$$e^x = m^2$$

$$e^y = s$$

$$m^2 \approx \frac{s_1 s_2}{s} \rightarrow y = g_1 + x + g_2$$

(Yes, this model is ignoring difference between m^2 and $m_{\perp}^2$, for the moment.)
 Where you say:

$$A_8 \approx (-s_1/s_0)^{d_1+1} (-s_2/s_0)^{d_2+1} A_{RR_2}(m^2)$$

↖ analytic in m^2
no L.H. cut

I will put in the following:

$$A_8 \approx P_1(g_1) A_{12}(x) P_2(g_2)$$

where $P_1(g_1) = \exp([x_1 + i] g_1)$ call it $e^{\beta_1 g_1}$. I will agree to hold g_1 fixed ($\approx$ hold s_1 fixed); the integral equation is then:

$$\int_0^{y-g_1} dx P_1(g_1) A_{12}(x) P_2(y-g_1-x) = P_1(g_1) A(y-g_1) + \text{inhom term, here ignored.}$$

Thus

$$\int_0^{\tilde{y}} dx A(x) P(\tilde{y}-x) = A(\tilde{y})$$

or

$$A(\tilde{y}) = \int_0^{\tilde{y}} dx A(x) e^{\beta_2(\tilde{y}-x)} = e^{\beta_2 \tilde{y}} \cdot \int_0^{\tilde{y}} dx A(x) e^{-x\beta_2}$$

↑ weight factor.

AHA! I must certainly agree: There is for sure an asymmetric weight factor in the finite mass unitarity integration. I agree thoroughly.

Upon applying $\int_0^{\infty} d\tilde{y} e^{-j\tilde{y}}$ to both sides of the above "bootstrap equation", one gets

$$A_j = A_j P_j + \text{inhom}$$

where

$$A_j = \int_0^{\infty} dx A(x) e^{-jx}$$

↑
no weight factor.

$$P_j = \int_0^{\infty} dg_2 e^{-jg_2} P(g_2)$$

So in the diagonalized bootstrap there is no weight factor in the Mellin projection, but there is a weight factor in the energy-plane integral equation.

A "good FMSR" will now say: $\int_0^{\infty} dx A(x) e^{-x\beta_2} = \text{Regge terms only, no constants.}$

I would read this as:

$A_{j=\beta_2} = 0$, a nonsense-zero (superconvergence, or whatever you want to call it)

So lets go back to $A_g = (-s_1)^{d_1+1} (-s_2)^{d_2+1} A_{RR}(m^2)$. I would be tempted to say:

"The reason there is a weight factor in the FMSR integral is that you are holding s_1 fixed. Because $s_2 \approx m^2/s_1$, the outside factor $(-s_2)^{d_2+1}$ yields a weight factor $(m^2)^{d_2+1}$ " When I hold the "group variable" g_1 fixed, same result!

I would say this even if I knew nothing about Steinmann relations. If I like, I can probe down into the structure of A_{RR} to conclude something like [faking the A_g case]:

$$A_g = (-s_1)^{d_1+1} (-s_2)^{d_2+1} \left\{ \left(\frac{s_1 s_2}{s} \right)^{-d_1-1} A_{RR}^{(1)} + \left(\frac{s_1 s_2}{s} \right)^{-d_2-1} A_{RR}^{(2)} \right\}$$

$$= (-s_2)^{d_2-d_1} (-s)^{d_1+1} A^{(1)} + (-s_1)^{d_1-d_2} (-s)^{d_2+1} A^{(2)}, \text{ on whatever.}$$

And I can say: Look, no overlapping singularities. [The same form occurs in your Weiz-quoted eqn. (2) in first FJ paper, when the Jth contour is shifted]

But it seems to me that the Steinmann-relations have really nothing to do with the presence of that weight factor.

② What's this noise about "asymmetric RR amplitudes?"

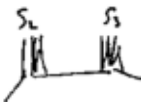
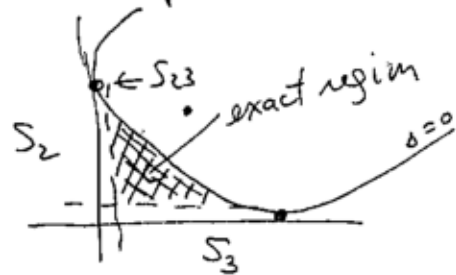
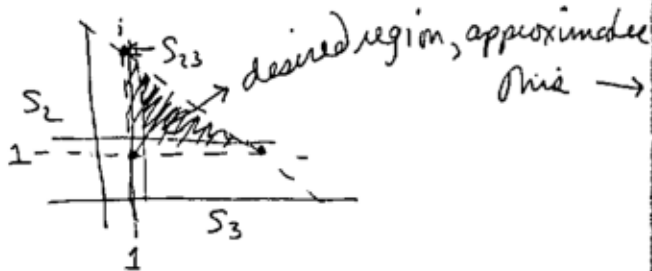
Lets now keep the distinction between m^2 and $m_\perp^2$ and go right to your one-twist cylinder equation [all at $\tau=1$]:

$$M(s_{23}) = \int d\phi_2 \int ds_2 (s_2^\pm)^{d_1+1} A_{12}(s_2, s_2^\pm)$$

$$\times \int ds_3 (s_3^\pm)^{d_3+1} A_{23}(s_3, s_3^\pm) \cdot (s_{23})^{d_2-d_1-d_3-2} \Theta(s_{23} - s_2 s_3). \quad (*)$$

Digression: By the way, I would use $s_2 = 1$ and not $s_2 = 0$ as lower end point ~ for several related reasons: (a) s_2 is really s_2/s_0 , which integration begins at, say, $s_2 = 1.369$, the lowest threshold of $A_{12}(s_2)$; (b) in sense that

$S_2 = e^{X_2}$, $S_2 = 0$ makes you think of $X_2 = -\infty$ but usually lower end is $X_2 = 0$; (c) if S_2 is allowed > 1 , then you get the illusion that S_3 is allowed $> S_{23}$. Compare:



rapidity $\Rightarrow \Theta(S_{23} - S_2 S_3)$

$P_{\text{res}} > 0 \Rightarrow \Theta[\Delta(S_{23}, S_2, S_3)]$.

Yes, I know you know all this ~ its just that the Mellin diagonalization requires 1 as lower end point.

→ end of digression

So, I now apply $\int_1^\infty dS_{23} (S_{23})^{-j-1}$ to both sides of (*).

Then define j by $j = j - \alpha_1 - \alpha_3 - 2$ to get: (**)

$$(\text{**}) \quad \tilde{C}_1(j - \alpha_1 - \alpha_3 - 2) = \int d\phi_2 S_2 \tilde{A}_{12}(j - \alpha_1 - \alpha_2 - 2) \left[\frac{1}{j - \alpha_2} \right] \tilde{A}_{23}(j - \alpha_2 - \alpha_3 - 2)$$

where $\tilde{C}_1(j) \equiv \int_1^\infty dS_{23} (S_{23})^{-j-1} C_1(S_{23})$

but, $\tilde{A}_{12}(j) \equiv \int_1^\infty dS_2 (S_2^+)^{\alpha_1+1} (S_2)^{-j-\alpha_1-2} A_{12}$

$$\tilde{A}_{23}(j) \equiv \int_1^\infty dS_3 (S_3^+)^{\alpha_3+1} (S_3)^{-j-\alpha_3-2} A_{23}$$

If we ignore difference between S_2 and S_2^+ (hi-energy approx), then these are all regular Mellin projections and in effect this analysis is like the thing back on page 2.

However, even without the HE approx, it seems to me that the $\tilde{C}_1$ equation still diagonalizes. The price paid is that the projected RR amplitudes are asymmetric.

I imagine you might agree with the last page because among other things, you do get your asymmetric-weighted good FMSR, which is to say,

$$\tilde{A}_{12}(J = \overset{-S_{max}-1}{-(d_1+1)}-1) = \int_1^\infty ds_2 (s_2^\pm)^{d_1+1} A_{12}(s_1, s_2^\pm) = 0.$$

This "nonsense zero" has the same content as your FMSR (2.2) of your cylinder paper. [Compare Eq's (4) and (10) of Dolan Horn + Schmidt]

So this is why I sometimes worry about asymmetric (projected) reggeon-reggeon amplitudes. Yes, I agree that the energy-plane RR amplitude is always symmetric.

[I seem to recall that people (other than myself) have run into such asymmetric projections before. In particular I am now staring at Eq. (4.16) of your paper with Darni ~ This thing may not be a "RR amplitude", but it is certainly an asymmetric projection of something + it has a nonsense zero $(J-d_1)$ but no $(J-d_2)$. No doubt if I restudied your paper things would become clear ~ it just seems to me that there have been papers using asymmetric projection...]

Anyway, getting back to my diagonalized C1 equation (**) [which I am sure you do not approve of...], I would say: The apparent cut being caused by $[\frac{1}{J-d_2}]$

is removed by the fact that the projected amplitudes $\tilde{A}$ have nonsense-zeros precisely at $j = d_2$. In the energy-plane analysis, it is the "good FMSR" that remove the cut ~ all the same thing. This is the basis of my "typical rapidity cylinder" section of Thesis.

[What do you think? (Check me)]
 — a) I am right
 — b) This is a pile of baloney.]

→ I think, then, that the use of nonsense-zeros whether in exact j or Mellin j is in tacit agreement with your good FMSR.

What is still peculiar is this: when I do the exact diagonalization of the C1 equation, I get a result very much like $(**)$ above [(11.27) of Thesis]. But, I think the (Froissart-Gibon) projective which appear in this equation are somehow symmetric again under $R_1 \leftrightarrow R_2$:

$$q_{j\mu}^j(1,2) \equiv \int dg_1 q_{j\mu}^j(g_1) a_{12}(g_1)$$

$$\stackrel{\text{like } z}{z=du_1} = \int d(du_1) q_{j\mu}^j(du_1) \cdot \int \frac{ds_1}{2\pi} e^{-u s_1} \cdot \int ds'_1 e^{-s'_1} a_{12}(s_1, u, s'_1)$$

and $q_{j\mu}^j = q_{j\mu}^j$. It seems that $q_{j\mu}^j(1,2) = q_{j\mu}^j(2,1)$.

It seems strange (but pleasing) that the exact projections ~~are~~ symmetric in this sense which appear in the exactly diag'd C1 equation are $1 \leftrightarrow 2$ symmetric (just like the energy-plane RL amp's), whereas the approx projections which appear in the approx C1 equation are asymmetric.

Nevertheless, I am sure your PMSR survive because they correspond to nonsense-zeros of the ^{exact} symmetric projected amplitude at an "asymmetric" value of j . This gives rise to that asymmetric weighting factor in the PMSR.

③ Concerning "trivial Mellin diagonalization"

You were commenting on the Mellin transform,

$$\tilde{A}(j) = \int ds s^{-j-1} A(s) \quad (1)$$

$$A(s) = \frac{1}{2\pi i} \oint dj s^j \tilde{A}(j). \quad (2)$$

I agree that if you take, say, $\tilde{A}(j) \approx \frac{(j-\alpha_1)(j-\alpha_2)}{(j-\alpha)}$,

the $A(s)$ so generated will be disastrously singular

at $s=1$, derivatives of delta's etc. The Mellin expansion is complaining vehemently because we have instructed it

to generate a reasonable threshold but we have withheld the required ingredients, i.e., complex pole sequence(s). If you don't feed it ~~properly~~ properly, it won't jump through the hoop.

I think I would say that implicit in the projection (1) of the discontinuity $A(s)$ is the fact that $A(s)$ has a lowest threshold at $s_{th} > 1$ (recall 1.369) + therefore the integration "misses" the double at $s=1$. The threshold is of course generated by those ~~neglected~~ complex pole terms in the true $\tilde{A}(j)$ which are neglected when one says $\tilde{A}(j) \propto \frac{\text{polynomial}}{(j-1)}$.

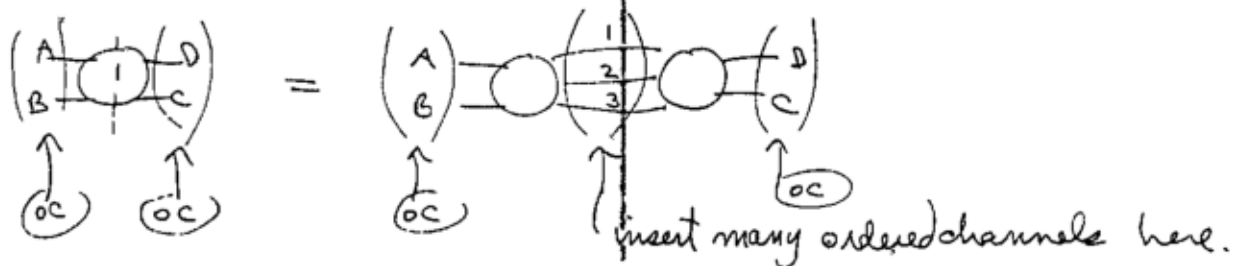
The point: The fact that some approximate $\tilde{A}(j)$ does not look good at $s=1$ is no reason not to proceed with a Mellin diagram where of course the exact (Mellin) $\tilde{A}(j)$ is implied.

In particular, I agree that my ~~old~~ projection (11.9) is not going to regenerate via (2) the threshold at " $s=0$ " appearing in (11.7). But I can still diag. (11.6) to get (11.13). [yes, I have looked at your appendix in F2].

④ Ordered amplitudes + "orientation"

I'm not sure I understand your question here, but I think you are saying: "why not characterize an "ordered amplitude" by its "adjacent-channel structure" and then no need to worry about the orientation, whether CW or CCW. When you encounter two flip-related terms, just drop one of them."

Somewhat I think ~~you~~ you want the idea of an ordered channel $\begin{pmatrix} A \\ B \end{pmatrix}$ because this idea leads to the correct "ordered bootstraps" + discontinuity formulas, eg,



Once you accept "ordered channels", you can easily prove "ordered crossing" which then implies a significance only to the overall sequence around the ring.

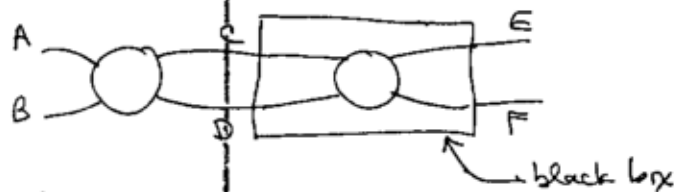
OK, but you say fine, why not identify "sequence" with "channel structure" + not choose an orientation... i.e., why have a CW Hilbert space of ordered channels, and also a CCW H-S of order chns.

The reason is, I think, that you want to obtain the sense in your equations;

$$\bigcirc \rightarrow = \bigcirc \leftarrow \bigcirc \leftarrow$$

otherwise, if you ignore orientation & look only at channel structure, you will accidentally make cylinder terms in planar $\mathcal{V}$.

I.e., consider:



If I only look at channel structure, I can replace the black box amplitude with its channel-equivalent:



But clearly this is a cylinder topology not be sitting in planar bootstrap.

$$\bigcirc \rightarrow \bigcirc \rightarrow = \bigcirc \rightarrow + \text{should}$$

You might say: just drop this

$$\bigcirc \rightarrow \bigcirc \rightarrow + \text{keep only } \bigcirc \rightarrow \bigcirc \rightarrow$$

I guess I would say: which one should you drop? What reasonable rule can you make? In effect, the orientation idea provides a "reasonable rule" and yields a scheme that works:

$$\bigcirc \leftarrow = \bigcirc \rightarrow \times \bigcirc \rightarrow$$

(Counterclockwise bootstrap) • equivalent.

$$\bigcirc \rightarrow = \bigcirc \rightarrow \times \bigcirc \rightarrow$$

(clockwise bootstrap)

level.

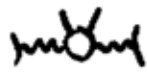
Summary of this lengthy letter

- a) I am not so sure there really is a conflict between the group variables and analytic structure/Steinmannism. Before I wrote this letter I thought there might be, but now I see no "washing out" of anything by the group analysis.
- b) In particular, the famous asymmetric weight factor arises not so much from the internal Steinmann structure of the n -point function as from the way you setup the bootstrap equation. You can hold s_1 fixed, I can hold g_1 fixed (more precisely, my helicity-pole propagator variable ξ_1). (Maybe I am missing an important point here ...?)
- c) Your "good FMSR" are equivalent to nonsense zeros in either the Mellin projection or the exact Froissart projection. These zeros occur at points like $j = \alpha_1 + \alpha'_1 - 1$ and $j = \alpha_2 + \alpha'_2 - 1$; the choice of one or the other of these nonsense points corresponds to your FMSR having one or the other asymmetric weight factor.
- d) Unless I have made some mistake, I am able to Mellin diagonalize the (C1) equation so that it looks like
"C(j) = A(j) $\frac{1}{j-\alpha}$ A(j), without making the HE approx of saying $(M^2_\perp) = M^2$. I.e., I do not make low-energy FMSR error. The cost is asymmetric projections. Of course there is implicit low-energy error in Mellin analysis because $Q_j(z) \neq z^{j-1}$, etc, if you want to identify $j = \text{something}$ to do with Regge trajectories (i.e., ang. m.m.). Moreover, I am encouraged by this nice diagonal form of the C1 because the same form results in the exact diag.

Best regards,

Phil

Jones/Freeman papers:

1. "Toward a General Proof..." . Think of A_6 in this way:  . It looks like an A_5 amplitude, and you know there are two Steinman terms. In my notes for this, the first FJ paper, I have shown how these two Steinman terms are contained in Weis' formula (2). If you are doing some kind of unitarity calc, you want as part of your integration this: $\int dM^2 \text{disc } A_6(M^2)$. I.e, we are sort of faking things by considering this picture:



By not bothering to Reggeize the upper particles, they keep it simple. But it is not clear to me that you really want the integral quoted. Accepting that you do want this quantity, you can expand A_6 in terms of two 2/reggeon/2-particle amplitudes, one for each Steinman term. These Q 's are each normal analytic in M^2 , but for fixed s, s_1 say are multiplied by certain $(-M^2)^{\dots}$ factors which cause kinematic cuts in the M^2 -plane. Thus, in any integrals you encounter, they claim that your M^2 integrals of reggeon amplitudes will always be weighted by such kinematic factors. This perhaps leads to asymmetric projections. Why don't I encounter these things in my method?

2. "Criteria for good FMSR..." . Here they do it again, consider A_6 using slightly different notation with V_1 and V_2 Steinman terms. My notes there summarize it well. Again you want a FM integral of disc A_6 ; you write out the two terms; the left hand cut thing has to go away to get cut killing. Remember that when the FM integral of something gives you only regge powers, the Mellin projection of that something has nonsense-zeros. Still I am not very clear on how to connect this stuff to me.

3. "Cancellation of..." Third paper of the sequence. Here they are finally considering the 4-reggeon ring discontinuity of Fig. 2. You are clearly going to have some kind of M^2 integral as part of your unitarity phase space. That integration appears in (2.1). Again, you see that your 4-reggeon amplitude is weighted by a certain factor. This is that problematic asymmetrizing factor.

Comments: none of these 3 papers makes any mention of Mellin projections. They are doing energy-plane bootstrap, so all they care about is finite mass integrals in M^2 . But now I will go look at FZV and F papers.

4. Freeman and Zarmi: Here I suddenly get the idea that the "weighting" factor arises from the asymmetric NDC. I can see this arise in my own rapidity notes at the end of the FZ paper. That is why I would say: asymmetric projection comes from asymmetric NDC. By the way, the three FJ papers never actually do a bootstrap, but refer to FZ and FZV.

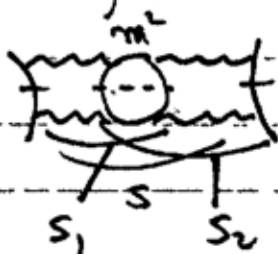
Idea: can perhaps identify the weight factor with threshold factor?

5. FZV: they show on page 8 that, when you write a Mellin projection of $\text{Disc} A_8$ (with no weight), and when you hold s and s_1 fixed, that weight factor magically appears. It is of course caused by the form of the limit of A_8 in terms of A_{rr} . I.e., the limit given in (2.5). There, you see that A (amplitude with no prime) is analytic in M^2 but you pick up extra $(M^2)^{..}$ factor from the regge behaviors. This seems to be true without a list of Steinamn terms.

Then I guess the question becomes: why do you want to integrate A_8 rather than A_{RR} directly?

Quick Way to see Origin of Weight Factor:

- ① Consider doing a bootstrap integral near right side of this object



So you really do want:

[right side with s_1, s_2 fixed] of $DA_{RR}(m^2)$

i.e., this is the object which enters your bootstrap calc.

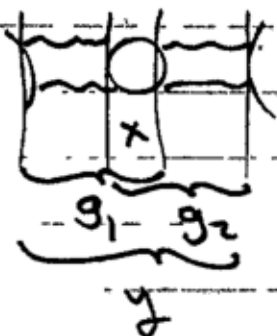
- ② Assume $t=0$ so $s_1 = s_1'$ etc. Claim: the A_8 involved here has this limit:

$$A_8 = \left(-\frac{s_1}{s_0}\right)^{d_1+1} \left(-\frac{s_2}{s_0}\right)^{d_2+1} A_{RR}(m^2).$$

Claim: $A_{RR}(m^2)$ is analytic in m^2 , no LHC. Reason: A_8 is analytic in all invariants.

- ③ Constant as invariant is $\frac{s_1 s_2}{s} = m^2$.

- ④ Convert everything to rapidity.



$$s_1 s_2 = m^2 s \rightarrow g_1 + g_2 = y + x$$

$$A_8 = e^{g_1(d_1+1)} e^{g_2(d_2+1)} A_{RR}(x).$$

analytic in x .

so, what is the integration!

$$\int_0^y dg_1 \int_0^y dg_2 \int_0^y dx \delta(g_1 + g_2 - y - x) \dots$$

But do the g_2 integral:

$$\int_0^y dg_2 \delta(g_2 - \underbrace{[y+x-g_1]}_{\equiv \tilde{g}_2}) = \Theta(\tilde{g}_2) \Theta(y - \tilde{g}_2)$$

$$\Theta(y+x-g_1)$$

$$\Theta(x > g_1 - y = -|y - g_1|) \quad \text{trivial}$$

$$\Theta(g_1 - x) \Rightarrow x < g_1$$

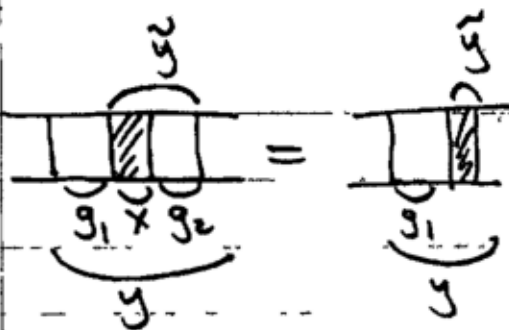
obvious.

Now hold g_1 fixed. Then

$$\text{int RS} | g_1 \text{ fix} = \int_0^{g_1} dx \Theta^{g_1(\kappa_1+1)} \Theta^{\tilde{g}_2(\kappa_2+1)} A_{RR}(x)$$

$$= e^{g_1(\kappa_1+1)} e^{(y-g_1)(\kappa_2+1)} \int_0^{g_1} dx e^{x(\kappa_2+1)} A_{RR}(x)$$

here is your asymmetric
"weight factor."



Hold g_1 fixed.

$$\int_0^{y-g_1} dx P(g_1) A(x) P(y-g_1-x) = P(g_1) A(y-g_1)$$

OR $\int_0^{\tilde{y}} dx A(x) P(\tilde{y}-x) = A(\tilde{y})$

Let $P(x) = e^{\beta x}$. Then

$$A(\tilde{y}) = \int_0^{\tilde{y}} dx A(x) e^{\beta(\tilde{y}-x)} = e^{\beta \tilde{y}} \cdot \int_0^{\tilde{y}} dx A(x) e^{-\beta x}$$

weight!

Now diag:

$$\int_0^{\tilde{y}} dx A(x) P(\tilde{y}-x) = A(\tilde{y})$$

$$\int_0^{\infty} d\tilde{y} e^{-j\tilde{y}} A(\tilde{y}) \equiv A_j = \int_0^{\infty} d\tilde{y} e^{-j\tilde{y}} \int_0^{\infty} dx A(x) P(\tilde{y}-x)$$

$$= \int_0^{\infty} dx A(x) e^{-jx} \cdot \int_0^{\infty} d\tilde{y} e^{-j(\tilde{y}-x)} P(\tilde{y}-x) \Theta(\tilde{y}-x)$$

$\Theta(\tilde{y}-x)$
 $x \leq \tilde{y}$

Let $\tilde{y}-x = g_2$. So

$$\int_0^{\infty} dg_2 e^{-jg_2} P(g_2) \Theta(g_2)$$

P_j

So: $A_j = A_j P_j$

no weight!

Now include asym. NDC.

$$A(\tilde{y}) = \int_0^{\tilde{y}} dx A(x) P(\tilde{y}-x) \Theta(\tilde{y}-\frac{\Delta}{2})$$

$$= \left[\int_0^{\tilde{y}} dx A(x) e^{\beta(\tilde{y}-x)} \right] \Theta(\tilde{y}-\frac{\Delta}{2})$$

correlate to the right.

$$= \Theta(\tilde{y}-\frac{\Delta}{2}) e^{\beta \tilde{y}} \cdot \left[\int_0^{\tilde{y}} dx A(x) e^{-\beta x} \right]$$

weight, as before.

Next, diagonalize:

$$A_j = \int_0^{\infty} d\tilde{y} e^{-j\tilde{y}} \Theta(\tilde{y}-\frac{\Delta}{2}) e^{\beta \tilde{y}} \int_0^{\infty} dx A(x) e^{-\beta x} \Theta(\tilde{y}-x)$$

$$= \int_0^{\infty} dx A(x) e^{-jx} \cdot \underbrace{\int_0^{\infty} d\tilde{y} e^{-j(\tilde{y}-x)} \Theta(\tilde{y}-x) \Theta(\tilde{y}-\frac{\Delta}{2}) e^{\beta(\tilde{y}-x)}}_{\text{weight, as before.}}$$

For small x , this integral begins at $\Delta-x$ and is x dependent. Hence, does not diagonalize!

$$\int dg_2 e^{-jg_2} \Theta(g_2) \Theta(g_2+x-\frac{\Delta}{2}) e^{\beta g_2}$$

$$= \int_{[\frac{\Delta}{2}-x]}^{\infty} dg_2 e^{-jg_2} e^{\beta g_2}$$

$\tilde{y} > x, \frac{\Delta}{2}$

So try again as:

$$A_j = \int_0^{\infty} dx A(x) e^{-jx} e^{-\beta x} \cdot \underbrace{\int_0^{\infty} d\tilde{y} e^{-j(\tilde{y}-x)} \Theta(\tilde{y}-x) \Theta(\tilde{y}-\frac{\Delta}{2}) e^{\beta \tilde{y}}}_{\text{still } x\text{-dependent.}}$$

So cannot diag at this level.

Gist of Letter to Reuben.

- ① You note that unitarity / bootstrap integrals always involve a certain weight factor. I agree. Example:

$$A(\tilde{y}) = K(\tilde{y}) + e^{\beta \tilde{y}} \int_0^{\tilde{y}} dx (e^x)^{\beta} A(x)$$

↑
weight.

- ② Rewrite above eqn. as

$$[A(\tilde{y}) e^{-\beta \tilde{y}}] = [K(\tilde{y}) e^{-\beta \tilde{y}}] + \int_0^{\tilde{y}} dx [A(x) e^{-\beta x}]$$

Now Mellin diagonalize to get

$$A^{2s}(j) = K^{2s}(j) + A^{2s}(j) K^{2s}(j)$$

where

$$A^{2s}(j) = \int_0^{\infty} d\tilde{y} e^{-j\tilde{y}} \cdot [A(\tilde{y}) e^{-\beta \tilde{y}}]$$

↑
weight.

This is what I call on page 60 an "asymmetric" projected amplitude.

- ③ The weight factor is caused by fact that g_1 is held fixed during integration, so that $e^{g_2 \beta} \rightarrow e^{(\tilde{y}-x)\beta}$ has x dependence.

④ So it is holding g_1 & s_1 fixed that gives rise to the factor

$$e^{-\beta x} = (e^x)^{-\beta} = (ze^x)^{-\beta}$$

which is the infamous "weight".

⑤ Now consider:

$$A(g_1, x, g_2) = e^{g_1 x} e^{g_2 x^2} A(x)$$

or

$$A_g = (-s_1)^{d_1+1} (-s_2)^{d_2+1} A_{\text{rel}}(z)$$

Even if I were unaware of Steinman terms here, I would still say: there is a weight factor due to holding s_1 fixed.

If I like, I can "probe" down into the Steinman terms to get:

$$\begin{aligned} A_g &= (-s_1)^{d_1+1} (-s_2)^{d_2+1} \left[\left(\frac{s_1 s_2}{s} \right)^{d_1} A^{(1)} + \left(\frac{s_1 s_2}{s} \right)^{d_2} A^{(2)} \right] \\ &= (-s_1)^{d_1} (-s)^{d_2} A^{(1)} + (-s_2)^{d_2} (-s)^{d_1} A^{(2)} \end{aligned}$$

and I can say: look, no analyzing drives. But, the conclusion that there is a weight is undisputed by this Steinman examination.

⑥ Now, go back to earlier equation:

$$A(\tilde{y}) = K(\tilde{y}) + e^{A\tilde{y}} \int_0^{\tilde{y}} dx (e^{-x})^p A(x)$$

↑ weight.

Diagonalize directly by applying $\int_0^{\infty} d\tilde{y} e^{-j\tilde{y}}$ and you will get:

$$A(j) = K(j) + A(j) K(j)$$

where $A(j) = \int_0^{\infty} d\tilde{y} A(\tilde{y}) e^{-j\tilde{y}}$

↑ no weight.

⑦ This is what I call a symmetric projection; I see no need for asymmetric (weight) projection! So this theorem seems false to me:

$$\left[\begin{array}{l} \text{Sturmian,} \\ \text{analyticity} \\ \text{etc} \end{array} \right] \Rightarrow \left[\begin{array}{l} \text{necessity of using} \\ \text{asymmetric Mellin proj.} \end{array} \right].$$

Integrals dx_1 and dx_2 really start above $x_1 = 0$
and $x_2 = 0$ because $A = \text{discont} + \text{have a}$
lowest threshold. These thresholds do not
pick up singularity of $A(x_i)$ at $x_i = 0$
(ie, $s_i = 1$). If same threshold $\Rightarrow$ complex
case $\Rightarrow$ (11.7) is not exactly correct.

Claim: when I start with the C1 equation in Reubens paper, I am able to successfully diagonalize the thing at the cost of having to use "asymmetric" Mellin projections. These projections then have nonsense zeros which correspond to Reubens "Good FMSR". Thus, I am able to cast the C1 equation into the usual diagonal-j form, but at a price.

However, in my exact kinematic analysis, I am again able to exactly diagonalize onto j; moreover, the projections involved here are perfectly symmetric. I have made no "high energy approximation".

Reuben would say, regarding his analysis: look, if you want to properly take account of the low-energy behavior of amplitudes, you cannot use the high-energy Mellin approximation, identifying s_2 with s_2^+ . If you do this and thereby avoid using my FMSR, you will get the wrong answer for your C1 term.

Reuben would go on to say: look, my FMSR are based on strict observation of analyticity; when I examine analyticity, I find no constant terms in the FMSR and therefore I will have cut-killing.

Dear Phil

March 78

Greetings! I hope you won't mind reading through the enclosed pages that deal with the diagonalization problem. My claim is that simply interchanging variables as is commonly done to generate a diagonalized equation is not always justified.

I have not done the two-lobed cylinder contribution calculation here but if you are willing to agree that the enclosed calculation is O.K., then surely the "instant" diagonalization of the cylinder equation is suspicious.

I am curious whether the delicate manipulation that I describe have an analog in your more precise Fourier-Gribov projection calculation?

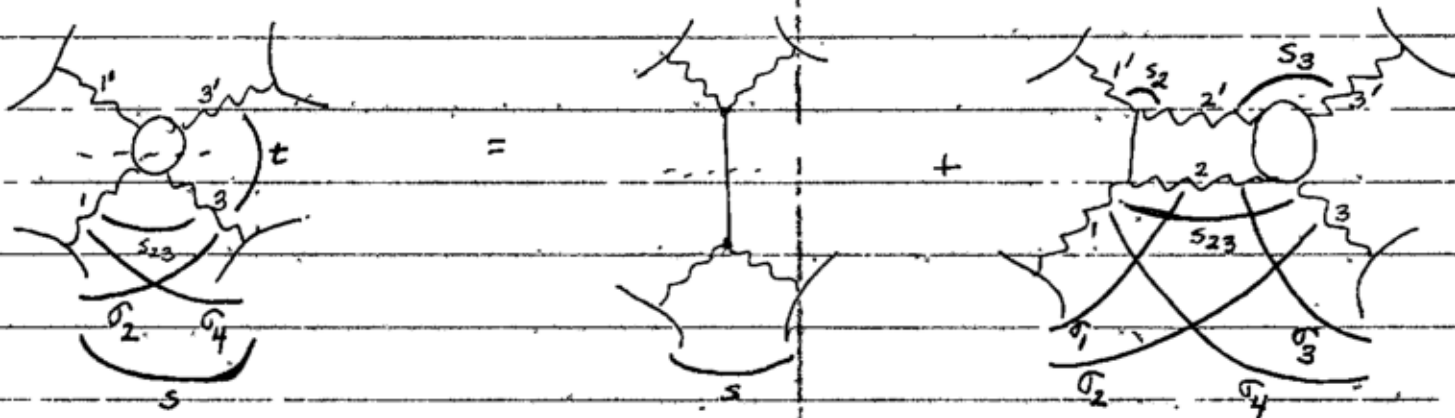
Keep well.

Benker

①

partially

Notation follows that of FZV, Nucl. Phys. B120 (1977) 477



$$s_2^+ \equiv \frac{\sigma_1 s_{23}}{\sigma_2} = s_2 + f(t_1, t_1', t_2, t_2', t)$$

$$\frac{s_{23}^+}{s} \equiv \frac{\sigma_2 \sigma_4}{s} = s_{23} + f(\dots)$$

$$s_3^+ \equiv \frac{\sigma_3 s_{23}}{\sigma_4} = s_3 + f(t_2, t_2', t_3, t_3', t)$$

trip off the external particle legs

$$\frac{1}{2i} \Delta A_{R_1 R_3}(s_{23}, s_{23}^+, \dots) = \text{diagram} + \int d\phi \eta_2(s_{23})^{d_{c,2} - d_{c,1} - d_{c,3} - 2}$$

$$(s_2^+)^{d_{c,1}+1} V_{R_1 R_2}(s_2, s_2^+, \dots) \int_{s_0}^{s_{23}} ds_3 (s_3^+)^{d_{c,3}+1} \frac{1}{2i} \Delta A_{R_2 R_3}(s_3, s_3^+, \dots) \quad (1)$$

A distinction between s_i and s_i^+ must be made in order to get the physical discontinuity

For example, lumping together s_{23} and s_{23}^+ then taking the overall discontinuity across s_{23} will not give a factorized result

$$\epsilon \leq 1$$

asymptotic

But taking the physical discontinuity gives

(2)

$$\frac{1}{2i} \Delta A_{R_1 R_3}(s_{23}, s_{23}^+, \dots) \rightarrow (s_{23})^{\alpha(t) - \alpha_{c,1} - \alpha_{c,3} - 2} \frac{g_1(\dots) g_3(\dots) \pi}{\Gamma(\alpha(t) + 1)} \quad (2)$$

$$\text{Now if } \int_{s_0}^{\infty} ds_3 (s_3^+)^{\alpha_{c,3} + 1} \frac{1}{2i} \Delta A_{R_2 R_3}(s_3, s_2^+, \dots) = 0 \quad (3)$$

then eq. (1) will be satisfied self-consistently for large s_{23}

$$\text{L.H.S.} = \int d\phi_2 \eta_2(s_{23})^{\alpha_{c,2} - \alpha_{c,1} - \alpha_{c,3} - 2} \int_{s_{23}}^{\infty} ds_3 (s_3^+)^{\alpha_{c,3} + 1} \frac{1}{2i} \Delta A_{R_2 R_3}(s_3, s_2^+, \dots) (s_2^+)^{\alpha_{c,1} + 1} V_{R_1 R_2}(\dots) \quad (4)$$

$$= (s_{23})^{\alpha(t) - \alpha_{c,1} - \alpha_{c,3} - 2} \int d\phi_2 \eta_2 \in \frac{\alpha(t) - \alpha_{c,2}}{\alpha(t) - \alpha_{c,2}} \frac{\pi g_2(\dots) g_3(\dots) (s_2^+)^{\alpha_{c,1} + 1} V_{R_1 R_2}(\dots)}{\Gamma(\alpha(t) + 1)} \quad (5)$$

so that matching terms on both sides of (1) requires

$$g_3(\dots) = \int d\phi_2 (s_2^+)^{\alpha_{c,2} + 1} V_{R_1 R_2}(s_2, s_2^+, \dots) \in \frac{\alpha(t) - \alpha_{c,2}}{\alpha(t) - \alpha_{c,2}} g_2(\dots) \quad (6)$$

Now that the problem is solved, let me try to re-derive the result using Mellin transforms:

Write eqn (4) as
$$(s_3^+)^{\alpha_{c,3} + 1} \frac{1}{2i} \Delta A_{R_2 R_3}(s_3, \dots) \equiv \int_{-i\infty}^{+i\infty} d\left(\frac{s_3}{j}\right) s_3^{j' - \alpha_{c,2} - 1} \frac{1}{2i} \tilde{\Delta A}_{R_2 R_3^*}(j' - \alpha_{c,2} - 1, \dots)$$

$$\text{L.H.S.} = \int d\phi_2 \eta_2(s_{23})^{\alpha_{c,2} - \alpha_{c,1} - \alpha_{c,3} - 2} (s_2^+)^{\alpha_{c,1} + 1} V_{R_1 R_2}(\dots) \int_{s_{23}}^{\infty} ds_3 \quad \text{do this next.}$$

$$\int_{-i\infty}^{+i\infty} \frac{dj'}{2\pi i} g_3^{j' - \alpha_{c,2} - 1} \frac{1}{2i} \Delta A_{R_2 R_3^*}(j' - \alpha_{c,2} - 1, \dots) \quad (7)$$

where the j' contour may be safely closed to the left

(3)

$$\text{L.H.S.} = \int d\phi_2 \eta_2 \int_{-i\infty}^{+i\infty} \frac{dj'}{2\pi i} (s_{23})^{j'-\alpha_{c,1}-\alpha_{c,3}-2} (s_2^+)^{\alpha_{c,1}+1} V_{R_1 R_2}(s_2, s_2^+, \dots)$$

$$\in \frac{j'-\alpha_{c,2}}{(j'-\alpha_{c,2})} \frac{1}{2i} \widetilde{\Delta A}_{R_2 R_3^*}(j'-\alpha_{c,2}-1) \quad (8)$$

Again closing to the right gives oblivion while closing to the left picks up the poles of $\frac{1}{2i} \widetilde{\Delta A}$ at $j' = \alpha, \dots$ (the zero at $j' = \alpha_{c,2}$ in $\frac{1}{2i} \widetilde{\Delta A}_{R_2 R_3^*}$ kills the pole at $j' = \alpha_{c,2}$ in the denominator of (8).) regenerating the correct result.

Now perform the M.T. $\int_{\frac{s}{s_2}}^{\infty} ds_{23} (s_{23})^{j+\alpha_{c,1}+\alpha_{c,3}+2-1}$

$$\widetilde{R}_{R_1 R_3}(j-\alpha_{c,1}-\alpha_{c,3}-2, \dots) = \widetilde{A}_{R_1 R_3}^0(j-\alpha_{c,1}-\alpha_{c,3}-2) + \int d\phi_2 \eta_2 \int_{-i\infty}^{+i\infty} \frac{dj'}{2\pi i} \frac{s_2^{j-j'}}{-(j'-j)}$$

$$(s_2^+)^{\alpha_{c,1}+1} V_{R_1 R_2}(s_2, s_2^+, \dots) \in \frac{j'-\alpha_{c,2}}{j'-\alpha_{c,2}} \frac{1}{2i} \widetilde{\Delta A}_{R_2 R_3^*}(j'-\alpha_{c,2}-1, \dots) \quad (9)$$

closing to the left still generates the correct result but closing to the right gives

$$\text{L.H.S.} = \int d\phi_2 \eta_2 (s_2^+)^{\alpha_{c,1}+1} V_{R_1 R_2}(s_2, s_2^+, \dots) \in \frac{j-\alpha_{c,2}}{j-\alpha_{c,2}} \frac{1}{2i} \widetilde{\Delta A}_{R_2 R_3^*}(j-\alpha_{c,2}-1, \dots) \quad (10)$$

which is also what one gets by the naive interchange of integration in Mellin transforming eq. (1).

If one neglects the distinction between s_3^+ and s_3 , one can symmetrize things, i.e. in eq. (4)

$$(s_3^*)^{\alpha_{c,3}+1} \frac{1}{2i} \widetilde{\Delta A}_{R_2 R_3}(s_3, s_2^+, \dots)$$

$$= (s_3)^{\alpha_{c,3}+1} \int_{-i\infty}^{+i\infty} ds_3 s_3^{j'-\alpha_{c,3}-\alpha_{c,2}-2} \frac{1}{2i} \widetilde{\Delta A}_{R_2 R_3}(j'-\alpha_{c,2}-\alpha_{c,3}+2) \text{ etc. but this won't alter the basic result}$$

explicit example

Set $s_3^+ \approx s_3$

$$\text{Take } \frac{1}{2\pi} \widetilde{\Delta A}_{R_2 R_3} (j - \alpha_{c,2} - \alpha_{c,3} - 2) = \frac{(j - \alpha_{c,2})(j - \alpha_{c,3})}{(\alpha(t) - \alpha_{c,2})(\alpha(t) - \alpha_{c,3})} \frac{h(j)}{(j - \alpha(t))} g_2^{(\infty)} g_3^{(\infty)} \quad (11)$$

$$\text{where } h(\alpha(t)) = 1$$

This has the appropriate "nonsense zeros"

the contain closing 1 eqn (9) to the left gives

$$\int d\phi_2 \eta_2 \frac{\bar{s} - j + \alpha(t)}{j - \alpha(t)} (s_2^+)^{\alpha_{c,1}+1} \bigvee_{R_1 R_2} (s_2^+, s_2^+, \dots) \in \frac{\alpha(t) - \alpha_{c,2}}{\alpha(t) - \alpha_{c,2}} g_2^{(\infty)} g_3^{(\infty)} \quad (12)$$

while closing to the right (as in eq. 10) gives

$$\int d\phi_2 \eta_2 (s_2^+)^{\alpha_{c,1}+1} \bigvee_{R_1 R_2} (s_2^+, s_2^+, \dots) \in \frac{j - \alpha_{c,2}}{j - \alpha_{c,2}} \frac{(j - \alpha_{c,2})(j - \alpha_{c,3})}{(\alpha - \alpha_{c,2})(\alpha - \alpha_{c,3})} \frac{h(j)}{j - \alpha(t)} g_2 g_3 \quad (13)$$

which is not the same as (12)

Responser to Reuben Freeman's Letter of March 28.

1. He says that if you go through a certain procedure and then shift the j' contour to the right, you get a result which is equivalent to the diagonalized-equation result. Whereas if you shift the contour to the left, you get a different result.

He then gives a little example and shows how the two shifts give different results.

My response is this: it is correct to shift to the right. Reason: the projection in the j' integrand is expo decaying in the right half j' -plane, so ~~one~~ you pick up the pole at $j' = j$, there are no more singularities out there and you the residual contour then vanishes. Thus, the diagonalized equation is correct.

If you shift the contour to the left, and pick up only the first regge pole and then ignore the background contour, you get a wrong result. Reason: just look at your own example. You assumed a specific projection with a nonsense zero ~~ina~~ at $j = \alpha_2$. In your result shifting to the right, than nonsense-zero is faithfully reproduced on both sides of the equation; if you shift to the left, that nonsense zero only appears on the left but not on the right.

If you were to continued to shift the contour to the left and pick up all the lower regge pole contributions, you would eventually regenerate that nonsense zero.

Of course in practice either method is OK. Ie, the two expressions you give agree when j is near α .

Reuben:

-1-

I.

① As shown in my own cylinder note, you can show that:

$$A(S_{23}) = \int dT_2 \int_{\tilde{S}_2}^{S_{23}} dS_3 (\tilde{S}_2^{\alpha_2+1}) (S_3^{\alpha_3+1}) (S_{23})^{\alpha_2-\alpha_1-\alpha_3-2} V(\tilde{S}_2) A(S_3) \quad (1)$$

where V included a delta $S_2 - \tilde{S}_2$, so $\tilde{S}_2$ is a hit value. I agree with this equation. Now, suppose this is true:

$$\int_{S_0}^{\infty} dS_3 (S_3^{\alpha_3+1}) A(S_3) = 0 \quad (3)$$

If true, then we can replace $\int_{S_0}^{S_{23}} = \int_{S_0}^{\infty} - \int_{S_{23}}^{\infty} = - \int_{S_{23}}^{\infty}$. This leaves

$$A(S_{23}) = \int dT_2 (\tilde{S}_2^{\alpha_2+1}) V(\tilde{S}_2) (S_{23})^{\alpha_2-\alpha_1-\alpha_3-2} \cdot (-1) \int_{S_{23}}^{\infty} dS_3 (S_3^{\alpha_3+1}) A(S_3).$$

Now assume that

$$A(S_3) = \frac{g_2 g_3 \pi}{\Gamma(\alpha+1)} (S_3)^{\alpha-\alpha_2-\alpha_3-2} \quad (2)$$

If S_{23} is large, then:

$$\begin{aligned} & (-1) \int_{S_{23}}^{\infty} dS_3 (S_3)^{\alpha_3+1} (S_3)^{\alpha-\alpha_2-\alpha_3-2} \frac{g_2 g_3 \pi}{\Gamma} = - \frac{g_2 g_3 \pi}{\Gamma} \int_{S_{23}}^{\infty} dx \cdot x^{\alpha-\alpha_2-1} \\ & = \frac{+g_2 g_3 \pi}{\Gamma} \frac{(S_{23})^{\alpha-\alpha_2}}{(\alpha-\alpha_2)} = \frac{-(S_{23})^{\alpha-\alpha_2-1}}{(\alpha-\alpha_2)} \end{aligned}$$

then we get

$$A(S_{23}) = \frac{g_1 g_3 \pi}{\Gamma} (S_{23})^{\alpha-\alpha_1-\alpha_3-2} = \int dT_2 (\tilde{S}_2^{\alpha_2+1}) V(\tilde{S}_2) (S_{23})^{\alpha-\alpha_1-\alpha_3-2} \frac{g_2 g_3 \pi}{\Gamma(\alpha-\alpha_2)}.$$

We then end up with

$$g_1 = \int dT_2 \cdot (\tilde{s}_2^+)^{\alpha_1+1} \frac{V(\tilde{s}_2^+)}{[\alpha - \alpha_2]} g_2 \quad (6)$$

This then becomes the vertex bootstrap. Counting aside, I agree so far.

II. How would I describe all this in Mellin? I did this in my last letter, but let's go again: Let me set

$$V(\tilde{s}_2^+) = \int ds_2 V(s_2) \delta(s_2 - \tilde{s}_2).$$

Then equation is:

$$A(s_{23}) = \int dT_2 \cdot \int_1^\infty ds_2 (s_2^+)^{\alpha_1+1} \{V(s_2) \delta(s_2 - \tilde{s}_2)\} \\ \times \int_1^\infty ds_3 (s_3^+)^{\alpha_3+1} A(s_3) \cdot (s_{23})^{\alpha_2 - \alpha_1 - \alpha_3 - 2} \Theta(s_{23} - s_2 s_3).$$

I would do what I said in my letter + would get

$$A(j - \alpha_1 - \alpha_3 - 2) = \int dT_2 \cdot A(j - \alpha_1 - \alpha_3 - 2) \frac{1}{j - \alpha_2} \cdot A(j - \alpha_2 - \alpha_3 - 2)$$

So I have no problem digging this thing. Here is my projection:

$$A_{23}(j) = \int_1^\infty ds_3 \left[(s_3^+)^{\alpha_3+1} A_{23}(s_3, s_3^+) \right] (s_3)^{-j - \alpha_3 - 2}$$

$$\text{or } A_{23}(j - \alpha_2 - \alpha_3 - 2) = \int_1^\infty ds_3 \left\{ (s_3^+)^{\alpha_3+1} A_{23}(s_3, s_3^+) \right\} (s_3)^{-j + \alpha_2}$$

How would I inverse this?

Let $\sigma = j - \alpha_3 - 2 = -j - 1$. Then $\sigma = j + 1 - \alpha_3 - 2$
 $= j - \alpha_3 - 1.$

So

$$A_{23}(j - \alpha_3 - 1) = \int_1^\infty ds_3 (s_3)^{-j-1} \{ (s_3^+)^{\alpha_3+1} A_{23}(s_3) \}$$

$$\Rightarrow \{ (s_3^+)^{\alpha_3+1} A_{23}(s_3) \} = \frac{1}{2\pi i} \oint ds_3 (s_3)^j A_{23}(j - \alpha_3 - 1)$$

This does not agree with this form:

Suppose instead I define

$$\tilde{A}_{23}(\sigma) \equiv \int_1^\infty ds_3 (s_3)^{-\sigma-1} \{ (s_3^+)^{\alpha_3+1} A_{23}(s_3) \}$$

Then

$$\{ (s_3^+)^{\alpha_3+1} A_{23}(s_3) \} = \frac{1}{2\pi i} \oint d\sigma (s_3)^\sigma \tilde{A}_{23}(\sigma).$$

Then I could let $\sigma = j' - \alpha_2 - 1$ so that

$$= \frac{1}{2\pi i} \oint dj' (s_3)^{j' - \alpha_2 - 1} \tilde{A}_{23}(j' - \alpha_2 - 1)$$

This, then, is what he wants to use. OK. Try the expansion in the equation:

$$A(s_3) = \int dT_2 \int_1^\infty ds_2 (s_2^+)^{\alpha_1+1} \{ V(s_2) \delta(s_2 - \tilde{s}_2) \}.$$

$$\times \int_1^\infty ds_3 \phi(s_{23})^{\alpha_2 - \alpha_1 - \alpha_3 - 2} \Theta(s_{23} - s_2 s_3) \frac{1}{2\pi i} \oint dj' (s_3)^{j' - \alpha_2 - 1} \tilde{A}_{23}(j' - \alpha_2 - 1)$$

Then I can do the S_3 :

$$\int_1^{\infty} ds_3 \Theta(S_{23} - S_2 S_3) (S_3)^{j' - d_{c2} - 1}$$

$$= \int_{S_{23}/S_2}^{S_{23}/S_2} ds_3 (S_3)^{j' - d_{c2} - 1} = \left\{ \frac{(S_{23}/S_2)^{j' - d_{c2} - 1} - 1}{j' - d_{c2}} \right\}$$

I then have:

$$A(S_{23}) = \int dT_2 \cdot \int_1^{\infty} ds_2 (S_2^+)^{d_{c1}+1} \left\{ \sqrt{S_2} \delta(S_2 - S_2^+) \right\} (S_2)^{d_{c2} - d_{c1} - d_{c3} - 2}$$

$$\times \frac{1}{2\pi i} \oint dj' \tilde{A}_{23}(j' - d_{c2} - 1) \left\{ \frac{(S_{23}/S_2)^{j' - d_{c2} - 1} - 1}{j' - d_{c2}} \right\}$$

But lets use the nonsense zero idea first. Then:

$$A(S_{23}) = - \int dT \cdot \int_1^{\infty} ds_2 (S_2^+)^{d_{c1}+1} \left\{ \sqrt{S_2} \delta(S_2 - S_2^+) \right\} (S_2)^{d_{c2} - d_{c1} - d_{c3} - 2}$$

$$\int_{S_{23}/S_2}^{\infty} ds_3$$

So I then get:

$$\int_{(S_{23}/S_2)}^{\infty} ds_3 \cdot (S_3)^{j' - d_{c2} - 1} = \frac{(S_{23}/S_2)^{j' - d_{c2}}}{j' - d_{c2}}$$

$\Rightarrow$ same result but no 1 !!

Now close j' contour left, lets see what happens.

$$\tilde{A}_{23}(s) \equiv \int_1^\infty ds_3 (s_3)^{-s-1} \left\{ (s_3^+)^{\alpha_3+1} \frac{g_2 g_3 \pi}{p} (s_3)^{\alpha - \alpha_{c2} - \alpha_{c3}} \right\}$$

$$\Rightarrow \tilde{A}_{23}(j - \alpha_{c2} - 1) \approx \frac{g_2 g_3 \pi}{p} \int_1^\infty ds_3 (s_3)^{-j + \cancel{\alpha_{c2}} + \cancel{\alpha_{c3}} + 1 + \alpha - \cancel{\alpha_{c2}} - \cancel{\alpha_{c3}}}$$

$$= \frac{g_2 g_3 \pi}{p} \frac{1}{(j - \alpha)}$$

OK, so shift the j contour left to get

$$A(s_{23}) \approx \int dT_2 \int_1^\infty ds_2 \dots \{ \dots \delta(\cdot) \}$$

$$\frac{g_2 g_3 \pi}{p} \frac{(s_{23}/s_2)^{\tilde{\alpha} - \alpha/2}}{(\alpha - \alpha_{c2})} (s_{23})^{\alpha/2 - \alpha_{c1} - \alpha_{c3} - 2}$$

$$= ? \quad g_1 g_3 \frac{(s_{23})^{\tilde{\alpha} - \alpha_{c1} - \alpha_{c3} - 2}}{p}$$

OK, this seems to "duplicate" earlier vertex bootstrap.

What has been done to this point? Reuben has taken a bootstrap equation with the left cluster delta'd at some fixed mass, hence no counting problem. He then takes the large s_{23} limit of both sides and requires that they be consistent. Assuming the nonsense zero or FMSR or whatever you want to call it, he is able to get consistency with a certain bootstrap condition. Fine.

Then, he tries to redo the same thing by first expanding the object $A(s_3)$ onto its Mellin projections, and then he executes the s_3 integral. Finally, he closes the j contour and takes the large s_{23} limit. This procedure leads to the same little bootstrap condition.

Thus, he has shown that if you are doing something asymptotic, you can freely expand onto Mellin projections, do your energy integral, then close your contour and regain the same little vertex bootstrap.

So far so good. No contradictions claimed or noticed by me. It is OK for him to define his own Mellin projection however he wishes.

III. Next, project the whole equation:

$$A(S_{23}) = -\int dT \cdot V(\tilde{S}_2) (\tilde{S}_2^\perp)^{\alpha_1+1} (S_{23})^{\alpha_2-\alpha_1-\alpha_3-2}$$

$$\int_{S_{23}/\tilde{S}_2}^{\infty} dS_3 (S_3^\perp)^{\alpha_3+1} A(S_3, S_3^\perp)$$

Apply to both sides $\int_1^{\infty} dS_{23} (S_{23})^{-j+\alpha_1+\alpha_3+2-1}$

$$\text{LHS} = \int_1^{\infty} dS_{23} (S_{23})^{-j+\alpha_1+\alpha_3+2-1} A(S_{23})$$

He calls this $\tilde{A}(j-\alpha_1-\alpha_3-2)$ which is inconsistent with his earlier notation. Nevertheless, I will go ahead & call this $A(j, \cdot)$. So

RHS = ? OH, we are also supposed to expand.

III, version 2. Start with:

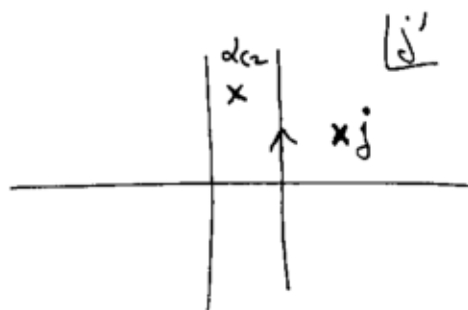
$$A(S_{23}) = \int dT_2 (\tilde{S}_2^\perp)^{\alpha_1+1} V(\tilde{S}_2) \frac{1}{2\pi i} \oint dj' \tilde{A}_{23}(j'-\alpha_2-1) \frac{(S_{23}/\tilde{S}_2)^{j'-\alpha_2}}{j'-\alpha_2} \times (S_{23})^{\alpha_2-\alpha_1-\alpha_3-2}$$

Now project as he says:

$$A(j-\alpha_1-\alpha_3-2) = \dots \underbrace{\int_1^{\infty} dS_{23} (S_{23})^{-j+\alpha_1+\alpha_3+2-1+j'-\alpha_2}}_{\frac{1}{(j'-j)} \text{ if } (-j+j') \leq 0} \Rightarrow \text{Re } j \geq \text{Re } j'$$

This leaves:

$$A(j-d_1-d_3-z) = \int dT_2 (\tilde{S}_2^+)^{d_1+1} V(\tilde{S}_2) \frac{1}{2\pi i} \oint dj' \tilde{A}_{23}(j'-d_2-1) (\tilde{S}_2^+)^{d_2 j'} \\ \times \frac{1}{(j'-j)} \times \frac{1}{(j'-d_2)}.$$



Close to the left, hence $j' = d_2$,

$$\text{so } A(j-d_1-d_3-z) = \int dT_2 (\tilde{S}_2^+)^{d_1+1} V(\tilde{S}_2) \cdot \frac{g_2 g_3 \pi}{f} (\tilde{S}_2^+)^{d_2-d} \\ \times \frac{1}{(d-j)} \frac{1}{(d-d_2)}$$

Looks ok.

You are not allowed to close right, I would guess.

If you go ahead + close to the right, you get:

$$A(j-d_1-d_3-z) = \int dT_2 \left[(\tilde{S}_2^+)^{d_1+1} V(\tilde{S}_2) (\tilde{S}_2^+)^{d_2-j} \right] \frac{1}{j-d_2} \left[\tilde{A}_{23}(j-d_2-1) \right]$$

= the usual divergized result.

A condensation of Reubens letter.

1. First, let's set $a_{c1}=a_{c3}=-1$, and let's get rid of the transverse integration, put in $V = \text{some constant for the potential}$. Then the bootstrap equation becomes: $S_{23} = S$.

$$S_3 = S_4$$

$$A(s) = V \cdot s^{\alpha_{c2}} \cdot \int_1^s ds_2 A(s_2)$$

2. Now let us assume that there is a nonsense zero as follows:

$$\int_1^\infty ds_2 A(s_2) = 0. \quad \text{Then,} \quad A(s) = -V \cdot s^{\alpha_{c2}} \cdot \int_s^\infty ds_2 A(s_2) (*)$$

Now assume these asymptotic limits:

$$A(s) \rightarrow g_1 g_3 s^\alpha$$

$$A(s_2) \rightarrow g_2 g_3 s_2^{\alpha - \alpha_{c2} - 1}$$

$$/ \text{ since } \alpha_{c1} = \alpha_{c3} = -1$$

$$/ \text{ since } \alpha_{c3} = -1, \alpha_{c2} \neq -1.$$

$$\Rightarrow g_1 g_3 s^\alpha = -V s^{\alpha_{c2}} \cdot \int_s^\infty ds_2 [g_2 g_3 s_2^{\alpha - \alpha_{c2} - 1}]$$

$$= \frac{g_2 g_3 (s)^{\alpha - \alpha_{c2}}}{(\alpha - \alpha_{c2})} \times (-1)$$

$$\Rightarrow g_1 = + \frac{V \cdot g_2}{(\alpha - \alpha_{c2})};$$

This then is the resultant bootstrap condition, Reubens equation (6).

3. Now let's simulate Reuben's second move, which is this: expand $A(s_2)$ in equation (*) above:

$$A(s_2) = \frac{1}{2\pi i} \oint_{\mathcal{H}_R} dj' (s_2)^{j' - \alpha_{c2} - 1} \tilde{A}(j' - \alpha_{c2} - 1)$$

Why not be simpler about it:

$$A(s_2) = \frac{1}{2\pi i} \oint dJ (s_2)^J \tilde{A}(J).$$

$$\text{Then: } A(s) = -V s^{\alpha_{c2}} \cdot \int_s^\infty ds_2 \cdot \frac{1}{2\pi i} \oint dJ (s_2)^J \tilde{A}(J)$$

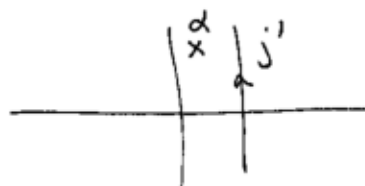
$$= -V s^{\alpha_{c2}} \cdot \frac{1}{2\pi i} \oint dJ \tilde{A}(J) \cdot \underbrace{\int_s^\infty ds_2 (s_2)^J}_{\Phi(s)} \frac{J+1}{J+1}$$

Well, shift to j since $A(s_2) \rightarrow A(j)$ has pole at $j = \alpha$:

$$A(s) = -V s^{\alpha_2} \frac{1}{2\pi i} \oint dj' \tilde{A}(j' - \alpha_2 - 1) \frac{(s)^{j' - \alpha_2}}{(j' - \alpha_2)}$$

Now shift left. Note that:

$$\tilde{A}(j' - \alpha_2 - 1) = \underbrace{\int_1^\infty ds_2 (s_2)^{-j' - \alpha_2 - 1} g_2 g_3(s_2)^{\alpha_2 - 1}}_{\frac{g_2 g_3}{j' - \alpha}}$$



So

$$A(s) = g_2 g_3 s = +V s^{\alpha_2} \frac{g_2 g_3 (s)^{\alpha_2 - \alpha_2}}{(\alpha - \alpha_2)}$$

= The same bootstrap.

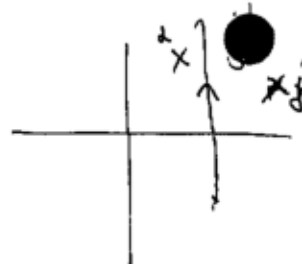
Now finally we come to Reubens third move. We start with equation (**) above and project the whole damn thing: what happens? How do you deal with the two contours j and j' , ie, where are these variables relative to each other? At the moment, the j' contour lies to the right of the leading singularity in $A(s_2)$. Maybe I should be more exact about this. The contour must be to the right of $j = \alpha$, because that is the leading pole in the projection shown in the equation. Now we are allowed to project with some very large positive j to get good convergence. Thus:

Apply $\int_1^\infty ds(s)^{j-1} A(s) = A(j)$, so

$$A(j) = -V \frac{1}{2\pi i} \oint dj' \tilde{A}(j' - \alpha_2 - 1) \underbrace{\int_1^\infty ds s^{\alpha_2} s^{-j-1} s^{j' - \alpha_2}}_{\frac{1}{j-j'} \quad \text{Re } j > \text{Re } j'}$$

$$\Rightarrow A(j) = V \cdot \frac{1}{2\pi i} \oint dj' \frac{\tilde{A}(j' - \alpha_2 - 1)}{(j' - \alpha_2)(j' - j)}$$

(***).



I would say: you must close to the ~~right~~ left. So:

$$A(j) = \frac{\tilde{A}(j - \alpha_{c_2} - 1)}{(j - \alpha_{c_2})(j - \alpha_{c_1})} = \frac{V \cdot g_2 g_3}{(j - \alpha_{c_2})(j - \alpha_{c_1})}$$

$$= \frac{V \cdot g_2 g_3}{(j - \alpha_{c_2})(j - \alpha_{c_1})} = ? \frac{g_1 g_3}{(j - \alpha)} \quad \checkmark \quad \text{same bootstrap.}$$

I would never close this thing to the right; Reuben merely points out that if you do so, you get an equation which looks to him like the diagonalized Mellin equation that I propound so vehemently. Namely:

$$A(j) = \frac{V \cdot \tilde{A}(j - \alpha_{c_2} - 1)}{(j - \alpha_{c_2})}$$

He then arrives at this conclusion: since this result disagrees with the result you get by doing direct integrations, it must be wrong; and since this result looks like my diagonalized equation, my diagonalized equation must be wrong.

However, I would claim that in this situation the diagonalized equation is wrong. But that is only because his bootstrap equation is wrong. If he had the right bootstrap, I think it would Mellin diagonalize in this way.

$$A(s) = V \cdot (s)^{\alpha_{c_2}} \int_1^s ds_2 A(s_2)$$

I think the fact is that this thing is not a convolution equation, and therefore it will not diagonalize in the "trivial" way. Lets see what happens:

$$A(s) = V \cdot (s)^{\alpha_{c_2}} \int_1^\infty ds_2 A(s_2) \Theta(s - s_2)$$

$$\int_1^\infty ds s^{-j-1} A(s) = \tilde{A}(j) = V \cdot \int_1^\infty ds_2 A(s_2) \int_{s_2}^\infty ds s^{-j-1} (s)^{\alpha_{c_2}}$$

$$= V \cdot \int_1^\infty ds_2 (s_2)^{j-1} A(s_2) \cdot \int_1^\infty ds \left(\frac{s}{s_2} \right)^{j-1} (s)^{\alpha_{c_2}}$$

$\uparrow$ add $(s_2)^{\alpha_{c_2}}$ $\uparrow$ add $(s_2)^{-\alpha_{c_2}}$ $\uparrow$ cooked!!

$$= V \cdot \tilde{A}(j) \cdot \frac{1}{j - \alpha_{c_1}}$$

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