

PION NUCLEON

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Non Nucleon Notations

MHS: $T_{++}, T_{+-}, F_{++}, F_{+-}, F^{\pm}$ (p361) A, B of CNOT.

A, B, A' (p456)

$f_{\mu\nu}, T_{\mu\nu}, T_{\pm}, f_{\pm}, \eta_{\pm}, \delta_{\pm}, (f_1, f_2), f_g$ (p250f)

Gre: $A_{op}^{ab}, B_{op}^{ab}, A^{(+)}A^{(-)}, B^{(+)}B^{(-)}$

$F^{(+)}F^{(-)}, F^{3/2}, F^{1/2}, f_{et}f_{el} \leftrightarrow f_{er}$

$\delta_{et}, \delta_{er}, F_{\pi^+}, F_{\pi^-}$

Goldwasser: $S_{\pm}, S_{\pm}^T, f, g, \delta_{\pm}, f_1, f_2, b_1, b_2, b_3, a, b$

Wall: $A_{\pm}, (g, h), a, b (= f_3, f_4)$

Reduction for PI-N scattering. Polarization.

The full equations are:

$$\frac{d\sigma}{d\Omega} = \sum_{m_1, m_1', m_3} f_{m_3, m_1} f_{m_3, m_1'}^* S_{m_1, m_1'}^{(1)} \quad S_{m_1}^{(1)} = \frac{1}{2} (1 + \vec{P}_i \cdot \langle \vec{\sigma} \rangle)$$

$$\frac{d\sigma}{d\Omega} t_{LM}^{(3)} = \sum_{m_1, m_1', m_3, m_3'} \langle \frac{1}{2} m_3', LM | \frac{1}{2} m_3 \rangle f_{m_3', m_1} f_{m_3, m_1'}^* S_{m_1, m_1'}^{(1)}$$

Obviously the only surviving polarization tensor is the $L = 1$, since the final beam 3 is spin- $\frac{1}{2}$. The P_y for this final beam is given by,

$$\frac{d\sigma}{d\Omega} [-iP_y] = \sum_{m_1, m_1', m_3} \sqrt{(\frac{1}{2} + m_3)(\frac{1}{2} - m_3 + 1)} S_{m_1, m_1'}^{(1)} \{ f_{m_3, m_1} f_{m_3-1, m_1'}^* - f_{m_3-1, m_1} f_{m_3, m_1'}^* \}$$

We can see that only $m_3 = +\frac{1}{2}$ contributes in this sum, for which the SQRT is unity. Thus,

$$\frac{d\sigma}{d\Omega} [-iP_y] = \sum_{m_1, m_1'} S_{m_1, m_1'}^{(1)} [f_{+, m_1} f_{-, m_1'}^* [-] f_{-, m_1} f_{+, m_1'}^*]$$

Of course as long as the initial beam has some obscure polarization, P_x and P_z are also present. These explicit formulas are still a bit messy.

$$\frac{d\sigma}{d\Omega} [P_z] = 2 \sum_{m_1, m_1', m_3} m_3 S_{m_1, m_1'}^{(1)} f_{m_3, m_1} f_{m_3, m_1'}^*$$

For an unpolarized input beam, these equations reduce to:

$$\frac{d\sigma}{d\Omega} [-iP_y] = \frac{1}{2} \sum_{m_1} [f_{+, m_1} f_{-, m_1}^* [-] f_{-, m_1} f_{+, m_1}^*] = \frac{1}{2} \sum_{m_1} \left[\frac{2 \text{Re}}{2i \Delta m} \right] [f_{+, m_1} f_{-, m_1}^*]$$

$$\frac{d\sigma}{d\Omega} [-iP_y] = [i \Delta m] \{ f_{++} f_{-+}^* + f_{+-} f_{--}^* \}$$

$$\frac{d\sigma}{d\Omega} [P_z] = \sum_{m_1, m_3} m_3 |f_{m_3, m_1}|^2$$

$$\frac{d\sigma}{d\Omega} = \frac{1}{2} \sum_{m_1, m_3} |f_{m_3, m_1}|^2 = \frac{1}{2} \{ |f_{++}|^2 + |f_{+-}|^2 + |f_{-+}|^2 + |f_{--}|^2 \}$$

Finally, since parity is conserved in such a strong interaction, we get:

$$\frac{d\sigma}{d\Omega} = \{ |f_{++}|^2 + |f_{+-}|^2 \}$$

$$\begin{aligned} \frac{d\sigma}{d\Omega} P_y &= -\Delta m [f_{++} f_{-+}^* + f_{+-} f_{--}^*] = \Delta m [f_{++} f_{+-}^* - f_{+-} f_{++}^*] \\ &= 2 \Delta m [f_{++} f_{+-}^*] \end{aligned}$$

These are correct. See (5.84) MS.

Gasirowics PI-N normalizations.

Looking at p 364 and p 370 and p 34 of Gas, we can figure out that:

$$T = \frac{1}{16\pi^5} \bar{u} [-A + \not{Q} B] u \quad \bar{u} u = 1$$
$$f = \frac{-16\pi^5 m}{\sqrt{s}} T$$

However, after converting to the BD metric, we find that CGLN defined their A and B according to:

$$M_{fi} = \bar{u} [-A - B \not{Q}] u \quad \bar{u} u = 2m$$
$$f = -\frac{1}{8\pi\sqrt{s}} M_{fi}$$

Putting these facts together, (f is the same in both cases) we find:

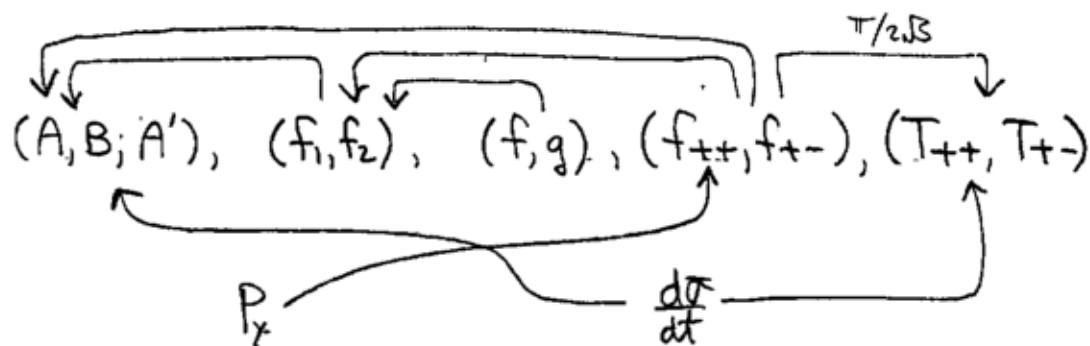
$$A^{GAS} = \frac{1}{4\pi} A$$
$$B^{GAS} = -\frac{1}{4\pi} B$$

Status of the Various Pion-Nucleon Amplitudes so Far. (Excluding partial waves.)

Here, an arrow means we have an explicit equation for x in terms of y :

$$x \longrightarrow y$$

Thus, here is where we stand:



I think that now I can get the polarization of the DCS in terms of any amplitudes with complete accuracy. Also, can relate any amplitudes to any other amplitudes. Now am prepared to play.

Information Hunting.

Concerning the nuclear science abstracts: the library staff has gathered together all the five year indices, the last 5-year period being 1967-1971. For all years covered by these indices, the annual author and subject indices have been discarded. Thus, they have only these annual indices for 1972, 1973.

These 5-year indices are extremely thorough. They have both subject and author indices: this is surely the place to look for anything if you do not have a clean journal reference.

Example: I was looking for an AEC report titled LAMS-2740 by A.K.Kerman. There as no journal reference for this. I should have then looked up Kerman in the appropriate 5-year index. That would have given me the publication reference if there ever was one. Now it turned out in this case that since the thing was so long that it wa never published. Then what do you do?

Then you go check one of the files in the locked room to see if UCB has a copy either up here, in the depository in Doe, or on microcards. There are two kinds of microcards. The positives do not make copies, but from the microfiche you can make copies directly from them in the library.

Crossing: S, U with neutral pion only.

This is the case that is considered in all the books. For example, see Gasiorowicz page 363f. Using the reduction formalism of LSZ (as discussed in B+D volume 2, say) we know that we can write the S-matrix amplitude for the s-channel process

$$\pi^b N \leftarrow \pi^a N \quad (S)$$

in the following way:

$$\begin{aligned} \langle p'q' | p q \rangle &= M_s(p'q' \leftarrow p q) = T_s(p, q, p', q') \\ &= \left(\frac{i}{\sqrt{2}}\right)^2 \iint d^4x d^4y \frac{e^{-iq \cdot x}}{\sqrt{1}} \vec{K}_x \langle p' | T(\phi^b(y) \phi^a(x)) | p \rangle \vec{K}_y \frac{e^{iq' \cdot y}}{\sqrt{1}} \end{aligned}$$

Here we only need extract the pion fields from the asymptotic states, since the nucleon fields will not be needed in the comparison that is to follow. By a similar application of the reduction formulae, we can write the S-matrix amplitude for the u-channel process

$$\pi^a N \leftarrow \pi^b N \quad (u)$$

in the following way:

$$\begin{aligned} \langle p'q' | p q \rangle &= M_u(p'q' \leftarrow p q) = T_u(p, q, p', q') \\ &= \left(\frac{i}{\sqrt{2}}\right)^2 \iint d^4x d^4y \frac{e^{-iq \cdot y}}{\sqrt{1}} \vec{K}_y \langle p' | T(\phi^b(y) \phi^a(x)) | p \rangle \vec{K}_x \frac{e^{iq' \cdot x}}{\sqrt{1}} \end{aligned}$$

By simply comparing these two expressions, we can quickly conclude that:

$$T_u(p, q = -q', p', q' = -q) = T_u(p, (-q'), p', (-q)) = T_s(p, q, p', q') \quad \leftarrow \text{crossing statement}$$

This says that the amplitude for the physical u-channel process (where $-q$ and $-q'$ are positive!) is given by the s-channel amplitude continued to a particular unphysical point. This point is unphysical for the s-channel because the q and q' appearing inside T_s are both negative.

Now, in the special case of neutral pion scattering, the s-channel is the same channel as the u-channel, because a channel is defined by the type of particles involved. Thus, in this case (only) we can say:

$$T_u(p, -q', p', -q) = T_s(p, -q', p', -q) = T_s(p, q, p', q') \quad \leftarrow \text{crossing}$$

same channel

If the T_s function is written out in the standard parity-invariant way,

$$T_s(p, q, p', q') = \bar{u}(p') [A(s, u) + \frac{1}{2}(\not{q} + \not{q}') B(s, u)] u(p)$$

Then we see that the identity of the s- and u-channels implies that:

$$T_s(p, -q', p', -q) = \bar{u}(p') [A(u, s) - \frac{1}{2}(\not{q} + \not{q}') B(u, s)] u(p)$$

(Again, this is only true for A and B describing the neutral pion-nucleon amplitude.)

$$\begin{aligned} A(s, u) &= A(u, s) \\ B(s, u) &= -B(u, s) \end{aligned}$$

Crossing. S, T case.

For the s-channel process: $\pi N \leftarrow \pi N$ (S)

we write the S-matrix amplitude as:

$$\mathcal{M}_s(p'q' \leftarrow pq) = \langle p'q' | p q \rangle = \bar{u}(p', s') T_s(p, q, p', q')$$

$$= \left(\frac{i}{\sqrt{2}}\right) \left(\frac{-i}{\sqrt{2}}\right) \iint d^4x d^4y \frac{e^{-iq \cdot x}}{\sqrt{1}} \overrightarrow{K}_x \frac{e^{ip' \cdot y}}{\sqrt{1}} \bar{u}(p', s') \overrightarrow{D}_y \langle q' | T(\psi(y) \phi(x)) | p \rangle$$

For the t-channel process: $\pi \pi \leftarrow \bar{N} N$ (T)

we write the S-matrix amplitude as: $\mathcal{M}_t(Qq' \leftarrow pP) = \langle Qq' | pP \rangle = \bar{v}(P, s) T_t(p, P, Q, q)$

$$= \left(\frac{i}{\sqrt{2}}\right) \left(\frac{+i}{\sqrt{2}}\right) \iint d^4x d^4y \frac{e^{-iP \cdot y}}{\sqrt{1}} \bar{v}(P, s) \overrightarrow{D}_y \langle q' | T(\phi(x) \psi(y)) | p \rangle \overleftarrow{K}_x \frac{e^{iQ \cdot x}}{\sqrt{1}}$$

Now a comparison of the objects T above shows that:

$$\mathcal{M}_t(-q'q' \leftarrow p(-p')) = \bar{v}(-p', s') T_t(p, -p', -q, q')$$

$$T_t(p, -p', -q, q') = -T_s(p, q, p', q')$$

ie, hanging index on T not shown.

This last line is the crossing statement. Notice that it applies to the amplitude with one of the spinors pulled off. There is no need to make any assumptions like,

$$\bar{u}(-p) = \bar{v}(p)$$

which really is not true.

If we express the s-channel amplitude in the usual way,

$$T_s(p, q, p', q') = [A(s, t) + \frac{1}{2}(q + q') B(s, t)] u(p, s)$$

Then we can write the t-channel amplitude like so:

$$\mathcal{M}_t(-q'q' \leftarrow p-p') = -\bar{v}(-p', s') [A(s, t) + \frac{1}{2}(q + q') B(s, t)] u(p, s)$$

↑ ↑ ↑
- - - ?

so $\bar{v}(-p')$ is not defined

Again, we should keep in mind that (here) q and p' are negative quantities. They lie in the past light cone. Here, t is positive and s negative. The idea of continuation is clear.

Now, unlike the S-U crossing case, the S and T channels are never the same (as defined here), so there is no equation analogous to $A(s, u) = A(u, s)$.

PI-N t-channel helicity amplitudes in terms of A and B via direct crossing.

Keeping in mind that p' and q are negative four vectors, we know from studying crossing that the invariant (Jackson) amplitude for the physical t-channel process $\pi\pi \leftarrow N\bar{N}$ is* given by:

$$M_t(-q' \leftarrow p - p') = -\bar{u}(-p', s') [-A - \not{Q} B] u(p, s) \leftarrow \text{usual CGLN def. of } A, B$$

$$= -\sqrt{-E'+m} \begin{pmatrix} v^\dagger & \frac{\vec{\sigma} \cdot (-\vec{p}')}{-E'+m} & -v^\dagger \end{pmatrix} (-A - \not{Q} B) \begin{pmatrix} u \\ \frac{\vec{\sigma} \cdot \vec{p}}{E+m} u \end{pmatrix} \sqrt{E+m}$$

where as before we are using:

$$Q \equiv \frac{1}{2}(q + q') \quad \bar{u}u = 2m = -\bar{v}v.$$

The A term may be tackled first. It gives:

$$A_{\text{term}} = (-A)(-\sqrt{-E'+m})(\sqrt{E+m}) \left(v^\dagger \left\{ \left(\frac{-\vec{\sigma} \cdot \vec{p}'}{-E'+m} \right) - \left(\frac{\vec{\sigma} \cdot \vec{p}}{E+m} \right) \right\} u \right)$$

Since we are in the cms, we of course know a few helpful things:

$$(-E') = E = \frac{\sqrt{s}}{2} \quad \text{and} \quad (-\vec{p}') = -(\vec{p})$$

With these aids, the A-term immediately reduces to: $A_{\text{term}} = -2A(v^\dagger \vec{\sigma} \cdot \vec{p} u)$

Next, the Q_0 term. This term vanishes for two different reasons, one of which is that the two terms which added above now cancel. Also, it turns out that Q_0 itself is 0 because:

$$Q_0 = \frac{1}{2}[q_0 - (-q'_0)] = \frac{1}{2}[\sqrt{s}/2 - \sqrt{s}/2] = 0.$$

$$\text{By the way: } \vec{Q} = \frac{1}{2}[\vec{q} - (-\vec{q}')] = \frac{1}{2}[\vec{q} - (-\vec{q})] = \vec{q}$$

All that remains is the momentum B term. We find:

$$B(\vec{Q})_{\text{term}} = -(E+m)(-B)(-\vec{Q} \cdot) \left(v^\dagger \left\{ \left(\frac{-\vec{\sigma} \cdot \vec{p}}{E+m} \right) \vec{\sigma} \left(\frac{\vec{\sigma} \cdot \vec{p}}{E+m} \right) + \vec{\sigma} \right\} u \right) \checkmark$$

$$= -2B \left(v^\dagger \left\{ E \vec{\sigma} \cdot \vec{q} - \frac{(\vec{\sigma} \cdot \vec{p})(\vec{p} \cdot \vec{q})}{E+m} \right\} u \right) \checkmark$$

Some algebra is required to simplify this, so here are some of the steps:

$$= v^\dagger \left[\left(-2A + \frac{2B}{E+m} \vec{p} \cdot \vec{q} \right) \vec{\sigma} \cdot \vec{p} + (-2BE) \vec{\sigma} \cdot \vec{q} \right] u \quad \vec{p} \cdot \vec{q} = -\vec{p} \cdot (-\vec{q}) = -|\vec{p}| |\vec{q}| \cos \theta$$

$$= v^\dagger \left[(-2Ap + 2Bm|\vec{q}| \cos \theta) \sigma_z + (+2BE|\vec{q}| \sin \theta) \sigma_x \right] u \quad \checkmark \quad \text{now } |\vec{q}| > 0.$$

Now we have assumed that $\vec{p}$ is along the $\hat{z}$ axis (the nucleon) and $\hat{n} = \hat{y}$. Notice by the way that the parity-allowed terms in the t-channel look like -- in some sense -- the parity-violating terms in the s-channel.

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Before going on to calculate the t-channel helicity amplitudes, it is necessary to understand clearly the spinor v of Dirac theory. First, consider a particle (fermion) (not an antifermion) travelling in the $+z$ direction with z -component of spin equal to $+\frac{1}{2}$. In other words, this particle has helicity $+\frac{1}{2}$. The question is: which spinor describes such a particle. As everybody knows, the answer is:

 $u(p,s) = u(p=p_z, s=+\frac{1}{2} \text{ in rest frame}) = \begin{bmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \frac{\sigma_z p_z}{E+m} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{bmatrix} \sqrt{E+m} \quad u = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Somehow, everyone knows that $(1,0)$ represents an electron or fermion with spin $+\frac{1}{2}$ along the z axis. This is the way σ_z is set up.

Now, consider an anti-fermion travelling in the $+z$ direction with spin in the plus z direction. This is a positive helicity anti-fermion. Everything is the same as before, except this is an antifermion. Question: which spinor represents such a particle? The answer is opposite what my intuition suggests. It is:

 $v(p,s) = v(p=p_z, s=+\frac{1}{2} \text{ in rest frame}) = \begin{bmatrix} \frac{\sigma_z p_z}{E+m} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{bmatrix} \quad v = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

This rather subtle point is discussed in B+d page 68. The point is simply this: this state $v(p,s)$ represents a negative-energy fermion with spin

down (along $-z$). But in hole theory, an anti-fermion is the absence of a negative energy fermion. It is a hole in the sea. The absence of something with spin down is a hole with spin up, just as in solid state physics.

In our present application, we have our anti-nucleon travelling in the $-z$ direction. If this anti-nucleon has $+$ helicity, then it has spin in the $-z$ direction. Thus, for the $+$ helicity antinucleon, one should use $v = (1,0)$. Naively, this is what one might expect in the first place, but we have seen that it is really a case of a double-negative.

Now, recalling the factor relating the Jackson (QED) M amplitude to the MS helicity amplitudes T, we have:

$$\mathcal{M} = - (4\pi^2) T_{00, \mu\mu'}^{(t)} = (-2Ap + 2Bm|q| \cos\theta) v^\dagger(\mu') \sigma_z u(\mu) + (+2BE|q| \sin\theta) v^\dagger(\mu') \sigma_x u(\mu)$$

where, as discussed above,

$$u(+)=\begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad u(-)=\begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad v^\dagger(+)=\begin{pmatrix} 1 & 0 \end{pmatrix} \quad v^\dagger(-)=\begin{pmatrix} 0 & 1 \end{pmatrix}$$

Thus, in particular we find:

$$\begin{aligned} -4\pi^2 T_{00,+-}^{(t)} &= + B \sqrt{t} |q| \sin\theta & E &= \left(\frac{\sqrt{t}}{2}\right) \\ -4\pi^2 T_{00,++}^{(t)} &= -2Ap + 2Bm|q| \cos\theta \end{aligned}$$

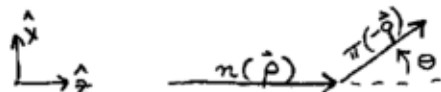
In order to compare these with the amplitudes quoted by MS pages 456, we have to do a little time-reversal. According to MS (5.51) p 232 we know that:

$$\begin{aligned} f_{\mu_a \mu_b; \mu_c \mu_d} &= (-1)^{(\mu_c - \mu_d) - (\mu_a - \mu_b)} f_{\mu_a \mu_b; \mu_c \mu_d} \\ \Rightarrow f_{00; \mu \mu'} &= (-1)^{-(\mu - \mu')} f_{\mu \mu'; 00} \\ \Rightarrow f_{00; ++} &= f_{++; 00} \\ \Rightarrow f_{00; +-} &= -f_{+-; 00} \end{aligned}$$

Then our final results should be:

$$\left[\begin{aligned} -4\pi^2 T_{++; 00}^{(+)} &= -2A p_\pi + 2B m |q| \cos \theta \\ -4\pi^2 T_{+-; 00}^{(+)} &= -B \sqrt{t} |q| \sin \theta \end{aligned} \right]$$

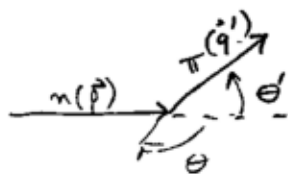
There remains a slight ambiguity concerning the meaning of θ . The way I have calculated things above, I have been using θ as the angle between the incident nucleon p and the final pion of momentum $-q$:



$$\vec{p} \cdot \vec{q} = -(\vec{p}) \cdot (-\vec{q}) = -p|q| \cos \theta$$

$$\vec{\sigma} \cdot \vec{q} = -\vec{\sigma} \cdot (-\vec{q}) = -\sigma_z |q| \cos \theta - \sigma_x |q| \sin \theta$$

Suppose instead that we had used as our scattering angle the angle θ' defined as the angle to the other pion, ie, the one with momentum q' . Then we'd have:



$$\vec{p} \cdot \vec{q} = \vec{p} \cdot \vec{q}' = p q' \cos \theta'$$

$$\vec{\sigma} \cdot \vec{q} = \vec{\sigma} \cdot \vec{q}' = \sigma_z q' \cos \theta' \pm \sigma_x q' \sin \theta'$$

But θ and θ' are supplements: $\cos \theta' = -\cos \theta$ and $\sin \theta' = \sin \theta$

In terms of this other angle, all the sin and cos terms in the helicity amplitudes would have their signs changed. So the question that remains is this: do MS use the angle from p to the crossed pion, or do they use the angle to the pion that was in the final state in the s -channel as well. My inclination is that they use the former (called θ -above). This is because:

$$\begin{aligned} s = (p+q)^2 &= (p - (-q))^2 = t_c \\ (p-p')^2 &= t_s \end{aligned} \quad \left. \vphantom{\begin{aligned} s = (p+q)^2 \\ (p-p')^2 \end{aligned}} \right\} \text{compare.}$$

Thus, the formulae above stand as they are. The lower one disagrees by a sign from MS's.

SU Crossing with Isospin.

Again, go slow and stay clean. Start with:

$$T_s^{\beta\alpha}(p', q'; p, q) \sim \int dx dy (e^{-iq \cdot x} \vec{k}_x \hat{\phi}_i^{(\alpha)}) \langle p' | T \{ \phi_j(y) \phi_i(x) \} | p \rangle (e^{iq' \cdot y} \vec{k}_y \hat{\phi}_j^{(\beta)})^*$$

Now rewrite shuffling stars and using fact that:

$$[\hat{\phi}^{(\alpha)}]^* = \hat{\phi}^{(\alpha*)}$$

Thus get,

$$= \int dx dy (e^{iq \cdot x} \vec{k}_x \hat{\phi}_i^{(\alpha*)}) \langle p' | T \{ \phi_j(y) \phi_i(x) \} | p \rangle (e^{iq' \cdot y} \vec{k}_y \hat{\phi}_j^{(\beta*)})^*$$

Now, reverse positions of the two (...) objects, switch x and y , switch i and j . Result is:

$$= \int dx dy (e^{iq' \cdot x} \vec{k}_x \hat{\phi}_i^{(\beta*)}) \langle p' | T \{ \phi_j(x) \phi_i(y) \} | p \rangle (e^{iq \cdot y} \vec{k}_y \hat{\phi}_j^{(\alpha*)})^*$$

Since the fields are inside a TOP, their order does not matter. Comparison of this last expression with the first shows that:

$$T_s^{\beta\alpha}(p', q'; p, q) = T_s^{\alpha*\beta*}(p', -q; p, -q')$$

Here I have been thinking in terms of charged pions. Of course the thing also goes through with cartesian pions. For cartesian pions, however, $(a^*) = (a)$ so we would forget the stars in the last line above.

$$\hat{\phi}^{(a)} = [\hat{\phi}^{(a)}]^* \text{ for cartesian pions: in fact } \hat{\phi}_i^{(a)} = \delta_{i,a}.$$

Time Reversal with Isospin.

For some reason this keeps goofing me up. Go in slow motion:

$$T_s^{\beta\alpha}(p', q'; p, q) \sim \int dx dy (e^{-iq \cdot x} \vec{K}_x \hat{\phi}_i^{(\alpha)}) \langle p' | T(\phi_j(y) \phi_i(x)) | p \rangle (\hat{\phi}_j^{(\beta)} \vec{K}_y e^{-iq' \cdot y})$$

$$[T_s^{\beta\alpha}(p', q'; p, q)]^* \sim \int dx dy (e^{-iq' \cdot x} \vec{K}_x \hat{\phi}_i^{(\alpha)})^* \langle p' | T(\phi_j(y) \phi_i(x)) | p \rangle^* (\hat{\phi}_j^{(\beta)} \vec{K}_y e^{-iq' \cdot y})$$

The matrix elements yields itself with p and p' reversed, so get:

$$\sim \int dx dy (e^{-iq' \cdot x} \vec{K}_x \hat{\phi}_i^{(\alpha)})^* \langle p | T(\phi_j(y) \phi_i(x)) | p' \rangle (\hat{\phi}_j^{(\beta)} \vec{K}_y e^{-iq' \cdot y})$$

Now simulataneously switch the (...) projectors and also switch x and y and also swithh i and j :

$$\sim \int dx dy (e^{-iq' \cdot x} \vec{K}_x \hat{\phi}_i^{(\beta)}) \langle p | T(\phi_i(x) \phi_j(y)) | p' \rangle (\hat{\phi}_j^{(\alpha)} \vec{K}_y e^{-iq' \cdot y})$$

Above we used the fact the the fields are hermitian and also the fact that the ordering of fields is unimportant inside a time ordered product. So, comparing this last expression with the first we find that:

$$[T_s^{\beta\alpha}(p', q'; p, q)]^* = T_s^{\alpha\beta}(p, q; p', q')$$

This result is valid for both charged and cartein pions.

Crossing Relations for the A and B amplitudes with Isospin.

The following statements are all made in terms of cartesian pion isospin labels only. The general form of the PI-N amplitude may be written as:

$$T_{ba}^{\beta\alpha}(p', q'; p, q) = \bar{u}(p') \left[A_{ba}^{\beta\alpha}(s, u) + \frac{1}{2} (\not{q} + \not{q}') B_{ba}^{\beta\alpha}(s, u) \right] u(p)$$

All isospin indices are shown explicitly. For the cartesian labels we have shown that each amplitude can be written as:

$$A_{ba}^{\beta\alpha} = \delta^{\beta\alpha} \delta_{ba} A^{(+)} + \frac{1}{2} [3^{\beta}, 3^{\alpha}]_{ba} A^{(-)}$$

$$B_{ba}^{\beta\alpha} = \delta^{\beta\alpha} \delta_{ba} B^{(+)} + \frac{1}{2} [3^{\beta}, 3^{\alpha}]_{ba} B^{(-)}$$

From crossing we found that,

$$T^{\beta\alpha}(p', q'; p, q) = T^{\alpha\beta}(p', -q; p, -q')$$

and from time reversal we found that,

$$T^{\beta\alpha}(p', q'; p, q) = [T^{\alpha\beta}(p, q; p', q')]^*$$

Combining these two relations, we can show that:

$$T^{\beta\alpha}(p', q'; p, q) = [T^{\beta\alpha}(p, -q'; p', -q)]^*$$

This is equation (23.16) of Gas. Gas called this relation "crossing" but it seems to me to be a combination of crossing and time-reversal. In order to apply this relation to the form for T above, we first need a little Dirac algebra:

$$\begin{aligned} [\bar{u}(p') \not{\delta}^{\mu} u(p)]^* &= [u^\dagger(p') (\not{\delta}^0 \not{\delta}^{\mu})_{\alpha\beta} u(p)_{\beta}]^* = u^\dagger(p')_{\alpha} (\not{\delta}^0 \not{\delta}^{\mu})_{\alpha\beta}^* u(p)_{\beta}^* \\ &= u^\dagger(p)_{\beta} (\not{\delta}^0 \not{\delta}^{\mu})_{\beta\alpha}^{\dagger} u(p')_{\alpha} = u^\dagger(p)_{\beta} \not{\delta}^{\mu\dagger} \not{\delta}^0 u(p')_{\alpha} \\ &= [\bar{u}(p) \not{\delta}^{\mu} u(p')] \end{aligned}$$

Thus, the relation may be written as:

$$A^{\beta\alpha}(s, u) + \frac{1}{2} (\not{q} + \not{q}') B^{\beta\alpha}(s, u) = [A^{\beta\alpha}(u, s)]^* - \frac{1}{2} [\not{q}' + \not{q}] [B^{\beta\alpha}(u, s)]^*$$

$$\Rightarrow A^{\beta\alpha}(s, u) = [A^{\beta\alpha}(u, s)]^* \quad \text{and} \quad B^{\beta\alpha}(s, u) = -[B^{\beta\alpha}(u, s)]^*$$

If we use (b*) notation on one side of these equations, they become also true for charged pions. Keeping with the cartesians, we see that:

$$A^{(\pm)}(s, u) = \pm [A^{(\pm)}(u, s)]^*$$

$$B^{(\pm)}(s, u) = \mp [B^{(\pm)}(u, s)]^*$$

This result is independent of the "type" of pions used. It is really a statement about t-channel isospin amplitudes, eg,

$$\Delta^{(+)}, \quad \Delta^{(I=0)}, \quad \Delta^{(I=2)}$$

(see (23.17) for)

Comment on Crossing.

I have been using the reduction formulae of axiomatic field theory to prove these crossing relations. Field theory happens to be a model in which crossing is true. But really crossing is more general. It is the analyticity statement that the 3 channel amplitudes are just 3 boundary values of one hugongous analytic function! In this general frame of reference, all these crossing relations are trivial and in fact belong on the same level as kinematics.

For example, to show S-U crossing relations, we would simply observe that $T(s,t,u)$ is supposed to give the scattering amplitudes in all three channels. Here the various spin, helicity, isospin etc indices are suppressed. Now, if the s and the u channel are the same channel, then it must be true that $T(s,t,u) = T(u,t,s)$. It could be no other way. Then all the other details follow like $B(s,u) = -B(u,s)$ simply from the definition of B. If we add isospin, the statement is a little more complicated, but the results are all the same (see Martin Spearman equation (5.100) for a complete statement of crossing with isospin shown explicitly. See equation (7.34) for a complete statement of crossing with helicity shown explicitly.)

About A and B.

When these functions appear without isospin indices, one says that he is dealing with a particular process with a known, given set of particles (isospin labels). It must be understood that when one does a crossing and of some sort that the isospin labels of the crossed process are determined by the labels of the uncrossed process. For example, back on the neutral s-u page we had:

$$T_s(p, q, p', q') \sim \bar{u}(p') A(s, u)$$

$$T_u(p, q, p', q') \sim \bar{u}(p') A(u, s)$$

The indices must be tacked on intelligently, not haphazardly:

$$T_s^{\beta\alpha}(p, q, p', q') \sim \bar{u}(p') A^{\beta\alpha}(s, u)$$

$$T_u^{\alpha^*\beta^*}(\dots) \sim \bar{u}(p') A^{\alpha^*\beta^*}(u, s)$$

Similarly, in the s-t crossing situation we had:

$$T_t(\dots) \sim \bar{v}(-p', s) [A(s, t) \dots]$$

$$T_s(\dots) \sim \bar{u}(p') [A(s, t) \dots]$$

Again, we can add isospin indices:

$$T_{sba}^{\beta\alpha} \sim \bar{u}(p') A_{ba}^{\beta\alpha}(s, t)$$

$$T_{t(b^*a)}^{(\beta\alpha^*)} \sim \bar{v}(-p') A_{b^*a}^{\beta\alpha^*}(s, t)$$

The meaning here of (b^*) is a little complicated. The isospin of a crossed nucleon is usually taken to be opposite the uncrossed, so $(b^*) = (-b)$. This allows CG manipulation, etc etc.

Crossing: With an eye toward finding a more direct way to express the t -channel $[\pi\pi \rightarrow N\bar{N}]$ helicity amplitudes in terms of A and B , I am hunting for references which describe s -ch $\leftrightarrow$ t -ch crossing in terms of A and B .

Burkhop Series, Vol I p214 They discuss the $s \leftrightarrow u$ crossing and how $A^{(+)}(s,u) = A^{(+)}(u,s)$ and all that. The phrase "crossing" is attributed to Gell-Mann and Goldberger (1954) {Proc. 4th Ann. Rochester Conf. p26}. But no use of $\bar{u}(p)$ here. Later (p256) They suggest that $T(s,t)$ inside $\bar{u}_2 t u_1$ is in fact the same in all channels, including the t . The reference here is the "substitution law" of Tanchi + Polids. They do not say what you should do with $\bar{u}_2$!

Gell Mann + Goldberger (under Rochester): A humorous theoretical session, but only passing mention here of crossing. Old too (1954), 20 years old!

Caruthers + Krish - This paper (Ann. Phys. 33,1) does the isospin crossing matrices. If one could get $T_{++}^{(s)}$ and $T_{+-}^{(s)}$ in terms of $T^{(1/2)}$ and $T^{(3/2)}$, then cross these, then unfold to get $T_{++}^{(t)}$ and $T_{+-}^{(t)}$. This might be a reasonable method. (No need to deal with spinors $\bar{u}(p)$ etc). The crossing matrix is no doubt given in MS.

Caruthers (book on unitary symmetry). He basically reviews his own paper and jazzes it up a little, then he generalizes to SU_3 . The starting point is always something like $(cd|Ms|ab)_c = \sum \xi_{\alpha} (\bar{c}\bar{b}|M_{\alpha}|a\bar{d})$ and similarly $(cd|Ms|ab)_c = (\bar{b}\bar{d}|M_{\alpha}|a\bar{c}) \sum \xi_{\alpha}$ where ξ_{α} are phases. The question here is what does $|ab\rangle$ mean if one (or both) have spin.

The index - This is like "a needle in a haystack" - no way I could ever locate a paper on crossing by so vague an approach. I need direct references.

Isospin and the t-channel. $\rightarrow F^{(+)} \text{ and } F^{(-)} \text{ etc.}$

Carefully define the s- and t-channels as follows:



The crossing statement with isospin is this:

$$\langle I_c m_c; I_d m_d | T^{(s)} | I_a m_a; I_b m_b \rangle = \langle I_c m_c; (I_a m_a)_c | T^{(t)} | (I_d m_d)_c; I_b m_b \rangle$$

where $(I_d m_d)_c$ means $C |I_d m_d\rangle$, the charge conjugated state (antiparticle). $= |I_d m$

Unfortunately, these physical antiparticle isospin states do not transform according to the usual angular momentum (isospin) algebra. This is simply due to the fact that C does not commute with I . In fact,

$$C I_1 = -I_1 C \quad C I_2 = +I_2 C \quad C I_3 = -I_3 C$$

Observing that $(I, G) = 0$, MS suggest defining a new kind of antiparticle isospin state:

$$|\bar{I} m\rangle \equiv \eta_I C |I m\rangle = \eta_I C e^{-i\pi I_2} |I m\rangle = \eta_I (-1)^{I-m} |I, -m\rangle_c$$

These barred isospin states do couple in the usual way, and they are related to the physical antiparticle states as shown above or below:

$$|\bar{I} m\rangle = \eta_I (-1)^{I-m} |I, -m\rangle_c \quad |I m\rangle_c = \eta_I^* (-1)^{I+m} |\bar{I} -m\rangle$$

The idea is to do all manipulations with the barred states, then return to the physical states at the end of a calculation.

As an example of the meaning of these states, we can write the isospin state of a proton as:

$$|p\rangle = |\frac{1}{2} \frac{1}{2}\rangle$$

$$|\bar{n}\rangle = |\frac{1}{2} -\frac{1}{2}\rangle$$

The anti-proton state physical is:

$$|\bar{p}\rangle = |\frac{1}{2} \frac{1}{2}\rangle_c$$

$$|n\rangle = |\frac{1}{2} -\frac{1}{2}\rangle_c$$

But the state that we will use in all the algebra is:

$$|p\rangle = |\frac{1}{2} \frac{1}{2}\rangle_c = \eta_{\frac{1}{2}}^* (-1)^{\frac{1}{2}-\frac{1}{2}} |\frac{1}{2} -\frac{1}{2}\rangle$$

You see that the I_3 components change sign in these barred states, and some of the states themselves pick up an overall minus sign.

$$|n\rangle = |\frac{1}{2} -\frac{1}{2}\rangle_c = \eta_{\frac{1}{2}}^* (+1)^{\frac{1}{2}+\frac{1}{2}} |\frac{1}{2} \frac{1}{2}\rangle$$

It would be nice if we could avoid these extra overall minus signs.

Recalling that the charge parity of a charged state is arbitrary (as opposed to the charge parity of a neutral state), we can avoid the above minus signs by including them in the definition of $|\bar{p}\rangle$ and $|\bar{n}\rangle$. In other words, choose:

$$\left. \begin{aligned} C|p\rangle &= +|\bar{p}\rangle \\ C|n\rangle &= -|\bar{n}\rangle \end{aligned} \right\} \Rightarrow \left. \begin{aligned} |\bar{p}\rangle &= +|\frac{1}{2} -\frac{1}{2}\rangle \\ |\bar{n}\rangle &= +|\frac{1}{2} \frac{1}{2}\rangle \end{aligned} \right\}$$

$$\eta_{\frac{1}{2}} = -1$$

Thus, we choose the phase so that the states come out with + signs. Now the only confusion is the reversal of the I_3 sign.

A similar analysis can be done for the pion states. Here, we have some conditions imposed. First, we know that the neutral pion has $C = +$. Second, since the pion is self conjugate, we would like to have the barred states identical to the unbarred. Then:

$$\text{choose } \eta_1 = -1, \quad C|\pi^{\pm}\rangle = -|\pi^{\mp}\rangle \Rightarrow |\pi^{\mp}\rangle = -C|\pi^{\pm}\rangle = -|\pi^{\pm}\rangle_c$$

Lets go over this pion once more. We choose the various phases and charge parities so that:

$$|\bar{p}\rangle = +|\frac{1}{2}, -\frac{1}{2}\rangle$$

$$|\pi_+\rangle = +|1, -1\rangle \quad (=|\pi_-\rangle = |1, -1\rangle)$$

This is all convention. However, since the antipluspion is a minuspion, we can add to the bottom line above as known.

With all these preliminaries out of the way, we now write the t-channel scattering amplitude with isospin shown explicitly and with helicity indices suppressed along with the invariants s and t:

$$\langle I_c m_c (I_a m_a)_c | T^{(t)} | (I_d m_d)_c I_b m_b \rangle = (-1)^{I_a + m_a} (-1)^{I_d + m_d} \langle I_c m_c \bar{I}_a - m_a | T^{(t)} | \bar{I}_d - m_d I_b \rangle$$

$$\textcircled{*} = c + \bar{a} \leftarrow \bar{d} + b$$

$$\textcircled{S} = c + d \leftarrow a + b$$

Now drop the bars on the a and d isospin states, since they are standard states. We will keep in mind that a and d enter as antiparticles. Now we can expand in terms of isospin amplitudes T(I) using usual CG coefficients:

$$= (-1)^{I_a + m_a} (-1)^{I_d + m_d} \sum_I \langle I_c m_c \bar{I}_a - m_a | IM \rangle \langle I_d - m_d I_b m_b | IM \rangle \cdot T^{(t)}(I)$$

Let's apply this immediately to the πN -N t-channel. We have:

$$\textcircled{t} = \begin{array}{c} \pi \quad \bar{\pi} \leftarrow \bar{N} \quad N \\ c + \bar{a} \leftarrow \bar{d} + b \end{array}$$

$$\textcircled{S} = \begin{array}{c} \pi \quad N \leftarrow \pi \quad N \\ c + d \leftarrow a + b \end{array}$$

Now let a, b, α , β be defined as the isospin labels in the s-channel (not in the $\bar{s}$ -channel). That is,

$$m_a = \alpha \quad m_b = a \quad m_c = \beta \quad m_d = b$$

In terms of these labels, we would write as the s-channel amplitude:

$$\langle b\beta | T^{(s)} | a\alpha \rangle$$

Using these same labels, the t-channel amplitude is:

$$\begin{aligned} \langle \beta \bar{\alpha} | T^{(t)} | \bar{b} a \rangle &= (-1)^{1+\alpha} (-1)^{\frac{1}{2}+b} \sum_I \langle 1\beta, 1-\alpha | IM \rangle \langle \frac{1}{2}-b, \frac{1}{2}a | IM \rangle T^{(t)}(I) \\ &= (-1)^{1+\alpha} (-1)^{\frac{1}{2}+b} \times \left[\langle 1\beta, 1-\alpha | 00 \rangle \langle \frac{1}{2}-b, \frac{1}{2}a | 00 \rangle T^{(t)}(I=0) + \langle 1\beta, 1-\alpha | 1M \rangle \langle \frac{1}{2}-b, \frac{1}{2}a | 1M \rangle T^{(t)}(I=1) \right] \end{aligned}$$

We do not include I=2 because the $\frac{1}{2}\frac{1}{2}$ CG vanishes. From either the s-channel or the $\bar{s}$ -channel one can see that:

$$\textcircled{t} \quad m_c - m_a = m_d + m_b$$

$$\textcircled{S} \quad m_a + m_b = m_c + m_d$$

$$\textcircled{t} \quad \beta - \alpha = -b + a$$

$$\textcircled{S} \quad \alpha + a = \beta + b$$

Now lets return to the general t-channel formalism where we had,

$$\langle I_c m_c; I_d m_d | T^{(s)} | I_a m_a; I_b m_b \rangle = (-1)^{I_a + m_a} (-1)^{I_d + m_d} \sum_I T^{(t)}(I) \cdot \langle I_c m_c I_a - m_a | IM \rangle \langle I_d - m_d I_b m_b | IM \rangle$$

If we use the properties of the GC coefficients given in MS appendix or anywhere else, we can show that:

$$\langle I_c m_c I_a - m_a | IM \rangle = (-1)^{I_c - m_c} \sqrt{\frac{2I+1}{2I_a+1}} \langle I_c m_c I - M | I_a m_a \rangle$$

$$\langle I_d - m_d I_b m_b | IM \rangle = (-1)^{I_d + m_d} \sqrt{\frac{2I+1}{2I_b+1}} \cdot (-1)^{I_b - I_d - I} \langle I_d m_d IM | I_b m_b \rangle$$

Recalling the definition of the multipole tensors used in polarization, we can rewrite these as:

$$\langle I_c m_c; I_a - m_a | IM \rangle = (-1)^{I_c - m_c} \sqrt{\frac{2I+1}{2I_a+1}} \langle I_c m_c | T^{I, -M} | I_a m_a \rangle$$

$$\langle I_d - m_d; I_b m_b | IM \rangle = (-1)^{I_d + m_d} \sqrt{\frac{2I+1}{2I_b+1}} \langle I_d m_d | T^{I, M} | I_b m_b \rangle (-1)^{I_b - I_d - I}$$

Inserting these into the above expansion we find:

$$\begin{aligned} \langle cd | T^{(s)} | ab \rangle &= (-1)^{I_c + I_a + m_a - m_c} \frac{1}{\sqrt{(2I_a+1)(2I_b+1)}} \\ &\times \sum_I T^{(t)}(I) \cdot (-1)^{I_b - I_d - I} (2I+1) \times \sum_M \langle I_c m_c | T^{I, -M} | I_a m_a \rangle \langle I_d m_d | T^{I, M} | I_b m_b \rangle \end{aligned}$$

We can see, for example, that unless $I_a = I_c$, there will be no $I=0$ "wave" in the t-channel, which of course makes sense.

We can write the above expansion in a direct product sense, where the first operator operates in the c-a manifold and the second operates in the d-b manifold:

$$\langle cd | T^{(s)} | ab \rangle = \sum_{I, M} f(I) \langle c | T^{I, -M} | a \rangle \langle d | T^{I, M} | b \rangle$$

$$T^{(s)} = \sum_I f(I) \left(\sum_M T^{I, -M} \otimes T^{I, M} \right) (-1)^{I_c + I_a + M}$$

The object within the parentheses is the direct product of two irreducible tensor operators. We know how to combine such things using more CG coefficients. We get:

$$\left(\sum_M T^{I, -M} \otimes T^{I, M} \right) = \sum_M \sum_{I'} T^{I', 0} \langle I' 0 | I - M; I M \rangle = \sum_{I'} T^{I', 0} \sum_M \langle I' 0 | I - M; I M \rangle$$

$(-1)^{I_c + I_a + M}$
 $(-1)^{I_c + I_a + M}$
 $(-1)^{I_c + I_a + M}$

By using the CG symmetry properties, we can find a relation between I and I' :

$$\sum_{all m} \langle I' 0 | I - m I m \rangle = \sum_{all m} \langle I' 0 | I m I - m \rangle = \sum_{all m} (-1)^{I' - 2I} \langle I' 0 | I - m I m \rangle$$

$$\Rightarrow (-1)^{I' - 2I} = +1 \Rightarrow I' - 2I = \text{even integer.}$$

$(-1)^{I' - 2I} (-1)^{2I + 2I_a} = +1$

$$\Rightarrow \begin{matrix} I' = 0, 2, 4, \dots, 2I & \text{if } I = \text{integer} \\ I' = 1, 3, 5, \dots, 2I & \text{if } I = \text{half integer} \end{matrix} \quad \left. \vphantom{\begin{matrix} I' = 0, 2, 4, \dots, 2I \\ I' = 1, 3, 5, \dots, 2I \end{matrix}} \right\} \text{ still OK.}$$

The point of all this detail is that the direct product of operators shown in the parenthesis on the preceding page is not necessarily an isotopic scalar operator as I once thought. If I is an integer, then it is an isoscalar operator plus the $M=0$ piece of an $I=2$ operator, plus so on.

In the case that $I=1$ we have in particular:

$$\left(\sum_m T^{1,m} \otimes T^{1,m} \right) = \left\{ \sum_m \langle 00 | 1, -m | 1 m \rangle \right\} T^{0,0} + \left\{ \sum_m \langle 20 | 1, -m | 1 m \rangle \right\} T^{2,0}$$

$$= \left(\sqrt{\frac{1}{3}} T^{0,0} + \sqrt{\frac{2}{3}} T^{2,0} \right) \quad (\text{by explicit calc.})$$

In the application to the PI-N t-channel, this second term above will give nothing because the NN system cannot yield $I=2$. In this special case only, the sum shown turns out to be an isoscalar.

Application: Now consider again the t-channel of PI-N scattering. The setups are described back on page 2. We find:

$$\langle \beta b | T^{(s)} | \alpha a \rangle = (-1)^{1+\beta} \frac{1}{\sqrt{3} \cdot \sqrt{2}} \sum_I T^{(t)}(I) (-1)^{\frac{1}{2} - \frac{1}{2} - I} (2I+1) \times \sum_m \langle 1\beta | T^{I-m} | 1\alpha \rangle \langle b | T^{I,m} | a \rangle (-1)^m$$

The second matrix element (remember, these things are just CG coefficients) shows that I is restricted to 0 and 1. First, we can calculate the $I=0$ term:

$$I=0 \text{ term} = (-1)^{\alpha-\beta} \cdot \frac{1}{\sqrt{6}} \cdot T^{(t)}(0) \cdot \langle 1\beta | 1\alpha \rangle \langle b | a \rangle = \frac{1}{\sqrt{6}} T^{(t)}(0) \delta_{\alpha\beta} \delta_{ba}$$

The $I=1$ term is a bit more complicated:

$$I=1 \text{ term} = (-1)^{\alpha-\beta} \cdot \frac{1}{\sqrt{6}} \cdot T^{(t)}(1) \cdot (-1)(3) \cdot \sum_m \langle 1\beta | T^{1-m} | 1\alpha \rangle \langle b | T^{1,m} | a \rangle (-1)^m$$

$\neq (-1)^m$

We have here the dot product of two spherical vectors. This can be converted into a useful cartesian vector dot product by recalling the relation to the angular momentum operators:

$$T_{10} = \frac{1}{\sqrt{I(I+1)}} I_z \quad T_{1,\pm 1} = \mp \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{I(I+1)}} [I_x \pm i I_y]$$

Here, I refers to the isospin of the manifold in which you are representing the operator T . For the pions, $I=1$ and for the nucleons $I=\frac{1}{2}$. Then:

$$\sum_m \langle \beta | T^{1,m} | \alpha \rangle \langle b | T^{1,m} | a \rangle^{(-1)^m} = \frac{1}{\sqrt{1 \cdot 2}} \cdot \frac{1}{\sqrt{\frac{1}{2}(\frac{3}{2})}} \times \left\{ \underbrace{1 \langle \beta | I_z | \alpha \rangle \langle b | I_z | a \rangle}_{m=0} + \right.$$

$$\left. + \underbrace{\left(-\frac{1}{2}\right) \langle \beta | I_x - i I_y | \alpha \rangle \langle b | I_x + i I_y | a \rangle}_{m=1} + \underbrace{\left(-\frac{1}{2}\right) \langle \beta | I_x + i I_y | \alpha \rangle \langle b | I_x - i I_y | a \rangle}_{m=-1} \right\}$$

$$= \left\{ \langle \beta | I_z | \alpha \rangle \langle b | I_z | a \rangle + \langle \beta | I_x | \alpha \rangle \langle b | I_x | a \rangle + \langle \beta | I_y | \alpha \rangle \langle b | I_y | a \rangle \right\} \sqrt{\frac{2}{3}}$$

$$= \left\{ \cancel{\langle \beta | I_z | \alpha \rangle \langle b | I_z | a \rangle} + \langle \beta | \vec{I} | \alpha \rangle \cdot \langle b | \vec{I} | a \rangle \right\} \sqrt{\frac{2}{3}}$$

Now at last we can connect to the Pauli matrices. The full $I=1$ term is now:

$$(I=1) = \left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \right)^{\alpha\beta} \cdot \frac{1}{\sqrt{6}} \cdot T^{(4)}(1) \cdot (-3) \cdot \frac{2}{\sqrt{6}} \cdot \left[2 \langle \beta | I_z | \alpha \rangle \langle b | I_z | a \rangle - \langle \beta | \vec{I} | \alpha \rangle \cdot \langle b | \vec{I} | a \rangle \right]$$

$$\langle b | \vec{I} | a \rangle = \frac{1}{2} \langle b | \vec{\sigma} | a \rangle = \frac{1}{2} \vec{\sigma}_{ba}$$

$$\langle \beta | \vec{I} | \alpha \rangle = \vec{\tau}_{\beta\alpha}$$

$$(I=1 \text{ term}) = \left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \right)^{\alpha\beta} (1) T^{(4)}(1) \left[\vec{\tau}_{\beta\alpha} \cdot \frac{1}{2} \vec{\sigma}_{ba} - \left(\frac{2}{\sqrt{6}} \tau_{\beta\alpha}^{(3)} \frac{1}{\sqrt{6}} \sigma_{ba}^{(3)} \right) \right]$$

Putting these two terms together we get:

$$\langle \beta | T^{(4)} | \alpha \rangle = \frac{1}{\sqrt{6}} T^{(4)}(0) \delta_{\alpha\beta} \delta_{ba} + \left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \right)^{\alpha\beta} T^{(4)}(1) \left[\frac{1}{2} \vec{\tau}_{\beta\alpha} \cdot \vec{\sigma}_{ba} - \tau_{\beta\alpha}^{(3)} \sigma_{ba}^{(3)} \right]$$

All indices here are physical, isospin indices. I.e., this is the answer for a physical scattering process. The matrix t' is the one Gasiorowicz calls t' . The eigenvectors of t'_3 are the physical pions like $(1 \ 0 \ 0)$. Notice that the first term (the $I=0$ term) does not contain all the "diagonal" amplitude. (By diagonal is meant $\alpha = \beta$ and $a = b$.)

$$\left[\langle \beta | T^{(4)} | \alpha \rangle = \frac{1}{\sqrt{6}} T^{(4)}(0) \delta_{\alpha\beta} \delta_{ba} - \frac{1}{2} T^{(4)}(1) \vec{\tau}_{\beta\alpha} \cdot \vec{\sigma}_{ba} \right]$$

However, if we wanted, we could perform a unitary transformation on these isospin states of the pion only to get "cartesian" isospin states. These unphysical pion states are labelled with indices 1,2,3. Such a transformation takes the matrices t' into new matrices t . These new t matrices are characterized by the fact that they have no diagonal elements anywhere (not even in t_3). So, for the 123 type pions which I indicate by i, j indices we have:

$$\langle j | T^{(3)} | i a \rangle = \frac{1}{\sqrt{6}} T^{(t)}(0) \delta_{ji} \delta_{ba} - \frac{1}{2} T^{(t)}(1) \vec{t}_{ji} \cdot \vec{3}_{ba}$$

where $\vec{t}_{ji}^{(k)} = -i \epsilon_{jik}$ So: $\vec{t}_{ji} \cdot \vec{3}_{ba} = -i \epsilon_{jik} \vec{3}_{ba}^{(k)}$
 $= -\frac{1}{2} [3_j, 3_i]_{ba}$ ✓

Thus, using 123 type pion states, our final answer is:

$$[\langle j | T^{(3)} | i a \rangle = \frac{1}{\sqrt{6}} T^{(t)}(0) \delta_{ji} \delta_{ba} + \frac{1}{4} T^{(t)}(1) [3_j, 3_i]_{ba}] \quad \text{see last page for def of } |i\rangle.$$

And so the long sought after question is answered: Yes, the amplitudes $F(+)$ and $F(-)$ used by Gas. are proportional to the t -channel isospin amplitudes.

In fact, directly comparing this with Gas (23.20):

$$[\langle j | F | i \rangle = \delta_{ji} F^{(+)} + \frac{1}{2} [3_j, 3_i] F^{(-)}]$$

we may conclude that:

$$F^{(+)} = \frac{1}{\sqrt{6}} T_{I=0}^{(t)} \quad T^{(0)} = \sqrt{6} F^{(+)} \\ T^{(1)} = 2 F^{(-)}$$

$$F^{(-)} = \frac{1}{2} T_{I=1}^{(t)}$$

On the following sheets we showed that:

$$F^{(+)} = \frac{1}{3} [2 T^{3/2} + T^{1/2}] \quad T^{3/2} = F^{(+)} - F^{(-)} \\ F^{(-)} = \frac{1}{3} [F^{1/2} - F^{3/2}] \quad T^{1/2} = F^{(+)} + 2 F^{(-)}$$

Based on the above findings, we would predict that:

$$T^{(0)} = \sqrt{\frac{2}{3}} T^{(1/2)} + \sqrt{\frac{2}{3}} T^{(3/2)} \quad T^{(1/2)} = \sqrt{\frac{1}{6}} T^{(0)} + 1 \cdot T^{(1)} \\ T^{(1)} = \frac{2}{3} T^{(1/2)} - \frac{2}{3} T^{(3/2)} \quad T^{(3/2)} = \sqrt{\frac{1}{6}} T^{(0)} - \frac{1}{2} T^{(1)}$$

These agree perfectly with (5.116) and (5.117) of MS (the isospin crossing matrices.)

Let's now reexamine the charged pion results. On page 5 we found that:

$$[\langle \beta b | T^{(s)} | \alpha a \rangle = F^{(+)} \delta_{\beta\alpha} \delta_{ba} - F^{(-)} \vec{t}'_{\beta\alpha} \cdot \vec{t}'_{ba}]$$

where F^{+} and F^{-} were just defined on the last page, and where t' is the usual angular momentum 3-dimensional representation (see p 247 Gas. or p 203 Schiff). In other words, these t' matrices satisfy the required commutation relations and t'_3 has eigenvectors corresponding to the charged, physical pions.

However, we know another expansion for the above amplitude: in direct s-channel isospin amplitudes. The result is:

$$\langle \beta b | T^{(s)} | \alpha a \rangle = \langle \frac{1}{2} m | | \alpha \frac{1}{2} a \rangle \langle \frac{1}{2} m | | \beta \frac{1}{2} b \rangle T^{1/2} + \langle \frac{3}{2} m | | \alpha \frac{1}{2} a \rangle \langle \frac{3}{2} m | | \beta \frac{1}{2} b \rangle T^{3/2}$$

To derive the relations quoted on page 6, we look at two terms in both expansions. Namely,

$$\langle \frac{1}{2} | T^{(s)} | \frac{1}{2} \rangle = F^{(+)} - F^{(-)} \cdot 1 \cdot 1 = T^{3/2}$$

$$\langle 0 \frac{1}{2} | T^{(s)} | 0 \frac{1}{2} \rangle = F^{(+)} + 0 = \frac{1}{3} T^{1/2} + \frac{2}{3} T^{3/2}$$

Thus we have derived :

$$F^{(+)} = \frac{1}{3} [T^{1/2} + 2T^{3/2}] \quad \text{and} \quad F^{(-)} = \frac{1}{3} [T^{1/2} - T^{3/2}]$$

Charged vs. neutral pions.

Suppose one has a matrix element of an operator taken between charged pion states β and α . Then, if the set of states $|i\rangle$ forms an orthonormal set, we can reexpress that matrix element.

$$\langle \beta | T | \alpha \rangle = \sum_{ij} \langle \beta | i \rangle \langle i | T | j \rangle \langle j | \alpha \rangle$$

For the pion system, the set of 9 (complex) numbers $\langle i | \alpha \rangle$ forms a unitary transformation connecting two representations. So far, we have not specified what representation $|i\rangle$ is. Let's define the representation $|i\rangle$ such that the unitary transformation $\langle i | \alpha \rangle$ performs the following service:

$$\vec{t}_{\beta\alpha} = \sum_{ij} \langle \beta | i \rangle \vec{t}_{ij} \langle j | \alpha \rangle$$

Here, t is the isospin operator. We know that $t'_{\beta\alpha}$ is the standard isospin representation, and we want t_{ij} to be:

$$t_{ij}^{(a)} = -i \epsilon_{ijk}$$

This then determines the transformation. We find that:

$$\langle i | \alpha \rangle = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ i/\sqrt{2} & -i/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{matrix} i=1 \\ i=2 \\ i=3 \end{matrix} \equiv \hat{g}_i^{(\alpha)}$$

$\alpha = \pi^+ \quad \alpha = \pi^- \quad \alpha = \pi^0$

In terms of this "unit vector notation" we can say:

$$\hat{g}^{(\pi^+)} = \begin{bmatrix} 1/\sqrt{2} \\ i/\sqrt{2} \\ 0 \end{bmatrix} \quad \hat{g}^{(\pi^-)} = \begin{bmatrix} 1/\sqrt{2} \\ -i/\sqrt{2} \\ 0 \end{bmatrix} \quad \hat{g}^{(\pi^0)} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Note that these things are vectors in the cartesian space, not in the charged pion space. In other words, the unit vectors are:

$$|1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad |2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad |3\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Since the matrices t do represent isospin in this 1234 representation, we expect and find that:-

$$t^{(3)} |\pi^\pm\rangle = t_{ij}^{(3)} \hat{g}_j^{(\pi^\pm)} = \pm \hat{g}_i^{(\pi^\pm)} = \pm |\pi^\pm\rangle$$

Also, we can show that:

$$\vec{t}_{\beta\alpha} = -i \hat{g}^{*(\beta)} \times \hat{g}^{(\alpha)}$$

Connection between $f_1 f_2$ and $f_{++} f_{+-}$.

The trick here is knowing how to manipulate helicity states and spin states in the same breath. When one rotates a helicity state, the helicity stays the same, but when one rotates a spin state, the rotation matrices must be called in. The page following shows exactly how spin states go under a rotation, along with some irrelevant gimmickry. In what follows (follows) I will use a greek μ for helicity indices, and m for spin labels. Of course at zero angle, both types of states coincide (a fact we make use of!)

So, start off with helicity states all round and make use of the helicity projection operators:

$$f_{\mu'\mu} = \langle \theta \phi \mu' | f | 00 \mu \rangle = \langle \theta \phi \mu' | f_1^{\sigma} + f_2^{\sigma} \hat{\sigma} \cdot \hat{R}_1 \hat{\sigma} \cdot \hat{R}_2 | 00 \mu \rangle \\ = \langle \theta \phi \mu' | f_1^{\sigma} + 2\mu' 2\mu f_2^{\sigma} | 00 \mu \rangle$$

We cannot just close down the states because μ' is quantized in some wierd direction. We can close after inserting a unity on the left, however:

$$f_{\mu'\mu} = \sum_m \langle \theta \phi \mu' | \theta \phi \frac{1}{2} m \rangle \langle \theta \phi \frac{1}{2} m | f_1^{\sigma} + 4\mu\mu' f_2^{\sigma} | 00 \frac{1}{2} \mu \rangle \\ = \sum_m \langle \theta \phi \mu' | \theta \phi \frac{1}{2} m \rangle \delta_{m\mu} [f_1(\theta, \phi) + 4\mu\mu' f_2(\theta, \phi)]$$

That dude on the left may be calculated by extracting a rotation operator from the left side and applying it to the right side, where it rotates those spin states:

$$\langle \theta \phi \mu' | \theta \phi \frac{1}{2} m \rangle = \langle 00 \mu' | R^\dagger | \theta \phi \frac{1}{2} m \rangle = \sum_{m'} D_{m'm'}^{1(\frac{1}{2})}(\phi, \theta, \phi) \langle 00 \mu' | 00 \frac{1}{2} m' \rangle \\ = D_{\mu'm}^{-1(\frac{1}{2})}$$

Thus, the helicity amplitudes are,

$$f_{\mu'\mu} = D_{\mu'\mu}^{-1} [f_1 + 4\mu\mu' f_2] = D_{\mu\mu'}^* [] = e^{i[\mu-\mu']} d_{\mu\mu'}^{1/2}(\theta) [].$$

$$\begin{aligned} f_{++} &= \cos \frac{\theta}{2} [f_1 + f_2] &= f_{--} \\ f_{+-} &= \sin \frac{\theta}{2} [f_1 - f_2] e^{-i\phi} &= -e^{-2i\phi} f_{-+} \end{aligned} \quad \left. \vphantom{\begin{aligned} f_{++} \\ f_{+-} \end{aligned}} \right\} \text{parity.}$$

Rotation of spin states.

Exercise: to illustrate the manipulation of non-helicity plane-wave states with spin.

What is the effect of a rotation operator on the state shown:

$$R \otimes | \vec{K}, S, \nu \rangle \quad | \vec{K}, S, \nu \rangle = | K \otimes \phi S \nu \rangle$$

Naively we would say it was simply:

$$R | \vec{K}, S, \nu \rangle = | R \vec{K}, \tilde{S}, \tilde{\nu} \rangle = \sum_{\nu'} D_{\nu' \nu}^{(S)} | R \vec{K}, S, \nu' \rangle$$

But this can be achieved by the following set up manipulations:
(all repeated indices are to be summed)

$$\begin{aligned} | \vec{K}, S, \nu \rangle &= | \vec{K} \rangle \otimes | S, \nu \rangle = | \ell m \rangle \langle \ell m | \vec{K} \rangle \otimes | S, \nu \rangle = \langle \ell m | \vec{K} \rangle | \ell m S \nu \rangle \\ &= \langle \ell m | \vec{K} \rangle | J M \rangle \langle J M | \ell m S \nu \rangle \\ R | \vec{K}, S, \nu \rangle &= \langle \ell m | \vec{K} \rangle D_{M' M}^J | J M' \rangle \langle J M' | \ell m S \nu \rangle \\ &= \langle \ell m | \vec{K} \rangle | J M' \rangle \langle J M' | \ell m S \nu \rangle \\ &= \langle \tilde{\ell} m | \tilde{\vec{K}} \rangle | J M' \rangle \langle J M' | \tilde{\ell} m S \nu \rangle \\ &= \langle \tilde{\ell} m | \tilde{\vec{K}} \rangle | \tilde{\ell} m S \nu \rangle = | \tilde{\ell} m \rangle \langle \tilde{\ell} m | \tilde{\vec{K}} \rangle \otimes | S \nu \rangle \\ &= | \tilde{\vec{K}} \rangle \otimes | S \nu \rangle = | \tilde{\vec{K}}, S, \nu \rangle \end{aligned}$$

Here, a twiddle on any state means that state rotated by operator R .
In fact, using this twiddle notation, we can do the whole thing without ever using the rotation matrices explicitly:

$$\begin{aligned} | \vec{K}, S, \nu \rangle &= \langle \ell m | \vec{K} \rangle | J M \rangle \langle J M | \ell m S \nu \rangle \\ R | \vec{K}, S, \nu \rangle &= \langle \ell m | \vec{K} \rangle | \tilde{J} M \rangle \langle \tilde{J} M | \ell m S \nu \rangle \\ &= \langle \tilde{\ell} m | \tilde{\vec{K}} \rangle | J M \rangle \langle J M | \tilde{\ell} m S \nu \rangle \quad (\text{see below}) \\ &= \langle \tilde{\ell} m | \tilde{\vec{K}} \rangle | J M \rangle \langle J M | \tilde{\ell} m S \nu \rangle \\ &= \langle \tilde{\ell} m | \tilde{\vec{K}} \rangle | \tilde{\ell} m S \nu \rangle = | \tilde{\vec{K}}, S, \nu \rangle \end{aligned}$$

We are using these shorthand facts throughout:

$$\langle A | B \rangle = \langle A R^\dagger R B \rangle = \langle \tilde{A} | \tilde{B} \rangle$$

$$1 = | J M \rangle \langle J M | = | \tilde{J} M \rangle \langle \tilde{J} M |$$

$$| \tilde{J} M \rangle \langle J M | a \rangle = R | J M \rangle \langle J M | a \rangle = R | a \rangle = | J M \rangle \langle J M | R | a \rangle = | J M \rangle \langle \tilde{J} M | \tilde{a} \rangle$$

Relations between f_1 and f_2 and the functions f, g etc.

The f_1 and f_2 functions are defined the same way by everyone, I think. They are used with spin states, not helicity states. Namely:

$$\frac{d\sigma}{d\Omega} = \frac{1}{2} \sum_{\chi_f} |\chi_f^\dagger M \chi_i| = \frac{1}{2} \sum_{\nu, \nu'} \langle \frac{1}{2} \nu' | M | \frac{1}{2} \nu \rangle$$

$$\{M = f_1 I + f_2 \vec{\sigma} \cdot \hat{k}_f \vec{\sigma} \cdot \hat{k}_i\} \equiv f_1, f_2$$

The relation of these to the set of functions (f, g) of MS p 251 is:

$$\{M = f I + i g \vec{\sigma} \cdot \hat{n}\} \equiv f, g \quad \text{where} \quad \hat{n} = \widehat{k_i \times k_f} = \frac{\hat{k}_i \times \hat{k}_f}{\sin \theta}$$

$$\begin{cases} f = f_1 + f_2 \cos \theta \\ g = -f_2 \sin \theta \end{cases}$$

Our old friend Williams defines some functions (g, h) according to:

$$\{M = g - i h \vec{\sigma} \cdot \hat{n}\} \equiv g, h \quad [g_W = f_{ms}, h_W = -g_{ms}]$$

$$\begin{cases} g = f_1 + f_2 \cos \theta \\ h = +f_2 \sin \theta \end{cases}$$

Goldberger and Watson define a third set of functions (f, g) by:

$$\{M = f + i g \sin \theta \vec{\sigma} \cdot \hat{n}\} \equiv f, g \quad [g_{GW} \sin \theta = g_{ms}]$$

$$\begin{cases} f = f_1 + f_2 \cos \theta \\ g = -f_2 \end{cases}$$

$$\sin \theta g^{GW} = g^{ms}$$

Pauli Reduction of the A'B amplitudes. Relating A and B to f_1 and f_2 .

From the original paper of CGLN, we had the use of the Pauli gamma matrices and the Pauli metric. Converting to the BD metric and gammas, I find that CGLN's expression for the Jackson "invariant amplitude" to be:

$$M = \bar{u}_f T u_i = \bar{u}_f [-A - \not{Q} B] u_i$$

→ let's off one * here. Hope it makes no diff.

$$Q = \frac{1}{2} [q_1 + q_2]$$

$$= \frac{E+m}{2m} \left[u_f^\dagger \quad -u_f^\dagger \frac{\vec{\sigma} \cdot \vec{p}_f}{E+m} \right] [-A - \not{Q} B] \begin{bmatrix} u_i \\ \frac{\vec{\sigma} \cdot \vec{p}_i}{E+m} u_i \end{bmatrix}$$

$$E_i = E_f = E_{\text{cms}}$$

The A term is simply:

$$(-A) \frac{E+m}{2m} \langle f | 1 - \frac{\vec{\sigma} \cdot \vec{p}_f \vec{\sigma} \cdot \vec{p}_i}{(E+m)^2} | i \rangle$$

where i and f are standard spin states, not helicity states. We can see clearly the A contribution to f_1 and f_2 . The second term must be split into two parts. The energy part gives:

$$Q_0 (-B) \frac{E+m}{2m} \langle f | 1 + \frac{\vec{\sigma} \cdot \vec{p}_f \vec{\sigma} \cdot \vec{p}_i}{(E+m)^2} | i \rangle \quad Q_0 = \frac{1}{2} [q_1 + q_2] = \omega$$

while the momentum part is

$$(-\vec{Q} \cdot \vec{\sigma}) (-B) \frac{E+m}{2m} \cdot \frac{1}{E+m} \langle f | \vec{\sigma} (\vec{\sigma} \cdot \vec{p}_i) + (\vec{\sigma} \cdot \vec{p}_f) \vec{\sigma} | i \rangle$$

Algebra suggests:

$$\begin{aligned} (\vec{Q} \cdot \vec{\sigma}) (\vec{\sigma} \cdot \vec{p}_i) + (\vec{\sigma} \cdot \vec{p}_f) (\vec{\sigma} \cdot \vec{Q}) &= (\vec{q}_i \cdot \vec{\sigma}) (\vec{\sigma} \cdot \vec{p}_i) + (\vec{\sigma} \cdot \vec{p}_f) (\vec{\sigma} \cdot \vec{q}_i) \\ &= \left\{ -|\vec{p}_i|^2 - (\vec{\sigma} \cdot \vec{p}_f) (\vec{\sigma} \cdot \vec{p}_i) \right\} \quad \vec{q}_i = -\vec{p}_i \end{aligned}$$

So momentum part is:

$$\frac{1}{2m} (-B) \langle f | \vec{p}_i^2 + (\vec{\sigma} \cdot \vec{p}_f) (\vec{\sigma} \cdot \vec{p}_i) | i \rangle$$

Now we can read off the expressions for f_1 and f_2 where,

$$M_1 = -\frac{(E+m)}{2m} [A + B(W-m)]$$

$$M = \langle f | M_1 + M_2 \vec{\sigma} \cdot \hat{p}_f \vec{\sigma} \cdot \hat{p}_i | i \rangle$$

$$M_2 = -\frac{(E-m)}{2m} [-A + B(W+m)]$$

Note: to interpret M as Jackson's M, we must use $\bar{u}u = 2m$. Since CGLN use $\bar{u}u = 1$, we cannot use their T for our M. Thus, all my $2m$'s should not be there.

To get the real f_1 and f_2 we have to convert from M-type to f-type amplitudes:

$$f = -\frac{1}{8\pi W} M \Rightarrow \left. \begin{aligned} f_1 &= +\frac{(E+m)}{8\pi W} [A + B(W-m)] \\ f_2 &= +\frac{(E-m)}{8\pi W} [-A + B(W+m)] \end{aligned} \right\} \text{And these are exactly those}$$

Relating A and B to f_{++} and T_{++} .

So far, this is what we know:

$$f_{++} = \cos \frac{1}{2} \theta (f_1 + f_2)$$

$$f_{+-} = \sin \frac{1}{2} \theta (f_1 - f_2) \quad \phi = 0$$

also in
Singh

$$\begin{cases} f_1 = \frac{E+m}{8\pi W} [A + B(W-m)] \\ f_2 = \frac{E-m}{8\pi W} [-A + B(W+m)] \end{cases}$$

$$f_{\mu\nu} = \frac{\pi}{2\sqrt{s}} T_{\mu\nu}$$

Lets make sure we know exactly what all these amplitudes are. The f_{++} babies are defined on p 247 of MS so that their square gives the DCS. The f_1 type are defined with the same normalization. The A and B are exactly those of CGLN where:

$$S = \delta_{fi} - i(2\pi)^4 \delta^4(\text{ext}) \cdot \sqrt{\frac{1}{2E_1 2E_2 W_2 W_1}} \mathcal{M}$$

$$\mathcal{M} = \bar{u}_f [-A - \frac{1}{2} [\not{q}_i + \not{q}_f] B] u_i \quad \bar{u} u = 2m$$

$$\mathcal{M} = -8\pi W f$$

It is then a simple matter to show that:

$$f_{++} = \cos \frac{1}{2} \theta \left[\frac{1}{8\pi\sqrt{s}} \right] (2mA + B(s-m^2-\mu^2))$$

$$f_{+-} = \sin \frac{1}{2} \theta \left[\frac{1}{8\pi\sqrt{s}} \right] (A[s+m^2-\mu^2] + mB[s-m^2+\mu^2]) / \sqrt{s}$$

$$T_{++} = \cos \frac{1}{2} \theta \left[\frac{1}{4\pi^2} \right] (\quad)$$

$$T_{+-} = \sin \frac{1}{2} \theta \left[\frac{1}{4\pi^2} \right] (\quad) / \sqrt{s}$$

does not quite agree with MS # (7.91).

Pion Nucleon Catalog of Partial Wave Amplitudes.

In what follows we will be using the following symbols:

$$\begin{array}{cccc} (T_{++}^J, T_{+-}^J) & (T_{J+}, T_{J-}) & (S_{J+}, S_{J-}) & (N_{J+}, \delta_{J+}, N_{J-}, \delta_{J-}) \\ (f_{++}^J, f_{+-}^J) & (f_{J+}, f_{J-}) & & \end{array}$$

All of these notations refer to partial wave amplitudes. The host of notations referring to the full PI-N amplitudes is discussed elsewhere. For example,

$$T^0, T^1, T^{1/2}, T^{3/2}, F^{(+)}, F^{(-)}, A, B, A', f_1, f_2, f_{++}, T_{++}, f, g$$

One possible source of confusion is the relation between f and T amplitudes. For the full amplitudes, the relation is:

For the partial wave amplitudes, the f amplitudes are defined differently (and arbitrarily) as:

$$f_{++} = \frac{\pi}{2\sqrt{s}} T_{++}^{(s)}$$

$$f_{++}^J \equiv \frac{1}{2q_i} T_{++}^J \quad f_{J\pm} \equiv \frac{1}{2q_i} T_{J\pm} \quad q_i = q_f = q$$

The phase shifts and inelasticities are used to represent S . It can be shown that $|S_{J\pm}| \leq 1$ so it is appropriate to use:

$$S_{J\pm} = N_{J\pm} e^{2i\delta_{J\pm}}$$

The connection between these $S_{J\pm}$ amplitudes and the $T_{J\pm}$ amplitudes is the MS standard relation:

$$\begin{aligned} S_{J\pm} &= 1 + iT_{J\pm} \Rightarrow iT_{J\pm} = N_{J\pm} e^{2i\delta_{J\pm}} - 1 \\ &\Rightarrow \quad \quad \quad = 2iq_i f_{J\pm} \end{aligned}$$

The amplitudes T_{++}^J are defined by:

$$f_{\mu'\mu}^J = \frac{1}{2\sqrt{q_i q_f}} e^{i(\mu-\mu')\phi} \sum_{\gamma} (2J+1) T_{\mu\mu}^J d_{\mu\mu'}^J(\theta) \quad \text{note!}$$

Thus, if we can define what we mean by the amplitudes $T_{J\pm}$, then the meaning of all the other amplitudes follows.

The motivation for defining the amplitudes $T_{J\pm}$ is as follows. Consider the partial wave S matrix elements:

$$S_{\mu'\mu}^J = 1 + iT_{\mu'\mu}^J = \langle JM; \mu' | S | JM; \mu \rangle$$

For a particular J , we are showing the matrix elements of S inside a sort of helicity manifold. The idea is simply this: since S is unitary within this manifold, we know that if it were also diagonal, then we could represent each diagonal element as a phase. This theorem is proved in section "group". Thus, we want to avoid representations in which S is non-diagonal. But

unfortunately S is non-diagonal in the usual helicity representation. In fact what we have is:

$$T_{\mu\mu'}^J = \begin{bmatrix} T_{++}^J & T_{+-}^J \\ T_{-+}^J & T_{--}^J \end{bmatrix} = \begin{bmatrix} T_{++}^J & T_{+-}^J \\ T_{+-}^J & T_{++}^J \end{bmatrix}$$

The last step was done with the help of parity. Recall that in general

$$P |JM, \mu_1 \mu_2\rangle = \eta_1 \eta_2 (-1)^{J-S_1-S_2} |JM, -\mu_1 -\mu_2\rangle$$

and in our special case of PI-N we have

$$P |JM, \mu\rangle = -(-1)^{J-\frac{1}{2}} |JM, -\mu\rangle$$

which means that

$$\langle JM, \mu' | T | JM, \mu \rangle = [-(-1)^{J-\frac{1}{2}}]^2 \langle JM, -\mu' | T | JM, -\mu \rangle$$

OR $T_{\mu'\mu}^J = + T_{-\mu' -\mu}^J$

The task is now to find some states in which T^J is diagonal. Then the diagonal elements of the corresponding S^J can be written as phase shifts (or, when the possibility of other channels is taken into account, as phase shifts and inelasticities).

Toward this end, let us define two explicit states a and b as:

$$\begin{aligned} |JM; a\rangle &= \frac{1}{\sqrt{2}} [|JM, +\frac{1}{2}\rangle + |JM, -\frac{1}{2}\rangle] & \text{or} & \quad |a\rangle = \frac{1}{\sqrt{2}} [|+\rangle + |-\rangle] \\ |JM; b\rangle &= \frac{1}{\sqrt{2}} [|JM, +\frac{1}{2}\rangle - |JM, -\frac{1}{2}\rangle] & & \quad |b\rangle = \frac{1}{\sqrt{2}} [|+\rangle - |-\rangle] \end{aligned}$$

Here the $\frac{1}{2}$'s refer of course to helicity (not "spin" wrt fixed axis, etc). It is easily shown that these states are parity eigenstates:

$$P|a\rangle = (-1)^{J+\frac{1}{2}} |a\rangle \quad P|b\rangle = (-1)^{J-\frac{1}{2}} |b\rangle$$

If one is +, then the other is -, but we cannot say which is which unless we know the value of J . Earlier, this caused me some confusion, but now I forget why.

What are the matrix elements of T in this parity eigenstate representation? Straightforward computation reveals:

$$\begin{aligned} \langle a | T | a \rangle &= T_{++}^J + T_{+-}^J \equiv T_{\bar{J}-} \\ \langle b | T | b \rangle &= T_{++}^J - T_{+-}^J \equiv T_{\bar{J}+} \\ \langle a | T | b \rangle &= \langle b | T | a \rangle = 0 \end{aligned}$$

We see that the off-diagonal elements vanish. We know this without any calculation because the operator T is parity invariant. It cannot connect states of opposite parity. Naturally, that is why we chose such states in the first place.

Now that we have a diagonal T, we can go ahead and define the inelasticities and phase shifts. This really completes the "catalog" of partial wave amplitudes, but there are a few more details to be worked out.

First, let's show explicitly why a diagonal S allows for a phase. Here I will use only the elastic channel so there will be no inelasticity. Start with:

$$S^\dagger S = 1 \Rightarrow \langle \theta \phi \mu_f | 00 \mu_i \rangle = \langle \theta \phi \mu_f | S^\dagger S | 00 \mu_i \rangle$$

$$\sum_{\mu_n} \int d\Omega_n \langle \theta \phi \mu_f | S^\dagger | \theta_n \phi_n \mu_n \rangle \langle \theta_n \phi_n \mu_n | S | 00 \mu_i \rangle$$

Rather than laboriously convert this to the JM representation by inserting unity four times, let's just start over in the JM:

$$S^\dagger S = 1 \Rightarrow \langle j m \mu_f | j m \mu_i \rangle = \langle j m \mu_f | S^\dagger S | j m \mu_i \rangle$$

$$= \sum_{\mu_n} \langle j m \mu_f | S^\dagger | j m \mu_n \rangle \langle j m \mu_n | S | j m \mu_i \rangle$$

We have used the fact that S conserves j (but does not "conserve" helicity). Although I show only one helicity index, we can imagine each as a full set for the two particles in the channel. Continuing:

$$\langle \mu_f | \mu_i \rangle = \sum_{\mu_n} S_{\mu_n \mu_f}^{j*} S_{\mu_n \mu_i}^j = \delta_{\mu_f \mu_i}$$

If these S matrices are diagonal we can add:

$$\delta_{\mu_f \mu_i} = S_{\mu_f \mu_f}^{j*} S_{\mu_i \mu_i}^j \delta_{\mu_f \mu_i} \quad \text{or} \quad 1 = |S_{\mu_f \mu_f}^j|^2 \quad \forall \mu_f.$$

Thus, each diagonal element is just a phase.

Next, let's repeat the argument including channels other than the elastic channel. In each state, then, we will add a channel index. We consider scattering that begins and ends in the same channel (channel e), but we allow for lots of intermediate channels.

$$\langle j m \mu_f e | j m \mu_i e \rangle = \sum_{\mu_n} \langle j m \mu_f e | S^\dagger | j m \mu_n e \rangle \langle j m \mu_n e | S | j m \mu_i e \rangle$$

$$+ \sum_{\delta} \sum_{\mu_n} \langle j m \mu_f e | S^\dagger | j m \mu_n \delta \rangle \langle j m \mu_n \delta | S | j m \mu_i e \rangle$$

other channels

Picking off the diagonal term (i = f):

$$1 = \sum_{\mu_n} |S_{\mu_n \mu_f}^j|^2 + \sum_{\delta} \sum_{\mu_n} |S_{\mu_n \mu_f}^j|^2$$

If S is diagonal in the elastic channel, we can say:

$$|S_{\mu_f \mu_f}^j|^2 = 1 - \sum_{\mu_n} |S_{\mu_n \mu_f}^j|^2$$

Thus, the elements of the diagonal elastic submatrix may be written as:

$$eS_{\mu\mu}^J = A(\mu) e^{i\phi(\mu)} \quad \text{with} \quad 0 \leq A \leq 1$$

In our particular case we write (the 2 is arbitrary):

$$eS_{J\pm} = \mathcal{N}_{J\pm} e^{2i\delta_{J\pm}}$$

Next, to explain the origin of the notation $S_{J\pm}$. We know that parity invariance tells us that:

$$-(-1)^{L_i} = -(-1)^{L_f} \quad \text{pion intrinsic}$$

And angular momentum says: $J_i = J_f$

For a given J, the possibilities for the initial and final L are:

$$L_i = J \pm \frac{1}{2} \quad L_f = J \pm \frac{1}{2}$$

They can be the same, or they can differ by one. If they differed by one, parity would be violated, thus they are the same. In pion-nucleon scattering then we have a very simple situation:

$$J_i = J_f = J \quad L_i = L_f = L = J \pm \frac{1}{2}$$

We showed that the state "a" had parity:

$$P|a\rangle = (-1)^{J+\frac{1}{2}} |a\rangle$$

But we know that its parity can also be written:

$$\rightarrow = -(-1)^L |a\rangle$$

$$\Rightarrow L = J - \frac{1}{2} \text{ for state a} \rightarrow T_{J-}$$

$$\text{and } L = J + \frac{1}{2} \text{ for state b} \rightarrow T_{J+}$$

Thus, the subscript + or - is a reminder to tell us what value L takes for the state in question.

Next, we tackle the problem of writing the standard partial wave expansion in terms of these parity eigenstate partial wave amplitudes. The partial wave amplitude appearing on page 1 may be expanded as:

$$T_{\mu'\mu}^J = \langle \mu' | T^J | \mu \rangle = \langle \mu' | a \rangle \langle a | T^J | a \rangle \langle a | \mu \rangle + \langle \mu' | b \rangle \langle b | T^J | b \rangle \langle b | \mu \rangle$$

The elements of the transformation matrix are:

$$\langle \mu' | a \rangle = \frac{1}{\sqrt{2}} [\langle \mu' | \frac{1}{2} \rangle + \langle \mu' | -\frac{1}{2} \rangle] = \frac{1}{\sqrt{2}} = \langle a | \mu \rangle$$

$$\langle \mu' | b \rangle = \frac{1}{\sqrt{2}} [\langle \mu' | \frac{1}{2} \rangle - \langle \mu' | -\frac{1}{2} \rangle] = \frac{1}{\sqrt{2}} (-1)^{\mu'-\frac{1}{2}} \leftarrow \text{a little tricky.}$$

$$\langle b | \mu \rangle = \frac{1}{\sqrt{2}} (-1)^{\mu-\frac{1}{2}} = \frac{1}{\sqrt{2}} (-1)^{\frac{1}{2}-\mu}$$

Thus, we find:

$$T_{\mu'\mu}^J = \frac{1}{2} [T_{J-} + (-1)^{\mu'-\mu} T_{J+}]$$

Next, stuff this into the partial wave expansion forthwith:

$$f_{\mu\mu'} = \frac{1}{2q} e^{i(\mu-\mu')\phi} \sum_{\mathcal{J}} (2\mathcal{J}+1) \left[T_{\mathcal{J}-} \times \frac{1}{2} d_{\mu\mu'}^{\mathcal{J}}(\theta) + T_{\mathcal{J}+} (-1)^{\mu-\mu'} d_{\mu\mu'}^{\mathcal{J}}(\theta) \right]$$

$$(\phi=0) \quad f_{\pm\pm} = \frac{1}{2q} \sum_{\mathcal{J}} (2\mathcal{J}+1) \cdot \frac{1}{2} d_{\pm\pm}^{\mathcal{J}}(\theta) [T_{\mathcal{J}-} \pm T_{\mathcal{J}+}]$$

Changing to the $f^{\mathcal{J}}$ amplitude just for fun this becomes:

$$f_{\pm\pm} = \sum_{\mathcal{J}} (2\mathcal{J}+1) \frac{1}{2} [f_{\mathcal{J}-} \pm f_{\mathcal{J}+}] d_{\pm\pm}^{\mathcal{J}}(\theta)$$

Now we look up some nasty properties (representations) of the d matrices, namely:

$$\begin{aligned} d_{++}^{\mathcal{J}}(\theta) &= \frac{1}{(\mathcal{J}+\frac{1}{2})} \cos \frac{1}{2}\theta \left[P'_{\mathcal{J}+\frac{1}{2}}(\cos\theta) - P'_{\mathcal{J}-\frac{1}{2}}(\cos\theta) \right] \\ d_{+-}^{\mathcal{J}}(\theta) &= \frac{1}{(\mathcal{J}+\frac{1}{2})} \sin \frac{1}{2}\theta \left[P'_{\mathcal{J}+\frac{1}{2}}(\cos\theta) + P'_{\mathcal{J}-\frac{1}{2}}(\cos\theta) \right] \end{aligned}$$

With these, we may show the mysterious formulae:

$$\begin{aligned} f_{++} &= \sum_{\mathcal{J}} [f_{\mathcal{J}-} + f_{\mathcal{J}+}] \cos \frac{\theta}{2} [P'_{\mathcal{J}+\frac{1}{2}} - P'_{\mathcal{J}-\frac{1}{2}}] &= (f_1 + f_2) \cos \frac{\theta}{2} \\ f_{+-} &= \sum_{\mathcal{J}} [f_{\mathcal{J}-} - f_{\mathcal{J}+}] \sin \frac{\theta}{2} [P'_{\mathcal{J}+\frac{1}{2}} + P'_{\mathcal{J}-\frac{1}{2}}] e^{-i\phi} &= (f_1 - f_2) \sin \frac{\theta}{2} e^{-i\phi} \end{aligned}$$

Those primes are derivatives with respect to z of usual Legendre functions. Note once again that the full helicity flip amplitude vanishes in the forward direction. (it does) The other two amplitudes can be obtained from parity in the usual way.

We could if we wanted rewrite the above as L sums, but that does not seem too useful. Instead, lets convert to the f_1 and f_2 amplitudes:

$$\begin{aligned} f_1 &= \sum_{\mathcal{J}} [f_{\mathcal{J}-} P'_{\mathcal{J}+\frac{1}{2}} - f_{\mathcal{J}+} P'_{\mathcal{J}-\frac{1}{2}}] \\ f_2 &= \sum_{\mathcal{J}} [f_{\mathcal{J}+} P'_{\mathcal{J}+\frac{1}{2}} - f_{\mathcal{J}-} P'_{\mathcal{J}-\frac{1}{2}}] \end{aligned}$$

These are basically Singh's equations (2.5). Since we are using parity invariance, there is no need for 3 labels on each amplitude as used by Singh.

There is another common form for these expansions. Using the MS notation we can express the "f" and "g" full amplitudes as:

$$\begin{aligned} f &= f_1 + f_2 \cos\theta = \sum_{\ell=0} f_{\ell+} [P'_{\ell+1} - z P'_{\ell}] - \sum_{\ell=1} f_{\ell-} [P'_{\ell-1} - z P'_{\ell}] \\ g &= -f_2 \sin\theta = + \sum_{\ell=0} [f_{\ell+} - f_{\ell-}] P'_{\ell} \sin\theta \end{aligned}$$

Here we have used the rather obvious definitions:

$$f_{J\pm} = f_{J^{\pm} = J \pm \frac{1}{2}}^{\pm} = f_{J^{\pm} = l \pm \frac{1}{2}}^{\pm} = f_{l\mp}$$

Some Legendre identities are helpful (see Jackson p 59 eg):

$$\begin{aligned} P_{l+1}' - P_{l-1}' &= (2l+1) P_l \\ P_{l+1}' - z P_l' &= (l+1) P_l \end{aligned} \Rightarrow P_{l-1}' - z P_l' = P_{l+1}' - z P_l' - (2l+1) P_l = -l P_l$$

since $P_l' = -P_l'$ (see GR p1005 8.733.)

Thus, we may restate the expansions as:

$$\begin{aligned} f &= \sum_{l=0}^{\infty} [(l+1) f_{l+} + l f_{l-}] P_l(\cos\theta) \\ g^{ms} &= \sum_{l=1}^{\infty} [f_{l-} - f_{l+}] P_l'(\cos\theta) \Rightarrow g_r^{GW} = \frac{g^{ms}}{\sin\theta} = \sum_{l=1}^{\infty} [f_{l+} - f_{l-}] P_l'(\cos\theta) \end{aligned}$$

Sometimes this thing is written as a two-by-two matrix in spin space as:

$$M = f(I) + i g \hat{n} \cdot (\vec{\sigma})$$

When expressed as a two-by-two matrix, the result is independent of the way in which f and g were defined (and on these definitions the authors vary a lot). So we write:

$$\begin{aligned} f &= \sum_{l=0}^{\infty} [(l+1) f_{l+} + l f_{l-}] P_l \cdot 1 + i \hat{n} \cdot \vec{\sigma} \sum_{l=1}^{\infty} [f_{l-} - f_{l+}] P_l' \\ f &= \sum_{l=0}^{\infty} [(l+1) f_{l+} + l f_{l-}] P_l \cdot 1 + i \vec{\sigma} \cdot \hat{R} \times \hat{R} \sum_{l=1}^{\infty} [f_{l+} - f_{l-}] P_l' \end{aligned}$$

$\hat{n} \sin\theta = \hat{R} \times \hat{R}_+$

This last line agrees exactly with GW (p 363 #76). It is the same as Gas. (23.28) except Gas has the wrong sign on the last term. It looks like Williams (3.20) and (3.21) exact. Martin Spearman (5.91) exact.

This last result is much more elegantly arrived at by the method of projection operators. This idea is discussed in Gas. p 369 and in more detail by GW page 361ff. It is a child of the LS formalism. The projection operators are in a mixed representation: the spin part is in the angmomrep, but the orbital part is in the coordinate rep. I have appended a page of notes from GW which shows how it all works. Below is the result. You can see in a way how these operators project out of the "state" on the right the piece with $J=L+\frac{1}{2}$ and $J=L-\frac{1}{2}$.

$$f_{\nu\mu} = \langle \frac{1}{2}\nu' | \sum_l (2l+1) [f_{l-} \vec{L}_- + f_{l+} \vec{L}_+] P_l(\cos\theta) | \frac{1}{2}\nu \rangle$$

$$\vec{L}_- = \frac{l - \vec{\sigma} \cdot \vec{L}}{2l+1} \quad \vec{L}_+ = \frac{l+1 + \vec{\sigma} \cdot \vec{L}}{2l+1} \quad \vec{L} = \vec{R} \times [i \vec{\nabla}]$$

Notes on the spin-0 spin- $\frac{1}{2}$ partial wave expansion "How to get Eq 23.25"

Start by inserting unity everywhere: (four times!)

$$\langle \hat{k}_f, s', v' | S | \hat{k}, s, v \rangle = \sum_{\substack{l, m \\ l', m'}} \sum_{J, M} \langle \hat{k}_f, s', v' | l, m, s, v' \rangle \langle l, m, s, v' | J, M \rangle \langle J, M | S | J, M \rangle \times \\ \times \langle J, M | l, m, s, v \rangle \langle l, m, s, v | \hat{k}, s, v \rangle$$

Recall that $|J, M\rangle$ means $|J, M, 0\rangle$. Now use fact that:

$$\langle J, M | S | J, M \rangle = \langle J, M, 0 | S | J, M, 0 \rangle = \delta_{J,0} \delta_{M,0} \langle l, s' | S^z | l, s \rangle$$

Then it all boils down to the famous equation (26):

$$\langle \hat{k}_f, s', v' | S | \hat{k}, s, v \rangle = \sum_{\substack{l, m \\ l', m'}} \sum_{J, M} \langle \hat{k}_f, s', v' | l, m, s, v' \rangle \langle l, m, s, v' | J, M \rangle \times \\ \times \langle l, s' | S^z | l, s \rangle \langle J, M | l, m, s, v \rangle \langle l, m, s, v | \hat{k}, s, v \rangle \quad (26)$$

So far we are good for any scattering with initial spin state $|s, v\rangle$ and final spin state $\langle s', v'|$.

Now let's specialize to a situation where $(-1)^l = (-1)^{l'}$. In other words, this is always the case in elastic scattering where parity is conserved. Thus l, l' must be the same, or must differ by 2, 4, ... In the partial wave J , they can at most differ by one. [if $l = J + \frac{1}{2}$, $l' = J - \frac{1}{2}$]. Thus, they must be equal. We are now using $s = \frac{1}{2}$, so:

$$\langle l, s' | S^z | l, s \rangle = \delta_{l, l'} \langle l, \frac{1}{2} | S^z | l, \frac{1}{2} \rangle = \delta_{l, l'} S_l^z$$

For a given J , there are two possible S^z 's, $S_{l=J+\frac{1}{2}}^z$, $S_{l=J-\frac{1}{2}}^z$. Or, for a given l , there are two J 's that work: $J = l + \frac{1}{2}$ and $J = l - \frac{1}{2}$. In any case, we arrive at (56a):

$$\langle \hat{k}_f, \frac{1}{2}, v' | S | \hat{k}, \frac{1}{2}, v \rangle = \sum_{\substack{l, m \\ l', m'}} \sum_{J, M} \langle \hat{k}_f | l, m \rangle \langle l, m, \frac{1}{2}, v' | J, M \rangle \langle J, M | l, m, \frac{1}{2}, v \rangle \langle l, m | \hat{k} \rangle S_l^z$$

Here, we used $\langle s', v' | s, v \rangle = 1$ etc. In fact, we can get a simpler (26):

$$\langle \hat{k}_f, s', v' | S | \hat{k}, s, v \rangle = \sum_{\substack{l, m \\ l', m'}} \sum_{J, M} Y_l^{m'}(\hat{k}_f) Y_l^{m*}(\hat{k}) \langle l, m, s', v' | J, M \rangle \langle J, M | l, m, s, v \rangle \cdot S_{l, s, s'}^J \quad [26]$$

And then (56a) becomes:

$$\langle \hat{k}_f, \frac{1}{2}, v' | S | \hat{k}, \frac{1}{2}, v \rangle = \sum_{\substack{l, m \\ l', m'}} \sum_{J, M} Y_l^{m'}(\hat{k}_f) Y_l^{m*}(\hat{k}) \langle l, m, \frac{1}{2}, v' | J, M \rangle \langle J, M | l, m, \frac{1}{2}, v \rangle \cdot S_l^z$$

Rewrite this S₀ as follows:

$$\langle \hat{k} \frac{1}{2} \nu' | S | \hat{k} \frac{1}{2} \nu \rangle = \sum_{l, m, m'} Y_l^{m'}(\hat{k}) Y_l^m(\hat{k})^* \langle l \frac{1}{2} m' \nu' | \sum_M | J M \rangle \langle J M | l \frac{1}{2} m \nu \rangle S_l^J$$

and recognize the projection operators:

$$\sum_M | J M \rangle \langle J M | = \Lambda_J$$

$$\text{Thus obtain:} = \sum_{l, m, m'} Y_l^{m'}(\hat{k}) Y_l^m(\hat{k})^* \sum_J \langle l \frac{1}{2} m' \nu' | \Lambda_J | l \frac{1}{2} m \nu \rangle S_l^J$$

Next, we strip aside the spin states and go into a coord rep so that:

$$\langle l m' | \Lambda_J | l m \rangle = \sum_M \langle l m' | l J M \rangle \langle l J M | l m \rangle = \sum_M \sum_{l'} \langle l m' | l J M \rangle \langle l J M | l m \rangle.$$

Here we are showing that adding $\sum_{l'}$ adds no new terms; then:

$$\begin{aligned} \langle l m' | \Lambda_J | l m \rangle &= \langle l m' | \hat{q} \rangle \langle \hat{q} | \Lambda_J | \hat{q}' \rangle \langle \hat{q}' | l m \rangle \\ &= \int d\Omega_{\hat{q}} \langle l m' | \hat{q} \rangle \vec{\Lambda}_J \langle \hat{q} | l m \rangle \end{aligned}$$

Thus, we have:

$$\langle \dots | S | \dots \rangle = \langle \frac{1}{2} \nu' | \sum_{l, m, m'} Y_l^{m'}(\hat{k}) Y_l^m(\hat{k})^* \int d\Omega_{\hat{q}} Y_l^{m'}(\hat{q}) \sum_J [\vec{\Lambda}_J Y_l^m(\hat{q})] \cdot S_l^J | \frac{1}{2} \nu \rangle$$

Now l and m' occur only in the $Y_l^{m'}$, so use $\sum_{m'} Y_l^{m'} Y_l^{m'} = \delta_{l, l'}$ to get:

$$\langle \dots | S | \dots \rangle = \langle \frac{1}{2} \nu' | \sum_{l, m} Y_l^m(\hat{k}) \sum_J [\vec{\Lambda}_J Y_l^m(\hat{k})] S_l^J | \frac{1}{2} \nu \rangle$$

where we used $\int d\Omega_{\hat{q}} \delta_{\hat{k}\hat{q}} f(\hat{q}) = f(\hat{k})$. Now move Y_l^m to the right:

$$= \langle \frac{1}{2} \nu' | \sum_l \sum_J [\vec{\Lambda}_J^{\text{ref}} Y_l^m(\hat{k}) Y_l^m(\hat{k})] S_l^J | \frac{1}{2} \nu \rangle$$

and use sum rule to get P_l :

$$= \sum_l \frac{(2l+1)}{4\pi} \langle \frac{1}{2} \nu' | \sum_J \vec{\Lambda}_J^{\text{ref}} P_l(\hat{k} \cdot \hat{R}) S_l^J | \frac{1}{2} \nu \rangle$$

But there are only two possible J 's for each l , so:

$$= \sum_l \left(\frac{(2l+1)}{4\pi} \right) \langle \frac{1}{2} \nu' | \left(S_l^{l+\frac{1}{2}} \vec{\Lambda}_{l+\frac{1}{2}}^{\text{ref}} + S_l^{l-\frac{1}{2}} \vec{\Lambda}_{l-\frac{1}{2}}^{\text{ref}} \right) P_l(\hat{k} \cdot \hat{R}) | \frac{1}{2} \nu \rangle$$

At last we arrive at Gasparovic "crucial" (23.25) p 369.

Pion Nucleon DCS expressions.

We can start with an expression we know, which is:

$$\frac{d\sigma}{d\Omega} = \frac{\pi^2}{4s} \{ |T_{++}^{(s)}|^2 + |T_{+-}^{(s)}|^2 \}$$

We also know that,

(MS₉₃₆₁)

$$T_{++}^{(s)} = \cos \frac{1}{2} \theta (2mA + B(s - m^2 - \mu^2)) / 4\pi^2$$

$$T_{+-}^{(s)} = \sin \frac{1}{2} \theta (A[s + m^2 - \mu^2] + mB[s - m^2 + \mu^2]) / 4\pi^2 \sqrt{s}$$

In PI-N kinematics, the angles are given by:

$$z = 1 + \frac{t}{2q^2} \quad \sin^2 \frac{1}{2} \theta = \frac{1}{2}(1 - z) = -t/4q^2 \quad \cos^2 \frac{1}{2} \theta = \frac{1}{2}(1 + z) = (1 + t/4q^2)$$

Substituting and grouping terms we first get:

$$\frac{d\sigma}{d\Omega} = \left(\frac{m}{4\pi W}\right)^2 \left\{ |A|^2 \left[\left(1 + \frac{t}{4q^2}\right) + \frac{(m+\omega)^2}{s} \left(-\frac{t}{4q^2}\right) \right] + |B|^2 \left[\omega^2 \left(1 + \frac{t}{4q^2}\right) + \frac{[]^2}{4s} \left(-\frac{t}{4q^2}\right) \right] + (AB^* + c.c.) \left[\left(1 + \frac{t}{4q^2}\right) \omega + \frac{(-t)}{4q^2} \frac{(m+\omega)}{2s} [] \right] \right\}$$

Here, ω is the lab pion energy and we have used:

$$\omega = \frac{s - m^2 - \mu^2}{2m} \quad \omega + m = \frac{s + m^2 - \mu^2}{2m} \quad [] = s - m^2 + \mu^2 \quad W = \sqrt{s}$$

Now by some near miracle, the coefficient of $|A|^2$ reduces to $(1 - t/4m^2)$. This is true at any energy, not just in the H.E. limit. At this point, it seems more profitable to try to work backwards from Singh's result which is:

$$\begin{aligned} \frac{d\sigma}{d\Omega} &= \left(\frac{m}{4\pi W}\right)^2 \left[\left(1 - \frac{t}{4m^2}\right) \left\{ |A|^2 + \frac{\omega + t/4m}{1 - t/4m^2} (AB^* + B^*A) + \left(\frac{\omega + t/4m}{(1 - t/4m^2)}\right)^2 |B|^2 \right\} + \frac{t}{4m^2} \left(s - \frac{(m+\omega)^2}{1 - t/4m^2} \right) |B|^2 \right] \\ &= \left(\frac{m}{4\pi W}\right)^2 \left\{ |A|^2 \left[\left(1 - \frac{t}{4m^2}\right) \right] + |B|^2 \left[\frac{(\omega + t/4m)^2}{(1 - t/4m^2)} + \frac{t}{4m^2} \left(s - \frac{(m+\omega)^2}{1 - t/4m^2} \right) \right] + (AB^* + B^*A) \left[\omega + t/4m \right] \right\} \end{aligned}$$

It is only after some phenomenal algebra that we show that the first and third coefficients match, ie, that:

$$\begin{aligned} \left[\left(1 + \frac{t}{4q^2}\right) + \frac{(m+\omega)^2}{s} \left(-\frac{t}{4q^2}\right) \right] &= \left(1 - \frac{t}{4m^2}\right) \\ \left[\left(1 + \frac{t}{4q^2}\right) \omega + \left(-\frac{t}{4q^2}\right) \frac{(m+\omega)}{2s} [s - m^2 + \mu^2] \right] &= (\omega + t/4m) \end{aligned}$$

Finally, after a whole page of algebra, we can show that the coefficients of $|B|^2$ also match. Thus, we have verified the formula of Singh, which I now rewrite using his A' amplitude:

$$\frac{d\sigma}{d\Omega} = \left(\frac{m}{4\pi \sqrt{s}}\right)^2 \left[\left(1 - \frac{t}{4m^2}\right) |A'|^2 + \frac{t}{4m^2} \left(s - \frac{(m+\omega)^2}{1 - t/4m^2} \right) |B|^2 \right]$$

$$A' = A + \left(\frac{\omega + t/4m}{1 - t/4m^2} \right) B$$

Conclusion: these are indeed the exact A and B of CGLN. There must be an easier

Pion Nucleon. The t-channel amplitudes in terms of A and B. Parity.

First, let us state all we know about the parity relations. Assume $\phi = 0$ so we can use MS (5.37). Then,

$$\eta = \eta_a \eta_b \eta_c \eta_d (-1)^{j_c + s_d - s_a - s_b} = +1, \text{ s channel.}$$

$$f_{\mu_c \mu_d \mu_a \mu_b}^{(\phi)} = \eta (-1)^{\mu' - \mu} f_{-\mu_c - \mu_d - \mu_a - \mu_b}^{(\phi)} \quad \mu' = \mu_c - \mu_d \quad \mu = \mu_a - \mu_b$$

S channel: $N\pi \rightarrow N\pi$
 $f_{\mu_c 0 \mu_a 0} = (-1)^{\frac{1}{2} + \frac{1}{2} + (0-0)} f_{-\mu_c 0 -\mu_a 0} (-1)^{\mu_c - \mu_a}$
 $\Rightarrow f_{++}^{(s)} = f_{--}^{(s)} \quad \text{and} \quad f_{+-}^{(s)} = -f_{-+}^{(s)} \quad \checkmark$

t channel: $\pi\pi \rightarrow N\bar{N}$
 $f_{\mu_c \mu_d 0 0} = (-1)^{\frac{1}{2} + \frac{1}{2} - 0 - 0} (-1)^{\mu_c - \mu_d} f_{-\mu_c - \mu_d 0 0} (-)(-)(-)(+)$
 $\Rightarrow f_{++}^{(t)} = + f_{--}^{(t)} \quad \text{and} \quad f_{+-}^{(t)} = -f_{-+}^{(t)}$

It turns out that the parity relations are the same in either channel. There are several tricky minus signs here, one of which is due to the fact that the intrinsic parity of an anti-proton is minus.

The helicity crossing relations are (MS p 342):

$$T_{++}^{(s)} = \sin \chi T_{++}^{(t)} - \cos \chi T_{+-}^{(t)}$$

$$T_{+-}^{(s)} = \cos \chi T_{++}^{(t)} + \sin \chi T_{+-}^{(t)}$$

$$\cos \chi = \frac{-t(s+m^2-\mu^2)}{3d}$$

$$\sin \chi = \frac{2m\sqrt{\phi}}{3d}$$

($\sin \chi = 1$ at $t=0$)

$$T_{++}^{(t)} = \sin \chi T_{++}^{(s)} + \cos \chi T_{+-}^{(s)} \quad \checkmark$$

$$T_{+-}^{(t)} = \cos \chi T_{++}^{(s)} - \sin \chi T_{+-}^{(s)}$$

I was planning to actually execute this helicity crossing operation, but now it looks too painful. For example, I have to unload those s-channel scattering angle sin and cos factors present in the s-channel helicity amplitudes. At the same time, I have to use messy expressions for χ , the crossing angle. It seems like a ton of algebra, I'll bet there is a more direct approach.

Well, except for an overall phase, I think it will work. First, let's write down all we know:

$$d^2 = [s - (m - \mu)^2][s - (m + \mu)^2] = s^2 - 2s(m^2 + \mu^2) + (m^2 - \mu^2)^2 = \text{positive in s and t ch.}$$

$$3^2 = 3_n^2 = t(t - 4m^2) = \text{positive in s and t ch.}$$

$$3_\pi^2 = t(t - 4\mu^2)$$

$$2\sqrt{t} p_t = 3_n$$

$$2\sqrt{t} q_t = 3_\pi \quad E_t = \omega_t = \frac{\sqrt{t}}{2}$$

p_t = nucleon momentum in t

q_t = pion momentum in t ch.

We know still more:

$$\phi = t[su - (m^2 - \mu^2)^2] = 4s q_s^4 \sin^2 \theta_s = 4t p_t^2 q_t^2 \sin^2 \theta_t \quad (p175)$$

$$= t[-st - d^2] \checkmark \text{ it turns out. } = \text{positive in all channels.}$$

Those s-channel half angles expressions are (from p 366):

$$\cos \frac{1}{2} \theta_s = \sqrt{-\phi/t}/d \quad \sin \frac{1}{2} \theta_s = \sqrt{-st}/d$$

Finally, the s-channel helicity amplitudes are, as functions of A and B:

$$4\pi^2 T_{++}^{(s)} = \cos \frac{1}{2} \theta_s [2mA + (s - m^2 - \mu^2) B]$$

$$4\pi^2 T_{+-}^{(s)} = \sin \frac{1}{2} \theta_s [(s + m^2 - \mu^2) A + m(s - m^2 + \mu^2) B] / \sqrt{s}$$

Putting all these things together into the crossing equations, we get:

$$d^2 \int \sqrt{-t} \, 4\pi^2 T_{++}^{(t)} = A \{ 4m^2 \phi + t^2 (s + m^2 - \mu^2)^2 \}$$

$$+ B \{ 2m\phi (s - m^2 + \mu^2) + mt^2 (s + m^2 - \mu^2)(s - m^2 + \mu^2) \}$$

Using the following identity which brings in $\cos \theta_t$,

$$\left(\frac{t}{2} + s - m^2 - \mu^2 \right) = 2p_t q_t \cos \theta_t = \frac{1}{2}(s - u)$$

one is able to show after mucho algebra that:

$$d^2 \int \sqrt{-t} \, 4\pi^2 T_{++}^{(t)} = A \{ d^2 \int 2\sqrt{t} p_t \} + B \{ d^2 \int \sqrt{t} (-2m q_t \cos \theta_t) \}$$

$$\Rightarrow (\pm i) 4\pi^2 T_{++}^{(t)} = 2p_t A - 2m q_t \cos \theta_t B$$

If we set that phase factor on the left equal to +1, then this result is exactly that of MS (7.96). The fact that this did work out, up to a phase, tends to support the feeling that MS have a sign error in their (7.91) equations for the s-channel amplitudes.

The second t-channel helicity amplitude is given by:

$$\frac{d^2 \int 4\pi^2 T_{+-}^{(t)}}{\sqrt{\phi} \sqrt{-t}} = A \{ 0 \} + B \{ d^2 \} = d^2 B$$

$$\Rightarrow (\pm i) 4\pi^2 T_{+-}^{(t)} = -B q_t \sqrt{t} \sin \theta_t$$

To summarize, here is what we found (to be compared with (7.96):

$$\left[\begin{array}{l} -e^{+i\frac{\pi}{2}} 4\pi^2 T_{++}^{(t)} = -2p_t A + 2m q_t B \cos \theta_t \\ -e^{+i\frac{\pi}{2}} 4\pi^2 T_{+-}^{(t)} = q_t \sqrt{t} B \sin \theta_t \end{array} \right. \sim +i\pi \text{ shift}$$

The main point is that the relative phase came out right. Then rephase:

Derivation of Singh's DCS equation from t-channel amplitudes.

We have already done this using the s-channel amplitudes, and after much pain we got the right answer. It should be much cleaner here. As noted on p 341 MS, the helicity crossing matrix is orthogonal so a DCS can be written in either s or t channel amplitudes. Of course it is always the s-channel DCS. So, we start as before:

$$\frac{d\bar{\sigma}}{d\Omega} = \frac{\pi^2}{4s} \{ |T_{++}^{(t)}|^2 + |T_{+-}^{(t)}|^2 \}$$

$$\begin{aligned} -4\pi^2 T_{++}^{(t)} &= -2ptA + 2mq_t B \cos\theta_t = -2ptA' \quad \text{where } A' \equiv \left(A - m \frac{q_t}{pt} B \cos\theta_t \right) \\ -4\pi^2 T_{+-}^{(t)} &= q_t \sqrt{t} B \sin\theta_t \end{aligned}$$

At this point we can rewrite A' several ways using the identity we derived earlier:

$$2ptq_t \cos\theta_t = \frac{1}{2}(s-u) \Rightarrow A' = A + \frac{m(s-u)B}{[4m^2 - t]}$$

Adding u in there doesn't really help much. In terms of the pion lab momentum,

$$\omega = \frac{s - m^2 - \mu^2}{2m} \quad s - u = t + 4m\omega \quad A' = A + \frac{(\omega + t/4m)B}{(1 - t/4m^2)}$$

Regardless of how you write A', the DCS is:

$$\frac{d\bar{\sigma}}{d\Omega} = \frac{\pi^2}{4s} \cdot \left(\frac{1}{4\pi^2} \right)^2 \cdot \{ |-2ptA'|^2 + |q_t \sqrt{t} B \sin\theta_t|^2 \}$$

From this, Singh's equation follows at once. One must be careful to use $/p_t^2/$ for example because in the s-channel p_t^2 is negative, as is t. The result is:

$$\frac{d\bar{\sigma}}{d\Omega} = \left(\frac{m}{4\pi\sqrt{s}} \right)^2 \left[\left(1 - t/4m^2 \right) |A'|^2 + \frac{t}{4m^2} \left(s - \frac{(m+\omega)^2}{1 - t/4m^2} \right) |B|^2 \right]$$

$$\frac{d\bar{\sigma}}{dt} = \frac{\pi}{q_s^2} \frac{d\bar{\sigma}}{d\Omega} = \frac{1}{\pi s} \left(\frac{m}{4q_s} \right)^2 \left[\text{same} \right]$$

Using the trivially derived relation $k_L^2 = (m+\omega)^2 - s$, we can finally cast this thing into the exact MS form on page 456:

$$\frac{d\bar{\sigma}}{dt} = \frac{1}{\pi s} \left(\frac{m}{4q_s} \right)^2 \left[\left(1 - t/4m^2 \right) |A'|^2 + \frac{t}{4m^2} \left(\frac{k_L^2 + st/4m^2}{1 - t/4m^2} \right) |B|^2 \right] \quad (9.92)$$

Notice that these are still the original CGLN A and B amplitudes. I have not dealt with any other A and B functions in all these sheets. Remember that it is these A and B functions that are free of all kinematic singularities and constraints. They are where you should inject Regge poles, I suspect.

PI-N: optical theorem and polarization in terms of A and B.

We have already derived an expression for the DCS in terms of A' and B, the Singh equation. A variation of this equation (the same variation that appears in MS) appears in the Fit of Rarita, Riddell et al 1967.

The optical theorem refers to the optical theorem averaged on initial spins. In general one would say:

$$\begin{aligned} \sigma_{tot}^{(ab)}(s) &= \frac{2\pi^2}{(\text{flux})} \text{Im} [T_{ab;ab}] \quad \leftarrow \text{MS} \\ \sigma_{tot}(s) &= \frac{2\pi^2}{\text{flux}} \text{Im} \left[\frac{1}{(2S_a+1)(2S_b+1)} \sum_{ab} T_{ab;ab} \right] \\ (\text{flux}) &= \sqrt{s} q^* = m_2 p_1^{LAB} \end{aligned}$$

In the case of pion - nucleon we have merely,

$$\bar{\sigma}(s) = \frac{2\pi^2}{\text{flux}} \text{Im} \left[\frac{1}{2} (T_{++} + T_{--}) \right] = \frac{2\pi^2}{\text{flux}} \text{Im} [T_{++}(s, t=0)]$$

From the "notes" we know that:

$$T_{++}(s, t=0) = \frac{1}{4\pi^2} [2mA + (s-m^2-\mu^2)B] = \frac{2m}{4\pi^2} [A + \omega B] = \frac{2mA'(t=0)}{4\pi^2}$$

which follows from the definition of A'. Thus,

$$\bar{\sigma}(s) = \frac{2\pi^2}{m p_1^L} \cdot \frac{2m}{4\pi^2} \text{Im} A' \Rightarrow \left[\bar{\sigma}_{tot}(s) = (p_1^{LAB})^{-1} \text{Im} [A'(s, t=0)] \right]$$

Next, the polarization. Start from helicity amplitudes:

$$\begin{aligned} \frac{d\sigma}{d\Omega} P_y &= 2 \text{Im} [f_{++} f_{+-}^*] \\ &= \frac{2}{\sqrt{s}} \left(\frac{1}{8\pi\sqrt{s}} \right)^2 \sin^2\theta \cos^2\theta \text{Im} \left[(2mA + B(s-m^2-\mu^2)) (A^*(s+m^2-\mu^2) + mB^*(s-m^2+\mu^2)) \right] \\ &= \frac{\sin\theta}{64\pi^2 s \sqrt{s}} \text{Im} [2m^2 AB^*(s-m^2+\mu^2) + (s-m^2-\mu^2)(s+m^2-\mu^2) BA^*] \\ &= \frac{\sin\theta}{64\pi^2 s \sqrt{s}} \cdot [-s^2] \text{Im} [AB^*] = \frac{-\sin\theta \cdot 4s p^{*2}}{64\pi^2 s \sqrt{s}} \text{Im} [A'B^*] \end{aligned}$$

$$= \left(\frac{-\sin\theta p^{*2}}{16\pi^2 \sqrt{s}} \right) \text{Im} [A'B^*]$$

$$\frac{d\sigma}{dt} = \frac{\pi}{pp'} \frac{d\sigma}{d\Omega}$$

$$\left[\frac{d\sigma}{dt} P_y = \frac{-\sin\theta}{16\pi\sqrt{s}} \text{Im} (A'B^*) \right]$$

TT N misc.

Basic formulae for $\pi-N$

Assume large S and parity conservation. Then:

$$\sigma_{\text{tot}}(s) = \frac{4\pi^2}{s} \operatorname{Im} [T_{+0,+0}(s,0)]$$

$$\frac{d\sigma}{dt} = \frac{\pi^3}{s^2} \left(|T_{+0,+0}|^2 + |T_{+0,-0}|^2 \right)$$

$$P \cdot \frac{d\sigma}{dt} = \frac{2\pi^3}{s^2} \operatorname{Im} [T_{+0,+0} T_{+0,-0}^*]$$

$$P = \frac{2 \operatorname{Im} [T_{+0,+0} T_{+0,-0}^*]}{\{|T_{+0,+0}|^2 + |T_{+0,-0}|^2\}}$$

NB: parity says

$$T_{+0,+0} = T_{-0,-0}$$

$$T_{+0,-0} = -T_{-0,+0}$$

$$\eta = \eta_1 = 1$$

Problem: by looking only at Kallen, how do we connect the number .088 to the "residue of the pole" in $\pi\pi$ scattering?

The quantity I am looking for is this:

$$\underset{\text{ms}}{\text{ms}} \rightarrow T_{\pi\pi \rightarrow \pi\pi}(s, t) = \frac{\beta(\pi\pi)}{s - m_\pi^2} + \text{analytic.} \quad \beta = ??$$

* pole of $T(\pi\pi)$ with residue $\beta(\pi\pi)$.

Kallen deals with an amplitude for $\pi\pi$ called $T_+(\omega)$. How is this amp. normalized?

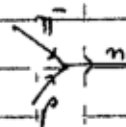
$$\left. \begin{aligned} \text{Def } T_+(\omega) &= 2M \rho_{\pi\pi} \sigma_{\pi\pi}^+(\omega) \\ \text{Def } T_{\pi\pi}(s, t) &= \frac{1}{2\pi^2} M \rho_{\pi\pi} \sigma_{\pi\pi}^+(s) \end{aligned} \right\} \Rightarrow T_+ = +4\pi^2 T_{\pi\pi}$$

So this T_+ is $-M$. Now on p. 137 Kallen says:

$$T_+(\omega) = - \frac{A}{\omega - (m^2/2M)} + \text{cut}$$

But we are now talking "nucleon pole" and $\pi\pi$ does not have this pole. Look instead at:

$$T_-(\omega) = \frac{+A}{\omega + (m^2/2M)} = \frac{2MA}{(s - M^2)}$$



And on p. 141 we learn:

$$A = 4M \left(1 - \frac{m^2}{4M^2}\right) f^2 \quad \text{where} \quad f^2/4\pi = .088$$

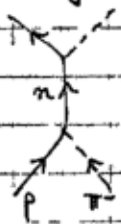
So, converting to the MS amplitude, we have:

$$\underset{\text{ms}}{\text{ms}} T_{\pi\pi \rightarrow \pi\pi}(s, t) = \frac{1}{4\pi^2} T_-(\omega) = \frac{\{2MA/4\pi^2\}}{s - M^2} = \frac{\{8M^2(1 - \frac{m^2}{4M^2}) \frac{f^2}{4\pi}\}}{s - M^2}$$

Thus, the value of the residue of the MS amplitude is:

$$T(\text{nucleon pole}) = \frac{[8(1 - \frac{m^2}{4M^2}) \frac{f^2}{4\pi}]}{(s/s_0) - 1} \quad \left. \begin{aligned} s_0 &= M^2 \\ \text{num} &= 8 \frac{f^2}{4\pi} = .70 \end{aligned} \right\}$$

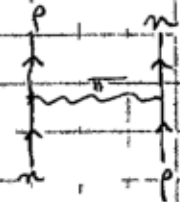
From factorization at this pole we have:



$$T(s,t) = \frac{(.70M^2)}{s-M^2} = \frac{[\beta(p\pi \rightarrow n)]^2 M^2}{s-M^2}$$

$$\Rightarrow \beta(p\pi \rightarrow n) = .84$$

I wonder if there is some connection between the residue of the neutron pole in πp , and the residue of the pion pole in $n\bar{n}$ scattering.



$$T(s,t) \cong (\beta_{p\pi n}(t))^2 f_{\pi}(t) \left(\frac{s}{s_0}\right)^{\alpha_{\pi}(t)}$$

$$f_{\pi}(t) = i - \cot\left(\frac{\pi \alpha_{\pi}}{2}\right) \approx i - \frac{(2/\pi)}{t - m_{\pi}^2}$$

The residue of the pion pole is therefore:

$$T(s,t) = \frac{[res]}{t - m_{\pi}^2}$$

$$[res] = \left(-\frac{2}{\pi}\right) (\beta_{p\pi n}(0))^2 \left(\frac{s}{s_0}\right)^{-m_{\pi}^2}$$

From my data analysis, got

$$\beta_{p\pi n}^2(0) = .14 \Rightarrow [res] = \left(-\frac{2}{\pi}\right) (.14) (\approx 1) = -.089$$

This thing is, however, slightly s dependent.

So, the pion pole residue $\neq$ neutron pole residue.

Basic formulae for $\pi-N$

Assume large S and parity conservation. Then:

$$\sigma_{\text{tot}}(s) = \frac{4\pi^2}{s} \text{Im} [T_{+0,+0}(s,0)]$$

$$\frac{d\sigma}{dt} = \frac{\pi^3}{s^2} \left(|T_{+0,+0}|^2 + |T_{+0,-0}|^2 \right)$$

$$P \cdot \frac{d\sigma}{dt} = \frac{2\pi^3}{s^2} \text{Im} [T_{+0,+0} T_{+0,-0}^*]$$

$$P = \frac{2 \text{Im} [T_{+0,+0} T_{+0,-0}^*]}{\{|T_{+0,+0}|^2 + |T_{+0,-0}|^2\}}$$

NB: parity says

$$T_{+0,+0} = T_{-0,-0}$$

$$T_{+0,-0} = -T_{-0,+0}$$

$$\eta = \eta_1 = 1$$

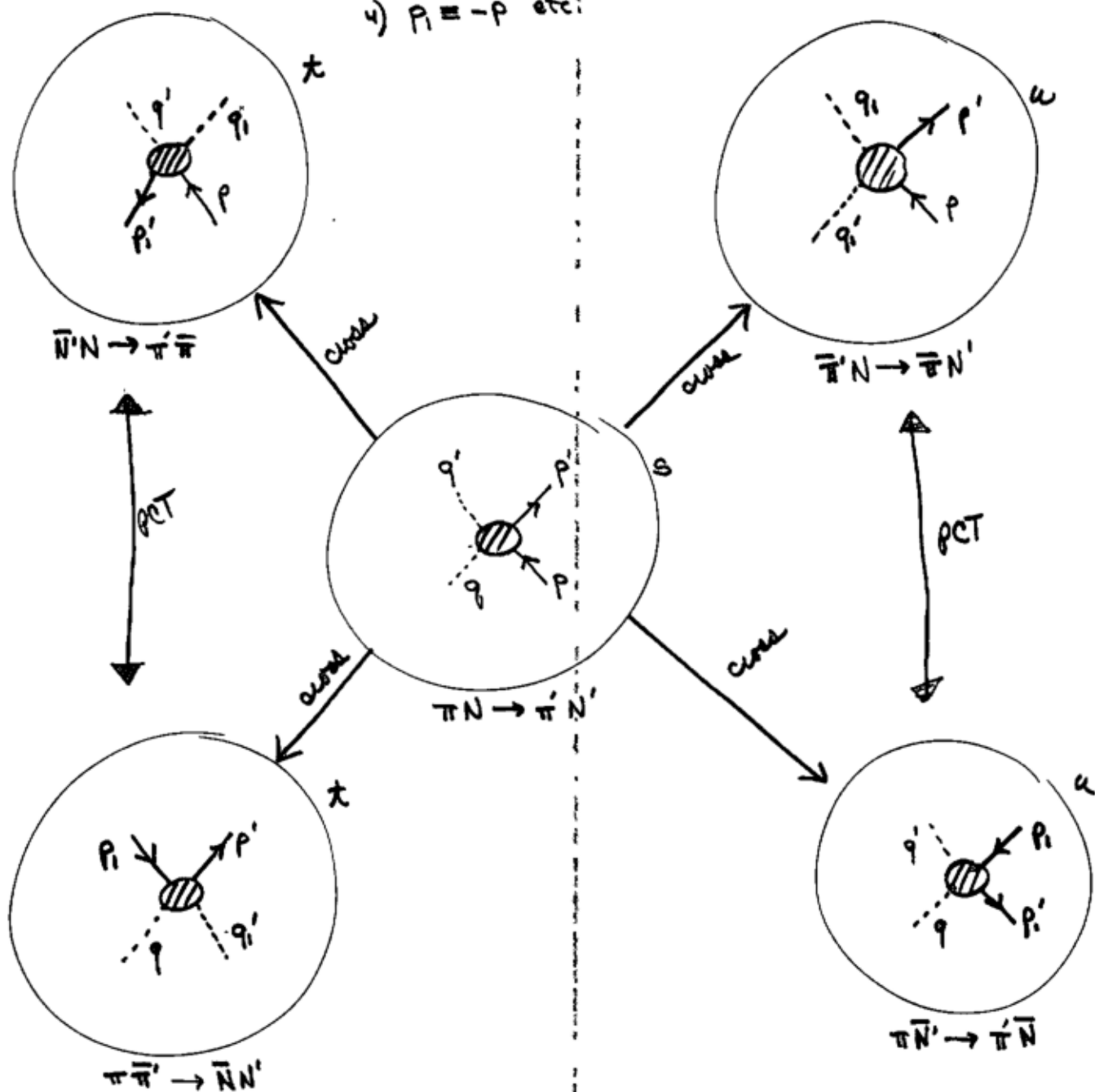
CROSSING:

- Rules:
- 1) s, t, u have a fixed meaning:
 - 2) crossed lines shown in red:
 - 3) all momenta are up and +:
 - 4) $p_1 = -p$ etc:

$$s = (p+q)^2$$

$$t = (q-q')^2$$

$$u = (q-p')^2$$



5) because of PCT, only 3 of these 5 related processes need be considered: i.e., one S, T, and U processes.

Application to πN crossing:

Reactions are: $S \left\{ \begin{matrix} N + \pi \rightarrow N + \pi \\ a \quad b \quad c \quad d \end{matrix} \right.$ $T \left\{ \begin{matrix} \bar{\pi} + \pi \rightarrow N + \bar{N} \end{matrix} \right.$

One and 3 objects are:

$$\begin{aligned} m_a = m_c = M_N = M & & s_{ab}^2 = s_{cd}^2 &= [s - (m + \mu)^2][s - (m - \mu)^2]^2 \\ m_b = m_d = \mu & & t_{ab}^2 = t_{cd}^2 &= t[t - 4M^2] \end{aligned}$$

Objects Δ , ϕ and u are:

$$\Delta = 0 \quad u = 2M^2 + 2\mu^2 - s - t$$

$$\phi = t[su - (M^2 - \mu^2)^2]$$

Only a and c have spin, so:

$$\sin w_a = \frac{2M\sqrt{\phi}}{2s} \quad \cos w_a = \frac{-(s + M^2 - \mu^2)t}{2s}$$

$$\sin w_c = \frac{2M\sqrt{\phi}}{2s} \quad \cos w_c = \frac{+(s + M^2 - \mu^2)t}{2s}$$

Thus we see that $w_c = \pi - w_a$. Let $w_a \equiv \chi$. Then:

$$T_{ca}^{(s)} = \sum_{c'a'} d_{aa'}^{(s)}(\chi) d_{cc'}^{(t)}(\pi - \chi) T_{c'a'}^{(t)}$$

Now write them out explicitly. For example:

$$T_{++}^{(s)} = (++) T_{++}^{(t)} + (-+)(++) T_{+-}^{(t)} + (++)(-+) T_{-+}^{(t)} + (-+)(-+) T_{--}^{(t)}$$

where:

$$(++) = \cos \frac{\chi}{2} \cdot \sin \frac{\chi}{2} = \frac{1}{2} \sin \chi$$

$$(-+)(++) = \sin \frac{\chi}{2} \cdot \sin \frac{\chi}{2} = \frac{1}{2} \sin^2 \frac{\chi}{2}$$

$$(++)(-+) = \cos \frac{\chi}{2} \cdot \cos \frac{\chi}{2} = \frac{1}{2} \cos^2 \frac{\chi}{2}$$

$$(-+)(-+) = \sin \frac{\chi}{2} \cdot \cos \frac{\chi}{2} = \frac{1}{2} \sin \chi$$

Using (5.37) for parity invariance:

$$T_{++} = T_{--} \quad T_{+-} = -T_{-+}$$

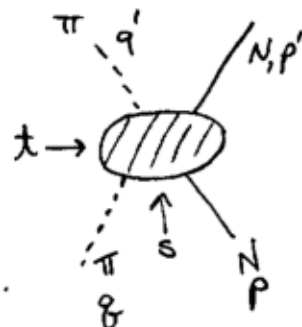
we get:

$$T_{++}^{(s)} = (\sin \chi) T_{++}^{(t)} - (\cos \chi) T_{+-}^{(t)}$$

As claimed, transform is unitary.

PION NUCLEON KINEMATICS

$$\begin{aligned} s &= (q+p)^2 & \pi N \rightarrow \pi' N \\ t &= (q-q')^2 & \pi \bar{\pi} \rightarrow \bar{N} N \\ u &= (q-p')^2 & \pi \bar{N} \rightarrow \pi \bar{N} \end{aligned}$$



$$s+t+u = 2M^2 + 2\mu^2$$

$$\nu \equiv \frac{s-u}{4M} = \frac{1}{2M} q \cdot (p+p')$$

$$\lambda(s, m^2, \mu^2) = 2m \sqrt{\nu^2 - \mu^2}$$

Specialize to forward direction:

$$s+u = 2M^2 + 2\mu^2$$

$$\nu = \frac{q \cdot p}{M} = \omega \leftarrow \text{pion lab energy.}$$

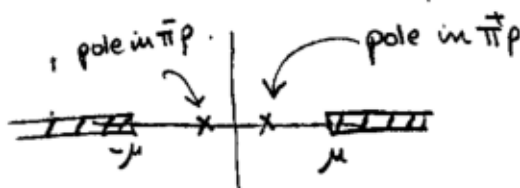
$$\nu = \omega = \frac{s - \mu^2 - M^2}{2M} = \frac{\mu^2 + M^2 - u}{2M}$$

$$\nu_B = \frac{\mu^2}{2M}$$

Position of cuts: $s = (M \pm \mu)^2 \Rightarrow \nu = \pm \mu!$

Position of poles: $\frac{g^2}{s-M^2} = \frac{[g^2/2M]}{\nu + \nu_B}$; $\frac{g^2}{u-M^2} = -\frac{[g^2/2M]}{\nu - \nu_B}$

Thus, picture looks like



Now produce two separate dispersion relations. But first general amplitude is

$$\mathcal{M}(p, q; p', q') = \bar{u}(p') [A(s, t) + \frac{1}{2}(q+q')B(s, t)] u(p)$$

In forward limit, in lab frame proton is at rest before + after, so

$$q=q', \quad \bar{u}(E=\mu) \gamma^\mu u = \begin{cases} 1 & \text{on } \gamma^0 \\ 0 & \text{on } \vec{x} \end{cases} \quad (\bar{u}u = 1 \text{ not.})$$

$$\text{Thus: } \mathcal{M}(M, q; M, q) = \bar{u}(M) [A(s, 0) + \omega B(s, 0)] u(M)$$

Define:

$$4\pi T(\nu) = A(s, 0) + \nu B(s, 0) = \mathcal{M}$$

spin flip
amp $\rightarrow 0$.

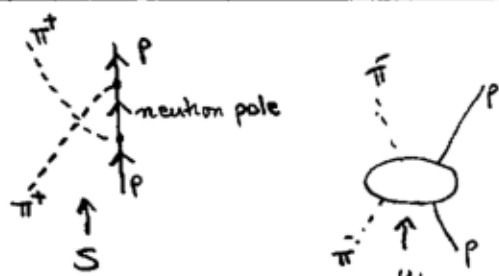
For low energy pions, $s \approx M^2$. Then

$$\left. \frac{d\sigma}{d\Omega} \right|_{\theta=0} = |f|^2 = \frac{|m|^2 (2M)^2}{64\pi^2 s} \underset{B \rightarrow 0 \text{ mm}}{\uparrow} = \frac{4M^2 |m|^2}{64\pi^2 (M+\omega)^2} \approx \frac{|m|^2}{16\pi^2} = |T(\nu)|^2$$

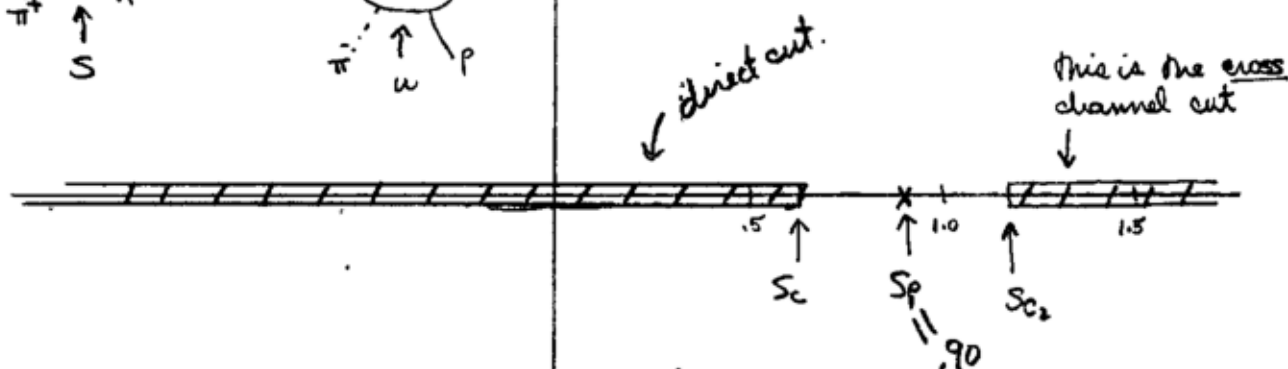
$B \rightarrow 0 \text{ mm.}$

forward scatt
amp.

non-flip.



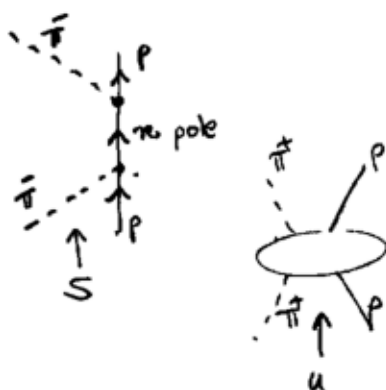
$\pi^+ p$ Analytic Structure of $T(s,0)$



$$s_p = M^2 + 2\mu^2 = .900 \leftarrow (u=M^2)$$

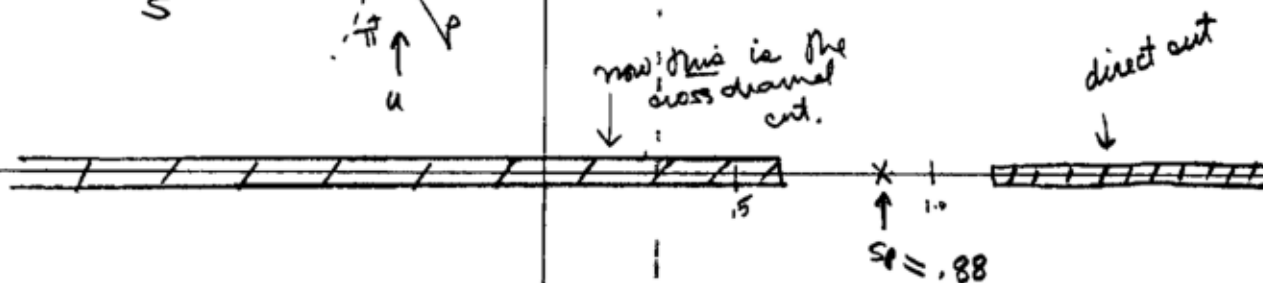
$$s_c = (M-\mu)^2 = .637 \leftarrow (u=(M+\mu)^2)$$

$$s_{c2} = (M+\mu)^2 = 1.16 \text{ GeV}^2$$



$\pi^- p$ Analytic Structure of $T(s,0)$

$$T(s,0) = A(s,0) + \sum B(s,0)$$



$$s_p = M^2 = .880$$

$$s_c = (M+\mu)^2 = 1.16$$

$$s_{c2} = (M-\mu)^2 = .637 \leftarrow u=(m+\mu)^2$$

Cuts are same cuts.

Pole's in slightly diff position

CROSSING STATEMENT

We know that:

$$\left. \begin{aligned} \text{amp}^{(\pi^+ p)}(s, u) &= \text{amp}^{(\pi^- p)}(u, s) \\ \text{amp}^{(\pi^- p)}(s, u) &= \text{amp}^{(\pi^+ p)}(u, s) \end{aligned} \right\} \text{crossing}$$

Thus:

$$\left. \begin{aligned} T^{(\pi^+ p)}(\nu) &= T^{(\pi^- p)}(-\nu) \\ T^{(\pi^- p)}(\nu) &= T^{(\pi^+ p)}(-\nu) \end{aligned} \right\}$$

Define crossing-even and crossing-odd amps:

$$T^{\pm}(\nu) = \frac{1}{2} [T^{(\pi^+ p)}(\nu) \pm T^{(\pi^- p)}(\nu)] = \frac{1}{2} [T^{(\pi^+ p)}(\nu) \pm T^{(\pi^+ p)}(-\nu)].$$

Then crossing says:

$$T^{\pm}(\nu) = \pm T^{\pm}(-\nu).$$

Kinematically Clean Amplitudes for $\pi\pi \rightarrow N\bar{N}$. And the Regge CEX DCS.

This is just an exercise in using the chart on page 364 of MS. The starting point for an annihilation process is:

$$F^{\pm}(\text{clean}) = (\sqrt{t})^n (\sqrt{t-4\mu^2})^{\beta_{\pm}} (\sqrt{t-4m^2})^{\delta_{\pm}} \hat{T}^{\pm}$$

Here, $\hat{T}$ is obtained from T by dividing by the sin and cos factors. We would do this division even if those factors are unphysical. I am thinking here of computing a t -channel clean amplitude, but in the s -channel high energy region. In other words, large z_t .

The + and - superscripts are telling us that we are dealing with special "parity conserving" linear combinations of helicity amplitudes. It is necessary to use such lincombs because they have fewer singularities for us to worry about. This arises because of the "kinematic constraints" on lincombs back in chapter 7 of MS.

$$\hat{T}^{\pm} \equiv \hat{T}_{cd,ab} \pm (-1)^{s_a+s_b-\mu} \hat{T}_{cd,-a-b}$$

By the way, at last we see why MS did their spin-regge theory with such weird amplitudes. You want "parity conserving" amplitudes to be chucked out by your regge machine; then you can suck out of these the proper kinematic singularities.

However, in the case here, $a=b=0$, $s_a = s_b = 0$ so we simply have:

$$\hat{T}_{++}^+ = 2\hat{T}_{++;\infty} \quad \hat{T}_{+-}^- = 2\hat{T}_{+-;\infty} \quad \text{others vanish.}$$

Now we have three numbers to compute (n , $\delta_{\pm}$, $\beta_{\pm}$) then we can compute the divide-bys for the hat-T's.

Compute n :

$$\varepsilon \equiv \eta_a \eta_b \eta_c \eta_d (-1)^{a+b+c+d} \quad \text{and} \quad \begin{aligned} \varepsilon = 1 &\Rightarrow n = 0 \\ \varepsilon = -1 &\Rightarrow n = 1 \end{aligned}$$

$$\varepsilon = (-1)(-1)(+1)(-1) = -(-1)^{c+d} = -(-1)^{c+d}$$

$$T_{++} \Rightarrow \varepsilon = -(-1)^{\frac{1}{2}+\frac{1}{2}} = -(-1)^1 = +1 \Rightarrow n = 0$$

$$T_{+-} \Rightarrow \varepsilon = -(-1)^{\frac{1}{2}-\frac{1}{2}} = -1 \Rightarrow n = 1 \Rightarrow \sqrt{t} \text{ factor present.}$$

Compute $\delta_{\pm}$:

$$\delta^+ \rightarrow \delta_- \text{ because } \eta_p = \eta_a \eta_b \eta_c \eta_d = (-1). \checkmark$$

$$\delta_- = [q^{(+)} - 1] \checkmark$$

$$q = s_c + s_d - \max(|\lambda|, |\mu|) = 1 - |\mu|.$$

$$\text{for } T_{++}, \mu = 0, q = 1, q^{(+)} = q + 1 \text{ since } q \text{ odd} = 2 \text{ so } \delta_- = 1$$

$$\delta_- \rightarrow \delta_+ = q^{(-)}, \# \quad q = 1 - \max(|\lambda|, |\mu|) = 1 - 1 = 0, \text{ so } q^{(-)} = 0 \text{ and } \delta_+ = 0 \text{ for } T_{+-}.$$

Conclusion: $T_{++} \sim (\sqrt{t-4m^2})'$ $T_{+-} \sim 1$

Compute $\beta_{\pm}$:

$$\beta_+ = p^{(-)}; \quad p = -1/4. \text{ For } T_{++}, |\mu|=0 \text{ so } p=0 \text{ so } \beta_+ = 0 \quad \checkmark$$

$$\beta_- = p^{(-)} - 1; \quad \text{For } T_{+-}, |\mu|=1 \text{ and } p=-1 \text{ so } p^{(-)} = p+1=0 \text{ so } \beta_- = (-1) \quad \checkmark$$

So far then we seem to have found that:

$$F_{++}^{(+)} = (\sqrt{t})^0 (\sqrt{t-4m^2})^1 (\sqrt{t-4\mu^2})^0 \hat{T}_{++}$$

$$F_{+-}^{(-)} = (\sqrt{t})^{1/2} (\sqrt{t-4m^2})^0 (\sqrt{t-4\mu^2})^{-1} \hat{T}_{+-} \quad \text{and}$$

Next, we expose the kinematic factors which make the hat-T:

$$T_{\lambda\mu} = b_{\lambda\mu}(z) \hat{T}_{\lambda\mu} \quad \text{where } b_{\lambda\mu}(z) = (\sin \frac{1}{2}\theta)^{|\lambda-\mu|} (\cos \frac{1}{2}\theta)^{|\lambda+\mu|}$$

$$\mu = c-d \quad \lambda = a-b = 0. \quad b_{0\mu} = (\sin \frac{1}{2}\theta)^{|\mu|} (\cos \frac{1}{2}\theta)^{|\mu|} = [\frac{1}{2} \sin \theta]^{|\mu|}$$

$$T_{++; \infty} = \hat{T}_{++; \infty}$$

$$T_{+-; \infty} = \frac{1}{2} \sin \theta \hat{T}_{+-; \infty}$$

Now let's write out the Regge asymptotic forms for the hat amplitudes:

$$\left\{ \hat{T}_{cd; ab} = \sum_{n \in i} \left[-\frac{1}{4} \pi (2\alpha+1) \frac{[1+d(-1)^{\alpha-\mu}]}{\sin \pi(\alpha-\mu)} \frac{f_{cd; ab}^{adn} (2\alpha)! \left(-\frac{z}{2}\right)^{\alpha-\mu}}{\sqrt{(\alpha-\mu)! (\alpha+\mu)! (\alpha-\lambda)! (\alpha+\lambda)!}} \right] \right\}$$

Specifically in our case we get:

$$\hat{T}_{++; \infty} = \sum_{n \in i} \left[-\frac{1}{4} \pi (2\alpha+1) \frac{[1+d(-1)^{\alpha}]}{\sin \pi \alpha} \frac{f_{++; \infty}^{adn} (2\alpha)! \left(-\frac{z}{2}\right)^{\alpha}}{(\alpha!) (\alpha!)} \right]$$

$$\hat{T}_{+-; \infty} = \sum_{n \in i} \left[-\frac{1}{4} \pi (2\alpha+1) \frac{[1+d(-1)^{\alpha}]}{\sin \pi(\alpha-1)} \frac{f_{+-; \infty}^{adn} (2\alpha)! \left(-\frac{z}{2}\right)^{\alpha-1}}{\alpha! \sqrt{(\alpha+1)! (\alpha-1)!}} \right]$$

Now rearrange some of the details:

$$\sin[\pi(\alpha \pm \mu)] = \sin(\pi\alpha \pm \mu\pi) = (-1)^{\mu} \sin(\pi\alpha) \quad \text{if } \mu = \text{integer, regardless of } \alpha$$

$$\Rightarrow \sin[\pi(\alpha-1)] = -\sin \pi\alpha.$$

$$\text{Also: } \sqrt{(\alpha+1)! (\alpha-1)!} = \sqrt{(\alpha+1)\alpha} \cdot (\alpha-1)! = \alpha! \sqrt{\frac{\alpha+1}{\alpha}}$$

$$\sqrt{\pi} (2\alpha)! = 2^{2\alpha} (\alpha - \frac{1}{2})! \alpha! \quad \left(-\frac{z}{2}\right)^{\alpha} = (-1)^{\alpha} \left(\frac{s}{pq}\right)^{\alpha} = (-1)^{\alpha} \left(\frac{s_0}{p_0 q_0}\right)^{\alpha} (s/s_0)^{\alpha}$$

Look first at T_{++} . Using MS definition of F , I find:

$$\hat{T}_{++;\infty} = \overbrace{\frac{1}{4} \cdot \frac{1}{2} \cdot (-1)^\alpha \cdot 1 \cdot f}^{(-2)_{++}} \cdot \left(\frac{s_0}{pq}\right)^\alpha \cdot \left(\frac{s}{s_0}\right)^\alpha \cdot F(\alpha)$$

where,

$$F(\alpha) \equiv \frac{\sqrt{\pi} (\alpha + \frac{1}{2})! \cdot [(-1)^\alpha + 1]}{\alpha! \sin \pi \alpha}$$

Several comments here. First, that factor $(-1)^\alpha$ came from the $(-z)$. If I were more careful with the meaning of $-z$, I might be able to make that factor go away. Second, also from z we are picking up some threshold factors (pq) . However, we already know explicitly the singularity structure of this amplitude, so we should absorb that apparent factor and exhibit the real factors.

$$\hat{T}_{++;\infty} = -\beta_{++} \left(\frac{s_0}{pq}\right)^\alpha \left(\frac{s}{s_0}\right)^\alpha F(\alpha)$$

$$\beta_{++} \left(\frac{s_0}{pq}\right)^\alpha = \frac{\gamma_{++}}{\sqrt{t-4m^2}} \int^{\text{real factors}} \quad \text{as page 2 suggests.}$$

$$\Rightarrow \hat{T}_{++} = -\frac{\gamma_{++}}{\sqrt{t-4m^2}} \cdot F(\alpha) \cdot \left(\frac{s}{s_0}\right)^\alpha$$

Now do T_{+-} . As before, we rearrange things so we can extract the function $F(\alpha)$

$$\hat{T}_{+-;\infty} = (-1)^\alpha \cdot F(\alpha) \cdot \frac{1}{t-} \cdot \left(\frac{s_0}{pq}\right)^{\alpha-1} \cdot \left(\frac{s}{s_0}\right)^{\alpha-1} \cdot \sqrt{\frac{\alpha}{\alpha+1}}$$

As before, we redefine the residue function to absorb the apparent threshold singularity and to display the known singularities. At the same time, we want to arrange to have the residue contain certain factors of α to kill those square roots of α . These square roots create unwanted cuts. $\alpha =$ Thus (and it certainly seems weak),

$$\hat{T}_{+-;\infty} = \beta_{+-} \cdot \left(\frac{s_0}{pq}\right)^{\alpha-1} \cdot \sqrt{\frac{\alpha}{\alpha+1}} \cdot F(\alpha) \cdot \left(\frac{s}{s_0}\right)^{\alpha-1}$$

$$\beta_{+-} \cdot \left(\frac{s_0}{pq}\right)^{\alpha-1} = \gamma_{+-} \cdot \sqrt{t-4\mu^2} \cdot (\sqrt{t})^{-1} \cdot \sqrt{\alpha} \cdot \sqrt{\alpha+1}$$

$$\Rightarrow \hat{T}_{+-} = \gamma_{+-} \cdot \sqrt{t-4\mu^2} \cdot (\sqrt{t})^{-1} \cdot \alpha F(\alpha) \cdot \left(\frac{s}{s_0}\right)^{\alpha-1}$$

The power of $\text{SQRT}(t)$ is wrong here, but I have followed carefully the rules given on MS p363. They do not allow $n = -1$. Lets compare to CS rules:

$$\begin{matrix} \text{MS} & \text{CS} \\ (\sqrt{t})^n & \rightarrow (\sqrt{t})^{-\gamma} \end{matrix} \quad \gamma = \text{Max}_k \{1, \dots, l\} \quad p_{lll} = \text{nonnegative} \Rightarrow \gamma = +1,$$

Yes, MS have an error in their use of n . As shown on p 266 of Annals of Physics (long and complete article by Cohen et al 1968 about these kinematic singularities), MS should really have $n=0$ or $n=-1$, not $+1$. So here let's restate the two amplitudes with this correction:

$$\hat{T}_{++;\infty} = -\frac{\gamma_{++}}{\sqrt{t-4m^2}} \cdot F(\alpha) \cdot \left(\frac{s}{s_0}\right)^\alpha$$

$$\hat{T}_{+-;\infty} = 2\gamma_{+-} \cdot \sqrt{t(t-4m^2)} \cdot \alpha F(\alpha) \cdot \left(\frac{s}{s_0}\right)^{\alpha-1}$$

Back on page 2 we showed the connection between the hat and no-hats, so:

$$T_{++;\infty}(\text{full}) = \hat{T}_{++;\infty} \quad T_{+-;\infty}(\text{full}) = \frac{1}{2} \sin \theta_t \hat{T}_{+-;\infty}$$

$$\text{where } \sin \theta_t = \sqrt{1-z^2} \approx \sqrt{-z^2} \approx i z_t \rightarrow i \cdot \left(\frac{s}{2p_t q_t}\right).$$

When this factor of $\cos \theta$ is included, we see that both full helicity amplitudes have the same regge power behavior. This is the general situation, of course. All helicity amplitudes have the same regge poles and the same power behavior. It is only the residues that are different. Also, of course, the kinematic singularities hidden within the residues are different, as we just found explicitly here in the PI-N case. Also, the ALPHA dependence is different as we can see clearly from the general Regge formula back on page 2.

While we are here, why not write out the DCS that results from the above amplitudes. Notice that here we avoid having to do the explicit crossing to the s-channel amplitudes because of orthogonality!

The DCS is:

$$\frac{d\sigma}{dt}(\text{s-channel}) = \frac{\pi^2}{4s q_s^2} \{ |T_{++}(t)|^2 + |T_{+-}(t)|^2 \}$$

$$\frac{d\sigma}{dt} = \frac{\pi^3}{s^2} \cdot \left(\frac{s}{s_0}\right)^{2\alpha} |F(\alpha)|^2 \cdot \frac{1}{t-4m^2} \left\{ |\gamma_{++}(t)|^2 + t(t-4m^2)(t-4m^2)/\sin^2 \theta_t \alpha^2 \left(\frac{s}{s_0}\right)^2 |\gamma_{+-}(t)|^2 \right\}$$

Now get rid of that sine:

$$-\sin^2 \theta_t \approx +z^2 \approx \left| \frac{s}{2p_t q_t} \right|^2 = \frac{4s^2}{(t-4m^2)(t-4m^2)}$$

$$|p_t| = \frac{1}{2} \sqrt{t-4m^2}$$

$$|q_t| = \frac{1}{2} \sqrt{t-4m^2} \Rightarrow \{ \} = \{ |\gamma_{++}(t)|^2 - 4t s_0^2 |\gamma_{+-}(t)|^2 \alpha^2 \}$$

So the high energy s-channel PI-N does should be:

$$\frac{d\sigma}{dt} = \pi^3 s_0^2 \left(\frac{s}{s_0}\right)^{2\alpha-2} |F(\alpha)|^2 \cdot \frac{1}{4m^2-t} \left\{ |\gamma_{++}(t)|^2 - 4t\alpha^2 s_0^2 |\gamma_{+-}(t)|^2 \right\}$$

Here we have used only one pole for the amplitudes, so this is most appropriate to the CEX reaction where only the RHO can occur. I suppose the pomeron could be used along and then this would work for the elastic case.

Two things to note: 1) at $t=0$ the second term vanishes so there will be a dip at $t=0$ in the DCS. There is no mystery about this. Everybody knows that the helicity flip amplitude vanishes at $t=0$. 2) at t such that $a(t)=0$ there should also be a dip because again the second term vanishes. This is the famous dip in the CEX DCS and I suppose this should be a good victory for a regge fit with only one pole. However, the arguments which lead to that power of ALPHA in there are a little weak. But even if that is not the "right" power of ALPHA, it seems possible that some power of alpha will enter the second term via the rotation functions, and the dip at $t=-.6$ still supports the whole idea.

For more details see :

- 1) Hohler et al, Phys Letters 20,79 (1966)
- 2) Arbab and Chiu, Phys Rev 147, 1045 (1966)

"Choosing sense and nonsense: PI-N scattering."

Here we are discussing the possible presence of zeroes in the residue functions at places where the trajectory hits some integer. In the case of pion-nucleon scattering, we must deal with the Regge expansion in the t-channel to get out the high energy PI-N behavior. The t-channel process is:

$$\pi\pi \rightarrow N\bar{N} \quad T_{cd,ab}^{(t)} = T_{cd,ab}^{(t)} \bigg|_{\infty}, \quad \lambda = c-d \quad \mu=0.$$

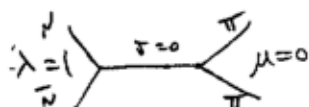
For the T_{++} and T_{--} amplitudes, $J=0$ is a sense-sense point, so there are no mysterious factors yet (but wait till below). For the amplitude T_{+-} we have $\lambda = 1$. For this amplitude, $J=0$ is sense-nonsense. Thus, for sure this amplitude has a square root, zero:

$$\beta_{+-} \sim \sqrt{\alpha-0}$$

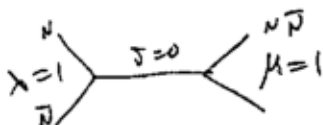
For this same amplitude, $J=-1$ is nonsense-nonsense, since $\lambda=1$ and $\mu=0$. However, $T_{-+} = -T_{+-}$ and for T_{-+} it turns out that $J=-1$ is a sense-nonsense because $\lambda=-1$ and $\mu=0$. Thus, we must include a square root zero at $J=-1$ also:

$$\beta_{+-} \sim \sqrt{\alpha-0} \sqrt{\alpha-(-1)} = \sqrt{\alpha(\alpha+1)}$$

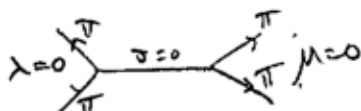
The business of "sense-nonsense" choosing does not come into play in our application here because the initial and final channels are different. Consider $J=0$. Then we can draw three pictures:



$$\beta_{ns}(\pi\pi \rightarrow N\bar{N}) = \gamma_{ns}^{\pi\pi} \cdot \gamma_{ns}^{N\bar{N}} \sim \sqrt{J}$$



$$\beta_{nn}(N\bar{N} \rightarrow \pi\pi) = (\gamma_{nn}^{N\bar{N}})^2$$



$$\beta_{ss}(\pi\pi \rightarrow \pi\pi) = (\gamma_{ss}^{\pi\pi})^2$$

Here we have a ss, sn, and nn residue. Factorization here does not lead us to the conclusion that ss or nn has a zero because these three residues do not apply to the same process. In the example given in MS p 448 with NN scattering, there all residues do apply to the same vertices. Then, factorization forces either ss or nn to pick up a factor J. If nn picks up the zero, then you have "chosen sense". You have decoupled from the nonsense at $j=0$.

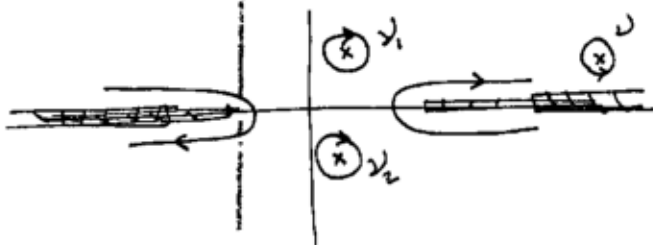
Note: another source of confusion. MS seem to define their residue β to exclude the various rotation function square roots, whereas here I am thinking of these squareroots as being included in β . Thus, MS in effect ~~exclude~~ expose these factors twice.

$$e, \quad \frac{1}{\sqrt{(\alpha-\mu)(\alpha+\mu)}} \dots$$

PI-N dispersion relations.

1. This is one of those things where there is so much technical detail that every time you make a mistake you have to start over. There are several subtle details which must be kept in mind:
2. Here are some of the details:
 - a) the symbol "s" always stands for the energy in either the $\text{PI} \rightarrow \text{P}$ or the $\text{PI} \rightarrow \text{p}$ process. When you write the dispersion integral for one process, the other gets mixed in due to the everpresent left hand cut.
 - b) both of these scattering amplitudes have one pole. In the full analytic function that describes both processes it is of course the same pole, but when you look at the separate scattering amplitudes this pole appears at different places in the v-plane. The comments made to this regard in PI-N Kinematics are correct.
 - c) The disprels that I have derived are twice-subtracted. One subtraction point is $+u$, the other is $-u$. There should really be two subtraction constants in each equation but the crossing property relates these two constants so there is only one constant in each equation.
 - d) However, these subtraction constants do not occur in the same way in both equations. This just comes out in the algebra.
 - e) the Born terms also are not the same; in particular, they do not cancel in the $(-)$ equation as you might think.
 - f) in folding over the left hand cut integral onto the right, there is a tricky extra minus sign that arises in going from $T(v+ie)$ to $T(-v-ie)$ then to $-T(-v+ie)$. This causes the bare v to be extracted from the $(-)$ integral, not the $(+)$.
3. It seems that you cannot really do a dispersion relation without subtractions and then just "add" the subtractions. You have to include them from the start to get it all right.
4. Eventually I should compare my result with the result in Jackson's notes for 227 which I can trust is correct. There certainly are other options for choosing the subtraction points. Also, usually one does not need any subtractions at all for the $(-)$ disprel. Whereas, usually you need not one but two subtractions for the $(+)$ one. The reason for this lies in the crossing property. This crossing property causes the $(-)$ disprel to converge faster by an extra power. Moreover, usually you say that the cross section difference goes to zero, so that gains you another power of convergence! This means two powers worth of difference.
5. In the form I have used, the disprel is independent of the scale of the amplitude used, except for the Born constant. However, once you use the optical theorem to get cross sections in there, you have to make a choice of scale.

Dispersion Relations



$$f(\nu) = \frac{1}{2\pi i} \oint_{\gamma} \frac{f(\nu') d\nu'}{\nu' - \nu} = -\frac{1}{2\pi i} \oint_{\gamma_1} \frac{f(\nu') d\nu'}{\nu' - \nu} - \frac{1}{2\pi i} \oint_{\gamma_2} \frac{f(\nu') d\nu'}{\nu' - \nu} + \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\text{Im} f(\nu') d\nu'}{\nu' - \nu}$$

$$= -\frac{1}{(\nu_1 - \nu)} \frac{1}{2\pi i} \oint_{\gamma_1} f(\nu') d\nu' - \frac{1}{(\nu_2 - \nu)} \frac{1}{2\pi i} \oint_{\gamma_2} f(\nu') d\nu' + \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\text{Im} f(\nu') d\nu'}{\nu' - \nu}$$

Try: $f(\nu) = \frac{T(\nu)}{(\nu - \nu_1)(\nu - \nu_2)}$ but let $T(\nu) = \frac{h}{\nu - \nu_m} + \dots$

This adds: $-\frac{1}{(\nu_m - \nu)} \frac{1}{2\pi i} \oint_{\gamma_m} f(\nu') d\nu'$, just like a subtraction pole.

But:

$$\frac{1}{2\pi i} \oint_{\gamma_1} f(\nu') d\nu' = \frac{1}{2\pi i} \oint_{\gamma_1} \frac{T(\nu') d\nu'}{(\nu' - \nu_1)(\nu' - \nu_2)} = \frac{T(\nu_1)}{(\nu_1 - \nu_2)}$$

$$\frac{1}{2\pi i} \oint_{\gamma_2} f(\nu') d\nu' = \frac{1}{2\pi i} \oint_{\gamma_2} \frac{T(\nu') d\nu'}{(\nu' - \nu_1)(\nu' - \nu_2)} = \frac{T(\nu_2)}{(\nu_2 - \nu_1)}$$

$$\frac{1}{2\pi i} \oint_{\gamma_m} f(\nu') d\nu' = \frac{1}{2\pi i} \oint_{\gamma_m} \frac{T(\nu') d\nu'}{(\nu' - \nu_1)(\nu' - \nu_2)} = \frac{h}{(\nu_m - \nu_1)(\nu_m - \nu_2)}$$

So, $f(\nu) = -\frac{1}{(\nu_1 - \nu)} \frac{T(\nu_1)}{(\nu_1 - \nu_2)} - \frac{1}{(\nu_2 - \nu)} \frac{T(\nu_2)}{(\nu_2 - \nu_1)} - \frac{1}{(\nu_m - \nu)} \frac{h}{(\nu_m - \nu_1)(\nu_m - \nu_2)} + \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\text{Im} f(\nu') d\nu'}{(\nu' - \nu)}$

Now set $\nu_1 = -\nu_2 = \mu$:

$$f(\nu) = -\frac{1}{(\mu - \nu)} \frac{T(\mu)}{-2\mu} - \frac{1}{(-\mu - \nu)} \frac{T(-\mu)}{(-2\mu)} - \frac{1}{(\nu_m - \nu)} \frac{h}{\nu_m^2 - \mu^2}$$

$$f(\nu) = \frac{1}{2\mu} \left[\frac{T(\mu)}{(\nu - \mu)} - \frac{T(-\mu)}{(\nu + \mu)} \right] - \frac{h}{(\nu_m - \nu)(\nu_m^2 - \mu^2)} + \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\text{Im} f(\nu') d\nu'}{\nu' - \nu}$$

Rewrite:

$$\frac{T(\nu)}{\nu^2 - \mu^2} = \frac{1}{2\mu} \left[\frac{T(\mu)}{\nu - \mu} - \frac{T(-\mu)}{\nu + \mu} \right] - \frac{h}{(\nu_+ - \nu)(\nu_+^2 - \mu^2)} + \frac{i}{\pi} \int \frac{\text{Im} T(\nu') d\nu'}{(\nu' - \nu)(\nu'^2 - \mu^2)}$$

Pause: do $T_{\text{HP}}(\nu)$ and $\bar{T}_{\text{HP}}(\nu)$ have the same pole term?!

$\bar{T}_{\text{HP}}(\nu)$ has pole at $s = m^2 \Rightarrow \nu_- = -\mu^2/2M$

$T_{\text{HP}}(\nu)$ has pole at $u = m^2 \Rightarrow \nu_+ = +\mu^2/2M$

$$\begin{aligned} \frac{\bar{\pi}^p}{T(s)} &= \frac{h}{\nu - \nu_-} \\ \frac{\pi^p}{T(s)} &= \frac{h}{\nu - \nu_+} \\ &= \frac{g}{s - m^2} \\ h &= g^2/2M \end{aligned}$$

Always, $s = (p_{\pi^+} + p_p)^2$. So $\nu_+ = \nu_+$ or ν_- . Let h be the residue = $g^2/2M$. Then:

$$\frac{T_+(\nu)}{\nu^2 - \mu^2} = \frac{1}{2\mu} \left[\frac{T_+(\mu)}{\nu - \mu} - \frac{T_+(-\mu)}{\nu + \mu} \right] - \frac{h}{(\nu_+ - \nu)(\nu_+^2 - \mu^2)} + \frac{i}{\pi} \int \frac{\text{Im} T_+(\nu') d\nu'}{(\nu' - \nu)(\nu'^2 - \mu^2)}$$

$$\frac{T_-(\nu)}{\nu^2 - \mu^2} = \frac{1}{2\mu} \left[\frac{T_-(\mu)}{\nu - \mu} - \frac{T_-(-\mu)}{\nu + \mu} \right] + \frac{h}{(\nu_+ + \nu)(\nu_+^2 - \mu^2)} + \frac{i}{\pi} \int \frac{\text{Im} T_-(\nu') d\nu'}{(\nu' - \nu)(\nu'^2 - \mu^2)}$$

Now subtract em:

$$\frac{T^{(\pm)}(\nu)}{\nu^2 - \mu^2} = \frac{1}{2\mu} \left[\frac{T^{(\pm)}(\mu)}{\nu - \mu} - \frac{T^{(\pm)}(-\mu)}{\nu + \mu} \right] - \frac{h}{(\nu_+^2 - \mu^2)} \left[\frac{1}{(\nu_+ - \nu)} \mp \frac{1}{(\nu_+ + \nu)} \right] + \frac{i}{\pi} \int \text{Im} T^{(\pm)}(\nu')$$

Now use $T^{(\pm)}(-\mu) = \pm T^{(\pm)}(\mu)$:

$$\begin{aligned} &= \frac{1}{2\mu} T^{(\pm)}(\mu) \left[\frac{1}{\nu - \mu} \mp \frac{1}{\nu + \mu} \right] - \frac{h}{\nu_+^2 - \mu^2} \left[\frac{1}{\nu_+ - \nu} \mp \frac{1}{\nu_+ + \nu} \right] + \frac{i}{\pi} \int \text{Im} T^{(\pm)}(\nu') \\ &= \left[\frac{2\mu}{2\nu} \right] \left[\frac{1}{\nu^2 - \mu^2} \right] - \frac{h}{(\nu_+^2 - \mu^2)} \left[\frac{2\nu}{2\nu_+} \right] \left[\frac{1}{\nu_+^2 - \nu^2} \right] + \frac{i}{\pi} \end{aligned}$$

$$\text{So: } T^{(+)}(\nu) = T^{(+)}(\mu) - h \frac{\nu^2 - \mu^2}{\nu_+^2 - \mu^2} \cdot \frac{2\nu}{\nu_+^2 - \nu^2} + \frac{(\nu^2 - \mu^2)}{\pi} \int \frac{\text{Im} T^{(+)}(\nu') d\nu'}{(\nu' - \nu)(\nu'^2 - \mu^2)}$$

$$T^{(-)}(\nu) = \frac{\nu}{\mu} T^{(-)}(\mu) - h \frac{\nu^2 - \mu^2}{\nu_+^2 - \mu^2} \cdot \frac{2\nu_+}{\nu_+^2 - \nu^2} + \frac{(\nu^2 - \mu^2)}{\pi} \int \frac{\text{Im} T^{(-)}(\nu') d\nu'}{(\nu' - \nu)(\nu'^2 - \mu^2)}$$

Finally lets write out these integrals.

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{\partial m T^{(+)}(v') dv'}{(v'-v)(v'^2-\mu^2)} &= \left(\int_{-\infty}^{-\mu} + \int_{\mu}^{\infty} \right) \frac{dv'}{v'^2-\mu^2} [\partial m T^{(+)}(v')] \left[\frac{1}{v'-v} \right] \checkmark \\ &= \int_{-\infty}^{-\mu} \frac{dv'}{v'^2-\mu^2} \partial m T^{(+)}(v') \frac{1}{v'-v} \checkmark + \int_{\mu}^{\infty} \frac{dv'}{v'^2-\mu^2} \partial m T^{(+)}(v') \frac{1}{v'-v} \checkmark \\ &= \int_{-\infty}^{-\mu} \frac{(-dv'')}{v''^2-\mu^2} \partial m T^{(+)}(-v'') \frac{1}{-v''-v} \checkmark \\ &= + \int_{\mu}^{\infty} \frac{dv''}{v''^2-\mu^2} \partial m T^{(+)}(v'') \frac{1}{v'+v} \checkmark \end{aligned}$$

So add: $\rightarrow \int_{\mu}^{\infty} \frac{dv'}{v'^2-\mu^2} \partial m T^{(+)}(v') \left[\frac{1}{v'-v} + \frac{1}{v'+v} \right] \checkmark$

$\frac{2v'}{v'^2-v^2} \checkmark$

$\frac{(-1)}{v'+v} - \frac{(-1)}{v'-v} = \frac{2v'}{v'^2-v^2}$

One other negative opp sign on second term

So $\frac{1}{v'-v} + \frac{1}{v'+v} = \frac{2v'}{v'^2-v^2}$

$$= 2 \int_{\mu}^{\infty} \frac{dv'}{(v'^2-\mu^2)(v'^2-v^2)} \partial m T^{(+)}(v')$$

So, here are my final results:

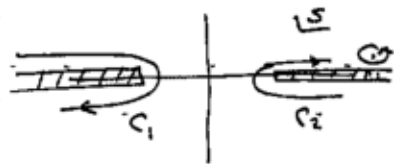
$$\begin{aligned} \text{Re } T^{(+)}(v) &= \text{Re } T^{(+)}(\mu) - \underbrace{\nu Q}_{\text{check}} + \frac{2}{\pi} (v^2-\mu^2) \int_{\mu}^{\infty} \frac{v' dv'}{(v'^2-\mu^2)(v'^2-v^2)} \partial m T^{(+)}(v') \\ \text{Re } T^{(-)}(v) &= \frac{\nu}{\mu} \text{Re } T^{(+)}(\mu) - \underbrace{\nu_+ Q}_{\text{check}} + \frac{2}{\pi} (v^2-\mu^2) \nu \int_{\mu}^{\infty} \frac{dv'}{(v'^2-\mu^2)(v'^2-v^2)} \partial m T^{(+)}(v') \\ Q &= (g/2m) \cdot \frac{v^2-\mu^2}{v_+^2-\mu^2} \cdot \frac{2}{v_+^2-v^2} \\ \nu_+ &= \mu^2/2m \end{aligned}$$

Using true amplitude with $\gamma = 1/(\epsilon\pi)^3$, $C_1 = (2\pi)^4$, $\text{and } C_1 = +1$, we know.

$$\partial m T_{\pm} = 2m \sqrt{v^2-\mu^2} \sigma_{\pm}(v)$$

$$T_S = \frac{T_{me}}{4m}$$

Pole Nucleon Forward Dispersion relations (earlier details)



$$C = C_1 + C_2$$

$$T(s) = \oint \frac{T(s') ds'}{s' - s} \cdot \frac{1}{2\pi i} = \frac{1}{2\pi i} \oint \frac{T(s') ds'}{(s' - s)}$$

$$\text{But } \int_{-C_1} \frac{T(s') ds'}{s' - s} = \int_{-\infty}^0 \frac{T(s' + i\epsilon) ds'}{s' - s} + \int_0^{\infty} \frac{T(s' - i\epsilon) ds'}{s' - s}$$

$$= \int_{-\infty}^0 \frac{ds'}{s' - s} [T(s' + i\epsilon) - T(s' - i\epsilon)] = \int_{-\infty}^0 \frac{ds'}{(s' - s)} \cdot 2i \operatorname{Im} T(s' + i\epsilon)$$

$$\text{And, } \int_{C_2} \frac{T(s') ds'}{s' - s} = \int_0^{\infty} \frac{T(s' + i\epsilon) ds'}{s' - s} + \int_{\infty}^0 \frac{T(s' - i\epsilon) ds'}{s' - s} = \int_0^{\infty} \frac{ds'}{s' - s} [2i \operatorname{Im} T(s' + i\epsilon)]$$

$$\text{Therefore, } T(s) = \frac{1}{\pi} \int_{-\infty}^0 \frac{ds'}{s' - s} \operatorname{Im} T(s' + i\epsilon) + \frac{1}{\pi} \int_0^{\infty} \frac{ds'}{s' - s} \operatorname{Im} T(s' + i\epsilon)$$

Now insert $s = s + i\epsilon$ so that

$$T(s + i\epsilon) = \frac{1}{\pi} \left[\int_{-\infty}^0 + \int_0^{\infty} \right] \frac{ds'}{s' - s - i\epsilon} \operatorname{Im} T(s' + i\epsilon)$$

$$\text{And: } \frac{1}{s' - s - i\epsilon} = P\left(\frac{1}{s' - s}\right) + i\pi \delta(s' - s)$$

$$\text{Thus: } \operatorname{Im} T(s + i\epsilon) = \frac{1}{\pi} \cdot \pi \cdot \operatorname{Im} T(s' + i\epsilon) \checkmark$$

$$\operatorname{Re} T(s + i\epsilon) = \frac{1}{\pi} \left[\int_{-\infty}^0 + \int_0^{\infty} \right] \frac{ds'}{s' - s - i\epsilon} \operatorname{Im} T(s' + i\epsilon)$$

$$\text{Or: } D(s + i\epsilon) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{ds' A(s' + i\epsilon)}{s' - s}$$

$$dv = \frac{1}{2M} ds$$

If we convert this to ν :

$$\nu = \frac{s - u}{4M} = \frac{(s - M^2 - \mu^2)}{2M}, t = 0.$$

$$D(\nu + i\epsilon) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{d\nu' A(\nu' + i\epsilon)}{\nu' - \nu}$$

However, suppose $T(s)$ has a pole at $s = s_0$. Then

$$T(s) \sim \frac{q}{s - s_0} \quad T(s + i\epsilon) = \frac{q}{s - s_0 + i\epsilon} \quad \operatorname{Im} T(s + i\epsilon) = -\pi q \delta(s - s_0)$$

Then: $\frac{1}{\pi} \oint_{\text{pole}} \frac{ds' A(s'+i\epsilon)}{s'-s} = \frac{1}{\pi} (-\pi g) \cdot \frac{1}{s_0-s} = \frac{g}{s-s_0} \checkmark$

Any pole in $T(s)$ can be handled like this, so

$$D(s+i\epsilon) = \sum_i \frac{g_i}{(s-s_i)} + \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{ds' A(s'+i\epsilon)}{(s'-s)} + \frac{1}{\pi} \int_{\infty}^{\infty} \frac{ds' A(s'+i\epsilon)}{(s'-s)}$$

In the ν variable, $\left. \begin{aligned} s &= 2M\nu + M^2 + \mu^2 \\ s_i &= 2M\nu_i + M^2 + \mu^2 \end{aligned} \right\} \quad s-s_i = 2M(\nu-\nu_i)$
so,

$$D(\nu+i\epsilon) = \sum_i \frac{g_i/2M}{(\nu-\nu_i)} + \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{d\nu' A(\nu'+i\epsilon)}{\nu'-\nu} + \frac{1}{\pi} \int_{\infty}^{\infty} \frac{d\nu' A(\nu'+i\epsilon)}{(\nu'-\nu)}$$

Now, let all these things refer to $\pi^{\pm}p$. Then crossing says:

$$D_{\pi^{\pm}p}(\nu+i\epsilon) = \sum_i \frac{g_i(\pi^{\pm}p)/2M}{\nu-\nu_i(\pi^{\pm}p)} + \frac{1}{\pi} \left(\int_{-\infty}^{\mu} + \int_{\mu}^{\infty} \right) \frac{d\nu'}{\nu'-\nu} A_{\pi^{\pm}p}(\nu'+i\epsilon)$$

But, $A_{\pi^{\pm}p}(\nu'+i\epsilon)$ = physical $\pi^{\pm}p$ region for $\nu > 6m^2$, say.
 $A_{\pi^{\mp}p}(\nu'+i\epsilon)$ = not in phys. region.

Now use: $A_{\pi^{\pm}p}(\nu) = A_{\pi^{\mp}p}(-\nu)$, ν = complex.

Thus: $\int_{-\infty}^{\mu} \frac{d\nu'}{\nu'-\nu} A_{\pi^{\pm}p}(\nu'+i\epsilon) = \int_{-\infty}^{\mu} \frac{d\nu'}{\nu'-\nu} A_{\pi^{\mp}p}(-\nu'-i\epsilon)$

Let $\nu' \rightarrow -\nu''$ so $\sim$ well $\nu'' = -\nu'$
 $= \int_{\infty}^{\mu} \frac{[-d\nu'']}{-\nu''-\nu} A_{\pi^{\mp}p}(\nu''-i\epsilon) = - \int_{\mu}^{\infty} \frac{d\nu''}{-\nu''+\nu} A_{\pi^{\mp}p}(\nu''-i\epsilon)$

But $A(\nu''-i\epsilon) = -A(\nu''+i\epsilon)$, so $= + \int_{\mu}^{\infty} \frac{d\nu''}{\nu''+\nu} A_{\pi^{\mp}p}(\nu''+i\epsilon)$

Then: $D_{\pi^{\pm}p}(\nu+i\epsilon) = \sum_i \frac{g_i(\pi^{\pm}p)/2M}{\nu-\nu_i} + \frac{1}{\pi} \int_{\mu}^{\infty} d\nu' \left[\frac{A_{\pi^{\pm}p}(\nu'+i\epsilon)}{\nu'-\nu} + \frac{A_{\pi^{\mp}p}(\nu'+i\epsilon)}{\nu'+\nu} \right]$
 $D_{\pi^{\mp}p}(\nu+i\epsilon) = \sum_i \frac{g_i(\pi^{\mp}p)/2M}{\nu-\nu_i} + \frac{1}{\pi} \int_{\mu}^{\infty} d\nu' \left[\frac{A_{\pi^{\mp}p}(\nu'+i\epsilon)}{\nu'-\nu} + \frac{A_{\pi^{\pm}p}(\nu'+i\epsilon)}{\nu'+\nu} \right]$

Now add these:

$$D_{\pi^{\pm}p} \pm D_{\pi^{\mp}p} = \sum_i \pm \frac{g_i}{\nu-\nu_i} + \frac{1}{\pi} \int_{\mu}^{\infty} d\nu' \left[A_{\pi^{\pm}p} \left(\frac{1}{\nu'-\nu} \pm \frac{1}{\nu'+\nu} \right) + A_{\pi^{\mp}p} \left(\frac{1}{\nu'+\nu} \pm \frac{1}{\nu'-\nu} \right) \right]$$

No Subs.

Now use: $-\frac{1}{2v'} + \frac{1}{v+v} = \frac{v'-v}{2v'}$

Ques: $D_{\pi p}^+ + D_{\pi p}^- = \sum + \sum + \frac{1}{2} \int_0^\infty dv' \cdot \frac{2v'}{v'^2 - v^2} (A_{\pi p}^+ + A_{\pi p}^-)$

Next: $-\frac{1}{2v'} - \frac{1}{v+v} = \frac{v'-v}{2v'}$

Next; rewrite: $[A_{\pi p}^+ \left(\frac{1}{v'-v} \pm \frac{1}{v+v} \right) + A_{\pi p}^- \left(\frac{1}{v'-v} \pm \frac{1}{v+v} \right)]$

$= A_{\pi p}^+ \left(\frac{1}{v'-v} \pm \frac{1}{v+v} \right) + A_{\pi p}^- \left(\frac{1}{v'-v} \pm \frac{1}{v+v} \right)$

$= \left(\frac{1}{v'-v} \pm \frac{1}{v+v} \right) (A_{\pi p}^+ \pm A_{\pi p}^-)$

Ques: $D_{\pi p}^+ \pm D_{\pi p}^- = \sum + \sum + \frac{1}{2} \int_0^\infty dv' \left(\frac{1}{v'-v} \pm \frac{1}{v+v} \right) (A_{\pi p}^+ \pm A_{\pi p}^-)$

Now do special case:

$\frac{1}{v'-v} + \frac{1}{v+v} = \frac{v'+v}{2v'}$

Ques: $[D_{\pi p}^+ + D_{\pi p}^- = \sum + \sum + \frac{1}{2} \int_0^\infty dv' \cdot \frac{v'}{v'^2 - v^2} (A_{\pi p}^+ + A_{\pi p}^-)]$

Ans: $\frac{1}{v'-v} - \frac{1}{v+v} = \frac{v'-v}{2v'}$, so

$[D_{\pi p}^+ - D_{\pi p}^- = \sum - \sum + \frac{1}{2} \int_0^\infty dv' \cdot \frac{v'}{v'^2 - v^2} (A_{\pi p}^+ - A_{\pi p}^-)]$

Now we need an optical theorem: Use $G_3 = G(\pi^0)$ norm?

$\sigma_{\text{tot}} = \pm G_1 \cdot \text{Im} T = \pm G_1 \cdot A = \frac{\pm \lambda(s, m_p^2)}{2 \cdot G_1 \cdot A} = G_1 \cdot A / \lambda$

Now, $\lambda = \sqrt{[s - (m + m_p)^2][s - (m - m_p)^2]}$

But: $s - (m + m_p)^2 = s - m^2 - m_p^2 - 2mm_p = 2m(v - m) - 2mm_p = 2m(v - m - m_p)$
 $s - (m - m_p)^2 = s - m^2 - m_p^2 + 2mm_p = 2m(v - m) + 2mm_p = 2m(v - m + m_p)$

So $\lambda = \sqrt{2m(v - m - m_p) \cdot 2m(v - m + m_p)} = 2m \sqrt{v^2 - m^2}$

Thus, $\sigma_{\pm\pm} = C_1 \cdot A / 2m \sqrt{v^2 - \mu^2}$ Choose $C_1 = +$ for now:

So: $A_{\pi p}^{\pm} = 2m \sqrt{v^2 - \mu^2} \sigma_{\pi p}^{\pm}$

Insert this into our relations:

no
subtractions

$$\begin{cases} D_{\pi p}^+ + D_{\pi p}^- = \sum_i + \sum_j + \frac{4m}{\pi} \int_{\mu}^{\infty} \frac{dv' \cdot v' \sqrt{v'^2 - \mu^2}}{v'^2 - v^2} (\sigma_{\pi p}^+ + \sigma_{\pi p}^-) \\ D_{\pi p}^+ - D_{\pi p}^- = \sum_i - \sum_j + \frac{4m}{\pi} \cdot v \int_{\mu}^{\infty} \frac{dv' \sqrt{v'^2 - \mu^2}}{v'^2 - v^2} (\sigma_{\pi p}^+ - \sigma_{\pi p}^-) \end{cases}$$

Now comes the subtractions. Define:

$$\tilde{T}(v) = \frac{T(v)}{v - m_{\pi}} \quad \text{Same for } D, A, \sigma:$$

Then: $(v - m_{\pi}) [\dots \times$ If we had dispensed $\tilde{T}$ instead of T , we would have obtained:

$$\tilde{D}_{\pi p}^+ + \tilde{D}_{\pi p}^- = \frac{\sum_i + \sum_j}{(v - m_{\pi})} + C_1 + \frac{4m}{\pi} \int_{\mu}^{\infty} \frac{dv' \cdot v' \sqrt{v'^2 - \mu^2}}{v'^2 - v^2} (\tilde{\sigma}_{\pi p}^+ + \tilde{\sigma}_{\pi p}^-)$$

one
subt.

Now unwind to get:

$$\begin{aligned} D_{\pi p}^+ + D_{\pi p}^- &= (v - m_{\pi}) (\sum_i + \sum_j) + (v - m_{\pi}) \cdot C_1 \\ &\quad + (v - m_{\pi}) \cdot \frac{4m}{\pi} \int_{\mu}^{\infty} \frac{dv' \cdot v' \sqrt{v'^2 - \mu^2}}{(v'^2 - v^2)(v' - \mu)} (\sigma_{\pi p}^+ + \sigma_{\pi p}^-) \end{aligned}$$

Well, instead, replace $v - m_{\pi}$ with $v^2 - m_{\pi}^2$ so that:

two
sub's.

$$\begin{aligned} D_{\pi p}^+ + D_{\pi p}^- &= (v^2 - m_{\pi}^2) (\sum_i + \sum_j) + (v^2 - m_{\pi}^2) [?] \\ &\quad + \frac{4m}{\pi} (v^2 - m_{\pi}^2) \int_{\mu}^{\infty} \frac{dv' \cdot v' \sqrt{v'^2 - \mu^2}}{(v'^2 - v^2)(v'^2 - \mu^2)} (\sigma_{\pi p}^+ + \sigma_{\pi p}^-) \end{aligned}$$

This is as if we had dispensed $\frac{T(v)}{v^2 - \mu^2}$. This then is two subtractions at the same point.

Clearly: $[?] = \frac{D_{\pi p}^+(v) + D_{\pi p}^-(v)}{v^2 - m_{\pi}^2} \quad ?$

Well: $D_{\pi p}^+(+v) + D_{\pi p}^-(+v) = \lim_{v \rightarrow +\infty} (v^2 - \mu^2) [?]$

Need more detail on this:

Define: $\bar{D}^{(+)} = D_{\pi^+}^+(\nu) + D_{\pi^0}^+(\nu)$.

Then: $\lim_{\nu \rightarrow \mu} (\nu^2 - \mu^2) [?] = D^{(+)}(\mu)$

$\lim_{\nu \rightarrow -\mu} (\nu^2 - \mu^2) [?] = D^{(+)}(-\mu) = D^{(+)}(\mu)$.

So, $[?] = \frac{D^{(+)}(\mu)}{\nu^2 - \mu^2}$

Hence: $D^{(+)}(\nu) = \frac{\nu^2 - m_\pi^2}{2} \left(\sum_i + \sum_j \right) + D^{(+)}(\mu)$
 $+ \frac{4m}{\pi} (\nu^2 - m_\pi^2) \int_\mu^\infty \frac{d\nu' \cdot \nu' \cdot \sqrt{\nu'^2 - \mu^2}}{(\nu'^2 - \nu^2)(\nu'^2 - \mu^2)} \sigma^{(+)}(\nu')$

However, we also could have dispensed:

$C \frac{1}{(\nu^2 - \mu^2)}$ for any constant C . Then:

$D^{(+)}(\nu) = \frac{\nu^2 - m_\pi^2}{2} \left(\sum_i + \sum_j \right) + \frac{\nu^2 - m_\pi^2}{C} [?] + \frac{4m}{\pi} \cdot \frac{\nu^2 - m_\pi^2}{C} \int \dots$

Then we'd conclude again that $C = D^{(+)}(\mu)$.

So I say: choose $C = \frac{4m}{\pi} = 4M$. Then:

$D^{(+)}(\nu) = \frac{\nu^2 - m_\pi^2}{4M} \left(\sum_i + \sum_j \right) + D^{(+)}(\mu) + \frac{\nu^2 - m_\pi^2}{\pi} \int \frac{d\nu' \cdot \nu' \cdot \sqrt{\nu'^2 - \mu^2}}{(\nu'^2 - \nu^2)(\nu'^2 - \mu^2)} \sigma^{(+)}(\nu')$

Now try repeating for $D^{(-)}$: again, choose $C = 4M$ so:

$D^{(-)}(\nu) = \frac{\nu^2 - m_\pi^2}{4M} \left(\sum_i - \sum_j \right) + \frac{\nu^2 - m_\pi^2}{4M} [?] + \frac{\nu(\nu^2 - m_\pi^2)}{\pi} \int \frac{d\nu' \sqrt{\nu'^2 - \mu^2}}{(\nu'^2 - \nu^2)(\nu'^2 - \mu^2)} \sigma^{(-)}(\nu')$

Now however,

$\lim_{\nu \rightarrow +\mu} \frac{\nu^2 - \mu^2}{4M} [?] = D^{(-)}(\mu)$

$\lim_{\nu \rightarrow -\mu} \frac{\nu^2 - \mu^2}{4M} [?] = D^{(-)}(-\mu) = -D^{(-)}(+\mu)$.

So we have to be more careful with $[?]$ here:

$$\begin{aligned} [?] &= \left\{ D^{(+)}(\mu) \cdot \frac{\nu+\mu}{2\mu} + D^{(+)}(-\mu) \cdot \frac{-\nu+\mu}{2\mu} \right\} \times \frac{4m}{\nu^2-\mu^2} \\ &= \frac{4m}{\nu^2-\mu^2} D^{(+)}(\mu) \left[\frac{\nu+\mu}{2\mu} - \frac{(-\nu+\mu)}{2\mu} \right] = \frac{4m}{\nu^2-\mu^2} \cdot \frac{1}{2\mu} \cdot D^{(+)}(\mu) \cdot [2\nu] \\ &= \frac{4m}{\nu^2-\mu^2} \cdot \left(\frac{\nu}{\mu}\right) \cdot D^{(+)}(\mu) \end{aligned}$$

Then: $\frac{\nu^2-\mu^2}{4m} [?] = \frac{\nu}{\mu} D^{(+)}(\mu)$

So: $D^{(+)}(\nu) = \frac{\nu^2-\mu^2}{4m} \left(\sum_i \rightarrow \sum_j \right) + \frac{\nu}{\mu} D^{(+)}(\mu) + \frac{\nu(\nu^2-\mu^2)}{\pi} \int \dots$

Now let's summarize in fullest possible notation these results:

→ First, definitions: $f_i = g_i/2M =$ residue in ν of a pole:

$$T(s) = \sum_i \frac{g_i}{s-s_i} = \sum_i \frac{f_i}{\nu+\nu_i} \quad \nu_i = (s_i - m^2 - \mu^2)/2m.$$

In particular, $f_i(\pi p)$ is a pole in the πp amplitude.

→ Define:

$$\begin{aligned} D^{(\pm)}(\nu) &= D_{\pi p}^{+}(\nu) \pm D_{\pi \bar{p}}^{+}(\nu) & \nu &= \text{complex.} & (\text{same for } \sigma^{(\pm)}) \\ A^{(\pm)}(\nu) &= A_{\pi p}^{+}(\nu) \pm A_{\pi \bar{p}}^{+}(\nu) & D &= \text{Re } T. \\ T^{(\pm)}(\nu) &= T_{\pi p}^{+}(\nu) \pm T_{\pi \bar{p}}^{+}(\nu) & A &= \text{Im } T \end{aligned}$$

→ Crossing says: $T^{(\pm)}(\nu) = \pm T^{(\mp)}(-\nu)$, ν complex!

The connection of $T(\nu)$ to A, B is not relevant here.

So:

$$\begin{aligned} D^{(+)}(\nu) &= D^{(+)}(\mu) + \left[\sum_i \frac{f_i(\pi p)}{\nu-\nu_i} + \sum_j \frac{f_j(\pi \bar{p})}{\nu-\nu_j} \right] \\ &\quad + \frac{\nu^2-\mu^2}{\pi} \int_{\mu}^{\infty} d\nu' \sigma^{(+)}(\nu') \frac{\nu' \sqrt{\nu'^2-\mu^2}}{(\nu'^2-\nu^2)(\nu'^2-\mu^2)} \\ D^{(-)}(\nu) &= \frac{\nu}{\mu} D^{(-)}(\mu) + \left[\sum_i - \sum_j \right] + \frac{\nu(\nu^2-\mu^2)}{\pi} \int_{\mu}^{\infty} d\nu' \sigma^{(-)}(\nu') \frac{\sqrt{\nu'^2-\mu^2}}{(\nu'^2-\nu^2)(\nu'^2-\mu^2)} \end{aligned}$$

↖ two subtractions at $\nu=\mu$ in each.