

Poincaré, SPIN, M-fctus, helicity...

I

Phil Lucht

Wigner '39

ON UNITARY REPRESENTATIONS OF THE INHOMOGENEOUS LORENTZ GROUP*

By E. WIGNER

(Received December 22, 1937)

1. ORIGIN AND CHARACTERIZATION OF THE PROBLEM

It is perhaps the most fundamental principle of Quantum Mechanics that the system of states forms a *linear manifold*,¹ in which a unitary *scalar product* is defined.² The states are generally represented by wave functions³ in such a way that φ and constant multiples of φ represent the same physical state. It is possible, therefore, to normalize the wave function, i.e., to multiply it by a constant factor such that its scalar product with itself becomes 1. Then, only a constant factor of modulus 1, the so-called phase, will be left undetermined in the wave function. The linear character of the wave function is called the superposition principle. The square of the modulus of the unitary scalar product (ψ, φ) of two normalized wave functions ψ and φ is called the transition probability from the state ψ into φ ; or conversely. This is supposed to give the probability that an experiment performed on a system in the state φ , to see whether or not the state is ψ , gives the result that it is ψ . If there are two or more different experiments to decide this (e.g., essentially the same experiment,

* Parts of the present paper were presented at the Pittsburgh Symposium on Group Theory and Quantum Mechanics. Cf. Bull. Amer. Math. Soc., 41, p. 306, 1935.

¹ The possibility of a future non linear character of the quantum mechanics must be admitted, of course. An indication in this direction is given by the theory of the positron, as developed by P. A. M. Dirac (Proc. Camb. Phil. Soc. 50, 150, 1934, cf. also W. Heisenberg, Zeits. f. Phys. 90, 209, 1934; 92, 623, 1934; W. Heisenberg and H. Euler, ibid. 98, 714, 1936 and R. Serber, Phys. Rev. 48, 49, 1935; 49, 545, 1936) which does not use wave functions and is a non linear theory.

² Cf. P. A. M. Dirac, *The Principles of Quantum Mechanics*, Oxford 1935, Chapters I and II; J. v. Neumann, *Mathematische Grundlagen der Quantenmechanik*, Berlin 1932, pages 19-24.

³ The wave functions represent throughout this paper states in the sense of the "Heisenberg picture," i.e. a single wave function represents the state for all past and future. On the other hand, the operator which refers to a measurement at a certain time t contains this t as a parameter. (Cf. e.g. Dirac, l.c. ref. 2, pages 115-123). One obtains the wave function $\varphi_s(t)$ of the Schrödinger picture from the wave function φ_H of the Heisenberg picture by $\varphi_s(t) = \exp(-iHt/\hbar)\varphi_H$. The operator of the Heisenberg picture is $Q(t) = \exp(iHt/\hbar)Q\exp(-iHt/\hbar)$, where Q is the operator in the Schrödinger picture which does not depend on time. Cf. also E. Schrödinger, Sitz. d. Kön. Preuss. Akad. p. 418, 1930.

The wave functions are complex quantities and the undetermined factors in them are complex also. Recently attempts have been made toward a theory with real wave functions. Cf. E. Majorana, Nuovo Cim. 14, 171, 1937 and P. A. M. Dirac, in print.

$$\int |f(s, x)|^2 d\mu(s) = \int |f(s)|^2 d\mu(s)$$

$$\Rightarrow (f, g) = (L_x f, L_x g)$$

$\Rightarrow$ unitarity of the L_x .

E. Wigner : On Unitary Representations of the Inhom. Lorentz Group
Ann. Math. 40, pp149-204 (1939)

Layout of paper:

1. Origin and characterization of the problem.
2. Comparison with previous treatments and some immediate simplifications.
 - A. previous treatments
 - B. immediate simplifications
 - C. classification of UR's according to Von Neumann and Murray (1936)
 - D. the same classification using infinitesimal operators
3. Summary of ensuing sections.
4. Description of the inhom. Lorentz group
 - A.
 - B. the theory of characteristic values and vectors of a homo LT.
 - C. decomposition of a LT into rotation and boost
 - D. the homo L group is simple.
5. Reduction of reps to 2-valued reps.
 - A.
 - B.
 - C.
 - D.
 - E.
6. Reduction of the reps of Poincare to reps of little group
 - A.
 - B.
 - C.
 - D.
7. Representations of the little groups
 - A. reps of rotation group
 - B. reps of 2-dim euclidean group
8. Representations of the extended Lorentz group
 - A.
 - B.
 - C.
 - D.

1. Origin and characterization of the problem.

1. States the invariants of matrix elements and shows that $D(L)$ form a group up to a phase factor which arises due to the arbitrary phase of wave functions in quantum mechanics.
2. The goal will be to find all the continuous unitary "ray" representations of the inhomogeneous IG. (But Wigner does not use the word ray).

2. A: Previous treatments.

1. Specifically, Dirac and Majorana did some work on the reps of Poincare group. They assumed t dependence of operators and made unnecessary math assumptions which Wigner will not have to make. They did use infinitesimals.

2.B: Some immediate simplifications

1. In searching for reps, we can ignore reps which are equivalent by similarity. Also, he notes that if a unitary rep has an invariant subspace, then the orthogonal complement is also invariant. Thus, you could decompose such a rep. He will not be interested in the reps which are just the sums of other reps (direct sum term is not used by Wigner).

2.C: classes of unitary reps.

1. First "class" of possible reps are the irreducible reps. He notes that the center of the group algebra of such a rep is multiple of identity, etc.
2. Next class of reps are the "factorials". Example is a simple multiple of an irreducible rep. Center is still identity, but you have a multiplicity. I wonder if this notion has evolved into primary factor concept?
3. There will be no "continuous" reps of Poincare, which means that any unitary rep can be broken down into factorial reps and each of these in turn can be broken down simply into irreducible reps. Continuous is a term of von Neumann which seems to perhaps be connected with direct integral concept? (Von Neumann invented the direct integral in 1938; published in 1949.)

2.D: use of infinitesimal operators.

1. Mentions that in 1932 Stone developed the notion of the generator of a one parameter subgroup and the exponentiation idea. There is some mention of pathological part of a rep, but it is then claimed that the Poincare reps have no such pathological parts.
2. Wigner has found a "new" rep of the Lorentz group which describes a particle with continuous spin. See section 7 later (precursor of regge theory?)

3. Summary of Ensuing Sections.

1. Section 4: discussion of Poincare group and theory of characteristic values (What the hell is this?)
2. Section 5: to show that that ray phase is really only $+1$ or -1 .
3. Sections 6,7,8 : will discuss the actual representations. The idea will be to use Frobenius' orbit method (not W's word). because translations form an invariant subgroup. Actually, von Neumann's unpublished "momentum vector" method is used so that the reps get classified into 4 types (timelike, lightlike, null, spacelike). In section 7 Wigner will only treat the $SU(2)$ and $E(2)$ cases. He hopes to return to the spacelike case in a future paper (but I gather than Bargmann beat him to it 8 years later).

4.A: the Poincare group (Wigner does not use word Poincare).

1. Wigner gives the def ofv what we now call the proper orthochronous Lorentz group. $\det = +1$ and time time element is positive. You move things around on the future light cone with such transformations. His convention is to perform the translation after the Lorentz rotation.

4.B: theory of characteristic values and characteristic vectors.

1. I skip this for now. By characteristic value and characteristic vector, Wigner refers to the eigenvalues and eigenvectors of the Lorentz transformation L . I.e., the eigenvalues are found from the secular equation as usual. It is not clear to me what this section has to do with anything so far.

4.C: decomposition of a homo Lorentz transformation:

1. Very simply, Wigner writes $L = RZS$ where Z is a z-boost and R and S are the pre and after rotations. The rotations are not uniquely determined but the size of the z-boost is determined. (by L).
2. Some remarks about the Lorentz group being doubly connected like the rotation group, so that there will be two-valued representations. Skip these.

4.D: the IG is simple.

1. Using the characteristic vector business as a tool, Wigner shows that any invariant subgroup of the IG must contain a rotation. But then he shows that if an invariant subgroup contains a rotation, then it contains all of the IG. In other words, he shows that the homo IG contains no non-trivial invariant subgroups. This is I presume the definition of a "simple" group.

5. Reduction of reps to 2-valued reps.

1. When you multiply two translations, you get a phase $w(a,b)$. Similarly, when you reverse the order of translation and LT, you get another phase $w(L,a)$. Finally, when you multiply two LT's you get a third phase. Wigner will cause the first two phases to be ± 1 , whereas the last phase will be ± 1 . This is a very famous sign, by the way, see Martin and Spearman.

5.A:

1. Here, Wigner shows that the various phases just mentioned can be made to be continuous functions of their two arguments (which are points in the group manifold of the Poincaré group).

5.B: Here it is shown that due to the continuity established in A, any two translations must commute with no phase factor.

5.C: And here is the argument why the phase is unity when you reverse order of a translation and LT (they do not of course commute; the LT is the same, but one of the translations has a going to La. The point is that there is no phase.

5.D: This is a huge 8 page section with the proof of the famous + and - 1 phase possibility. Very complicated, messy, and geometrical with application of all that characteristic value stuff. He breaks $L = MN$ for some reason and then exhausts all possible cases of eigenvalues.

5.E: something about showing that "renormalizing" the transformations did not hurt the continuity. Skip.

6. Reduction of Poincare reps onto little groups.

1. Wigner notes that the math rigor of this section relies on von Neumann's work which is not yet published.

6.A:

1. Here is presented the notion that also appears in Barut's book: that of writing the action of a LT on a wave function by breaking into two pieces, $d(L) = Q(L)P(L)$, where $P(L)$ only acts on 4-momentums in the wavefunction, and $Q(L)$ becomes a finite matrix which only acts on the spin indices of the wavefunction.

2. The four possible little groups are restated. However, Wigner's method of introducing the little group concept seems vague to me.

6.B: skip.

6.C: to get the UIR's of the Poincare group, you need only specify P^2 and the rep of the correct little group.

6.D: the names of the various little groups are given.

7. Representations of the Little Groups

7.A: rotation group.

1. Reps here are the well-known $D^{(j)}(R)$, says Wigner. He shows how you reduce some general rep of $SU(2)$ called $q(R)$ onto the UIR's, see Eq. (74). He does mention that you might consider a certain sign in this little group case to tell you whether you are on the future or past light cone, P_+ and P_- , but not much is made of this fact.

7.B: the euclidean group. a short piece here, theory is incomplete apparantly.

8. Representations of the "extended" Lorentz group.

A: OK, he now wants to remove the two restrictions placed earlier, but notes that the restricted Lorentz group connects continuously to the identity. Somehow the whole section is not as clear as a future discussion would be.

B,C,D: skipping these sections. just more detail on extended reps. Does not really mention time reversal and parity.

Moussa
Stora 164

- Three surveys, worked out for the explicit use of physicists, are: J. M. Jauch, CERN Seminar 5425/TH.06 (1959), part IV; G. Emch, Séminaire de l'Institut de physique théorique, Genève 1961, and appendix to ref. 5. See also the appendix written by Mackey for Segal's Book: Mathematical Problems of Relativistic Physics. Amer. Math. Soc. 1963.
22. F. Riesz et B. Sz.-Nagy, Leçons d'analyse fonctionnelle, Paris et Budapest 1955.
23. The definition of a position operator in a relativistic theory is not a trivial question, already in complex quantum mechanics. See for instance T. D. Newton and E. P. Wigner, Rev. Mod. Phys. 21, 400 (1949) (for other references, see Wightman's paper quoted below). A modern formulation of this problem was presented by A. S. Wightman, Rev. Mod. Phys. 34, 845 (1962); he uses there a mathematical object found by G. W. Mackey (see for instance Proc. N.A.S. 35, 537 (1949) and which is referred to as "system of imprimitivity for representations of groups"; the essential idea of Wightman's work is to define the position operator through its spectral family, and to impose, on this spectral family, the correct transformation laws with respect to the euclidian three-dimensional group, as they result from the considered irreducible representation of the Poincaré group. This definition of the position operator is trivially translated in quaternionic quantum mechanics.

SOME REMARKS ON THE PRODUCT OF IRREDUCIBLE REPRESENTATIONS OF THE INHOMOGENEOUS LORENTZ GROUP*

P. Moussa and R. Stora
Centre d'Etudes Nucléaires de Saclay
Service de Physique Théorique
B.P.n°2 GIF-sur-YVETTE (S & O)
- France -

INTRODUCTION

The problem of reducing the product of two irreducible representations of the inhomogeneous Lorentz group is an old one. The result is currently known and used and leads to the various phenomenological expansions of transition amplitudes: l -s type expansions, multipole expansions,¹⁾ helicity expansions. Whereas l -s type couplings have been known for a long time, at least in nonrelativistic situations, as well as multipole expansions which are usually derived from an analysis of fields rather than of particle configurations, the helicity type couplings have been discovered fairly late.

The need for defining such concepts independently from any semi-classical, or configuration space nonrelativistic argument has been recognized by several authors:

Chou Kuang Chao and M. I. Shirokov¹ have worked out practically all coupling schemes known at present. Their analysis relies however on a heavy use of the position operator² associated with irreducible representations, when it exists, in a way which is not completely clear because the frame of reference in which this operator is defined depends on the eigenvalue of the conjugate momentum which does not commute with it.

Jacob and Wick,³ on the other hand, have a formulation of the helicity coupling which requires some specification before it can be interpreted in relativistically invariant terms, specification which indeed was given in Wick's paper³ on the description of three particle states. Although the simplicity of relativistic effects is not emphasized in this paper, one result of which relies on Shirokov's analysis.

H. Joos's derivation⁴ of the coupling of massive particles,

*Presented at the LORENTZ GROUP SYMPOSIUM, Institute for Theoretical Physics, University of Colorado, Summer, 1964.

1) This name is usually reserved to situations where massless particles are involved.

Comments on Lee and Wick paper about P,C, and T.

1. The analysis of this paper is within the context of field theory. Recall that the action of P, C, and T operators on the various field operators is known in a definite way, see eg Bjorken and Drell. I think the idea is to insist that these operators do something simple to the Lagrangian operator that gives rise to the various gamma functions which act on the spinor fields. So, we have to accept the action of P C and T on the various fields as given.
2. Given these actions, Lee and Wick show several things:
 - a) when you square the PCT operator, you get $(-1)^{2J}$ and not just 1. This arises simply from the fermion difference from bosons which is of course just a minus sign. Toller will make something of this little sign.
 - b) although PCT commutes with all integral symmetry operators, these pairs only quasi-commute: (PCT,P); (PCT,T); (T,P). By quasi-commute, I mean simply that there is a factor $(-1)^J$ sitting in the commutator.
3. Remarks by me: I find this analysis interesting but not very general. In field theory the fields tend to get classified by their finite-dim Lorentz group representations rather than Poincare reps, so spin-1 is confused with vector, and so on. Somehow I cannot accept the comments of this paper as S-matrix definitions of these little commutation relations.

Obviously, the little $(-1)^{2J}$ is going to be the whole point of Toller 1967 for my purposes, so I had better learn exactly where this is coming from. Outside of field theory, ie.

Conclusions on the subject of representations of the Poincare Group as Induced Representations.

I do not fully understand this business, but now get the basic idea. The general notion of induced reps of a group G from a rep of a subgroup K is described well in Mackey 1968, along with the vector bundle interpretation. When there is only one orbit, you have a transitive G -space or homogeneous space and the Base of the bundle = $S = G/H$. The scalar product in the induced rep space is an integral over the base of scalar products in the various Hilbert Spaces which compose the bundle. The inducing rep is a rep which maps the reference fiber into itself. As Mackey points out, the bundle idea is not essential.

So we know that there are certain theorems about inducing, shown in Mackey, eg, direct summing and inducing commute; regular reps induce regular reps; and so on.

But the Poincare group is a direct product of two groups; one being commutative so we can go read Mackey's discussion of such groups. However, Mackey fails to be concrete enough on the Poincare -- even with his Appendix reference -- so I have to switch to moussa and Stora to see what is going on, ie, to see exactly how the Poincare reps are induced reps. It turns out that for the Poincare group, all the irreducible reps are induced by various subgroups, ie, are induced reps.

An obvious question which comes to mind is this: what are the subgroups K which induce reps of the full Poincare group P . The answer is that the inducing groups are essentially Wigner's old little groups (times the translation group). You then want to start with a specific representation of the inducing subgroup, which means, eg, a specific representation of the rotation group times a character of the little group vector $\vec{p}$.

The induced rep itself always looks like the regular representation. But, the catch is that it goes with a certain "covariance condition" which is simply the inducing rep, which is also a shift rep. When you combine these two ideas, you find that the induced representation of the Poincare group does look shown in equation II.5 of Moussa Stora. Ie, the action of the induced representation on a particular function of a particular group element gives you the function evaluated at some other group element times a D function of the Wigner rotation.

One thing to be kept in mind is that for the Poincare group there is more than just one orbit. You have one orbit for each value of P^2 . These are just the mass shells. So, given an orbit, I guess this means you can identify the base of a bundle with a mass shell. The induced rep scalar product will then involve an integral over the mass shell.

As shown in note on cosets, you can divide the group G into partition of cosets, and each coset is in effect labelled by the momentum p . If this momentum happens to be future timelike, then the little group vector will be a certain form and the little group will be rotation group. For each orbit (i.e. P^2 value), there will be a certain little group and a subgroup K , and as p moves on this orbit (mass shell), it labels the various cosets modulo K . So the coset space G/K gets identified with the orbit = base of a bundle.

So, the Wigner rotation business may be mathematically identified with the representation of a subgroup which induces the Poincare representation. Of course you can go to the Poincare representations without ever mentioning inducing. It's just that inducing provides the general mathematical framework. You would get the irreducible representations of other direct product groups like Galilean group in a similar fashion.

A good & quick repeat of the inducing analysis is given in book of Manfred Schaaf, which I have outlined as a book elsewhere. The stumbling block is to realize that the mass shell = orbit = coset space G/K . Actually, the scalar product does not seem to play much of a role.

To actually label the UIR's of Poincare, you specify the value of P^2 which then puts you on an orbit and determines the little group. Then in addition you specify the rep labels of the little group, whatever they may be. Thus, you have labels all the reps of Poincare.

Since one of the potential little groups is the Lorentz group, you actually have to face the messy problem of UIR's of that group. (not the finite dim IR's).

When you induce, what entities are present?

① group G and closed subgroup K .

② a representation L of K and its carrier space $\mathcal{H}(L)$.

Pause: This rep L of K is of course a s.u. rep., so there must be a homogeneous space S with invariant measure $\mu(S)$. Then the rep L will be a shift rep. on $\mathcal{H} = L^2(S, \mu)$.

③ Recall idea that $S = G/K$. This worked when only one orbit ("transitive") with origin S_0 .

④ In vector bundle language, this one orbit is the base, and $\mathcal{H}(L)$ is the reference fiber. $\mathcal{H}_{S_0} = \mathcal{H}(L)$.

⑤ In (I.1) of M-S, we see object $f(g) \in \mathcal{H}(L)$? It would seem that $S = G$! In (i) p23 M, same thing. The function $g(x)$ is not constant on cosets. So the M-space of the rep L is all of G :

for L :
 K = group represented
 $S = G$; the M-space;
 $\mathcal{H}(L)$ = rep. space for this rep.

⑥ What is the scalar product in $\mathcal{H}(L)$? If $g(x) \in \mathcal{H}(L)$, what is $\|g(x)\|$? You do not integrate over x . Perhaps just a finite component sum? We do know that the scalar product $\langle g(x) | g(x) \rangle$ will be constant on K cosets because: L is unitary.

⑦ Then integrate over $G/K = S$ to get induced rep s.p.

* cosets:

(11.3) is a statement made for each p on orbit of p_0 .

$$g = \tilde{g} \cdot k \quad \text{so} \quad g \in \tilde{g}K, \quad g \in \text{a certain coset.}$$

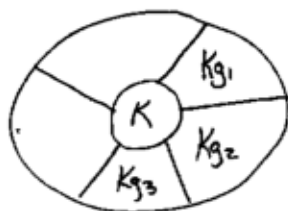
As p varies, $\tilde{g} = \{0, 1\}$ varies so we may regard p as labelling the various cosets of G . There happens to be a continuum of such cosets which partition the group.

So somehow identify p and $d\mu(p)$ with coset space.

About Induced Reps. and Cosets

- ① The usual induced rep says: $[U(g)f](g_1) = f(\tilde{g}^{-1}g_1)$. But there is also a covariance condition which says: $[\tilde{U}(k)f](g) = f(gk)$. Now,
- ② Any element $g \in G$ lies in a coset modulo K . I.e., given g , you can find some $\tilde{g}$ so that $g \in \tilde{g}K$, i.e., $g = \tilde{g}k$. But what freedom have we to choose $\tilde{g}$ and K ? How do the cosets fit into G ?

$Kg =$ right coset if $g \notin K$.



Even if K is not invariant, the cosets partition the group as shown.

So, every element $g \in G$ lies in some coset.

- ③ Choose $\tilde{g} = \{0, \Lambda_p\}$. Which elements of G may be written $g = \tilde{g}k$ where $k \in K$?
- $k = \{a, R_p\}$

As (II.3) shows, once you have chosen $\tilde{p}$ and Λ_p , then any $g = \tilde{g}k$.

- ④ So, let $R_p =$ invariance group of $\tilde{p}$. Then

$\{a, R_p\} =$ subgroup K .

$$\text{So: } [U(g)f](g_1) = f(\tilde{g}^{-1}g_1) = f(g_2) = f(\tilde{g}k) = [\tilde{U}(k)f](\tilde{g})$$

$$[U(g)f](\tilde{g}) = f(\tilde{g}^{-1}\tilde{g}) = f(g_2) = f(\tilde{g}k) = [\tilde{U}(k)f](\tilde{g})$$

$$g_2 = \tilde{g}^{-1}\tilde{g} = \tilde{g}k \quad \text{so} \quad k = (\tilde{g}^{-1})^{-1}\tilde{g}^{-1}\tilde{g}$$

$$\Rightarrow [U(g)f](\tilde{g}) = [\tilde{U}(\underbrace{\tilde{g}^{-1}\tilde{g}^{-1}\tilde{g}}_{=k})f](\tilde{g})$$

Specifically, what is this k when $\tilde{g} = \{0, \Lambda_p\}$. $\tilde{g}^{-1} = \{0, \Lambda_p^{-1}\}$

$$(\Lambda_p)^{-1} \text{ is well-defined. So } k = \tilde{g}^{-1}\tilde{g}^{-1}\tilde{g} = \{0, \Lambda_p^{-1}\}\{0, \Lambda_p^{-1}\}\{0, \Lambda_p\}$$

$$= \{0, \Lambda_p^{-1}\} \cdot \{-\Lambda_p^{-1}a + \Lambda_p^{-1}\phi, \Lambda_p^{-1}\Lambda_p\} = \{0, \Lambda_p^{-1}\}\{-\Lambda_p^{-1}a, \Lambda_p^{-1}\Lambda_p\} = \{-\Lambda_p^{-1}\Lambda_p^{-1}a, \Lambda_p^{-1}\Lambda_p^{-1}\Lambda_p\}$$

Example: Poincaré Group Induced Representations

$$G = T_4 \otimes L$$
$$K = T_4 \otimes R$$

$$f(g) \overleftarrow{L(k)} = f(gk)$$

$f(g)$ = vector in $\mathcal{H}(L)$ with $(2j+1)$ components.

$$L(k) = e^{-i\vec{p}\vec{a}'} \overleftarrow{D(R\vec{p})} \quad \text{where} \quad k = \{a', R\vec{p}\}$$

$f(g)$ still defined on $S = \text{all of } G$.

The Meaning of Equation (II.3).

- ① The Λ appearing in II.3 is not general, but must be some L.T. which maps $\tilde{p}$ into the vector p . Only then will $\Lambda^{-1}\tilde{p} \in R_{\tilde{p}}$.
- ② However, for any Λ there will exist a p such that $p = \Lambda\tilde{p}$. Example: suppose $\Lambda = \Lambda^{-1}\Lambda p$. Then $p = \Lambda^{-1}\Lambda p \tilde{p} = \Lambda^{-1}p$. So now look down at (II.5):

$$\{ -\Lambda^{-1}a, \Lambda^{-1}\Lambda p \} = \{ 0, \Lambda^{-1}p \} \cdot \{ -\Lambda^{-1}\Lambda^{-1}a, R_{\tilde{p}}(\Lambda^{-1}\Lambda p) \}$$

$$\text{where } R_{\tilde{p}} = \Lambda_{\Lambda p}^{-1} \cdot (\Lambda^{-1} \cdot \Lambda p). \text{ I.e.,}$$

$$(\Lambda^{-1}\Lambda p) \tilde{p} = (\Lambda^{-1}p)$$

$$\text{But } (\Lambda_{\Lambda p}^{-1}) \cdot (\Lambda^{-1}p) \propto \tilde{p}.$$

$$\Lambda_{\tilde{p}}^{-1} = \text{some standard form.}$$

$$\Lambda_{\Lambda p}^{-1} \text{ has general } \Lambda \text{ object.}$$

$$\textcircled{2} \text{ In older notation: } \tilde{g} = (0, \Lambda p)$$

$$[U(g)f](\tilde{g}) = f(\tilde{g}^{-1}\tilde{g}) = f(g_2) = f(\tilde{g}'k) = [\tilde{U}(k)f](\tilde{g}')$$

$$g = \tilde{g}'k, \text{ says (I.3)}$$

Every g is $= \tilde{g}'k$, but not any $\tilde{g}'$ will work. We'll contain Λp where $p = \Lambda\tilde{p}$.

So, $g_2 = \tilde{g}'k$ where $\tilde{g}' = \{0, \Lambda_{\Lambda p}\}$ as shown in (II.5).

MISC.

Reduction of Product of two UIR's of Poincaré Group.The Book of Manfred Schaaf

Lecture Notes in Physics #5, Springer Verlag 1970 (completed March 1969)

1. About the series: in 1970 there were only 5 volumes in this series. Of course the math series had over 100 volumes at that time. Not sure what has become of these Springer-Verlag babies. Recall that Collins and Squires in #45, so there must be a lot in between? Or is that still anteqhr series?

2. About the book: 120 pages absolutely crammed with equations, nearly incomprehensible. The author Schaaf works in the German tradition of fine detail. He was at the University of Munich Dept of Physics.

Contents:

- 1) first, we get a quick review of the UIR's of the Poincare via Mackeys inducing technique. No details, but ≈ 2 references[†] are given for more information on this (look em up). Then right away the representations of all four little groups are discussed, including $SL(2, C)$ for the null little group vector. Nothing is skipped here. This Chapter 1 is 24 pages.
- 2) in Chapter 2 Schaaf derives the matrix elements $SU(2)$, $SU(1,1)$ and $E(2)$ all in the usual discrete basis which he calls H_1 . He also does $SU(1,1)$ in the continuous basis ^{H_1} which fills 16 large pages of calculation. The answer seems to appear on p 49 and is of course a stupendous and useless mess. Then comes a section on the orthogonality and completeness of the various representations (expansion theorems). Chapter 2 is 50 pages.
- 3) Finally, this Chapter 3 discusses how you explicitly do the reduction of the direct product of two Poincare representations. Schaaf has completed the efforts of Moussa and Stora by doing certain extra cases (for example, when both reps are spacelike, or one is null, etc). Clebsh's are given in ghory detail. This chapter is also 50 pages.

† EMCH in Lec. Theo. Phys. VIII A, Boulder 1965 (just before Moussa-Stora)

† Guillot & Petit, Helv. Phys. Acta 39, 281 (1966)

1964

ΣmCH , Bldr

Emch., Section from his paper at the Boulder Conference (same as Moussa Stora)

He gives a little concise and good presentation of the Poincare UIR's as induced representations. The Radon deal is included, and he refers to a theorem as Mackey-Frobenius, which says that you get all UIR's of a semi-direct product group in this way.

Interestingly, this paper contains some new references on the subject:

- 1) two papers by Emch in Helvetia.... 1963
- 2) Jauch 1959 reference
- 3) Emch at another conference. 1961

The fiber bundle idea is not mentionned, but Mackye's 1955 notes are advertised as being didactic and complete, so good I have them on hold, maybe will copy some sections of same.

$$\left. \begin{aligned} \underline{I}f_+ &= if_+; \quad \underline{I}f'_+ = if'_+ \\ f &= f_+ + jf'_+ \end{aligned} \right\} \quad (5.5)$$

which suffices to prove the completeness of $\mathcal{H}_C^{(i)}$ in $\mathcal{H}_Q$. Carried out from any imaginary unit quaternion i the whole above construction gives rise to the family asserted in the theorem.

Theorem 5.1 suggests the following definitions: Let $\underline{t}$ be a linear mapping of $\mathcal{H}_Q$ onto itself. If $\underline{t}$ commutes with a unitary, anti-hermitian $\underline{I}$, it maps each of the corresponding $\mathcal{H}_C^{(i)}$ onto itself and one can therefore consider the restriction $\underline{t}^{(i)}$ of $\underline{t}$ to $\mathcal{H}_C^{(i)}$. Conversely, if $\underline{t}^{(i)}$ is a linear mapping of one of the $\mathcal{H}_C^{(i)}$ onto itself one can define a unique linear continuation $\underline{t}$ of $\underline{t}^{(i)}$ in $\mathcal{H}_Q$ by the conditions

- (i) $\underline{t}$ is linear
(ii) $\underline{t}f = \underline{t}^{(i)}f \quad \forall f \in \mathcal{H}_C^{(i)}$.

For instance, $i\underline{I}^{(i)}$ is the restriction of $\underline{I}$ to $\mathcal{H}_C^{(i)}$ and $\underline{I}$ is the linear continuation of $i\underline{I}^{(i)}$ in $\mathcal{H}_Q$. The following properties are attached to these concepts:

- (i) Let $\underline{t}$ be a linear mapping of $\mathcal{H}_Q$ which commutes with a given $\underline{I}$; if $\underline{t}$ is unitary, hermitian, or is a projector, so are also all its restrictions with respect to $\underline{I}$.
(ii) Let $\underline{t}$ and $\underline{s}$ be two linear mappings which commute among themselves and with a given $\underline{I}$; then $\underline{t}^{(i)}$ and $\underline{s}^{(i)}$ also commute in the space $\mathcal{H}_C^{(i)}$ where they are simultaneously defined.
(iii) Let $\{\underline{t}_n\}$ a sequence of commuting linear mappings with a strong limit $\underline{t}$ if every $\underline{t}_n$ commutes with a same given $\underline{I}$, then $\underline{t}^{(i)} = s \lim \underline{t}_n^{(i)}$.

The following theorem is now meaningful:

Theorem 5.2: Let $\underline{u}$ be an unitary, strongly continuous representation of a topological group $\underline{G}$, acting on the vectors of a quaternionic Hilbert space $\mathcal{H}_Q$. If each of the $\underline{u}_x$ (with x running over $\underline{G}$) commutes with the same unitary, anti-hermitian $\underline{I}$, then

- (i) for a given i , the set

$$\{\underline{u}_x^{(i)} \mid x \in \underline{G}\} \equiv \underline{u}^{(i)}$$

is an unitary, strongly continuous representation of $\underline{G}$, acting on the vectors of $\mathcal{H}_C^{(i)}$.

- (ii) If $\underline{u}$ is irreducible, so is also $\underline{u}^{(i)}$ (for every i)

- (iii) if $\underline{u}^{(i)}$ is an irreducible representation, of class +1 or 0 in the sense of Frobenius and Schur,¹⁸ $\underline{u}$ is also irreducible.

Assertions (i) and (ii) are easily proved from the given properties of restrictions and linear continuations. The proof of the third assertion is more involved and can be found in my paper.⁵

The last step will now be to show that an unique operator $\underline{I}$ is associated with every physically meaningful representation of the Poincaré group. This will justify the mathematical investment performed in this subsection, and will be the matter of my last lecture. Since I shall have to use properties of the complex representations of the Poincaré group, the next lecture will be devoted to a survey of this theory.

FOURTH LECTURE

6. Classification of the Irreducible Complex Representations of the Poincaré Group.

The aim of this paragraph is to recall rapidly the essential features of the classification of the irreducible representations of the Poincaré group in complex quantum mechanics. This problem was completely solved by the famous paper by Wigner¹⁹ and more recently by Bargmann, Wightman and Wigner.²⁰ Most of the arguments presented in the original papers are in fact particular (and particularly explicit!) expressions of general theorems stated and proved by Mackey in a series of mathematical papers.²¹ The use of the Mackey-Frobenius' method allows a very quick survey of the problem and this is precisely what I want to present to you in this lecture.

For sake of general agreement on notation and nomenclature, let me first redefine the different groups we will deal with.

The Poincaré group is the group of all the linear transformations which leave invariant the distance in Minkowski-space; the metric of this space will be chosen as $g_{00} = +1$, $g_{ii} = -1$ for $i = 1, 2, 3$, $g_{\mu\nu} = 0$ for $\mu \neq \nu$ (with $\mu, \nu = 0, 1, 2, 3$). (This group, which we will denote by $\underline{G}$ is also named: the inhomogeneous extended Lorentz group).

The proper Poincaré group (or inhomogeneous Lorentz group) is the connected component $\underline{G}_e$ of $\underline{G}$. Every element of $\underline{G}_e$ is the product of a translation by a real vector $\underline{b}$ (with component b^μ) and of a real homogeneous transformation $\underline{\Lambda}$ which does not contain any reflexion. (The translation is done after the homogeneous transformation). Every element of $\underline{G}_e$ can therefore be written as the pair $(\underline{b}, \underline{\Lambda})$ with the composition law:

$$\left. \begin{aligned} (\underline{b}, \underline{\Lambda}) (\underline{b}', \underline{\Lambda}') &= (\underline{b}'', \underline{\Lambda}'') \text{ where} \\ \underline{b}'' &= \underline{b} + \underline{\Lambda} \underline{b}' \quad \text{and} \quad \underline{\Lambda}'' = \underline{\Lambda} \underline{\Lambda}' \end{aligned} \right\} \quad (6.1)$$

($\underline{\Lambda} \underline{b}'$ is the translation by the transform through $\underline{\Lambda}$ of the vector $\underline{b}'$).

Mathematically the structure of $\underline{G}_e$ can be described as follows: the translation group $\underline{T}^4$ and the homogeneous group $\underline{H}^4$ are locally compact. $\underline{T}^4$ is moreover abelian. To each element $\underline{\Lambda}$ of $\underline{H}^4$

corresponds an automorphism of T^4 in such a way that this correspondence is an homomorphism of H^4 into the group of the automorphism of T^4 . Let us now denote by $\Lambda[\dots]$ the automorphism associated to $\underline{\Lambda}$ and consider all the pairs $(\underline{b}, \underline{\Lambda})$ with $\underline{b}$ and $\underline{\Lambda}$ running respectively through T^4 and H^4 . On this set one defines a composition law:

$$(\underline{b}, \underline{\Lambda}) (\underline{b}', \underline{\Lambda}') = (\underline{b} + \underline{\Lambda}[\underline{b}'], \underline{\Lambda}\underline{\Lambda}'). \quad (6.2)$$

Equipped moreover with the topology of the direct product of the topological space T^4 and H^4 , this set is a locally compact topological group: $\underline{G}_e \cdot T^4$ is isomorphic to the closed invariant subgroup of $\underline{G}_e$, consisting of all the elements of the form $(\underline{b}, \underline{1})$, and H^4 is isomorphic to the closed subgroup of all the elements of the form $(\underline{0}, \underline{\Lambda})$. This situation is summarized by saying that $\underline{G}_e$ is the semi-direct product of T^4 by H^4 ; as we shall see, this structure is essential for the theory of the representations. My last comment will be that T^4 is connected and simply connected whereas H^4 is connected and doubly connected. The universal covering group of H^4 is the group of all two-by-two complex, unimodular matrices, and is denoted by $SL(2, \mathbb{C})$.

a) Classification of the irreducible representations of the proper Poincaré group (in complex Hilbert spaces). The beginning of the subsection is a survey of Mackey's results, with the following aim: to extract from the very elegant and complete treatment given by Mackey the only tools which are of interest for most of the physicists.

The measure-theoretical trimmings are certainly essential as soon as one is dealing with topological groups; however, the principal features of the theory of induced representations are not (at least for our purpose) to be searched there: one can understand the principle of the method with discrete groups also, ignoring there the topological aspects of the theory. (In fact, induced representations were already studied by Frobenius in this particularly simple case.)

Let $\underline{G}$ be a separable, locally compact topological group, $\underline{H}$ a closed subgroup of $\underline{G}$, $\underline{L}$ a unitary, strongly continuous representation of $\underline{H}$ acting on the vectors of a complex Hilbert space $\mathcal{H}(\underline{L})$, μ a quasi-invariant Borel measure on $\underline{G}/\underline{H}$, $\mathcal{M}(\underline{U})$ the set of all the mappings f of $\underline{G}$ into $\mathcal{H}(\underline{L})$ which satisfy the following conditions:

- (i) $(f(\underline{x}), \varnothing)$ is a Borel function in $\underline{x}$ for every $\varnothing$ in $\mathcal{H}(\underline{L})$
- (ii) $f(\underline{h}\underline{x}) = \underline{L}_{\underline{h}} f(\underline{x})$ for all $\underline{x}$ in $\underline{G}$ and all $\underline{h}$ in $\underline{H}$.

$$(iii) \int_{\underline{G}/\underline{H}} \|f(\underline{x})\|^2 d\mu(\underline{s}) < \infty. \quad (6.3)$$

One observes that expression (iii) has a meaning only if condition (ii) is previously imposed; the reason is that condition (ii) has, as a consequence, that $\|f(\underline{x})\|$ is constant over cosets with respect to $\underline{H}$; this remains true for the function $(f(\underline{x}), g(\underline{x}))$; one can thus define a scalar product in $\mathcal{M}(\underline{U})$ which becomes now a complex Hilbert space:

$$(f, g) = \int_{\underline{G}/\underline{H}} (f(\underline{x}), g(\underline{x})) d\mu(\underline{s}). \quad (6.4)$$

We are now prepared to define the induced representation $\mu_{\underline{U}}^{\underline{L}}$ of $\underline{G}$: for all f in $\mathcal{M}(\underline{U})$, every pair $(\underline{x}, \underline{y})$ of elements of $\underline{G}$, consider the function:

$$(\mu_{\underline{U}}^{\underline{L}} f)(\underline{y}) = f(\underline{y}\underline{x}) \sqrt{\rho_{\underline{x}}(\underline{y})} \quad (6.5)$$

where $\rho_{\underline{x}}$ denotes the Radon-Nikodym derivative $(d\mu_{\underline{x}}/d\mu)$; for our purpose it is sufficient to know that $\rho_{\underline{x}}(\underline{s})$ is a Borel function on $\underline{G} \times \underline{G}/\underline{H}$

$$\left. \begin{aligned} \rho_{\underline{x}\underline{y}}(\underline{s}) &= \rho_{\underline{x}}(\underline{s}) \rho_{\underline{y}}(\underline{s}\underline{x}) \\ \text{for every } \underline{x}, \underline{y} \text{ in } \underline{G}, \text{ every } \underline{s} \text{ in } \underline{G}/\underline{H} \end{aligned} \right\} \quad (6.6)$$

It is quite easy to verify first that $(\mu_{\underline{U}}^{\underline{L}} f)$ is an element of $\mathcal{M}(\underline{U})$. The interesting feature is that the correspondence $\mu_{\underline{U}}^{\underline{L}} : f \rightarrow (\mu_{\underline{U}}^{\underline{L}} f)$ is a unitary, strongly continuous representation of $\underline{G}$, acting in the complex Hilbert space $\mathcal{M}(\underline{U})$. This representation is said to be induced by the representation $\underline{L}$ of the subgroup $\underline{H}$ of $\underline{G}$.

Mackey furthermore proved that the index μ of $\mu_{\underline{U}}^{\underline{L}}$ is superfluous for our purpose: the condition imposed to μ to be quasi-invariant is very strong and in fact determines μ up to an equivalence; now it appears that two representations $\mu_{\underline{U}}^{\underline{L}}$ and $\nu_{\underline{U}}^{\underline{L}}$ which differ only by equivalent measures are unitary equivalent. We shall therefore omit the index μ in the following.

In order to state Mackey's theorem in a form accessible to non-mathematicians, let us indicate some few further definitions.

Let $\underline{G}$ be a semi-direct product of two topological groups $\underline{T}$ and $\underline{H}$; suppose that $\underline{T}$ is moreover abelian. For every character ϑ on $\underline{T}$ (in the sense of Pontrjagin⁷ for instance) let us define the function $\chi = e^{i\vartheta}$ which we will also call (abusively!) a character. Two characters χ and χ' are said to be equivalent with respect to $\underline{H}$ if there

algebra

exists an element $\underline{\Lambda}$ in $\underline{H}$ such that

$$\left. \begin{aligned} x'(\underline{b}) &= x(\underline{\Lambda}[\underline{b}]) \\ \text{for every element } \underline{b} \text{ in } \underline{T} \end{aligned} \right\} \quad (6.7)$$

and we will then write for this:

$$x' = [x]\underline{\Lambda}. \quad (6.8)$$

A class of characters which are equivalent in the above sense is said to be an orbit under $\underline{H}$; the little group $\underline{H}_x$ of a character x is defined as:

$$\underline{H}_x = \{\underline{\Lambda} \in \underline{H} \mid x = [x]\underline{\Lambda}\}. \quad (6.9)$$

Let now x be any character on $\underline{T}$, $\underline{M}$ an irreducible, unitary, continuous representation of $\underline{H}_x$; then the set:

$$\underline{L}^{x, \underline{M}} = \left\{ \underline{L}_{\underline{b}, \underline{\Lambda}}^{x, \underline{M}} = x(\underline{b}) \underline{M}_{\underline{\Lambda}} \mid \underline{b} \in \underline{T}; \underline{\Lambda} \in \underline{H}_x \right\} \quad (6.10)$$

is an irreducible, unitary, continuous representation of the subgroup of $\underline{G}$, defined as the semi-direct product of $\underline{T}$ by $\underline{H}_x$.

Now comes our theorem!

This theorem is valid as soon as a "regularity" condition on the semi-direct product is satisfied (and it turns out that this is the case for the proper Poincaré group!). The "representations" which appear in the theorem are all unitary and strongly continuous representations.

Mackey-Frobenius theorem: Let $\underline{G}$ a topological group obtained as semi-direct product of two separable, locally compact group $\underline{T}$ and $\underline{H}$, where $\underline{T}$ is moreover abelian.

(i) If x is a character on $\underline{T}$, $\underline{M}$ an irreducible representation of the little group $\underline{H}_x$ of x , then the representation induced by the representation $\underline{L}^{x, \underline{M}}$ of the semi-direct of $\underline{T}$ by $\underline{H}_x$ is irreducible.

(ii) Every irreducible representation of $\underline{G}$ can be obtained with construction (i).

(iii) Two irreducible representations of $\underline{G}$, respectively induced by $\underline{L}^{x, \underline{M}}$ and $\underline{L}^{x', \underline{M}'}$ are equivalent if and only if x and x' belong to the same orbit and if $\underline{M}$ and $\underline{M}'$ can be transformed into equivalent representations.

This theorem provides a program for finding all the equivalence classes of irreducible, unitary, strongly continuous representations of a group $\underline{G}$ satisfying the hypothesis of the theorem (including that of regularity which is not reported here):

(i) Determine the characters on $\underline{T}$

- (ii) Determine the orbits under $\underline{H}$
- (iii) Choose arbitrarily a character in each orbit
- (iv) Determine the little group of each chosen character
- (v) Determine the equivalence classes of the irreducible representations for the above little groups.

Let us now apply this program to the proper Poincaré group: $\underline{T}$ is now $\underline{T}^4$; for $\underline{H}$, we will consider, instead of $\underline{H}^4$, its universal covering group $SL(2, \mathbb{C})$ (this because of the well-known result of the analysis of the phase reduction, ref. 19 and 12). Furthermore we will restrict our attention to representations which lead to positive energy and discrete spin.

(i) The characters on $\underline{T}^4$ are of the form:

$$x(\underline{b}) = e^{ip^\mu \underline{b}_\mu}$$

where $\underline{b}$ is the four-vector of the considered translation; $\underline{p}$ is a four-vector characteristic of x .

(ii) Each orbit is characterized by $m^2 = \underline{p}^\mu \underline{p}_\mu$, and for those corresponding to positive m^2 , by the sign of the zero-component p^0 of $\underline{p}$.

(iii) If $m^2 > 0$ and $p^0 > 0$, it is convenient to choose for each orbit the character $\underline{p} = m(1, 0, 0, 0)$.

If $m^2 = 0$, and $\underline{p} \neq (0, 0, 0, 0)$, we will choose $\underline{p} = (1, 0, 0, 1)$.

(iv) The little group of $(1, 0, 0, 0)$ is $SU(2, \mathbb{C})$; it is also quite easy to see that the little group of $(1, 0, 0, 1)$ is isomorphic to E_2 , the proper euclidian two-dimensional group.

(v) The representation of $SU(2, \mathbb{C})$ are well-known; they are characterized by a real parameter $s = 1/2, 1, 3/2, \dots$.

For E_2 , one observes that this group also satisfies the hypothesis of Mackey-Frobenius' theorem; it is then an amusing game to apply to it the above program; one finds that two kinds of representations appear, those with continuous spin which do not concern us, and those with discrete, integer or half-integer spin.

This is sufficient for our purpose to classify all the unitary, continuous, irreducible representations of the proper Poincaré group. (Details of the above discussion are given in papers by Jauch and myself;²¹ it is also interesting to compare it with Wigner's pioneer work, ref. 19.)

For the application to quaternionic quantum mechanics (see Theorem 5.2 iii) it is interesting to observe that none of the considered representations of the proper Poincaré group is equivalent to its complex conjugate, since (for the representation of physical interest) $\underline{p}$ and $-\underline{p}$ never belong to the same orbit. All these representations are therefore of "class zero" in the sense of Frobenius and Schur.

b) Classification of the irreducible representations of the Poincaré group (in complex Hilbert spaces). The Mackey-Frobenius' criteria are also very useful when one enlarges the proper Poincaré group by space- and time-reflexions. This was done with some details in the appendix of the second of my papers ⁽⁵⁾ and provides a short alternative approach to the so-called theory of types due to Bargmann, Wightman and Wigner. ⁽²⁰⁾ Since my main purpose is, after all, to speak on quaternionic quantum mechanics, I shall not insist on this point here. The only thing I would, however, like to mention is the following: when one introduces discrete symmetries, one has to take into account the existence of eventual superselection rules.

In particular, this is obviously so if one associates charge-conjugation to space-time reflexion. The following results can be proved very easily ⁽⁵⁾ with the help of Mackey-Frobenius' criteria: When the proper Poincaré group is enlarged with TCP, the elementary systems with respect to this new symmetry-group possess

- P-symmetry if the mass is non-zero, whereas this is not the case if the mass is zero (unless the spin is also zero).
- CP- and T-symmetries, and one has $T^2 = (-1)^{2s}$ for systems with non-zero mass, and $T^2 = +1$ when the mass is zero.

(These results apply only to systems for which the charge-conjugation C effectively passes from one coherent subspace to another, i.e., if at least one of the charges is non-zero; when all the charges are zero, the ordinary theory of types apply.)

FIFTH LECTURE

7. Special Relativity in Quaternionic Quantum Mechanics.

The aim of this last lecture is to inject special relativity into the frame of quaternionic quantum mechanics; this will be done through the representations of the Poincaré group. As in the complex case, one can proceed in three steps corresponding respectively to the introduction of the translation symmetry-group, of the proper Poincaré symmetry-group and of the Poincaré symmetry-group.

As we observed in the beginning of the fourth lecture, the proper Poincaré group $\underline{G}_E$ is connected; by Theorem 2.2, we can assert that every irreducible description with respect to $\underline{G}_E$ is "coherent," we shall therefore have no problem with superselection rules (as in the complex case!) at this stage. $\underline{G}_E$ is moreover doubly connected. From Theorem 4.4 we know that each quaternionic description with respect to $\underline{G}_E$ can be made by a strongly continuous, unitary, true or two-valued representation of $\underline{G}_E$, acting on the vectors of a quaternionic Hilbert space. The structure of $\underline{G}_E$ as a semi-direct product (see fourth lecture again) of $\underline{T}^4$ by $\underline{H}^4$, where $\underline{T}^4$ is simply connected (whereas $\underline{H}^4$ is doubly connected), allows to

assert that, from a given projective representation $\underline{U}$ of $\underline{G}_E$, it is always possible to construct a (true or two-valued) representation of $\underline{G}_E$ such that its restriction to $\underline{T}^4$ is a true representation of this subgroup. $\underline{T}^4$ is moreover abelian; one has therefore to study the representations of abelian groups in quaternionic Hilbert spaces.

A construction, analogous to that which led to the quaternionic corollary of Schur's lemma, and which uses only the abelian property of the group (without any reference to its topology), allows to prove the following fundamental lemma.

Lemma 7.1: Let $\underline{G}$ be an abelian group and $\underline{u}$ a unitary representation of $\underline{G}$, acting on the vectors of a quaternionic Hilbert space $\mathcal{H}_Q$. Then, there exists at least one unitary anti-hermitian operator on $\mathcal{H}_Q$ which commutes with every elements $\underline{u}_x$ of the representation $\underline{u}$. (Moreover, if there is no vector f in $\mathcal{H}_Q$, left invariant by the representation $\underline{u}$, the operator $\underline{J}$, constructed for the proof of the above lemma commutes with every operator commuting with each $\underline{u}_x$ of the representation.)

The introduction of $\underline{J}$ is quite essential here and one can now apply our Theorem 5.2 to the translation group $\underline{T}^4$. By the use of the procedures of restriction and linear continuation described in Section 5b, one can easily translate to the quaternionic case the complex SNAG-theorem (for a statement of this theorem see for instance ref. 22, pp. 387 ff.).

Theorem 7.1: Let $\underline{G}$ be an abelian, locally compact, connected and simply connected group, and $\underline{u}$ a unitary, continuous representation of $\underline{G}$, acting on the vectors of a quaternionic Hilbert space $\mathcal{H}_Q$. Then there exists at least one unitary anti-hermitian operator on $\mathcal{H}_Q$ which commutes with every $\underline{u}_x$ of the representation. To this $\underline{J}$ corresponds an unique spectral family $\{\underline{a}(S)\}$ on the character group Γ of $\underline{G}$ such that

$$(i) \quad \underline{u}_x = \int_{\Gamma} e^{J\theta(x)} d\underline{a}(S) \quad \text{for all } x$$

where $\theta(x)$ is a character on $\underline{G}$.

(ii) $\underline{a}(S)$ commutes with every operator commuting with each $\underline{u}_x$. This theorem applies in particular to our translation group $\underline{T}^4$ (which is by the way isomorphic to its character group). One can therefore write any unitary, continuous representation of $\underline{T}^4$ in the form

$$\underline{u}_{\underline{p}} = \int_{\Gamma} e^{J\underline{p} \cdot \underline{b}_{\underline{p}}} d\underline{a}(S) \quad (7.1)$$

The usual methods of functional analysis lead to the definition of hermitian operators $\underline{P}^{\mu}$ such that

$$\underline{u}_{\underline{p}} = e^{J\underline{P}^{\mu} \underline{b}_{\underline{p}}} \quad (7.2)$$

extension through space-reflexion and its extension through space-time reflexion.

This one-to-one correspondence was established with the help of a unitary, anti-hermitian operator J . This J commutes with all the space-symmetries not related to time-reversal, whereas it anti-commutes with the time-reversal operator. It therefore plays a role quite analogous to that of the ordinary imaginary unit, or of Stueckelberg's J . This analogy can even be pursued further: let us call charges, hermitian operators which are left invariant through all the transformations of the proper Poincaré group, and such that they commute with all the components of the four-momentum operator. One can easily verify that these physical postulates have as a mathematical consequence that all the charges so-defined commute with J , and can therefore be completely determined by their restrictions (see Section 5b) to complex Hilbert spaces, as it is already the case for all the generators of the proper Poincaré group. Phenomenologically, it does not seem to exist observables which are not expressed by means of the charges, the generators of the proper Poincaré group and eventually of position;²³ if this is effectively so, all that we said above on the charges and on the generators of the proper Poincaré group, applies generally to any observable. This would be a good reason for the understanding of the importance of complex Hilbert spaces in the description of physics.

A temptation still exists, however, to investigate further the structure of quaternionic quantum mechanics, and I will formulate it in the form of a question: "To what extent isospin is it related to the continuous family of complex-Hilbert spaces that can be extracted from a quaternionic Hilbert space by means of an unitary, anti-hermitian J ?"

References

1. C. Piron, thèse, Lausanne 1964.
2. G. Birkhoff and J. von Neumann, Ann. Math. **37**, 823 (1936).
3. E. C. G. Stueckelberg et al., Helv. Phys. Acta **33**, 727 (1960); **34**, 621 (1961); **34**, 675 (1961); **35**, 673 (1962).
4. D. Finkelstein, J. M. Jauch and D. Speiser, CERN 59-7, 59-9; 59-17; Journ. Math. Phys. **4**, 136 (1963). D. Finkelstein, J. M. Jauch, S. Schiminovich and D. Speiser, Journ. Math. Phys. **3**, 207 (1962); **4**, 788 (1963).
5. G. Emch, Helv. Phys. Acta **36**, 739 (1963); **36**, 770 (1963).
6. G. Emch and C. Piron, Journ. Math. Phys. **4**, 469 (1963).
7. L. Pontrjagin, Topological Groups, Princeton 1958.
8. M. H. Stone, Trans. Am. Math. Soc. **40**, 37 (1936), see also: L. H. Loomis, Bull. Am. Math. Soc. **53**, 757 (1947).
9. A pedagogical textbook on projective geometry is that of E. Artin, Geometric Algebra, New York 1957. In short, a projective

geometry is a lattice which is complete, modular, complemented, atomic and satisfies $\wedge$ -continuity; modularity means that if $x \leq z$, then for every y , one has $x \vee (y \wedge z) = (x \vee y) \wedge z$. The above definition of projective geometry is presented in the language of lattice theory in M. L. Dubreil-Jacotin, L. Lesieur and R. Croisot, Leçons sur la théorie des treillis..., Paris 1953.

10. We make here a clear distinction between (closed) subspaces and linear manifolds. A set $\mathcal{M}$ of vectors is called a linear manifold if it contains all the linear combinations of its elements; this concept is therefore purely algebraic. The concept of subspace needs a topology: a linear manifold is called a (closed) subspace if it is moreover closed with respect to strong convergence. The closure property can be expressed by means of orthocomplementation: a linear manifold is a subspace if and only if it satisfies $\mathcal{M} = (\mathcal{M}^\perp)^\perp$.

11. E. P. Wigner, Group theory..., New York 1959.
12. V. Bargmann, Ann. Math. **54**, 1 (1954).
13. J. M. Levy-Leblond, Journ. Math. Phys. **4**, 776 (1963).
14. J. M. Jauch, Gauge-Invariance as a consequence of Galilei-Invariance, preprint 1964.
15. N. Bourbaki, Eléments de Mathématiques, Groupes et algèbres de Lie Fasc. XXVI, Paris 1960.
16. L. Michel, Lecture notes of the Istanbul Summer School 1962.
17. G. W. Mackey, Acta Mathematica **99**, 265 (1958).
18. Let $\mathcal{H}_C$ be a complex Hilbert space, K a conjugation (i.e., an antiunitary, involutive mapping) of $\mathcal{H}_C$ onto itself, and u a unitary representation of a group G . Let us now define the representation $\tilde{u} = \{\tilde{u}_x = K u_x K | x \in G\}$. Then three cases and only three can occur.

(i) $\tilde{u}$ is equivalent to u and the unitary operator C (defined by $\tilde{u}_x = C^{-1} u_x C$) satisfies $CKCK = I$.

(ii) $\tilde{u}$ is equivalent to u , with $CKCK = -I$.

(iii) $\tilde{u}$ is not equivalent to u .

u is then said to be respectively of class (+1), (-1) or zero.

For details, see ref. 11, p. 285. Note that this classification does not depend on a special choice of the conjugation K .

19. E. P. Wigner, Ann. of Math. **40**, 149 (1939).
20. V. Bargmann, A. S. Wightman and E. P. Wigner, Princeton preprint; see also: A. S. Wightman, Ecole des Houches, Session de 1960, pp. 181-194 and N. C. Suppl. XIV, 81 (1959); L. Michel and A. S. Wightman, Princeton University Lecture notes (unpublished); E. P. Wigner, Lecture notes of the Istanbul Summer School 1962.
21. G. W. Mackey, Proc. N.A.S. **35**, 537 (1949); Ann. of Math. **55**, 101 (1952); for a didactic and complete account of this method see: G. W. Mackey, Chicago Lecture notes, Summer 1955.

not just
a,b ∈ S
→ ab ∈ S.

Three surveys, worked out for the explicit use of physicists, are: J. M. Jauch, CERN Seminar 5425/TH.06 (1959), part IV: G. Emch, Séminaire de l'Institut de physique théorique, Genève 1961, and appendix to ref. 5. See also the appendix written by Mackey for Segal's Book: Mathematical Problems of Relativistic Physics. Amer. Math. Soc. 1963.

22. F. Riesz et B. Sz.-Nagy, Leçons d'analyse fonctionnelle, Paris et Budapest 1955.
23. The definition of a position operator in a relativistic theory is not a trivial question, already in complex quantum mechanics. See for instance T. D. Newton and E. P. Wigner, Rev. Mod. Phys. 21, 400 (1949) (for other references, see Wightman's paper quoted below). A modern formulation of this problem was presented by A. S. Wightman, Rev. Mod. Phys. 34, 845 (1962); he uses there a mathematical object found by G. W. Mackey (see for instance Proc. N.A.S. 35, 537 (1949) and which is referred to as "system of imprimitivity for representations of groups"; the essential idea of Wightman's work is to define the position operator through its spectral family, and to impose, on this spectral family, the correct transformation laws with respect to the euclidian three-dimensional group, as they result from the considered irreducible representation of the Poincaré group. This definition of the position operator is trivially translated in quaternionic quantum mechanics.

SOME REMARKS ON THE PRODUCT OF IRREDUCIBLE REPRESENTATIONS OF THE INHOMOGENEOUS LORENTZ GROUP*

P. Moussa and R. Stora
Centre d'Etudes Nucléaires de Saclay
Service de Physique Théorique
B.P. n°2 GIF-sur-YVETTE (S & O)
- France -

INTRODUCTION

The problem of reducing the product of two irreducible representations of the inhomogeneous Lorentz group is an old one. The result is currently known and used and leads to the various phenomenological expansions of transition amplitudes: ℓ -s type expansions, multipole expansions,¹⁾ helicity expansions. Whereas ℓ -s type couplings have been known for a long time, at least in nonrelativistic situations, as well as multipole expansions which are usually derived from an analysis of fields rather than of particle configurations, the helicity type couplings have been discovered fairly late.

The need for defining such concepts independently from any semi-classical, or configuration space nonrelativistic argument has been recognized by several authors:

Chou Kuang Chao and M. I. Shirokov¹ have worked out practically all coupling schemes known at present. Their analysis relies however on a heavy use of the position operator² associated with irreducible representations, when it exists, in a way which is not completely clear because the frame of reference in which this operator is defined depends on the eigenvalue of the conjugate momentum which does not commute with it.

Jacob and Wick,³ on the other hand, have a formulation of the helicity coupling which requires some specification before it can be interpreted in relativistically invariant terms, specification which indeed was given in Wick's paper³ on the description of three particle states. Although the simplicity of relativistic effects is not emphasized in this paper, one result of which relies on Shirokov's analysis.

H. Joos's derivation⁴ of the coupling of massive particles,

*Presented at the LORENTZ GROUP SYMPOSIUM, Institute for Theoretical Physics, University of Colorado, Summer, 1964.

1) This name is usually reserved to situations where massless particles are involved.

Guillot +
Petit '66

Nouvelles formes des représentations unitaires irréductibles du groupe de Poincaré

I

par J. C. Guillot¹⁾ et J. L. Petit¹⁾

Institut de Physique Théorique, Université de Genève, Suisse

(29 1 66)

Abstract. In this paper we study different forms of representations of the Poincaré group and establish their interrelations. Starting with the induced representations by Mackey, we construct explicitly the isomorphisms to the forms preferred by physicist defined on the space of functions on a homogeneous space. As a particular case we find the representations of Wigner and specify the arbitrary parameters involved. We then give different possible forms of representations defined on either the Poincaré group or the Lorentz group. Finally, we specify the different conventions which are used for different physical situations.

I. Introduction

WIGNER, dans son célèbre article [1]²⁾, a classé toutes les représentations unitaires irréductibles du groupe de Poincaré ou groupe de Lorentz inhomogène. Elles sont repérées par deux nombres: la masse et le spin. La manière dont ces représentations sont réalisées est importante, par exemple dans les problèmes de cinématique relativiste, car on sait qu'une particule libre de masse m et de spin j se transforme suivant une représentation irréductible repérée par m et j . WIGNER considéra une réalisation particulière dans un espace d'Hilbert de fonctions définies sur l'hyperboloïde de masse $p^2 = m^2$, à valeur dans l'espace de spin. Une autre réalisation s'exprime à l'aide de fonctions d'ondes tensorielles ou spinorielles satisfaisant des équations d'onde (par exemple l'équation de Dirac et l'équation de Klein-Gordon). MACKEY [2] a ensuite généralisé à une grande classe de groupes la théorie de WIGNER en définissant la notion de représentations induites et nous nous proposons, à l'aide de cette théorie, d'étudier une nouvelle forme des représentations irréductibles du groupe de Poincaré réalisée dans un espace d'Hilbert de fonction définies sur le groupe. Plus précisément les fonctions sont définies soit sur le groupe de Poincaré soit sur le groupe de Lorentz.

Cette forme nouvelle présente un certain nombre d'avantages. D'abord elle fait jouer un rôle plus «dynamique» au spin, ce qui fournirait peut-être une introduction naturelle à la théorie des pôles de REGGE: cette idée a été développée par F. LUKAT [3]. Ensuite elle permet d'effectuer des opérations mathématiques caractéristiques des fonctions définies sur un groupe. Ainsi WIGNER [12], pour traiter un problème de diffusion multiple, considère une équation définie non dans l'espace des états de la particule mais sur un groupe pour obtenir une équation comportant un produit de

¹⁾ Boursiers O.T.A.N.

²⁾ Les chiffres entre crochets renvoient à la Bibliographie, page 299.

labelled

representations

either a

involving

Comitato di redazione

Physique de l'Université de Genève
Institut der ETH Zürich
Physique de l'Université de Lausanne

Institut der Universität Basel
Institut der Universität Zürich
Physique de l'Université de Fribourg

Physique Théorique, 21, pl. du
Lausanne

PHYSICA ACTA sont
Société Suisse de Physique et
publiée en 6 à 8 fascicules.

manuscrits au professeur G. Wanders
ces mémoires ne doivent pas être
imprimées. Dans la règle, ils
sont d'ordre d'arrivée à la rédaction.
Ils doivent être accompagnés de 50 tirés à
l'appui. Les auteurs plus, de
avant au début du premier fasci-
cule volume.

numéro: sFr. 148.-
25.-

envoyer vos commandes aux li-

BASEL

Birkhäuser Verlag

Zeitschrift für
Geophysik (PAGEOPH)
of Geofisica pura e applicata
paläontologische Abhandlungen
Zeitschrift für Hydrologie, Hy-
drobiologie, Fischereiwissen-
senschaften

Zeitschrift für moderne Pro-
phetischen und angewandten
national-technische Zeitschrift
der Naturforschenden Gesell-

angewandte Mathematik und

© by Birkhäuser AG.

The operation of rotation through an angle θ about an axis $\hat{n}$ is represented by a unitary operator $R(\hat{n}, \theta)$. One can represent this unitary operator in the form

$$R(\hat{n}, \theta) = e^{-i\theta H}, \quad (2-34)$$

where H is a hermitian operator, $H = H^\dagger$. The angular momentum operator J is defined by setting (see, for example, Rose [12, chap. 2], or Wigner [13])

$$H = \theta \hat{n} \cdot J. \quad (2-35)$$

A state $|J, m\rangle$ having angular momentum J and z -component $J_z = m$ is transformed under rotation into a linear combination of states $|J, m'\rangle$; that is,

$$R(\hat{n}, \theta) |J, m\rangle = \sum_{m'} \mathcal{D}_{m'm}^J(\theta, \hat{n}) |J, m'\rangle. \quad (2-36)$$

The notation $\mathcal{D}_{m'm}^J$ is a conventional one.

We shall discuss these $\mathcal{D}$ matrices and their properties later, but the point we wish to make now is more general: The $2J + 1$ objects $|J, m\rangle$ having a given J but all possible m are transformed among themselves under rotations. A state $|J, m\rangle$ is transformed into a linear combination of states $|J, m'\rangle$ with the same J . Moreover, none of the coefficients $\mathcal{D}_{m'm}^J(\hat{n}, \theta)$ is zero for all $\hat{n}, \theta$; that is, all $2J + 1$ states $|J, m'\rangle$ are mixed in by a rotation. Physicists usually call the set of states $|J, m\rangle$ a *multiplet*. In the terminology of group theory, the states are said to form the basis of a $(2J + 1)$ -dimensional *irreducible representation* of the rotation group. A shorter and more suggestive name for such a set of states is an *elementary system* with respect to rotations.

We are now able to state one attribute of an elementary particle: an elementary particle is an elementary system with respect to all symmetry operations. The physical meaning of this statement is very simple for the case of rotations: an elementary particle has a unique spin. Clearly, this is a reasonable minimum requirement for a particle to deserve the name *elementary*. Unfortunately, it is not possible at this time to give a complete and noncontroversial definition of the term *elementary particle*. Some physicists advance theories in which a certain few particles or fields are fundamental (for example, Schwinger [14] or Sakurai [15]); at the other extreme, Chew and Frautschi [16] have advocated "nuclear democracy" in which all nuclear species, from the pion to uranium isotopes, are equally fundamental.

Returning to the statement at the beginning of the preceding paragraph, we must explain the meaning of the term *symmetry operation*. A symmetry operation is an operation which leaves transition probabilities invariant. For example, we make the very reasonable assumption that no physical observable (transition probability) is affected by rotating the laboratory in space. We restrict our attention to symmetry operations when we look for quantum numbers, because only symmetry operations are associated with

conservation laws (for example, rotational symmetry implies conservation of angular momentum). Without angular momentum conservation, it would be impossible to infer from scattering experiments a unique spin for each particle.

The assumption of rotational invariance is indeed very reasonable, and we would be amazed if it were not correct, but we should not lose sight of the fact that it is an assumption. Parity nonconservation, which will be discussed in Chapter 8, showed physicists the danger of regarding some invariance principles as obvious *a priori*. They must ultimately be established experimentally. Rotational invariance can be tested by looking for reactions which would violate angular momentum conservation. So far, it has passed all tests.⁶

2.5 Implications of Rotational Invariance in Two-Body Collisions; Partial-wave Analysis

The tool which has been used for measuring the spins of almost all the particles is the scattering experiment. In order to see how spins can be inferred from scattering processes, we must first examine the consequences of rotational invariance for the description of these processes. In particular, we consider in this section collisions of the type $A + B \rightarrow C + D$ for arbitrary spins and masses of the four particles.

The reader is already familiar with the consequences of rotational invariance in the scattering of spinless particles: If the differential cross section in the CM (center of mass) system is written

$$\frac{d\sigma}{d\Omega} = |f(\theta)|^2, \quad (2-37)$$

where θ is the scattering angle and $f(\theta)$ is called the *scattering amplitude*, then the usual partial-wave expansion is

$$f(\theta) = (1/2ip) \sum_J (2J + 1) [S^J(E) - 1] P_J(\cos \theta), \quad (2-38)$$

where, for elastic scattering below inelastic thresholds,

$$S^J(E) = e^{2i\delta_J(E)}, \quad (2-39)$$

and where p is the incident momentum in the CM system. A derivation of Eq. (2-38) is given in Schiff [17], but it is based on the Schrödinger equation and does not emphasize that the validity of the expansion Eq. (2-38) rests only on the assumption of rotational invariance. Moreover, Eq. (2-39) follows from the unitarity condition, which we shall discuss in Sec. 2.13.

For the general problem $A + B \rightarrow C + D$ with arbitrary spins, Eq. (2-38) has been shown by Jacob and Wick [18] to retain its simple form, but with the P_J replaced by one of the $\mathcal{D}$ functions of Eq. (2-36). Jacob and Wick

⁶See the review article by Feinberg and Goldhaber [20].

were able to achieve this simplicity by abandoning the old method of specifying the components of the spins along an arbitrary "z axis," combining spins into a resultant S , then adding S to the orbital angular momentum L . Instead, they specify the component of the spin along the direction of motion of the particle, which is called the *helicity*, λ . The helicity has the useful property of being invariant under rotations.

This invariance of the helicity enables one to define a state of two particles in their CM system by means of a rotation. We start with a state $\psi_{p\lambda_1\lambda_2}$ in which particle 1 travels in the positive z direction, and particle 2 in the negative z direction, with relative momentum p , where

$$p = |p|, \quad \mathbf{p} = (\mathbf{p}_1 - \mathbf{p}_2)/2. \quad (2-40)$$

A state $|p\theta\varphi; \lambda_1\lambda_2\rangle$ with $\mathbf{p}$ in the direction θ, φ may be obtained by applying an appropriate rotation to the state $\psi_{p\lambda_1\lambda_2}$. First rotate by the angle θ about the y axis, then by φ about the z axis. The corresponding rotation operator is

$$e^{-i\varphi J_z} e^{-i\theta J_y}.$$

Actually, Jacob and Wick use the operator

$$R_{\varphi,\theta,-\varphi} \equiv e^{-i\varphi J_z} e^{-i\theta J_y} e^{+i\varphi J_z}, \quad (2-41)$$

which differs only by a phase, as we shall see in the next equation. Then one can define

$$|p\theta\varphi; \lambda_1\lambda_2\rangle = R_{\varphi,\theta,-\varphi} \psi_{p\lambda_1\lambda_2} = e^{i\lambda\varphi} R_{\varphi,\theta,0} \psi_{p\lambda_1\lambda_2}, \quad (2-42)$$

$$\text{where } \lambda = \lambda_1 - \lambda_2. \quad (2-43)$$

Now we come to the heart of the problem, the calculation of the transformation matrix

$$\langle p\theta\varphi; \lambda_1\lambda_2 | JM; \lambda'_1\lambda'_2 \rangle = \delta_{\lambda_1\lambda'_1} \delta_{\lambda_2\lambda'_2} \langle p\theta\varphi; \lambda_1\lambda_2 | JM; \lambda_1\lambda_2 \rangle \quad (2-44)$$

which will enable us to expand in a state of given J and M . In order to calculate this matrix we first simplify it by an appropriate rotation,

$$\langle p\theta\varphi; \lambda_1\lambda_2 | JM; \lambda_1\lambda_2 \rangle = \langle p\theta\varphi; \lambda_1\lambda_2 | R_{\varphi,\theta,-\varphi} R_{\varphi,\theta,-\varphi}^{-1} | JM; \lambda_1\lambda_2 \rangle. \quad (2-45)$$

Now recall from Eq. (2-36) that

$$R_{\varphi,\theta,-\varphi} |Jm\rangle = \sum_{m'} \mathcal{D}_{m'm}^J(\varphi, \theta, -\varphi) |Jm'\rangle \quad (2-46)$$

$$\text{where } \mathcal{D}_{m'm}^J(\varphi, \theta, -\varphi) = \langle Jm' | R_{\varphi,\theta,-\varphi} | Jm \rangle. \quad (2-47)$$

Hence making use of the unitarity of the rotation operator we find

$$\begin{aligned} R_{\varphi,\theta,-\varphi}^{-1} |Jm\rangle &= \sum_{m'} |Jm'\rangle \langle Jm' | R_{\varphi,\theta,-\varphi}^{-1} | Jm \rangle \\ &= \sum_{m'} |Jm'\rangle \langle Jm' | R_{\varphi,\theta,-\varphi}^\dagger | Jm \rangle \\ &= \sum_{m'} |Jm'\rangle \langle Jm | R_{\varphi,\theta,-\varphi} | Jm' \rangle^* \\ &= \sum_{m'} \mathcal{D}_{m'm}^{J*}(\varphi, \theta, -\varphi) |Jm'\rangle \end{aligned} \quad (2-48)$$

Therefore it follows from Eq. (2-48) and Eq. (2-45) that

$$\begin{aligned} \langle p\theta\varphi; \lambda_1\lambda_2 | JM; \lambda_1\lambda_2 \rangle &= \sum_{M'} \mathcal{D}_{M'M}^{J*}(\varphi, \theta, -\varphi) \langle p00; \lambda_1\lambda_2 | JM'; \lambda_1\lambda_2 \rangle \\ &= \mathcal{D}_{M\lambda}^J(\varphi, \theta, -\varphi) \langle p00; \lambda_1\lambda_2 | J\lambda; \lambda_1\lambda_2 \rangle, \end{aligned} \quad (2-49)$$

where $\lambda = \lambda_1 - \lambda_2$. The last step follows from equating the z components of angular momentum.⁷ Equation (2-49) exhibits the angular dependence of the transformation matrix.

The remaining factor on the right-hand side of Eq. (2-49) can be determined by normalization requirements. We assume the usual normalizations

$$\langle J'M'\lambda'_1\lambda'_2 | JM\lambda_1\lambda_2 \rangle = \delta_{J'J} \delta_{M'M} \delta_{\lambda_1\lambda'_1} \delta_{\lambda_2\lambda'_2}, \quad (2-50)$$

$$\langle p\theta'\varphi'; \lambda'_1\lambda'_2 | p\theta\varphi; \lambda_1\lambda_2 \rangle = \delta(\varphi - \varphi') \delta(\cos\theta - \cos\theta') \delta_{\lambda_1\lambda'_1} \delta_{\lambda_2\lambda'_2}. \quad (2-51)$$

In addition, we need to use some further properties of the $\mathcal{D}$'s. From Eq. (2-47) it follows that

$$\mathcal{D}_{M'M}^J(\varphi, \theta, -\varphi) = e^{i(M-M')\varphi} d_{M'M}^J(\theta) \quad (2-52)$$

$$\text{where } d_{M'M}^J(\theta) = \langle JM' | e^{-iJ_y\theta} | JM \rangle. \quad (2-53)$$

These functions are conventionally normalized according to

$$\int_{-1}^1 d \cos\theta [d_{mn}^J(\theta)]^2 = 2/(2J+1). \quad (2-54)$$

Now if one makes use of all these relations to calculate the quantity

$$\begin{aligned} \langle JM\lambda_1\lambda_2 | JM\lambda_1\lambda_2 \rangle \\ = 1 = \sum_{\lambda'_1\lambda'_2} \int d\varphi \int d \cos\theta \langle JM; \lambda_1\lambda_2 | p\theta\varphi\lambda'_1\lambda'_2 \rangle \langle p\theta\varphi\lambda'_1\lambda'_2 | JM; \lambda_1\lambda_2 \rangle \end{aligned} \quad (2-55)$$

one finds that Eq. (2-49) takes the final form

$$\langle p\theta\varphi; \lambda_1\lambda_2 | JM; \lambda'_1\lambda'_2 \rangle = N_J \delta_{\lambda_1\lambda'_1} \delta_{\lambda_2\lambda'_2} \mathcal{D}_{M\lambda}^{J*}(\varphi, \theta, -\varphi) \quad (2-56)$$

$$\text{where } |N_J|^2 = (2J+1)/4\pi. \quad (2-57)$$

Equation (2-56) is the basic formula which we require in order to derive the expansion of the S -matrix in terms of eigenstates of angular momentum.⁸ It tells us the transformation coefficients between plane-wave states $|p\theta\varphi; \lambda_1\lambda_2\rangle$ and angular momentum states $|JM; \lambda_1\lambda_2\rangle$. Making use of these coefficients

⁷ The helicity λ is the component of J along the direction of motion. The orbital angular momentum L does not contribute in this direction (classically, $L = \mathbf{r} \times \mathbf{p}$, which is perpendicular to $\mathbf{p}$). For the state $|p00; \lambda_1\lambda_2\rangle$ the direction of motion is the z direction, so $M = \lambda$.

⁸ Recall that the physical significance of the S -matrix is that $P_{fi} = |\langle f | S | i \rangle|^2$ is the probability of a transition from the initial state i to the final state f . For a complete treatment of scattering theory see Goldberger and Watson [2].

we expand the plane-wave S -matrix elements as follows:

$$\langle p' \theta \varphi \lambda_c \lambda_d | S | p \theta_0 \varphi_0 \lambda_a \lambda_b \rangle = \sum_{J, M} \frac{2J+1}{4\pi} \mathcal{D}_{M\mu}^{J*}(\varphi, \theta, -\varphi) \\ \times \langle \lambda_c \lambda_d | S^J(E) | \lambda_a \lambda_b \rangle \mathcal{D}_{M\lambda}^J(\varphi_0, \theta_0, -\varphi_0),$$

where $\lambda = \lambda_a - \lambda_b$, $\mu = \lambda_c - \lambda_d$. (2-58)

Here we made use of the reduction

$$\langle J' M' \lambda_c \lambda_d | S(E) | J M \lambda_a \lambda_b \rangle = \delta_{J'J} \delta_{M'M} \langle \lambda_c \lambda_d | S^J(E) | \lambda_a \lambda_b \rangle. \quad (2-59)$$

Equation (2-59) really contains most of the physics of the partial-wave expansion we are deriving, so it's worthwhile to pause long enough to establish what it is based on. It expresses the conservation of angular momentum in the collision, which in turn follows from the assumption of the rotational invariance of the collision operator S ,

$$R(\hat{n}, \theta) S R^{-1}(\hat{n}, \theta) = S, \quad (2-60a)$$

$$\text{or,} \quad [R, S] = 0. \quad (2-60b)$$

Since $R = \exp(-i\hat{n} \cdot \mathbf{J}\theta)$, it then follows that the angular momentum operator $\mathbf{J}$ commutes with S . Hence the state $S|JM\rangle$ is again a state of total angular momentum J and z component M , and is orthogonal to the state $|J'M'\rangle$ unless $J = J'$ and $M = M'$. Moreover, the reduced matrix element on the right-hand side of Eq. (2-59) cannot depend on M , since this quantity is obviously not invariant under rotations.

Further simplification results if we choose the incident direction along the z axis ($\theta_0 = 0$) since from Eq. (2-52) and Eq. (2-53)

$$\mathcal{D}_{M\lambda}^J(\varphi_0, 0, -\varphi_0) = e^{i(\lambda-M)\varphi_0} \mathcal{D}_{M\lambda}^J(0) = \delta_{M\lambda}. \quad (2-61)$$

We obtain then

$$\langle p' \theta \varphi \lambda_c \lambda_d | S | p 0 0 \lambda_a \lambda_b \rangle = \sum_J \frac{2J+1}{4\pi} \langle \lambda_c \lambda_d | S^J | \lambda_a \lambda_b \rangle d_{\lambda\mu}^J(\theta) e^{i(\lambda-\mu)\varphi}, \quad (2-62)$$

where $\lambda = \lambda_a - \lambda_b$, $\mu = \lambda_c - \lambda_d$. We can write down the correctly normalized cross section by comparison with the familiar formula Eq. (2-38) for the spinless case:

$$d\sigma/d\Omega = |\mathcal{f}_{\lambda_c \lambda_d, \lambda_a \lambda_b}(\theta, \varphi)|^2, \quad (2-63)$$

$$\mathcal{f}_{\lambda_c \lambda_d, \lambda_a \lambda_b}(\theta, \varphi) = (1/p) \sum_J (2J+1) \langle \lambda_c \lambda_d | T^J(E) | \lambda_a \lambda_b \rangle e^{i(\lambda-\mu)\varphi} d_{\lambda\mu}^J(\theta), \quad (2-64)$$

$$\text{where} \quad \langle \lambda_c \lambda_d | S^J | \lambda_a \lambda_b \rangle = \delta_{\lambda_a \lambda_c} \delta_{\lambda_b \lambda_d} + 2i \langle \lambda_c \lambda_d | T^J | \lambda_a \lambda_b \rangle. \quad (2-65)$$

The angle θ is the angle included between the direction of A and C . The Jacob-Wick formula, Eq. (2-64), is remarkably simple. The functions $d_{mn}^J(\theta)$ have been investigated by a number of authors. Appendix A lists some properties of the d functions.

Before we can apply Eq. (2-64) to a practical problem, we must investigate the restrictions imposed on the S^J by the unitarity condition, parity invariance, and time-reversal invariance.

In concluding this section we note the connection between the states $|JM; \lambda_1 \lambda_2\rangle$ and the customarily employed states $|JM; LS\rangle$ formed by adding the spins s_1, s_2 to a resultant S , then combining $\mathbf{J} = \mathbf{L} + \mathbf{S}$. The transformation coefficients are

$$\langle JM; LS | JM; \lambda_1 \lambda_2 \rangle = \left(\frac{2L+1}{2J+1} \right)^{1/2} C(LS J; 0, \lambda) C(s_1 s_2 s; \lambda_1, -\lambda_2). \quad (2-66)$$

where we use the notation of Rose [12] for the Clebsch-Gordon coefficients; $\langle j_1 j_2 m_1 m_2 | j_1 j_2 j m \rangle = C(j_1 j_2 j; m_1 m_2) \delta_{m, m_1+m_2}$. The derivation of this formula can be found in Jacob and Wick [18, appendix B].

2.6 Parity Invariance

Before the work of Lee and Yang and the subsequent experiments, parity invariance (invariance under inversion of the space coordinates) was regarded as a self-evident requirement. Since the discovery of its violation in weak interactions, to which we shall return in Chapter 8, its consequences have been the subject of many experiments. It is now known that, at least to a very good approximation, parity is a valid symmetry operation for the strong and electromagnetic interactions.

In order to see what limitations parity imposes on the submatrices S^J we must go back to the beginning and evaluate its effect on the states $\psi_{p\lambda_1\lambda_2}$. These are product states

$$\psi_{p\lambda_1\lambda_2} = \psi_{p\lambda_1}(1) \chi_{p\lambda_2}(2), \quad (2-67)$$

where particle 1 travels in the $+z$ direction with helicity λ_1 , and particle 2 travels in the $-z$ direction with helicity λ_2 . The state $\psi_{p\lambda}$ is formed, for the case of nonmassless particles, by applying a pure Lorentz transformation in the z direction, Λ_z , to the state $\psi_{0\lambda}$. The relative phases of these states are specified by the convention

$$(J_z \pm iJ_y) \psi_{0\lambda} = [(s \mp \lambda)(s \pm \lambda + 1)]^{1/2} \psi_{0, \lambda \pm 1}, \quad (2-68)$$

where s is the spin of the particle.

Now the parity operator P leaves the spin direction unchanged, as the reader can easily convince himself by a classical argument (for example, consider the effect on $\mathbf{L} = \mathbf{r} \times \mathbf{p}$); hence,

$$P \psi_{0\lambda} = \eta \psi_{0\lambda} \quad (2-69)$$

where η is a phase factor independent of λ (since P commutes with J). It is convenient also to consider the operation Y of reflection in the xz plane,

$$Y = e^{-i\pi J_y} P \quad (2-70)$$

From Eq. (2-46)

$$e^{-isJ_z} \psi_{0\lambda} = \sum_{\lambda'} d_{\lambda'\lambda}^s(\pi) \psi_{0\lambda'} = (-1)^{s-\lambda} \psi_{0,-\lambda}, \quad (2-71)$$

where we have used Eq. (A-4) from Appendix A. Combining the last three equations and applying Λ_z (Λ_z and Y commute), we find

$$Y \psi_{p\lambda} = \eta(-1)^{s-\lambda} \psi_{p,-\lambda}. \quad (2-72)$$

Similarly, one finds that

$$Y \chi_{p\lambda} = \eta(-1)^{s+\lambda} \chi_{p,-\lambda}. \quad (2-73)$$

The reversal of the sign of λ in the exponent occurs because the state $\chi_{p\lambda}$ has momentum p in the negative z direction; hence, z component of spin $m = -\lambda$. Then it follows that

$$Y \psi_{p\lambda_1\lambda_2} = \eta_1\eta_2(-1)^{s_1+s_2-\lambda_1+\lambda_2} \psi_{p,-\lambda_1,-\lambda_2}, \quad (2-74)$$

and hence that

$$P \psi_{p\lambda_1\lambda_2} = \eta_1\eta_2(-1)^{s_1+s_2-\lambda_1+\lambda_2} e^{isJ_z} \psi_{p,-\lambda_1,-\lambda_2}. \quad (2-75)$$

Now we wish to evaluate $P|JM; \lambda_1\lambda_2\rangle$. Physically we know that

$$P|JM; \lambda_1\lambda_2\rangle = e^{i\alpha}|JM; -\lambda_1, -\lambda_2\rangle,$$

so our task is to find what phase factor α is implied by the way we have defined our states. The reader who would rather not go through the derivation can skip to the result, Eq. (2-82). The task is greatly simplified if we observe that α must be independent of M since P commutes with any rotation. Therefore we can without loss of generality confine our attention to the case $M \equiv \lambda = \lambda_1 = \lambda_2$. We then specialize to the case $\theta = \varphi = 0$ and calculate $P|p00; \lambda_1\lambda_2\rangle$. From Eq. (2-75) and (2-42) we find

$$\begin{aligned} P|p00; \lambda_1\lambda_2\rangle &= \eta_1\eta_2(-1)^{s_1+s_2-\lambda} R_{0,-\pi,0}|p00; -\lambda_1, -\lambda_2\rangle \\ &= \eta_1\eta_2(-1)^{s_1+s_2-\lambda}|p, -\pi, 0; -\lambda_1, -\lambda_2\rangle. \end{aligned} \quad (2-76)$$

Now we use the transformation matrix Eq. (2-56) to expand this state. With the help of Eq. (A-5), we find

$$\begin{aligned} |p, -\pi, 0; -\lambda_1, -\lambda_2\rangle &= \sum_{JM\lambda'_1\lambda'_2} |JM; \lambda'_1\lambda'_2\rangle \\ &\quad \times \langle JM; \lambda'_1\lambda'_2 | p, -\pi, 0; -\lambda_1, -\lambda_2 \rangle \end{aligned} \quad (2-77)$$

$$= \sum_J N_J^* (-1)^{\lambda-J} |J\lambda; -\lambda_1, -\lambda_2\rangle. \quad (2-78)$$

Substituting this result in Eq. (2-76) we find

$$P|p00; \lambda_1\lambda_2\rangle = \sum_J N_J^* \eta_1\eta_2 (-1)^{s_1+s_2-J} |J\lambda; -\lambda_1, -\lambda_2\rangle. \quad (2-79)$$

On the other hand, we could use Eq. (2-56) to expand the left-hand side of Eq. (2-79) directly

$$\begin{aligned} P|p00; \lambda_1\lambda_2\rangle &= \sum_{JM} P|JM; \lambda_1\lambda_2\rangle N_J^* D_{M\lambda}^J(0, 0, 0) \\ &= \sum_J N_J^* P|J\lambda; \lambda_1\lambda_2\rangle. \end{aligned} \quad (2-80)$$

Comparing Eq. (2-79) and Eq. (2-80), we find

$$P|J\lambda; \lambda_1\lambda_2\rangle = \eta_1\eta_2(-1)^{s_1+s_2-J} |J\lambda; -\lambda_1, -\lambda_2\rangle. \quad (2-81)$$

Noting that $J - s_1 - s_2$ is necessarily an integer, and using the independence of M remarked earlier, we finally obtain

$$P|JM; \lambda_1\lambda_2\rangle = \eta_1\eta_2(-1)^{J-s_1-s_2} |JM; -\lambda_1, -\lambda_2\rangle. \quad (2-82)$$

If parity is conserved, then

$$\langle f | S | i \rangle = \langle f | P^{-1} S P | i \rangle. \quad (2-83)$$

Applying this to the S matrix in the $JM; \lambda_1\lambda_2$ representation, we find

$$(-\lambda_c, -\lambda_d | S^J | -\lambda_a, -\lambda_b) = \eta_a \langle \lambda_c \lambda_d | S^J | \lambda_a \lambda_b \rangle \quad (2-84)$$

$$\eta_a = (\eta_c \eta_d \eta_a \eta_b) (-1)^{s_c+s_d-s_a-s_b}. \quad (2-85)$$

The quantities $\eta_a, \eta_b, \eta_c, \eta_d$ are the *intrinsic parities* of the particles A, B, C, D which are taken equal to ± 1 . We shall return later to several examples of their determination. Using Eq. (A2) we find for the scattering amplitude the analogous property,

$$f_{-\lambda_c, -\lambda_d; -\lambda_a, -\lambda_b}(\theta, \varphi) = \eta_a f_{\lambda_c, \lambda_d; \lambda_a, \lambda_b}(\theta, \pi - \varphi). \quad (2-86)$$

In discussions of the effect of parity transformations it is often convenient to revert to the $JM; LS$ representation. By using Eq. (2-82), the transformation coefficient, Eq. (2-66), and symmetries of the Clebsch-Gordon coefficients, one obtains the result

$$P|JM; LS\rangle = \eta_1\eta_2(-1)^L |JM; LS\rangle. \quad (2-87)$$

Thus the LS states are more convenient in discussions of parity because they are states of definite parity, whereas one must form linear combinations of the helicity states to obtain parity eigenstates.

Parity conservation imposes restrictions on the change in L which can occur in a reaction. Making use of Eq. (2-87) and parity conservation, we find

$$\begin{aligned} \langle JMLS' | S | JMLS \rangle &= \langle JMLS' | P^{-1} S P | JMLS \rangle \\ &= (-1)^{L'-L} \eta_c \eta_d \eta_a \eta_b \langle JMLS' | S | JMLS \rangle \end{aligned} \quad (2-88)$$

It follows that the S -matrix element must vanish unless

$$\eta_c \eta_d \eta_a \eta_b (-1)^{L'-L} = 1. \quad (2-89)$$

For example, if there is no change in intrinsic parities ($\eta_c \eta_d = \eta_a \eta_b$) then L and L' must be either both even or both odd.

2.7 Time Reversal

Experimental tests have not yet uncovered in nature any reactions which violate time-reversal invariance.⁹ Physically it is clear that time reversal

⁹ The experiment of Christenson, Cronin, Fitch, and Turlay [21] probably implies a small violation in the weak interactions. See Chapter 8.

reverses both momenta and spins, thus leaving helicities unchanged, and it is also clear that time reversal relates $\langle f|S|i\rangle$ to $\langle i^T|S|f^T\rangle$, where the T denotes the time-reversed state. Hence up to a phase factor time-reversal invariance implies that

$$\langle \lambda_c \lambda_d | S^J | \lambda_a \lambda_b \rangle = \langle \lambda_a \lambda_b | S^J | \lambda_c \lambda_d \rangle. \quad (2-90)$$

In fact, the phase factor is unity. The proof, which can be found in Jacob and Wick [18], is rather lengthy and will not be reproduced here.

2.8 Detailed Balancing; Spin of the Charged Pion

An important application of time-reversal invariance is relating the cross section for the reaction $C + D \rightarrow A + B$ to the cross section for the reaction $A + B \rightarrow C + D$. Equation (2-90) tells us that the S -matrix elements are the same. Let us calculate the *unpolarized* cross section; that is unpolarized beam and target, and final polarizations not observed;

$$\frac{d\sigma}{d\Omega}(AB \rightarrow CD) = \frac{1}{(2S_A + 1)(2S_B + 1)} \sum_{(\lambda)} |f_{\lambda_c \lambda_d, \lambda_a \lambda_b}(\theta, \varphi)|^2, \quad (2-91)$$

where (λ) is an abbreviation for the four helicities. The cross section for the inverse reaction is

$$\frac{d\sigma}{d\Omega}(CD \rightarrow AB) = \frac{1}{(2S_C + 1)(2S_D + 1)} \sum_{(\lambda)} |f_{\lambda_a \lambda_b, \lambda_c \lambda_d}(\theta, \varphi)|^2. \quad (2-92)$$

But Eq. (2-90) and Eq. (2-64) imply that

$$P_A^2 \sum_{(\lambda)} |f_{\lambda_c \lambda_d, \lambda_a \lambda_b}(\theta, \varphi)|^2 = P_C^2 \sum_{(\lambda)} |f_{\lambda_a \lambda_b, \lambda_c \lambda_d}(\theta, \varphi)|^2 \quad (2-93)$$

Therefore we find

$$\frac{d\sigma/d\Omega(AB \rightarrow CD)}{d\sigma/d\Omega(CD \rightarrow AB)} = \frac{P_C^2(2S_C + 1)(2S_D + 1)}{P_A^2(2S_A + 1)(2S_B + 1)}. \quad (2-94)$$

Both cross sections are to be evaluated at the same total energy and angle in the CM system. This equation is called the *principle of detailed balancing*.

It was suggested in 1951 by Marshak [22] and Cheston [23] that this relation could be applied to the reactions

$$\pi^+ + D \rightleftharpoons p + p \quad (2-95)$$

to determine the spin of the π^+ . Equation (2-94) yields for this case

$$\frac{d\sigma}{d\Omega}(\rightarrow) / \frac{d\sigma}{d\Omega}(\leftarrow) = \frac{4p^2}{3(2S_\pi + 1)q^2}, \quad (2-96)$$

where p and q are the CM momenta of proton and pion, respectively. For the total cross sections,

$$\sigma_T(\rightarrow) / \sigma_T(\leftarrow) = \frac{2p^2}{3(2S_\pi + 1)q^2}. \quad (2-97)$$

The factor $\frac{1}{2}$ occurs because an integration over all solid angles counts each pp final state twice. (See Golgberger and Watson [2, p. 144] for a complete discussion of these extra factors which arise when identical particles are present.) Both reactions were measured [24] and found consistent with $S_\pi = 0$. This result is, however, open to the criticism that if $S_\pi = 1$ and if the pion beam were polarized, it might be possible to obtain the same result by accident.

2.9 Massless Particles

The Jacob-Wick formalism is applicable without modification to scattering of massless particles. Although some of the steps in Sec. 2.6 were applicable only for $m \neq 0$ because of the use of the rest frame, all the results are correct as long as one remembers that for $m = 0$ the only possible states of polarization are $\lambda = \pm s$. This fact is already familiar to the reader for the case of the photon. For the neutrino only the state $\lambda = -s$ is realized in nature. The statement $\lambda = \pm s$ is generally valid for massless particles, but we shall not prove this here. (See Wigner's [25] lucid discussion which provides further references.)

The specification of states in Sec. 2.6 being inapplicable, we must find another way to specify the relative phases of the two possible states $\psi_{p,s}$ and $\psi_{p,-s}$. This is provided by Eq. (2-72) which for $\lambda = s$ reads

$$Y_{\psi_{p,s}} = \eta \psi_{p,-s}. \quad (2-98)$$

For the photon it is customary to define the $\lambda = \pm 1$ solutions as having a vector potential

$$\mathbf{A} = \mp (\mathbf{e}_x \pm i\mathbf{e}_y) e^{ipz}. \quad (2-99)$$

With this convention one finds easily that $\eta = -1$.

2.10 Identical Particles; Spin and Parity of the π^0

Having discussed in Sec. 2.8 some evidence that the spin of the π^+ is zero, we should now consider the evidence that the spin of the π^0 is also zero. In Sec. 2.14 we shall discuss isotopic spin, according to which the π^+ , π^0 , π^- are different charge states of the same multiplet. This theory would, of course, be untenable if the charged and neutral pions had different spins or parities.

Consider these state vectors and normalizations:

$$\begin{cases} \langle JM \lambda_1 \lambda_2 | \\ \langle J' M' \lambda'_1 \lambda'_2 | JM \lambda_1 \lambda_2 \rangle = \delta_{JJ'} \delta_{MM'} \delta_{\lambda_1 \lambda'_1} \delta_{\lambda_2 \lambda'_2} \\ 1 = \sum_{JM \lambda \lambda'} |JM \lambda \lambda' \rangle \langle JM \lambda \lambda'| \end{cases}$$

$$\begin{cases} \langle p \theta \phi; \lambda_1 \lambda_2 | \\ \langle p \theta' \phi'; \lambda_1 \lambda_2 | p \theta \phi \lambda_1 \lambda_2 \rangle = \delta(\phi - \phi') \delta(\cos \theta' - \cos \theta) \delta_{\lambda_1 \lambda'_1} \delta_{\lambda_2 \lambda'_2} \\ 1 = \sum_{\lambda \lambda'} \int d(\cos \theta) d\phi |p \theta \phi \lambda \lambda' \rangle \langle p \theta \phi \lambda \lambda'| \end{cases}$$

The curiosity here is that the state $\langle p 0 0; \lambda_1 \lambda_2 |$ has $J_z = \lambda_1 - \lambda_2 = \lambda$, but it is not a state of definite J . In fact, as will be shown below,

$$|p 0 0; \lambda_1 \lambda_2 \rangle = \sum_{J \geq |\lambda|} N_J |p J \lambda; \lambda_1 \lambda_2 \rangle$$

Oh, as long as J is in the range of λ (i.e., $J \geq |\lambda|$) we have:

$$\langle p J \lambda \lambda_1 \lambda_2 | p 0 0 \lambda_1 \lambda_2 \rangle = N_J = \sqrt{\frac{(2J+1)}{4\pi}}$$

The basic idea of Jacob + Wick is to calculate this element:

$$\langle p \theta \phi \lambda_1 \lambda_2 | p J M \lambda_1 \lambda_2 \rangle$$

First, let's review the meaning of "rotation matrix".

$$\begin{aligned} R_{\phi, \theta, \psi} |Jm\rangle &= |Jm, \text{quantized along a new, rotated } z \text{ axis}\rangle \\ \uparrow \\ \text{operator} &= \sum_{m'} D_{mm'}^J(\phi, \theta, -\phi) |Jm'\rangle \quad \text{original } z\text{-axis quant.} \\ \langle Jm' | R_{\phi, \theta, -\phi} |Jm\rangle &= D_{mm'}^J(\phi, \theta, -\phi) \end{aligned}$$

Thus, we can say:

$$\begin{aligned} \langle p \theta \phi | J M \rangle &= \langle p \theta \phi | R_{\phi, \theta, -\phi} \tilde{R}_{\phi, \theta, -\phi}^{-1} |J M \rangle \quad \text{note: } \tilde{R} = R^\dagger \\ &= \langle p 0 0 | \sum_{m'} D_{mm'}^{J*}(\phi, \theta, -\phi) |J m' \rangle \\ &= N_J D_{M\lambda}^{J*}(\phi, \theta, -\phi) \quad N_J^2 = (2J+1)/4\pi. \end{aligned}$$

A lot of algebra has been omitted here. m' is clamped to λ because with $\theta \phi = 0$, $J_z = S_z = \lambda = \lambda_1 - \lambda_2$. The other detail is done on the next page. Note $\lambda_1 \lambda_2$ indices are suppressed here.

Back up one step:

$$\langle p \theta \phi \lambda_1 \lambda_2 | \mathcal{J} m \lambda, \lambda_2 \rangle = D_{m\lambda}^{\mathcal{J}*}(\phi, \theta, -\phi) \langle p 0 0; \lambda_1 \lambda_2 | \mathcal{J} \lambda; \lambda_1 \lambda_2 \rangle$$

A careful analysis would show that this last m.e. is dependent only on $\mathcal{J}$, not on λ . Basically, if it depended on λ , something would not be rotationally invariant. Thus, the element comes out = $n_{\mathcal{J}}$ in order to make normalizations work.

More generally;

$$\langle p \alpha \beta \delta \lambda_1 \lambda_2 | \mathcal{J} m \lambda, \lambda_2 \rangle = n_{\mathcal{J}} D_{m\lambda}^{\mathcal{J}}(\alpha \beta \delta)$$

Completeness on these $D^{\mathcal{J}}$ functions is:

$$\underbrace{\int_0^{2\pi} d\alpha \int_{-1}^1 d\beta \int_0^{2\pi} d\delta}_{\int du} D_{\mu_1 m_1}^{\mathcal{J}_1*}(\alpha \beta \delta) D_{\mu_2 m_2}^{\mathcal{J}_2}(\alpha \beta \delta) = \frac{8\pi^2}{2\mathcal{J}+1} \delta_{\mu_1 \mu_2} \delta_{m_1 m_2} \delta_{\mathcal{J}_1 \mathcal{J}_2}$$

Rewrite as:

$$\int du D_{mn}^{\mathcal{J}*}(\alpha \beta \delta) D_{mn}^{\mathcal{J}}(\alpha \beta \delta) = \frac{8\pi^2}{2\mathcal{J}+1}$$

Or, the following statements: IF

$$\langle p \alpha \beta \delta | \mathcal{J} m \rangle = \tilde{n}_{\mathcal{J}} D_{m\lambda}^{\mathcal{J}*}(\alpha, \beta, \delta) \quad |\tilde{n}_{\mathcal{J}}|^2 = \frac{2\mathcal{J}+1}{8\pi^2}$$

Then the D completeness in full Euler angles is:

$$\int du |\alpha \beta \delta\rangle \langle \alpha \beta \delta| = 1$$

When you use θ, ϕ ; you use $\int d\Omega$ not $\int du$; they differ by factor 2π which sets $\tilde{n}_{\mathcal{J}}$ apart from $n_{\mathcal{J}}$.

Close onto $|\mathcal{J} m\rangle$ to get:

$$\begin{aligned} |\mathcal{J} m\rangle &= \int du \langle \alpha \beta \delta | \mathcal{J} m \rangle |\alpha \beta \delta\rangle \\ &= \tilde{n}_{\mathcal{J}} \int du D_{m\lambda}^{\mathcal{J}*}(\alpha, \beta, \delta) |\alpha \beta \delta\rangle \end{aligned}$$

This is (16) of the original JW paper. There they use:

$$\frac{\mathcal{N}}{2\pi} = \tilde{n}_{\mathcal{J}} = \frac{n_{\mathcal{J}}}{2\pi} \Rightarrow \mathcal{N} = n_{\mathcal{J}}$$

$$|\alpha \beta \delta\rangle = R_{\alpha \beta \delta} \Psi_{p \lambda \lambda_2}$$

So far we have shown that

$$\langle p \rightarrow \phi \lambda_1 \lambda_2 | JM p \lambda_3 \lambda_4 \rangle = \delta_{\lambda_1 \lambda_3} \delta_{\lambda_2 \lambda_4} \cdot \sqrt{\frac{2J+1}{4\pi}} \cdot D_{M\lambda}^{J*}(\phi \theta - \phi)$$

and if $\theta \neq \pi$, then we must have $m = \lambda$.

The Punch of Jacob + Wick is this: The particles in a $2 \rightarrow 2$ enter and exit as plane waves; but, J is conserved, so:

$$\begin{aligned} \langle p \rightarrow \phi \lambda_c \lambda_d | S | p \rightarrow \phi \lambda_a \lambda_b \rangle & \quad \begin{aligned} \lambda &= \lambda_a - \lambda_b \\ \mu &= \lambda_c - \lambda_d \end{aligned} \\ &= \sum_{J, M} \frac{(2J+1)}{4\pi} D_{M\mu}^{J*}(\phi, \theta, -\phi) \langle \lambda_c \lambda_d | \tilde{S}(E) | \lambda_a \lambda_b \rangle D_{M\lambda}^J(\phi, \theta, -\phi) \end{aligned}$$

Take incident beam as z axis to get:

$$\langle p \rightarrow \phi \lambda_c \lambda_d | S | p \rightarrow \phi \lambda_a \lambda_b \rangle = \sum_J (2J+1) \cdot \frac{1}{4\pi} \langle \lambda_c \lambda_d | \tilde{S}(E) | \lambda_a \lambda_b \rangle e^{i(\lambda-\mu)\phi} d_{\lambda\mu}^J(\theta)$$

This decomposition into partial waves of course applies to any amplitude S , be it A, M, SM , or whatever. The new item here is "helicity partial waves." The phase could be absorbed into S^* .

The remaining results I quote from Collins, altho they are also in S-W and Frazer:

$$\begin{aligned} \left\{ \begin{aligned} \frac{d\sigma}{d\Omega} &= \frac{q_{s_{34}}}{q_{s_{12}}} \cdot \frac{1}{(8\pi)^2 S} \cdot \left[\frac{1}{(2S_1+1)(2S_2+1)} \sum_{\substack{\lambda_1, \lambda_2 \\ \lambda_3, \lambda_4}} |\langle \lambda_3 \lambda_4 | A | \lambda_1 \lambda_2 \rangle|^2 \right. \\ \langle \lambda_3 \lambda_4 | A(s, t) | \lambda_1 \lambda_2 \rangle &= 16\pi \sum_J (2J+1) \langle \lambda_3 \lambda_4 | A_J(s) | \lambda_1 \lambda_2 \rangle d_{\lambda\mu}^J(\theta) \\ \text{parity} \left\{ \begin{aligned} \langle -\lambda_3 -\lambda_4 | A_J(s) | -\lambda_2 -\lambda_1 \rangle &= \eta_g \langle \lambda_3 \lambda_4 | A_J(s) | \lambda_2 \lambda_1 \rangle \\ \eta_g &= \eta_1 \eta_2 \eta_3 \eta_4 (-1)^{S_3+S_4-S_1-S_2} \\ P | JM \lambda_1 \lambda_2 \rangle &= \eta_1 \eta_2 (-1)^{J-S_1-S_2} | JM -\lambda_1 -\lambda_2 \rangle \\ P | JM LS \rangle &= \eta_1 \eta_2 (-1)^L | JM LS \rangle \leftarrow \text{eigenstate of } P! \end{aligned} \right. \\ \text{time reversal} \left\{ \begin{aligned} T | JM; \lambda_1 \lambda_2 \rangle &= (-1)^{J-M} | J, -M; \lambda_1 \lambda_2 \rangle \\ \langle \lambda_3 \lambda_4 | A_J(s) | \lambda_1 \lambda_2 \rangle &= \langle \lambda_2 \lambda_1 | A_J(s) | \lambda_3 \lambda_4 \rangle \end{aligned} \right. \end{aligned} \right. \end{aligned}$$

We are assuming that, e.g., time reversal + parity are invariances.

Simple properties of the $d_{\lambda\mu}(\theta)$ functions.

Definition:

$$d_{m'm}^{J}(\theta) = \langle Jm' | e^{-iJ_y\theta} | JM \rangle \quad (\text{p 23 FRA})$$

↖ operator.

Clearly at once we have:

$$d_{\mu\mu}^{J}(0) = \langle J\mu | J\lambda \rangle = \delta_{\mu\lambda} \quad (\text{assume } J \text{ is in proper range})$$

There is a messy general formula (eg, Tinkham p 110) for $d_{\lambda\mu}^{J}(\theta)$ but why bother. These symmetries exist:

$$d_{\lambda\mu}^{J}(\theta) = d_{-\mu, -\lambda}^{J}(\theta) = (-1)^{\lambda-\mu} d_{\mu\lambda}^{J}(\theta) = (-1)^{J+\lambda} d_{\lambda, -\mu}^{J}(\pi-\theta)$$

↑
negate both

↑
reverse order.

↑
change sign of one.

at π you get,

$$\left. \begin{aligned} d_{\lambda\mu}^{J}(\pi) &= (-1)^{J+\lambda} d_{\lambda, -\mu}^{J} \\ d_{\lambda\mu}^{J}(-\pi) &= (-1)^{\lambda-J} d_{\lambda, -\mu}^{J} \end{aligned} \right\} \begin{aligned} &\text{must have } \mu = -\lambda \text{ in order to get something} \\ &\text{at } \theta = \pi. \quad \text{at } \theta = 0 \text{ need } \mu = +\lambda. \end{aligned}$$

The normalization is:

$$\int_{-1}^1 d(\cos\theta) [d_{m\mu}^{J}(\theta)]^2 = \frac{2}{2J+1}$$

The function d is part of the full rotation matrix:

$$D_{n'n}^{J}(\alpha\beta\gamma) = e^{-i\alpha n} \cdot e^{-i\beta n'} \cdot d_{n'm}^{J}(\gamma)$$

Although there is always the problem of $\alpha n - \alpha$ (active vs passive) in all these things.

Another simple observation:

$$d_{00}^{J}(\theta) = P_J(\cos\theta) \quad (\text{Will p 49})$$

$$d_{m0}^{J}(\theta) = \sqrt{\frac{(J-m)!}{(J+m)!}} P_J^m(\cos\theta)$$

I have not been careful on these signs!

References:

1. Angular Momentum. This is a highly written-about subject. It includes such things as connection to group theory and representations and basis functions, to the Lie Algebras that go with the group generators. One always calculates the matrix elements in the operators J , learns how to combine angular momenta using Clebsch-Gordan's, plays with the actual representations (rotation matrices) and the basis vectors $|JM\rangle$ (sometimes called reps also). In the end one does tensor operators and Wigner-Eckart and observes that in the coord rep the Ylm's have a special group-theoretical meaning.
 - A. Williams pp 24 to 66. This is my favorite place to go to unravel questions involving angmom. Williams is an excellent writer and is logical.
 - B. Messiah pp 507 to 581. A big chapter with the same stuff but with some examples from atomic physics. See p 534 for sign convention in R .
 - C. Schiff pp 194 to 224. Again, the same old stuff. However, Schiff takes a slightly more general view and includes $SU(n)$ description. The straight ang mom discussion is too concise for a good reading, though. Schiff never struck me as a clear and logical writer. Too straight.
 - D. Condon and Shortley 1935. They were the original dudes to set the state phase conventions and have a big detailed chapter on ang mom in their atomic spectra book.
 - E. Tinkham has a chapter on FGR and group theory. Again all the usual results are obtained.
- In addition to these books which I possess, there are some monographs:
- F. Rose "Elementary Theory of Angular Momentum" 1957
 - G. Edmonds "Angular momentum in QM" 1957
 - H. Wigner "Group Theory and QM Apps" 1959
 - I. Brink and Stohler "Angular Momentum" 1962

2. Partial Wave analysis.

- a. Williams Chapter 3. Agin, this is my place-to-go. First he does the partial waves elastique. Then, inelasticity is added on. The $\frac{1}{2}-\frac{1}{2}$ and then the $0-\frac{1}{2}$ cases are done. Actually, $\frac{1}{2}-\frac{1}{2}$ is only mentioned, reference given.
- b. Schiff mentions the PW method, but as usual is too brief to be of any help to me. I am never happy about that book.
- c. Collins and Squires at least define what they are talking about in the partial wave expansion, see page 40
- d. Messiah does a little bit, pa 385. Jackson has $e^{ik \cdot r}$ expansion.

3. The helicity formalism

1. Frazer has the best account of it, period.
2. Jacob and Wick paper.
3. Collins and Squires chapter on spin quotes main results in their notation.

July 16, 1977

Stapp 1962 paper: Derivation of CPT and Spin-Statistics in S-matrix Theory.

I am now rereading this paper quickly for second or third time. Something I see now is that this paper treats topics which are in themselves detailed technologies and it is not reasonable to try to think about all of them at once. These technologies are always combined in Henry's later papers. Some of them are:

Postulatesland: Sections 1,2,3

M-function-land: notation and spinor conventions

Analyticityland: things required by unitarity, Landau, physical sheets, bubbles

Antiparticleland: continuations, crossing, CPT theorem

Henry uses so many words always in "most general form" that you always have trouble hearing him. He distains simple examples, something I thrive on.

1. Introduction. Henry will prove CPT and spin-statistics without using fields or even spacetime.

2. Postulates. Quantum theory amplitude and probability, Lorentz invariance, analyticity.

3. Remarks about postulates.

4. M-functions: in $M(K', \tilde{K}'')$ first argument describes final channel I think. Things are done with products of 2×2 matrices instead of the (o,s) which I like. Unitarity stated.

5. Complex Lorentz Transformations. Hall Wightman invoked.

6. CPT Theorem. Equation (6.2) just negates the 4-momenta. I think Henry's "crossing" is implicit in the notation (continue to different physical regions), so the momentum negation together with the implicit crossing gives CPT theorem. Not stated in a very physical way.

7. Spin Statistics. Proof uses symbol $(-1)^N$ where N = number of spin indices in channel K = number of fermions in channel $K \bmod 2$ so is like my symbol σ . Proof also involves reordering of same type particles. I do not follow the proof, but get the point. My proof will always be that of Froissart Taylor since so much crisper, shorter, cleaner. and overall better, even though Henry came first.

8. Unitarity. Talks about $R = S - 1$.

9. Lorentz invariance. A long section with no content I can discern.

10. Antiparticles. The usual remarks.

11. Order of Variables. Long diatribe about M-functions changing sign under interchange of like fermions. But no rules for different types. No rules even if spin indices differ.

12. Analytic structure. The unitarity singularity generation program.

(over for appendices)

A: Unitarity proof.

B: massless particles

C: Taylor like formalism

D: Higher Spins

E: P.C. and T.

F: the spin- $\frac{1}{2}$ /spin-0 case

G: system of two identical spin- $\frac{1}{2}$ particles

H: location of singularities of unitarity. Derives Landau, same as perturbation theory.

I: the physical sheet.

J: notes on Spinor Analysis

Stapp, 1962-- CPT paper.

I. Introduction: Discusses the "unwholesome air" which surrounds field theory and its dependence on the existence of spacetime points, and its associated problems. S-matrix theory is starting to survive on its own, and the time has come to get rid of unnecessary baggage. In this paper, Henry will prove the two major successes of field theory without using field theory but entirely within the framework of S-matrix theory. These successes are: (1) the CPT theorem and (2) connection of spin and statistics.

II. The Postulates . Just a listing, explanation given in next section:

- a) quantum theory
- b) Lorentz invariance (strong and weak form called B' and B)
- c) particles (make up experiment)
- d) conservation laws (up to here, you can define M-functions)
- e) maximal analyticity (E' is minimal analyticity)
- f) the physical connection (stuff actually applies to real world) ("persistence")

III. Remarks on Postulates. Key new idea is this: when everything goes complex, the physical sheets have only singularities needed by unitarity, and if you go to any physical boundary, the function will describe a physical process. Much more discussion, but I will skip this for now. More interested in M-functions at the moment. *Have done second reading, see added comments ↓.*

IV. The M-functions properties. A quick run down here. He has chosen several conventions

- a) lower undotted = particle index
- b) lower dotted = antiparticle
- c) the unitarity product has some funny spin matrices sitting in it
- d) statistics sign if you change order of two variables
- e) antiparticle if you reverse 4-momentum etc.

Obviously I will have to study these things more later. For now, Henry is giving us a bird's eye view. Notice the fancy relation between $-K$ and K . This change changes momenta, spin dottedness, and particle type to antiparticle.

V. Extension to Complex LT. The Lorentz transformation properties of the M-functions are stated in super compact notation as (5.1). Remember that each particle has its own matrix, there are spin index sums, and so on. Normally this thing is thought of in terms of proper Lorentz transformations, but Hall and Wightman showed that it is reasonable to analytically continue the equation which is the invariance condition of the M-functions to the complex Lorentz group. No details given, sounds fine to me.

VI. CPT theorem. By choosing a specific complex LT (which I do not understand at the moment), Henry shows that you can equate the M-function for an amplitude with the M-function with the CPT-related amplitude. All you get is a phase. In other words, the amplitude for a process and its CPT dual are the same, modulo a phase; this is what the CPT theorem says. I do not quite follow Stapp's use of twiddles, though. Once you accept complex LT's, the CPT theorem is immediate. No proof needed.

VII. Spin and Statistics. Henry gives a very short proof based on the simple unitarity equation satisfied by the M-functions. Has to do with orders, etc. Since the notation looks strange to me, I will postpone the proof. I will accept that it is very simple.

The point is: both the CPT Theorem and the Spin-Statistics Theorem are trivial once you accept the basic S-matrix tenets. Now Henry must show that these unitarity equations are correct, and so on.

VIII. Unitarity. Henry writes unitarity in terms of S , and then in terms of $R = S - 1$. He wants to remove the highly singular straight through stuff. Apparently, here is some of his notation:

$$S(K', -\tilde{K}) = \text{[Diagram: A circle with two incoming lines from the left labeled } K \text{ and } -\tilde{K} \text{, and two outgoing lines to the right labeled } K' \text{ and } -\tilde{K}'. \text{ Above the circle is a wavy line.]} \quad \text{mom cons.}$$

So the minus sign indicates that the things is "standardized" with all particles outgoing. The twiddles remind us that in the initial side, particles are anti'd

$$\leftarrow K' \quad \text{[Diagram: A circle with two incoming lines from the left labeled } K \text{ and } -\tilde{K} \text{, and two outgoing lines to the right labeled } K' \text{ and } -\tilde{K}'. \text{ Above the circle is a wavy line.]} \quad \leftarrow \tilde{K}'$$

$K' = +\tilde{K}'$ says mom cons.

relative to the "standard" form of all particles outgoing.

Unitarity for the M-functions is not derived in this section.

IX. Lorentz Invariance. Oh oh. Trouble. In order to talk about spin, Stapp wants to use some kind of projection operators. Clearly he is only thinking about spin-0 and spin-1/2 particles. I hate this because I don't see how to generalize it neatly other than saying the usual "we'll combine spin-1/2 particles to build up anything you want." Henry's M-functions are thus going to be matrices in a 2x2 space for each spin-1/2 particle; in the same sense that you have matrices in Dirac space. I guess the thing to remember is this:

$$D^{(0, \frac{1}{2})}(\tilde{B}(\nu)) = e^{-i \vec{v} \cdot (\vec{\sigma} \cdot \frac{1}{2} \vec{\sigma})} = e^{-i \vec{v} \cdot (\vec{\sigma} \cdot \vec{K})} = \left(\frac{\vec{p} \cdot \vec{\sigma}}{m} \right)^{\frac{1}{2}}$$

$$\text{where } \vec{p} = m\vec{v} \text{ and } p^0 = \sqrt{m^2 + \vec{p} \cdot \vec{p}}$$

In other words, for a spin-1/2 particle, it happens that the (0,S)-representation function for a boost has this simple form. I think Henry is going to define his single particle states in the canonical sense as boosted "direct" from the rest states, so that the above object tends to appear.

This restriction to spinors seems silly to me, I like Taylor's approach much more going with general (0,s) functions. Why does Henry stick to the spin-1/2 stuff???

Comments: I stop reading at this point because I now see that Stapp is not going to tell me what I want to know about the M-functions. Taylor is much more relevant. There are of course other interesting points made in this paper. The very recent book of Jagolnitzer in fact accepts Taylor's square states and so will I. (Iag 1974).

Conversation with Henry Stapp, July 12, 1977.

1. Why did he use $(0, \frac{1}{2})^S$ stuff instead of simply $(0, s)$? Answer: he was used to using the σ, ρ notation in the spin- $\frac{1}{2}$ case, so he kept on using it. Also, it is obvious that this extra factor which occurs in unitarity products does not damage any analyticity.

2. What is difference between $(\frac{1}{2}, \frac{1}{2})$ and $(0, 1)$? Although they are not unitary equivalent, you can go from one to the other by raising, lowering or dotting/undotting indices.

3. What is the status of the proof that you can do the analytic continuations? Answer: it is not rigorously proved that there can not exist something which can block your path. But it is extremely probably that it is OK. There is a non-relativistic theory counterexample where two branch points pinch right at the point where you want to do your crossing proof, but none in relativistic theory. The philosophy Henry accepts is that you will have only those singularities which are generated by the normal thresholds through unitarity. These are the Landau's and are therefore the same ones people encounter in perturbation field theory. There, you can do the continuations.

These continuations are necessary for the spin-statistics theorem, etc. Also crossing.

4. Does antiparticle have negative spin projection or not? First, he agrees that C should commute with Poincare generators. However, the combined CPT changes your physical spin (just think of particle at rest), even though it does not change the index on the spinor M -function.

5. What does Henry think of Barut's book and Barut? People have told him that they do not find the book that useful, Henry thinks it was perhaps a rushed job. Barut is perhaps too group centralized.

Exercise: what is Hermiticity in Taylor's notation?

$$\begin{aligned}
 \begin{array}{c} 5 \\ 4 \\ 3 \end{array} \begin{array}{c} \leftarrow \\ \leftarrow \\ \leftarrow \end{array} \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} \begin{array}{c} 1 \\ 2 \end{array} \begin{array}{c} \leftarrow \\ \leftarrow \end{array} = M^{+m_1 m_2 m_3 m_4 m_5} (p_1, p_2; p_3, p_4, p_5) \\
 \\
 = [p_5^{m_5}] \otimes [p_4^{m_4}] \otimes [p_3^{m_3}] S_c^+ [p_1^{m_1}] \otimes [p_2^{m_2}] / \bar{A} \\
 \\
 = \{ (p_1^{m_1}] \otimes (p_2^{m_2}] S_c [p_5^{m_5}] \otimes [p_4^{m_4}] \otimes [p_3^{m_3}] \}^* / \bar{A} \\
 \\
 = \frac{\{ [p_1^{m_1'}] \otimes [p_2^{m_2'}] S_c [p_5^{m_5'}] \otimes [p_4^{m_4'}] \otimes [p_3^{m_3'}] \}}{A} \cdot D_{m_1' m_1}^{(0, s_1)} (\gamma^{-1} L^2(p_1))^* \\
 \dots D_{m_5' m_5}^{(0, s_5)} (L^2(p_5) \gamma) \dots \}^* \\
 \\
 = \left[\begin{array}{c} 1 \\ 2 \end{array} \begin{array}{c} \leftarrow \\ \leftarrow \end{array} \begin{array}{c} \text{---} \\ \text{---} \end{array} \begin{array}{c} 5 \\ 4 \\ 3 \end{array} \begin{array}{c} \leftarrow \\ \leftarrow \\ \leftarrow \end{array} \right]^* \times \prod_{i=1}^2 D(\gamma^{-1} L^2(p_i)) \cdot \prod_{i=3}^5 D(L^2(p_i) \gamma)^*
 \end{aligned}$$

1. Thus in Taylor's notation there is a price to be paid. In order to relate a minus bubble amplitude to a plus bubble amplitude, you can't just star as in Stapp. You have to add some D-functions.

But this is not Hermiticity or Hermitian Analyticity. The process above is unnamed. You are basically just converting a S_c^+ operator to a S_c operator

2. Minus sign? Notice that the way I chose to define my M^+ object, you do not also get a minus sign in the above equation. If I had related M^+ to S^+ in the same way as M to S , then there would be an extra minus sign in the above! The bubble is always supposed to be the M -function. This extra minus sign explains why Stapp has a plus sign on the left side of his 1962 unitarity equations.

3. I think both Taylor's and Stapp's M -functions have Hermitian analyticity, ie:

$$\left[M^{m_1 m_2} (p_1, p_2, \dots) \right]^* = M^{m_1 m_2} (p_1^*, p_2^*, \dots)$$

Stapp '62

...within a considerably shorter interval of time ...heretofore and so indeed to consider occurrences ...most anywhere to be of a contemporary nature. In any ...the structure of the equations of motion which

have emerged even for "free particles" may perhaps provide us with other conceptual possibilities the starting point of which may lead to a theory with less startling consequences.

PHYSICAL REVIEW

VOLUME 125, NUMBER 6

MARCH 15, 1962

Derivation of the CPT Theorem and the Connection between Spin and Statistics from Postulates of the S-Matrix Theory*

HENRY P. STAPP

Lawrence Radiation Laboratory, University of California, Berkeley, California

(Received September 20, 1961)

The CPT theorem and the normal connection between spin and statistics are shown to be consequences of postulates of the S-matrix approach to elementary particle physics. The postulates are much weaker than those of field theory. Neither local fields nor any reference to space-time points are used. Quantum commutation relations and properties of the vacuum play no role. Completeness of the asymptotic states and positive definiteness of the metric are not required, though certain weaker asymptotic conditions prevail. The proofs depend on unitarity, macroscopic relativistic invariance, and a very weak analyticity requirement on the mass-shell scattering functions. The proofs are in the framework of the new S-matrix approach to elementary particle physics, which is established on a formal basis.

I. INTRODUCTION

THE two most important general physical consequences of relativistic field theory are the CPT theorem¹⁻⁴ and the connection between spin and statistics.⁵⁻¹⁰ The CPT theorem states that for every process occurring in nature there is an allowed dual process in which the particles of the first are replaced by their respective antiparticles, all spins are reversed, and paths are changed to their images under inversion through the origin in space-time. Relationships between probabilities are stated to be the same for a process and its dual. The proved connection between spin and statistics is that wave functions are symmetric under the interchange of variables referring to two identical integral-spin particles and antisymmetric for the half-integral-spin case.

These important results are derived from the postulates of local field theory, which, however, are subject to considerable doubt. In the first place it is not known whether the postulates are sufficiently realistic to

include any theories except trivial ones in which the scattering matrix is unity.

Secondly, the postulates are very specialized and restrictive, in that they assign a fundamental role to hypothetical local field operators defined over the field of space-time points. Experience does not entail the existence of such points, and the restriction to theories in which they play a fundamental role may immediately exclude all theories connected to physical reality. Because space-time points are experimentally inaccessible both in practice and in principle, their introduction runs counter to the philosophy of quantum mechanics. This philosophic inconsistency appears to have its analog in the mathematical structure in which related inconsistencies seem to arise.¹¹⁻¹³

Even within the general framework of local field theories, some of the postulates are so restrictive that many reasonable theories are excluded. In particular, the requirements of the completeness of the asymptotic states and the positive definiteness of the metric are assumed to hold, not only asymptotically, but also throughout the course of the interaction. But added states of negative metric not among those observed asymptotically seem to be exactly what are needed to remove the apparent inconsistencies from field theory. A theory based on this possibility is among those being

*This work was performed under the auspices of the U. S. Atomic Energy Commission.

¹G. Lüders, Kgl. Danske Videnskab. Selskab Mat.-fys. Medd. 28, No. 5 (1954).

²W. Pauli, Ed., *Niels Bohr and the Development of Physics* (Pergamon Press, New York, 1955).

³G. Lüders, Ann. Phys. 2, 1 (1957).

⁴R. Jost, Helv. Phys. Acta 30, 409 (1957).

⁵W. Pauli, Phys. Rev. 58, 716 (1940); Progr. Theoret. Phys. (Kyoto) 5, 526 (1950).

⁶R. P. Feynman, Phys. Rev. 76, 749 (1949).

⁷J. Schwinger, Phys. Rev. 82, 914 (1951).

⁸G. Lüders and B. Zumino, Phys. Rev. 110, 1450 (1958).

⁹N. Burgoyne, Nuovo cimento 8, 607 (1958).

¹⁰R. Jost, *Theoretical Physics in the Twentieth Century*, edited by Fierz and V. F. Weisskopf (Interscience Publishers, Inc., New York, 1960).

¹¹G. Källen, Kgl. Danske Videnskab. Selskab Mat.-fys. Medd. 27, 12 (1953).

¹²I. Pomeranchuk, Doklady Akad. Nauk S.S.S.R. 103, 1005 (1955); 104, 51 (1955); 105, 461 (1955); A. A. Abrikosov, A. D. Galanin, L. P. Gorkov, L. D. Landau, I. Pomeranchuk, and K. A. Ter-Martirosyan, Phys. Rev. 111, 321 (1958).

¹³L. D. Landau, in *Niels Bohr and the Development of Physics*, edited by W. Pauli (Pergamon Press, New York, 1955), p. 52.

Wick '62

where

$$\xi^2 = - \left(2 \frac{b_{11}}{\alpha^2} + 1 \right) \tau^2$$

RECEIVED: January 3, 1962

REFERENCES

1. G. COSTA AND B. T. FELD, *Ann. Phys. (NY)* **9**, 354 (1960).
2. B. T. FELD, *Ann. Phys. (NY)* **1**, 58 (1957).
3. B. T. FELD, *Ann. Phys. (NY)* **4**, 189 (1958).
4. J. M. BLATT AND V. F. WEISSKOPF, "Theoretical Nuclear Physics." Wiley, New York, 1952.
5. J. L. GAMMEL AND R. M. THALER, "Progress in Elementary Particle and Cosmic Ray Physics," Vol. V, Chap. II. North-Holland, Amsterdam, 1960.
6. M. TAKETANI, S. MACHIDA, AND S. OKUMA, *Progr. Theoret. Phys. (Kyoto)* **6**, 638 (1951); *ibid.* **7**, 45 (1952).
7. T. HAMADA, *Progr. Theoret. Phys. (Kyoto)* **24**, 1033 (1960).
8. R. A. BRYAN, *Nuovo cimento* **16**, 805 (1960).
9. S. GARTENHAUS, *Phys. Rev.* **100**, 900 (1955); *ibid.* **107**, 291 (1957).
10. J. L. GAMMEL AND R. THALER, *Phys. Rev.* **107**, 291 (1957).
11. K. A. BRUECKNER AND K. M. WATSON, *Phys. Rev.* **92**, 1023 (1953).
12. M. SUGAWARA AND S. OKUBO, *Phys. Rev.* **117**, 605 (1960).
13. H. FESHBACH, E. LOMON AND A. TUBIS, *Phys. Rev. Letters* **6**, 635 (1961).
14. P. SIGNELL AND R. E. MARSHAK, *Phys. Rev.* **109**, 1229 (1958).
15. S. OKUBO AND R. E. MARSHAK, *Ann. Phys. (NY)* **4**, 166 (1958).
16. R. CIRELLI AND G. STABILINI, *Suppl. Nuovo cimento* **20**, 157 (1961).

ANNALS OF PHYSICS: 18, 65-80 (1962)

Angular Momentum States for Three Relativistic Particles*

G. C. WICK

Brookhaven National Laboratory, Upton, New York

A complete set of angular momentum states for three free relativistic particles is described, the main purpose being to write formulas which are simple and explicit enough to be used in detailed calculations. The three particles may have arbitrary masses and spins. The scheme is a generalization of the states introduced by Dalitz in the study of π -meson decay. An angular-momentum recoupling coefficient is computed explicitly.

INTRODUCTION

Relativistic three-body systems at present attract a great deal of attention. It seems desirable to have some reasonably simple formalism to deal with angular momentum in such systems. The fundamental principles are, of course, well known, but relativistic kinematics introduces certain complications.

In this paper we discuss various ways in which a complete orthonormal set of angular momentum states can be constructed for three free particles (with or without spin); the unitary matrix, which transforms the different sets into one another, and various formulas that can be useful in calculations with these states are given. The problem is merely the practical one of setting up things in such a way that calculations do not become ugly and unmanageable. The method we shall use is, in a sense, an extension of that employed in a previous paper by Jacob and the author (1), to which the reader is referred for notations; some changes in notation will be explained as needed.

It would seem most natural to start from some convenient state in the center-of-mass system, for example, a state in which the particles have definite momenta q_1, q_2, q_3 satisfying the condition

$$q_1 + q_2 + q_3 = 0 \quad (1)$$

and apply to it the Wigner projection operator, compare JW, Eq. (16). If in addition, the three particles have definite helicities in the original state, they still do so after the projection, for the same reasons as in the two-body case.

With this procedure, however, it is not quite obvious how one obtains a complete and orthonormal system; I was led, therefore, sometime ago to consider

* Work performed under the auspices of the U.S. Atomic Energy Commission.

a somewhat different method which is an obvious relativistic adaptation of the procedure followed by Dalitz in the discussion of τ -meson decay (2). In this way, a simple discussion was given of various three-body decays (3). In the meantime, Shirokov (4) has given independently a discussion of three-body reactions and decays based essentially on the same generalization of the Dalitz scheme. Nevertheless our discussion of these states and some of our results may be sufficiently different from Shirokov's to warrant publication. We have omitted some parts of our work to avoid overlap as much as possible.¹

The relativistic generalization of the Dalitz procedure requires the following steps. We start with two particles, say 1 and 2, in their center of mass system Σ_{12} , and construct with the Wigner projection operator, a state with angular momentum quantum numbers j, m . If, more specifically, the procedure of JW Eqs. (16) and (18) is followed precisely, the two particles will have definite helicities λ_1, λ_2 and a total energy w in the Σ_{12} system. This state may thus be treated as the rest state ψ_{0m} of JW for a particle "12" of mass w and spin j . A Lorentz transformation in the z -direction can be used to transform it into a state with momentum q in the z -direction and "helicity" m , a state which plays the same role as ψ_{0m} in JW and may be designated provisionally as $\psi_{qm}(12)$. A subsequent rotation $R_{\Phi, \theta, \phi}$ will transform it into a state having a total momentum q in an arbitrary direction Θ, Φ . After the Lorentz transformation, of course, the spin state of particles 1 or 2 is a linear combination of different helicity states. The Lorentz transformation of helicity states is discussed in the Appendix; it is a trivial modification of the transformation law given by various authors (5-7) for other definitions of the spin states of a particle (8).

At this stage particle 3 may be introduced. The product $\psi_{qm}(12)\chi_{\lambda_3}(3)$ is formed, which then plays the role of JW Eq. (14). A new application of the Wigner projection operator will generate a three-particle state $|WJM\rangle$ where

$$W = \sqrt{w^2 + q^2} + \sqrt{m_3^2 + q^2} \quad (1')$$

is the total energy in the over-all rest system. A more complete symbol for the state is $|WJM; wjm \lambda_1 \lambda_2; \lambda_3\rangle$ which is self-explanatory. It should be noted that λ_1, λ_2 and λ_3 do not occur on the same footing since they are helicities in different reference systems.

What we have described may be called the $|(12)3\rangle$ coupling scheme. Another possible scheme is, of course, $|(23)1\rangle$, for example. Since both schemes are of interest in certain calculations (for instance when certain symmetrizations are necessary) we shall compute the "recoupling coefficient" $\langle (23)1 | (12)3 \rangle$.

¹ I wish to thank Dr. A. J. Macfarlane for sending me quite recently an abbreviated version of his thesis on "Relativistic Partial Wave Analysis," which pursues aims closely related to the present work. Macfarlane considers more general coupling schemes than we do, and his methods are rather different from ours.

ONE- AND TWO-PARTICLE STATES

We shall use the notation $|p; \lambda\rangle$ for the "helicity" states of JW Eq. (6). Here p denotes the four-momentum of the particle. We also depart slightly from JW in the phase and normalization of these states (see also the Appendix) that is, we multiply the states of JW by a factor $(2\omega)^{1/2} e^{-i\phi}$. In this way we have

$$\langle p' \lambda' | p \lambda \rangle = \delta_{\lambda\lambda'} \delta(p', p) \quad (2)$$

where δ is the invariant δ -function on the mass shell $\delta(p, p') = 2\omega \delta_3(\mathbf{p} - \mathbf{p}')$. This corresponds, of course, to using the invariant volume element on the mass-shell

$$d^3p = (2\omega)^{-1} d^4p \approx \delta(p^2 - m^2) d^4p \quad (2')$$

in place of the ordinary phase space element.

Consider now a Lorentz transformation l , which transforms the four-momentum p into say $p' = lp$. The corresponding Lorentz operator L transforms the state $|p; \lambda\rangle$ into a state of four-momentum p' ; this will be, in general, a linear combination of states $|p'; \mu\rangle$ with various helicities μ . Because of the normalization we have chosen, the linear combination is given by a unitary matrix U , the precise form of which is given in the Appendix. This matrix depends on both l and p ; we can also express it, of course, as a function of l and $p' = lp$. We write therefore:

$$U = U(l; p) = U(\overset{\text{initial } p}{p'}; \overset{\text{final } p}{p}) \quad (3)$$

The convention used is that the order of the variables indicates whether we use the initial or final momentum. Thus the transformation law may be written, for example (5-7):

$$L |p; \lambda\rangle = \sum_{\mu} U_{\mu\lambda}(l; p) |lp; \mu\rangle \quad (4)$$

A formula of this type will hold, no matter which rule is used to label the spin states. We shall use the "helicity" convention, see Appendix, Eq. (A.2). Let us notice that with the convention (A.2) the state does not have a definite limit (independent of φ) as θ tends to 0 or π . In these two limiting cases the meaning of the symbol $|p; \lambda\rangle$ has to be specified by an appropriate convention.²

² If one uses "helicity states," it is impossible to define the phase of the symbol $|p; \lambda\rangle$ so that it is continuous over the whole unit sphere. This means that some discontinuity of $U(l; p)$ as a function of l and p has to be accepted. These discontinuities will cause no trouble, provided one pays attention to them. With the definition (A.2), which has certain other advantages, we have not attempted, as we did in JW Eq. (6), to minimize the discontinuities. While JW Eq. (6) is only discontinuous at the "south pole," Eq. (A.2) is also discontinuous at the "north pole," and moreover, if s is a half integer, along the meridian $\varphi = \pm \pi$.

note on discontinuity in phase of states.

Specifically, for $\mathbf{p}$ parallel and respectively antiparallel to the z -axis we shall set, by definition, $|p; \lambda\rangle$ equal to the limit of Eq. (A.2) for $\vartheta = 0, \varphi = 0$ and respectively $\vartheta = \pi$ and $\varphi = \pi$.

We can now replace the state $\psi_{p\lambda_1\lambda_2}$; JW Eq. (14), with the symbol $|p_1^0\lambda_1; p_2^0\lambda_2\rangle$, which represents the direct product of the states $|p_1^0; \lambda_1\rangle$ and $|p_2^0; \lambda_2\rangle$ for particles 1 and 2 respectively. Here $\mathbf{p}_1^0 = -\mathbf{p}_2^0$ is a vector pointing in the positive z -direction.

The angular momentum states for 2 particles in the c.m. system may now be introduced as in JW Eqs. (16) and (18). Let j, m be the angular momentum quantum numbers, $w = (p^2 + m_1^2)^{1/2} + (p^2 + m_2^2)^{1/2}$, the total energy in the c.m. system. Changing the notation of JW somewhat (for the normalization see later) we write:

$$|wjm; \lambda_1 \lambda_2\rangle = (p/4w)^{1/2} \mathfrak{N}_j \int d\Omega \mathfrak{D}_{\lambda m}^j(r^{-1}) R |p_1^0 \lambda_1; p_2^0 \lambda_2\rangle \quad (5)$$

$$\lambda = \lambda_1 - \lambda_2; \mathfrak{N}_j = [(2j+1)/4\pi]^{1/2}; \quad d\Omega = \sin \vartheta d\vartheta d\varphi$$

where r is the rotation with Euler angles $\varphi\vartheta\gamma$, $R = R_{\varphi\vartheta\gamma}$ the corresponding operator. We have used the unitarity of $\mathfrak{D}^j$ to change $\mathfrak{D}_{\lambda m}^j(r)$ to $\mathfrak{D}_{\lambda m}^j(r^{-1})$. Owing to the condition $\lambda = \lambda_1 - \lambda_2$, the integrand of (5) does not really depend on the third Euler angle, so that we were able to omit the integration over γ , compare JW Eq. (18).

Let now h be a Lorentz transformation belonging to the three-parameter set of transformations such that

$$hP_{12}^0 = P_{12} \quad (6)$$

where $P_{12}^0 = (w, 0)$ and P_{12} is a specified value of the four momentum with the same invariant mass as P_{12}^0 . In particular let h be the special Lorentz transformation $h(P_{12})$ defined in the Appendix. The corresponding Lorentz operator $H \equiv H(P_{12})$ is then of the form

$$H = R_{\varphi, 0, \vartheta} Z \quad (7)$$

According to the general convention used, e.g. the "helicity" convention, the states of the system "12" with mass w and spin j and with momentum $\mathbf{P}_{12} \neq 0$ can be generated by applying the operator $H \equiv H(P_{12})$ to the states (5), so that

$$|P_{12}; jm; \lambda_1 \lambda_2\rangle = H(P_{12}) |wjm; \lambda_1 \lambda_2\rangle \quad (8)$$

$$= (p/4w)^{1/2} \mathfrak{N}_j \int d\Omega \mathfrak{D}_{\lambda m}^j(r^{-1}) L |p_1^0 \lambda_1; p_2^0 \lambda_2\rangle$$

* The purpose of this convention is to make $|p^0 \lambda\rangle$, for example, completely equivalent to JW Eq. (13). Thus none of the formulas of JW have to be modified (consider in particular the symmetry properties discussed in JW, Section V).

where $r, d\Omega$ have the same meaning as before and

$$L = H(P_{12}) R_{\varphi\vartheta\gamma} = R_{\varphi, 0, \vartheta} Z R_{\varphi\vartheta\gamma} \quad (9)$$

is the operator for a transformation l which also belongs to the set (6).

When P_{12} is given, the set of transformations l obeying (6) is specified. Picking l out of this set is the same as picking $\varphi, \vartheta, \gamma$ since we have

$$\mathbf{r} = h^{-1}(P_{12})l \quad (10)$$

In this sense, the integration over $d\Omega$ in (8) is an integration over l varying over the set (6). Provided $\mathbf{r}$ is tied to l by (10), l and $\mathbf{r}$ need only be specified "up to γ " and a convenient way to do this is to give, instead of l , the two vectors

$$\mathbf{p}_1 = l\mathbf{p}_1^0; \quad \mathbf{p}_2 = l\mathbf{p}_2^0 \quad (11)$$

These, of course, must be chosen to lie on the mass-shells of 1 and 2, respectively, and to satisfy the condition

$$\mathbf{p}_1 + \mathbf{p}_2 = P_{12} \quad (12)$$

Defining the invariant volume elements on the mass-shell $\bar{d}p_i$ in the usual way (Eq. (2')) while the invariant mass w and relative momentum p are, of course, given by

$$w^2 = (p_1 + p_2)^2; \quad 4w^2 p^2 = \Delta(w^2, m_1^2, m_2^2); \quad (13)$$

$$\Delta(a, b, c) = a^2 + b^2 + c^2 - 2ab - 2bc - 2ca$$

one can see, by an easy transformation of variables, that

$$\bar{d}p_1 \bar{d}p_2 = (p/4w) d^4(p_1 + p_2) d\Omega \quad (14)$$

where $d\Omega$ is the solid angle for the direction of the relative momentum in a c.m. system.

We can now write (8) in a compact form if we express the effect of L on the two one-particle states by means of Eq. (4) and use (14) to replace $\int \dots d\Omega$ by an integration over p_1 and p_2 . Finally:

$$|P_{12}; jm; \lambda_1 \lambda_2\rangle = \mathfrak{N}_j (4w p)^{1/2} \int \bar{d}p_1 \bar{d}p_2 \delta_4(p_1 + p_2 - P_{12}) \quad (15)$$

$$\cdot \mathfrak{D}_{\lambda m}^j(r^{-1}) \sum_{\lambda_1 \lambda_2} U_{\lambda_1 \lambda_1}^{(1)}(p_1; l) U_{\lambda_2 \lambda_2}^{(2)}(p_2; l) |p_1 \lambda_1; p_2 \lambda_2\rangle$$

A more symmetric and compact form is obtained if we notice that, according to the conventions, $h(P_{12}^0)$ is the unit transformation, so that comparing with Appendix, Eq. (A.3) we can write

$$\mathbf{r}^{-1} = h^{-1}(P_{12}^0) l^{-1} h(P_{12}); \quad \mathfrak{D}^j(r^{-1}) = U^{(12)}(\Gamma^{-1}; P_{12}) \quad (16)$$

BMW '63

possesses a continuous symmetry group under which the ground or vacuum state is not invariant, that state is, therefore, degenerate with other ground states. This implies a zero-mass boson. Thus, the solid crystal violates translational and rotational invariance, and possesses phonons; liquid helium violates (in a certain sense only, of course) gauge invariance, and possesses a longitudinal phonon; ferro-magnetism violates spin rotation symmetry, and possesses spin waves; superconductivity violates gauge invariance, and would have a zero-mass collective mode in the absence of long-range Coulomb forces.

It is noteworthy that in most of these cases, upon closer examination, the Goldstone bosons do indeed become tangled up with Yang-Mills gauge bosons and, thus, do not in any true sense really have zero mass. Superconductivity is a familiar example, but a similar phenomenon happens with phonons; when the phonon frequency is as low as the gravitational plasma frequency, $(4\pi G\rho)^{1/2}$ (wavelength $\sim 10^4$ km in normal matter) there is a phonon-graviton interaction: in that case, because of the peculiar sign of the gravitational interaction, leading to instability rather than finite

mass.¹² Utiyama¹³ and Feynman have pointed out that gravity is also a Yang-Mills field. It is an amusing observation that the three phonons plus two gravitons are just enough components to make up the appropriate tensor particle which would be required for a finite-mass graviton.

Spin waves also are known to interact strongly with magnetostatic forces at very long wavelengths,¹⁴ for rather more obscure and less satisfactory reasons. We conclude, then, that the Goldstone zero-mass difficulty is not a serious one, because we can probably cancel it off against an equal Yang-Mills zero-mass problem. What is not clear yet, on the other hand, is whether it is possible to describe a truly strong conservation law such as that of baryons with a gauge group and a Yang-Mills field having finite mass.

I should like to thank Dr. John R. Klauder for valuable conversations and, particularly, for correcting some serious misapprehensions on my part, and Dr. John G. Taylor for calling my attention to Schwinger's work.

¹² J. H. Jeans, *Phil. Trans. Roy. Soc. London* 101, 157 (1903).

¹³ R. Utiyama, *Phys. Rev.* 101, 1597 (1956); R. P. Feynman (unpublished).

¹⁴ L. R. Walker, *Phys. Rev.* 105, 390 (1957).

PHYSICAL REVIEW

VOLUME 130, NUMBER 1

1 APRIL 1963

Construction of Invariant Scattering Amplitudes for Arbitrary Spins and Analytic Continuation in Total Angular Momentum*

ASIM O. BARUT,[†] IVAN MUZINICH,[‡] AND DAVID N. WILLIAMS

Lawrence Radiation Laboratory, University of California, Berkeley, California

(Received 9 November 1962)

From group-theoretical considerations, invariant scattering amplitudes for two-body reactions of particles with arbitrary spins and nonzero masses are constructed in various forms, including helicity amplitudes and amplitudes free of kinematical singularities. They are linear combinations of spin basis functions with scalar coefficients. In the process of construction the Pauli spin matrices are generalized for arbitrary spin. On the basis of a Mandelstam representation for the scalar coefficients, the unique analytic continuation of the amplitudes in total angular momentum is obtained. Possible kinematical singularities of the scalar amplitudes at the boundary of the physical region are discussed.

I. INTRODUCTION

THE basic quantities of S -matrix theory are the Lorentz-invariant scattering matrix elements (S functions), which depend on the spins and types of incoming and outgoing particles and on the mass shell values of their four-momenta. From the S functions, invariant scattering amplitudes (M functions) that have simpler transformation properties and that are expected to be free of kinematical singularities can be defined.¹ A general procedure has been given to con-

struct the invariant amplitudes in terms of the irreducible unitary representations of the inhomogeneous proper Lorentz group, based on a two-component spinor formalism.²

Although the invariant scalar amplitudes for which the Mandelstam representation is expected to be valid have been known for some time in the simpler cases such as those of the pion-nucleon³ and nucleon-nucleon⁴

Theory [W. A. Benjamin, Inc., New York (to be published)].

¹ A. O. Barut, *Phys. Rev.* 127, 321 (1962).

² G. F. Chew, M. L. Goldberger, F. E. Low, and Y. Nambu, *Phys. Rev.* 106, 1337 (1957).

³ M. L. Goldberger, M. T. Grisaru, S. W. MacDowell, and D. Y. Wong, *Phys. Rev.* 120, 2250 (1960), (referred to hereafter as GGMW); D. Amati, E. Leader, and B. Vitale, *Nuovo Cimento* 17, 68 (1960).

* Work done under the auspices of the U. S. Atomic Energy Commission.

[†] Present address: University of Colorado, Boulder, Colorado.

[‡] Present address: University of Washington, Seattle, Washington.

¹ H. P. Stapp, *Phys. Rev.* 125, 2139 (1962); *Lectures on S-Matrix*

big deal on proving CPT etc.

very short

scattering
atic con-
spins.³ T
the invar
reactions
to constr
(for exam
scalar am
Second, i
Mandela
amplitude
momentu
cases.⁷ I
tioned, i
have no
this que
separate
that one
to scalar

In the
systema
but ma
pate.

Apart
sideration
of pract
higher s
analytic
Many of
volving
restrict
matrices
Pauli ma
other ap
projection
spaces of

II. D

The fo
are rather
in this se
but first
various i
is only a
appears
of compli

⁶ A. C. I
amplitudes
theory f

⁷ A const
point of vi
Departme

⁸ S. C. P
Rev. 126, 2
Proceedi
Energy P
1962) and
Singh, Pl

A. The Invariant Functions

We consider scattering processes for outgoing particles and incoming antiparticles with spins and four-momenta S_i, k_i , and incoming particles and outgoing antiparticles with spins and four-momenta S_j, k_j , all with nonzero rest masses. The invariant scattering functions (or S-matrix elements) have the following transformation property under representations of the inhomogeneous orthochronous proper Lorentz group^{2,8}:

$$S(K) = \otimes_i \mathcal{D}^{S_i}[A'(-k_i)] \otimes_j \mathcal{D}^{S_j}[A'(k_j)]^* S[A^{-1}(A)K], \quad (2.1)$$

where

$$A'(k) = B_{k \leftarrow p}^{-1} A B_{q \leftarrow p} \quad \text{and} \quad \Lambda(A^{-1})k = q.$$

Here K stands for the set of incoming and outgoing four-momenta, k_n , with $\sum_n k_n = 0$ from momentum conservation; and ΛK stands for the set of transformed momenta, Λk_n . Elements of the orthochronous proper homogeneous Lorentz group L_{+1} are denoted by $\Lambda(A)$, where $\pm A$ are the corresponding elements of the two-by-two unimodular group. The spin indices of the S function, which have been suppressed, are transformed by direct products of the unitary matrices $\mathcal{D}^{S_i}$ and $\mathcal{D}^{S_j*}$, which are the well-known $[(2S_i+1), (2S_j+1)]$ -dimensional irreducible representations of the three-dimensional proper real orthogonal group. An index transforming according to $\mathcal{D}^S$ corresponds to an outgoing particle or incoming antiparticle and one transforming according to $\mathcal{D}^{S*}$ corresponds to an incoming particle or outgoing antiparticle.⁹ In the argument $A'(k)$ of $\mathcal{D}^S$ or $\mathcal{D}^{S*}$, the unimodular matrices B are so defined that

$$\Lambda(B_{k \leftarrow p})p = k, \quad k = \vec{B}p$$

and similarly for $B_{q \leftarrow p}$. The Lorentz transformation corresponding to the unitary-unimodular matrix $A' = B_{k \leftarrow p}^{-1} A B_{q \leftarrow p}$ transforms the vector p into itself (it is an element of the little group of the vector p), where $p = (m, 0, 0, 0)$ is the rest-frame value of k ; hence, this transformation is a rotation.¹⁰

From the definition of p and Eq. (A1.1) in Appendix 1, we have, in terms of Pauli matrices, σ_μ ,⁸

$$B \vec{B} = B_{k \leftarrow p} B_{k \leftarrow p}^\dagger = k^\mu \sigma_\mu / m. \quad (2.2)$$

The general solution of this equation can be written in the form $B_{k \leftarrow p} = A_{k \leftarrow p} U$, where $A_{k \leftarrow p}$ is the Hermitian matrix $(k \cdot \sigma / m)^{1/2}$ and U is an arbitrary unitary matrix corresponding to the freedom of arbitrary rotations in the rest system of the particle. We use this freedom later in the construction of helicity amplitudes. An

scattering systems, there is to our knowledge no systematic construction of such amplitudes for arbitrary spins.⁵ The purpose of this paper is, first, to construct the invariant M functions of arbitrary spin for two-body reactions (two particles in, two particles out), and also to construct the S functions in various representations (for example, the helicity representation) in terms of scalar amplitudes and explicitly given basis functions.⁶ Second, it is our purpose to define, on the basis of a Mandelstam representation for the two-body scalar amplitudes, an analytic continuation in total angular momentum that generalizes the recent work on simpler cases.⁷ In pion-nucleon scattering, as already mentioned, there exist scalar amplitudes that are known to have no kinematical singularities. An investigation of this question for arbitrary spin will be reported in a separate paper. We proceed here on the assumption that one among a large class of possible bases will lead to scalar amplitudes without poles.

In this paper, we ignore isotopic spin and give no systematic discussion of C , P , and T transformations, but make only occasional comments where appropriate.

Apart from their theoretical interest, the considerations involving higher spins will be, we believe, of practical importance in connection with the new higher spin resonances, and perhaps in the problem of analytic continuation in spin of the S -matrix elements. Many of these considerations apply to processes involving arbitrary numbers of particles and are not restricted to two-body systems. For example, the spin matrices introduced in this paper generalizing the Pauli matrices to higher spins may be of interest in other applications. From these matrices we obtain the projection operators for the irreducible invariant subspaces of the tensors of arbitrary rank.

II. DEFINITION OF INVARIANT FUNCTIONS AND GENERAL PROCEDURE

The formulas developed in the succeeding sections are rather involved. To facilitate the reading, we outline in this section the procedure that we have followed; but first we define the transformation laws of the various invariant functions. It is often said that spin is only an inessential complication. Nevertheless, it appears that except in simple cases a certain amount of complication is, if not essential, at least unavoidable.

⁵ A. C. Hearn, *Nuovo Cimento* **21**, 333 (1961), discusses the amplitudes for spin $\frac{1}{2}$ and photon processes, in a perturbation theory framework.

⁶ A construction of the M functions from a somewhat different point of view has been given by Willard Miller, Mathematics Department, University of California, Berkeley, California.

⁷ S. C. Frautschi, M. Gell-Mann, and F. Zachariasen, *Phys. Rev.* **126**, 2204 (1962). For the N - N system, see V. N. Gribov, in, *Proceedings of the 1962 Annual International Conference on High-Energy Physics at CERN*, edited by J. Prentki, (CERN, Geneva, 1962) and I. Muzinich, *Phys. Rev.* (to be published). See also V. Singh, *Phys. Rev.* **129**, 1889 (1963).

⁸ For a list of conventions, notation, and various important relations involving two-component spinors, two-by-two matrices, and group representation matrices, see Appendix I.

⁹ This choice is purely conventional, especially since the two representations are equivalent. It agrees with the usual convention in the four-component formalism, as will be seen incidentally in Appendix II. See also the references in footnote 1.

polar decomposition

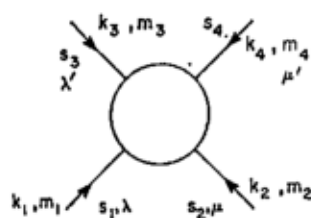


FIG. 1. Two-body scattering parameters.

important characteristic of the invariant M functions defined below is that their transformation property is independent of B , and hence of nonanalytic S stuff!

The transformation law (2.1) also holds for the R functions,

$$R = S - I. \quad (2.3)$$

There is a natural way of simplifying this transformation law. Because the matrix $A'(k) = B_{k \leftarrow p}^{-1} A B_{q \leftarrow p}$ is unitary, we have the identity

$$\mathcal{D}^S[A'(k)] = \mathcal{D}^{(S,0)}[A'(k)] = \mathcal{D}^{(0,S)}[A'(k)], \quad (2.4)$$

where $\mathcal{D}^{(S,S')}$ are the irreducible, in general, nonunitary, representations of dimension $(2S+1)(2S'+1)$ of $L_+\uparrow$. We can, then, use the group property of $\mathcal{D}^{(S,0)}$ and obtain

$$\mathcal{D}^S(B_{k \leftarrow p}^{-1} A B_{q \leftarrow p}) = \mathcal{D}^{(S,0)}(B_{k \leftarrow p})^{-1} \times \mathcal{D}^{(S,0)}(A) \mathcal{D}^{(S,0)}(B_{q \leftarrow p}). \quad (2.5)$$

Thus, if we introduce M functions defined by

$$M(K) = \otimes_i \mathcal{D}^{(S_i,0)}(B_{-k_i \leftarrow p_i}) \otimes_j \mathcal{D}^{(S_j,0)}(B_{k_j \leftarrow p_j})^* R(K), \quad (2.6)$$

we see from (2.1) that they have the simple transformation law under $L_+\uparrow$,

$$M(K) = \otimes_i \mathcal{D}^{(S_i,0)}(A) \otimes_j \mathcal{D}^{(S_j,0)}(A)^* M[A(A^{-1})K]. \quad (2.7)$$

It is simpler to construct the solutions of this equation than those of (2.1). Equations (2.3), (2.6), and (2.7) are the basic formulas from which the construction of the M and S functions begins. For spin $\frac{1}{2}$ these are just the M functions introduced by Stapp.¹

B. The Scalar Amplitudes

For practical purposes, such as the application of the Mandelstam representation, it appears convenient to use a representation of the invariant functions in which all of the dynamics is contained in a set scalar amplitudes. In a sense this removes spin from the problem. Our problem is, thus, to find a simple, explicit set of basis functions, $Y^{(i)}(K)$, in the spin space which have the same transformation property (2.7) as the M functions. Then we write

$$M(K) = \sum_{(i)} A^{(i)}(K) Y^{(i)}(K), \quad (2.8)$$

where the $A^{(i)}(K)$ are Lorentz scalars and must, therefore, be functions of the scalar invariants formed from

the four-momenta (and possibly of the signs of the energies). One can also require that the basis functions $Y^{(i)}(K)$ have definite transformation properties under P and T . Thus, if P and T are conserved, the total number of independent scalar amplitudes will be smaller than the $\prod_i (2S_i+1) \prod_j (2S_j+1)$ resulting from (2.7) and (2.8).

The essential requirement on the scalar amplitudes is that they shall have only the singularities of the M function itself, which on the basis of perturbation theory or of a pure S -matrix theory are expected to be only dynamical.¹ Furthermore, we wish to require that the basis functions themselves have no singularities. The simplest possibility is that the basis functions should be polynomials in the components of the linear momenta. To require that the basis functions have this form is not enough, however, for the scalar amplitudes could still have kinematical poles at various degenerate points where the basis functions become linearly dependent. Indeed, the question of whether there exists a set of basis functions that never induces kinematical poles in the scalar amplitudes already involves considerable subtlety in the case of two-body reactions; and therefore, we shall restrict ourselves primarily to this case in any discussion where the singularities are important.

The question of to what extent these various requirements determine a set of basis functions is not settled in this paper. Rather we seek to establish a basic formalism for arbitrary spins that can be used in the construction of a large class of basis functions. We follow a procedure that is natural and systematic, and that yields the usual analytic amplitudes in special cases. It consists first of building up in Sec. III a set of higher spin matrices from the spin- $\frac{1}{2}$ matrices, σ_μ , by using Clebsch-Gordan coefficients in a process corresponding to the addition of spins. For two-body reactions we then, in Sec. IV, combine the spin matrices with tensors formed from the four-momenta to obtain a set of basis functions, $Y^{(i)}(K)$; and we give a brief discussion of the question of kinematical poles in the resulting scalar amplitudes. (If preliminary results are substantiated, a second paper showing how to eliminate the kinematical poles will be submitted by one of us (DNW).)

C. Angular Momentum

In Sec. V we define an analytic continuation in total angular momentum for the scattering functions shown in Fig. 1. For this purpose it is convenient to use helicity amplitudes. Having constructed $Y^{(i)}(K)$ and, therefore, $M(K)$ by (2.8), we obtain the helicity amplitudes $H(K)$ from (2.6) by making the appropriate choice for B in the expression

$$R(K) = \mathcal{D}^{(S_4,0)}(B_{-k_4 \leftarrow p_4})^{-1} \otimes \mathcal{D}^{(S_2,0)}(B_{-k_2 \leftarrow p_2})^{-1} \otimes \mathcal{D}^{(S_1,0)}(B_{k_1 \leftarrow p_1})^{-1} \otimes \mathcal{D}^{(S_3,0)}(B_{k_3 \leftarrow p_3})^* \times \sum_{(i)} A^{(i)}(K) Y^{(i)}(K), \quad (2.9)$$

where r
Mandel

with s -
are defi

$B_{k \leftarrow p} \Rightarrow$

where q
from th
rotation

With
of-mass
for any
angular
amplitu
function

where Z
by the

indices
The p
of $H(\lambda)$

where s
are the

final pa
Wenc
expansi

where v
 $= d^i(\theta)_0$

If we
 $d^i(\theta)_0$ in
integrati

we obta

where Z
of Clebs
values at

From

¹⁰ M. Ja
For rotat
dotted inu

where now the $A^{(i)}(K)$ can be taken as functions of the Mandelstam variables,

$$s = (k_1 + k_2)^2, \quad t = (k_1 + k_3)^2, \quad u = (k_1 + k_4)^2,$$

with $s + t + u = \sum_i m_i^2$. The helicity amplitudes $H(K)$ are defined to be $R(K)$ when

$$B_{k \rightarrow p} = (k \cdot \sigma / m)^{1/2} \exp(-i\phi\sigma_3/2) \exp(-i\theta\sigma_2/2) \times \exp(i\phi\sigma_3/2), \quad (2.10)$$

$$= \exp(-i\phi\sigma_3/2) \exp(-i\theta\sigma_2/2) \times \exp(i\phi\sigma_3/2) (q \cdot \sigma / m)^{1/2},$$

where $q^\mu = (k_0, 0, 0, |k|)$, i.e., a velocity transformation from the rest frame to the z direction followed by a rotation to the direction (θ, ϕ) of k .¹⁰

Without loss of generality we can put, in the center-of-mass frame of the s channel, $\phi = 0$. It turns out that for any among a large class of basis functions the angular dependence (θ dependence) of the helicity amplitudes can be factored into a product of $d^S(\theta)$ functions in the form

$$H_{(\lambda)}(K) = \sum_{(i), R} A^{(i)}(s, t, u) Z_{(\lambda), R}^{(i)} d^R(\theta), \quad (2.11)$$

where $Z^{(i)}$ does not depend upon θ , and R is determined by the spins of the particles. Here (λ) stands for the indices $(\mu', \lambda', \mu, \lambda)$ and $d^S(\theta) \equiv \mathcal{D}^{(S, 0)}[\exp(i\theta\sigma_2/2)]$.

The projection over the total angular momentum J of $H_{(\lambda)}$ is defined by¹⁰

$$h_{(\lambda)}^J(s) = \frac{1}{2} (qq')^{1/2} \int_{-1}^1 dz d^J_{\Delta\lambda, \Delta\lambda'} H_{(\lambda)}, \quad (2.12)$$

where $z = \cos\theta$, $\Delta\lambda = \lambda - \mu$, $\Delta\lambda' = \lambda' - \mu'$, and q and q' are the magnitudes of the momenta of the initial and final particles, respectively.

We now write for the scalar amplitudes a partial-wave expansion in the s channel, for example,

$$A^{(i)}(s, t, u) = \sum_l (2l+1) A^{(i)}(l, s) d^l(\theta) d^l(\theta_0), \quad (2.13)$$

where we put for the Legendre polynomials, $P_l(z) = d^l(\theta) d^l(\theta_0)$.

If we insert this into $H_{(\lambda)}$ and combine $d^R(\theta)$ with $d^l(\theta) d^l(\theta_0)$ into a single d function and perform the angular integration, which is of the form

$$\frac{1}{2} \int_{-1}^1 dz d^J(\theta) d^R(\theta) d^l(\theta) d^l(\theta_0) = \frac{1}{2J+1} \delta_{JR}, \quad (2.14)$$

we obtain

$$h_{(\lambda)}^J(s) = \sum_{i, (l)} A^{(i)}(l, s) \bar{Z}_{(\lambda)}^{(i)}, \quad (2.15)$$

where $\bar{Z}$ contains a sum of the original Z times a number of Clebsch-Gordan coefficients. In the above sum the l values are restricted by the given J .

From the fixed-energy dispersion relation for

¹⁰ M. Jacob and G. C. Wick, *Ann. Phys. (N. Y.)* **7**, 404 (1959). For rotations, an upper undotted index transforms as a lower dotted index.

$A^{(i)}(s, t, u)$ we express $A^{(i)}(l, s)$ in terms of the absorptive parts $A_i^{(i)}$ and $A_u^{(i)}$ of the amplitudes in the crossed channels and obtain

$$h_{(\lambda)}^J(s) = \sum_{i, (l)} \bar{Z}_{(\lambda)}^{(i)} \left[\int_{-1}^1 dz Q_l(z) A_i^{(i)}(s, z) + (-1)^l \int_{-1}^1 dz Q_l(z) A_u^{(i)}(s, z) \right], \quad (2.16)$$

where the $Q_l(z)$ are Legendre functions of the second kind. Assuming that the absorptive parts A_i and A_u are uniformly bounded in l and u by l^N (or u^N), we see that the expression (2.16) defines an analytic function of J for $\text{Re} J > N'$, where N' is displaced from N by some integer determined by the spins of the particles.

Details are given in Sec. V.

III. CONSTRUCTION OF SPIN MATRICES

It is convenient to separate into two parts the construction of the basis functions $Y^{(i)}(K)$ for arbitrary spin. In this section we construct a set of matrices which span the spin space and which contain most of the complications in the transformation law due to spin. These matrices are independent of the four-momenta in the problem, except under special circumstances to be mentioned later; they have essentially no effect on the singularity structure of the scalar amplitudes. The results of this section apply to M functions that describe arbitrary numbers of particles.

The matrices that span a given spin space are labeled with tensor indices in addition to spin indices labeling their matrix elements. A complete set of basis functions $Y^{(i)}(K)$ is obtained by contracting the tensor indices of the spin matrices with a complete set of tensor functions which are polynomials in the components of the four-momenta. Given a spin basis, it is the construction of a basis for the space of tensor functions that can lead to possible kinematical poles in the scalar amplitudes. This question is discussed in Sec. IV.

A. Spin- $\frac{1}{2}$ Matrices

The basis for general spin is constructed from two-component Pauli spinors. Since the total number of incoming and outgoing fermions in any scattering process must be even,¹¹ the simplest case that we can consider involves two spin- $\frac{1}{2}$ particles, one incoming, the other outgoing.

Equation (2.7) then becomes

$$M(K) = A \otimes A^* M[A(A^{-1})K] = AM[A(A^{-1})K]A^\dagger, \quad (3.1)$$

¹¹ Because $\mathcal{D}^{50}(-A) = (-1)^{2S} \mathcal{D}^{50}(A)$ and $\Lambda(-A) = \Lambda(A)$, we have

$$M(K) = M[\Lambda(-I)K] = (-1)^{2S} \mathcal{D}^{50}(I) M(K) = (-1)^{2S} \mathcal{D}^{50}(I) M(K).$$

Hence $\sum_i S_i$ must be an integer if $M(K)$ is not to vanish identically.

$$M = g m g^\dagger$$

or, writing the spinor indices,

$$M_{\alpha\dot{\beta}}(K) = \lambda_{\alpha\alpha'} A_{\dot{\beta}\dot{\beta}'} M_{\alpha'\beta'} [\Lambda(A^{-1})K]. \quad (3.2)$$

As usual, the dotted index (incoming particle or outgoing antiparticle) transforms according to A^* and the undotted index (outgoing particle or incoming antiparticle) according to A .¹² Any two-by-two matrix can be written as a linear combination of Pauli matrices, σ_μ . Hence, we can put

$$M(K) = f^\mu(K) \sigma_\mu. \quad (3.3)$$

From the transformation law of σ_μ given by (A1.1), it is clear that we must have

$$\Lambda_\mu^\nu f_\nu(\Lambda^{-1}K) = f_\mu(K), \quad (3.4)$$

if (3.1) is to be satisfied. *ve, $\Sigma_\mu = 4vec$.*

The four-vector function $f^\mu(K)$ can be expanded in terms of the four-momenta K , but that construction is reserved for Sec. IV.

If we define

$$\rho_\mu = (1/\sqrt{2})\sigma_\mu = \sigma'_\mu \text{ of } \text{Bade-Jullien} \quad (3.5)$$

and

$$\bar{\rho}_\mu = (1/\sqrt{2})\bar{\sigma}_\mu,$$

where $\bar{\sigma}_\mu$ is defined in Appendix I, the orthogonality relations (A1.6) in Appendix I become

$$\rho_{\mu\alpha\dot{\beta}} \bar{\rho}^{\mu\dot{\beta}\alpha'} = \delta_{\alpha\alpha'} \delta_{\dot{\beta}\dot{\beta}'}, \quad (3.6)$$

and

$$\rho^{\mu\alpha\dot{\beta}} \bar{\rho}_{\mu\alpha'\dot{\beta}'} = C_{\alpha\alpha'} C_{\dot{\beta}\dot{\beta}'},$$

where C is the "lowering" spinor defined in (A1.2).

The general formalism of the theory also requires basis spinors with two undotted or two dotted indices. Such spinors can be obtained in several different ways. For example, the matrices $\rho_\mu \bar{\rho}_\nu C^{-1}$ have lower undotted indices, and they certainly span the space. There is a choice, however, introduced by Stapp,¹ that is natural and especially convenient for a discussion of crossing relations. It consists in defining the special spinors

$$g_{\alpha\dot{\beta}}(k) = k \cdot \sigma_{\alpha\dot{\beta}} / m, \quad (3.7)$$

which can be used to change a dotted index into an undotted one and vice versa, where k is taken to be the four-momentum of the particle whose spin index is to be operated upon.

We then define basis spinors

$$\begin{aligned} \omega_\mu(k)_{\alpha\beta} &= g_{\alpha\dot{\alpha}} \bar{\rho}_\mu^{\dot{\alpha}\beta} = (k \cdot \bar{\sigma} \bar{\rho}_\mu C^{-1} / m)_{\alpha\beta}, \\ \bar{\omega}_\mu(k)_{\dot{\alpha}\dot{\beta}} &= \bar{\rho}_\mu^{\alpha\dot{\alpha}} g_{\alpha\dot{\beta}} = (C \bar{\rho}_\mu k \cdot \sigma / m)_{\dot{\alpha}\dot{\beta}}. \end{aligned} \quad (3.8)$$

These spinors transform according to $A \otimes A$ and $A^* \otimes A^*$, respectively. For example,

$$\Lambda \omega^\mu(k) A^T = \Lambda_\nu^\mu(A) \omega^\nu[\Lambda(A)k]. \quad (3.9)$$

¹² A review of spinor calculus with conventions for dotted and undotted indices is included in Appendix I.

They satisfy orthogonality relations

$$\omega^\mu(k)_{\alpha\beta} \omega_\mu(k)_{\alpha'\beta'} = \delta_{\alpha\alpha'} \delta_{\beta\beta'} \quad (3.10)$$

and

$$\omega^\mu(k)_{\alpha\beta} \bar{\omega}_\mu(k)_{\alpha'\beta'} = C_{\alpha\alpha'} C_{\beta\beta'},$$

with corresponding formulas for dotted indices.

A spin basis for arbitrarily many spin- $\frac{1}{2}$ particles is obtained by taking direct products of matrices chosen from among ρ_μ , $\omega_\mu(k)$, and $\bar{\omega}_\mu(k)$, depending on the desired index types.

B. Properties of Matrices for Arbitrary Spin

Many of the characteristics of the spin matrices for higher spin are a straightforward generalization from the spin- $\frac{1}{2}$ matrices and can be understood without going through the details of a somewhat involved construction. Before proceeding to the actual construction, we shall, therefore, describe the essential results.

As already indicated, the fermion spin indices can always be paired; and we can also pair the boson indices by adding a dummy spin-0 index whenever the total number of particles is odd. Thus, we require a basis for matrices with two fermion or two boson spin indices; any spin space can be spanned with direct products of these. This basis is given by a set of rectangular matrices $\rho^{\mu_1 \dots \mu_{2M}}(SS')$, $\bar{\rho}^{\mu_1 \dots \mu_{2M}}(SS')$, $\omega^{\mu_1 \dots \mu_{2M}}(SS'; k)$, and $\bar{\omega}^{\mu_1 \dots \mu_{2M}}(SS'; k)$, where $M = \max(S, S')$, which span the spin- S , spin- S' space, and which reduce to (3.5) and (3.8) when $S = S' = \frac{1}{2}$. Here S and S' are the spins of the pair of bosons or fermions. The spin indices labeling the matrix elements have $2S+1$, $2S'+1$ values, respectively, ranging through $S, S-1, \dots, -S$ and $S', S'-1, \dots, -S'$.

1. Transformation Properties

The spin matrices just described are classified according to the representations of L_{+1} of the types $\mathcal{D}^{(S,0)}(A)$, $\mathcal{D}^{(S,0)}(A^*)$, or the respective contragredient representations $\mathcal{D}^{(S,0)}(A^{-1}T)$, $\mathcal{D}^{(S,0)}(A^{-1}T^*)$. The whole apparatus of the spinor calculus can be taken over for arbitrary spin. The spin indices will be written as lower undotted, lower dotted, upper undotted, and upper dotted, respectively, corresponding to the four representations listed above. The contraction of an upper with a lower index of the same type is then an invariant operation.

The raising of a spin- S index is accomplished by contracting on the right with the matrix

$$\mathcal{D}^{(S,0)}(C^{-1})^{\alpha\beta} = \mathcal{D}^{(S,0)}(C^{-1})^{\dot{\alpha}\dot{\beta}} = (-1)^{S-\alpha} \delta_{\alpha,-\beta}, \quad (3.11)$$

and lowering by contracting on the right with

$$\mathcal{D}^{(S,0)}(C)_{\alpha\beta} = \mathcal{D}^{(S,0)}(C)_{\dot{\alpha}\dot{\beta}} = (-1)^{S-\beta} \delta_{\alpha,-\beta}. \quad (3.12)$$

The spinor for changing dotted to undotted indices

$$(\omega^\mu(k) C^{-1})^{\alpha\beta} \text{ itself}$$

McK '64

Deane

Relativistic Two- and Three-Particle States.

A. McKERRELL

Department of Natural Philosophy, The University - Glasgow

(ricevuto l'8 Giugno, 1964)

Summary. — Two-particle states transforming according to standard representations of the inhomogeneous Lorentz group are constructed which, in the centre-of-mass system, are eigenstates of orbital and spin angular momentum operators. The connection between these and helicity states is derived and a recoupling coefficient for three-particle states in the former scheme found.

1. - Introduction.

The unitary representations up to a factor of the inhomogeneous Lorentz group were studied by WIGNER⁽¹⁾, who showed that any such representation is equivalent to a two-valued representation. The components of the generator of translations (the energy-momentum four-vector) commute and their eigenvalues label the wave-functions (or states) of the representation, which is built from a representation of the «little group», the group of transformations which leave invariant a given four-momentum. For positive mass (a timelike four-momentum) the little group is isomorphic to the rotation group and we can construct, for each positive m and nonnegative integral or half-integral s , an irreducible representation of the inhomogeneous Lorentz group describing a particle of rest-mass m and spin s .

The direct product of two such representations describes the system of two noninteracting particles and is reducible. The irreducible components are labelled by the energy and angular momentum in the centre-of-mass system and correspond to the original type of single-particle representation. However, in general any such representation may appear several times and additional labels

⁽¹⁾ E. P. WIGNER: *Ann. Math.*, 40, 149 (1939).

Summary of McKernell Results.

① J_i and K_i obvious. Then S_i defined in fancy way, then $L_i \equiv J_i - S_i$.

$$[L_i, S_j] = 0 \quad \vec{P} \cdot \vec{L} = 0 \quad [P_i, S_j] = 0 \quad \left[L(P)^{-1} S_i L(P) = J_i \right]_{\text{on rest state.}}$$

② $|\vec{p}m\rangle \equiv \text{rest state.}$

③ $|pm\rangle_c \equiv L(P) |\vec{p}m\rangle$

$$R^S |pm\rangle_c = |pm\rangle_c D(R)$$

$$S_3 |p,m\rangle_c = m |p,m\rangle_c$$

$$R^L |pm\rangle_c = |p,m\rangle_c$$

$$R |p,m\rangle_c = |p,m\rangle_c D(R)$$

④ $|pm\rangle_H \equiv H(P) |\vec{p}m\rangle = |pm\rangle_c D(R(P))$

$$R^S |pm\rangle_H = |p,m\rangle_H D[R(P)^{-1} R R(P)]$$

$$R^L |p,m\rangle_H = |p,m\rangle_H D[R(P)^{-1} R(P)]$$

$$R |p,m\rangle_H = |p,m\rangle_H D[R(P)^{-1} R R(P)]$$

$$\vec{J} \cdot \vec{P} |p,m\rangle_H = m |p,m\rangle_H$$

Comments on paper of McKerrell on Spin States etc.

1. This is the best reference I know of (Nov 1976) about this subject.
2. Mc opens, as he should, with discussion of the 10 generators of Poincare, the usual P^μ and the six $M^{\mu\nu}$. In terms of these generators one constructs the famous vector W^μ which gives the second "spin" Casimir of the Poincare. Then, the objects J_i, K_i are given to replace the $M^{\mu\nu}$. In terms of J and K we get a fancy looking definition of the objects S_i . Finally, L_i is defined in terms of $L = J - S$. So, at last we have all the objects J, L, S all fitted into the relativistic Poincare framework.
3. The first defined state is the single particle at rest, called $|\tilde{p}\rangle$. For such a state, the objects J and S are the same and $L = 0$ since it is their difference. A single particle at rest has no orbital angmom. The action of rotations (generated either by J or S) gives the trivial rotation matrix shuffle of these rest states.
4. Now comes the definition of single particle states not at rest and the distinction between canonical and helical states. Let $L(p)$ be the direct boost out to p from rest. I.e, $L(p)$ is a boost off in some strange direction. Clearly $L(p)$ acting on the rest state gives a state associated with particle moving at p. Alternatively, you could first z-boost $Z(p)$, and then rotate into position. This combination is called $H(p) = R(p) Z(p)$. So here then are the two kinds of single particle states not at rest:

$$|p\rangle_c = L(p) |\tilde{p}\rangle \quad |p\rangle_h = H(p) |\tilde{p}\rangle$$

5. Canonical states: these are eigenstates of S^2 and S_3 , even though they are not at rest. In general, S acts on these canonical states in the same way that J acts on rest states. So, R^S (rotation generated by S) gives simple rotation matrices. R^J gives same rotation matrices, but shifts p to Rp . I.e, J somehow also rotates the three-momentum. Finally, R^L only moves the momentum, there are no rotation matrices at all. Thus, these are intuitively very pleasant states to deal with.

6. Helicity states: Mc does not give detailed action of various R's on these states, but is easy to deduce from the relation to canonical states. The difference between the H-state and the C-state is simply a mixing of helicities via the ~~rotation~~ rotation matrix of the rotation called $R(p)$ which you used after the z-boost to get your helicity state. Both states of course have the same momentum. Helicity state is eigenstate of $J \cdot P$, ~~XXXXXX~~.

7 Now some vague intuitive remarks: somehow, the C-state "never got rotated".

The spin is still doing the same thing it did in the rest frame; only the momentum got rotated because you did the "direct symmetric boost" $L(p)$.

On the other hand, the H-state is the same as the C-state except that to get the H-state you pre-rotate by $R(p)$ before applying $L(p)$. This preorients the spin, say, along the p direction so that helicity is preserved.

This explains the relation between the two kinds of states shown in (3.30). Starting with an H state, you must rotate with $D^S(R(p))$ to get to H-state.

McKerrell's Construction of Two-Particle States.

1. It is obvious how to form the direct product, momentum-space two particle state. It is this:

$$|m_1, \vec{p}_1; s_1, m_1\rangle \otimes |m_2, \vec{p}_2; s_2, m_2\rangle.$$

The problem is that this state is not very interesting. By this I mean that this state is an eigenstate of P with eigenvalue $(p_1 + p_2)$ and that is all. It is not an eigenstate of S or L or J or any combination of these.

2. McKerrell's first step is this: consider the above state in the cms frame with particle 1 going in the plus z . Then combine these states with Clebsch's to get eigenstates of total spin operator. So:

$$|\vec{p}, s, m\rangle = \sum_{m_1, m_2} C(s_1, s_2, s, m, m_1, m_2) |p_1, m_1; p_2, m_2\rangle$$

We now have states which are eigenstates of S^2 , S_3 , and P . Under S -rotation, this state does just what you would think.

3. Next, these z -states are L -rotated, and weighted with spherical harmonics and integrated all round. This results in states which are eigenstates of L^2 and L_3 and S^2 and S_3 . A very standard operation. If you consider the action of a full J rotation on these states $|l_1 s s_3\rangle$, you find that the two pieces rotate independently. This is because L and S commute.

4. Finally, the L and S are combined with another round of Clebsches so that you end up with states that diagonalize all these operators: J, L, S, P . Also J_3 . This last state "looks like" a single particle of spin J and mass w , and it rotates under J accordingly. Then you can boost it out with $L(p)$ just as you do for a single particle state. In this last step you can do either the helicity or canonical type boost.

5. Remark: in atomic physics of a two-electron atom, the above procedure would be called L - S coupling. The difference here is that there is only one orbital angular momentum, not two, because there are only two particles not three.

proof of (4.11)

$$① \quad |\tilde{p} l l_3 s s_3\rangle = \int d(\cos\theta) d\phi \cdot D_{0l_3}^l(\phi, \theta, \phi') R^L(-\phi', -\theta, 0) |\tilde{p} s s_3\rangle.$$

From (4.9) we know that $L_3 |\tilde{p} s s_3\rangle = 0$ so $R^L(-\phi) |\tilde{p} s s_3\rangle = |\tilde{p} s s_3\rangle.$

This is because in $|\tilde{p} s s_3\rangle$, $\vec{p}_1, \vec{p}_2$ are along the z -axis. Thus, we can drop the above to get:

$$= \int d\phi d\phi' D_{0l_3}^l(\phi, \theta, \phi') R^L(-\phi', -\theta, -\phi) |\tilde{p} s s_3\rangle.$$

Since the integrand does not depend on ϕ , we add $\int d\phi: \frac{1}{2\pi}$. Then we go to group notation:

$$|\tilde{p} l l_3 s s_3\rangle = \int_G dg D_{0l_3}^l(g) R^L(g^{-1}) |\tilde{p} s s_3\rangle$$

② Now rotate this state & look what happens:

$$\begin{aligned} R^L |\tilde{p} \dots\rangle &= \int_G dg D_{0l_3}^l(g) R^L(g, g') |\tilde{p} \dots\rangle \\ &= \int_G dg D_{0l_3}^l(gg') R^L(g') |\tilde{p} \dots\rangle \quad \text{Then} \quad \begin{matrix} g \rightarrow gg' \\ g' \rightarrow g'g' \end{matrix} \\ &= \sum_{l'_3} \underbrace{D_{l'_3 l_3}^l(g_1) \int_G dg D_{0l'_3}^l(g) R^L(g') |\tilde{p} \dots\rangle}_{|\tilde{p} l l'_3 s s_3\rangle} \end{aligned}$$

$$\text{Thus: } R^L(g_1) |\tilde{p} l l_3; s s_3\rangle = \sum_{l'_3} D_{l'_3 l_3}^l(g_1) |\tilde{p} l l'_3; s s_3\rangle.$$

● (3.15) In (3.12), $L(p) = e^{-iS K \cdot \hat{p}} =$ boost in direction $\hat{p}$. From my notes "Lorentz" this yields:

$$L(p)^{\mu}_{\nu} = B_{\mu\nu} = \begin{pmatrix} \dots & \dots \\ \vdots & \vdots \end{pmatrix}$$

where $B_{\mu\nu}$ = symmetric matrix and

$$B_{00} = \text{shr}$$

where $\text{shr} = \frac{E}{m}$ $\text{sho} = \frac{p}{m}$

$$B_{0i} = \text{sho } n_i$$

$$B_{ij} = (\text{shr} - 1)n_i n_j + \delta_{ij}$$

$$p^i = n^i p$$

Clearly we can rewrite this as

$$B_{00} = \frac{p_0}{m}$$

$$B_{0i} = \frac{p_i}{m}$$

$$B_{ij} = \frac{p_i p_j}{m(m+p_0)} + \delta_{ij}$$

So $L(p)^{\mu}_{\nu} = B_{\mu\nu}$

● (3.16) Let V^{μ} = a vector operator. Then in general, one knows that

$$L^{-1} V^{\mu} L = L^{\mu}_{\nu} V^{\nu}$$

Apply this to S^{μ} . So $L^{-1}(m S^i) L = L^i_{\nu} S^{\nu}$

But: $m S^{\mu} = W^{\mu} - \frac{W^0}{(m+p_0)} p^{\mu}$

So $L^{-1}(m S^i) L = L^i_{\mu} W^{\mu} - \frac{W^0}{(m+p_0)} L^i_{\mu} p^{\mu}$

● (2.9) This reads: $L^{-1} p^{\mu} L = L^{\mu}_{\nu} p^{\nu}$ Here are some extensions:

$$L^{-1} p^{\mu} p^{\mu} L = (L^{-1} p^{\mu} L)(L^{-1} p^{\mu} L) = (L^{\mu}_{\nu} p^{\nu})(L^{\mu}_{\alpha} p^{\alpha}) = (L^{\mu}_{\nu} p^{\nu})^2$$

$$L^{-1} (p^{\mu})^n L = (L^{\mu}_{\nu} p^{\nu})^n$$

$$L^{-1} f(p^{\mu}) L = f(L^{\mu}_{\nu} p^{\nu}) = f(p^{\mu})$$

↑ here, f = an operator function of p^{μ}

example: $\bar{L}^{-1} \left(\frac{1}{m+p^0} \right) L = \frac{1}{m+p^0} = \frac{1}{m+l_{\mu}^0 p^{\mu}}$

more: $\bar{L}^{-1} \begin{pmatrix} V_1^{\mu} & V_2^{\nu} \end{pmatrix} L = V_1'^{\mu} V_2'^{\nu} \quad V_1, V_2 = \text{vectors.}$

$\bar{L}^{-1} (f(V_1, V_2)) L = f(V_1', V_2' \dots)$

Very general: $\boxed{\bar{L}^{-1} f(V_1, V_2, \dots) L = f(V_1', V_2', \dots)}$

Here, f = an operator-function of operator-variables V_i , which happen to be vectors-operators.

example: $\bar{L}^{-1} (P^i W^0) L = P^i{}' W^0{}'$

(3.16) We use the above results applied to (2.12)

$$\bar{L}^{-1} (m S^i) L = \bar{L}^{-1} (W^i - (m + P^0)^{-1} P^i W^0) L$$

$$= W^i{}' - (m + P^0{}')^{-1} P^i{}' W^0{}'$$

$$= l_{\mu}^i W^{\mu} - (m + l_{\mu}^0 P^{\mu})^{-1} l_{\nu}^i P^{\nu} l_{\mu}^0 W^{\mu} \quad \checkmark$$

Properties of rest states:

$$P^{\mu} |\tilde{p}, \nu\rangle = \delta_{\mu 0}^{\mu} m |\tilde{p}, \nu\rangle$$

$$P^2 |\tilde{p}, \nu\rangle = m^2 |\tilde{p}, \nu\rangle$$

$$J^2 |\tilde{p}, \nu\rangle = s(s+1) |\tilde{p}, \nu\rangle$$

$$J_3 |\tilde{p}, \nu\rangle = \nu |\tilde{p}, \nu\rangle$$

$$W^{\mu} |\tilde{p}, \nu\rangle = \frac{1}{2} \epsilon^{\mu\nu\sigma\rho} P_{\nu} M_{\sigma\rho} |\tilde{p}, \nu\rangle$$

$$= \frac{1}{2} \epsilon^{\mu\nu\sigma\rho} \{ M_{\sigma\rho} P_{\nu} + [P_{\nu}, M_{\sigma\rho}] \} |\tilde{p}, \nu\rangle$$

$$= \frac{1}{2} \epsilon^{\mu\nu\sigma\rho} \{ M_{\sigma\rho} P_{\nu} + i (g^{\nu\sigma} P_{\rho} - g^{\nu\rho} P_{\sigma}) \} |\tilde{p}, \nu\rangle$$

Well, note:

$$\begin{aligned}\epsilon^{\mu\nu\sigma\rho} P_\nu M_{\sigma\rho} &= \epsilon^{\mu\nu\sigma\rho} \{ M_{\sigma\rho} P_\nu + [P_\mu, M_{\sigma\rho}] \} \\ &= \epsilon^{\mu\nu\sigma\rho} \sum_A \{ M_{\sigma\rho} P_\nu + i g_{\nu\sigma}^A P_\rho - i g_{\nu\rho}^A P_\sigma \} \\ &= \epsilon^{\mu\nu\sigma\rho} M_{\sigma\rho} P_\nu\end{aligned}$$

Thus: $\boxed{\epsilon^{\mu\nu\sigma\rho} P_\mu M_{\sigma\rho} = \epsilon^{\mu\nu\sigma\rho} M_{\sigma\rho} P_\nu}$

Now then:

$$\begin{aligned}W^\mu |\tilde{p}, \nu\rangle &= \frac{1}{2} \epsilon^{\mu\sigma\rho\tau} M_{\sigma\rho} \tilde{m} |\tilde{p}, \nu\rangle \\ &= \frac{1}{2} m \epsilon^{\mu\sigma\rho\tau} M_{\sigma\rho} |\tilde{p}, \nu\rangle\end{aligned}$$

Thus $W^0 |\tilde{p}, \nu\rangle = 0$

$$W^i |\tilde{p}, \nu\rangle = \frac{1}{2} m \epsilon^{ijk} M_{jk} |\tilde{p}, \nu\rangle = m J^i |\tilde{p}, \nu\rangle$$

$$\boxed{\begin{aligned}W^i |\tilde{p}, \nu\rangle &= m J^i |\tilde{p}, \nu\rangle \\ W^0 |\tilde{p}, \nu\rangle &= 0\end{aligned}}$$

$$m S^i |\tilde{p}, \nu\rangle = \frac{W^i |\tilde{p}, \nu\rangle}{m J^i} - (m + P^0)^{-1} P^i \underbrace{W^0 |\tilde{p}, \nu\rangle}_0$$

$$\Rightarrow \boxed{S^i |\tilde{p}, \nu\rangle = J^i |\tilde{p}, \nu\rangle}$$

$$\Rightarrow \boxed{L^i |\tilde{p}, \nu\rangle = 0}$$

[(3.17)]^T Using last two results:

$$\begin{aligned}L^i (m S^i) L |\tilde{p}, \nu\rangle &= \underline{L^i_\mu W^\mu} |\tilde{p}, \nu\rangle - (m + L^0_\mu P^\mu) L^i_\nu P^\nu L^0_\alpha W^\alpha |\tilde{p}, \nu\rangle \\ &= L^i_j W^j |\tilde{p}, \nu\rangle - (m + L^0_\mu P^\mu) L^i_\nu P^\nu L^0_j W^j |\tilde{p}, \nu\rangle \\ &= L^i_j m J^j |\tilde{p}, \nu\rangle - (m + L^0_\mu P^\mu) L^i_\nu P^\nu L^0_j m J^j |\tilde{p}, \nu\rangle\end{aligned}$$

Claim: $P^\nu J^j |\tilde{p}, \nu\rangle = J^j P^\nu |\tilde{p}, \nu\rangle$

Reason: difference is $[P^\nu, J^j] |\tilde{p}, \nu\rangle$

$$= [P^\nu, \frac{1}{2} \epsilon^{jkl} M^{kl}] |\tilde{p}, \nu\rangle$$

$$= \frac{1}{2} \epsilon^{jkl} [P^\nu, M^{kl}] |\tilde{p}, \nu\rangle = \frac{1}{2} \epsilon^{jkl} i (g^{\nu k} P^l - g^{\nu l} P^k) |\tilde{p}, \nu\rangle = 0.$$

So: $[P^\nu, J^j] |\tilde{p}, \nu\rangle = 0.$

Therefore:

$$L^{-1} (S^i) L = L^{-1} \gamma^0 \gamma^i \gamma^0 \gamma^j L = \gamma^0 \gamma^i \gamma^0 \gamma^j - (\gamma^0 + L^0 \gamma^0)^{-1} \gamma^0 \gamma^i \gamma^0 \gamma^j |\tilde{p}, \nu\rangle$$

or: $L^{-1} (S^i) L = \gamma^0 \gamma^i \gamma^0 \gamma^j - (1 + L^0)^{-1} \gamma^0 \gamma^i \gamma^0 \gamma^j |\tilde{p}, \nu\rangle$

$$= \left\{ \gamma^0 \gamma^i \gamma^0 \gamma^j - \frac{\gamma^0 \gamma^i \gamma^0 \gamma^j}{(1 + L^0)} \right\} \gamma^0 \gamma^j |\tilde{p}, \nu\rangle.$$

$$= \left\{ \delta^{ij} + \frac{p^i p^j}{m(m+p_0)} - \frac{p^i}{m} \frac{p^j}{m} \frac{m}{(m+p_0/m)} \right\} = \delta^{ij}$$

Therefore: $L^{-1} (S_i) L |\tilde{p}, \nu\rangle = J_i |\tilde{p}, \nu\rangle$

where $L = L(p)$ = the direct boost out to p .

(3.20) From (3.17) we get:

$$L(p)^{-1} R^S L(p) = R^J$$

Thus $R^S |p\rangle_c = R^S L(p) |\tilde{p}, \nu\rangle = L(p) R^J |\tilde{p}, \nu\rangle$

$$= L(p) \sum_{\mu} D_{\mu\nu}^S(R) |\tilde{p}, \mu\rangle = \sum_{\mu} D_{\mu\nu}^S(R) |p, \mu\rangle_c \quad \square$$

(3.19) We know that

$$L^{-1} P^\mu L = L^\mu_\nu P^\nu$$

$$\begin{aligned} \text{So } P^\mu |p, \nu\rangle_c &= P^\mu L(p) |\tilde{p}, \nu\rangle = L(p) L^\mu_\nu P^\nu |\tilde{p}, \nu\rangle \\ &= L(p) L^\mu_{\nu=0} m |\tilde{p}, \nu\rangle = L(p) p |\tilde{p}, \nu\rangle = p |p, \nu\rangle_c. \end{aligned}$$

(3.22) This is a special case of our (2.9). Recall:

$$L^{-1} f(p) L = f(p') \quad \rightarrow \quad \tilde{R}^{-1} f(p) R = f(\tilde{R}p) \quad \checkmark$$

$$\begin{aligned} (3.23) \quad R |\nu\rangle_c &= R L(p) |\tilde{p}, \nu\rangle = L(rp) R |\tilde{p}, \nu\rangle \\ &= L(rp) \sum_\mu D_{\mu\nu}^S(R) |\tilde{p}, \mu\rangle = \sum_\mu D_{\mu\nu}^S(R) |rp, \mu\rangle_c \quad \square \end{aligned}$$

$$(3.27) \quad L |\nu\rangle_c = L L(p) |\tilde{p}, \nu\rangle = ?$$

$$\text{Consider: } L_1 \equiv L(l_p)^{-1} L L(p) \quad L = \text{arbitrary.}$$

$$\begin{aligned} \text{Well: } L_1 \tilde{p} &= l(l_p)^{-1} \cdot l \cdot l(p) \cdot \tilde{p} = l(l_p)^{-1} \cdot l \cdot p \\ &= l(l_p)^{-1} \cdot (l p) = \tilde{p}. \end{aligned}$$

Notation is a bit weak: consider $\Lambda_{p'}^{-1} \Lambda_p$ where:

$$\Lambda_p = L(p). \quad \text{Apply to } \tilde{p}. \quad \Lambda_p \tilde{p} = p. \quad \Lambda_{p'} = p'. \quad \Lambda_{p'}^{-1} p' = \tilde{p}.$$

$$\text{Thus, } \Lambda_{p'}^{-1} \Lambda_p = \text{a rotation} = R(\Lambda_p, p) = \text{Wigner.}$$

$$\text{So } L(l_p)^{-1} L L(p) = \bar{R}(L, p)$$

$$\begin{aligned} \text{Then: } L |\nu\rangle_c &= L L(p) |\tilde{p}, \nu\rangle = L(l_p) \bar{R}(L, p) |\tilde{p}, \nu\rangle \\ &= L(l_p) D_{\mu\nu}^S(\bar{R}(L, p)) |\tilde{p}, \mu\rangle = D_{\mu\nu}^S(\bar{R}(L, p)) |lp, \mu\rangle_c. \end{aligned}$$

Then translate.

Helicity states: Why are they helicity eigenstates?

$$\vec{J} \cdot \vec{L} (\vec{L} \cdot \vec{L})^{-\frac{1}{2}} |p, \nu\rangle_H$$

$$= \vec{J} \cdot \vec{L} (\vec{L} \cdot \vec{L})^{-\frac{1}{2}} H(p) |p, \nu\rangle$$

$$= \vec{J} \cdot \vec{L} (\vec{L} \cdot \vec{L})^{-\frac{1}{2}} R(p) \vec{L}(p) |p, \nu\rangle$$

OK recall that:

$$L^i f(v_1, v_2, \dots) L = f(v_1^i, v_2^i, \dots)$$

Restrict this to:

$$\vec{R}^i f(\vec{J}, \vec{P}) R = f(\vec{J}', \vec{P}')$$

two 3-vectors.

In particular, let R be defined by $\vec{P}' = R \vec{P}$ where $\vec{P} = (0, 0, p)$. I.e., in frame " ν ", $P = P'$, $J = J'$.
Then:

$$\vec{R}^i(p) f(\vec{P}', \vec{J}') R(p) = f(P, J)$$

$$\text{On } f(P, J) = R(p) f(\vec{P}', \vec{J}') \vec{R}^i(p)$$

$$\text{Now let's take: } f(P, J) = (\vec{J} \cdot \vec{L}) \cdot (\vec{L} \cdot \vec{L})^{-\frac{1}{2}}$$

$$\text{Thus: } \boxed{\vec{J} \cdot \vec{L} \cdot (\vec{L} \cdot \vec{L})^{-\frac{1}{2}} |p, \nu\rangle_H}$$

$$= f(J, P) |p, \nu\rangle_H = R(p) f(\vec{P}', \vec{J}') \vec{R}^i(p) \cdot \vec{L}(p) |p, \nu\rangle$$

$$= R(p) J_3 \frac{1}{|\vec{p}|} \cdot \vec{L}(p) |p, \nu\rangle$$

$$\text{But } [J_3, K_3] = 0 \text{ so } = \frac{R(p)}{|\vec{p}|} \vec{L}(p) J_3 |p, \nu\rangle =$$

$$= R(p) \vec{L}(p) \nu |p, \nu\rangle = \nu H(p) |p, \nu\rangle = \boxed{\nu |p, \nu\rangle_H}$$

Thus, we have shown that

$$\underbrace{\mathcal{J} \cdot \mathcal{L}}_{\equiv h} (\mathcal{R} \cdot \mathcal{L})^{-\frac{1}{2}} |p, v\rangle_H = v |p, v\rangle_H$$

where $|p, v\rangle_H = H(p) |p, v\rangle = R(p) Z(p) |p, v\rangle$. We see also why this would not be true for canonical states, because you do not have that $R(p)$ to cancel $R^{-1}(p)$.

$$\begin{aligned} \text{In fact: } h |p, v\rangle_c &= h \cdot \sum_{\mu} \bar{D}_{\mu\nu}^S(\tilde{R}(p)) |p, \mu\rangle_H \\ &= \sum_{\mu} \mu D_{\mu\nu}^S(\tilde{R}(p)) |p, \mu\rangle_H \end{aligned}$$

$$= \sum_{\mu} \mu D_{\mu\nu}^S(\tilde{R}(p)) \cdot \sum_{\nu'} \bar{D}_{\nu'\mu}^S(R(p)) |p, \nu'\rangle_c$$

$$h |p, v\rangle_c = \sum_{\mu, \nu'} \mu D_{\mu\nu}^S(\tilde{R}(p)) D_{\nu'\mu}^S(R(p)) |p, \nu'\rangle_c$$

● (3.113) How do we know that:

$$R(p) Z(p) R^{-1}(p) = L(p).$$

$$\text{I.e., } e^{-i\theta \cdot \mathcal{J}} e^{-i\sigma K_3} e^{+i\theta \cdot \mathcal{J}} = e^{-i\sigma \hat{R} \cdot \mathbf{K}} \quad \checkmark$$

$$\begin{aligned} \text{Really: } & e^{-i\phi \mathcal{J}_3} e^{-i\theta \mathcal{J}_2} e^{-i\sigma K_3} e^{+i\theta \mathcal{J}_2} e^{+i\phi \mathcal{J}_3} \\ &= e^{-i\phi \mathcal{J}_3} e^{-i\sigma [K_3 \cos\theta + K_1 \sin\theta]} e^{+i\phi \mathcal{J}_3} \\ &= e^{-i\sigma \{ e^{-i\phi \mathcal{J}_3} [K_3 \cos\theta + K_1 \sin\theta] e^{+i\phi \mathcal{J}_3} \}} = e^{-i\sigma \vec{K} \cdot \hat{p}} \end{aligned}$$

$$\{ \} = \cos\theta \underbrace{\left(e^{-i\phi \mathcal{J}_3} K_3 e^{+i\phi \mathcal{J}_3} \right)}_{K_3} + \sin\theta \underbrace{\left(e^{-i\phi \mathcal{J}_3} K_1 e^{+i\phi \mathcal{J}_3} \right)}_{K_1 \cos\phi + K_2 \sin\phi}$$

$$= K_3 \cos\theta + \sin\theta \cos\phi K_1 + \sin\theta \sin\phi K_2 = \vec{K} \cdot \hat{p}$$

$$\text{where } \hat{p} = (\sin\theta \cos\phi, \sin\theta \sin\phi, \cos\theta).$$

So, using the boost theorem, I have verified that.

$$R(p) Z(p) \tilde{R}^{-1}(p) = \bar{L}(p).$$

1. These comments are in regard to M's equation (3.16) which is an application of earlier equation (2.9) in a more complicated way. Equation (2.9) is a statement about how the momentum operator transforms under LT. However, I think we could consider this to be the transformation property of any 4-vector operator. Let A^μ be such a four-vector operator, maybe the momentum of some particle. Then we know several things, for example:

$$L^{-1} A^3 L = \lambda^3_\mu A^\mu = A^3$$

$$L^{-1} (A^3)^2 L = (\lambda^3_\mu A^\mu)^2$$

Since we can define the action of $L^{-1}(\dots)L$ on any power of any component of a vector operator A , we can extend the definition to operator-valued functions of A . These functions need not be Lorentz scalars or anything like that.

Moreover, we can extend the idea to more than one four-vector operator.

Thus:

$$L^{-1} F(A^\mu, B^\nu) L = F(\lambda^\mu_\alpha A^\alpha, \lambda^\nu_\beta B^\beta) = F(A^\mu, B^\nu)$$

If we happened to know that $F(A, B)$ was a Lorentz invariant, then in addition we would have $F(A, B) = F(A', B')$.

McKerrell

"Relativistic 2- and 3-particle states"

NS. 34, pp 1289-1305 (1964)

1. Introduction. This piece will show the relation between two formalisms

A). The LS of McFarlane (1963) "M."

B). The helicity of Wick (1962) "W."

Their derivations will be simplified!

2. The Poincaré Group. $A =$ a translation operator $= e^{-i q_\mu P^\mu}$
 $L =$ a L.T. $= e^{-\frac{i}{2} \omega_{\mu\nu} M^{\mu\nu}}$

Define: $P^\mu, M^{\mu\nu}, L, S, W^\mu$, $\frac{1}{2} W^\mu W_\mu = \text{scalar} = -m^2 s(s+1)$

$\vec{J}, \vec{K}, \vec{S}$

$$m \vec{S} = \vec{W} - (m + P^0) \vec{L} W^0$$

copy this paper!

All the significant Poincaré stuff is here.

3. Single-Particle States.

$$|m, s, \vec{p}, v\rangle$$

$$\vec{p} = (m, \vec{0})$$

$$v = m_s$$

$$? = |\vec{p}, v\rangle \text{ so short.}$$

First shows:

$$R|\vec{p}, v\rangle = \sum_{\tilde{m}} D_{\tilde{m}v}^s(R) |\tilde{p}, \tilde{m}\rangle \checkmark$$

He uses: $D_{\mu\nu}^s(\alpha, \beta, \delta) = e^{+i\alpha J_3} e^{+i\beta J_2} d_{\mu\nu}^s(\beta)$ of Edmonds

Usual sym rules $\checkmark$

Now get ready to boost up:

$$D(p) = e^{-i\vec{p} \cdot \vec{K}}$$

In 4x4 rep:

$$D(p) \tilde{p} = m (d\vec{s}, 0, 0, s\vec{g}) \checkmark$$

Next:

$$R(p) = e^{-i\vec{p} \cdot \vec{J}_3} e^{-i\vec{p} \cdot \vec{J}_2} = \text{operator}$$

What does this $\uparrow$ do to $p^\mu = m$ wait!

Let

$$p = m (d\vec{g}, s\vec{g}) (S_0, S_0, S_0, C_0) \leftarrow \text{This is "p"}$$

$$D(p) = e^{-i\vec{p} \cdot \vec{K}} = \text{operator}$$

Now, if $p = m(\text{dir}, 0, 0, -\text{dir})$, he claims ~ well, what is $\Theta \phi$?

$$\vec{p} = (S_0 \text{ dir}, S_0 \text{ dir}, C_0)$$

$$= e^{-i\phi J_z} e^{-i\theta J_y} (0, 0, 1)$$

Why not set $\phi=0$, $\theta=\pi$. $e^{-i\pi J_y} (0, 0, 1) = (0, 0, -1)$. Well, there is ambiguity in this special case, so he makes specific choice:

$$R(\vec{p} = p\hat{z}) = e^{-i\pi S} e^{-i\pi J_3} e^{-i\pi J_z} \quad \text{convention !!}$$

choose $s = \text{spin of particle}$

Now, clearly $\exists$ a pure boost such that

$$L(p) = e^{-i\vec{s} \cdot \vec{K} \cdot \hat{p}} \quad L(p) \vec{p} = p.$$

We can show that, in general: (on any vector).

$$L(p) = R(p) Z(p) R^{-1}(p)$$

Obviously this is correct when applied to $\vec{p} = (m, 0)$.
So:

$$H(p) \equiv R(p) Z(p) = L(p) R(p).$$

Next, claim: $L^{-1}(m\vec{S}) L = \dots \quad L = L(p).$

Comes up with: $L(p)^{-1} \vec{S} L(p) |\vec{p}, \nu\rangle = \vec{S} |\vec{p}, \nu\rangle.$

[This also says: choose any p you like. How about $p = \vec{p}$.
Then $\vec{s} = 0$, $L(p) = 1$. Then

$$\vec{S} |\vec{p}, \nu\rangle = \vec{S} |\vec{p}, \nu\rangle. \quad \checkmark \text{ since } \vec{L} = 0.$$

So define: $|p\nu\rangle_c \equiv L(p) |\tilde{p}\nu\rangle$

Then: $P |p\nu\rangle_c = p |p\nu\rangle$

$R^S |p\nu\rangle_c \text{ means } R^S(p) |p\nu\rangle_c$

$= e^{-i\phi S_3} e^{i\theta S_2} |p\nu\rangle_c \rightarrow = \sum_{\mu} D_{\mu\nu}^s(R) |p\mu\rangle_c$

So, the states $|p\nu\rangle_c$ transform under R^S just the way you want, and they are not at rest!

$S^2 |p\nu\rangle_c = s(s+1) |p\nu\rangle_c$

$S_3 |p\nu\rangle_c = \nu |p\nu\rangle_c$

[Angeli]

Now about application of $R^S = R$, a R^L ??

$R |p\nu\rangle_c = \sum_{\mu} D_{\mu\nu}^s(R) |p, \mu\rangle_c$
 $\left(e^{-i\phi S_3} e^{-i\theta S_2} \right) \quad \nwarrow \text{ie, rotated momentum.}$

$R = e^{i\theta \cdot \vec{S}} = e^{i\theta(\hat{L} + S)} = e^{i\theta \hat{L}} e^{i\theta S} = R^L R^S$

So $R^L |p\nu\rangle_c \equiv |p, \nu\rangle_c$

General result:

$AL |p\nu\rangle_c = AL L(p) |\tilde{p}\nu\rangle = A L(\tilde{L}_p) \bar{R}(L_p) |\tilde{p}\nu\rangle$

Result:

$AL |p\nu\rangle_c = e^{-i\eta \cdot (L_p)} \sum_{\mu} D_{\mu\nu}^s(\bar{R}(L_p)) |L_p, \mu\rangle_c$
 $\nwarrow \text{Wigner rotation.}$

T+ W 164

Crossing Relations for Helicity Amplitudes*

T. L. TRUMAN AND G. C. WICK

Brookhaven National Laboratory, Upton, New York

Crossing relations for helicity amplitudes for particles of arbitrary spin are formulated without recourse to the introduction of scalar amplitudes. The basic assumption is that the amplitudes are simply related by analytic continuation; the path of continuation is carefully specified. The relations are given a simple geometrical interpretation. The relation between $\pi N \rightarrow \pi N$ and $\pi\pi \rightarrow N\bar{N}$ obtained in this way agrees with that obtained by direct elimination of scalar amplitudes.

1. INTRODUCTION

The most common applications of crossing relations involve particles of spin 0 or at most $\frac{1}{2}$. The customary Dirac formalism allows one to express the reaction amplitude in terms of so-called scalar amplitudes ("A" and "B" in the case of $\pi - N$ scattering) and the crossing-relation then simply states that analytic continuation of a scalar amplitude from the physical region of a channel to that of a "crossed" channel yields the corresponding scalar amplitude in the crossed channel. The introduction of scalar amplitudes is not a simple matter in the general case, so that it would be technically advantageous to formulate the crossing relations in terms of some other amplitudes, which are more easily generalized, for example helicity amplitudes (1, 2).

A crossing relation for helicity amplitudes for a simple case, such as πN scattering versus $N\bar{N}$ annihilation into two pions, can, of course, be obtained indirectly by elimination of the scalar amplitudes A and B from the equations connecting A and B to the helicity amplitudes $F_{\lambda\mu}$ and $G_{\lambda\mu}$ for the two crossed reactions. But the relation obtained is not very transparent at first sight. How can it be generalized?

Recently we arrived at a very simple geometrical interpretation of these relations, which suggests an obvious generalization. Essentially the same interpretation has been arrived at independently and apparently somewhat earlier by M. S. Marinov and V. I. Roginskii (3) and by Ya. A. Smorodinsky (4). The results of these authors, however, are only similar to but not identical with ours.

* Work performed under the auspices of the U. S. Atomic Energy Commission.

† We are indebted to Prof. M. Goldberger for repeatedly drawing our attention to this question.

In particular, the formulas derived by these authors do not seem to agree with those obtained by the elementary "indirect" method for πN scattering, and must therefore be incorrect. We believe that this is due to certain complications arising from analytic continuation, complications which are not discussed in either of the above-mentioned papers. We hope, therefore, that the following remarks will contribute to the further elucidation of this problem. ✓

II. AN EXAMPLE: πN SCATTERING.

Let us first briefly recall the results of the calculation via the scalar amplitudes. The πN scattering amplitude is given by

$$\bar{u}(p_2) \hat{T} u(p_1) \quad (1)$$

$$T = -A + \frac{1}{2}\mu^{-1} \gamma \cdot (q_1 + q_2) B.$$

The notation is standard (5), except that the usual B is replaced by B/μ . The helicity amplitudes G_{++} and G_{+-} , say, are then obtained by an appropriate choice of the Dirac spinors $u(p_1)$ and $u(p_2)$. The formulas of this section are based on the phase conventions of ref. 1. Note that according to Eq. (13) of (1), there is a factor $(-1)^{s_2-s_1}$ in the definition of the helicity state for "particle 2." In the following equations the pion in both the initial and final state is taken as "particle 2," so that the factor is avoided. One has, ignoring an irrelevant overall phase factor,

$$G_{++} = \cos\left(\frac{\theta_s}{2}\right) \left(A + \frac{k^2 + \mu^2}{m\mu} B \right), \quad G_{+-} = \sin\left(\frac{\theta_s}{2}\right) \left(\frac{\epsilon}{m} A + \frac{\omega}{\mu} B \right) \quad (2)$$

where k and θ_s are c.m. momentum and scattering angle, $\epsilon = (m^2 + k^2)^{1/2}$ and $\omega = (\mu^2 + k^2)^{1/2}$ are c.m. energies of nucleon and meson respectively. For the relation of these variables to the Mandelstam variables s, t, u (or $\bar{s}$) and related notations we refer the reader to the papers of W. R. Frazer and J. R. Fulco (6). With the abbreviation

$$S^2 = [s - (m + \mu)^2][s - (m - \mu)^2] \quad (3)$$

we have

$$k^2 = S^2/4s; \quad \sin(\theta_s/2) = (-st)^{1/2}/S \quad (4)$$

$$\cos(\theta_s/2) = (S^2 + st)^{1/2}/S = [(m^2 - \mu^2)^2 - su]^{1/2}/S \quad (5)$$

$$\epsilon = (s + m^2 - \mu^2)/(4s)^{1/2}; \quad \omega = (s - m^2 + \mu^2)/(4s)^{1/2}. \quad (6)$$

These formulas are to be used in the physical region for the s-channel (πN scattering). In this region the square roots in (4)-(6) are all positive by definition. In the t-channel ($\pi\pi \rightarrow N\bar{N}$) the helicity amplitudes are given (6) by

OK. See Phil
lengthy
paper on
this

error of
others

$$F_{++} = -(p/m)A + (q/\mu) \cos \theta B, \quad F_{+-} = (Eq/m\mu) \cos \theta B \quad (7)$$

where:

$$p = (\frac{1}{4}l - m^2)^{1/2}, \quad q = (\frac{1}{4}l - \mu^2)^{1/2}, \quad \cos \theta = (s - u)/4pq, \quad (8)$$

and the functions $A(s, t)$, $B(s, t)$ in the t -channel region are analytic continuations of the corresponding functions in the s -channel region. In these equations, the nucleon in the final state is taken as "particle 2." If A and B have the kind of singularities that are postulated in the Mandelstam representation, the analytic continuation has to go from a point, say, $s = s_i + i\epsilon$, $t = t_i - i\epsilon$, where ϵ is infinitesimal and positive, $s_i > (m + \mu)^2$, $t_i < 0$, $su_i < (m^2 - \mu^2)^2$ to a point $s = s_f - i\epsilon$, $t = t_f + i\epsilon$ where $s_f < 0$ and $su_f > (m^2 - \mu^2)^2$. In these equations the variable u is, of course, $u = 2(m^2 + \mu^2) - s - t$. If we assume, for the sake of simplicity, that u stays real along the trajectory, then in order to reverse the sign of the imaginary parts of s and t , the trajectory must go through a point of the real s, t -plane. If this is the only real point of the trajectory, and if the real point lies within the triangle defined by the inequalities:

$$s < (m + \mu)^2, \quad u < (m + \mu)^2, \quad t < 4\mu^2, \quad (9)$$

then we will have insured that the endpoint is still on the "first sheet" where A and B are given by the Mandelstam formulae and have the correct values for the t -channel.

The coefficients of A and B in Eq. (2) have certain singularities (branch-singularities), namely, as can be seen from Eqs. (3)-(6), at $s = 0$, $s = (m \pm \mu)^2$, $t = 0$, and $su = (m^2 - \mu^2)^2$. The trajectory should, of course, avoid these singularities. We may assume that the imaginary part of $s(-t)$ remains small throughout so that the trajectory may be specified for our purposes by drawing a line in the s, t -plane, indicating by a cross the point where the trajectory crosses the real s, t plane. The lines $su = (m^2 - \mu^2)^2$, $s = (m - \mu)^2$, and $t = 0$ divide the triangle (9) into five pieces, see Fig. 1, and depending on where the crossing-point lies, one will get different determinations of the coefficients in Eq. (2) at the final point in the t -channel. Since an over-all change in sign of G_{++} and G_{+-} is unimportant, we have to distinguish only two cases: if the crossing point lies within the hyperbolic segment delimited by $t = 0$ and $su = (m^2 - \mu^2)^2$, then the final values of $\cos(\theta_s/2)$ and $s^{-1/2} \sin(\theta_s/2)$ are pure imaginary and of the same sign. In all other cases, they are pure imaginary and of opposite sign. Thus if we adopt the first alternative, we have

$$\cos(\theta_s/2) = -2ipq \sin \theta/S, \quad s^{-1/2} \sin(\theta_s/2) = 2iE/S, \quad (10)$$

where $2E = l^{1/2}$ is the total energy in the c.m. system. The minus sign in the expression for $\cos(\theta_s/2)$ is explained in the Appendix. After introducing these

CROSSING RELATIONS

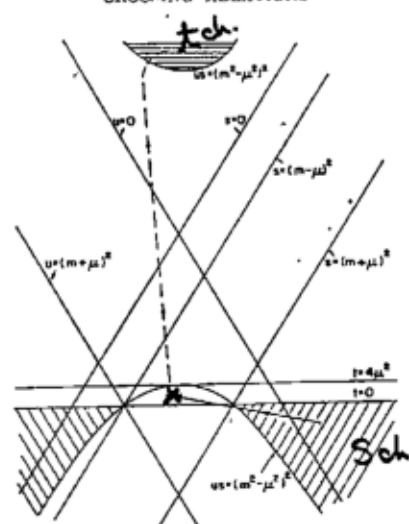


FIG. 1. The s, t plane for $\pi N \rightarrow \pi N$ and $\pi\pi \rightarrow NN$ (drawn for $m = 4\mu$)

values into Eq. (2), we may eliminate A and B from (2) and (7), obtaining

$$\begin{aligned} G_{++} &= (2i/S)[mq \sin \theta F_{++} + E(p - q \cos \theta)F_{+-}], \\ G_{+-} &= (2i/S)[E(p - q \cos \theta)F_{++} - mq \sin \theta F_{+-}], \end{aligned} \quad (11)$$

where, as pointed out before, it is irrelevant which of the two determinations for the square root of (3) is used, provided it is the same in the two equations.²

It is easy to verify the identity

$$(mq \sin \theta)^2 + E^2(p - q \cos \theta)^2 = \frac{1}{4}S^2 \quad (12)$$

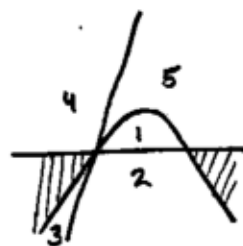
so that, apart from the uninteresting factor i , the transformation matrix between the two sets of helicity amplitudes is an orthogonal matrix. But the meaning of this transformation is not immediately apparent.

Owing to the orthogonal nature of the transformation we are, however, tempted to write Eq. (11), disregarding the factor i , in the form

$$G_{++} = \sin \chi F_{++} + \cos \chi F_{+-}, \quad G_{+-} = \cos \chi F_{++} - \sin \chi F_{+-} \quad (11')$$

where χ is determined by

² The opposite sign for S in the two equations corresponds, however, to the alternative choice of the crossing point!



$$\tan \chi = \frac{mq \sin \theta}{E(p - q) \cos \theta} \quad (13)$$

In order to interpret this formula, we must re-examine carefully the process of analytic continuation.

III. GEOMETRICAL INTERPRETATION

In the customary presentation of the crossing relations one rewrites the conservation law for πN scattering

$$q_1 + p_1 = q_2 + p_2 \quad (14)$$

in the form

$$q_1 - q_2 = -p_1 + p_2 \quad (15)$$

and reinterprets $-q_2$ and $-p_1$ as initial four-momentum of a pion and final four-momentum of an antinucleon respectively. It is clear that in this interpretation the values of the four-momenta are *not the same* in Eqs. (14) and (15), since q_2 and p_1 are positive timelike in Eq. (14) and negative timelike in Eq. (15). In fact, the values of Eq. (14) correspond to the initial point of the trajectory of Fig. 1, those of (15) to the final point. Moreover one sees that also the values of q_1 and p_2 have to vary along the trajectory since

$$M^2 = (q_1 - p_2)^2 \quad (16)$$

cannot remain constant. However, this variation of q_1 and p_2 is often disregarded, since in the end the two four-vectors revert to the real positive timelike mass shell. In fact, if the initial point of the trajectory of Fig. 1 lies in the $u < 0$ part of the physical region, we may indeed assume that the final values of q_1 and p_2 are identical with the initial ones. Henceforward we shall make this assumption for simplicity.

Thus in the latter case indicating by primes the values of $q_1, \dots$ etc. at the end of the trajectory, we may write

$$q_1' = q_1, \quad p_2' = p_2, \quad q_2' = -Q_2, \quad p_1' = -P_1, \quad (17)$$

where q_1, Q_2 are the pion momenta in the initial state, and p_2, P_1 are the nucleon and antinucleon momenta in the final state of the reaction in the t -channel. It should be noted that P_1 and Q_2 must be different from p_1 and q_2 in Eq. (14). In this respect the usual notation is apt to lead to confusion.

We may clearly assume that at every stage of the analytic continuation the vectors p_1, p_2, q_1, q_2 (whether real or complex) lie in the xz plane, since this gives us sufficient freedom to vary s and t at will. This assumption avoids phase

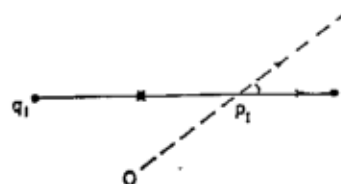


FIG. 2. Velocity space diagram. The ends of the segments p_1 and q_1 represent the velocity points of p_1 and q_1 ; X represents the velocity point of the c.m. of p_1 and q_1 ; O is an arbitrary point. The arrows indicate the directions in which the spin components are measured.

factors connected with the azimuthal variable ϕ ,³ and furthermore it means that the y axis is not affected in any of the Lorentz transformations we shall encounter in the following discussion.

Let us now introduce, in addition to the customary helicity amplitudes $G_{\lambda\lambda}(s, t)$ so far employed, the notion of "generalized" helicity amplitudes $G_{\lambda\lambda}(p_1, q_1; p_2, q_2)$ by which we mean matrix elements of the scattering matrix T between helicity states satisfying condition (14) but not subject to the c.m. condition: $p_1 + q_1 = 0$. The phases of the nucleon helicity states are defined by Eq. (6) of (1), with $\phi = 0$ and θ unrestricted. The discontinuity at the "south pole" causes no trouble since the state obtained with $\theta = \pi$ is simply $(-1)^{2s}$ times the state with $\theta = -\pi$, independent of the helicity. Let β be the velocity of the center of mass

$$\beta = (p_1 + q_1)/(e_1 + \omega_1) \quad (18)$$

where $e_1 = (m^2 + p_1^2)^{1/2}$, $\omega_1 = (\mu^2 + q_1^2)^{1/2}$, and denote by L_β^{-1} the Lorentz transformation (in the xz -plane) which transforms the c.m. to rest, so that $p_1 = l_\beta p_1^0$, $q_1 = l_\beta q_1^0$, where p_1^0, q_1^0 is an initial state satisfying the c.m. condition. Then according to the transformation law of helicity states⁴

$$L_\beta |p_1^0 \lambda\rangle = \sum_{\lambda'} U_{\lambda\lambda'}(p_1; l_\beta) |p_1 \lambda'\rangle \quad (19)$$

where U is a spin-rotation matrix corresponding to a rotation about the y -axis. The rotation angle is indicated in the diagram in velocity space; see Fig. 2. Besides the velocity points corresponding to p_1 and q_1 , the diagram has a point representing the velocity of the c.m. (a cross) and a point O representing the velocity of the arbitrary system, in which the momenta are p_1 and q_1 . From Eq. (19) and the Lorentz invariance of the T -matrix one easily derives the connection

³ See, for example, ref. 1.

⁴ See, for example, ref. 2, especially the Appendix.

$\rightarrow \left\{ \begin{array}{l} p_1^0 = 4 \text{ mm of } l \text{ in rest frame of } l \\ (\text{i.e., } p_1^0 = 0) \\ p = 4 \text{ mm of } l \text{ in c.m.} \end{array} \right.$

$$G_{\mu\lambda}(s, t) = \sum_{\lambda'} U_{\mu\lambda'}(l_0^{-1}; p_2) G_{\mu'\lambda'}(p_2 q_2; p_1 q_1) U_{\lambda'\lambda}(p_1; l_0) \quad (20)$$

A similar formula holds for an arbitrary Lorentz transformation l , except that in this case we would have on the left-hand side again a generalized amplitude for the values $l^{-1}p_2, \dots$ etc. of the four-momenta.

We may perform the analytic continuations of $G_{\mu\lambda}(s, t)$ by continuing each of the three factors on the right hand side of Eq. (20). Let us assume, for the sake of simplicity, that the values (17) at the end of the trajectory satisfy the c.m. condition for the crossed reaction

$$q_1 + q_2 = p_1 + p_2 = 0. \quad (21)$$

It then turns out that the end-value of the generalized helicity amplitude $G_{\mu'\lambda'}(p_2 q_2; p_1 q_1)$ coincides up to a phase factor (see Eq. (31)) with a helicity amplitude F for the crossed reaction. Such a relationship has in fact been conjectured by other authors (3, 7) with the difference that here, contrary to their results, the helicity does not change sign in the crossing process. Finally, the spin-rotation matrices U , or rather their analytic continuation, give rise to the orthogonal transformation E_{q_1} (11), as has also been pointed out by Ya. Smorodinsky (4) and by Marinov and Roginskii (3).

Let us first examine the behavior of the generalized amplitude $G_{\mu\lambda}(p_2 q_2; p_1 q_1)$. We can do this in two ways. To begin with, we may say that G is given by Eq. (1) when the spinors are chosen to be "helicity-spinors" $u_\mu(p_2)$ and $u_\lambda(p_1)$. This means that, for example, $u_\mu(p_2)$ must satisfy, in addition to the Dirac equation $(i\gamma p_2 + m)u(p_2) = 0$ also a helicity condition²

$$\sigma \cdot p_2 u_\mu(p_2) = 2\lambda(p_2^2)^{1/2} u_\mu(p_2) \quad (22)$$

where $\lambda = \pm 1/2$.

Since we have made the customary assumption that there is no problem in continuing the matrix T , Eq. (1) along a path such as that of Fig. 1, the whole question reduces to the behavior of the helicity spinors. Since p_2 reverts to the positive real (i.e., p_2 real) mass shell in the end, it is not hard to see that the helicity μ remains unchanged in the process. The case of p_1 requires more care. At the end of the process, u will, of course, become a negative energy spinor, $v(p_1)$ satisfying a Dirac equation $(i\gamma p_1 - m)v(p_1) = 0$. If we assume, moreover, that along the trajectory p_1^2 never becomes zero, then since Eq. (22) is always satisfied by analytic continuation, it follows that the final $v(p_1)$ must satisfy

² Equation (22) does not, of course, determine the phase and normalization of $u_\mu(p_2)$. We assume that the latter are chosen according to the convention explained above when p_2 is real. They are then determined for the other values by analytic continuation.

$$-\sigma \cdot P_1 v_\lambda(P_1) = 2\lambda(P_1^2)^{1/2} v_\lambda(P_1) \quad (22')$$

where the sign of the square-root on the right hand side is determined unambiguously by continuity along the path. As we shall see presently, the sign of the square-root is determined to be positive, whereupon it follows that the antinucleon state described by v_λ has helicity $+\lambda$, as one can see immediately by examining the charge-conjugate spinor $C\bar{v}$ (or by the more elementary hole-theory argument: a missing particle of spin $-\lambda$ in the P_1 -direction corresponds to an antiparticle of spin $+\lambda$ in the same direction). Thus the helicity does not change sign in the analytic continuation.

The crux of the argument, it will be seen, is the behavior of the square-root of P_1^2 . This is not trivial, since the vector P_1 becomes complex along the path. Let, however, e_1 be the time component of p_1 at any point of the trajectory so that $(P_1^2)^{1/2}$ is the end value of

$$(e_1^2 - m^2)^{1/2}. \quad (23)$$

The whole question hinges on the path of the representative point for e_1 in the complex e_1 -plane, Fig. 3. Now assume for simplicity that q_1 and p_2 are exactly real at the endpoint. For the initial value of p_1 we write instead $p_1 + i\eta$ where p_1 and η are real, and $n = (\eta_0, n)$ is infinitesimal. Thus $p_1^2 = m^2$ and $p_1 \cdot \eta = 0$. Calculating in the c.m. system for $p_1 + q_1$, i.e., assuming $p_1 = -q_1$, the condition for $s = (p_1 + i\eta + q_1)^2$ to have a positive imaginary part is

$$0 < \eta \cdot q_1 = \eta_0 \omega_1 - n \cdot q_1 = \eta_0 \omega_1 + n \cdot p_1 = \eta_0(\omega_1 + e_1).$$

This implies $\eta_0 > 0$, i.e., the initial value of e_1 is $e_1 + i\eta_0$; it has a small positive imaginary part.

Similarly the condition that $t = (p_1 - p_2)^2$ has a small positive imaginary part at the endpoint of the trajectory implies that the end-value of e_1 has a negative imaginary part. Thus the endpoints are as indicated in Fig. 3, and the remaining question is whether the e_1 -trajectory cuts the real axis between $-m$ and $+m$ (as indicated in the figure) or not. If it does, the end-value of (23) is positive, as we have assumed. Now this can certainly be arranged, if we assume that the time components of the four-vectors p_1 and q_2 are everywhere complex along the trajectory except at the crossing point. Since the latter is inside the triangle

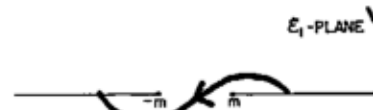


FIG. 3. The complex e_1 -plane

how to continue naturally.

Helicity Amplitudes
and
Crossing
etc.

A summary of Helicity State Information.

(Note: all this stuff is contained in the book of Martin and Spearman)

1. The single particle helicity eigenstate is defined simply and clearly in (3.50).
2. Then the CMS partner state is defined in (3.58) with the Jacob Wick phase choice.
3. The direct product of these states is called /p00...)
4. The connection between the angle representation and the JM representation is given in (3.71); the inverse in (3.72).
5. At this point, the full store of data about the rotation matrices and their symmetry properties must be assimilated. Cannot do without.
6. The technique for extracting the momentum dependence and getting to the states within the little Hilbert space is shown in (3.85). Thus, (3.87 and 3.88).
7. The action of an arbitrary transformation (within the RIHL group) on the two particle JM rep state is shown in (3.93).
8. The action of the parity operator with standard phase 1 on the single particle state and its upsidedown partner is shown in (5.22 and 5.23). The effect on the two particle JM state is shown in (5.28). The symmetry properties of the helicity amplitudes which result if the S-matrix is parity conserving are then displayed in (5.35) and (5.37) for zero azimuth. Essentially, you pick up a phase and all the helicities change their signs.
9. The action of time reversal on the single particle state and its partner is shown in (5.40) and (5.41). Similar to the effect of parity but a different phase and the helicity does not change sign as it does in the parity case. The effect of T on the two particle JM state is shown in (5.47). You get a phase and M changes sign (the helicities are the same). The symmetry properties of the helicity amplitudes that result if T is an invariance are shown in (5.50) and (5.51) for zero azimuth. the discussion here is slightly complicated by the fact that T is antiunitary, but nothing topfancy. The resulting symmetry is a phase and an interchange of the initial and final helicities (but no change in any signs).
10. Then we move to chapter 7 on crossing. The generalized helicity amplitudes, not just in the cms frame, are defined in (7.13). Then the crossing claim is made in equation (7.14) which shows that the helicities stick right with their particles, no sign changes. Finally, you have to to some transforming to get from one cms frame to the other. The end result is stated in (7.46). Then some simple examples are given.

Crossing Relations for the Helicity amplitudes.

1. Show how the single particle state is generated from particle at rest.
2. Show how simply this one particle helicity amplitude behaves under rotations.
3. Pose the question of what this amplitude might do under a boost B .
4. State that the coefficients are simply rotation matrices for a special rotation, Wigner.
5. Explain what the Wigner rotation is in terms of the other stuff. Amazing.
6. Conclude that this helicity state transforms trivially under R 's and B 's.
7. Do simple direct product to show behavior of a two particle state, not nec. cms.
8. Show how a helicity amplitude transforms under a boost. There are four D 's.
9. Restrict to $y=0$ plane so D replaced with d .
10. Write out arbitrary transformation for (s) and (t) channel amplitudes. Emphasize how different the various parameters of the d 's are because all sorts of helicities and momenta got switched around.
11. In fact, write out in entirety the s and t channel helicity amplitudes using p and q .
12. Now make the amazing claim that, if you do the right thing, the d 's are exactly the same!!! Therefore the helicity amplitudes are exactly equal. This is the generalized statement of crossing.
13. Remark that these two equal amplitudes must necessarily be in different reference frames. Show with picture of particles and momenta arrows. Frames are say S_0 for the t channel and S_1 for the s , where S_0 happens to be a cms frame.
14. Write a transform relating S cms to S in frame S_1 .
15. Replace S amp in S_1 with S amp in S_0 .
16. Thus, we have final relation between s channel cms helamps and t channel cms helamps. Just a technical detail to find the Wigner angles in terms of sbabbbb the various momenta.
17. Go back to claim 12. that the d 's are exactly the same. In trying to show this, you have to deal with a boost by a negative 4-momentum. What does that mean???
18. The elements in the boost matrix depend on p^0 and $|p|$. We know that p^0 is going to change sign, but we cannot say whether or not $|p|$ changes sign. Show why. Thus there is ambiguity in the matrix. Track it down!
19. Show how $|p|$ depends on E and draw E plane with cuts. Show that whether or not the sign changes depends on whether or not you go onto the other sheet. If you snake between the cuts, you get the result I gave above. If you have a massless particle cannot do this; but massless only have two helicity states anyway, easy to deal with. If you cut through cut, all that happens is a sign change on the crossed helicities.

①
helicity
amplitude

②
amplitude

③
Transform
with
formula

④
ambiguity
down

The Role of the S.T. in crossing relations

and

The Origin of all those d factors.

Firstly, when you go from one frame to another by doing a z-boost, and when $\phi = 0$ everywhere, helicity amplitudes transform in a special way:

$$\langle p_c p_d \mu_c \mu_d | T | p_a p_b \mu_a \mu_b \rangle$$

$$= \sum_{a' b' c' d'} (-1)^{b-b'+d'-d}$$

$$p'_i = \Lambda p_i$$

$$R_i = \Lambda_{p'_i}^{-1} \Lambda p_i$$

$$\times d_{cc'}^{S_c}(\omega_c) d_{dd'}^{S_d}(\omega_d) d_{aa'}^{S_a}(\omega_a) d_{bb'}^{S_b}(\omega_b)$$

$$\times \langle p'_c p'_d \mu'_c \mu'_d | T | p'_a p'_b \mu'_a \mu'_b \rangle$$

Each particle thus transforms within its own little subspace. Assuming this rule is correct and that we can determine all those little Wigner angles, what should we do with it?

One thing we know is that the scm system is not the same system as the xcm system. If $\vec{p}_a + \vec{p}_b = 0$, then $\vec{q}_b + \vec{q}_d = +\vec{p}_a - \vec{p}_d$ and there is no reason why these frames should be the same

$$\vec{P}^{(s)} = \vec{p}_a + \vec{p}_b = 0 \text{ say.}$$

$$\vec{P}^{(t)} = \vec{p}_a - \vec{p}_d = \text{something else.}$$

However, these two frames are connected by a boost. If we select the $\hat{z}$ axis to be in this special direction, then we can say

$$T_{cd;ab}^{(s)}(s,t) = \sum_{a' b' c' d'} (-1)^{b-b'+d'-d} d_{cc'}^{S_c} d_{dd'}^{S_d} d_{aa'}^{S_a} d_{bb'}^{S_b}(\omega_b) \underbrace{\langle p'_c p'_d \mu'_c \mu'_d | T | a' b' \dots \rangle}_{\text{in xcm}}$$

Now use the crossing result: (7:14)

$$\begin{aligned} \langle p_c p_d \mu_c \mu_d | T^{(s)} | p_a p_b \mu_a \mu_b \rangle &= \langle q_c q_a \mu_c \mu_a | T^{(t)} | q_d q_b \mu_d \mu_b \rangle \\ &= T_{ca;db}^{(t)}(s,t) \end{aligned}$$

Thus, (7:46) is proven.

So, why do those d-factors appear? The generalized crossing statement involves two referentials:

$$\begin{aligned} & \left(\langle \sim | T^{(s)} | \sim \rangle \right) && \text{frame } S \\ & = && \\ & \left(\langle \sim | T^{(t)} | \sim \rangle \right) && \text{frame } t \end{aligned}$$

If we choose frame S to be the S.C.M., frame t will not be the t.c.m.

$$\begin{aligned} & \left(\langle \sim | T^{(s)} | \sim \rangle \right) && \text{frame S.C.M.} \\ & = && \\ & \left(\langle \sim | T^{(t)} | \sim \rangle \right) && \text{frame } \Sigma \end{aligned}$$

Thus, crossing equates S.C.M. amplitudes for the s process to Σ amplitudes for the t.

But, we want to connect S.C.M. s to t.c.m. t. Thus, we have to transform the t's:

$$\begin{aligned} & \left(\langle \sim | T^{(s)} | \sim \rangle \right) && \text{frame S.C.M.} \\ & = && \\ & \left(\langle \sim | T^{(t)} | \sim \rangle \right) && \text{frame } \Sigma \\ & = && d d d d \left(\langle \sim | T | \sim \rangle \right) \text{ frame T.C.M.} \end{aligned}$$

Now the entire study is reduced to discrete parts:

- 1) Show equation (7.12) including phase details
- 2) Show crossing (7.14) with details on
 - a. path of continuation
 - b. non flip of helicity.
- 3) Show how to find those angles, ie, (7.42).

Stapp '65

July 15, 1977

Stapp Trieste: Part I: Analytic S-Matrix Framework

1. Introduction. The idea is to extend Mandelstam analyticity to multiparticle amplitudes. What happens to crossing if you have no "fields"? In S-matrix theory, started by Heisenberg in 1947, all particles stay on their mass sheets, another point of difference with field theory.

2. Basic Quantum Principles and Fundamental Observables.

A. Postulates of quantum Theory. There are probabilities which add to one, and a prob is some amplitude squared, plus superselection. Theorem then follows: the s-matrix which related experimental amplitudes is unitary, details in appendix.

Henry wants to avoid concept of Hilbert Space with bras and kets. He thinks it is better to focus on the functions themselves and not get sidetracked into mathematics of operators in some space etc, especially from the analytic point of view.

B. Fundamental Observables. This section makes no sense to me. I think it is about density matrices and projection operators for the spin.

C. The covariant spin operators and M-functions. Complex (A,B) Lorentz transformations are mentioned. A few spinor facts are mentioned. The spin sum of a projection operator is identified with the thing $G = L^2$ which raises or lowers spinor indices.

The S-functions are in canonical basis, as I see it, and the M-functions are in spinor basis, so you need some G-functions in the new unitarity equation, 2.53.

3. Lorentz Invariance. As usual, Henry starts with the probabilities, but finally ends up with Lorentz transformation property of the M-functions. A certain phase is set to one, a la Wigner. Things are not too specific in this transformation law.

4. Spin and Angular Momentum. Somehow spin is interpreted and treated as a four vector like regular momentum. Spin is much harder to conceptualize when you are now outside the bra/ket state framework. You have to do more work and say more words, and still the ideas are not really clear.

5. Energy momentum conservation. Big deal.

6. The decomposition principle and bubble diagrams. Gives cluster decomposition and allows a phase for each term. If you keep particles in same order, then all phases are one. When you put the CD into unitarity, you get "bubble diagrams". But only a column of plus bubbles and a column of minus bubbles. More generally, you can define a huge number of columns bubble diagram.

When you set each bubble to a constant, you get a "generalized phase space integral". These things are identified with "Landau diagrams", ie, you contract each bubble to a point.

7. The analyticity postulate. This postulate says: the M-functions are analytic except for the singularities generated by singularity. The idea is to define by iteration of the unitarity equations the maximal set of singularities. All singularities come from the phase space factors. Perhaps you can extend analyticity to other sheets (unstable particles) or to infinity (polynomial growth bound). See 4-4 E.

end of body of text, start of appendices for part I.

Appendix A: proof that S is unitary.

Appendix B: proof of relation between spin indices and rotations.

Appendix C: Identities of Spinor Formalism. Things are very different from my formalism. First, H talks about a complex (A,B) LT acting on spinors, he seems to show $(\frac{1}{2}, \frac{1}{2})$ rep in C.3, Also, notice that dotted's have the transformation matrix following, undotted's proceeding. The metric spinor is called C, and he uses $C = -$ my G. His C lowers on the left, mine, lowers on the right, etc. Finally, H constructs the higher matrices by clebsh induction, gets a general metric spinor.

Appendix D: Derivation of Lorentz transformation rule from physical Lorentz invariance .

Appendix E: Singularities generated by Unitarity and the Physical Sheet. Here at last is a clear statement of the idea of bubble diagrams. Your first draw unitarity with only two columns of bubbles, you then contract .. well, you then allow this bubble structure on the right to be inside one of the bubbles on the left. Then you iterate that structure back into the right side, etc etc until you get all possible bubble diagrams with any number of columns (but ordered in a certain way). Then you contract bubbles to a point and you see your phases space diagrams which are Landau diagrams. The Landau positive alpha thing is just a way to say where these phase space diagrams have singularities.

Then the claim is this: these singularities are precisely the same ones which appear in perturbation theory. Some of the problems associated with this singularity generating program are: you need HA to get disc formulas from unitarity formulas; you need extended U to get onto other sheets; there are some dangerous points to worry about.

But the idea is plain as can be.

.end of Part I.

Stapp Trieste. Part II: Antiparticles in S-matrix Theory.

1. Introduction. How are you going to show the existence of antiparticles without using fields? In 1962 plan was given by Gunson, then reworked by Olive in 1964, then further by Stapp (unpub). This section contains Stapp's work on the subject. Olives work is generalized.

2. Pole-Factorization Theorem. This is a key step in the proof of existence of antiparticles. Lets agree to assume (1) cluster decomposition and unitarity; (2) you have analyticity near $k^2 = m^2$. Singularities on manifold $k^2 = m^2$ but which lie on none of the other singularity surfaces go with Landau pole graphs. (3) the +ie rule.

In fancy box notation, Henry shows how to draw all unitarity terms containing the pole on the diagonal, see (2.4). Actually, these are terms containing the delta function, not the pole. At this point I lose the argument and do not now wish to pursue it. The argument roughly says, as I have said before: assume a certain pole form, then show that it works. Then show that no other form works.

The main result is that, given the assumptions noted above, particle implies pole and pole implies particle. Then the antiparticle pole will imply existence of actual live antiparticle.

3. Crossing and Antiparticles. Long piece on path continuations to show how you get from one pole diagram to the other. Notice that the little symbol  is not used. A major assumption here is that the only Landau diagrams (from general bubble diagrams) which can yield singularities as $k^2 = m^2$ are the obvious ones which have a "key bubble" which divides the diagram into two parts. Then somehow you can do continuations on the two sides separately (vague notion to me now). I think this argument was brought into question by Branson later, and finally re-resolved by Stapp in positive-alpha language in the 1968 paper.

Once you continue to the position of the antiparticle pole, you have to show it is really there. If you can continue by an uninterrupted path, then it must be there due to persistence of a pole in analytic function (again, vague to me). If you had to jump across some cuts, not obvious any more.

4. Discussion of the assumptions. Considers possibility of a pinch occurring at $k^2 = m^2$ so you path is really blocked. Also, the question of "accidental" Landau singularities at $k^2 = m^2$ is rediscussed. I think this was assumption 2 above: there are no accidentals. In the Haag-Ruelle field theory, these do not exist.

3/77

ANALYTIC S-MATRIX THEORY

H.P. STAPP*
INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS,
TRIESTE, ITALY

PART I

THE ANALYTIC S-MATRIX FRAMEWORK

1. INTRODUCTION

Analytic S-Matrix theory is an attempt to provide a theoretical basis for Mandelstam-type calculations that will allow them to be extended to general cases and used as the basic tool of a general dynamical theory of elementary particles. As is well known, Mandelstam's formulas not only reproduce, when solved by iteration, the appropriate terms of the perturbation expansion, in automatically renormalized form [1], but also give the basis of the most practically-successful, non-perturbative, dynamical calculations [2]. However, they are based on an analyticity assumption for two-particle scattering amplitudes that far transcends what has been proved on the basis of field theoretic principles. Since these principles themselves have little experimental basis, while the Mandelstam calculations appear successful, the idea naturally presents itself of extending Mandelstam's analyticity assumption in a precise way to multi-particle scattering amplitudes, and adopting it as a basic postulate of an alternative dynamical theory. This should permit Mandelstam-type calculations to be developed in a systematic and rigorous way from general principles that can be judged on the basis of their consistency, practicability, and apparent success.

That field theoretic principles are not the most appropriate basis for Mandelstam calculations is indicated also by the fact that Mandelstam's key formulas involve not the general Green's functions but only their mass-shell restrictions, the physical scattering amplitudes. Since Green's functions are a basic element of field theories, their unimportance in Mandelstam calculations suggests that the natural setting for these calculations is not field theory, but rather an S-matrix theory based, as proposed by HEISENBERG [3], directly on properties of the physical scattering amplitudes.

Though in the Mandelstam approach fields are needed neither for defining extensions of the scattering functions off the mass-shell (i.e. Green's functions) nor for carrying causality conditions into analyticity properties, it is not clear that they can be completely eliminated. For crossing relations play a central role in the theory, and these seem to arise from the intimate connection between particles and their conjugate antiparticles residing in the fact that both are described by a single local field. It is not

* Permanent address: Lawrence Radiation Laboratory, Berkeley, Calif., United States of America

clear that these crossing relations would naturally persist if the field theoretic substructure were eliminated.

In order to save crossing one might construct a hybrid theory that retains the parts of field theory involving asymptotic fields but eliminates the causality constraint on the interpolating (or Heisenberg) fields. This course is logically objectionable because the connection between the asymptotic fields at plus and minus infinity, would be carried by unconstrained interpolating fields that serve no other purpose. The particle-antiparticle connection would, in effect, be merely a prescription based, in a disguised way, on the rejected locality condition.

While the hybrid approach is perhaps adequate for practical calculations, it would not provide a satisfactory basis for proofs of general theorems, such as CPT-invariance or the connection between spin and statistics, for it imposes on the theory artificial constraints that are simply carry-overs from field theory. Though theorems might be derived within such a framework, the deeper and more pertinent question is whether a satisfactory theory can be obtained without these constraints, and if so whether the theorems would still hold true - a CPT-theorem or a proof of a connection between spin and statistics has little significance if based on special conditions that are not integral and necessary parts of the general theoretical structure.

Stated differently, the question of possibilities of unusual statistics, or parastatistics, or other pathologies (such as those involved in relativistic versions of SU(6)) inevitably raises the question whether the exotic connections could be accommodated in a more general theoretical framework that would still accommodate the basic experimental facts. In order to eliminate unnecessary constraints, and obtain the most general theoretical framework consistent with the generally accepted experimental facts, we shall build upon postulates that are, as far as possible, direct assertions of specific, testable, experimental relationships. Though this procedure is not as elegant or economical as the alternative of building on sweeping general postulates referring to abstract entities, it provides a very secure theoretical foundation and illuminates the experimental basis of the theoretical structure. The concomitant elimination of unnecessary entities, such as field operators and vacuum states, simplifies not only the logical structure, but practical calculations as well.

To this general S-matrix structure is added a mass-shell analyticity postulate. This provides the general theoretical framework of analytic S-matrix theory. The question of whether this framework entails antiparticles and crossing relations is discussed in Part II, where it is shown that these emerge in an unambiguous and well-defined way.

2. BASIC QUANTUM PRINCIPLES AND FUNDAMENTAL OBSERVABLES

A. Quantum theory postulates

Quantum theory deals with correlations between the probabilities of possible results of possible experiments. A class C of complete experi-

ments is characterized as follows: Let $\mathcal{P}(R', E'; R, E)$ be the probability that experiment E' gives result (R', E') if the probability that experiment E gives a result different from (R, E) is zero. Then for all (R, E) and $(R'E')$ with E and E' in C

$$A1 \quad \mathcal{P}(R', E'; R, E) \geq 0.$$

$$A2 \quad \sum_{R'} \mathcal{P}(R', E'; R, E) = 1$$

A3 There is a set of complex numbers $a(R', E'; R, E)$ such that

$$\mathcal{P}(R', E'; R, E) = |a(R', E'; R, E)|^2$$

where the amplitudes $a(R', E'; R, E)$ are linearly related: for any E'' in C

$$a(R', E'; R, E) = \sum_{R''} \frac{S(R', E'; R'', E'')}{S(R'', E''; R'', E'')} a(R'', E''; R, E).$$

(Postulates A1 and A2 are expressions of the fact that $\mathcal{P}(R', E'; R, E)$ is a probability. Postulate A3 expresses the basic characteristic of quantum theory - probabilities are expressed as the squares of absolute values of amplitudes that are linearly related. This is the superposition principle. In its formulation given above the usual apparatus of quantum theory, involving the abstract concepts of Hilbert spaces and operators, and the concept of completeness in terms of eigenvectors of complete sets of commuting operators has not been introduced. Rather the essential physical relationships have been stated directly.

The amplitudes a and the transformation matrix S are arrays of complex numbers depending only on their displayed arguments. Although these two sets $a(R', E'; R, E)$ and $S(R', E'; R, E)$ are in principle different, they are obvious numerically equal to within a phase factor, which can be set to unity by taking the phase of $a(R, E; R, E)$ to be zero for all (R, E) . This can be done by trivial re-definitions that leave the above postulates unaltered). Definition Let E' be in C. Results (R', E') and (R'', E') belong to different superselection classes (relative to C) if and only if

$$\mathcal{P}(R', E'; R, E) \mathcal{P}(R'', E'; R, E) = 0 \quad (2.1)$$

for all (R, E) with E in C.

A4 For any E' in C and any set of numbers $a(R', E')$ there is an (R, E) with E in C such that

$$a(R', E') \simeq a(R', E'; R, E), \quad (2.2)$$

provided $a(R', E')$ satisfies the (clearly necessary) conditions

Taylor '66

Spin and Isospin in S -Matrix Theory

JOHN R. TAYLOR

Palmer Physical Laboratory, Princeton, New Jersey

(Received 7 July 1965)

The Gunson-Olive technique for exploiting physical region poles and the unitarity equations to derive the analytic properties of S -matrix elements is applied to particles which belong to isospin multiplets and to particles with spin. In the former case it is shown that, in the framework of S -matrix theory, crossing symmetry leads to the requirement that particles and antiparticles transform according to complex conjugate representations; and it is possible to clarify the origin of the familiar isospin crossing matrices. In the case of particles with spin the construction of appropriate "spinor" basic states is discussed and the analytic properties of the corresponding spinor S -matrix elements or " M -functions" are derived.

I. INTRODUCTION

DURING the last two years great progress has been made towards the construction of a self-contained theory of the S -matrix based on physically realistic postulates. As a result of the work of Gunson¹ and Olive,² it appears that the essential ingredients of such a theory are the following five postulates:

- (1) The superposition principle, *Hilbert Space*.
- (2) The existence of a unitary S -matrix,
- (3) Lorentz invariance,
- (4) The decomposition law (or "connectedness structure"), *cluster decomposition*.
- (5) Analyticity on the mass-shell.

Using these postulates the attempt has been made to prove such familiar properties as crossing symmetry, Hermitian-analyticity, the correspondence between particles and poles, and the existence of antiparticles; all of which properties appeared as somewhat unnatural postulates in previous attempts at a theory of the S -matrix.³

Unfortunately, the proofs of these properties are not completely satisfactory. This is because of their dependence on a knowledge of the singularities of the S -matrix, the complete enumeration of which is still an unsolved problem. At present all that is known is that the exact location of singularities should not appear explicitly in the postulate of analyticity; rather, this postulate should be formulated in such a way that all singularities are derivable from the other postulates (in particular, from unitarity and the decomposition law). Thus far no one has shown how this can be done without using the very properties it was hoped to prove (i.e., crossing symmetry, etc.).

Until these difficulties can be overcome, the S -matrix theory "proofs" of crossing symmetry, etc., can only be considered as explanations. (Given, for example, that crossing symmetry is true, there is no question but that the Gunson-Olive "proof" explains why it is true.) However, even to have explained such properties is a considerable achievement, since previously their only justification was their occurrence in perturbation theory. And it would seem valuable to extend the Gunson-Olive discussion of spinless particles to the case of particles which belong to isospin multiplets and to particles with spin.⁴

The principal aim of this paper, then, is to discuss the S -matrix theory of particles with isospin and—much more complicated problem—particles with spin. Because the most recent developments in S -matrix theory are probably not very widely known and the notation is not familiar, the next section is devoted entirely to a review of the S -matrix theory of spinless particles. I first discuss the five postulates listed above and then, having defined the necessary notation, review briefly the relevant parts of the Gunson-Olive work. In Sec. III, I am then free to consider isospin invariance and its effect on the form of crossing symmetry. The last two sections are concerned with the S -matrix theory of particles with spin. More space is given to these sections because of the greater complexity of the topic. With the great number of possible labellings of spin states, one must first decide which matrix elements would be expected to have the desired analytic properties. Accordingly Sec. IV is devoted entirely to the construction of the appropriate basic spin states—the

¹ J. Gunson, J. Math. Phys. 6, 827 (1965).

² D. I. Olive, Phys. Rev. 135, B745 (1964).

³ See, in particular, H. P. Stapp, Phys. Rev. 125, 2139 (1962).

⁴ Many of the techniques used here to include spin and isospin in the Gunson-Olive work have been known for some time. See, in particular, A. O. Barut, Phys. Rev. 130, 436 (1963); A. O. Barut and B. C. Uti, Nuovo Cimento 28, 112 (1963); these articles discuss crossing symmetry for particles with spin and isospin, respectively, but both take the view that crossing is a fundamental postulate.

so-called spinor states. Then in Sec. V, I discuss the analytic properties of the corresponding spinor matrix elements (or "M-functions").

II. S-MATRIX THEORY OF SPINLESS PARTICLES

The first postulate, the superposition principle, specifies the nature of the system of interest long before and long after interactions take place. The most natural statement of the principle is in terms of Hilbert space; namely, that long before and long after interactions a system can be described by "states" and that these states are found to correspond to rays⁵ in a Hilbert space which has the following properties:

- (i) The space is composed of orthogonal "superselection subspaces" within which any linear superposition of vectors represents a physically realizable state.
- (ii) If φ and ψ are any two realizable states represented by unit vectors $|\varphi\rangle$ and $|\psi\rangle$, then $|\langle\varphi|\psi\rangle|^2$ is the probability that a system known to be in the state ψ be observed in the state φ .
- (iii) The space is a Fock space spanned by an orthonormal basis of (improper) vectors representing states of definite numbers of particles with definite momenta. The notation used here for these states is as follows:

The states of a single (spinless) particle are labelled by their four-momentum⁶ p and particle type⁷ t ; the two variables p and t can conveniently be denoted by a single label $k = (p, t)$ and the corresponding state vector by $|k\rangle = |p, t\rangle$. These vectors are normalized such that

$$\langle p', t' | p, t \rangle = p^0 \delta_3(p' - p) \delta_{t't}. \quad (2.1)$$

The n -particle states are labelled by the n -tuple $K = (k_1, \dots, k_n)$, where $k_i = (p_i, t_i)$, and the corresponding state vectors are just tensor products of one-particle vectors,

$$|K\rangle = |k_1, \dots, k_n\rangle = |k_1\rangle \otimes \dots \otimes |k_n\rangle,$$

(symmetrized according to the appropriate statistics

⁵ A ray ψ corresponding to a vector $|\psi\rangle$ is the set $\psi = \{z|\psi\rangle\}$ where z ranges over the whole complex plane. Within any superselection subspace the rays are in one-to-one correspondence with physically realizable states, and the same label can therefore be used for both.

⁶ The space component p ranges over the whole of three-dimensional Euclidean space; the energy component p^0 is of course redundant being defined by the mass-shell condition, $p^0 = + (p^2 + m^2)^{1/2}$. The label t runs over all stable particles of the theory.

if there are identical particles⁷). It is also convenient to introduce the n -tuples $P = (p_1, \dots, p_n)$ and $T = (t_1, \dots, t_n)$ and to write the state K as $K = (P, T)$. (No serious ambiguity arises from these two ways of writing the state K .)

The reason for introducing the relatively abstract and unphysical concept of Hilbert space is that it does provide the only natural expression of the superposition principle. Many recent papers on S -matrix theory⁸ have tried to bypass discussion of the Hilbert space of initial and final states by defining the S -matrix elements as transition amplitudes (i.e. numbers whose squared moduli are transition probabilities). This procedure fails to settle the important question of how the phase of S -matrix elements can be consistently defined—a question which can only be answered using the superposition principle. It seems to be essential that this principle applies to both initial and final states [i.e. a superposition of two initial (or final) states in one superselection class is a realizable initial (or final) state] and that the corresponding Hilbert space enter, at least implicitly, in the theory. Indeed, if one examines the only complete attempt at formal S -matrix postulates⁹ which do not include Hilbert space, one can easily see that they contain sufficient information to construct it.

The second postulate, which defines the S -matrix, does not in fact have to state that S is unitary. It states only the experimental observation that there is a one-to-one correspondence between possible initial and final states and that this correspondence is such as to preserve the superposition principle (i.e. if initial states φ and ψ lead to final states φ' and ψ' , then $|\langle\varphi|\psi\rangle| = |\langle\varphi'|\psi'\rangle|$, where $|\varphi\rangle, \dots$ are unit vectors representing $\varphi, \dots$). A famous theorem due to Wigner¹⁰ ensures that such a correspondence between states (i.e., between rays in the Hilbert space of the first postulate) can be represented by a correspondence S between vectors in Hilbert space with S either unitary or antiunitary. That S is actually unitary follows because the relationship between

⁷ The famous connection between statistics and spin has been discussed in the S -matrix framework by Stapp.⁸ In this paper I shall for the most part avoid these complications by considering only states with all different particles. In the last section, however, the question of ordering the variables has to be discussed and the normal connection between spin and statistics will be assumed.

⁸ See, for example, Ref. 3 and P. V. Landshoff, *The S-matrix without Field Theory*, Cambridge preprint, 1964 (to be published).

⁹ H. P. Stapp, *Studies in the Foundations of S-matrix Theory*, California preprint UCRL-10343 (to be published). The construction mentioned here has been described in detail by J. R. Taylor, *Phys. Rev.* **140**, B187 (1965).

¹⁰ E. P. Wigner, *Group Theory* (Academic Press Inc., New York, 1959), p. 233.

Added data:

1. Rest states differ slightly as follows:

$$D_{mm}^{(s)}(\gamma) = (-1)^{s-m} \delta_{m,-m'}$$

$$L(\tilde{p}) = H(\tilde{p}) = 1$$

$$|\tilde{p}^m\rangle = |\tilde{p}, m\rangle$$

$$|\tilde{p}_m\rangle = (-1)^{s+m} |\tilde{p}, -m\rangle$$

$$|\tilde{p}^m\rangle = (-1)^{s-m} |\tilde{p}, -m\rangle$$

$$|\tilde{p}_m\rangle = |\tilde{p}, m\rangle$$

2. Dot conversion:

$$|p^m\rangle [p_m] = |p_m\rangle (p^m)$$

$$\text{or } x^m y_m = x_m y^m$$

1. How are the spinor states related to the Helicity States?

$$|p^m\rangle = |p, m\rangle_H D(H(p)^{-1})$$

$$|p^m] = |p, m\rangle_H D(H(p)^{\dagger} Y)$$

$$|p_m\rangle = |p, m\rangle_H D(H(p)^{-1} \bar{Y}^{-1})$$

$$|p_m] = |p, m\rangle_H D(H(p)^{\dagger})$$

2. How are the spinor states related to the Canonical states?

$$|p^m\rangle = |p, m\rangle_C D(L(p)^{-1})$$

$$|p^m] = |p, m\rangle_C D(L(p)^{\dagger} Y)$$

$$|p_m\rangle = |p, m\rangle_C D(L(p)^{-1} \bar{Y}^{-1})$$

$$|p_m] = |p, m\rangle_C D(L(p)^{\dagger})$$

3. How do the spinor states transform under Lorentz transformations?

$$L|p^m\rangle = |Lp^m\rangle D(L)$$

$$L|p^m] = |Lp^m] D(L)^{\dagger}$$

$$L|p_m\rangle = |Lp_m\rangle D(L^{-1})$$

$$L|p_m] = |Lp_m] D(L^{-1})^{\dagger}$$

4. How are the spinor states related to each other?

$$|p^m\rangle = |p_m\rangle D(Y)$$

$$|p^m] = |p_m] D(Y)$$

$$|p_m] = |p_m\rangle D(Y L^2(p))$$

$$|p^m] = |p^m\rangle D(L^2(p) Y)$$

$$|p_m] = |p^m\rangle D(L^2(p))$$

$$|p^m] = |p_m\rangle D(Y L^2(p) Y)$$

5. What is the completeness relation?

$$\begin{aligned} 1 &= |p, m\rangle_H \langle p, m|_H = |p, m\rangle_C \langle p, m| \\ &= |p^m\rangle D[L^2(p)] \langle p^m| \\ &= |p^m\rangle [p_m] \end{aligned}$$

6. Related information: the spinor states are the standard spinor fields, as in appendix to paper. Other facts are:

$$|p, m\rangle_H = |p, m\rangle_C D[R(p)]$$

$$L(p)^{\dagger} = L(p)$$

$$Y L Y^{-1} = L^{-1T}$$

$$H(p) = R(p) Z(p) = L(p) R(p)$$

$$H(p) H(p)^{\dagger} = L(p)^2$$

$$L(Lp)^{-2} L L^2(p) = L^{-1T}$$

$$R(p) H^{\dagger}(p) = L(p) \quad \Leftarrow$$

$$L|p, m\rangle_H = |Lp, m\rangle_H D[R_H]$$

The spinor states of Taylor.

Support for data sheet.

1. These equations may be found directly in Taylor's paper:

$$1) |p^m\rangle = |p, m\rangle_H D^{\circ 5}(H(p)^{-1})$$

$$2) L|p^m\rangle = |lp^m\rangle D(L)$$

$$3) |p_m] = |p, m\rangle_H D(H(p)^{\dagger})$$

Taylor really doesn't say much else. He does not use canonical states at all. I must make sure any results I get and use agree with the above three equations.

2. I will start out by conjecturing that these are the correct states:

$$|p^m\rangle = |p, m\rangle_H D[H(p)^{-1}]$$

$$|p^m] = |p, m\rangle_H D[H(p)^{\dagger} \gamma^{-1}]$$

$$|p_m\rangle = |p, m\rangle_H D[H(p)^{-1} \gamma^{-1}]$$

$$|p_m] = |p, m\rangle_H D[H(p)^{\dagger}]$$

I have copied two from Taylor, and have guessed the other two from the lowering tensor:

$$\rightarrow |p_m\rangle = |p^{m'}\rangle G_{m'm} =$$

$$|p^m\rangle D(\gamma^{-1}) = |p_m\rangle$$

$$|p_m] = |p^{m'}] G_{m'm} =$$

$$|p^m] D(\gamma^{-1}) = |p_m]$$

$$\Rightarrow \rightarrow |p^m] = |p_m] D(\gamma)$$

3. Now let's compute how each of these four states transforms under L:

$$\begin{aligned} L|p^m\rangle &= |lp, m\rangle_H D(H(p)^{-1}) = |lp, m\rangle_H D(H(lp)^{-1} \cdot L \cdot H(p)) D(H(p)^{-1}) \\ &= |lp, m\rangle_H D(L) \end{aligned}$$

$$\begin{aligned} L|p_m\rangle &= |lp, m\rangle_H D(H(lp)^{-1} \cdot L \cdot H(p)) \cdot D(H(p)^{-1} \gamma^{-1}) \\ &= |lp, m\rangle_H D(H(lp)^{-1} \gamma^{-1}) D(\gamma L \gamma^{-1}) \\ &= |lp_m\rangle D(L \gamma^{-1}) \end{aligned}$$

$$\begin{aligned}
 L|p^m] &= |p, m\rangle_H D(H(p)^{-1} \cdot L \cdot H(p)) D(H(p)^T \cdot Y) \\
 &= |p, m\rangle_H D(H(p)^T Y) D(Y^{-1} \cdot H(p)^{-1}) D(H(p)^{-1} L H(p)) D(H(p)^T Y) \\
 &= |p^m] \underbrace{D(Y^{-1} \{H(p) H(p)^T\}^{-1} L \{H(p) H(p)^T\} Y)}_{= D(Y^{-1} \cdot L(p)^{-2} \cdot L \cdot L(p)^2 \cdot Y)} \\
 &= D(Y^{-1} \cdot L^{T^{-1}} \cdot Y) = D(L^{T^{-1}})^{-T} = D(L^*) \\
 &= |p^m] D(L)^*
 \end{aligned}$$

$$\begin{aligned}
 L|p_m] &= |p, m\rangle_H D(H(p)^{-1} \cdot L \cdot H(p)) H(p)^T \\
 &= |p, m\rangle_H D(H(p)^T) D(H(p)^T H(p)^{-1} \cdot L(p)^2) \\
 &= |p, m] \cdot \underbrace{D(L(p)^2 \cdot L \cdot L(p)^{-2})}_{= D(L^{T^{-1}})} \\
 &= |p, m] \cdot D(L^{T^{-1}})^*
 \end{aligned}$$

To summarize:

$L p^m] = p^m] \cdot D[L]$	$L p^m] = p^m] D(L)^*$
$L p_m] = p_m] D[L^{T^{-1}}]$	$L p_m] = p_m] D(L^{T^{-1}})^*$
<i>first-kind</i>	<i>second-kind</i>

Good, things are as expected. Next, what about canonicals?

$$\begin{aligned}
 |p, m\rangle_H &= R(p) |p, m\rangle_c = |p, m\rangle_c D[R(p)] \\
 H(p) &= R(p) Z(p) = L(p) R(p)
 \end{aligned}$$

So add $R(p)$ on left of every H

Note: $H^T(p) = R^{-1}(p) L(p) \Rightarrow R(p) H^T(p) = L(p)$

$$\text{Then } |p^m\rangle = |p, m\rangle_c D[R(p)H(p)^{-1}] = |p, m\rangle_c D(L(p)^{-1})$$

$$|p_m\rangle = |p, m\rangle_c D[L(p)^{-1}Y^{-1}]$$

$$|p^m\rangle = |p, m\rangle_c D[R(p)H(p)^{\dagger}Y] = |p, m\rangle_c D(L(p)Y)$$

$$|p_m\rangle = |p, m\rangle_c D[R(p)H(p)^{\dagger}] = |p, m\rangle_c D(L(p))$$

So:

$ p^m\rangle = p, m\rangle_c D[L(p)^{-1}]$	$ p^m\rangle = p, m\rangle_c D[L(p)^{\dagger}Y]$
$ p_m\rangle = p, m\rangle_c D[L(p)^{-1}Y^{-1}]$	$ p_m\rangle = p, m\rangle_c D[L(p)^{\dagger}]$

This means we can deduce all these results from the helicity ones by replacing H with C and replacing $H(p)$ with $L(p)$. Recall that $L(p) = L(p)^{\dagger}$ but so what.

5. Curse relation:

$$|p_m\rangle = |p, m\rangle_H D[H(p)^{-1}Y^{-1}] = |p_m\rangle D(H(p)^{\dagger}) D(H(p)^{-1}Y^{-1})$$

$$= |p_m\rangle D[L^2(p)Y^{-1}]$$

$$|p_m\rangle = \bigotimes \bigotimes |p^m\rangle \frac{D[Y^{-1}H(p)^{\dagger}]}{D[Y^{-1}L^2(p)Y^{-1}]} D[H(p)^{-1}Y^{-1}]$$

$$|p_m\rangle = |p, m\rangle_H D[H(p)^{\dagger}] = \bigotimes |p^m\rangle D(H(p)) D(H(p)^{\dagger})$$

$$= |p^m\rangle D(L(p)^2)$$

So:

$$|p_m\rangle = |p_m\rangle D(L^2(p)Y^{-1}) \Rightarrow$$

$|p$

$$|p^m\rangle = |p, m\rangle_H D(H(p)^{\dagger}Y)$$

$$= |p^m\rangle D(H(p)) D(H(p)^{\dagger}Y) = |p^m\rangle D(L^2(p)Y)$$

$ p_m\rangle = p_m\rangle D(Y L^2(p))$
$ p^m\rangle = p_m\rangle D(Y L^2(p)Y)$
$ p_m\rangle = p^m\rangle D(L^2(p))$
$ p^m\rangle = p^m\rangle D(L^2(p)Y)$

Next, $1 = |p, m\rangle_H \langle p, m|_H = ?$

$$|p, m\rangle_H = |p^m\rangle D(H(p))$$

$$\langle p, m|_H = \langle p^m| D(H(p))^\dagger$$

$$= D(H(p)^*) \langle p^m|$$

$$\text{So } 1 = |p^m\rangle D(H(p)) D(H(p)^*) \langle p^m|$$

$$\Rightarrow 1 = |p^m\rangle D(L(p)^2) \langle p^m|$$

But also,

$$|p, m\rangle_H = |p, m\rangle D[H(p)^*]^{-1}$$

$$\text{so } 1 = |p^m\rangle D(H(p)) \cdot D(H(p)^*)^{-1} \langle p, m| = |p^m\rangle \langle p, m|$$

So.

$$\begin{aligned} 1 &= |p, m\rangle_H \langle p, m|_H \\ &= |p^m\rangle D[L(p)^2] \langle p^m| \\ &= |p^m\rangle \langle p, m| \end{aligned}$$

A quick Review of Taylor States.

oops found!

7/1/77

① First, we define how the two first-kind spinors transform:

$$\hat{\chi}^m = \Lambda^{m'}_{m'} \chi^{m'} = \chi^{m'} \cdot D_{m'm}^{(0,s)}(a)$$

$$\hat{\chi}_m = \Lambda_m^{m'} \chi_{m'} = \chi_{m'} \cdot D_{m'm}^{(0,s)}(a^{-1})$$

Thus you get $\hat{\chi}^m \hat{\chi}_m = \chi^m \chi_m = \text{Lorentz Scalar.}$

② We know that $U(A)|p, m\rangle_c = |p, m'\rangle_c D_{m'm}^{(0,s)}[L(a)^{-1} \cdot A \cdot L(p)]$, Wigner.

Thus, if we define:

$$|p, m\rangle \equiv |p, m'\rangle_c D_{m'm}(L(p)^{-1})$$



$$\text{Then } A|p, m\rangle = \{A|p, m'\rangle_c\} D_{m'm}(L(p)^{-1})$$

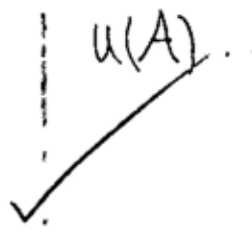
$$= |p, m''\rangle_c D_{m''m'}[L(a)^{-1} A L(p)] D_{m'm}(L(p)^{-1})$$

$$= |p, m''\rangle_c D_{m''m'}(L(a)^{-1}) \cdot D_{m'm}^{(0,s)}(A)$$

$$= |p, m'\rangle D_{m'm}^{(0,s)}(a)$$

$$\Rightarrow U(A) \dots$$

$$\Rightarrow L|p, m\rangle = |p, m'\rangle \cdot D_{m'm}^{(0,s)}(a)$$



Therefore, $|p, m\rangle$ is a first-kind spinor field, like $A^\mu(x)$ is a tensor field.

③ Therefore we know:

$$|p, m\rangle_c \equiv |p, m'\rangle \cdot D_{m'm}(L(p))$$



④ Completeness:

$$1 = \sum_m |p, m\rangle_c \langle p, m| = |p, m'\rangle D_{m'm}(L(p)) \cdot \langle p, m''| D_{m''m}(L(p))^*$$

$$= |p, m'\rangle D_{m'm''}(L(p) L(p)^\dagger) \langle p, m''|$$

$$\Rightarrow 1 = |p, m'\rangle D_{m'm}(L^2(p)) \langle p, m'|$$



⑤ This suggests that we define: $[p, m'] = D_{m'm} (L^2(p)) (p, m) \checkmark$

$$a \quad [p, m'] = (p, m) D_{mm'}^{(0,0)} (L^2(p))^T \checkmark$$

$$a \quad [p, m'] = (p, m) D_{mm'}^{(0,0)} (L^2(p))$$

How does such a state transform?

$$L[p, m'] = \{ L[p, m] \} D_{mm'} (L^2(p))$$

$$= \{ |l p, m'' \rangle \} D_{m''m} (a) D_{mm'} (L^2(p))$$

$$= \{ |l p, m'' \rangle D_{m''\tilde{m}} (L^2(l p)) \cdot D_{\tilde{m}m'} (L^2(l p)) \cdot \} D_{m''m'} (a \cdot L^2(p))$$

$$= |l p, \tilde{m} \rangle D_{\tilde{m}m'} [L^2(l p) \cdot L^2(p)]$$

$$= |l p, \tilde{m} \rangle D_{\tilde{m}m'} [L^2(l p) \cdot L^2(p)]$$

$$= |l p, \tilde{m} \rangle D_{\tilde{m}m'} [a \cdot L^2(p) \cdot L^2(l p)]$$

So far:

$$L[p, m'] = |l p, \tilde{m} \rangle D_{\tilde{m}m'} [L^2(l p) \cdot L \cdot L^2(p)]$$

Too hard to understand.

⑥ Instead, suggests

$$[p, m'] = D_{m'm} (L^2(p)) (p, m)$$

$$= [p, m] D_{mm'} (L^2(p))^T$$

$$= (p, m) D_{mm'} (L^2(p))^{\dagger} = (p, m) D_{mm'} (L^2(p))^*$$

$$\Rightarrow \boxed{[p, m] = [p, m'] D_{m'm} (L^2(p))} \checkmark$$

But, $|p, m'\rangle = |p, m''\rangle_c D_{m''m'}(L(p))$

So $|p, m\rangle = |p, m''\rangle_c D_{m''m}(L(p)) \cdot D_{m'm}(L^2(p))$

$\Rightarrow |p, m\rangle = |p, m'\rangle_c D_{m'm}(L(p))$ ✓

So:

$L|p, m\rangle = L|p, m'\rangle_c D_{m'm}(L(p))$

$= |p, m''\rangle_c D_{m''m'} [L(p)^{-1} \cdot L \cdot L(p)] D_{m'm}(L(p))$

$= |p, m'\rangle D_{m'm} [\underbrace{L(p)^{-2} \cdot L \cdot L(p)^2}_{= (L^+)^{-1}}]$ / same as in ②

$\Rightarrow L|p, m\rangle = |p, m'\rangle D_{m'm}^{(0,s)}(L^{+1})$ ✓

But second-kind spins do this

$\hat{y}^m = \Lambda^m_{m'} y^{m'} = y^{m'} \cdot D_{m'm}^{(s,s)}(a)^*$

$\hat{y}_m = \Lambda_m^{m'} y_{m'} = y_{m'} \cdot D_{m'm}(a^{+1})$

Thus: $|p, m\rangle$ is a lower index, second-kind spin.

① Now let's adopt metric spin as in notes (BD-like). Thus:

$G_{\mu\nu} = G_{\nu\mu} = G^{\mu\nu} = G^{\mu\nu} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = i\sigma_2.$

Then

$D^{(0,s)}(G)$

Question: when you go to Tayloers round and square states, what happens to this
idea: $(f/T^\dagger/i) = (i/T/f)^*$?

1. In terms of the usual states, this equation simply serves to define the operator $T^\dagger$. Once we have so-defined this operator, we cannot redefine it, so we have to check to see if things come out in a nice way.

2. Checking with Taylor data sheet, here is what you find to be consistent:

$$\boxed{[p_1^{m_1} | T^\dagger | p_2^{m_2}] = [p_2^{m_2} | T | p_1^{m_1}]^*}$$

$$[p_1^{m_1} | T^\dagger | p_2^{m_2}] \neq [p_2^{m_2} | T | p_1^{m_1}]^*$$

In other words, when you switch initial and final states, you must keep the square with the state in question, the "squareness" is a part of the state and must be moved with the rest of the state.

3. The first line quoted in 2 above is true for all index combinations. Eg,

$$[p_{1,m_1} | T^\dagger | p_{2,m_2}] = [p_{2,m_2} | T | p_{1,m_1}]^*$$

✓

How are $|p, m\rangle$ and $|p, m\rangle$ related?

① From Taylor we know that:

$$|p, m\rangle = |p, m\rangle_H D_{mm}(H(p)^{-1}) \quad (1)$$

$$|p, m\rangle = |p, m\rangle_H^* D_{mm}^*(H(p)^{-1}) \quad (2)$$

Therefore, solve (1). First write as matrix mult:

$$\begin{aligned} |p, m\rangle &= |p, m\rangle_H D(H(p)^{-1}) \rightarrow |p, m\rangle D(H(p)) = |p, m\rangle_H \\ |p, m\rangle &= |p, m\rangle_H D(H(p)^{-1})^* \rightarrow |p, m\rangle D(H(p)^*) = |p, m\rangle_H \end{aligned}$$

Therefore:

$$|p, m\rangle = |p, m\rangle D(H(p))^* D(H(p)^{-1})$$

② Now we need details on these functions!

$$D_{mm}^{(0,0)}(a)$$

$a \in \text{L.G.}$

but really $a = \text{SL}(2, \mathbb{C})$ matrix.

From Taylor page 191 we know

$$\begin{aligned} D(a^T) &= D(a)^T \\ D(a^*) &= D(a)^* \\ D(a^\dagger) &= D(a)^\dagger \end{aligned}$$

$$a = \begin{matrix} R_x(\phi) & R_y(\phi) & R_z(\phi) & B_x(v) & B_y(v) & B_z(v) \end{matrix}$$

$$a^* = \begin{matrix} \phi^{-1} & \phi & \phi^{-1} & v & v^{-1} & v \end{matrix}$$

$$a^T = \begin{matrix} \phi & \phi^{-1} & \phi & v & v^{-1} & v \end{matrix}$$

$$a^\dagger = \begin{matrix} \phi^{-1} & \phi^{-1} & \phi^{-1} & v & v & v \end{matrix}$$

So make a chart:

Better chart:

$a =$	R_x	R_y	R_z	B_x	B_y	B_z
$a =$	a	a	a	a	a	a
$a^* =$	a^{-1}	a	a^{-1}	a	a^{-1}	a
$a^T =$	a	a^{-1}	a	a	a^{-1}	a
$a^\dagger =$	a^{-1}	a^{-1}	a^{-1}	a	a	a

Spinor Conventions

① Baade-Jahlesay: (1953)

first-kind spin: $\bar{\xi}^a = \Lambda^a_b \bar{\xi}^b$ where $\bar{\Lambda}^a_b = D^{(0,0)}(g)_{ab} = g_{ab}$

second-kind spin: $\bar{\xi}^{\dot{a}} = \Lambda^{\dot{a}}_{\dot{b}} \bar{\xi}^{\dot{b}}$ where $\Lambda^{\dot{a}}_{\dot{b}} = D^{(0,1)}(g)_{\dot{a}\dot{b}} = g_{\dot{a}\dot{b}}^*$

Thus: $\Lambda^{\dot{a}}_{\dot{b}} = (\Lambda^a_b)^*$

Hermiticity says: Raiser & lower:

$$A^{\dot{a}b} = (\gamma)^{bb'} A^{\dot{a}}_{b'} \quad \gamma^{ab} = \gamma_{ab} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\text{Hermiticity: } (A^{\dot{a}b})^\dagger = (A^{\dot{a}b})^{T*} = (A^{b\dot{a}})^* = \dots$$

② Taylor: (1965)

$$(x')^\alpha = D^{(0,\epsilon)}_{\beta\alpha}(g) X^\beta$$

where $D^{(0,\epsilon)}(g) \equiv \begin{pmatrix} e^{a \cdot \underline{J}} & e^{-ib \cdot \underline{J}} \\ e^{a \cdot \underline{J}} & e^{-ib \cdot \underline{J}} \end{pmatrix}$ where $\underline{J}$ are "usual"!

$$\text{Thus, } D^{(0,\frac{1}{2})}(g) = \begin{pmatrix} e^{a \cdot \frac{1}{2} \underline{J}} & e^{-ib \cdot (\frac{1}{2} i \underline{\sigma})} \\ e^{a \cdot \frac{1}{2} \underline{J}} & e^{-ib \cdot (\frac{1}{2} i \underline{\sigma})} \end{pmatrix}$$

$$\text{I usually say: } \begin{pmatrix} e^{ia \cdot \underline{J}} & e^{-ib \cdot \underline{J}} \\ e^{ia \cdot \underline{J}} & e^{-ib \cdot \underline{J}} \end{pmatrix}$$

$$\text{So expect: } D^{(0,0)}(g) = \begin{pmatrix} e^{-ia \cdot \underline{J}} & e^{-ib \cdot \underline{J}} \\ e^{-ia \cdot \underline{J}} & e^{-ib \cdot \underline{J}} \end{pmatrix}$$

$$\text{Then: } D^{(0,\frac{1}{2})}(g) = \begin{pmatrix} e^{-ia \cdot \underline{J}} & e^{-ib \cdot i \underline{J}} \\ e^{-ia \cdot \underline{J}} & e^{-ib \cdot i \underline{J}} \end{pmatrix} =$$

$$\text{If } g = b \cdot u = a \cdot b \begin{matrix} \swarrow \text{rotation} \\ \uparrow \text{boost} \end{matrix} \text{, then } \begin{pmatrix} e^{-ia \cdot (i \underline{\sigma})} & e^{-ib \cdot \underline{J}} \\ e^{-ia \cdot (i \underline{\sigma})} & e^{-ib \cdot \underline{J}} \end{pmatrix} = e^{a \cdot \underline{J}} e^{-ib \cdot \underline{J}} = g$$

$$\text{So Taylor takes } D^{(0,0)}(g) = g$$

$$\begin{aligned} D^{(0,0)}(g) &= [D^{(0,0)}(g)]^{+1} = \begin{pmatrix} e^{ia \cdot \underline{J}} & e^{-ib \cdot \underline{J}} \\ e^{ia \cdot \underline{J}} & e^{-ib \cdot \underline{J}} \end{pmatrix}^+ \\ &= (e^{ib \cdot \underline{J}} e^{a \cdot \underline{J}})^{-1} = e^{-a \cdot \underline{J}} e^{-ib \cdot \underline{J}} \\ &= D^{(0,0)}(g^{+1}) \end{aligned}$$

$$\text{Note: } D^{(0,0)}(u) = D^{(0,0)}(u)$$

So:

$$(x')^\alpha = \sum_{\beta} D_{\beta\alpha}^{(0,s)}(g) x^\beta$$

← basic object

$$(x')^\alpha = \sum_{\beta} x^\beta D_{\beta\alpha}^{(0,s)}(g^*)$$

raise lower:

$$x^\beta C_{\beta\alpha} = x_\alpha$$

$$x_\beta C^{\beta\alpha} = x^\alpha$$

$$C \equiv D^{(0,s)}(c = \begin{pmatrix} 0 & -1 \\ +1 & 0 \end{pmatrix})$$

$$x^\beta C_{\beta\alpha} = x_\alpha$$

$$x_\beta C^{\beta\alpha} = x^\alpha$$

Well,

$$D^{(0,s)}(c = \begin{pmatrix} 0 & -1 \\ +1 & 0 \end{pmatrix}) = e^{-i\pi J_z}$$

③ Barut.

Quick comments about spinor conventions.

1. Nobody seems to agree on the exact meaning of $D^{(s,0)}(g)$. Among other problems, there is no absolute meaning to the order of the indices in the exponent. Usually people just pick their favorite order, and then set $D^{(\frac{1}{2},0)}(g) = g$ where g is some "standard" form of the elements of $SL(2, \mathbb{C})$. I have my own favorite standard form. Nice to write $g = hu$ and then use standard forms for boost h and rotation u . This first object that everybody defines controls the transformation of the contravariant (upper index) first kind or undotted spinors. Everybody agrees that this is the case.

2. Everybody also seems to agree that the dotted contravariant spinors should transform according to the complex conjugate of however the undotted contravariants transform. Biedenharn, Jehle, Taylor, Barut all agree on this. However, the real question arises: with what D object should you associate these dotted contravariant spinors? We know they go with g^* in the $s=\frac{1}{2}$ case. Everybody agrees they go with g^* . The question is with what do you associate your g^* :

$$g = D^{(0, \frac{1}{2})}(g) \quad g^* = \begin{cases} D^{(\frac{1}{2}, 0)}(g) & \text{for BJ} \\ D^{(\frac{1}{2}, 0)}(g^{*-1}) & \text{for T} \end{cases}$$

The choice here seems to be this: if you take BJ, then very nice because you associated the dotted's with the "other" basic rep. But if you take T, then you associate $D(s,0) = D(0,s)$ for rotations, with no factors. I gather this is a nice property to have.

$U(x,a)$ is unitary

More on the Spinor Basis used by Taylor.

May 27, 1977

1. Recall that in the beginning, the operator $U(a)$ representing a Lorentz transformation called a is a unitary operator belong to a representation (spin, mass) of the Poincare group. The general 10 parameter representing operator is $U(x,a)$ where you do your Lorentz transformation first, then you translate. We know this to be true:

$$U(x,a) |p,m\rangle_H = e^{-ix \cdot (ap)} \sum_{m'=-s}^s D_{mm'}^s(R(a,p)) |ap,m'\rangle_H$$

The representation space of this representation of the Poincare Group is infinite dimensional because the rep is unitary and the group is non-compact. The various basis vectors are labelled by a finite number of spin components and an infinite number of momenta. For a fixed 3-momentum, we may regard the space as being of dimension $(2s+1)$.

2. It is in this $(2s+1)$ dimensional subspace that we can if we want redefine or reshuffle the basis vectors. In this way we can generate the round and square states. The roundness or squareness simply tells you which basis you are using. We find:

$$U(x,a) |p,m\rangle = e^{-ix \cdot (ap)} \sum_{m'=-s}^s D_{m'm}^{(a,s)}(a) |ap,m'\rangle$$

Here is the point I wish to make: although we have changed the basis in the $(2s+1)$ part of the space, the operator $U(x,a)$ is still representing an element of the Poincare Group and belongs to a particular (spin, mass) representation of the Poincare Group. The operator $U(x,a)$ really has nothing to do with any representations of the Lorentz Group, even though matrix elements of these representations are appearing.

3. How would you show that $U(x,a)$ is unitary?

$$\begin{aligned} \langle p',m' | p,m \rangle_H &= 2E \delta(p-p') \delta_{mm'} = \langle p',m' | 1 | p,m \rangle_H = \langle p',m' | U^\dagger(x,a) U(x,a) | p,m \rangle_H \\ &= \langle p',m' | U^\dagger(x,a) U(x,a) | p,m \rangle \quad \text{IF YES UNITARY} \\ &= e^{+ix \cdot (ap')} \sum_{m''} D_{m''m'}^s [R(a,p')]^* \cdot e^{-ix \cdot (ap)} \sum_{m'''} D_{m''m}^s [R(a,p)] \\ &\quad \times \langle ap',m'' | ap,m''' \rangle \end{aligned}$$

$$\begin{aligned} \text{But } \langle ap',m'' | ap,m''' \rangle &= 2E \delta^{(3)}(p''-p''') \delta_{m''m'''} = \langle p'',m'' | p''',m''' \rangle = \langle p',m'' | p,m''' \rangle \\ &= ? \end{aligned}$$

$$\langle p | p' \rangle = 2E \delta^3(p-p') = \int dp^2 \delta^4(p-p') = \int d(ap)^2 \delta^4(ap-ap') = \langle ap | ap' \rangle$$

due to this

(continuing)

$$\begin{aligned} \text{So: } {}_H \langle p', m' | p, m \rangle_H &= e^{-ix \cdot (ap) - (ap')} \sum_{m''} D_{m' m''}^{(s)} [R(a, p')]^\dagger D_{m'' m}^{(s)} [R(a, p)] 2E \delta^3(p - p') \\ &= 2E \delta^3(p - p') \delta_{m, m'} \end{aligned}$$

So in this way we have verified that the operator $U(x, a)$ is in fact unitary. Along the way we have used the Lorentz invariance of the object (p/p') , the fact that D is a unitary matrix ,.

4. How would we show unitarity in the Round basis? In order to do this we will have to involve the square basis. In the square basis we have:

$$U(x, a) |p, m\rangle = e^{-ix \cdot (ap)} \sum_{m'=-s}^s D_{m' m}^{(0, s)} (\bar{a})^\dagger |a p, m'\rangle.$$

$$\Rightarrow \langle p, m | U^\dagger(x, a) = e^{+ix \cdot (ap)} \sum_{m'=-s}^s D_{m' m}^{(0, s)} (\bar{a})^T [a p, m']$$

Now once again let's verify that $U(x, a)$ is unitary. We have:

$$\begin{aligned} \langle p, m | p', m' \rangle &= 2E \delta_{m, m'} \delta^3(p - p') = \langle p, m | U^\dagger(x, a) U(x, a) | p', m' \rangle \quad \text{IF UNITARY} \\ &= e^{+ix \cdot (ap)} e^{-ix \cdot (ap')} \sum_{m''} D_{m'' m}^{(0, s)} (\bar{a})^T \sum_{m'''} D_{m'' m'}^{(0, s)} (a) \underbrace{\langle a p, m'' | a p', m''' \rangle}_{2E \delta^3(p - p') \delta_{m'', m'''}} \\ &= \sum_{m''} D_{m'' m}^{(0, s)} (\bar{a}) D_{m'' m'}^{(0, s)} (a) 2E \delta^3(p - p') \\ &= \delta_{m, m'} 2E \delta^3(p - p'). \end{aligned}$$

5. Warning: although we usually shorten $U(x, a)$ to $U(a)$, do not forget that this operator belongs to a representation of the Poincaré group and is always unitary for real particles.

Comments on the Toller M-function.

1. In the Cozenza PAPER, we find the connection between the Toller M-function and Stapp's M-function. I have several comments to make about this connection.

2. First, Stapp's M-function is a function of the momenta, but is usually regarded as a function of complex momenta, but each still constrained to mass shell. Now, normally if you define your standard momenta $Q^{(\pm)}$ to be say $(m, 0, 0, 0)$ for incoming particle, say, then to get a complex momentum you are going to need a complex Lorentz transformation. Thus, if you are going to talk about the Stapp function in the complex or analytic sense, then you are really talking about the Toller M-function in the sense that the a_i are complex Lorentz transformations. Such transformations are discussed by Cozenza et al, you can think of such an a_i as being in $SL(2, \mathbb{C}) \times SL(2, \mathbb{C})$. Complex Lorentz transformation allows for a smooth connection between future timelike vectors and past timelike ones, I think. However, parity operation s still has to be added by hand. When parity has been added, the group of the a_i is called $\tilde{A}_c$ where c is for complex.

3. After reading Taylor, I think I know what "crossing" says for the Stapp M-function. The one function gives you all channel processes. It is just a question of which momenta are future and which are past timelike. The spin does not really complicate things very much if you use Stapp functions. ~~Similarly~~ Thus, two crossed reactions are the Stapp function evaluated at two points in the huge R^{4n} complex space. Similarly, two crossed reactions are given by the same M-function evaluated at two points in the space $\tilde{A}_c^n$. Usually, however, we prefer to write the various a_i as multiples of a_i in the real $SL(2, \mathbb{C})$ group, and then add s or t as needed to get the momentum to be future or past timelike as needed. So by adding various factors of t , you cause M to describe reactions in different channels. However, in this case you cannot go smoothly between crossed channels. To do that you have to think in terms of complex Lorentz transformations, just like you have to think in terms of complex s and t .

4. There is still some confusion about the exact form of the covariance conditions for the M-functions. I.e, having to do with the $*$ or no $*$ on final states. Look at Cozenza page 268, or Var. p 347. Clearly, Toller wants to use the same $D^{0,s}(a)$ function for all particles. He does not want some to be starred, and others not. This at once implies that there are never stars on the covariance conditions. This seems to contradict my natural feelings on starring, so what is going on here????

In response, I think that Taylor has provided an answer to this question. Recall that Taylor wanted his "bras" as well as his "kets" to transform under the same object D , he did not want his bras to go under D^* . In order to achieve this end, he invented the "square bra" idea. This achieved an invariance condition for the M -functions which looks just like the one Toller wants to use, and gives a uniform covariance condition.

Before working out the details here, I want to know what Stapp actually did and said. To the Libe!

Relation between Round, Square, and Regular Rest States.

1. Look first at Taylor (4.8) which I convert to my notation:

$$|p, m\rangle = \sum_{m'} D_{m'm}^{(0,s)} (H(p)^{-1}) |p, m'\rangle$$

For any $p \neq \tilde{p}$, $H(p)$ is unambiguous, i.e., you positive z-boost out, then rotate. But for $p = \tilde{p}$, $H(p)$ is ambiguous. It is some rotation ω . So

$$|\tilde{p}, m\rangle = \sum_{m'} D_{m'm}^{(0,s)} [\omega^{-1}] |\tilde{p}, m'\rangle \Rightarrow H(\tilde{p}) = H(\tilde{p}) = R(\omega)$$

Is there some way to determine or arbitrate this? Suppose we apply a general L.T. to both sides. Then, apply $U(a)$:

$$U(a) |\tilde{p}, m\rangle = U(a) |\tilde{p}, m'\rangle D_{m'm}^{(0,s)} [\omega^{-1}]$$

Let $a = \text{rotation}$. Then:

$$D_{m'm}^{(0,s)}(a) |\tilde{p}, m''\rangle$$

In this case, you learn nothing!

② Do the general a :

$$D_{m'm}^{(0,s)}(a) |p, m''\rangle = D_{m'm'}^s(R(a, \tilde{p})) D_{m'm'}^{(0,s)}(\omega^{-1}) |p, m''\rangle$$

$$\text{But, } R(a, \tilde{p}) = H(p)^{-1} \cdot U(a) \cdot H(\tilde{p}).$$

$$\Rightarrow D(a) |p, m\rangle = |p, m\rangle D(H(p)^{-1}) D(U(a)) D(H(\tilde{p})) D(H(\tilde{p})^{-1})$$

$$\text{So } |p, m\rangle D(a) = |p, m\rangle D(H(p)^{-1}) D(a)$$

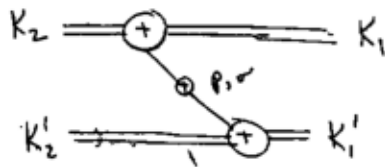
$$\Rightarrow |p, m\rangle = |p, m\rangle D(H(p)^{-1}).$$

Conclusion: you can choose $H(\tilde{p})$ to be anything you want.

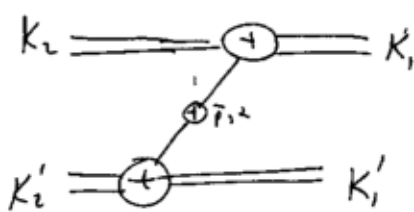
Problem: if you choose $H(\tilde{p}) = 1$, then how does $U(a) |p, m\rangle$ know whether it is $>$ or $>$ state?

Question: to what extent is Taylor's "solution" of the crossing problem unique?

1. Here is the crossing equation that you want to solve:



$$= \sum_{\sigma} [K_2 \downarrow | T | K_1 \uparrow; p_1, \sigma] [K_2' \downarrow, p_1, \sigma | T | K_1' \uparrow] \quad \text{reverse.}$$



$$= \sum_{\sigma} [K_2 \downarrow; \bar{p}_1, \sigma | T | K_1 \uparrow] [K_2' \downarrow | T | \bar{p}_1, \sigma; K_1' \uparrow]$$

Notice that the "tilts" here are not arbitrary. Particle in a final state has spin index lowered, period. Now we can tilt in the second equation introducing a sing factor:

$$= \sum_{\sigma} [K_2 \downarrow; \bar{p}_1, \sigma | T | K_1 \uparrow] [K_2' \downarrow | T | \bar{p}_1, \sigma; K_1' \uparrow] (-1)^{2s}$$

So here is the solution suggested by Taylor:

$$[K_2 \downarrow | T | K_1 \uparrow; \underline{p_1, \sigma}] = [K_2 \downarrow; \underline{\bar{p}_1, \sigma} | T | K_1 \uparrow] \lambda_T$$

$$[\underline{p_1, \sigma}; K_2' \downarrow | T | K_1' \uparrow] = [K_2' \downarrow | T | \underline{\bar{p}_1, \sigma}; K_1' \uparrow] \underline{(-1)^{2s}} (\lambda_t)^*$$

Notice that each of these equations is Lorentz-consistent, ie, both sides of the equation transform in the same way. We would not want contravariant spinor = covariant spinor. That would not be an acceptable solution to the "crossing equation".

2. Obviously the $(-1)^{2s}$ factor could conventionally be associated with the first equation rather than the second. In fact, fooling around and using hermitian analyticity you can show that at most there is a type-dependent phase as shown above, so I could set that phase = $(-1)^{2s}$ and move the factor from one side to the other.

3. Conclusion: Taylor's crossing solution is unique up to the above phase factor which we set to one.

Taylor: Why was it necessary to use the spinor basis?

Look at Taylor's paper. I think we could have obtained a statement like (5.1) in the usual helicity basis without any problem. And I suspect we could even use Taylor's little math theorem to conclude equation (5.3). I suspect that the need for the spinor basis comes at this point, when you want to show that the crossing matrix in (5.3) is in fact proportional to the identity matrix.

2. Here is how the little identity proof goes. Start with (5.3). Take each matrix element in (5.3) and "expand it" according to the rule (4.11). This gives:

$$\sum_{a_2 a_1} \sum_{a'} [g K_2 \quad \text{stop.} \quad]$$

$$\text{Rewrite: } [K_2 | M | K_1; p^d) = [K_2^{a_2} | M | K_1^{a_1}; p^d)$$

to show "set" of spin indices a_2, a_1 . Then

$$[K_2^{a_2} | M | K_1^{a_1}; p^d) = \sum_{a'_2 a'_1 a'} [g K_2^{a'_2} | M | g K_1^{a'_1}; p_g^{a'}] D_{a'_2 a_2}(g) D_{a'_1 a_1}(g) D_{a' a}^s(g)$$

Here I am just showing what rule (4.11) looks like for one of the matrix elements. The D is of course a product of D-functions, one for each particle. Now, (5.3) becomes:

$$\begin{aligned} & \sum_{a'_2 a'_1 a'} [g K_2^{a'_2} | M | g K_1^{a'_1}; p_g^{a'}] D_{a'_2 a_2}(g) D_{a'_1 a_1}(g) D_{a' a}^s(g) \\ &= \sum_{\beta} \sum_{a''_2 a''_1 a''} [g K_2^{a''_2} | M | g K_1^{a''_1}; p_g^{a''}] D_{a''_2 a_2}(g) D_{a''_1 a_1}(g) \\ & \quad \times D_{a'' \beta}(g) \Lambda_{\beta a}(p) \end{aligned}$$

Now, invert, use $\sum_{a''_2} D_{a'_2 a_2}(g) D_{a''_2 a_2}^{-1}(g^{-1}) = \delta_{a'_2 a''_2}$, then $\sum_{a'_2} \delta_{a'_2 a''_2} \rightarrow 1$.
also $\sum_{a'_1} D_{a'_1 a_1}(g^{-1})$. Also, $\sum_{a''_1} D_{a''_1 a_1}(g^{-1})$. Thus:

$$\begin{aligned} [g K_2^{a''_2} | M | g K_1^{a''_1}; p_g^{a''}] &= \sum_{\beta} \sum_{a''_2 a''_1 a''} \sum_{a'_2 a'_1 a'} \overbrace{D_{a'_2 a_2}(g^{-1}) D_{a''_2 a_2}(g)}^{\delta_{a'_2 a''_2}, \delta_{a'_1 a''_1}} \\ & \times \underbrace{D_{a'_1 a_1}(g^{-1}) D_{a''_1 a_1}(g)}_{\delta_{a'_1 a''_1}, \delta_{a'_2 a''_2}} \times D_{a'' \beta}(g) \Lambda_{\beta a}(p) \cdot D_{a' a}^s(g^{-1}) \\ & \times [g K_2^{a''_2}; -p_g^{a''} | M | g K_1^{a''_1}) \end{aligned}$$

(go to p. 2.)

Then $[g_{K_2}^{a_2} | M | g_{K_1}^{a_1}; p_3^{a_1}] =$

$$\sum_{\beta \neq \alpha} [g_{K_2}^{a_2} | p_3^{a_1} | M | g_{K_1}^{a_1}] \cdot D_{\alpha\beta}^{a_2}(R(g, p)) \Lambda_{\beta\alpha}(\tilde{p}) D_{\alpha\alpha'}(R(g, p))$$

Then I think you would conclude that

$$\Lambda(p_g) = D^S[R(g, p)] \Lambda(p) D^S[R(g, p)]^{-1}$$

Then again you would take $p = \tilde{p}$. Then $R(g, \tilde{p}) = ?$

$$R(g, p) = \Lambda_p^{-1} g \cdot \Lambda_p$$

What is Λ_p ? It is the lorentz transformation taking $\tilde{p}$ out to p in some standard way.

Suppose we are working in the helicity basis. Then if we set $g =$ rotation, we know that the Wigner rotation is identity because we know that a rotation does not mix helicity. In this case, the above equation reduces to

$$\Lambda(p_g) = \Lambda(p) \text{ tells you nothing..}$$

No doubt Λ still is the identity, but the same proof does not work.

If you were in the canonical basis, then if $g =$ rotation, $R =$ same rotation because you know that a rotation does mix helicity in this basis. In this case the proof would work.

So, not totally clear that Taylor had to go to the spinor basis to do his crossing.

AHA! I forgot something! Suppose you are in helicity or canonical basis again. Then you do have the same problem that arises in MS, namely, after you cross taking p to $-p$, you are going to have to

go to $\tilde{p}$?

$$= \sum_{\beta \alpha'' \alpha} D_{\alpha'' \beta}(g) \Lambda_{\beta \alpha}(p) D_{\alpha \alpha''}(g^{-1}) [g K_2^{a_2''} ; -p_g^{\alpha''} | M | g K_1^{a_1''} ; p_g^{\alpha''}]$$

$$= [g K_2^{a_2''} | M | g K_1^{a_1''} ; p_g^{\alpha''}]$$

Now get rid of α'' on a_2, a_1 . Set $\alpha'' = \alpha'$. Then

$$\sum_{\beta \alpha \alpha''} [g K_2^{a_2} ; -p_g^{\alpha''} | M | g K_1^{a_1} ; p_g^{\alpha''}] D_{\alpha'' \beta}(g) \Lambda_{\beta \alpha}(p) D_{\alpha \alpha''}(g^{-1}) \quad (1)$$

$$= [g K_2^{a_2} | M | g K_1^{a_1} ; p_g^{\alpha'}]$$

Now use (5.3) again:

$$= \sum_{\alpha''} [g K_2^{a_2} ; -p_g^{\alpha''} | M | g K_1^{a_1} ; p_g^{\alpha''}] \Lambda_{\alpha'' \alpha'}(p_g)$$

We now have this situation:

$$\sum_{\alpha''} V^{\alpha''} (D(g) \Lambda(p) D(g^{-1}))_{\alpha'' \alpha'} = \sum_{\alpha''} V^{\alpha''} \Lambda_{\alpha'' \alpha'}(p_g)$$

OR: $\vec{V} \cdot \vec{A} = \vec{V} \cdot \vec{B}$

Now apply $\sum_{\alpha'} \Lambda_{\alpha' \alpha''}(p_g)^{-1}$ to both sides. Get $\delta_{\alpha'' \alpha''}$ in Λ .

$$\sum_{\alpha''} V^{\alpha''} [D(g) \Lambda(p) D(g^{-1}) \Lambda(p_g)^{-1}]_{\alpha'' \alpha''} = V^{\alpha''}$$

$\vec{V} \cdot \vec{M} = \vec{V}$ or $A \vec{v} = \vec{v}$.

In order to conclude that A is the unit operator, at some point you have to assume that by varying say K_2 you can make v an arbitrary vector. This, assuming this you end up with statement (5.5). You then choose p to be rest and g = rotation to get next equation. From this you conclude that you have multiple of unit matrix.

Comments: In order to make this proof work, it was necessary to use the transformation properties of the M-functions in the spinor basis. You take a crossing equation and you transform both matrix elements, and compare this to the new crossing equation. I.e., you cross, transform, then cross back.

Now, could all this have been done in the helicity basis? We know that the transformation properties are messier, as in Martin Spearman. Would the same proof have worked without further ado?

$$[K_2^{a_2} | M | K_1^{a_1}, p^d] = \sum_{a'_2, a'_1, a'} [g K_2^{a'_2} | M | g K_1^{a'_1}, p^{d'}] \\ \times D_{a'_2 a_2} (R(g, K_2)) D_{a'_1 a_1} (R(g, K_1)) D_{a' p}^s (R(g, p))$$

/go to back of sheet.

Reconsider the proof in the canonical or helicity basis. How does it go?

$$[K_2 | M | K_1, p] = [g K_2 | M | g K_1; p_g] D(R(K_2, g)) D(R(K_1, g)) D(R(g, p))$$

But,

$$[K_2, -p | M | K_1] = [g K_2, -p_g | M | g K_1] D(R(g, K_1)) D(R(g, K_1)) D(R(g, -p))$$

The D's will cancel ok. Crossing says:

$$[K_2 | M | K_1, p] = [K_2, -p | M | K_1] \Lambda(p)$$

Then we get:

$$[g K_2 | M | g K_1; p_g] D(R(g, p)) = [g K_2, -p_g | M | g K_1] D(R(g, -p)) \Lambda(p)$$

Solve:

$$(K_2' | M | K_1', p') = (K_2', -p' | M | K_1') D[R(g, -p)] \Lambda(p) D^{-1}[R(g, p)]$$

$$= (K_2', -p' | M | K_1') \Lambda(p')$$

cross

$$\Rightarrow \boxed{\Lambda(p') = D[R(g, -p)] \Lambda(p) D^{-1}[R(g, p)]}$$

This is the statement in helicity or canonical which corresponds to Taylor's (5.5). Notice the $R(g, -p)$! It is a tricky business to relate this to $R(g, p)$, and I'll bet that is why Taylor went to the spinor basis. In spinor, this problem simply never enters! Now, in MS you have to decide exactly how you are going to get to $-p$ from p . Using their decision, then find that:

$$D[R(g, -p)] = (-1)^{m-m'} D[R(g, p)]^*$$

$$\text{Now let } F_{mm'} = (-1)^{m-m'} \delta_{m, m'} \text{ so } D[R(g, -p)] = F \cdot D[R(g, p)]^*$$

$$\text{Then: } \Lambda(p') = F \cdot D[R(g, p)]^* \Lambda(p) D^{-1}[R(g, p)]$$

Now take $g = \text{rotation}$ again so that Wigner rotations ~~want~~ becomes identity in helicity formalism. You would then conclude that:

$$\Lambda(p) = F \Lambda(p)$$

a ~~$A_{\alpha\beta} = \dots$~~

a $\Lambda_{mm'} = (-1)^{m-m'} \Lambda_{mm'}$

This certainly is consistent with the solution that $\Lambda \cdot I$, but does not prove it. Somehow, the whole issue is complicated by the helicity states which are not very sharp when $p = \text{rest momentum}$.

Conclusions: By using the M-functions, you somehow manage to avoid all the messy questions which involve Wigner rotations. All states transform in a way that simply does not involve the momenta, so you don't have to do much thinking about what happens when the momentum is at rest.

I wonder where the ambiguity went? Probably it is connected with the fact that the helicity states themselves are a little ambiguous when $p = \text{rest}$, so that $L(p)$ could or could not be a rotation.

To explain why $x^u y_u = (-1)^{2s} x_u y^u$

1. The finite-dimensional representations of the 6-parameter Lorentz group are labelled by two labels (j_1, j_2) . For example, there is one vector representation $(\frac{1}{2}, \frac{1}{2})$, but there are two spinor representations, $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$. [The size of the representation (j_1, j_2) is as one expects $(2j_1+1)(2j_2+1)$.] The two spinor representations are not equivalent. For Stapp's M-functions, the interesting representations are $(s, 0)$ and $(0, s)$.

2. The Lorentz group $SO(3,1)$ is almost the same as $SL(2, \mathbb{C})$, the group of matrices $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ with unit determinant. We denote such matrices by letter g . Obviously these g form a two-dimensional representation of the Lorentz group, so matrices g must be one of the two spinor representations. We identify:

$$g_{ab} = D_{ab}^{(0, \frac{1}{2})}(g) \quad , \quad g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \quad \det g = 1$$

If g is just a rotation, then $D^{(0, \frac{1}{2})} = D^{\frac{1}{2}}$ the usual rotation matrix. The matrices $D^{(0, s)}(g)$ represent the representation $(0, s)$, and $(0, \frac{1}{2})$ is just a special case.

3. The object which transforms according to $D^{(0, s)}(g)$ is called a contravariant, undotted spinor. It is written like so:

$$(x')^\mu = \sum_{\nu} D_{\nu}^{\mu (0, s)}(g) (x)^\nu \quad \text{contravariant, undotted}$$

There is a unique way to define from this a covariant undotted spinor. The goal is to make sure that $x^u y_u$ is a Lorentz scalar. One finds easily that y_μ must be:

$$(y')_\mu = \sum_{\nu} D_{\nu}^{\mu (0, s)}(g^{-1T}) (y)_\nu \quad \begin{array}{l} \text{covariant, undotted} \\ T = \text{transpose} \end{array}$$

I.e., with the covariant (lower index) object so defined, you can show that:

$$(x')^\mu (y')_\mu = (x)^\mu (y)_\mu$$

These two representations are equivalent. For example, in the case $(0, \frac{1}{2})$ this means that you can find a matrix c such that $c^{-1} g c = (g^{-1})^T$. This matrix is $c = \sigma_2$ or a multiple thereof by some phase. The matrix c is antisymmetric! Since this matrix connects the x^u representation to the x_u representation, the matrix c is to be identified with the object that raises and lowers indices, like the metric tensor g_{uv} encountered in the tensor algebra. The matrix c is called a metric spinor and we have seen that for the rep. $(0, \frac{1}{2})$, the metric spinor is antisymmetric. This has the immediate consequence that

$$x^\mu y_\mu = - x_\mu y^\mu \quad \text{for } (0, \frac{1}{2})$$

4. What is the "metric spinor" for the representation $(0,s)$? It is this:

$$C_{\mu\nu} = D_{\mu\nu}^{(0,s)}(c) \quad \text{where} \quad c = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \text{antisym.}$$

This general $(0,s)$ metric spinor can be shown to have this property:

$$C_{\mu\nu} = C_{\nu\mu} \cdot (-1)^{2s}$$

Ie, it is symmetric or antisymmetric depending on value of s . If one thinks of the rep $(0,s)$ as a product of representations $(0, \frac{1}{2})$, then this is what you would expect. The metric spinor for any s -half-integral representation will be antisymmetric. This yields the final result:

$$x^\mu y_\mu = (-1)^{2s} x_\mu y^\mu \quad x, y \in (0,s) \text{ rep.}$$

Proof: $x^\mu y_\mu = (x_\nu C^{\nu\mu}) y_\mu = x_\nu (y_\mu C^{\mu\nu}) (-1)^{2s} = x_\nu y^\nu \cdot (-1)^{2s}$

5. Just as a reminder, we note that each representation (j_1, j_2) has its own metric tensor/spinor. The metric tensor for $(\frac{1}{2}, \frac{1}{2})$ is the symmetric $g_{\mu\nu}$.

6. We have shown the desired result and have traced the fermion minus sign to the fact that the matrix c which solves $c^{-1} g c = (g^{-1})^T$ is antisymmetric. For the sake of completeness we mention the dotted representations. The contravariant, dotted is:

$$(x')^{\dot{\mu}} = \sum_{\nu} D_{\nu\mu}^{(0,s)}(g)^* (x)^\nu \quad \text{contravariant, dotted}$$

$$(x')_{\dot{\mu}} = \sum_{\nu} D_{\nu\mu}^{(0,s)}(\bar{g})^* (x)_\nu \quad \text{covariant, dotted}$$

The relation between upper and lower is the same as before. We can identify $D^{(0,s)}(g)^*$ with $D^{(s,0)}(g)$, so the undotted reps are the $(0,s)$, the dotted reps are the $(s,0)$. The rep. matrices are simply related by complex conjugation. The reps $(0, \frac{1}{2})$ and $(\frac{1}{2}, 0)$ are not equivalent because there exists no solution matrix a to this: $a^{-1} g a = g^*$. If g were just a rotation, there would be a solution, which is why the rotation group has only one kind of spinor rep namely $j = \frac{1}{2}$.

Refs: Taylor "Spin and Isospin in S-matrix Theory," JMP 7, 181 (1966).
Bade & Jehle "Introduction to Spinors" RMP 25, 714 (1953).

Motivation: I want to know what the exact statement of crossing is in terms of Stapp's M-functions or spinorial amplitudes, with spin. Then I can use this to find the corresponding statement of crossing in terms of the Toller M-function. Maybe. In any case, I should certainly know this for the Stapp functions, since probably all you have to do is flip some 4-momenta around and nothing else. We shall see

II. S-matrix th. At time of writing, 1965, S-matrix theory was a new subject, so Taylor takes time to explain things, lucky for later readers like me. He sees it as 5 postulates.

- 1) superposition-principle. Basically, this is thought to be equivalent to assuming there is a Hilbert space of asymptotic states with superselection subspaces. In any subspace, you can superpose to get a new realizable state. Multiparticle states make a Fock space.
- 2) existence of an S-matrix. If it exists in the sense that physical transition amplitudes correspond to matrix elements in the above Hilbert space, then unitarity of the S-matrix is nothing new, follows from Wigner theorem. I.e, once you have Hilbert Space, no need to assume that $\bar{S}$ is unitary. It is.
- 3) Lorentz invariance. Taylor wants to have usual Lorentz invariance relating actual amplitudes with no phase at all. Can do this, he says, see (2.5).
- 4) cluster decomposition. This enables you to isolate connected parts, so you can write unitarity in terms of these connected parts, the famous "bubbles".
- 5) analyticity. The amplitudes should be boundary values of functions which are analytic in the momenta. Perhaps related to causality. Only singularities in these analytic functions are those required by unitarity.

From these postulates of S-matrix theory, you should be able to prove other properties, such as

- a) correspondence of particles and poles in the S-matrix
- b) Hermitian-analyticity
- c) discontinuity equations
- d) existence of antiparticles
- e) crossing symmetry

It is this last property that Taylor wants to show still works even with spin and isospin.

For the rest of this section, Taylor reviews how you prove the above list of things in the spinless case. (1) why pole corresponds to particle (look at A_6) (2) why hermitian analytic (3) why crossing symmetry. In (2.10), you are crossing one particle, and you can set the "crossing phase" equal to unity by adjusting the phase of the single particle states at rest.

Finally, it is noted that you need a good notation to express all this; he likes the Stapp M-function.

III. The Isospin Problem. Due to charge superselection, you cannot simply rotate states. But, you can relate two physically possible (charge possible) amplitudes using the isospin D functions, as in (3.1).

The crossing problem is at once approached. Again there is some crossing phase. The game is to find a solution for this phase. By fiddling around, Taylor shows that (3.5) is a feasible phase, and does not require shuffling the standard phases of the D-functions (of Rose). Page 189/bottom shows example of how this phase might work for a pion exchange.

Finally, in the A_1 case you have to cross two particles at once. This is suggested by the figure 6, and the result is (3.7) with the same phase factor obtained before, applied twice.

Then you can make isospin amplitudes and derive the isospin crossing matrices. Taylor is elated that you can do this without referring to an "isoscalar Lagrangian" etc. etc. A real fan of the trade, I must say.

IV. Setting up nice M-functions. Taylor starts with the usual JW helicity states (single particle) and shows that they transform with Wigner rotations which mix in the momentum and are therefore undesirable. See (4.4).

To get around this, you need some way of "exploding" the Wigner rotations into its three components, if you will. This explosion can be done if you extend to the $SL(2, C)$ finite dimensional representations (o, s).

A quick review of these representations is given. There are four kinds of spinors, you can raise and lower, etc etc.

The new "M-function ket" is defined in (4.8) so that it transforms as $D^{0, s}(g)$ as shown in line after (4.8). This is an undotted, raised index spinor (contravariant?)

In order to do the crossing proof, Taylor will want the corresponding antiparticle bra to transform in exactly this way also. This necessitates the definition of a new kind of state, bra and ket, which is given by a square bracket. It turns out that the square ket must transform as $D^{s, 0}(g^{-1})$, which happens to be a lower dotted object. Going from ket to bra causes dotted to go to undotted, and so on.

So these new states are defined, and I gather he wants to use them all the time for the bras, but use the usual kets, so everything transforms with $D(g)$. For example, Lorentz invariance is now given by (4.11) which really gets the point across.

In this "mixed basis", all the relations you think are true are, like orthogonality and completeness. Thus, unitary equations will come out just as you would think without any Mickey Mouse.

V. Crossing with spin. Interesting problems along the way, but the result is simply given by (5.6). There is slight question about statistics, but now is not the time to worry about it. When you cross a particle with spin, it goes to antiparticle, momentum negates, and it keeps its "spinor index".

Spin and Isospin in S-matrix Theory

Taylor, JMP 7, 181 (1966)

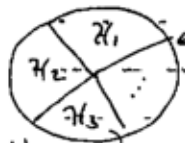
[John R., PhD UC8, 1963; 1972 Prof @ Colorado]

I. INTRO. Speaks of the "Gunn-Oline" version of SMT. This "theory" is reviewed in Sec. II. Contents of paper are:

- | | |
|-----------------------------------|-----------------------------------|
| II. The G-O S-matrix Theory (SMT) | ref: Barut: spin. PR130, 436 (63) |
| III. add isospin to SMT | B+Unal: isospin |
| IV. spin states (M-functions) | NS, 112 (1963) |
| V. add spin to SMT. | |

II. SPINLESS SMT. There are 5 postulates:

① There exists an asymp. description of system as "states". State = ray in a H.S.



shows "orthog. superselection subspaces"

- (i) Within such a subspace, any superposition of vectors = realizable state!
- (ii) $|\langle \psi | \phi \rangle|^2$ = probability
- (iii) space is "Fock"

Fock (MMP p455): single particle states span a H.S. called $\mathcal{H}_1$
 n -particle H.S. is $\mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \dots \oplus \mathcal{H}_n = \mathcal{H}_n$

Then construct a hyper-hilbert space $\mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \dots$ to $n = \infty$.
 This thing is a "Fock Space"

$$|k\rangle \equiv |p, t\rangle_{\text{mom}} \quad \text{type}$$

$$|K\rangle = |k_1\rangle \otimes |k_2\rangle \otimes \dots \otimes |k_n\rangle$$

Mentions some work which tries to evade notion of H.S.

② superposition $\Rightarrow$ unitarity (ref: Wigner book p233)

② Lorentz Invariance: $|\langle \phi | S | \psi \rangle| = |\langle \phi' | S | \psi' \rangle|$

From Wigner, $U(L) = LT$ on HS = unitary

$$U^\dagger(L) S U(L) = \text{phase} \cdot S$$

But argument $\Rightarrow$ phase = 1.

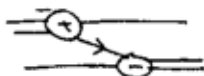
Once: $\langle P', T' | S | P, T \rangle = \langle LP', T' | S | LP, T \rangle$ in general!
 (type vector.)

- ④ cluster decomposition ⑤ analyticity.

aside: with spin, it is natural to have analyticity in momenta, not invariants (Refs given)

Now, assuming the above, what do you want to prove?

- 1) particles = poles
- 2) hermitian analyticity
- 3) discont rel's.
- 4) existence of antiparticles
- 5) crossing

① Discusses the pole in T_6  $T_6 = T_4 \times T_4 \times \frac{1}{(p^2 - m^2)}$.

You could physically go onto this pole! Not factorization.
 So claim is this:

$$\text{unitarity} + \text{cluster} \Rightarrow (\text{particle} = \text{pole})$$

② Prove that $M(2+) = M^\dagger(2-)$ (normal) by examining double pole in T_8 . This lets you get rid of $M^\dagger$ everywhere.

③ crossing symmetry = substitution rule. Usual argument gives us antiparticles. Stapp gets some credit here.

III. Isospin. Since you cannot make a state $R(27^0) | \pi^+ \rangle$, the statement of isospin invariance is different from that of rotational invariance.

$$\langle f | S | i \rangle = \langle R f | S | R i \rangle \quad \text{has no meaning for isospin due to "isospin superselection".}$$

$$\text{But go one more step} = \sum_{\alpha' \beta'} D^{(I)}(R) \dots \dots \langle f' | S | i' \rangle$$

← again? physical states

Taylor shows that isospin invariance is in fact compatible with crossing. Might not have been the case. I.e., not really obvious.

I have not read this in detail.

IV. PARTICLES WITH SPIN: SPINOR BASIS

$$g = hu$$

$u = \text{rotation}$

$$h = h(\vec{v}) = \text{boost} = e^{\frac{1}{2} \lambda \hat{v} \cdot \vec{\sigma}} = \sqrt{\frac{p^0 + \vec{p} \cdot \vec{\sigma}}{m}} \quad \text{one may show.} \quad \in SL(2, \mathbb{C})$$

i.e., $e^{\frac{1}{2} \lambda \hat{v} \cdot \vec{\sigma}} = \dots$ acc. to my theorem. (fill in)

$$g = hu$$

$$g^\dagger = u^\dagger h^\dagger$$

$$gg^\dagger = h u u^\dagger h^\dagger = h h^\dagger = h^2$$

$u = \text{unitary}$

$h = \text{hermitean}$

Given $g \in SL(2, \mathbb{C})$,

$$h = [gg^\dagger]^{\frac{1}{2}}, \text{ then } u = h^{-1}g$$

$$L(g) = \text{the } 4 \times 4$$

$$U(g) = \text{unitary H.S. transform}$$

Let $D(p) = u(\phi, \theta, -\phi) h(r \hat{z})$. Then $|p, \alpha, t\rangle = U[V(p)] |p_0, \alpha, t\rangle$

No pre-rotation $\Rightarrow$ usual singleparticle helicity state

Let $g_p = p_g = \text{new momentum}$

Parametrize:

$$g = u(p_g) u(g, p) v^{-1}(\vec{p})$$

$\uparrow \quad \quad \uparrow \quad \quad \uparrow$
boost. Wigner rotate boost to rest

defines Wigner in terms of g and $-p$.

Problem: unpleasant to have $D[u(g, p)]$ depend on p . You would rather have only the group object in there.

Representation Theory:

Let $g = e^{\frac{1}{2} a \cdot \vec{\sigma}} \cdot e^{-\frac{1}{2} i b \cdot \vec{\sigma}} \in SL(2, \mathbb{C})$ obviously $a, b = \text{real}$.

Define:

$$D^{0,s}(g) \equiv e^{a \cdot \vec{J}} \cdot e^{-i b \cdot \vec{J}}$$

where $\vec{J}$ are $(2s+1)^2$ generators.

Then:

$$D^{s,0}(g) \equiv D^{0,s}(g^\dagger)^{-1} = ([D^{0,s}(g)]^\dagger)^{-1}$$

For $g \in SU(2)$, these both agree & either one "extends" SU_2 to $SL(2, \mathbb{C})$.

two more reps: (1.) $D^{0,s}(g)$ (3.) $D^{0,s}(g^{-1})$
 (2.) $D^{0,s}(g^{T^{-1}})$ (4.) $D^{0,s}(g^*)$

Thus, there are four kinds of "spinors".

$$X^\alpha \rightarrow \sum_\beta D_{\beta\alpha}^{0,s}(g) X^\beta \quad \text{/type 1.}$$

$$X_\alpha \rightarrow \sum_\beta D_{\beta\alpha}^{0,s}(g^{T^{-1}}) X_\beta \quad \text{/type 3}$$

Note:

$$\sum_\alpha y_\alpha X^\alpha \rightarrow \sum_{\beta\beta'} D_{\beta\alpha}^{0,s}(g^{T^{-1}}) y_\beta D_{\beta'\alpha}(g) X^{\beta'}$$

$$\begin{aligned} \text{But } \sum_\alpha D_{\beta\alpha}(g^{T^{-1}}) D_{\beta'\alpha}(g) &= \sum_\alpha \left([D_{\beta\alpha}(g)]^T \right)^{-1} \left(D_{\alpha\beta'}(g) \right)^T \\ &= \sum_\alpha D_{\beta\alpha}(g^{T^{-1}}) D_{\alpha\beta'}(g^T) \\ &= D_{\beta\beta'}(e) = \delta_{\beta\beta'}. \end{aligned}$$

Thus, $y_\alpha X^\alpha$ is invariant under $SL(2, \mathbb{C})$, hence use of two kinds of indices.

What matrix arises here?

$$c = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad (g^T)^{-1} = c^{-1} g c$$

So search for matrix st:

$$X^\alpha = \sum_\beta C^{\alpha\beta} X_\beta$$

Well

$$\sum_\gamma D_{\gamma\alpha}^{0,s}(g) X^\gamma = \sum_\beta C^{\alpha\beta} \sum_{\gamma'} D_{\gamma'\beta}^{0,s}(g^{T^{-1}}) X_{\gamma'}$$

Mult both sides by $\sum_\alpha D_{\alpha\gamma''}^{0,s}(g')$. Get

$$X^{\gamma''} = \sum_{\beta\gamma'} C^{\alpha\beta} D_{\alpha\gamma''}(g') D_{\gamma'\beta}(g^{T^{-1}}) X_{\gamma'}$$

To be consistent, we need: $C^{\alpha\beta} = \sum_{\gamma\delta} C^{\alpha\gamma} D_{\gamma\delta}(\vec{g}) D_{\delta\beta}(\vec{g}^{-1})$

Try: $C^{\alpha\beta} = D_{\alpha\beta}(c)$. Then

$$D_{\alpha\beta}(c) = \sum_{\gamma\delta} D_{\alpha\gamma}(\vec{g}^{-1}) D_{\gamma\delta}(c) D_{\delta\beta}(\vec{g})$$

$$= D((\vec{g}^{-1})^T c \vec{g})$$

True if $c = (\vec{g}^{-1})^T c \vec{g} \Rightarrow \vec{g} c \vec{g}^{-1} = (\vec{g}^{-1})^T$

Not quite right. OK,

$$C_{\alpha\beta} = D_{\alpha\beta}(c) \quad C^{\alpha\beta} = D_{\alpha\beta}(\vec{c})$$

Next:

$$Z^{\alpha} \rightarrow \sum_{\beta} Z^{\beta} D_{\beta\alpha}(\vec{g}^*) \quad \text{type 4}$$

$$Z_{\alpha} \rightarrow \sum_{\beta} Z_{\beta} D_{\beta\alpha}(\vec{g}^{*T-1}) \quad \text{type 2.}$$

These bear identical relation as Z^{α} to Z_{α} $C =$ same thing.

Factor the Wigner:

$$D^S[u(\vec{g}, P)] = D^S[\vec{v}^1(P) \vec{g} v(P)]$$

$$= D^{S,0}(\vec{v}^1(P)) \cdot D^{S,0}(\vec{g}) \cdot D^{S,0}(v(P))$$

Now, recombine helicity states:

$$|P, \alpha, t\rangle = U[v(P)] |P, \alpha, t\rangle \quad \text{recall.}$$

$$|P, \alpha, t\rangle = \sum_{\beta} |P, \beta, t\rangle D_{\beta\alpha}^{0,S}[\vec{v}^1(P)]$$

just a function

Then $|P, \alpha, t\rangle \rightarrow \sum_{\beta} \left\{ \sum_{\beta'} |P, \beta', t\rangle D_{\beta'\beta}^S[u(\vec{g}, P)] \right\} D_{\beta\alpha}^{0,S}[\vec{v}^1(P)]$

$$= \sum_{\beta} |P, \beta, t\rangle D_{\beta\alpha}^{0,S}(\vec{v}^1(P)) \cdot D^{0,S}(\vec{g}) \cdot D^{0,S}(v(P)) \cdot D^{0,S}(v^1(P))$$

So, by cleverly defining $|p,^\alpha, t\rangle$, you get

$$|p,^\alpha, t\rangle \rightarrow \sum_{\beta} |p,^\beta, t\rangle D_{\beta\alpha}^{0,s}(g)$$

no longer nasty p dependence here.

Two kinds of kets:

$$|k\rangle = |p,^\alpha, t\rangle d(y, g) = \sum_{\beta} |p,^\beta, t\rangle D_{\beta\alpha}^{0,s}[\tilde{v}^1(p)]$$

$$|k\rangle \equiv \sum_{\beta} |p,^\beta, t\rangle D_{\beta\alpha}^{0,s}[\tilde{v}^1(p)]$$

$|k\rangle$ transforms under $D(g)$

$|k\rangle$ " " $D(g^T)^{-1}$

$$\Rightarrow [k| \text{ " " } D(g^T)^{-1} \equiv [p,^\alpha, t|$$

One point of this is that the bra $[p,^\alpha, t|$ and ket $|p,^\alpha, t\rangle$ both transform as $D(g)$.

M-function Lorentz invariance:

$$[p',^A, T| S | p,^A, T]$$

$$= [" | U^\dagger(g) S U(g) | "$$

$$= \sum_{\beta, \beta'} [p',^\beta, T| S | p,^\beta, T] D_{\beta'A'}(g) D_{\beta\alpha}(g)$$

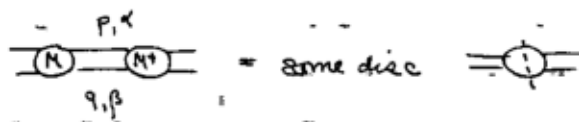
product of $D_{\beta\alpha}^{(0,s_i)}(g)$

spin of particle i

Now ready to read final Section II.

V. Particles with Spin; Crossing.

① Consider unitarity:



We add spin in this way.

$$\sum_{\alpha, \beta} [K_2 | M | p, \alpha, t; q, \beta, t] [p, \alpha, t | q, \beta, t | M^\dagger | K_1]$$

ie, you now sum on spin indices which appears in the "spinor" single particle states.

② How does a pole appear?



$$\sum_{\alpha=-s}^{+s} \langle \text{stuff} | M | \text{stuff}; p, \alpha, t \rangle \langle \text{stuff}; p, \alpha, t | M | \text{stuff} \rangle \cdot \frac{1}{\pi(p^2 - m^2)}$$

same except add spinor sum.

③ So each particle gets an index which is not just helicity what have you.

④ How would this pole look in helicity formalism:

$$\sum_{\alpha} |p, \alpha, t\rangle \langle p, \alpha, t|$$

$$= \sum_{\alpha} \left\{ \sum_{\beta} |p, \beta, t\rangle D_{\beta\alpha}^{0,s}(\vec{v}(p)) \right\} \left\{ \sum_{\beta'} \langle p, \beta', t| D_{\beta'\alpha}^{0,s}[\vec{v}(p)] \right\}$$

$$= ? \neq \sum_{\alpha} |p, \alpha, t\rangle \langle p, \alpha, t|$$

⑤ Why are things so simple in spinors, but not in helicity?

$$\langle p, \beta, t | M | p', \beta', t \rangle = \sum_{\alpha, \alpha'} D_{\beta\alpha}^{0,s}[\vec{v}(p)] D_{\alpha'\beta'}^{0,s}[\vec{v}(p')] \underbrace{\langle p, \alpha, t | M | p', \alpha', t \rangle}_{\text{analytic}}$$

These must generate kinematic singularities in helicity amplitudes!

The Standard Method of Making Spinorial Amplitudes, and Taylors Trick.

1. First, lets go to the notation of McKerrell and observe the effect of a Lorentz transformation on a single particle helicity state or canonical state:

$$L|p\nu\rangle_c = \sum_{\mu} D_{\mu\nu}^s(\bar{R}(L,p)) |q,\mu\rangle_c \quad \bar{R}(L,p) = L(q)^{-1} \cdot L \cdot L(p)$$

$$L|p\nu\rangle_H = \sum_{\mu} D_{\mu\nu}^s(R(L,p)) |q,\mu\rangle_H \quad R(L,p) = H(q)^{-1} \cdot L \cdot H(p)$$

2. At this point, let us adopt the convention that for rotations, $s = (0,s) = (s,0)$. This is a convention chosen by Taylor. I will find out how nice it is in a moment. Then we can "explode" the Wigner rotations in this way:

$$D_{\mu\nu}^s(\bar{R}) = D_{\mu\nu}^{(0,s)}(\bar{R}) = D_{\mu\nu}^{(0,s)}(L(q)^{-1} \cdot L \cdot L(p))$$

$$= \sum_{\nu',\nu''} D_{\mu\nu'}^{(0,s)}(L(q)^{-1}) \cdot D_{\nu'\nu''}^{(0,s)}(L) \cdot D_{\nu''\nu}^{(0,s)}(L(p)).$$

For the helicity thing you do the same thing but replace L with H on the sides.

3. Before proceeding, it is nice to know how the Lorentz invariance is stated in terms of helicity or canonical amplitudes. Using $S = L^{-1}SL$, and noting that L is a unitary operator in the QM Hilbert space, $S = L^+SL$. We do this inside a matrix element of S or T or whatever and get:

$$\langle \dots \langle p_1 \nu_1 | \dots | S | \dots | p_2 \nu_2 \rangle_H \dots \rangle$$

$$= \pi \cdot \sum_{\nu'_2} D_{\nu'_2 \nu_2}^{(0,s)}(R) \cdot \pi \sum_{\nu'_1} D_{\nu'_1 \nu_1}^{(0,s)}(R)^*$$

$$\times \langle \dots \langle p_1, \nu'_1 | \dots | S | \dots | p_2, \nu'_2 \rangle_H \dots \rangle$$

As usual, the Wigner rep matrices for all the particles in the final state get starred. This comes directly from the L^+ being a dagger.

4. Now, how should we define the spinorial states? Follow Taylor's hint and try:

$$|p,\nu\rangle \equiv \sum_{\nu'} D_{\nu'\nu}^{(0,s)}[H(p)^{-1}] |p,\nu'\rangle_H$$

$$a \equiv \sum_{\nu'} D_{\nu'\nu}^{(0,s)}[L(p)^{-1}] |p,\nu'\rangle_c$$

$$\begin{aligned}
 \text{Then: } L|p, v\rangle &= \sum_{v'} D_{v, v'}^{(0, s)} [H(p)^{-1}] L|p, v'\rangle_H \\
 &= \sum_{v''} D_{v'' v'}^{(p, s)} (R) L|p, v''\rangle_H \\
 &= \sum_{m, m'} D_{v'' m}^{(0, s)} (H(p)^{-1}) D_{m m'}^{(0, s)} (L) D_{m' v'}^{(0, s)} (H(p)) \\
 &= \sum_{m, m'} \left(\sum_{v'} D_{m' v'}^{(0, s)} (H(p)) D_{v, v'}^{(0, s)} (H(p)^{-1}) \right) D_{m m'}^{(0, s)} (L) \underbrace{\sum_{v''} D_{v'' m}^{(0, s)} (H(p)^{-1}) |p, v''\rangle_H}_{|p, m\rangle} \\
 &\quad \delta_{m, v}
 \end{aligned}$$

$$L|p, v\rangle = \sum_m D_{m v}^{(0, s)} (L) |p, m\rangle$$

$$S_0 \quad \boxed{L|p, v\rangle = \sum_{v'} D_{v, v'}^{(0, s)} (L) |p, v'\rangle}$$

This single particle state now transforms as a contravariant undotted spinor of the representation $(0, s)$ of the lorentz group.

5. Lorentz invariance now says:

$$\begin{aligned}
 &\langle \dots \langle p_1, v_1 | \dots | S | \dots | p_2, v_2 \rangle \dots \rangle \\
 &= \prod \dots D_{v_2' v_2}^{(0, s)} (L) \prod \dots D_{v_1' v_1}^{(0, s)} (L)^* \\
 &\times \langle \dots \langle p_1, v_1' | \dots | S | \dots | p_2, v_2' \rangle \dots \rangle
 \end{aligned}$$

As far as Barut is concerned (or Stapp) these amplitudes are the M-functions. But Taylor realizes that it is useful to go another step. But before doing that, ...

6. Lets try to stick with this and see how it might mess up unitarity. Note that so far it has not mattered one bit whether we use $(0, s)$ or $(s, 0)$ for all D functions. Unitarity is a question of the completeness relation of the single particle states.

$$1 = \sum_{\nu} |p, \nu\rangle_H \langle p, \nu|$$

Now, however, we invert result 4 to get:

$$|p, \nu\rangle_H = \sum_{\nu'} D_{\nu'\nu}^{(0,s)}(H(p)) |p, \nu'\rangle$$

$$\langle p, \nu| = \sum_{\nu''} D_{\nu''\nu}^{(0,s)}(H(p))^* \langle p, \nu''|$$

Then,

$$1 = \sum_{\nu', \nu''} D_{\nu'\nu}^{(0,s)}(H(p)) D_{\nu''\nu}^{(0,s)}(H(p))^* \langle p, \nu' | p, \nu'' \rangle$$

At this point, it is convenient to assume that conventions are set in such a way that you can pull the * inside the argument of the second D, this allowing you to do the NU sum. Then:

$$1 = \sum_{\nu', \nu''} D_{\nu'\nu}^{(0,s)}(H(p)) D_{\nu''\nu}^{(0,s)}(H(p)^*) |p, \nu'\rangle \langle p, \nu''|$$

But this is not enough. Lets also assume that the transpose symbol T can also be pulled in (Taylor shows in footnote that his D have these nice properties). Then:

$$1 = \sum_{\nu', \nu''} D_{\nu'\nu''}^{(0,s)} [H(p) \cdot H(p)^T] |p, \nu'\rangle \langle p, \nu''|$$

If $H(p)$ were a rotation (it is not!), you could get rid of this last factor and all would be well. But you cannot get rid of it, you are stuck with extra sum and extra factor for each particle! Clearly, this is the motivation for defining the square bracketed state!

7. Recall that $H(p) = R(p)Z(p)$. From our $g = uh$ studies, we know that $H \cdot STOP$. Better to put $H(p) = L(p)R(p)$ so we have $g = hu$ form. Recall $L(p)$ is pure boost. Then we know that $HH^+ = L(p)L(p)^+$ in either helicity or canonical case! But, for spin $\frac{1}{2}$ which Stapp always used, we know we can say that $HH^+ = LL^+ = \frac{p \cdot \sigma}{m}$ where p is any 4-momentum with velocity equal to vector parameter in the boost $L(p)$. Thus, Stapp would show a factor:

$$D_{\nu'\nu''}^{(0,s)} [H(p)H(p)^+] = \left[\frac{p \cdot \sigma}{m} \right]_{\nu'\nu''}$$

$$\Rightarrow 1 = \sum_{\nu', \nu''} |p, \nu'\rangle \left[\frac{p \cdot \sigma}{m} \right]_{\nu'\nu''} \langle p, \nu''|$$

Thus, I have finally identified the mystery factor that Stapp shows in his unitarity equations.

8. OK, now how do we get rid of this damn factor? Obviously you want

$$\sum_{v''} D_{v''v'}^{(0,s)} [L(p) L(p)^\dagger] |p, v''\rangle \equiv |p, v'\rangle$$

then you will have

$$1 = \sum_{v'} |p, v'\rangle \langle p, v'|$$

and unitarity will be saved. So

$$\begin{aligned} |p, v'\rangle &= \sum_{v'} D_{vv'}^{(0,s)} [LL^\dagger]^* |p, v'\rangle \\ &= \sum_{v'} D_{vv'}^{(0,s)} [H(p) H(p)^\dagger]^* \sum_{v''} D_{v''v'}^{(0,s)} (H(p)^{-1}) |p, v''\rangle_H \\ &= \sum_{v''} D_{vv''}^{(0,s)} [H(p) H(p)^\dagger] D_{v''v'}^{(0,s)} (H(p)^{-1}) |p, v''\rangle_H \\ &= \sum_{v''} D_{v''v}^{-(0,s)} [H(p)^\dagger] |p, v''\rangle_H \end{aligned}$$

$$\text{So } \boxed{|p, v'\rangle = \sum_{v''} D_{v''v}^{(0,s)} [H(p)^\dagger] |p, v''\rangle_H} \quad (4.9)$$

Now, this is what this thing has to do in order that the unitarity thing come out right. How does this baby transform under L ? Notice that we can rewrite the above boxed item as:

$$|p, v'\rangle = \sum_{v''} D_{v''v}^{(s,0)} [H(p)^{-1}] |p, v''\rangle_H$$

We repeat the manipulations in 4. but use $(s,0)$ to explode the Wigner rotation. Thus result will be:

$$\begin{aligned} L |p, v'\rangle &= \sum_{v''} D_{v''v}^{(s,0)} (L) |p, v''\rangle \\ &= \sum_{v''} D_{v''v}^{(0,s)} [L^\dagger] |p, v''\rangle \\ &\Rightarrow \text{lower dotted.} \end{aligned}$$

9. So here is the picture. You know that going from bra to ket and vice versa causes D functions to get starred. But this is same as saying that dotted indices go to undotted, and vice versa.

So, we have found that the square ket goes as a lower dotted. Therefore, the square bra goes as a lower undotted. Therefore, the square bra transforms in exactly the same way as the round ket. This is of course what you need to get nice unitarity completeness relations:

$$\begin{aligned}
 1 &= \sum_{\alpha} |p, \alpha\rangle [p, \alpha| = \sum_{\alpha} L |p, \alpha\rangle [p, \alpha| L^\dagger \\
 &= \sum_{\alpha' \alpha''} D_{\alpha' \alpha}^{(0, s)}(L) D_{\alpha'' \alpha}^{(0, s)}(L)^{-1} |p, \alpha'\rangle [p, \alpha''| \\
 &= \sum_{\alpha'} D_{\alpha' \alpha}^{(0, s)}(L) [L^\dagger]^{-1} |p, \alpha'\rangle \\
 &= \sum_{\alpha'} D_{\alpha' \alpha}^{(0, s)}(L) [L^\dagger]^{-1} [p, \alpha| \\
 &= \sum_{\alpha' \alpha''} D_{\alpha' \alpha}(L) D_{\alpha'' \alpha}(L)^{-1} |p, \alpha'\rangle [p, \alpha''| \\
 &= \sum_{\alpha} |p, \alpha\rangle [p, \alpha|
 \end{aligned}$$

Wait
so
So

To recover the thing you start with, you really have to show also the momentum integration which will reset lp to p in momentum entry. The point is really in the spin space though. The matching of the spin indices suggests that both sides are world scalars. This operator unity has a form which is invariant under any Lorentz transformation!

10. How, then, should we define good M-functions, and how do they transform? Taylor suggests using up-up and then everything does with the same D's, as in Toller's M-functions. The unitarity completeness insertions will of course always be facing up/down.

$$\langle \dots [p_1, \nu_1] \dots | S | \dots [p_2, \nu_2] \dots \rangle$$

$$= \pi \cdot D_{\nu_1 \nu_1}^{(0, s)}(L) D_{\nu_2 \nu_2}^{(0, s)}(L) \langle \dots [lp_1, \nu_1'] \dots | S | \dots [lp_2, \nu_2'] \dots \rangle$$

These are Taylors DYNOMITE m-functions. Notice there are no longer any stars.
These are the things I will use to make the Toller M-functions.

Summary of the Taylor M-function Theory.

$$\textcircled{1} u(L) | p, \dot{\nu} \rangle = \sum_{\nu'} D_{\nu \nu'}^{(0, s)}(L) | \varrho p, \nu' \rangle$$

$$\langle p, \dot{\nu} | u^\dagger(L) = \sum_{\nu'} D_{\nu' \nu}^{(0, s)}(L)^* \langle \varrho p, \nu' |$$

$$u(L) | p, \dot{\nu}] = \sum_{\nu'} D_{\nu \nu'}^{(0, s)}(L^*) | \varrho p, \nu']$$

$$[p, \dot{\nu} | u^\dagger(L) = \sum_{\nu'} D_{\nu' \nu}^{(0, s)}(L) [\varrho p, \nu' |$$

$$\textcircled{2} \text{lowering causes } L \rightarrow L^{\pi^{-1}}$$

This last line shows that you do not get a star when you "covariate" off to the left.