

Poincaré, SPIN, m-fctus... II

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Section 1: Introduction (and Abstract)

This paper is 40,000 words long, published in October 1968. At time of writing (1966) there were already proofs of the big three items:

- a) crossing b) hermitian analyticity c) spin-statistics.

Henry wanted to strengthen these proofs by reducing the assumptions in light of his own recent work on identifying all possible physical region singularities. Earlier proofs of all three of course appear in ELOP book. Olive, Gunson, Polkinghorn did their stuff in 1962-1964 when S-matrix theory was blooming into vogue.

To me, crossing and hermitian analyticity are roughly the same idea. There is an analytic function which does everything and you are free to continue around where you like. Olive's 1964 work considered only normal threshold singularities, whereas here Henry considers all singularities (all meaning the positive- ω stuff which you deduce by looking at the bubble diagrams, i.e., all the singularities required by unitarity). Henry's work on finding all physical region singularities is based on idea called macrocausality, about which H says much in later papers.

In terms of spin-statistics, H improves older proofs by avoiding the delicate idea of phases when you interchange unlike particles. However, Froissart and Taylor (note added) sort of reduce the need for Henry's improvement.

The pole factorization theorem is obvious, but needed repairs because faulty assumptions were used earlier.

In short, this is not a Beginner's paper. It is a specialists paper, where various proofs are refined. Hence, you should not read it except for the review of notation in Section 2. Probably better to read the Trieste notes for general overview. I think Trieste happened only 1 year before Henry finished this paper, so maybe not too much new between them.

Here is an outline of the sections:

1. Introduction 4p
2. General Formalism 10p
3. Structure Theorems (bubble diagrams) 4p
4. The Pole-Factorization Theorem 5p
5. Hermitian Analyticity 6p
6. How to connect the paths of: crossing, hermitian analyticity 4p
7. Connection between spin and statistics. 2p
8. Phase Factors in Crossing Relation 2

Appendices: 9p

Section 2. General Formalism.

General comments: this is a simply hugeongous section which summarizes all of S-matrix theory as Henry saw it in Nov 1966. There are about 10 pages and 24 subsections. I have not read all indetail, especially the Landau stuff I have skimmed.

- A: the sets $K = [p, m, t]$ are defined; all momenta are on shell, all masses positive
- B: Unitarity is stated in terms of $SS^\dagger = 1$ with $S(K'; K)$ in canonical spin basis. The first momentum set is the final particles.
- C: The S-functions are converted to M-functions using the usual $L(p)$ matrices. $L(p)$ is written out in different ways.
- D: Lorentz invariance of the M-functions is stated rather vaguely because Henry always has to worry about whether indices are of one kind or another. Single-particle states are normalized with $\delta = (2\pi)^3$ convention.
- E: ordering of variables in sets like K is "irrelevant", ie, does not change experiment.
- F: M-functions as $M(K)$ are analytic on some domain. This must be composed of "closed orbits" of the Lorentz transformations. Analyticity is Lorentz invariant concept.
- G: Symbol ϵ tells whether initial or final state, symbol λ tells whether index is dotted or undotted. For Henry's original M-functions (say all states lower square) you must have $\epsilon_\lambda = +1$ for obvious reasons, but of course later you can shuffle spinors from one kind to another and break this rule.
- H: Comment about how to contract indices in connection of S to M.
- I: Unitarity stated in terms of M-functions: $M G^{-1} M^\dagger = G$ roughly. $G = L(p)^2$.
- J: M-functions contain momentum delta function.
- K: physical momenta called p ; mathematical called k so that $\text{Sum}(k_i) = 0$.
- L: Cluster decomposition annotated; phase of relative terms is discussed, but I dont see the conclusions yet. Much reference back to Trieste paper.
- M: "scattering function" called $M_c(K)$ is a connected part defined by pulling overall delta and 2π fourth. Ie, this is the thing I call $A^\dagger$ function.
- N: A Landau diagram D is made of some lines L , vertices V , and directions ϵ_{ij} . Just an abstract graph idea. Can have free lines, no "tadpole" lines. If the external lines are in 1:1 with process, you can call diagram $D(K)$.
- O: A Landau surface $M[D]$ (script M) is a surface in the space of certain momenta p_i . Each line L of your mathematical Landau diagram gets a momentum P and an ϵ . The surface is defined by three conditions: p 's all on mass shell, p conserved at vertices, and the third one is some kind of sum involving the alphas. You can draw D in momentum space, where it is called $\bar{D}$.
 $M^+[D]$ is a positive alpha Landau surface of some diagram. If you add up these singularity surfaces for all possible diagrams, you get $M^+(K)$, the complete positive alpha singularity surface in the momentum space K .
- P: The function $M_c(K)$ is supposed to be analytic at any physical point in $\mathcal{O}(K)$, after you remove the set of positive alpha singularities $M^+(K)$. In simple case this means on the real axis, but not at actual normal threshold branchpoints.

Q: The $i\epsilon$ rules. I skipped most of this, but idea is the rule tells you how to go around a singularity in $M^+(K)$ if you hit one. A path along real axes in K -space which makes little detours around the positive alpha singularities is called an essentially real path. It is by means of such paths that different physical regions are connected.

R: The idea of "persistence" is that the physical significance of the function $M_c(K)$ cannot suddenly change as you move along an essentially real path. Seems obvious.

Second idea in this section is interchange of order for identical particles. The remaining sections all deal with this question. The conclusion is that you get the field theory result, usual statistics, but Henry goes out of his way to show this. EG, how do you know that an interchange does not cause a phase which depends on momentum, or on position in the set K , or on something else.

Crossing, Hermitian Analyticity, and the Connection between Spin and Statistics

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The analytic S -matrix framework is further developed. First some results of earlier works are collected and the physical-region analyticity properties recently derived from macroscopic causality conditions are described. These entail that scattering functions are analytic at physical points except on positive- α Landau surfaces, and that there they are $i\epsilon$ limits of analytic functions from certain well-defined directions, except possibly at certain points where four or more positive- α surfaces intersect. A general $i\epsilon$ rule that also covers these exceptional points is then stated. It is then shown that the scattering function defined by analytic continuation is either symmetric or antisymmetric under interchange of variables describing identical particles and that the sign induced by the interchange is independent of the particular scattering function in which the variables appear. The physical-region analyticity properties of bubble-diagram functions are then derived from the general $i\epsilon$ rule. These functions are products of scattering functions and conjugate scattering functions integrated over physical internal-particle variables, as in the terms of unitarity equations. They are shown to be analytic in the physical region except on Landau surfaces and, more specifically, except on those Landau surfaces that correspond to Landau diagrams that are supported by the bubble diagram in question, with the further restriction that the Landau α 's must be positive or negative for lines lying within positive or negative bubbles, respectively. Also, the basic rule for continuation around these singularities is derived. A new general derivation of the pole-factorization theorem is given, which is based on slightly weaker assumptions than earlier proofs. Particular attention is paid to the over-all sign. A general derivation of the crossing and Hermitian analyticity properties of scattering functions is then given. On the basis of the deduced general rules for constructing the paths of continuation that connect the crossed and Hermitian conjugate points, the various related points are found to be boundary values of a single physical sheet. In particular, a certain sequence of continuations is shown to take one back to the original point. From this fact it follows that abnormal statistics are incompatible with simultaneous unitarity in both the direct and crossed channels. The proof given here does not depend on the notion of interchange of variables other than those of identical particles. Earlier proofs depended on the unphysical notion of interchange of variables representing conjugate particles. Finally it is shown that the analytically continued M functions with normal-ordered variables are precisely the scattering functions: no extra signs are needed or permitted. Aside from the general i rule, the analyticity assumptions are these: (1) The discontinuity around a singularity of a bubble diagram B has no residue at a physical-particle mass value (in an appropriate variable) unless the singularity corresponds to a diagram that is supported by B and has the single-particle-exchange form that corresponds to a pole at that mass value. (2) The residue just described has the pole-factorization property. (3) Confluences of infinite numbers of singularity surfaces do not invalidate the results established by assuming that this number is locally finite. Assumption (1) entails that all relevant singularities of scattering functions lie on Landau surfaces. That is the basic assumption.

1. INTRODUCTION

An earlier S -matrix proof of the normal connection between spin and statistics given by this author¹ depended on an assumption that self-conjugate combinations of particle and antiparticle amplitudes were in principle observable. The assumption is objectionable because it has no experimental basis in the case of charged particle and in fact conflicts with a conjectured superselection rule.²

In that original paper, the beginning of a second proof, not depending on this special assumption, was also given.³ This alternative proof depended on an apparent conflict between abnormal statistics and the

crossing and Hermitian analyticity properties of scattering functions. The crossing property of (multi-particle) scattering functions is the property whereby the scattering function describing one reaction is connected by analytic continuation to the scattering function describing certain other reactions, called crossed reactions. The Hermitian analyticity property is the property whereby the scattering function representing a given process is analytically connected to the complex conjugate of the scattering function for the transposed process at certain real boundary points.

This second argument was not a full proof. In the first place, the statistics involved was the sign change under interchange of variables describing relative antiparticles, whereas the spin-statistics connection involves the sign change under interchange of two variables describing particles of the same type (identical particles). And in the second place, the required

¹ H. P. Stapp, Phys. Rev. 125, 2139 (1962).

² G. C. Wick, A. S. Wightman, and E. Wigner, Phys. Rev. 88, 101 (1952).

³ Reference 1, Appendix I. There, crossing was assumed to hold without any extra phase factors, but the no-scattering parts were allowed to be different from unity.

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⁴ H. P. S.
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⁵ J. Gussac,
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⁶ D. Olive.

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⁹ J. B. Boy.

Barut '66

Selected Xerox from Barut book:

- 2.3 The UIR's of Poincare Group
- 2.4 Reps of the Little Groups
- 2.5 S-Matrix and Amplitudes
- 2.6 Spinorial Amplitudes
- 4.2 Construction of Spin Amplitudes
- 4.3 Direct construction of Helicity Amplitudes
- 4.4 Photon amplitudes
- 4.5 Neutrino amplitudes
- 5.1 Discrete Symmetry Operations
- 5.2 Parity
- 5.3 Time Reversal
- 5.4 CPT, C, T and spinorial amplitudes
- 5.5 the CPT theorem and analyticity, complex LT
- Appendix 2: Poincare Group
- Appendix 3: FDRs of Lorentz Group

Chapter 2.

Section 2.3 Comments:

Barut is interested in having Poincaré transformations $U(a, A)$ act on functions of momenta which he calls $X(K)$, where K is a set of momenta, and X will have one index for each momentum. If you just consider the translations, you get $X(K) = X'(K)$ times the obvious phase. This result would be obvious in terms of the S-matrix. Now consider just the LT's called A . It is obvious that $U(A) X(K)$ is going to take the 4-momenta into AK , so Barut says: write $U(A) = Q(A)T(A)$ and let $T(A)$ do nothing but this action on the K variables. As he says, under $T(A)$, $X(K)$ transforms as a scalar.

Due to d^4k stuff, $T(A)$ is unitary. Since $U(A)$ is supposed to be unitary, $Q(A)$ must also be unitary.

Next, he shows that $Q(A)$ must commute with translations. If this is so, it then follows that $Q(A)$ must not affect the arguments K . This means that $Q(A)$ can only act on the spin type indices.

Finally, he shows that for each particle there is a $Q_i(A)$, and that this $Q_i(A)$ must form a representation of the little group of k . I should have said that $Q(A)$ is a product of $Q_i(k_i, A)$. By somehow mapping the general k_i back to the rest vector you ended up with a Wigner rotation.

All this stuff concludes with (2.26) which says that the function $X(K)$ transforms with the usual translation phases and a Wigner rotation product, as usual.

Section 2.4 comments: In Case a Barut does the usual rotation group case. Notation is as follows:

p = the rest vector $(m, 0, 0, 0)$

$B_{k \leftarrow p}$ = either $L(k)$, $H(k)$, or any other case

/ aside: if in $H(k)$ we allow any rotation $R(p)$, then this exhausts all cases /

$A_{k \leftarrow p} = L(k)$, the symmetric boost $= (\frac{k^\mu \sigma_\mu}{m})$, only!

Obviously, $BB^\dagger = L^2(k)$ as in (2.29').

More notation is this: (to compare to my own notation)

I say

$$\tilde{p} = (m, 0, 0, 0)$$

$$H(p)\tilde{p} = L(p)\tilde{p} = p$$

$$Lp = (Qp)$$

I don't see where Barut's momentum q fits in. It is too complicated. Overly notated. (over)

he says

$$p = (m, 0, 0, 0)$$

$$\Lambda(B_{k \leftarrow p})p = k$$

$$\Lambda(A)k = ?$$

I think his notation stinks because it

OK, I think he has changed his notation in mid course. Look at (2.31):

$$[U(a, A)X](K) = e^{i\sum k_i a_i} \prod D^j(B_{k \leftarrow p}^{-1} A B_{q \leftarrow p}) X[\Lambda(A^{-1})K]$$

How would I translate this? First, $p \rightarrow \tilde{p}$ and $A = L$.

$$[U(a, L)X](K) = e^{i\sum k_i a_i} \prod D^j(B_{k \leftarrow \tilde{p}}^{-1} L B_{q \leftarrow \tilde{p}}) X[LK]$$

Next,

$$B_{q \leftarrow p} = H(q) \quad \bar{B}_{k \leftarrow p}^{-1} = H(k)^{-1}$$

Therefore, it must be that $k = lq$. So, he should be using q 's

$$[U(a, L)X](q_i) = e^{i\sum (lq)_i a_i} \prod D^j(H(lq)^{-1} L H(q)) X[lq_i]$$

In order to make sense out of his notation, we have to correct an error: the momenta he calls K are really what he calls the q_i . Then $Aq_i = k_i$ and these are the ones which appear in the phase (as he says). So, comparing to my notation:

For (2.31):

m_e	m_{in}
$\tilde{p}$	p
p	q
lq	k
L	A
$L(p), H(p)$	$B_{q \leftarrow p}$
$\{n B_{p \leftarrow \tilde{p}}\}$	

So I regard (2.13), which I already know, as the final result of this section.

Case b I skipped deals with the lightlike little group.

To summarize this section, Barut has shown how the S-matrix elements transform, in the case of all massive or all massless particles. However, he has done this without ever mentioning single particle states.

Section 2.5: Finally the transformation rule is written in terms of the S-matrix. Barut's convention is: D^* goes with an incoming particle, D with outgoing. Also, in (2.43) he shows a specific example, so he is also saying: incoming particles have "up" indices, outgoing particles have "down" indices. Notice that he always tilts his indices on the D-functions in the correct way.

Barut mentions isospin in passing and notes that much simpler, though same D-functions. You can make amplitudes of total I , but not of total S , because S is not conserved in the theory whereas I is.

Finally, the two special cases are mentioned: the helicity amplitudes and the R-amplitudes which I call canonical.

Section 2.6: Spinor M-functions. Notice, by the way, that Barut was using both up and down indices for R-functions (canonicals). I never do this because rotations functions only are used, no need to go up and down. But certainly you could go up and down if you wanted. Barut wants to. Barut goes through the usual deal of converting helicity or canonical amplitudes to the M-functions. He shows how everything transforms in a nice compact notation, though the index attachments are not quite obvious. I skip the massless case, once again. This stuff is all clear to me.

Section 2.7: very short. Shows that number of fermions must be even using the transformation properties of the R-functions. Very nice, but obvious. Footnote is interesting: some people erroneously call this a superselection rule, it is not! It follows from rotation invariance.

Chapter 5: Discrete Symmetries and Analytic Continuation.

and 5.2

Section 5.1: Look at once at (5.7) and (5.8), which gives parity invariance. The arguments α are like momenta, so maybe $\alpha = p^\mu$ whereas $\alpha' = sp^\mu$, the parity inverted p . Then the combinations of 5.7 and 5.8 gives parity invariance. You see the $L(p)^2$ appearing inside the D-functions, just the way I want. You see the product of intrinsic ~~naaa~~ parity factors gathering in front. Presumably, the order of the initial of final state is not changed. I skip the rest of this section. Parity agrees with what I get for same.

Sections 5.3 and 5.4: To me these sections come across as very confused. The notation is not clearly explained. Worse, there is what I consider a mistake: Barut suggests that an anti particle should have a negative spin projection as well as a negative isospin projection. This is in conflict with "usual" understanding (eg, Sakurai book on P,C,T; Martin Spearman; Williams, etc) who say that the operator C should commute with all Poincare generators. So, this means that the J_3 eigenvalue of a particle at rest cannot be negated by C. Because Barut says that the index m should be negated, he gets the relation (5.20) for C-invariance: in Barut's notation, a dotted index is numerically the negative of an undotted index. Since he sees the spin as negating under C, he says that you should interchange dotted with undotted indices on the canonical amplitudes. Except for this interchange of the dotted with undotted locations, I could agree with (5.20).

Next, (5.21) gives the combination of PT. In the R-amplitudes, since initial states go with D^* and final states go with D (the way Barut has chosen to do things), clearly initial indices are dotted and final indices are undotted for an amplitude. Then PT, since it includes T, should change dotted to undotted, so I agree with (5.21), ignoring isospin. For T-alone, I then agree with (5.22).

The three equations, actually the four, namely (5.19) through (5.22), tell the action of all the symmetries on the canonical amplitudes called R. I agree with everything except the dotted-undotted interchange caused by C. In particular, there are no D-functions, which is the same result I get with the $(p,m)_B$ amplitudes.

Next, he wants to convert all these statements to the M-functions. In order to do this, he uses this bizarre fact:

$$G \equiv L(k) \bar{C}^1 \quad \text{and} \quad A = G(k) A^* \bar{G}(\bar{1}k)$$

If this is true, then also true with $D(A)$ etc. This allows him to go from D^{os} to D^{os*} in this way. These are inequivalent reprints. Well, somehow using this he gets that in (5.31) you can do T-invariance without and D-functions. I know that somehow the D-functions are hidden in the switch from one kind of index to the other kind.

In general, Baruts' handling of these discretes just leaves me cold. I will have to examine Stapp to see if the same kind of confusion arises. If Stapp also does things in this way, then I will go back and restudy things. Taylor does not seem to do things in this way.

Section 5.5: CPT theorem and analyticity. Claims that the CPT theorem is also obtained directly from the simple transformation law using the magic complex transformation. I think this is what Stapp said to me one day. Somehow I think of this as being combined with crossing to get the CPT theorem.

overall comments: Although Barut has lots to say about things I am interested in, somehow he just doesn't tell me things I want to know. We have a terrible notational difference, for one thing.

One thing that I don't like is his ignoring the single-particle states. Enough.

Appendices:

Appendix 2: Poicare and Lorentz groups. Nothing new here in terms of proper, restricted, and full Lorentz groups. However, the "second kind" complex Lorentz group is something I never heard of before.

Appendix 3: Barut does many things differently from me. First, he uses the symbol σ^μ differently than I do. Secondly, he defines the first kind spinor x_m to transform with the matrix A, whereas I have x^m transforming with A. He does use starring to go from dots to no dots, as in 3.18^{1?} ^($\delta^{\mu\nu}$ and $\epsilon^{\mu\nu}$) Another difference is his definition of the object $D^{(j^1, j^2)}$ which is given in 23. We agree as in line 3 of (24) that D^{s^1, s^2} is a combination of first kind and second kind, but he sets it up so that it is upper in one but lower in the other. Unpleasant to me. The result of his definitions is that (s,o) and (o,s) are related by $+^{-1}$ instead of just * .

In handling the metric spinor, he deals with the antisymmetry problem by noting that C^{-1} raises but C lowers. Actually, D(c). So at least we know what C is.

This is discussed in the next section. The discussion of the other groups is postponed to later chapters.

2.3 Determination of the Unitary Representations of the Poincaré Group

Let us turn back to the Poincaré group (a, A) (see first Appendices 2 and 3), and denote its unitary representations by $U(a, A)$, which we want to determine. From the group property we have

$$U(a_2 A_2) U(a_1 A_1) = U(a_2 + \Lambda(A_2) a_1, A_2 A_1). \quad (2.10)$$

Instead of Λ we use the 2×2 unitary-unimodular matrices A (see Appendix 3). Actually, in quantum mechanics one must consider representations up to a phase factor $\omega(1, 2)$ on the right-hand side of Eq. (2.10), but this factor can be chosen to be +1 when A is used.⁵

The translations T which we diagonalize form a commutative subgroup with

$$U(a_2, 1) U(a_1, 1) = U(a_2 + a_1, 1). \quad (2.11)$$

Their representations are all direct products of one-dimensional representations. For the functions $X(K_i; K_f)$ transforming under the representations of translations, we can then write, up to a unitary equivalence,

$$X'(K_i; K_f) = \exp \left(i \sum_j k_j a \right) X(K_i; K_f), \quad (2.12)$$

where the sum $\sum_j k_j a$ includes all the particles incoming and outgoing, if we draw in the S -matrix, for convenience, all particles as outgoing, for example.

Next we consider the representations $U(0, A)$ which we split into two parts

$$U(0, A) = Q(A) T(A), \quad (2.13)$$

where $T(A)$ is defined to act on the arguments K only:

$$[T(A)X](K) = X[\Lambda(A^{-1})K]. \quad (2.14)$$

Under $T(A)$, X transforms like a scalar. Because the unitarity condition involves an invariant integration, $T(A)$ is unitary. Hence $Q(A)$ in Eq. (2.14) must also be unitary if $U(0, A)$ is unitary. The definition (2.14) fixes the transformation property of $Q(A)$ which is supposed to affect X itself, not the arguments.

⁵ V. Bargmann, *Ann. Math.*, 59, 1 (1954). The phase ω can first be reduced to ± 1 using the continuity and associativity properties of the group. Then because the correspondence between A and Λ is 2-to-1; i.e., $\pm A \leftrightarrow \Lambda(A)$, the remaining phase can be taken to be +1. The groups with the phase factors properly taken care of, are called the *quantum mechanical groups*. Thus the quantum mechanical group of relativistic invariance is the group $SL(2, c)$, the 2×2 unimodular group which is the covering group of Λ .

The translations $U(a, 1)$ do not commute with $U(0, A)$. However, now that the $Q(A)$ part of $U(0, A)$ commutes with the translations, from the group property

$$U(0, A) U(a, 1) = U(\Lambda(A) a, 1) U(0, A)$$

we have

$$Q(A) T(A) U(a, 1) T^{-1}(A) Q^{-1}(A) = U[\Lambda(A) a, 1]. \quad (2.15)$$

By direct computation from its definition,

$$[T(A) U(a, 1) X](K) = T(A) e^{i \sum_j k_j a} X(K) \quad \checkmark$$

translation $\rightarrow \exp \left(i \sum_j k_j a \right) X[\Lambda(A^{-1}) K] = [U(\Lambda(A) a, 1) T(A) X](K)$ $\checkmark$

for all X ; hence

$$T(A) U(a, 1) = U(\Lambda(A) a, 1) T(A) \quad (2.16)$$

so that T and $U(a, 1)$ do not commute, but from Eq. (2.15) we now get

$$Q(A) U(\Lambda(A) a, 1) = U(\Lambda(A) a, 1) Q(A), \quad (2.17)$$

which is the desired result. $\square$

Any operator F which commutes with the translations cannot affect the momentum arguments of the function X , for if $[F, U(a, 1)] = 0$ we get $\exp \left(i \sum_j k_j a \right) F X = U(a, 1) F X$. Hence $Q(a)$ must act as follows⁶

$$[Q(A) X](K) = \left[\prod_i Q_i(k_i, A) \right] X(K), \quad (2.18)$$

where $Q_i(k_i, A)$ are operators acting on the additional indices of X , one for each k_i . These additional degrees of freedom on X will be interpreted as the "spin" indices. In this sense the separation given in Eq. (2.13) is one which separates spin and orbital angular momenta.

To determine $Q(A)$ or $Q(k, A)$ we use the group property

$$U(0, A_1) U(0, A_2) = U(0, A_1 A_2)$$

or

$$Q(A_1) T(A_1) Q(A_2) T(A_2) = Q(A_1 A_2) T(A_1 A_2). \quad (2.19)$$

Let us operate both sides of Eq. (2.19) on X and use Eq. (2.18). We have in

⁶ "Every bounded operator commuting with translations has the spectral representation $(FX)(k) = F(k)X(k)$, where $F(k)$ is a bounded operator." See SNAG (Stone-Neumark-Ambrose-Godement) theorem, in M. M. Stone, *Ann. Math.*, 33, 643 (1932), M. Neumark, *Iz. Akad. Nauk USSR*, 7, 237 (1943), W. Ambrose, *Duke Math. J.*, 11, 589 (1944), R. Godement, *C. R. Acad. Sci.*, 218, 901 (1944).

section 1.1 "The S-matrix"

- Credit Heisenberg 1942. Speaks of "primary quantities" like the masses etc., and derived quantities: cross sections, lifetimes, energy levels. The S-matrix elements are probability amplitudes, hence QM.

section 1.2 "The quantum principle"

- Reminds us that it is amplitude which flows, not probability. This is in contrast to classical analysis where you always linearly combine probabilities. Not clear what is a_i and w_i here.
- OK. Let a_i = amplitude that system is initially in state i . Then, why not say $a_i = \langle i | a \rangle$. Then since system must be in some state, we get $\sum_i |a_i|^2 = 1$. So:

$$- \text{out} \langle f | a \rangle = \sum_i \text{out} \langle f | i \rangle_{in} \langle i | a \rangle = \sum_i \langle f | S | i \rangle_{in} \langle i | a \rangle$$

$$\text{On: } a_f = \sum_i S_{fi} a_i \quad (1.5)$$

- This says: if you know the amplitude for your system to be in each initial label state i , then S will tell you amplitude to go into each final state.

- So the "quantum principle" says (1.5). Roughly, this is the fact of quantum interference. Usually we choose $|a\rangle$ such that $\langle i | a \rangle = \delta_{i,1}$ for a particular initial system 1.

section 1.3 "unitarity condition"

- First, we have a nice proof of $S^\dagger S = 1$ with more care than is usually taken.

Second, the actual "scattering amplitude R " is defined by $R = S - 1$. Unitarity is rewritten in terms of R . Then non-linearity is pointed out. Baut sees unitarity as "the most fundamental relation of S-matrix theory."

Third, the S-matrix gospel: all particles (composites) are treated on equal footing. "Unitarity + invariances + analyticity will, in principle, determine the S-matrix,"

section 1.4. The "diagonalization" of the S-matrix.

$$S = \begin{pmatrix} \boxed{\text{---}} & & \\ & \boxed{\text{---}} & \\ & & \boxed{\text{---}} \end{pmatrix} = \text{block form.}$$

Each submatrix is unitary + characterized by certain value of invariants (Casimirs). This simplifies the problem a bit.

section 1.5 "The definition of space-time"

Not really necessary in S-matrix theory to even have a space-time. Only needed when you must think about an actual experiment. "succession of events generates space-time."

Maybe space-time interpolated events only.

If you really want x^μ description, just do FT.

About the Canonical Single-Particle Spin State

1. I understnad what McKerrell has to say. I am a little confused however on how you experimentally know what you are dealing with. Let's start off with a particle of spin S at rest, with component m along a z -axis. Thus, the picture is this:

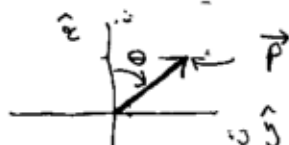


2. Now, we directly boost this rest state in a direction n such that:

$$|p, m\rangle_c \neq e^{-ig\hat{n} \cdot \vec{K}} |\tilde{p}, m\rangle$$

$$p = m(\underset{\substack{\uparrow \\ = x, y, z}}{shg}, 0, shg \sin \theta, shg \cos \theta)$$

Thus, we now have a new "state" with momentum pointing in the first quadrant:



$$\vec{p} = p_y \hat{y} + p_z \hat{z} = (shg \sin \theta m) \hat{y} + (shg \cos \theta m) \hat{z}$$

3. Now here is the problem: suppose an experimenter has this particle going at this momentum p . How does he figure out what "m" is for this single particle? How does he measure it? One way might be this: the experimenter runs along with the particle and view it at rest. However, he continues to xxx. Hopefully, he sees what I have drawn in figure 1 above. He then figures out what the component of spin is along the z axis. What z axis? ?

4. If the particle is in a helicity single-particle state, then it is completely clear to me what the experimenter does because there is no confusion about which direction is the relevant direction.

Transformation Property of the S-matrix.

1. In his equation (3.27) McKerrell shows the effect of an arbitrary Poincare transformation on the single particle canonical state. The translation part of this is clear and is done last. You pick up the usual rotation matrices with a certain Wigner angle. The slight subtlety is that here, the Wigner angle is generated from the $L(p)$ operation, whereas in the helicity formalism you get a different Wigner angle generated from the operation $H(p)$. Recall that these differ by a pre-rotation.
2. Now, consider an S-matrix element $\langle f|S|i\rangle$. Here, we imagine S to be a scalar operator in the super QM Hilbert Space. The statement that this operator is invariant under Poincare Transformations is the statement of Lorentz Invariance plus translations. Now, for the state $|i\rangle$ we can take the direct product of single particle states in either the canonical or helicity basis. Thus, for example, here is a statement of Lorentz invariance:

$$\langle p_3, m_3; p_4, m_4; p_5, m_5 | S | p_1, m_1; p_2, m_2 \rangle = \langle f | S | i \rangle$$

$$\langle f | S | i \rangle = \langle f | L^{-1} S L | i \rangle$$

$$\text{Now } L | p, m \rangle_c = \sum_{m'} D_{m/m'}^{(s)} (L(\vec{p})) L | p, m' \rangle_c$$

$$(L^{-1})^\dagger | p, m \rangle_c = \text{" " } (L^{-1})^\dagger (L^{-1})^\dagger L | p, m \rangle_c \text{" "}$$

$$\Rightarrow \langle p_3, m_3 | L^{-1} = \sum_{m'_3} D_{m'_3/m_3}^{(s_3)*} \underbrace{(L^{-1})^\dagger L}_{\text{Ia.}} | p_3, m'_3 \rangle$$

So just using appropriate rotations:

$$\langle f | S | i \rangle = \dots$$

$$\langle p_3, m_3; p_4, m_4; p_5, m_5 | S | p_1, m_1; p_2, m_2 \rangle$$

$$= \sum_{\substack{m'_1, m'_2 \\ m'_3, m'_4 \\ m'_5}} D_{m'_3/m_3}^{s_3*} D_{m'_4/m_4}^{s_4*} D_{m'_5/m_5}^{s_5*} D^{s_1} D^{s_2} D^f(L_i)$$

$$\langle p_3, m'_3; p_4, m'_4; p_5, m'_5 | S | p_1, m'_1; p_2, m'_2 \rangle$$

Connection to the M-functions or "spinorial amplitudes".

1. On the last page, we established the condition of Lorentz Invariance on the S-matrix elements. We pick up a D-function of a Wigner rotation for each particle involved. Actually, the Wigner rotation may be written as the product of three $SL(2, C)$ transformations, as usual.
2. From our study of Liubarski, we presumably know all about the finite dimensional representations of the Lorentz Group. These were there called $\tilde{\rho}_{PQ}$. Basically the 6 generators are combined so as to form two separating algebras, so you get roughly the direct product of two $SU(2)$'s. P and Q are really $j_1 j_2$. Each j takes on the usual integer or half-integer values. There are two spinor representations $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ which transform dotted and undotted spinors. From such objects you build up fancy higher spinors in the ~~xxxxxx~~ "spinor algebra". The familiar vector representation is $(\frac{1}{2}, \frac{1}{2})$ and powers of this representation form the tensor algebra.

What Liubarski did not do was construct the explicit representation matrices for $SL(2, C)$. These would be called $D_{mn}^{j_1 j_2}(A)$ where A is element of $SL(2, C)$. I.e, there are four lower subscripts. I think this thing is just the simple direct product of $D_{mn}^{j_1 0}(A)$ $D_{mn}^{0 j_2}(A)$. These objects are just extensions of the $SU(2)$ matrix elements. I.e, you take as basis monomials $x^{j-m} y^{j+m}$ and you let (x, y) go to (x', y') under an $SL(2, c)$ transformation, things all complex.

In the special case that A really is just a rotation, D^{0j} is exactly the same as D^j . However, the D^j themselves are not defined outside of $SU(2)$. See Barut appendix for a few details on this.

3. So now we come to the spinorial amplitudes. Recall that we had Wigner rotators $A' = L(p)^{-1} L(p) L(p')$ in McKerrell notation for canonical basis. $L = A =$ arbitrary Lorentz-Transformation. Now, $B^S(A') = D^{OS}(A')$ in $SL(2, C)$ = product of three factors = $D^{OS}(L(p)^{-1}) \times D^{OS}(L) \times D^{OS}(L(p))$. Each argument here is an $SL(2, C)$ group element. In general, none of these arguments is a simple rotation. So here we develop the rule and definition of the M-functions:

$$R_{m_3 m_4; m_1 m_2}(k_3, k_4; k_1, k_2) = \sum_{n_3 n_4 n_1 n_2} D_{m_1 m_1}^{(s_1)*}(W_1) D_{m_2 m_2}^{(s_2)*}(W_2) \\ \times D_{m_3 m_3}^{(s_3)}(W_3) D_{m_4 m_4}^{(s_4)}(W_4) \cdot R_{n_3 n_4; n_1 n_2}(\tilde{\Lambda}^1 k_3, \tilde{\Lambda}^1 k_4; \tilde{\Lambda}^1 k_1, \tilde{\Lambda}^1 k_2)$$

where $W_i = L(p_i)^{-1} \Lambda L(p_i)$ or $H(p_i)^{-1} \Lambda H(p_i)$

These are just the R-functions

Here is Barut's shorthand: *family* *starting*

$$R(k) = D^F \cdot R(\Lambda^1 k) \cdot D^{S+}$$

$$= \prod_f D^{(s_f)}(W_f) \cdot R[\Lambda^1 k] \cdot \prod_i D^{(s_i)}(W_i)$$

Now let's pick out one particle with spin (say rest are spinless)...

$$R_{m_3}(k \dots k_3 \dots k_N) = \sum_{m_3} D_{m_3 m_3}^{(s_3)}(W_3) R_{m_3}(\Lambda^1 k \dots \Lambda^1 k_3 \dots k_N)$$

$$= \sum_{\substack{m_3 \\ m_3' \\ m_3''}} D_{m_3' m_3}^{(s_3)}(L(p_3)) D_{m_3' m_3''}^{(s_3)}(L(A)) D_{m_3'' m_3}^{(s_3)}(L(p)) R_{m_3}(\Lambda^1 k \dots)$$

Now absorb the two

boosts.

somehow:

$$M \equiv D^{(s_3)}(L(p_3)) \cdot D^{(s_3)}(L(p)) \cdot R$$

How do you disrupt group ordering?!

$$= \sum_{\substack{m_3' m_3'' \\ m_3}} D_{m_3' m_3}^{(0, s_3)}(L=A) \cdot D_{m_3'' m_3}^{(0, s_3)}(L(p)) \cdot D_{m_3 m_3'}^{(0, s_3)}(L(p_3)) R_{m_3}(\Lambda^1 k \dots)$$

$$= \sum_{m_3' m_3''} D_{m_3' m_3''}^{(0, s_3)}(A) \times \left[\sum_{m_3} D_{m_3' m_3}^{(0, s_3)}(L(p)) D_{m_3 m_3'}^{(0, s_3)}(L(p_3)) R_{m_3}(\Lambda^1 k \dots) \right] \\ \equiv M_{m_3' m_3''}(\Lambda^1 k) ??$$

Let $A' = B_1 A B_2$.

Then: $R_{m_3}(\dots k_3) = \sum_{m_3'} D_{m_3 m_3'}^{(A')'} R_{m_3'}(\Lambda^1 k)$

$$= \sum = D_{m_3 m_3'}^{o s_3}(B_1) D_{m_3' m_3''}^{o s_3}(A) D_{m_3'' m_3'''}^{o s_3}(B_2) R_{m_3'''}(\Lambda^1 k)$$

So:

$$R_{m_3}(k_3) = D_{m_3 m_3}^{-os_3} \dots$$

$$\Rightarrow R_m(k) = D_{mm'}^{os}(B_1) D_{m'm''}^{os}(A) D_{m''m'''}^{os}(B_2) R_{m'''}(\tilde{A}k)$$

where we display only one index. Now define:

$$M_m(\tilde{A}k) = D_{mm'}^{os}(B_1) D_{m'm''}^{os}(B_2) R_{m''}(\tilde{A}k)$$

$$\text{Invert: } R_m(\tilde{A}k) = D_{mm'}^{os}(B_2) D_{m'm''}^{os}(B_1) M_{m''}(\tilde{A}k)$$

$$\text{Well: } R_m(k) = D_{mm'}^{os^{-1}}(B_2) D_{m'm''}^{os^{-1}}(B_1) M_{m''}(k)$$

$$\Rightarrow R_m(\tilde{A}k) = D_{mm'}^{os}(B_2) D_{m'm''}^{os^{-1}}(B_1) M_{m''}(\tilde{A}k) \quad \text{roughly mky.}$$

$$\text{Then we get: } D_{mm'}^{os^{-1}}(B_2) D_{m'm''}^{os^{-1}}(B_1) M_{m''}(k)$$

$$= D_{mm'}^{os}(B_1) D_{m'm''}^{os}(A) D_{m''m'''}^{os}(B_2) D_{m''m'''}^{-1}(B_1) M_{m'''}(\tilde{A}k)$$

OK, too sloppy. The idea is to end up with:

$$M_m(k) = D_{mm'}^{(os)}(L) M_{m'}(k')$$

I.e., such an M transforms as τ_{os} in $SL(2, \mathbb{C})$.

More generally, $\tau_{os_1} \otimes \tau_{os_2} \otimes \tau_{os_3} \otimes \dots \otimes \tau_{os_n}$.

Thus: M -functions are "covariant spinors" $\uparrow$.

This seems to say that: the M -functions transform simply as direct product representation of spin-representations of the $SL(2, \mathbb{C})$ group. Barut simply does not give enough information to be comprehensible.

Boyce '67

Relation of the $O(2, 1)$ Partial-Wave Expansion to the Regge Representation

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The general two-particle scattering amplitude is expanded in terms of partial waves corresponding to the crossed channel little group, $O(2, 1)$. Under the assumption of square integrability over the group manifold, the invariance of the S matrix under the complex Lorentz group, which follows from the Bargmann-Hall-Wightman theorem, enables this expansion to be identified with the Regge representation in the crossed channel, whenever no dynamical singularities occur to the right of $\text{Re } j = -\frac{1}{2}$. The identification requires the assumption of the fixed t dispersion relation necessary for the definition of the Regge representation.

1. INTRODUCTION

DUE to the crossing symmetry of the S matrix the two-particle scattering amplitude may be expressed in terms of two-particle helicity states which correspond to one incoming and one outgoing particle. The spacelike character of the total momentum of such a state permits its expansion in terms of eigenstates of the little group, $O(2, 1)$, maintaining, however, the reality of the masses of the component single-particle states. This expansion in turn enables the amplitude to be expanded in terms of the irreducible unitary representations of $O(2, 1)$, a subset of which forms a complete set for the expansion of any function which is square integrable over the group manifold. The manner of making the expansion which is adopted below is due to Dr. J. A. Strathdee, as is also the tenor of the approach.

The invariance of the S matrix under the complex Lorentz group enables this restricted expansion to be identified with the Regge continuation of the crossed channel $O(3)$ partial-wave expansion, subject to the condition that the partial-wave amplitudes have no dynamical singularities to the right of $\text{Re } j = -\frac{1}{2}$.

The chief results are the identification of the principal series of $O(2, 1)$ representations with the background integral of the Regge continuation, and the discrete series with the nonsense channel terms, which for $O(2, 1)$ are perfectly natural contributions.

The major assumptions are the square integrability of the amplitude over the group manifold and the absence of dynamical singularities to the right of $\text{Re } j = -\frac{1}{2}$ in the physical region of the s channel, together with the fixed t dispersion relation necessary for the definition of the Regge continuation of the amplitude.

We have tried to alleviate the complication due to spin and the presence of exchange forces by presenting the essential framework of the paper in Sec. 2. A

summary of the crossed channel $O(3)$ expansion in Sec. 3 precedes the definition of $O(2, 1)$ helicity states in Sec. 4 and the expansion of the S matrix in terms of them in Sec. 5. The analytic continuation in Sec. 6 enables its identification with the $O(3)$ expansion of Sec. 3.

This work may be regarded as the continuation of an enquiry suggested by Joos¹ and is complementary to recent works by Toller,² Hadjioannou,³ and Roffman.⁴ It draws heavily upon the properties of the representations of $SL(2, R)$ which have been established by Andrews and Gunson.⁵

2. CENTER-OF-MASS AND BRICK-WALL FRAMES

The invariance of the S matrix under the Poincaré group $\mathcal{P}$ enables the scattering amplitude to be expanded in terms of its unitary irreducible representations. The usual partial-wave expansion is based upon the representations of the little group $O(3)$, which corresponds to positive definite eigenvalues of the Casimir operator P^2 of $\mathcal{P}$. The general two-particle transition amplitude

$$\langle p_1 \lambda_1, p_2 \lambda_2 | T(s, t) | p_3 \lambda_3, p_4 \lambda_4 \rangle,$$

for the process represented in Fig. 1, is expanded by transforming to the center-of-mass frame, in which $(p_1 + p_2)$ is along the time axis; and by basing the definition of the scattering amplitude on the single particle helicity state

$$|p \lambda\rangle = U(L_p) |\lambda\rangle, \quad (2.1)$$

where

$$p = m(\cosh \gamma, \sinh \gamma \sin \theta \cos \varphi, \sinh \gamma \sin \theta \sin \varphi, \sinh \gamma \cos \theta),$$

$$U(L_p) = \exp(-i\varphi J_{12}) \exp(-i\theta J_{31}) \exp(-i\gamma J_{03}),$$

¹ H. Joos, in *Lectures in Theoretical Physics* (University of Colorado, Boulder, 1964), Vol. 7A.

² M. Toller, *Nuovo Cimento* 37, 631 (1965); Istituto di Fisica "G. Marconi", Note Interne 76 and 84.

³ F. T. Hadjioannou, *Nuovo Cimento* 44A, 185 (1966).

⁴ E. H. Roffman, *Phys. Rev. Letters* 16, 210 (1966).

⁵ M. Andrews and J. Gunson, *J. Math. Phys.* 5, 1391 (1964).

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also see next sheets

1. First, Boyce defines the usual $O(3)$ helicity states using the L_p xform. These states are then indicated without a prime, eg, $|p\lambda\rangle$. Next, for reasons as yet unknown (but see below), Boyce defines what he calls an $O(2,1)$ helicity state via xform L_p' and state is $|p\lambda'\rangle$. Given these definitions, it is easy to show that these states are related by the y-rotation given in 2.3. But why are these $O(2,1)$ states defined in this way???? Just wait...
2. Scene changes and we are talking about the Hall-Wightman theorem. This theorem says that if you have some M-functions which transform on $SL(2,C)$, then you can extend the transformation to the complex lorentz group if everything is nice. Boyce indicates M-functions by M inside the states, whereas helicity amplitudes of either kind have a T.

As we well know, the relation of the M functions to the s-channel bws amplitude with $O(3)$ helicity states is given above 2.15. You get products of $D(L_p)$ where L_p was used to define the $O(3)$ helicity states. Now, suppose we choose $\Lambda = R$, the imaginary z-boost, in the Hall-Wightman. Note that p's are converted into q's, so we are converting s-ch bws to t-ch cms! When this Hall-Wightman statement is converted into language of $O(3)$ helicity amplitudes, we get (2.15). This says that the s-channel bws $O(3)$ helicity amplitudes are related to the t-channel cms $O(3)$ helicity amplitudes by products of D-functions $D(W=S)$. But these are precisely the $D(S)$ which define the $O(2,1)$ helicity amplitudes, so you can get rid of all those D functions and end up with this statement:

$$(\star\star) \quad \langle p_1 \lambda_1, p_2 \lambda_2 | T(s,t) | p_3 \lambda_3, p_4 \lambda_4 \rangle = \langle q_1 \lambda_1, q_2 \lambda_2 | T(s,t) | q_3 \lambda_3, q_4 \lambda_4 \rangle$$

$\uparrow$
s-ch bws w/ $O(2,1)$ hel states

$\uparrow$
t-ch cms w/ $O(3)$ hel states

This equality is explicitly stated in Boyce by equating equations (2.17) and (2.18). This then shows why he defines those $O(2,1)$ helicity states in the way he does.

I will say this once again:

(**) says: if you look at the s-channel amplitude in the s-channel brick wall system with $O(2,1)$ helicity states; it is the same as if you take the s-channel amplitude to the t-channel cms with regular $O(3)$ helicity states.

An overview outline of Boyce's main idea.

$$\langle q_1 \lambda_1 q_2 \lambda_2 | T^{(s)} | q_3 \lambda_3 q_4 \lambda_4 \rangle$$

s-channel, $O(3)$ hel, t-channel

cross

$$\langle -q_4 \lambda_4, q_2 \lambda_2 | T^{(t)} | q_3 \lambda_3, -q_1 \lambda_1 \rangle$$

t-channel, $O(3)$ hel, t-channel

$$\hookrightarrow \equiv \bar{T}_{\lambda\mu}(z, t)$$

$$= \sum_{j=-\infty}^{\infty} (2j+1) \bar{T}_{\lambda\mu}(j, t) d_{\lambda\mu}^j(z)$$

$O(3)$ expansion

$$\begin{aligned} \bar{T}_{\lambda\mu}(j, t) &= \int_{-\infty}^{\infty} dz T_{\lambda\mu}(z, t) d_{\lambda\mu}^j(z) \\ &= \bar{A}_{\lambda\mu}(j, t) - e^{i\pi j} \bar{B}_{\lambda\mu}(j, t) \\ \text{where } \bar{A} &= \sum_{z=-1}^1 \rho_{\lambda\mu}^j(z) (X) \cdot g^k(z) \\ \bar{B} &= \sum_{z=-1}^1 \rho_{\lambda\mu}^j(z) (1) \cdot g^L(-z) \\ &= \text{poisson qubon projection} \end{aligned}$$

$$\langle p_1 \lambda_1 p_2 \lambda_2 | T^{(s)} | p_3 \lambda_3 p_4 \lambda_4 \rangle$$

s-channel, $O(2,1)$ hel, s-channel b.w.s.

cross

$$\langle -p_4 \lambda_4, p_2 \lambda_2 | T^{(t)} | p_3 \lambda_3, -p_1 \lambda_1 \rangle$$

t-channel, $O(2,1)$ hel, s-channel b.w.s.

$$\hookrightarrow \equiv \bar{T}_{\lambda\mu}(\xi, t)$$

$$= \sum_q (2q+1) \bar{T}_{\lambda\mu}(q, t) d_{\lambda\mu}^q(\xi)$$

$O(2,1)$ expansion

principle series:

$$\begin{aligned} \bar{T}_{\lambda\mu}(q, t) &= \tilde{A} - B \text{ sngt} \\ &= A - B \quad (6.8), (6.9) \\ &= \text{same as Pop.} \end{aligned}$$

← shift contour to cancel discrete series

$$= \oint_M d_{\lambda\mu}^j(z) d_{\lambda\mu}^j(\xi) [A - B]$$

M Exp

← inverse SWT

$$= \sum_M (2j+1) [A - B] d_{\lambda\mu}^j(\xi)$$

← compare to show equality in the end.

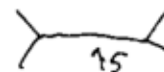
4

Discussion of overview outline:

The (t-channel cms amplitude) called $\bar{T}(z,t)$ is treated in the usual way. You use an $O(3)$ expansion and $O(3)$ helicity states.

What is new is the definition of the $O(2,1)$ helicity states so that the (s-channel bws) amplitude can be defined and is numerically equal to the (t-channel cms) amplitude. Of course as usual they cant both be physical at the same time.

If Boyce had not used these clever $O(2,1)$ helicity states, the (t-channel cms) amplitude would be related to the (s-channel bws) amplitude by a product of four d-functions, reflecting the complex LT which is needed to connect these two frames. (Called R) One of the necessities is showing how the (s-channel bws) amplitude is a function of $Q(2,1)$, and this is done very naturally via the $O(2,1)$ helicity states.



Are these $O(2,1)$ states really nessary? Think of the s-channel bws. The result of using "states@" is to get the $O(2,1)$ angles to appear in one of the states, analogous to the $/p_0...$ of $O(3)$ analysis. However, it seems to me that exactly the same thing is accomplished by using BCP frames. In this case, the group angles show up as you move across the horizontal bar. Ie, the group stuff appears as an operator on the same level with T, the S-matrix operator.

Thus, the use of BCP frames gives you the same amplitudes you would get using Boyces $O(2,1)$ helicity states. I dont think I have to ever explicitly construct Boyces two-particle helicity states.

Jan 11, 1977

Comments on Boyce Paper.

Boyce seems to claim that he has discovered the connection between nonsense regge terms and the discrete series. He notes the cancellation idea with amazement. However, if he had only derived the $SU(1,1)$ expansion theorem he would see at once why this cancellation occurs and where the discrete series terms come from.

The main gist of the paper is to compare $O(3)$ regge theory to the $O(2,1)$ theory. Boyce is very concerned with helicity crossing matrices, brick wall systems, etc. But the ideas are now so familiar to me that I won't bother to look more into his details (yet).

He does quote the Hall-Wightman theorem which seems to say that the M -functions defined on the Lorentz group may be continued onto the complex Lorentz group $SL(2,C) \times SL(2,C)$. He quotes PCT and all that book on this result. Also, shows how to relate M -functions to his various kinds of helicity amplitudes.

The $O(2,1)$ functions are taken from Andrews and Gunion.

To Boyce's credit, this is a very early paper, May 1966 submitted.

To get more detail, I would have to compare this to McKerrill.

Understanding Boyce

① Martin + Spearman. Look back at page 117

$$U(\Lambda) |m, j; \vec{p}, \mu\rangle = \sum_{\mu'} D_{\mu\mu'}^{(j)}(R) |m, j; \Lambda p, \mu'\rangle$$

where $R = \Lambda_p^{-1} \Lambda \Lambda_p = \text{"Wigner" rotation.}$

where $p = \Lambda_p p_0$
 $p' = \Lambda_{p'} p_0 = \Lambda p$

② Then, on page 326, if you let $\Lambda =$ L.T. which stays in the $x-z$ plane, $\phi = \theta$, you get:

$$\begin{aligned} \langle p_c \mu_c; p_d \mu_d | T | p_a \mu_a; p_b \mu_b \rangle &= \langle " | U^{-1}(\Lambda) T U(\Lambda) | " \rangle \\ &= \sum_{a' b' c' d'} (\text{phase}) \cdot D_{a'a}^{(s_a)}(R_a) D_{b'b}^{(s_b)}(R_b) D_{c'c}^{(s_c)}(R_c)^* D_{d'd}^{(s_d)}(R_d) \\ &\quad \times \langle p_c' \mu_c'; p_d' \mu_d' | T | p_a' \mu_a'; p_b' \mu_b' \rangle \end{aligned}$$

where $R_a = \Lambda_{p_a}^{-1} \Lambda \Lambda_{p_a}$, etc.

③ Comments: the above Martin and Spearman equation is only valid for the usual $O(3)$ helicity states. It derives simply from the equation on the top of this page which says an arbitrary LT on a single-particle helicity state always does a Wigner rotation. I do not yet know what a LT does to a " $O(2,1)$ helicity state". But, a quick scratch paper figuring seems to say that equations are exactly the same for $O(2,1)$ helicity state. Only the meaning of the symbol is changed:

$$\begin{array}{lll} p = \Lambda_p^{O(3)} \bar{p} \quad \downarrow \text{rest} & \Lambda_p^{O(3)} = R_z(\phi) R_y(\theta) R_z(\chi) & O(3) \\ p = \Lambda_p^{O(2,1)} \bar{p} \quad \downarrow \text{rest} & \Lambda_p^{O(2,1)} = R_z(\phi) B_x(\beta) B_z(\alpha) & O(2,1) \end{array}$$

Beware: although both these transformations take $\bar{p}$ to same p , they could and do differ by a pre-rotation! Boyce's symbols are:

$$\begin{array}{ll} \Lambda_p^{O(3)} = L_p & p = L_p \bar{p} \\ \Lambda_p^{O(2,1)} = L'_p & p = L'_p \bar{p} \end{array} \quad \Rightarrow \quad (L'_p)^{-1} L_p = S_p \in O(3).$$

In fact, the two transformations differ by a prerotation which is a Y-rotation by amount given in bottom of page 676, first column.

④ We now derive equation (2.3) of Boyce:

$$[|p\rangle \times |\alpha\rangle] = L_p [|\bar{p}\rangle \times |\alpha\rangle]$$

$$[|p\rangle \times |\alpha'\rangle] = L'_p [|\bar{p}\rangle \times |\alpha'\rangle]$$

where $|\alpha\rangle = |\alpha\rangle^{0(3)}$ and $L_p = L'_p S_p$
 $|\alpha'\rangle = |\alpha\rangle^{0(2,1)}$

Thus: $[|p\rangle \times |\alpha\rangle] = L'_p S_p [|\bar{p}\rangle \times |\alpha\rangle]$
 $= L'_p [|\bar{p}\rangle \times S_p |\alpha\rangle]$

But $|\alpha\rangle = |j, \mu\rangle$, so $S_p |\alpha\rangle = \sum_{\mu'} D_{\mu'\mu}^{(j)}(S_p) |j, \mu'\rangle$

Thus: $[|p\rangle \times |\alpha\rangle] = \sum_{\mu'} D_{\mu'\mu}^{(j)}(S_p) L'_p [|\bar{p}\rangle \times |\alpha'\rangle]$

New notation: $[|p\rangle \times |\alpha\rangle] = |p\mu\rangle$
 $[|p\rangle \times |\alpha'\rangle] = |p\mu'\rangle$

Thus we have shown that

$$|p\mu\rangle = L_p |\bar{p}\mu\rangle = L'_p S_p |\bar{p}\mu\rangle = L'_p \sum_{\mu'} D_{\mu'\mu}^{(j)}(S_p) |\bar{p}\mu'\rangle$$

$$= \sum_{\mu'} D_{\mu'\mu}^{(j)}(S_p) L'_p |\bar{p}\mu'\rangle = \sum_{\mu'} D_{\mu'\mu}^{(j)}(S_p) |p\mu'\rangle$$

or to avoid confusion

$$|p\lambda\rangle = \sum_{\mu} D_{\mu\lambda}^{(j)}(S_p) |p\mu\rangle = \sum_{\mu} |p\mu\rangle \langle j\mu | S_p | j\lambda \rangle \quad (2.3) \checkmark$$

What we learn from this calculation is this: the $O(3)$ single particle helicity states (SPHS) are very simply related to the $O(2,1)$ SPHS. In fact, equation 2.3 says that an $O(3)$ SPHS is just a linear combination of all the $O(2,1)$ SPHS with simple coefficients given by the rotation matrix of that y rotation mentioned above.

⑤ Now work on (2.15) + (2.16). First, consider $\Lambda = \text{arb L.T.}$ and we say:

$$\begin{aligned} \textcircled{5.1} \quad \langle p_1 \lambda_1; p_2 \lambda_2 | T | p_3 \lambda_3; p_4 \lambda_4 \rangle &= \langle p_1 \lambda_1; p_2 \lambda_2 | \bar{U}(\Lambda) T U(\Lambda) | p_3 \lambda_3; p_4 \lambda_4 \rangle \\ &= \sum_{\mu_1 \mu_2 \mu_3 \mu_4} D^{S_1}(R_1)_{\mu_1 \lambda_1}^* D^{S_2}(R_2)_{\mu_2 \lambda_2}^* D^{S_3}(R_3)_{\mu_3 \lambda_3} D^{S_4}(R_4)_{\mu_4 \lambda_4} \\ &\quad \times \langle p'_1 \mu_1; p'_2 \mu_2 | T | p'_3 \mu_3; p'_4 \mu_4 \rangle \end{aligned}$$

where $R_i = \bar{\Lambda}_i^{-1} \Lambda \Lambda_i$

$$\begin{aligned} p_i &= \Lambda p_i \quad \bar{p}_i \\ p'_i &= \Lambda p'_i \quad \bar{p}_i \end{aligned}$$

$$\begin{aligned} p'_i &= \Lambda p_i \\ &= \Lambda \Lambda p_i \quad \bar{p}_i \end{aligned}$$

⑤.2 Hold on this; now consider:

$$|p\mu\rangle = \sum_{\mu'} D_{\mu'\mu}^{(j)}(S_p) |p\mu'\rangle' \quad \text{relates } O(3) \text{ SPHS to } O(2,1) \text{ SPHS}$$

Then:

$$|p_3 \mu_3; p_4 \mu_4\rangle = \sum_{\mu'_3 \mu'_4} D_{\mu'_3 \mu_3}^{(S_3)}(S_3) D_{\mu'_4 \mu_4}^{(S_4)}(S_4) |p_3 \mu'_3; p_4 \mu'_4\rangle'$$

Then:

$$\begin{aligned} \langle p_1 \mu_1; p_2 \mu_2 | T | p_3 \mu_3; p_4 \mu_4 \rangle &= \sum_{\mu'_1 \mu'_2 \mu'_3 \mu'_4} D_{\mu'_1 \mu_1}^{(S_1)*}(S_1) D_{\mu'_2 \mu_2}^{(S_2)*}(S_2) \\ &\quad \times D_{\mu'_3 \mu_3}^{(S_3)}(S_3) D_{\mu'_4 \mu_4}^{(S_4)}(S_4) \cdot \langle p_1 \mu'_1; p_2 \mu'_2 | T | p_3 \mu'_3; p_4 \mu'_4 \rangle' \end{aligned}$$

Pause: this equation above merely related amplitudes in the two kinds of helicity states. S_i are the y -rotations of earlier. All the 4-momenta are the same on both sides, so these are two kinds of amplitudes seen in the same frame of reference.

This is not (2.15) however.

(5.3) Now look in on (2.14). Utmost care required here. These equations are for sure the most difficult to keep straight in particle physics.

In the s-channel bws we have four momenta called p_i , each is characterised by a specific boost α_i , and another boost which only appears in two of the four 4-vectors and is called β . This must be the central Toller variable ξ and plays the role of Θ in cms stuff.

In the t-channel cms we have the same particles (modulo antiparticles) with same masses. Now the four 4-momenta are called q_i . Each is characterized by a specific boost χ_i , and two of the particles (1 and 3) also have a rotation angle Θ .

-- Notice that in both cases, it is the particles 1 and 3 which have the extra variable. Very unlike the s-channel cms! I.e., particles 1 and 3 are in the same side of the reaction only in the t-channel. (Consistent with thinking of Toller BCP variables in the s-channel bws).

Now, as already worked out by me, these two "frames" can be connected by a complex Lorentz transformation which Boyce calls $\tilde{R}$, the imaginary z boost. Once this connection is made, a direct comparison of the variables follows. One finds that each α_i relates simply to a χ_i , and of course β relates simply to Θ . This is fairly straightforward, just compare invariants!

Now comes the skinny: notice that

$$p_i = p_i^{(sch\ bws)} = e^{-i\beta\gamma_0} e^{-i\alpha_i\gamma_3} \bar{p}_i = B_x(\beta) B_z(\alpha) \bar{p}_i$$

I.e., $p_i = L_p \bar{p}_i$, except we have taken $\phi=0$ in L_p

$$\text{Similarly, } q_i = q_i^{(t-ch\ cms)} = R_y(\Theta) B_z(\chi_i) \bar{q}_i$$

$$\text{I.e., } q_i = L_q \bar{q}_i$$

$$\begin{aligned} \text{But, } \bar{p}_i &= s\text{-ch incoming} = m_i (1, 0, 0, 0) \\ \bar{q}_i &= t\text{-ch outgo} = -m_i (1, 0, 0, 0) \end{aligned}$$

$$\begin{aligned} \text{Roughly: } p &= L_p \bar{p} & \bar{p} &= m(1, 0, 0, 0) \\ & & p &= m(\cosh\alpha \cosh\beta, \cosh\alpha \sinh\beta, 0, \sinh\alpha) \end{aligned}$$

$$\begin{aligned} q &= L_q \bar{q} & \bar{q} &= -m(1, 0, 0, 0) \\ & & q &= -m(\cosh\chi, \sinh\chi \sin\Theta, 0, \sinh\chi \cos\Theta) \end{aligned}$$

Boyce seems to have a little inconsistency with the overall sign of a vector. From now on I will take ... well look again:

$$p = Rq \Rightarrow L_p = R L_q \rightarrow \text{acting on } \bar{q}$$

↙ acting on $\bar{p}$

if you include
relativistic in (2.12).

OK consider the equation (2.14) which says: $L'_p = R L_q$. In words, here is how I read this thing: "In the t-channel, you would normally start with $\bar{q}$ as a past timelike vector, so $(-\bar{q})$ is a relevant FTL particle in the t-channel reaction. Then if you act on this with L_q , you get the particle as seen in the t-channel standard cms frame. It then you apply complex LT called R to this, you get a certain vector called p, which is the 4-momentum of the antiparticle in the crossed s-channel reaction, but brick wall, not cms. All of this is equivalent to applying the operator L'_p directly to $\bar{p}$ and doing everything in the s-channel brick wall system.

Upshot: (I am unable to follow Boyce's dialogue, also am unable to self-derive his equation (2.15). But I can get fairly close: the point is this: equation 2.15 is the usual rule. (say from Martin and Spearman) except applied to a certain complex Lorentz transform). What we find is that the Wigner angles are precisely the same angles (rotations) which appear bac, in my section 5.2 on page 2.

But, equation 2.15 relates the [s channel cms $O(3)$ helicity states amplitude] to the [t-channel cms $O(3)$ helicity states] amplitude. On the other hand, the other result which looked the same related [the s-channel $O(3)$ helicity states amplitude] to the (s-channel $O(2,1)$ helicity states amplitude]. (Either cms or bws):

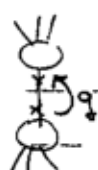
A complete summary can be found in equations (2.17) and (2.18).

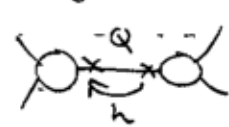
J.F. Boyce (IC)

"Relation of $O(2,1)$ p.w.e. to regge rep"

JMP 8, 675 (1967)

1. Introduction. Credit is given to J. Strathdee for the approach. The results are: principal $O(2,1)$ series = background regge term; discrete series = nonsense terms. Square integrability is needed, so all regge singularities must be left of $\text{Re } s = -\frac{1}{2}$. He relies on Andrews + Gunson for the $SL(2, R)$ representations.

2. "Cms" and "Brick-Wall Frames". A long section of a full 3 pages. We are reminded of the usual idea of using a cms-frame ($\sqrt{s}, 0, 0$) and then projecting the S-matrix onto the little group reps, $O(3)$.
I.e., we take: - then we can write $A = A(g)$, then we use
 The group theoretic expansion onto D_{mn} , which in turn $\Rightarrow$ states $|Jm\rangle$. In this case, one wants to know $|p\rangle = U(g) B_z(p) |1\rangle$ helicity states. I.e., you want to know how to $O(3)$ rotate helicity states.

However, you can just as well write $A = A(h)$ where $h \in SO(2,1)$ as in
 Then $A = A(h)$. In this case, both frames have $q = (q, 0, 0, \sqrt{t})$, i.e., they are "brick wall".
I.e., you want to know $|p\rangle' = U(h) B_z(p) |1\rangle$, i.e., you want to use $SO(2,1)$ "rotated" helical states.
Let's look more closely at L_p and L_p' .

L_p : start with $\hat{p} = (m, \vec{0}) + \text{spin } \lambda \text{ on } \hat{z}$. Boost z by γ , then rotate x by θ , then R_z by ϕ . This gives $|s, \theta, \phi, \lambda\rangle$ state, some $|p\rangle$.
The idea is that you can also go from $\hat{p}$ to the same $|p\rangle$ via L_p' .

L_p' : start $\hat{p} = (m, \vec{0})$, λ . Boost z by α this time. Then x -boost by β , and R_z by ϕ . To end up the same p^μ , need:

$$\begin{aligned} & m(\chi\lambda, \text{sh}\gamma, S_0\phi, \text{sh}\gamma, S_0\phi, \text{sh}\gamma, S_0\phi) \\ & = m(\chi\alpha\chi\beta, \chi\alpha\chi\beta\phi, \chi\alpha\chi\beta\phi, \chi\alpha\chi\beta\phi, \text{sh}\alpha) \end{aligned}$$

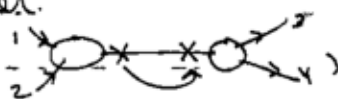
So $L_p \hat{p} = p = L'_p \hat{p}$. This does not mean that $L_p = L'_p$, because they could differ by a pre-rotation $\alpha(3)$. Such a pre-rotation would not affect $\hat{p}$, of course. I.e.,

$$L_p = L'_p \times g_1 \quad \text{for some } g_1 \in \alpha(3).$$

a. $(L'_p)^{-1} L_p = S_p = g_1 \in \alpha(3)$. You can of course compute this particular g_1 in terms of α, β etc. It turns out to be a y-rotation. See p626 for explicit value of Θ .

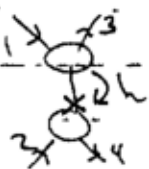
Now, let $T(p)$ be any in some arbitrary frames. Then L_1 and L_2 takes us to cms and bws frames. The scalarity of the S-matrix then justifies (2.4). We do not leave the s-ch process even, however.

In the particular cms of interest



p_1 and p_2 appear at O as in

(2.5). He calls $p_i^{(cms)} \equiv \hat{q}_i$



In the particular bws,

1 and 3 appear off the

t-z plane. $p_i^{(bws)} \equiv p_i$, as in (2.8).

Of course, from s and t you can compute the $\hat{\alpha}_i, \hat{\Theta}$ of the cms, and the α_i, β of the bws. Then you can compare them. Result:

$$\hat{\Theta}(s,t) = +i\beta(t,s)$$

$$\hat{\alpha}_i(s,t) = \alpha_i(t,s) - \frac{1}{2}i\pi \quad \text{etc.} \quad \text{as in (2.9)}$$

③ Next, after all this, we finally discuss t-ch cms momenta, see (2.10). How do you go from t-ch cms to s-ch bws? Not a proper L.T., certainly. The answer must take $(\sqrt{E}, 0, 0, 0)$ to $(0, 0, 0, \sqrt{E})$.

Now we can show, again from st ,

$$\beta(s,t) = i\Theta(s,t)$$

$$\xi = \chi\beta = \cos\Theta = z$$

$$\gamma_i = \alpha_i + \frac{1}{2}i\pi, \text{ etc.}$$

We require an R such that:

s-ch bws.

$$p = Rq \leftarrow \text{t-ch cms}$$

Solving for R using these p 's + q 's: $R = e^{\frac{i}{2}\pi J_{03}}$, a non-proper z -boost (an "imaginary z boost").

④ Now we start transforming amplitudes, as in M+S. First, do a "real Lorentz xform" $\Lambda(A)$, $A \in SL(2, \mathbb{C})$. This causes rotation matrix spin sums via $D^{(3,0)}_{\frac{1}{2}}(\Lambda)_{\lambda_1}^{\lambda_2}$. I.e., if $\Lambda \in O(3)$, then this would be a usual rotation matrix.

Notice that $SL(2, \mathbb{C})$ is like $SO(4)$ or $SO(3,1) \sim$ a 6-parameter group $\sim$ the proper Lorentz group. But, if you are going to allow all 6 parameters to go complex, then you are dealing with $SL(2, \mathbb{C}) \otimes SL(2, \mathbb{C})$, whose elements are, say, (A, B) .

The fact that you can in fact transform scat. amps on this 12 parameter group is the Bargmann-Hall-Wightmann Theorem.

Well, discussion becomes unclear, then emerges with the usual Toller-Treiman Wick business. I.e., (2.15) relates the s-ch. bus to the t-ch. cms via certain separate rotations on each momenta, namely those $S_p = L_p^{-1} L_p$ used earlier.

The point is: we can do either $O(2,1)$ or $O(3)$; then compare. Maybe the supp. series will become significant.

In closing, he sums his argument to come: take $T_{\mu\nu}(s, t)$, the $O(2,1)$ type amp, and expand on $O(2,1)$ as in (2.19) using principle plus discrete series. The rep functions used are called d 's. (projection shown in (2.20).) Then do an inverse SWT, also do crossing, and your result is (hopefully) the t-ch. p.w.e. The discrete series is actually necessary!

3. Normal t-ch partial-wave expansion

① I think $\bar{T}(s, t)$ refers to the t-channel process amplitude, whereas $\langle q \dots | T(s, t) | \dots \rangle$ refers to the s-ch. process amp continued to the t-ch. cms system. These two objects are related by crossing matrices as in (3.1) [The Treiman-

Wide helicity crossing matrices.

② Now given $\bar{T}$ (s.t.), you can do standard $\alpha(3)$ pwa in terms of $d_{j\mu}^i(\cos\theta)$. Inversion is (3.3). Note that, as always, A_{μ} has no azimuthal dependence in the helicity formalism, so all D^j reduce to d^j .

Somehow, $\bar{T}$ is broken into Left + Right pieces $\bar{A}, \bar{B}$, and each gets a FGP. Notice how $\alpha \rightarrow \alpha_{\mu}$ in these FGP's. See (3.6) and (3.7).

4. Definition of $\alpha(2,1)$ helicity states.

① He opens by defining $\alpha(2,1)$ two particle helicity states in analogy with $\alpha(3)$, as in MS. Instead of connecting these $|p_1, \lambda_1; p_2, \lambda_2\rangle$ to $|0M\rangle$ states, we connect them to $|q, m\rangle$ $\alpha(2,1)$ up basis. The main result here is:

$$|p_1, \lambda_1; p_2, \lambda_2\rangle = N \sum_{q, m} N(q) |\alpha; \lambda_1, \lambda_2; q, m\rangle D_{m, \lambda_1, \lambda_2}^q(\beta, \beta, 0)$$

where $D_{mm'}^q(\beta, \beta, \psi) = e^{-im\psi} d_{mm'}^q(\chi\beta) e^{-im'\psi}$ and

you get

these $d_{mm'}^q(\chi\beta)$ by merely continuing the $d_{mm'}^j(\cos\theta)$ to z above 1. That is why $d_{00}^q(\chi\beta) = P_q(\chi\beta)$, eq.

② The rest of this section deals with computing normalizations.

Eg,

$$\langle \alpha; \lambda_1, \lambda_2; q, m | \alpha'; \lambda_1', \lambda_2'; q', m' \rangle = 2\pi(\alpha - \alpha') \delta_{qq'} \delta_{mm'}$$

I think α is some extra continuum index.

5. $\alpha(2,1)$ decomposition in the s-channel.

First, $z = \beta$, but she keeps them separate anyway. Then (5.2) shows how crossing relates t and s-ch processes. Then you can insert $\sum_q |q, m\rangle \langle q, m|$ twice into $\langle s | T | i \rangle$ to get your $\alpha(2,1)$ expansion. Define $T^q = T(q)$ as the "pwa", see (5.3).

The main "expansion" result now is shown

in (5.5) as: $\bar{T}_\mu(s,t) = \sum_q (2q+1) \bar{T}_\mu(q,t) d_{j\mu}^q(s)$

② Now he gets into these $\alpha(2,1)$ reps. His favorite reference seems to be not Bargmann but:

→ Barut + Fronsdal, Proc. Roy. Soc. (London) A287, 532 (1965)

He shows how the $\alpha(2,1)$ reps fit into the j -plane, like $j = -\frac{1}{2} + i\epsilon$. This may be a good place to look!

③ The Peter-Weyl Theorem is a statement of the completeness of the group rep. functions with regards to expanding a square-integrable function. This theorem, 1934, was designed for compact groups, like $O(3)$. On page 639, Bargmann gave the analog of Peter-Weyl for $\alpha(2,1)$. The result is that you only need the principle continuous series, plus the $j > \frac{1}{2}$ part of the discrete series. I.e., function set is complete ~~or~~ without the exceptional series.

So we may omit:

a) The trivial rep at $j = -1$

b) The supplementary series in $j = (-1, -1/2)$

c) the $D^{(2)}(-\frac{1}{2})$ discrete

Aside: This is the only place I have heard this so clearly stated. Hours of work are worth this one result. ✓

6. Analytic Continuation of $\alpha(2,1)$ decomposition.

① First, for usual signature reason, let $\bar{T} = \bar{T}^D(s,t) + \bar{T}^E(-s,t)$. Then let $\bar{A}(j,t)$ be projection of $\bar{T}^D$, $\bar{B}$ be that of $\bar{T}^E$. Then the idea is:

$$\bar{T}(s) = \int dj \dots [\bar{A}(j) + \bar{B}(j)] d_{j\mu}^j(s) \quad \leftarrow \begin{array}{l} \text{for } \alpha(2,1) \\ \text{principle} \\ \text{series} \\ \text{piece} \end{array}$$

+ discrete terms.

There is some tricky business here & these projections look like so:

$$\bar{A}(j) \approx \sin \pi(j-1) \int_0^\infty ds \, e_{j\mu}^j(s) \bar{T}^D + \int_0^1 ds \, d_{j\mu}^j(s) \bar{T}^E$$

a signature-like fact?

② Now notice: your $O(3)$ & d. series wants to start at $\text{Re } j = M; \frac{1}{2}$ not $\text{Re } j = -\frac{1}{2}$, because we do not want to include nonsense stuff. Therefore, the $O(2,1)$ contour must be shifted right from its $\text{Re } j = -\frac{1}{2}$ position.

③ As the contour is shifted to the right, we pick up contributions from j -plane kinematic poles present in the $C_{j\mu}$ functions. The sum of the residues of these poles exactly cancels our discrete series $O(2,1)$ terms!

④ Then, modulo details, the remaining contour closes down and duplicates the $O(3)$ series!

2. Conclusion.

The usual AS problem is noted that you cannot use this analysis for regge poles to the right of $j = -\frac{1}{2}$. Thus, Boyce restricts to large negative t .

Can I make the Casimir Theory work?

① Let's first review his basic idea. There are two parts:

$$a_l = \int_{-\infty}^{\infty} dx \underbrace{P_l(x)}_{\text{even}} \underbrace{\pi \cot(\pi x)}_{\text{odd}} \quad \tilde{a}_l = \int_{-\infty}^{\infty} dx Q_l(x) a(x)$$

Now, use this fact: $\pi \cot(\pi x) P_l(x) = Q_l(x) - Q_{l-1}(x)$

$$a_l = \tilde{a}_l - \tilde{a}_{l-1} \quad \leftarrow \text{relates these parts.}$$

Define: $\bar{a}_l \equiv \tilde{a}_l + \tilde{a}_{l-1}$ Note: $a_l + a'_l = 2\bar{a}_l$

Now: $a'_l = \int_{-\infty}^{\infty} dx a(x) [Q_l(x) + Q_{l-1}(x)]$

Note: $a'_{l-1} = a'_l$

If $l = \text{critical or } \frac{1}{2}$ then $a'_l = a'_l^*$
Let $l = -\frac{1}{2} + i\sigma$. Then

$$a'(\sigma) = a'(-\sigma)$$

Thus, $a'(\sigma)$ is an even function of σ .

$$l = -\frac{1}{2} + i\sigma \quad 2l+1 = 2i\sigma$$

② The expansion is: $F(z) = \sum_{-\frac{1}{2}+i\sigma}^{\frac{1}{2}+i\sigma} dl \frac{(2l+1) P_l(z) a_l}{\tanh \pi l}$

In terms of σ , this is:

$$F(z) = \int_{-\infty}^{+\infty} d\sigma \underbrace{(2i\sigma)}_{\text{even}} \underbrace{P_{-\frac{1}{2}+i\sigma}(z)}_{\text{even}} \underbrace{\frac{1}{\coth(\pi\sigma)}}_{\text{even}} a(\sigma)$$

So only even $a(\sigma)$ can contribute.

misc!

How are the various spin amplitudes related?

1. Types I know of are:

- a) helicity amplitudes
- b) canonical amplitudes
- c) Stappian M-functions (spinorial amplitudes)
- d) Toller M-functions.

2. For the first two, my reference is McKensell:

$$L|p, \nu\rangle_H = D_{\nu\nu'}(R(L, P)) |p, \nu'\rangle_H$$

Invariance says:

$$\langle p_3 \nu_3, p_4 \nu_4 | S | p_1 \nu_1, p_2 \nu_2 \rangle_H = \prod_{i=1}^4 D_{\nu_i \nu'_i}(R(L, P_i)) \langle p_3 \nu'_3, p_4 \nu'_4 | S | p_1 \nu'_1, p_2 \nu'_2 \rangle_H$$

General amplitude = $\langle H(p_3) \tilde{p}_3 \nu_3, H(p_4) \tilde{p}_4 \nu_4 | S | H(p_1) \tilde{p}_1 \nu_1, H(p_2) \tilde{p}_2 \nu_2 \rangle$

Why not think of $H(p_i)$ as Toller's $U(a_i)$? As a result, above, you actually figure out:

$$U(a) = \{H(p) \equiv R(p) Z(p) \equiv L(p) \equiv \text{asid boost}\}$$

helicity
amplitude

So far it looks as if the canonical and helicity formalisms are two special cases of Toller's formalism.

Let $U(a_i) \equiv H(p_i)$ etc. then

$$a_3 = R(p_3) Z(p_3) \text{ etc.}$$

$$\langle p_3 \nu_3, p_4 \nu_4 | S | p_1 \nu_1, p_2 \nu_2 \rangle_H = M_{\nu_3 \nu_4 \nu_1 \nu_2}(a_3, a_4, a_1, a_2)$$

$$= \langle U(a_3) p_3 \nu_3, U(a_4) p_4 \nu_4 | S | U(a_1) p_1 \nu_1, U(a_2) p_2 \nu_2 \rangle_H = M_{\nu_3 \nu_4 \nu_1 \nu_2}(a a_3, a a_4, a a_1, a a_2)$$

$$= \prod D_{\nu_i \nu'_i}(R(p_i, a_i)) \langle p'_3 \nu'_3, p'_4 \nu'_4 | S | p'_1 \nu'_1, p'_2 \nu'_2 \rangle_H$$

$$= \prod D_{\nu_i \nu'_i}(R) : M_{\nu'_3 \nu'_4 \nu'_1 \nu'_2}(a a'_3, a a'_4, a a'_1, a a'_2) ??$$

Catch must be that $a'_3 \neq a a_3$.

$$R(p_3) Z(p_3) \neq U(a) R(p_3) Z(p_3) ?$$

$$= M_{\nu_3 \nu_4 \nu_1 \nu_2}(R(p_3) Z(p_3), R(p_4) Z(p_4), R(p_1) Z(p_1), R(p_2) Z(p_2))$$

I see something now. Although it is true that any helicity amplitude can be written as a single Toller function element, the reverse is not true. On bottom of last page I have shown the first part of this statement. The catch is that general a_i will contain pre-rotations!

$$U(a_i) = R(u_i) Z(\xi_i) R(u_i')$$

Thus:

$$\begin{aligned} & \mathcal{M}_{\nu_1 \nu_2 \nu_3 \nu_4}(a_1, a_2, a_3, a_4) \\ &= \prod_{i=1}^4 D_{\nu_i \nu_i'}(u_i') \cdot \mathcal{M}_{\nu_1' \nu_2' \nu_3' \nu_4'}(u_1 \xi_1, u_2 \xi_2, u_3 \xi_3, u_4 \xi_4) \\ &= \prod_{i=1}^4 D_{\nu_i \nu_i'}(u_i') \langle p_3 \nu_3' p_4 \nu_4' | S | p_1 \nu_1' p_2 \nu_2' \rangle_u \end{aligned}$$

As a function on $SL(2, \mathbb{C})^4$, the M-function of Toller is free of all singularities, whereas as a function of momenta R^{16} or whatever, the helicity amplitudes do have singularities! The Toller M-functions are defined on a better "surface".

Stapp: the Stapp functions are shown on page 270 of the Toller Conspir. paper with Cozenza etc. Claim is that both Stapp and Toller are singularity-free.

If you want, you could view the helicity amplitudes as Toller functions restricted to a certain submanifold of $SL(2, \mathbb{C})^4$. In each $SL(2, \mathbb{C})$, you only allow elements of the form $R(p)Z(p)$. I.e., you never allow pre-rotations.

If you restrict to this submanifold, then to express Lorentz invariance you have to reprocess the amplitude to get it back on this manifold. This reprocessing then brings out the p-dependent Wigner rotations and resultant kinematic singularities.

Problem: what does elastic unitarity look like in terms of Toller M-functions?

$$\delta(\text{ext}) M_{m_1 m_2 m_3 m_4}(a_1, a_2, a_3, a_4)$$

$$= \langle U(a_3) \phi_{m_3} \otimes U(a_4) \phi_{m_4} | S^{-1} U(a_1) \phi_{m_1} \otimes U(a_2) \phi_{m_2} \rangle$$

① Seek completeness relation for these states

② First, go back to helicity basis problem use integration for the sum over

$$\delta(\text{ext}) f^{\dagger}(p_1, p_2, p_3, p_4) = \langle p_1 p_2 | S^{-1} | p_3 p_4 \rangle = \langle p_1 p_2 | T | p_3 p_4 \rangle$$

$$\text{For single-particle states } \frac{1}{\sqrt{m_1}} \int d^3 p \delta^3(p - m_1) \phi(p) = 1$$

$$\langle p_1 p_2 | = \int d^3 p \delta^3(p - m_1) \phi(p)$$

$$\# \frac{1}{\sqrt{m_1}} \int d^3 p \delta^3(p - m_1) \phi(p) = \int d^3 p \delta^3(p - m_1) \phi(p)$$

③ where $|p_1 p_2\rangle = |p_1\rangle \otimes |p_2\rangle$ Now add $\int d^3 p \delta^3(p - m_1) \phi(p)$

$$1 = \int d^3 p \delta^3(p - m_1) \phi(p) \int d^3 p \delta^3(p - m_1) \phi(p)$$

$$\text{So } 1 = \int d^3 p \delta^3(p - m_1) \phi(p) \int d^3 p \delta^3(p - m_1) \phi(p)$$

④ Try this in $S^{\dagger} S = 1$ Let $S^{-1} = i T$ then: $i(T^{\dagger} - T) = i T^{\dagger}$

$$\text{So } \langle p_1 p_2 | (T^{\dagger} - T) | p_3 p_4 \rangle = \int d^3 p \delta^3(p - m_1) \phi(p) \int d^3 p \delta^3(p - m_1) \phi(p) \langle p_1 p_2 | T | p_3 p_4 \rangle$$

Now show delta:

$$\delta(\text{ext}) f^{\dagger}(p_1, p_2, p_3, p_4) = \delta(\text{ext}) f^{\dagger}(p_1, p_2, p_3, p_4) = \int d^3 p \delta^3(p - m_1) \phi(p) \int d^3 p \delta^3(p - m_1) \phi(p)$$

$$\int d^3 p \delta^3(p - m_1) \phi(p) \int d^3 p \delta^3(p - m_1) \phi(p) = \int d^3 p \delta^3(p - m_1) \phi(p) \int d^3 p \delta^3(p - m_1) \phi(p)$$

is it like is you usual thing

$$\int d^3 p \delta^3(p - m_1) \phi(p)$$

8. Werle(1966): the ultimate book on partial wave analysis. ~~See~~ contents sheet in Books binder. ~~Werle~~ does partial wave expansions in both JWick and-JLS formalism, plus much much more.

9. Barut(1967): does not really have much on partial wave expansions except for the specific cases of PI-PI and PI-N and N-N. On the other hand, the M-amplitudes are defined and use in a general sense, whereas Werle does not use these.

Martin and Spearman: helicity versus spin.

1. Unfortunately, MS never bother to compare the two formalisms. They have built helicity into their machine at a very early stage. In particular, look back at equation (3.28). Remember that Λ_p in general is first a z boost, then a rotation. If the state $|\alpha\rangle$ where a "spin" state referred to a fixed z axis, then you would expect that a rotation would cause mixing. But there is no mixing shown in (3.28), so we are forced to interpret the α in the RHS as component of the spin on a new z axis. Thus, we are automatically into the helicity formalism before we know it. The Wigner rotation is predicated, therefore, on helicity notation.
2. The fact that there is mixing in (3.47) seems pure accident. At $p=0$ there is some question about what happens in the Wigner rotation. If you let $p \rightarrow 0$ then the Wigner rotation is always just the identity, as shown in (3.51). But suddenly, as p goes to 0, this Wigner rotation suddenly jumps to be equal to the rotation itself.
3. A regular spin formalism would have no Wigner rotation. See attached sheet for regular spin formalism.

Regular Spin Formalism

① SPS is $|m, \vec{p}\rangle \otimes |j, \mu\rangle$. Consider this as direct product of two Hilbert spaces. For example,

$$|m, \vec{p}\rangle = \sum_{LM} |pLM\rangle \langle pLM|m, \vec{p}\rangle$$

② Apply a rotation to $|LM\rangle \otimes |j, \mu\rangle$ and what do you get?

$$J_3 = J_3^{(1)} + J_3^{(2)} \rightarrow e^{i\alpha J_3} = e^{i\alpha J_3^{(1)}} \otimes e^{i\alpha J_3^{(2)}}$$

$$\Rightarrow R[|LM\rangle \otimes |j, \mu\rangle] = R|LM\rangle \otimes R|j, \mu\rangle$$

$$\begin{aligned} \text{Now, } R|LM\rangle &= \sum_{LM'} |LM'\rangle \langle LM'|R|LM\rangle = |LM'\rangle D_{MM'}^L(R) \\ R|j, \mu\rangle &= \sum_{j, \mu'} |j, \mu'\rangle \langle j, \mu'|R|j, \mu\rangle = |j, \mu'\rangle D_{\mu\mu'}^j(R) \end{aligned}$$

$$\text{Thus: } R[|LM\rangle \otimes |j, \mu\rangle] = D_{MM'}^L(R) D_{\mu\mu'}^j(R) [|LM'\rangle \otimes |j, \mu'\rangle]$$

③ Now go back to:

$$\begin{aligned} R|\vec{p}\rangle &= \sum_{LM} R|pLM\rangle \langle pLM|\vec{p}\rangle = \sum_{LM'} D_{MM'}^L(R) |pLM'\rangle \langle pLM|\vec{p}\rangle \\ &= \sum_{LM'} R|pLM\rangle \langle LM|\vec{p}\rangle \langle pLM|\vec{p}\rangle = R|\vec{p}\rangle = |\vec{p}'\rangle \end{aligned}$$

OK, so result is:

$$R|m, \vec{p}; j, \mu\rangle = \sum_{\mu'} D_{\mu\mu'}^j(R) |m, \vec{p}; j, \mu'\rangle$$

Spin is always referred to a fixed axis, as particles are actually rotated.

$$\begin{aligned} \text{④ Also, } \langle m, \vec{p}; j, \mu | R^\dagger &= \sum_{\mu'} D_{\mu\mu'}^j(R)^* \langle m, \vec{p}; j, \mu' | \\ &= \langle m, \vec{p}; j, \mu | R^\dagger = \langle m, \vec{p}; j, \mu | \quad (\text{ie, } U(R)) \end{aligned}$$

$$\text{So } \langle m, \vec{p}; j, \mu | R^\dagger = \sum_{\mu'} D_{\mu\mu'}^j(R)^* \langle m, \vec{p}; j, \mu' |$$

⑤ Now apply these rules to an amplitude: -----

$$\begin{aligned}
 & \langle p_3 m_3; p_4 m_4 | T | p_1 m_1; p_2 m_2 \rangle \\
 & \quad \quad \quad \uparrow \quad \uparrow \\
 & \quad \quad \quad R^{-1} \quad R \\
 & = \sum_{m'_3 m'_4 m'_1 m'_2} D_{m'_3 m_3}^{(S_3)}(R) D_{m'_4 m_4}^{(S_4)}(R) D_{m'_1 m_1}^{(S_1)}(R) D_{m'_2 m_2}^{(S_2)}(R) \\
 & \quad \quad \quad \times \langle p'_3 m'_3; p'_4 m'_4 | T | p'_1 m'_1; p'_2 m'_2 \rangle
 \end{aligned}$$

where $p'_i = R p_i$. This is statement of rotational invariance, in the spin formalism. Lots of mixing.

→ ⑥ Compare to helicity result:

$$\langle p_3 m_3; p_4 m_4 | T | p_1 m_1; p_2 m_2 \rangle = \langle p'_3 m_3; p'_4 m_4 | T | p'_1 m_1; p'_2 m_2 \rangle$$

(True as long as no "winding" of a particle about z axis)
(rotations only; boosts do mix)

Origin of the Wigner Rotation

1. Consider two states in the direct product Hilbert Space $\mathcal{H} = \mathcal{H}_p \otimes \mathcal{H}_s$. We define these states relative to the standard momentum $\vec{p}$.

$$|p\rangle \otimes |\alpha\rangle = U(\Lambda_p) [|\vec{p}\rangle \otimes |\alpha\rangle]$$

$$|p_2\rangle \otimes |\alpha_2\rangle = U(\Lambda_{p_2}) [|\vec{p}\rangle \otimes |\alpha_2\rangle]$$

The idea here is that as you boost and rotate out to your desired momentum p , the little hilbert space state is not changed. I.e., $|\alpha\rangle$ means whatever it meant in the standard momentum state. For example, perhaps $\vec{p} = (m, 0, 0, 0)$ and $|\alpha\rangle$ referred to a spin state of particle at rest.

2. The space $\mathcal{H}$ is closed, so we expect that,

$$U(\Lambda) [|\vec{p}\rangle \otimes |\alpha\rangle] = \text{some } [|\vec{p}'\rangle \otimes |\alpha'\rangle].$$

$$\text{So: } U(\Lambda) [|\vec{p}\rangle \otimes |\alpha\rangle] = [|\vec{p}'\rangle \otimes |\alpha'\rangle]$$

$$\hookrightarrow U(\Lambda_p) [|\vec{p}\rangle \otimes |\alpha\rangle] \hookrightarrow U(\Lambda_p) [|\vec{p}\rangle \otimes |\alpha\rangle]$$

$$\text{So: } U(\Lambda) U(\Lambda_p) [|\vec{p}\rangle \otimes |\alpha\rangle] = U(\Lambda_p') [|\vec{p}\rangle \otimes |\alpha\rangle]$$

$$\Rightarrow U(\Lambda_p^{-1} \Lambda \Lambda_p) [|\vec{p}\rangle \otimes |\alpha\rangle] = [|\vec{p}\rangle \otimes |\alpha\rangle].$$

3. If this transformation takes you from one rest state to another, it must be a rotation, call it R_W , the Wigner rotation. This is where the $|\alpha\rangle$ states have their meaning, so we conclude,

$$U(R_W) |\alpha\rangle = |\alpha'\rangle = D(R_W) |\alpha\rangle$$

Of course we have in mind that $|\alpha\rangle$ = a spin state, so we could write this out in terms of rotation matrices.

4. What is the meaning or interpretation of a Wigner rotation? Somehow, it is the transformation "felt" by the spin state when the full state undergoes some arbitrary LT. Notice that, for fixed p , each LT Λ maps into a rotation R_W .
5. However, if Λ itself happens to be a rotation, then the corresponding Wigner rotation is the identity. So, when you look at,

$$U(\Lambda) [|\vec{p}\rangle \otimes |\alpha\rangle] = \sum_{\mu} D_{\mu\alpha}^{(s)}(R_W) [|\vec{p}'\rangle \otimes |\alpha_\mu\rangle]$$

$$= [|\vec{p}'\rangle \otimes D(R_W) |\alpha\rangle]$$

you can verify that helicities are not "mixed" by a rotation. This is also obvious more directly since,

$$\begin{aligned} R(|p m s\rangle) &= U(R) \underbrace{U(R(\vec{p}))}_{\Lambda_p} \sum_p |m \vec{0} s\rangle \\ &= U(R \cdot R(\vec{p})) \sum_p |m \vec{0} s\rangle = |m p' s\rangle \end{aligned}$$

Martin and Spearman: constraints

M+S chose to follow the method of Trueman(1968). Prior to that paper, the kinematic singularities of the helicity amplitudes were located via the helicity crossing relations. It turns out that this method is rather circuitous and, as was shown by Trueman, the singularities are really caused by the helicity state definitions themselves.

It might be helpful to review the presentation of Collins and Squires on this. By looking directly at the partial wave expansion with the d_{ab} functions, they quickly isolate the $z = \pm 1$ singularities. These are in fact the only singularities in z , but there are others in s and t . These are obtained in OS by the crossing method and all results are stated. One sees in these formulae all the various threshold and pseudo threshold and $s=0$ singularities.

Now we can review Martin and Spearman's review of Trueman's 1968 unified approach. First, they write down all relevant kinematics. Then, they divide the singularities into four categories:

- 1) initial and final thresholds and pseudothresholds. These are values of s at which the sine and cosine of the scattering angle both blow up. (For fixed t .)
- 2) At $s = 0$. Here, depending on the mass structure, the initial and final cms momenta can blow up, along with their energies. Also, the sine of the scattering angle has a square root zero.
- 3) At $\phi(s,t) = 0$. Again, the sine has a SQRT zero in ϕ . These singularities are called boundary singularities.
- 4) In the variable t , the only singularities occur when $\phi(s,t) = 0$. Then there is the same situation as in 3 above.

3.1 Threshold and Pseudothreshold — General Mass Case

Final state threshold. Here, we find that the singularity has the form:

$$T_{cd,ab}(s,t) \sim a_{nJ,ab} \times \left(\sqrt{s - (m_c + m_d)^2} \right)^{n-J}$$

Of course we want the largest value of J and the smallest n to get the real power of the singularity. The maximum value of J in this context is $s_c + s_d$ and the minimum value of n is zero. However, the singularity may not be all that bad because parity may cause the leading a_{nJ} to vanish. The parity condition is:

$$a_{nJ,a-b}(s,t) = \eta e^{i\pi\lambda} (-1)^{n+J-s_c-s_d} a_{nJ,ab}(s,t)$$

It turns out that this condition which applies to the all-different-masses case only actually has a different effect. It allows one to relate the actual values of the different helicity amplitudes near the singularity. Thus, this condition will later lead to kinematic "constraints." In the elastic case a similar condition will lead to the effect mentioned above of changing the power of the singularity.

Final State Pseudothreshold. Similar analysis and similar results:

$$T_{cd,ab}(s,t) \sim a_{nJ,ab} \times \left(\sqrt{s - (m_c - m_d)^2} \right)^{n-J} \quad m_d > m_c$$

$$a_{nJ,a-b} = (-1)^{2s_c} \times [\text{same factors}] a_{nJ,ab}$$

Initial threshold. Similarly,

$$T_{cd,ab} \sim a_{n\sigma cd} (\sqrt{s - (m_a + m_b)^2})^{n-\sigma}$$

with $a_{n\sigma-c-d} = \eta e^{i\pi\mu} (-1)^{n+\sigma-s_a-s_b} a_{n\sigma;cd} (-1)^{2s_c+2s_d}$

Initial pseudothreshold:

$$T_{cd,ab} \sim \overline{a_{n\sigma cd}} (\sqrt{s - (m_a - m_b)^2})^{n-\sigma}$$

with $a_{n\sigma-c-d} = [abme] (-1)^{2s_a} \overline{a_{n\sigma cd}}$

3.2 Elastic Scattering: threshold and pseudothreshold.

Here, the a and $\bar{a}$ type amplitudes have the same parity and time reversal restrictions, so let's treat the two thresholds at once. We find:

$$T_{cd,ab} \sim a_{n\sigma,s's} (\sqrt{s - (m_a \pm m_b)^2})^{n-\sigma} \quad m_b > m_a$$

with

Parity: $a_{n\sigma,s's} = \eta (-1)^{\sigma+n-2s_a-2s_b} a_{n\sigma,s's}$

Time Reversal: $a_{n\sigma,s's} = (-1)^{s+s'+n-2s_a-2s_b} a_{n\sigma,s's'} \uparrow$

In the special elastic case of equal masses, the pseudothreshold hits at $s=0$. In the above elastic equations, s and s' represent possible values generateable from s_a and s_b . In turn, J are possible values obtainable from s and s' .

3.3 Singularities at $s = 0$.

- (a) with all masses different, it turns out there are no $s=0$ type singularities
- (b) in the AA BB mass situation, there is no singularity, but there is a constraint condition on the amplitudes at $s=0$. In the NN case, this condition is known as the NN conspiracy. Also happens in equal mass. No mention of elastic here.

3.4 Physical Region Boundary Singularities.

These are quickly identified and removed in the standard way:

$$T_{cd;ab} = (\cos \frac{1}{2}\theta)^{|\lambda+\mu|} (\sin \frac{1}{2}\theta)^{|\lambda-\mu|} \left(\begin{array}{c} \uparrow \\ T_{cd;ab} \end{array} \right) \leftarrow \text{free of sing.}$$

3.5 Here, we learn the general procedure for constructing clean amplitudes, including the need to use parity eigenstates in the general mass case. Good, examples, and the general results are concisely stated. Hint at MacDowell.

A Review of Final State Thresholds, MSip348

I think a better understanding of this one case will help. We have a state $|q\theta\theta, m\rangle$. Some obvious boosts exist.

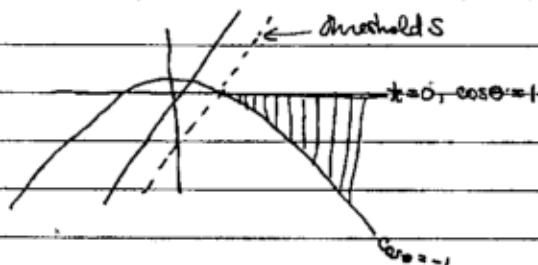
At threshold, both p_c and $p_d \rightarrow 0$
This leads to isolation of singularity:

$$p_c = m_c \sinh \xi_c$$

$$p_d = m_d \sinh \xi_d$$

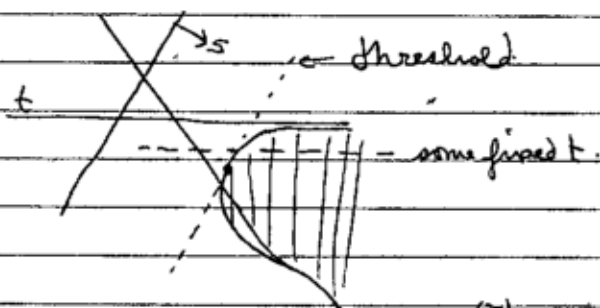
$$* T_{cd,ab} = \sum_{c,d} d_{c,c}^{(S_c)}(\theta) \underbrace{d_{d,d}^{(S_d)}(\theta)}_{\text{here they are}} \underbrace{B_{cd,ab}}_{\text{free of sing.}}$$

Why do these d's have singularities? Because, for fixed t , threshold causes $\cos\theta \rightarrow 0$. This in turn causes $\sin\theta \rightarrow \infty$ and $\cos\theta \rightarrow \infty$, naively;



If we go to threshold along constant $t=0$, then I do not see why there should be any problem. (πN case! But any other fixed t is trouble.

More generally (general mass case)



here, for some fixed t , as you go to "threshold", you are not in the physical region and it does seem possible that $\cos\theta \rightarrow \infty$!

So we write:

$$d_{j_1 j_2}^{(j)}(\theta) \approx Z \sum_{n=0}^{\infty} \alpha_{n j_1 j_2} \left(\frac{M}{Z}\right)^n \quad \text{as } Z \rightarrow \infty.$$

But near threshold, recall that

$$Z = \cos\theta \approx \frac{f(s,t)}{\sqrt{s - (m_1 + m_2)^2}}$$

So, at the worst, your singularity goes as $\left(\frac{1}{\sqrt{s - m_0^2}}\right)^j$.

Notice that in the above $*$ we had ≥ 2 d functions. There is a CG way to reduce this to one.

So let $B_{\mu\nu'; ab}(s, t) = CG \Sigma$ on $B_{cd'; ab} = \text{clean}$
 then

$$T_{ed, ab} = \sum_j CG \times \underbrace{d_{\mu\nu'}^j(\Theta)}_{\text{here is the blowup, isolated.}} \times B_{\mu\nu'; ab}$$

This says that $J \subset S_c \otimes S_d$, so at most, $J = S_c + S_d$.

So going back,

$$d_{\mu\nu'}^j(\Theta) \sim \sum_j \sum_n \alpha_{n\mu\nu'} \left(\frac{\mu}{z}\right)^n \\ \sim \sum_j \left[\alpha_{0\mu\nu'} + \alpha_{1\mu\nu'} \frac{\mu}{z} + \alpha_{2\mu\nu'} \frac{\mu^2}{z^2} + \dots \right]$$

We seek, therefore, the lowest non-vanishing term $\alpha_{n\mu\nu'}$ in this series. It will tell you how bad the sing is.

Parity: $T_{ed, ab} = \mathcal{N}(-1)^{\lambda+\mu} T_{-c-d, -a-b}$

so: $\sum_{c'd'} d_{c'e}^{S_c} d_{-d'-d}^{S_d} B_{c'd'; ab} = S_{J, J}(-1)^{\lambda-d}$

$$= \mathcal{N}(-1)^{\lambda+\mu} \sum_{c'd'} \left(d_{c'e}^{S_c} \right) \left(d_{-d'-d}^{S_d} \right) B_{c'd'; -a-b}$$

$$= d_{c'e}^{S_c} \times (-1)^{c'-e}$$

$$= \mathcal{N}(-1)^{\lambda+\mu} (-1)^{c'-e} (-1)^{d'-d} \sum_{c'd'} d_{c'e}^{S_c} d_{-d'-d}^{S_d} B_{c'd'; -a-b}$$

$$(-1)^{c'-e+d'-d} = (-1)^{c'-d'} (-1)^{d'-d} \frac{c-d}{x} = \mathcal{N}(-1)^{\lambda+\mu}$$

By carefully tracing the effect of PCas, we end up with

$$T_{ed, ab} = \sum_j \sum_n \frac{\alpha_{n\mu\nu', ab}}{(s-s_0)^{n-\frac{1}{2}}}$$

Again, we seek the smallest n coef.

So far, then, we get

$$T_{cd,ab} = \sum_n \frac{a_{n\bar{c},ab}}{(\sqrt{s-s_0})^{J-n}}$$

Now lets form parity eigenstates here

$$|ab\rangle_p = |ab\rangle \pm \eta_a \eta_b (-1)^{J-s_0} |a-b\rangle.$$

$$\text{Then: } T_{cd,(ab)_p} = \sum_n \left[\frac{a_{n\bar{c},ab} \pm \eta_a \eta_b (-1)^{J-s_0} a_{n\bar{c},a-b}}{(\sqrt{s-s_0})^{J-n}} \right]$$

Well, the idea is to form some lincomb of the $T_{cd,ab}$ such that the structure is:

$$LC(T_{cd,ab})_{\pm} = \sum_n \frac{[a_{n\bar{c},ab} \eta_a \eta_b (-1)^{J-s_0} \pm a_{n\bar{c},a-b}]}{(\sqrt{s-s_0})^{J-n}}$$

You want this exact object to appear up top because then, for a given J , one of the lincombs always vanishes.

$$\begin{aligned} \text{Then, maybe } LC(T_{cd,ab})_+ &\approx (J=\max, n=0) \sim \left(\frac{1}{\sqrt{s-s_0}}\right)^J \\ LC(T_{cd,ab})_- &\approx (J=\max, n=1) \sim \left(\frac{1}{\sqrt{s-s_0}}\right)^{J-1} \end{aligned}$$

Thus, one lincomb always has better behavior than the other by one half power. ✓

* The point of these 2 pages is to show why certain lincombs of the helicity amplitudes are nice from a kinematic viewpoint. Eg, T_{++} and T_{+-} in $\pi\pi N$ might separately blow up at threshold, whereas $T_{++} + i T_{+-}$ might vanish at threshold! [kinematic constraints].