

POLARIZATION

Lucht MAY 1974

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add

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Exercise: relating the PI-N helicity amplitudes to the non-helicity f_1 and f_2

Start with the helicity amplitudes and helicity states:

$$f_{\mu\mu} = \langle \theta\phi\mu' | f_{op} | 00\mu \rangle = \sum_{\nu} \langle \theta\phi\mu' | \theta\phi \frac{1}{2} \nu \rangle \langle \theta\phi \frac{1}{2} \nu | f_{op} | 00\frac{1}{2}\mu \rangle$$

$\uparrow$ helicity $\uparrow$ spin

The rightmost element has pure spin states, as opposed to the helicity states we started with. And f_{op} is the transition operator, a constant times T_p . We can replace the last element with its usual spin representation:

$$\langle \theta\phi \frac{1}{2} \nu | f_{op} | 00\frac{1}{2}\mu \rangle = \langle \frac{1}{2} \nu | f_1(\theta\phi) I + f_2(\theta\phi) \hat{\sigma} \cdot \hat{k}_z \hat{\sigma} \cdot \hat{r} | \frac{1}{2}\mu \rangle = \chi_{\mu}^T M \chi_{\mu}$$

Luckily, the right ket is an eigenstate of the ~~first~~ dot product, since we assume that the incident direction is the z axis. This dot product is just the helicity operator and the right ket is a helicity eigenstate. The other dot product is messier. So far we have:

$$\langle \theta\phi \frac{1}{2} \nu | f_{op} | 00\frac{1}{2}\mu \rangle = \langle \frac{1}{2} \nu | f_1 + 2\mu f_2 \hat{\sigma} \cdot \hat{k}_z | \frac{1}{2}\mu \rangle \quad \hat{k}_z = (\theta, \phi) \quad \hat{\sigma} \cdot \hat{k}_z = \begin{bmatrix} \cos\theta & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & -\cos\theta \end{bmatrix}$$

[We now consider the cases of initial spin separately:] ~~WZ~~, here is a gimmick to avoid doing things separately:

$$\hat{\sigma} \cdot \hat{k}_z | \frac{1}{2}\mu \rangle = \hat{\sigma} \cdot \hat{k}_z R^{-1}(\hat{k}_z) R(\hat{k}_z) | \frac{1}{2}\mu \rangle = \hat{\sigma} \cdot \hat{k}_z \tilde{R}^{-1}(\hat{k}_z) | \frac{1}{2}\mu \rangle \quad \text{error? OK.}$$

$$= \hat{\sigma} \cdot \hat{k}_z D_{\lambda\mu}^{-1}(\hat{k}_z) | \frac{1}{2}\lambda \rangle = 2\lambda D_{\lambda\mu}^{-1}(\hat{k}_z) | \frac{1}{2}\lambda \rangle$$

$$= 2\lambda D_{\lambda\mu}^{-1}(\hat{k}_z) R(\hat{k}_z) | \frac{1}{2}\lambda \rangle = 2\lambda D_{\lambda\mu}^{-1}(\hat{k}_z) D_{\lambda\lambda}(\hat{k}_z) | \frac{1}{2}\lambda \rangle$$

$$\Rightarrow \langle \theta\phi \frac{1}{2} \nu | f_{op} | 00\frac{1}{2}\mu \rangle = \langle \frac{1}{2} \nu | f_1 + 2\mu f_2 \sum_{\lambda} D_{\lambda\mu}^{-1} \cdot D_{\lambda\lambda} \cdot | \frac{1}{2}\lambda \rangle$$

$$= \delta_{\nu\mu} f_1 + 2\mu f_2 D_{\lambda\mu}^{-1} \cdot D_{\lambda\lambda} \quad = f_1 + f_2 \sum_{\lambda} 2\mu D_{\lambda\mu}^{-1} D_{\lambda\lambda}$$

Now lets work on the first element up top:

$$\begin{aligned} \langle \theta\phi\mu' | \theta\phi \frac{1}{2} \nu \rangle &= \langle 00\mu' | R^{\dagger} R | 00\frac{1}{2}\nu \rangle = \langle 00\mu' | 00\frac{1}{2}\nu \rangle = \delta_{\mu'\nu} \\ \Rightarrow f_{\mu\mu} &= f_1 + f_2 \mu \sum_{\lambda} D_{\lambda\mu}^{-1} \cdot D_{\lambda\lambda} \cdot \lambda = f_1 + f_2 \mu \sum_{\lambda} \lambda D_{\lambda\mu}^{-1} \cdot D_{\lambda\lambda} \\ &= f_1 + f_2 \mu \exp[i\phi(\mu-\mu')] \sum_{\lambda} \lambda d_{\mu\lambda} d_{\lambda\mu} \end{aligned}$$

error.

We can evaluate this last object for the two cases of interest:

$$\begin{aligned} \sum_{\lambda} \lambda d_{\lambda\mu} d_{\lambda\mu} &= \frac{1}{2} [d_{\lambda\mu}^2 - d_{-\lambda\mu}^2] = \frac{1}{2} [\cos^2 \frac{\theta}{2} - \sin^2 \frac{\theta}{2}] = \frac{1}{2} \cos \theta \\ \sum_{\lambda} \lambda d_{\lambda\mu} d_{-\lambda\mu} &= \frac{1}{2} [d_{\lambda\mu} d_{-\lambda\mu} - d_{-\lambda\mu} d_{\lambda\mu}] = \frac{1}{2} [2 d_{\lambda\mu} d_{-\lambda\mu}] = \cos \frac{\theta}{2} \sin \frac{\theta}{2} = \frac{1}{2} \sin \theta \\ \Rightarrow f_{++} &= f_1 + \frac{1}{2} f_2 \cdot \frac{1}{2} \cos \theta \\ f_{+-} &= f_1 - \frac{1}{2} e^{-i\phi} \cdot \frac{1}{2} \sin \theta f_2 \end{aligned}$$

error.

However, these are the wrong results. I think the error is shown above, so lets back up and write:

Two potential errors. Check them out one at a time. First,

$$\begin{aligned}
 |\frac{1}{2}\mu\rangle &= \bar{R}^\dagger R |\frac{1}{2}\mu\rangle = \bar{R}^\dagger |\frac{1}{2}\tilde{\mu}\rangle = D_{\tilde{\mu}\mu}^{-1} |\frac{1}{2}\lambda\rangle? \\
 &= D_{\tilde{\mu}\mu}^{-1} \cdot R |\frac{1}{2}\lambda\rangle = D_{\tilde{\mu}\mu}^{-1} \cdot D_{m\lambda} |\frac{1}{2}m\rangle = D_{m\lambda} D_{\tilde{\mu}\mu}^{-1} |\frac{1}{2}m\rangle \\
 &= 1_{m\mu} |\frac{1}{2}m\rangle = |\frac{1}{2}\mu\rangle.
 \end{aligned}$$

This looks OK. In other words, you are allowed to rotate a z' -referenced JM state in the same way you rotate a z -referenced JM state.

The other error was surely an error.

$$\begin{aligned}
 \langle \theta\phi\mu' | \theta\phi\frac{1}{2}\nu \rangle &= \langle 00\mu' | \bar{R}^\dagger | \theta\phi\frac{1}{2}\nu \rangle = \langle 00\mu' | 00\frac{1}{2}\nu \rangle \leftarrow R \text{ rotates the spin reference axis!} \\
 &= \sum_m \langle 00\mu' | D_{m\nu}^{-1} | 00\frac{1}{2}m \rangle = D_{\mu'\nu}^{-1} \\
 \Rightarrow f_{\mu\mu} &= \sum_{\nu} D_{\mu'\nu}^{-1} [\delta_{\mu\mu'} f_1 + f_2 \mu \sum_{\lambda} D_{\lambda\mu}^{-1} D_{\nu\lambda, \lambda}] \\
 &= f_1 D_{\mu'\mu}^{-1} + f_2 \mu \sum_{\lambda} \lambda \delta_{\mu\lambda} D_{\lambda\mu}^{-1} \\
 &= f_1 D_{\mu'\mu}^{-1} + \mu \mu' f_2 D_{\mu'\mu}^{-1} \\
 &= D_{\mu'\mu}^{-1} [f_1 + (2\mu)(2\mu') f_2] = D_{\mu'\mu}^{-1} [f_1 + (2\mu)(2\mu') f_2]
 \end{aligned}$$

Now at last we can compute em up:

$$\begin{aligned}
 f_{++} &= D_{\frac{1}{2}\frac{1}{2}}^{-1} [f_1 + f_2] = (f_1 + f_2) \cos \frac{\theta}{2} \\
 f_{+-} &= D_{\frac{1}{2}-\frac{1}{2}}^{-1} [f_1 - f_2] = D_{\frac{1}{2}\frac{1}{2}}^* (f_1 - f_2) = e^{i\phi} \sin \frac{\theta}{2} (f_1 - f_2)
 \end{aligned}$$

And these are exactly right.

Now lets derive these relations in a faster way. Essentially, go backwards:

$$\chi_{\nu}^\dagger M \chi_{\mu} = \langle \theta\phi\frac{1}{2}\nu | M^{\text{op}} | 00\frac{1}{2}\mu \rangle = \sum_{\lambda} \langle \theta\phi\frac{1}{2}\nu | \theta\phi\lambda \rangle \langle \theta\phi\lambda | M^{\text{op}} | 00\mu \rangle$$

$$\textcircled{1} \langle \theta\phi\frac{1}{2}\nu | \theta\phi\lambda \rangle = \langle \theta\phi\frac{1}{2}\nu | R | 00\lambda \rangle = \sum_m D_{\nu m} \langle 00\frac{1}{2}m | 00\lambda \rangle = D_{\nu\lambda}$$

$$\textcircled{2} \langle \theta\phi\lambda | M | 00\mu \rangle = \langle \theta\phi\lambda | f_1^{\text{op}} + f_2^{\text{op}} (2\lambda)(2\mu) | 00\mu \rangle = D_{\lambda\mu}$$

$$\begin{aligned}
 &= \langle \theta\phi\lambda | \theta\phi\frac{1}{2}m \rangle \langle \theta\phi\frac{1}{2}m | f_1 + 4\mu\lambda f_2 | 00\frac{1}{2}\mu \rangle \\
 &= D_{\lambda m}^{-1} \cdot \delta_{m\mu} \cdot (f_1 + 4\mu\lambda f_2) = D_{\lambda\mu}^{-1} (f_1 + 4\mu\lambda f_2)
 \end{aligned}$$

$$\text{So, } \chi_{\nu}^\dagger M \chi_{\mu} = \sum_{\lambda} D_{\nu\lambda} \cdot D_{\lambda\mu}^{-1} [f_1 + 4\mu\lambda f_2]$$

must be first.

Unfortunately, that accomplished nothing. New idea, more direct:

$$\begin{aligned}
 f_{\mu'\mu} &= \langle \theta\phi\mu' | f_{op} | 00\mu \rangle = \langle \theta\phi\mu' | f_1^{op} + f_2^{op} \sigma(\hat{k}_3) \cdot \hat{k} | 00\mu \rangle \\
 &= \langle \theta\phi\mu' | f_1^{op} + 2\mu' 2\mu f_2^{op} | 00\mu \rangle \\
 &= \langle \theta\phi\mu' | \theta\phi \pm m \rangle \langle \theta\phi \pm m | f_1^{op} + 2\mu' 2\mu f_2^{op} | 00 \pm \mu \rangle \\
 &= \langle \theta\phi\mu' | \theta\phi \pm m \rangle [\delta_{m\mu}] [f_1 + 4\mu\mu' f_2]
 \end{aligned}$$

$$\begin{aligned}
 \langle \theta\phi\mu' | \theta\phi \pm m \rangle &= \langle 00\mu' | \hat{R}^\dagger | \theta\phi \pm m \rangle = \sum_{\lambda} \hat{D}_{\lambda m}^{-1} \langle 00\mu' | 00 \pm \lambda \rangle \\
 &= \hat{D}_{\mu' m}^{-1}
 \end{aligned}$$

$$\begin{aligned}
 f_{\mu'\mu} &= \hat{D}_{\mu' m}^{-1} [f_1 + 4\mu\mu' f_2] = \hat{D}_{\mu\mu'}^* [] \\
 &= \exp[i(\mu - \mu')] d_{\mu\mu'}^{1/2}(\theta) [f_1 + 4\mu\mu' f_2]
 \end{aligned}$$

$$\begin{aligned}
 f_{++} &= \cos \frac{\theta}{2} [f_1 + f_2] = f_{--} \\
 f_{+-} &= e^{-i\phi} \sin \frac{\theta}{2} [f_1 - f_2] = -e^{-2i\phi} f_{-+} \\
 f_{-+} &= -e^{i\phi} \sin \frac{\theta}{2} [f_1 - f_2]
 \end{aligned}$$

Clearly this is the fastest and most efficient way to get these relations. We get to use the helicity operators to their best advantage. The cost is that we have to know how to rotate a spin (non-helicity) state. This was covered in another exercise. I got it from Goldberger and Watson, because Martin and Spearman only use helicity states.

$$\begin{aligned}
 f_{++} &= (f_1 + f_2) \cos \frac{\theta}{2} = f_{--} \\
 f_{+-} &= (f_1 - f_2) \sin \frac{\theta}{2} e^{-i\phi} \\
 f_{-+} &= (f_2 - f_1) \sin \frac{\theta}{2} e^{+i\phi}
 \end{aligned}$$

Exploding a Myth about norms.

One might think that an operator of unit norm evaluated between normalized states would yield a matrix element of unit norm. This is simply not true. By "operator of unit norm" I mean only that the operator has eigenvalues of unit magnitude.

For example, if an operator is self adjoint and unitary, then the operator is its own inverse, the operator squared is the identity, and any eigenvalues must be + or - one. But it is easy to show that the matrix element in question does not have magnitude one:

$$\sigma = \sigma^\dagger = \sigma^{-1} \quad \sigma^2 = I \quad \sigma|\psi_a\rangle = a|\psi_a\rangle \quad \sigma^2|\psi_a\rangle = a^2|\psi_a\rangle = 1 \cdot |\psi_a\rangle \Rightarrow a = \pm 1.$$

$$|\langle x|\sigma|x\rangle|^2 \neq 1 \text{ ! as we show:}$$

$$\begin{aligned} 1 &= \langle x|x\rangle = \langle x|\sigma^\dagger\sigma|x\rangle \text{ assuming only unitary } \sigma. \\ &= \langle x|\sigma^\dagger|x\rangle\langle x|\sigma|x\rangle + \sum_{n \neq x} \langle x|\sigma^\dagger|n\rangle\langle n|\sigma|x\rangle \end{aligned}$$

Thus:

$$1 = |\langle x|\sigma|x\rangle|^2 + \sum_{n \neq x} |\langle n|\sigma|x\rangle|^2$$

Here, $|x\rangle$ was one of a complete set of states $|n\rangle$.

Thus we see that in general, the magnitude of the matrix element lies between zero and one. It equals one only in the special case that the operator does not connect to any other states.

This is reminiscent of the idea of the norm of an operator as discussed in Stakgold where the definition I think is:

$$\|\sigma\| = \max_x \frac{|\langle x|\sigma|x\rangle|}{|\langle x|x\rangle|} \text{ or l.u.b.}$$

In the above example (a unitary operator), we see that the norm is in fact unity by Stakgold's definition also, because at most, the numerator is equal to the denominator.

This discussion was motivated by consideration of the final polarization in spin-0:spin- $\frac{1}{2}$ scattering. I was wondering why it was possible for the magnitude of the final polarization to be less than one:

$$\vec{P}_f = \frac{(A_f|\vec{\sigma}|A_i)}{(A_f|A_i)} \quad \text{where} \quad A_i = f_{\uparrow}|\uparrow\rangle$$

Now it is clear why unity is just the maximum value of $(P)^2$. In general, P will be less than one. Of course here one must define the norm appropriately to take into account the vector sense of the operator.

Question: why is $P_i^2 = 1$ but $P_f^2 \leq 1$?

Here we are considering the initial state to be a pure spin- $\frac{1}{2}$ state. This state is completely polarized in some direction, so it is obvious that $|P_i| = 1$. In particular, note that we are not dealing with mixed states or density matrix stuff.

So how is it possible that the "thing" called A not be completely polarized in some direction (if not the same direction as the initial beam). The reason is that if you operate on a 100% polarized state with some operator A , that state remains 100% polarized only if that operator A happens to be a constant multiple of a rotation operator R . Now there is no reason in the world why the scattering operator f should be a multiple of the rotation operator. So,

$|\nu\rangle = 100\% \text{ pole.}$ $|A\nu\rangle = f|\nu\rangle = \text{not } 100\% \text{ pole.}$ Thus

$$\frac{|\langle \nu | \vec{\sigma} | \nu \rangle|}{\langle \nu | \nu \rangle} = 1 \quad \text{but} \quad \frac{|\langle A\nu | \vec{\sigma} | A\nu \rangle|}{\langle A\nu | A\nu \rangle} \leq 1$$

Remember, although the initial fermion is all polarized in some direction, the scattering operator contains flip and non-flip type terms. For example, it would be possible for a beam initially "up" to go into a final state with no polarization in any direction. I.e., $P_f = 0$. That is of course something you cannot do with a rotation operator!

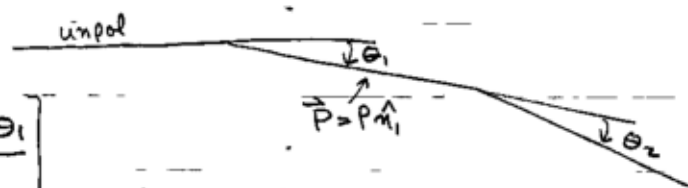
Formally, we can show why the initial 100% polarized beam has $P = 1$ according to the above definition of P . The trick is simply to define your z -axis to be in the direction of that polarization. Then the x and y matrix elements vanish because they can be written in terms of raising and lowering operators, which operators always kill diagonal matrix elements:

$$\begin{aligned} |\vec{P}_i|^2 &= |\langle \nu | \vec{\sigma} | \nu \rangle|^2 = \langle \nu | \vec{\sigma} | \nu \rangle \cdot \langle \nu | \vec{\sigma} | \nu \rangle \\ &= |\langle \nu | \sigma_z | \nu \rangle|^2 + 0 + 0 \\ &= (\pm 1)^2 \\ &= 1. \end{aligned}$$

now choose z axis:

"Derivation of 7-233 on double scattering."

Consider the double scattering process sketched below. It is assumed that the same interaction is responsible for both scatterings. Also, assume the initial incoming beam is unpolarized. We know that after the first scattering the beam will acquire a polarization in the $\hat{n}_1$ direction. This (partially) polarized beam is used for input to the second scattering event:



$$\vec{P} = \hat{n}_1 \left[\frac{2 \operatorname{Im}(g^* f) \sin \theta_1}{\left(\frac{d\bar{\sigma}}{d\Omega_1} \right)} \right]$$

$$\frac{d\bar{\sigma}}{d\Omega_1} = |f|^2 + |g|^2 \sin^2 \theta_1$$

Now, as an aside, rewrite (227) using (231) for P :

$$(227) \quad \frac{d\sigma(P_i)}{d\Omega} = \{ |f|^2 + |g|^2 \sin^2 \theta \} \times [1 + \vec{P}_i \cdot \hat{n} P]$$

$$= \frac{d\bar{\sigma}}{d\Omega} [1 + P_i \vec{P}_i \cdot \hat{n}]$$

This interesting equation looks like Williams (3.48). Now apply this to the second process using for P_i the result of the first process:

$$\left. \frac{d\bar{\sigma}}{d\Omega_2} \right|_{\text{2scatt}} = \frac{d\bar{\sigma}}{d\Omega_2} [1 + P(\theta_2) [P(\theta_1) \hat{n}_1] \cdot \hat{n}_2] = \frac{d\bar{\sigma}}{d\Omega_2} [1 + P(\theta_1) P(\theta_2) \hat{n}_1 \cdot \hat{n}_2]$$

where

$$P(\theta_i) = \left[\frac{2 \operatorname{Im}(g^*(\theta_i) f(\theta_i)) \sin \theta_i}{|f(\theta_i)|^2 + |g(\theta_i)|^2 \sin^2 \theta_i} \right]$$

This is GW's (233). It seems to me that there is something fishy about the magnitude of this double scattering cross section. Some intuition says that it should be much smaller than the single scatter cross section, but here it is of the same order of magnitude. In this experiment, every particle that hits the first scattering center also hits the second scattering center. The second center is in exactly the right place so it gets "hit". Yeah, there is some flux stuff here that has been fudged over. Nevertheless, his point about the asymmetry is the same regardless of the normalization of the double scatter cross section, so let it rest at that.

In fact, we have merely written the DCS for the second scattering only. Its just that the beam for the second process was prepared in a certain way. The point is that if you know all about the second process, you know P_2 exactly and can deduce P_1 from the asymmetry. No need to "know" the first process.

Exercises with the Density Matrix.

Look at the original definition in GW (238) which shows the thing as an ensemble sum of something. Just looking at it we can abstract away the operator density matrix (of course in its operator form it is not a matrix):

$$\langle \lambda | \rho | \lambda' \rangle = \frac{1}{N} \sum_{\alpha=1}^N \langle \lambda | \alpha \rangle \langle \alpha | \lambda' \rangle \quad \rho = \frac{1}{N} \sum_{\alpha=1}^N |\alpha\rangle \langle \alpha|$$

We have to be a little careful with the notation here. The object $|\alpha\rangle$ is the exact state of one of the systems in the ensemble, the one carrying the label α . On the other hand, $\{|\lambda\rangle\}$ is a complete set of states for any of the systems in the ensemble. The index λ tells what state you have (say that λ runs from 1 to 5), whereas the index α tells you which system $|\alpha\rangle$ is the state of. (And maybe $\alpha = 1, 10,000$ whereas $N = 10,000$)

Now if you know the exact state of every system, then you know the density matrix exactly (just look at the above formula). And then you would also be able to find the set of states $\{|\delta\rangle\}$ that diagonalizes ρ . This leads us to a little exercise:

$$\begin{aligned} |\alpha\rangle &= \sum_{\delta} |\delta\rangle \langle \delta | \alpha \rangle \Rightarrow \rho = \frac{1}{N} \sum_{\alpha} \sum_{\delta \delta'} |\delta\rangle \langle \delta | \alpha \rangle \langle \alpha | \delta' \rangle \langle \delta' | \\ &= \sum_{\delta \delta'} |\delta\rangle \langle \delta | \frac{1}{N} \sum_{\alpha} |\alpha\rangle \langle \alpha | \delta' \rangle \langle \delta' | \\ &= \sum_{\delta \delta'} |\delta\rangle \langle \delta | \rho | \delta' \rangle \langle \delta' | \\ \rho &= \sum_{\delta} P(\delta) |\delta\rangle \langle \delta| \quad \Leftarrow \quad = \sum_{\delta} |\delta\rangle P(\delta) \langle \delta| \quad P(\delta) = \langle \delta | \rho | \delta \rangle \end{aligned}$$

What is a bit strange is that in this last expression for the density matrix as an operator, there are no "ensemble" indices. There is no reference at all to the ensemble. One must keep in mind that this expression is only correct if those states $|\delta\rangle$ do in fact diagonalize ρ . This is where you need information about the ensemble!

Now let's explore a two index notation: $|\delta\alpha\rangle \equiv |\delta(\alpha)\rangle \equiv |\delta\rangle_{\alpha} = |\alpha\rangle$
 $\equiv |\delta\alpha\rangle$

possible that $|1\rangle = |2\rangle$, i.e., that $|\delta 1\rangle = |\delta 2\rangle$

In this notation, α is just a number from 1 to N , whereas $\delta(\alpha)$ tells you about the nature of the state. Thus, for example, we can say:

$$|\delta(\alpha)\rangle = \sum_{\delta'} |\delta'\rangle \langle \delta' | \delta(\alpha) \rangle$$

$$\rho = \frac{1}{N} \sum_{\alpha=1}^N |\delta(\alpha)\rangle \langle \delta(\alpha)| \quad (= \sum_{\delta} |\delta\rangle P(\delta) \langle \delta| \text{ if } \{|\delta\rangle\} = \text{diag. rep.})$$

true for any $\{|\delta\rangle\}$.

General Results with statistical beams.

Recall that the results of pages 399 and 400 applied to an initial beam that was pure, ie, it was 100% polarized in some direction. Now, having read through the bottom of page 407, we are able to generalize those equations to the statistical beam.

The basic results of the pre-density matrix analysis were these: the initial beam is described by some vector P_i of unit magnitude. With such an initial beam, the final polarization is a very complicated function of P_i , see (230). However, if the initial beam is "unpolarized", then the equation for the differential cross section and polarization finally simplify:

$$\text{pol. } \vec{P}_i: \begin{cases} \left(\frac{d\sigma}{d\Omega}\right)(\vec{P}_i) = |\vec{g}|^2 + |\vec{g}|^2 \sin^2 \theta + 2 \hat{n} \cdot \vec{P}_i \sin \theta \operatorname{Im}(g^* f) \\ \vec{P}_f = \left\{ \frac{d\sigma}{d\Omega}(\vec{P}_i) \right\}^{-1} \times \left\{ [\text{wavy}] \hat{n} + [\text{wavy}] \vec{P}_i + [\text{wavy}] \vec{P}_i \times \hat{n} \right\} \end{cases} \quad (227)$$

$$\text{unpol.} \begin{cases} \frac{d\sigma}{d\Omega} = |\vec{g}|^2 + |\vec{g}|^2 \sin^2 \theta \\ \vec{P}_f = \left\{ \frac{d\sigma}{d\Omega} \right\}^{-1} \times \left\{ 2 \operatorname{Im}(g^* f) \sin \theta \right\} \end{cases} \quad (230)$$

Now here are the results for a statistical beam. The statistical beam, first of all, is fully described by the initial density matrix:

$$\rho_i = \frac{1}{2} [1 + \vec{P}_i \cdot \vec{\sigma}] \quad \vec{P}_i = h(\rho_i \cdot \vec{\sigma})$$

Now, the vector P_i can have magnitude less than unity. This formalism of course allows for the incident beam to be partially polarized in any direction we like. Then, the relevant equations are:

$$\begin{cases} \overline{\left(\frac{d\sigma}{d\Omega}\right)} = \operatorname{Tr}(f \rho_i f^\dagger) = \text{total DCS into all final states with beams characterized by } \rho_i \\ \overline{\vec{P}_f} = \operatorname{Tr}(\rho_f \vec{\sigma}) = \left\{ \overline{\left(\frac{d\sigma}{d\Omega}\right)} \right\}^{-1} \times \left\{ \operatorname{Tr}(f \rho_i f^\dagger \vec{\sigma}) \right\} \end{cases}$$

In this notation, the long bar $\overline{\quad}$ indicates the ensemble average, whereas a short bar over the σ indicates an unpolarized initial beam (not used here yet). Now, being more explicit:

$$\begin{aligned} \overline{\left(\frac{d\sigma}{d\Omega}\right)} &= \operatorname{Tr}(f \rho_i f^\dagger) = \frac{1}{2} \operatorname{Tr}(f f^\dagger) + \frac{1}{2} \vec{P}_i \cdot \operatorname{Tr}(f \vec{\sigma} f^\dagger) \\ &= \left(\frac{d\sigma}{d\Omega}\right) + 2 \sin \theta \operatorname{Im}(g^* f) \vec{P}_i \cdot \hat{n} \end{aligned}$$

$$\begin{aligned} \text{Note: } (\vec{\sigma} \cdot \hat{n}) \vec{\sigma} &= \vec{n} + i \vec{\sigma} \times \hat{n} \\ \vec{\sigma} (\vec{\sigma} \cdot \hat{n}) &= \vec{n} - i \vec{\sigma} \times \hat{n} \\ (\vec{\sigma} \cdot \hat{n}) \vec{\sigma} (\vec{\sigma} \cdot \hat{n}) &= 2(\vec{n} \cdot \vec{\sigma}) \vec{n} - \vec{\sigma} \\ h(f \rho_i f^\dagger \vec{\sigma}) &= h(f f^\dagger \vec{\sigma}) \end{aligned}$$

Notice that this equation is the same as the non-statistical equation except that now P_i has the fully general meaning, ie, can have $|P_i| \leq 1$. The final polarization is given by: FIRST:

$$\rho_f = \frac{f \rho_i f^\dagger}{\operatorname{Tr}(f \rho_i f^\dagger)} \quad \text{so } h(\rho_f) = 1 \quad \text{If } \rho_f = A + \vec{B} \cdot \vec{\sigma}, \text{ then } 2A = h(\rho_f) = 1$$

$$\text{If } \vec{P}_f = h(\rho_f \vec{\sigma}) \text{ then } 2\vec{B} = \vec{P}_f \text{ So}$$

$$\rho_f = \frac{1}{2} (1 + \vec{P}_f \cdot \vec{\sigma})$$

My guess is that the final polarization is exactly given by (230) for any P_i .

Conclusions on the density matrix formalism.

In Goldberger and Watson, the equations (227) and (230) and (231) were all first derived with pure beam and before any mention of the density matrix and the statistical beam. However, it turns out that these equations are correct for the statistical beam also. The only difference is that for the statistical beam the magnitude of the vector $\vec{P}_i$ can be less than one.

Comparisons: Williams produces the same kind of equations, except he chooses to normalize his final density matrix differently. To connect to Williams notation, use:

$$\frac{Williams}{\langle \sigma \rangle, \langle \sigma \rangle} \rightarrow \frac{G+W}{\vec{P}_i \vec{P}_f}$$

$$\frac{d\sigma}{d\Omega} \rightarrow \frac{d\bar{\sigma}}{d\Omega}$$

$$\frac{d\sigma}{d\Omega} \rightarrow \frac{d\sigma}{d\Omega}(\vec{P}_i)$$

$$g \rightarrow f$$

$$h \rightarrow -g \sin \theta$$

$$\frac{M}{\hbar} \rightarrow \frac{f}{\hbar}$$

$$\rho_2 \rightarrow \frac{d\sigma}{d\Omega}(\vec{P}_i) \rho_f$$

$$(\vec{P}_i \cdot \vec{n})(\vec{P}_f \cdot \vec{n}) = (\vec{P}_i \cdot \vec{n}) \rho_f$$

$$= \vec{P}_i \cdot \vec{P}_f$$

Now let's summarize again the primary results of G+W:

$$\frac{d\sigma}{d\Omega}(\vec{P}_i) = \left(\frac{d\sigma}{d\Omega} \right) = |f|^2 + |g|^2 \sin^2 \theta + 2 \vec{P}_i \cdot \vec{n} \sin \theta \operatorname{Im}(g^* f) = (227)$$

$$= \left(\frac{d\bar{\sigma}}{d\Omega} \right) + 2 \vec{P}_i \cdot \vec{n} \sin \theta \operatorname{Im}(g^* f) = \frac{d\bar{\sigma}}{d\Omega} + \frac{d\bar{\sigma}}{d\Omega} \vec{P}_i \cdot \vec{n} \vec{P}_f \cdot \vec{n}$$

$$= \frac{d\bar{\sigma}}{d\Omega} [1 + \vec{P}_i \cdot \vec{n} \vec{P}_f \cdot \vec{n}]$$

$\vec{P}_f =$ final pole that you would have if it were unpol.

$\vec{P}_f =$ mass in (230) (depends on $\vec{P}_i$)

$\vec{P}_f^0 = (231)$

$$= \frac{d\bar{\sigma}}{d\Omega} [1 + \vec{P}_i \cdot \vec{P}_f]$$

Obviously: $\left(\frac{d\bar{\sigma}}{d\Omega} \right) = |f|^2 + |g|^2 \sin^2 \theta = \frac{d\bar{\sigma}}{d\Omega}$

Discussion of GW's statistical analysis of general 2-to-2 reaction.

The problem is this. In a reaction $A + B \rightarrow C + D$, how do you statistically characterize the initial system? Given such a characterization or specification, what is the statistically averaged differential cross section (into all final states)? If A and/or B were unpolarized, again what would be the DCS? Also, how would you characterize the final system? After the reaction, given a particular characterization of the initial system, what are the final system parameters statistically (as functions of the initial parameters?) If A and/or B are unpolarized, what are these final parameters.

Obviously one could try to generalize this beyond the 2-to-2 sense. I can see how that would go, but let's not worry about it.

It turns out that the initial system can be decomposed into beam X target. If A has spin S_A , then there will be a certain finite number of parameters which completely describe the beam. That number is $4S(S+1)$. For spin $\frac{1}{2}, 1, 3/2, 2, \dots$ the number is $3, 3+5, 3+5+7, 3+5+7+9, \dots$. As the spin is increased by $\frac{1}{2}$, a new "multipole (spherical) tensor" of parameters is added. The lowest tensor is always the polarization vector. The higher tensors do not in my mind have any physical interpretation. What is a "multipole polarization tensor"? They are just parameters which describe the way the ensemble is distributed in the various spin states.

Similarly, the target is described by its set of multipole parameters. Then the initial average system is described by these two sets of parameters. It is up to the experimenter to determine all these parameters so he knows what his initial system is, otherwise there is no point in doing the scattering experiment.

Of course the final system of $C + D$ is also characterized by two sets of parameters. If we knew all about the interaction (ie, if we knew all the functions like f and g in that simple spin- $\frac{1}{2}$ /spin-0 case) then given those initial parameter sets, we should be able to predict the final parameter sets at a given scattering angle. Conversely, the experiment can be used to determine those functions in the interaction matrix f . That is the whole point!

Now it turns out that those initial parameters are all numbers in the initial density matrix. They are the coefficients you get when you expand the initial density matrix in terms of all the independent matrices (the basis matrices). So to know the initial state is to know the initial density matrix. Then the standard density matrix formalism tells you the answers to all your questions: (of course you have to know f):

$$\left(\frac{d\sigma}{d\Omega}\right)_{f_i} = \text{tr}(f g_i f^\dagger) \quad p_f = (f g_i f^\dagger) / \text{Tr}(f g_i f^\dagger)$$

initial params $\rightarrow g_i$
 $g_i + f \rightarrow p_f$
 $g_i \rightarrow$ final parameters

The initial density matrix is of course a cross product object. If one of the two particles is unpolarized, then the IDM reduces to just that of the other particle. If they are both unpolarized, then the IDM is just the identity matrix. Obviously either of these situations will simplify tremendously the equations involved.

Application to elastic nucleon-nucleon scattering.

Consider this experiment: an unpolarized beam of spin- S_2 particles is shot at a polarized spin- $\frac{1}{2}$ target. This situation is analyzed on page 412, 413 of GW. In particular, they give the DCS and final recoil P explicitly:

$$S_i = \frac{1}{2(2S_2+1)} (1 + \vec{P}_i \cdot \vec{\sigma}) \otimes 1$$

$$\text{DCS} \rightarrow \left(\frac{d\sigma}{d\Omega} \right) = \text{Tr}(f S_i f^\dagger) = \left(\frac{d\sigma}{d\Omega} \right) [1 + \vec{P}_u \cdot \vec{P}_i]$$

where

$$\left(\frac{d\sigma}{d\Omega} \right) = \frac{1}{2(2S_2+1)} \text{Tr}(f^\dagger f)$$

$$\vec{P}_u = \frac{\text{Tr}(f^\dagger \vec{\sigma} f)}{\text{Tr}(f^\dagger f)}$$

$$\text{FP} \rightarrow \vec{P}_f = \text{Tr}(f \vec{\sigma}) = (2S_2) = \frac{\vec{P}_u + \vec{\sigma} \cdot \vec{P}_i}{1 + \vec{P}_u \cdot \vec{P}_i}$$

$$\text{where } \vec{\sigma} = \frac{\text{Tr}(\vec{\sigma} f \vec{\sigma} f^\dagger)}{\text{Tr}(f f^\dagger)}$$

We see that the size of S_2 does not make much difference in these formulas. However, the value of S_2 does determine the dimensionality of the scattering matrix f . A large S_2 makes f have lots of independent functions. If, as in N-N scattering, $S_2 = \frac{1}{2}$, I think f has 5 independent functions and is a 4-dimensional object.

We know that the above experiment (with a polarized target) can be used to determine P_u via the asymmetry. Of course, P_u is a quantity related to a different experiment. So they measure P_u in, say, proton-proton scattering. If we wanted to make, say, a Regge fit to this experimental P_u , we would have to fit those 5 functions in f . Of course these 5 functions are related to the 5 independent helicity amplitudes. Thus, by fitting the various helicity amplitudes, and by worrying about all that sense-nonsense stuff and the kinematic singularities etc, we can use the above formula for P_u to fit the data. Now I have the data out to very large t , so lets go!

Possible Separation of Density Matrix Formalism

It certainly seems reasonable that the IDM (initial density matrix) separates into a direct product of IDM's, one IDM describing the state of each incident particle:

$$\rho_i = \frac{1}{(2S_1+1)(2S_2+1)} \sum_{\lambda=1}^N \langle \Omega^\lambda \rangle_i \Omega_\lambda \quad \text{where } N = (2S_1+1)^2 (2S_2+1)^2$$

$$\rho_i = \rho_i^{(1)} \otimes \rho_i^{(2)} \quad \rho_i^{(1)} = \frac{1}{(2S_1+1)} \sum_{\lambda_1=1}^{N_1} \langle \omega^{\lambda_1} \rangle_i \omega^{\lambda_1} \quad N_1 = (2S_1+1)^2$$

$$\rho_i^{(2)} = \frac{1}{(2S_2+1)} \sum_{\lambda_2=1}^{N_2} \langle \mu^{\lambda_2} \rangle_i \mu^{\lambda_2} \quad N_2 = (2S_2+1)^2$$

In general, these two IDM's will have different dimensionality, and thus a different set of tensors. One of the tensors for S_1 is ω^5 and one for S_2 is μ^2 .

Now it also seems reasonable that the scattering matrix itself can be expanded in a direct product using these same canonical tensors:

$$f = \sum_{\lambda_1, \lambda_2} A_{\lambda_1, \lambda_2} \omega^{(\lambda_1)} \otimes \mu^{(\lambda_2)} \quad \begin{array}{l} \omega = \text{set of } (2S_1+1) \times (2S_1+1) \text{ matrices} \\ \mu = \text{set of } (2S_2+1) \times (2S_2+1) \text{ matrices} \end{array}$$

Two obvious examples of such an expansion suggest themselves:

① πN : $N_1=1$ $N_2=4$ $f = \sum_{\lambda_2} A_{0, \lambda_2} 1 \otimes \mu^{(\lambda_2)}$
 $\{\mu\} = \{1, \sigma_x, \sigma_y, \sigma_z\}$ $f = A_{00} 1 \otimes 1 + \sum_j A_{0j} 1 \otimes \sigma_j$
 $= 1 \otimes [A_{00} 1 + \vec{B} \cdot \vec{\sigma}] \quad (B_j = A_{0j})$
 $A_{00} = f(\theta) \quad \vec{B} = i g(\theta) \sin \theta \vec{n} = A_{0j}$

② NN : $N_1=4$ $N_2=4$ $f = A_{00} 1 \otimes 1 + \sum A_{ij} \sigma_i \otimes \sigma_j$ etc.

Using such a factorized or direct product form, we can write the DCS as:

$$(1.1) \quad \frac{d\sigma}{d\Omega}(\vec{q};) = \text{tr}(f \rho_i f^\dagger) = \sum_{\lambda_1, \lambda_2} A_{\lambda_1, \lambda_2} A_{\lambda_1, \lambda_2}^* \text{tr}(\omega^{(\lambda_1)} \rho_i^{(1)} \omega^{(\lambda_1)}) \text{tr}(\mu^{(\lambda_2)} \rho_i^{(2)} \mu^{(\lambda_2)})$$

Then, by inserting these DM's, you could get an explicit expression for the DCS in terms of the amplitude functions (like $f, g, \dots$) and the initial polarization parameters. We have of course done this reduction explicitly in the case of spin-0/spin- $\frac{1}{2}$ and we get:

$$\frac{d\sigma}{d\Omega}(\vec{p};) = |f|^2 + |g|^2 \sin^2 \theta + \vec{n} \cdot \vec{p}_i \text{Im}(g^* f) \cdot 2 \sin \theta$$

In addition to the DCS, we also want to know the final polarization parameters of each of the final beams. Using this idea of a separable operator f we can say

$$f_{\beta}^{\dagger} f_{\beta} = \sum_{\substack{\lambda_1, \lambda_2 \\ \lambda_1', \lambda_2'}} A_{\lambda_1 \lambda_2} A_{\lambda_1' \lambda_2'}^* [\omega_{\beta}^{(\lambda_1)} s_{\beta}^{\dagger} \omega_{\beta}^{(\lambda_1')}] \otimes [\mu_{\beta}^{(\lambda_2)} s_{\beta}^{\dagger} \mu_{\beta}^{(\lambda_2')}]$$

$$\langle \Omega_{\beta}^{\lambda} \rangle = \text{tr}(\rho_{\beta} \Omega_{\beta}^{\lambda}) = \frac{\text{tr}[f_{\beta} \rho_{\beta} f_{\beta}^{\dagger} \Omega_{\beta}^{\lambda}]}{\text{tr}[f_{\beta} \rho_{\beta} f_{\beta}^{\dagger}]} = \left[\frac{d\sigma}{d\Omega}(s_{\beta}) \right]^{-1} \cdot \text{tr}(f_{\beta} \rho_{\beta} f_{\beta}^{\dagger} \Omega_{\beta}^{\lambda})$$

$$\frac{d\sigma}{d\Omega}(s_{\beta}) \cdot \langle \Omega_{\beta}^{\lambda} \rangle = \text{tr}[f_{\beta} \rho_{\beta} f_{\beta}^{\dagger} \Omega_{\beta}^{\lambda}] \quad \lambda = ab \text{ or } (\lambda_1, \lambda_2)$$

Now write the direct product version of this last equation. Note that $f_{\beta} \rho_{\beta} f_{\beta}^{\dagger}$ is expressible as a direct product. Naturally this must be so since $f_{\beta} \rho_{\beta} f_{\beta}^{\dagger}$ is, except for a constant, equal to ρ_{β} which we know factorizes in this sense.

$$\frac{d\sigma}{d\Omega}(s_{\beta}) \cdot \langle \omega_{\beta}^{(a)} \rangle \otimes \langle \mu_{\beta}^{(b)} \rangle = \sum_{\substack{\lambda_1, \lambda_2 \\ \lambda_1', \lambda_2'}} A_{\lambda_1 \lambda_2} A_{\lambda_1' \lambda_2'}^* \text{tr} \left[(\omega_{\beta}^{\lambda_1} s_{\beta}^{\dagger} \omega_{\beta}^{\lambda_1'}) \otimes [\mu_{\beta}^{\lambda_2} s_{\beta}^{\dagger} \mu_{\beta}^{\lambda_2'}] \right] (\omega_{\beta}^a \otimes \mu_{\beta}^b)$$

$$(2.1) \quad = \sum_{\substack{\lambda_1, \lambda_2 \\ \lambda_1', \lambda_2'}} A_{\lambda_1 \lambda_2} A_{\lambda_1' \lambda_2'}^* \text{tr}[\omega_{\beta}^{\lambda_1} s_{\beta}^{\dagger} \omega_{\beta}^{\lambda_1'}] \otimes \text{tr}[\mu_{\beta}^{\lambda_2} s_{\beta}^{\dagger} \mu_{\beta}^{\lambda_2'}]$$

Now we have a slight problem, however. We cannot isolate the multipole parameters for the separate final particles because of the sums on the right side. But still the mess could be solved for those separate parameters as a set of coupled algebraic equations.

Now I have my doubts about this whole idea of separability in the general case of (s_1, s_2) goes to (s_3, s_4) . This is because now that matrix f is rectangular. You might be able to write it as a direct product, but note in general as a direct product of square matrices:

$$f = \left[\begin{array}{c} \leftarrow (2s_1+1)(2s_2+1) \rightarrow \\ \uparrow (2s_3+1)(2s_4+1) \\ \downarrow \end{array} \right] = ? \left[\begin{array}{c} (2s_1+1) \\ \downarrow \\ (2s_3+1) \\ \downarrow \\ (2s_2+1) \end{array} \right]$$

$$f = \sum_i A_i \otimes B_i \quad \text{where } A \text{ and } B \text{ are not square + thus cannot be the "canonical" tensors.}$$

By assuming that f is so-called factorizeable, I am assuming the following nonsense:

$$\langle s_3 s_3, s_4 s_4 | f | s_2 s_2, s_1 s_1 \rangle = \sum_{ij} A_{ij} \langle s_3 s_3 | f^i | s_2 s_2 \rangle \langle s_4 s_4 | f^j | s_1 s_1 \rangle$$

This really makes no sense at all. It would have more sense to combine S1 and s2 to get some s(initial) and then maybe combine that with L to get J and then use a diagonalized version of f, namely f^J . Well, maybe it does make some sense after all. If it does, then we can write:

$$\omega^{\lambda_1} g_1^i \omega^{(\lambda_1)} = \left[\begin{array}{c} \leftarrow (2S_1+1) \\ \uparrow \\ (2S_3+1) \\ \downarrow \end{array} \right] \left[\begin{array}{c} 2S_1+1 \\ + \\ 2S_1+1 \end{array} \right] \left[\begin{array}{c} \uparrow \\ (2S_1+1) \\ \leftarrow (2S_1+1) \end{array} \right] = \left[\begin{array}{c} 2S_3+1 \\ \times 2S_3+1 \end{array} \right] = \text{square.}$$

$\omega_5 = (2S_3+1)(2S_3+1)$ also, so $\text{tr} [\omega g_1^i \omega^{\lambda_1} \omega_5]$ is sensible

So maybe the idea has been rescued. For a while there it looked as if we were taking traces of rectangular matrices. Now equations (1.1) and (2.1) seem to be OK. The traces are always in the final spin spaces.

Now, if we want to know the final parameters of one of the final particles, we can set the other index to 1 to get it. I.e., looking at (2.1),

$$\frac{d\sigma}{d\Omega}(g_i) \langle \omega_5 \rangle^a = \sum_{\lambda_1 \lambda_2 \lambda_1' \lambda_2'} A_{\lambda_1 \lambda_2} A_{\lambda_1' \lambda_2'}^* \text{tr} [\omega^{\lambda_1} g_1^i \omega^{(\lambda_1)} \omega_5^a] \text{tr} [M^{\lambda_2} g_2^i M^{(\lambda_2')}]$$

As expected, the multipole parameters of one of the final particles (here #3) depend on both sets of incident beam parameters through g_1 and g_2 .

Now we know all there is to know. Turning the crank we can find whatever we want. So let's consider the special case where the beam 2 is completely unpolarized. This yields:

$$g_1^i = g_1^i \quad \text{but} \quad g_2^i = \frac{1}{(2S_1+1)(2S_2+1)} 1$$

$$\text{tr} [M^{\lambda_2} g_2^i M^{(\lambda_2')}] \rightarrow \frac{1}{(2S_1+1)(2S_2+1)} \text{tr} [M^{\lambda_2} M^{(\lambda_2')}]$$

It is probably possible to choose the rectangular matrices ($M^{(\lambda)}$) such that they are normalized as suggested by GW (268). Thus,

$$\text{tr} [M^{\lambda_2} M^{(\lambda_2')}] = (2S_1+1)(2S_2+1) \delta_{\lambda_2 \lambda_2'} \quad \text{Then}$$

$$\left\{ \begin{array}{l} \frac{d\sigma}{d\Omega}(g_2=1) = \frac{(2S_4+1)}{(2S_1+1)} \sum_{\lambda_1 \lambda_2 \lambda_1'} A_{\lambda_1 \lambda_2} A_{\lambda_1' \lambda_2}^* \text{tr} [\omega^{(\lambda_1)} g_1^i \omega^{(\lambda_1')}] \\ = \sum_{\lambda_1 \lambda_1'} \left\{ \frac{(2S_4+1)}{(2S_1+1)} \sum_{\lambda_2} A_{\lambda_1 \lambda_2} A_{\lambda_1' \lambda_2}^* \right\} \text{tr} [\omega^{(\lambda_1)} g_1^i \omega^{(\lambda_1')}] \\ \frac{d\sigma}{d\Omega}(g_2=1) \langle \omega_5 \rangle^a = \sum_{\lambda_1 \lambda_1'} \left\{ \frac{(2S_4+1)}{(2S_1+1)} \sum_{\lambda_2} A_{\lambda_1 \lambda_2} A_{\lambda_1' \lambda_2}^* \right\} \text{tr} [\omega^{(\lambda_1)} g_1^i \omega^{(\lambda_1')} \omega_5^a] \end{array} \right.$$

If both initial beams are unpolarized these things become:

$$\begin{cases} \frac{d\sigma}{d\Omega} (p_1 = p_2 = 1) = \frac{(2S_4+1)(2S_3+1)}{(2S_1+1)(2S_2+1)} \sum_{\lambda_1 \lambda_2} |A_{\lambda_1 \lambda_2}|^2 \\ \frac{d\sigma}{d\Omega} \langle w_4 \rangle^a = \sum_{\lambda_1' \lambda_1} (2S_4+1)(2S_3+1) \sum_{\lambda_2'} A_{\lambda_1' \lambda_2} A_{\lambda_1 \lambda_2'}^* \int d\Omega [\underline{w}^{(m)} \underline{w}^{+(\lambda_1)} w_4^a] \end{cases}$$

These probably are not accurate, but I can see how the general theory would go. Perhaps some book goes into the full case.

On the Relation between the Spin and the Helicity Rep. in simple case.

First of all, in a spinless scattering process we know that the scattering operator f is an operator in the state Hilbert Space. We say something like:

$$\langle p\theta\phi | f^{op} | p00 \rangle = f(p, \theta, \phi)$$

Here we are using a cms two-particle formalism. The f inside is an operator, whereas the f on the right is just a function.

Now let us go to the case of spin-0: spin- $\frac{1}{2}$. Now the operator f is an operator in two distinct manifolds. It is an operator in the above sense (if you like, in the little hilbert space h it is T_p . Here, h is a space spanned by applying rotations to the state $|00\rangle$. Forget about p , it can be removed.) But it is also an operator in the spin-space. Here, that means it is a 2×2 matrix. So,

$$\langle p\theta\phi | f^{op} | p00 \rangle = \langle p\theta\phi | f_1^{op} | p00 \rangle + \langle p\theta\phi | f_2^{op} \hat{\sigma} \cdot \hat{k} \hat{\sigma} \cdot \hat{k}_f | p00 \rangle$$

$$f = f_1 \cdot 1 + f_2 \cdot \hat{\sigma} \cdot \hat{k} \hat{\sigma} \cdot \hat{k}_f$$

$$= \langle p\theta\phi | f_1^{op} | p00 \rangle \langle b | 1 | a \rangle + \langle p\theta\phi | f_2^{op} | p00 \rangle \langle b | \hat{\sigma} \cdot \hat{k} \hat{\sigma} \cdot \hat{k}_f | a \rangle$$

$$= f_1(\theta, \phi) \langle b | 1 | a \rangle + f_2(\theta, \phi) \langle b | \hat{\sigma} \cdot \hat{k} \hat{\sigma} \cdot \hat{k}_f | a \rangle$$

We have not yet specified exactly what the spin indices a and b mean. Of course each index has two values corresponding to up and down in some direction, but the whole question is what direction do we use. In the "spin formalism" we choose the same direction for both spin states (initial and final) and we often choose this direction to be n . On the other hand, in the "helicity formalism" we choose k for the direction of quantization of the initial spin, and we choose k_f for the final spin.

Lets continue the above reduction in both cases. In the "spin" case we get:

$$\hat{\sigma} \cdot \hat{k} \hat{\sigma} \cdot \hat{k}_f = \hat{k} \cdot \hat{k}_f + i \hat{\sigma} \cdot \hat{k} \times \hat{k}_f = \cos\theta + i \sin\theta \hat{\sigma} \cdot \hat{n}$$

$$\langle b | 1 | a \rangle = \langle \hat{n} b | 1 | \hat{n} a \rangle = \delta_{ba}$$

$$\hat{\sigma} \cdot \hat{n} | \hat{n} a \rangle = 2a | \hat{n} a \rangle$$

$$\langle b | \hat{\sigma} \cdot \hat{k} \hat{\sigma} \cdot \hat{k}_f | a \rangle = \cos\theta \delta_{ba} + i \sin\theta \langle b | \hat{\sigma} \cdot \hat{n} | a \rangle$$

$$= \cos\theta \delta_{ba} + i \sin\theta 2a \delta_{ba}$$

$$\Rightarrow \langle p\theta\phi | f^{op} | p00 \rangle = \{ f_1(\theta, \phi) + f_2(\theta, \phi) [\cos\theta + 2a i \sin\theta] \} \delta_{ab}$$

$$\text{OR} \quad \langle p\theta\phi \pm | f^{op} | p00 \pm \rangle = f_1 + f_2 \cos\theta \pm 2 f_2 i \sin\theta$$

$$\langle p\theta\phi \mp | f^{op} | p00 \pm \rangle = 0$$

Parity invariance of the amplitude restricted us to the form f_1 and f_2 . We see that, when the spin quantization is chosen along the n direction, we see that a spin-flip is not allowed at any scattering angle.

Next, still staying in the "spin" formalism, what would have happened had we chosen the k axis for the quantization? Watch:

$$\langle b || a \rangle = \langle \hat{k}_i | 1 | \hat{k}_i \rangle = \delta_{ab}$$

$$\langle \hat{k}_i | \vec{\sigma} \cdot \hat{k}_i \vec{\sigma} \cdot \hat{k}_i | \hat{k}_i \rangle = 2b \langle \hat{k}_i | \vec{\sigma} \cdot \hat{k}_i | \hat{k}_i \rangle$$

We have used one dot product as a helicity operator to pull out the $2b$, but now we have to do some work: choose $\hat{z} = \hat{k}_i$

$$= 2b \{ k_{fx} \langle b | \sigma_x | a \rangle + k_{fy} \langle b | \sigma_y | a \rangle + k_{fz} (2a) \delta_{ba} \}$$

$$\langle \text{pol } b | f^{\text{op}} | \text{pol } a \rangle = [f_1 + f_2 \hat{k}_f \cdot \hat{k}_i (2a)(2b)] \delta_{ba} + f_2 (2b) [\hat{k}_f \cdot \hat{x} \langle b | \sigma_x | a \rangle + \hat{k}_f \cdot \hat{y} \langle b | \sigma_y | a \rangle]$$

So, when we quantize along the k direction, now there is a flip possibility, as evidenced by those last two terms.

Now let us repeat the reduction in the "helicity" formalism. Then,

$$\begin{aligned} \langle b || a \rangle &= \langle b \hat{k}_f | 1 | a \hat{k}_i \rangle = \langle b \hat{k}_f | R \bar{R} | a \hat{k}_i \rangle = \langle b \hat{k}_f | R | \sum_{a'} \bar{D}_{a'a}^{-1} | a' \hat{k}_i \rangle \\ &= \sum_{a'} \bar{D}_{a'a}^{-1} \langle b \hat{k}_f | R | a' \hat{k}_i \rangle \quad \text{Let } R = R(\hat{k}_i \rightarrow \hat{k}_f, \text{ i.e., } \theta, \phi) \\ &= \sum_{a'} \bar{D}_{a'a}^{-1}(\phi, \theta, -\phi) \langle b \hat{k}_f | a' \hat{k}_f \rangle = \sum_{a'} \bar{D}_{a'a}^{-1} \delta_{ba'} = \bar{D}_{ba}^{-1} \end{aligned}$$

$$\langle b | \vec{\sigma} \cdot \hat{k}_i \vec{\sigma} \cdot \hat{k}_f | a \rangle = \langle b \hat{k}_f | (2b)(2a) | a \hat{k}_i \rangle = 4ab \bar{D}_{ba}^{-1}$$

Thus:

$$\langle \text{pol } b | f^{\text{op}} | \text{pol } a \rangle = f_1 \cdot \bar{D}_{ba}^{-1} + f_2 4ab \bar{D}_{ba}^{-1} = \bar{D}_{ba}^{-1} [f_1 + 4ab f_2]$$

Because the 2×2 matrix D^{-1} has off diagonal elements (at least when $\theta \neq 0$), again there is the flip possibility. This last equation is the one that allows us to connect the helicity amplitudes with the functions f_1 and f_2 . (Another sheet).

Important Point: this discussion has served to illustrate the following point: that the operator f (the "scattering operator") is a thing that is independent of the type of spin states you choose. Thus, the trace structures of the density matrix formalism are also independent. For example, the unpol DCS is given by the following formula independent of choice of spin states.:

$$\frac{d\sigma}{d\Omega} = \frac{1}{2} h(ff^\dagger) = \frac{1}{2} \sum_a \langle a | ff^\dagger | a \rangle = \frac{1}{2} \sum_{ab} | \langle a | f | b \rangle |^2$$

$$\frac{d\sigma}{d\Omega} \vec{P}_a = \frac{1}{2} h(ff^\dagger \vec{\sigma}) = \frac{1}{2} \sum_a \langle a | ff^\dagger \vec{\sigma} | a \rangle = \frac{1}{2} \sum_{abc} \langle a | f | b \rangle \langle b | f^\dagger | c \rangle \langle c | \vec{\sigma} | a \rangle$$

Comment on the way things are done from here on out: In the "spin" formalism, the basic units we deal with are the various functions like f and g . These amplitudes cannot cleanly be identified with "spin-flip" and non-flip amplitudes and all that. They are just a convenient parametrization. Thus, in the spin formalism, you stop right there at the trace level, put in the operator form of f (including its operator form in the spin space) and then you compute the traces directly. For example,

$$\frac{d\vec{\sigma}}{d\Omega} = \frac{1}{2} \text{tr}(ff^\dagger) = \frac{1}{2} \text{tr}(f + ig \hat{\sigma} \cdot \vec{n})(f^\dagger - ig \hat{\sigma} \cdot \vec{n}) = |f|^2 + |g|^2 \sin^2 \theta$$

$$\frac{d\vec{\sigma}}{d\Omega} \cdot \vec{p} = \frac{1}{2} \text{tr}(ff^\dagger \vec{\sigma}) = \vec{n} 2 \text{Im}(g^* f) \sin \theta$$

One obvious problem with extending this method (the "spin" method) to more complicated spin systems is that you always have to hunt up a set of functions like f and g .

Now contrast this with the helicity method. Here, you use the actual helicity amplitudes themselves as the independent functions. You do expand the traces in terms of spin states as noted on the bottom of page 2. But, now, keeping in mind that the helicity spin states are defined in special directions, we can say:

$$\frac{d\vec{\sigma}}{d\Omega} = \frac{1}{2} \sum_{a,b} |\langle b | f | a \rangle|^2 = \frac{1}{2} \sum_{a,b} |\langle \hat{k}_b | f | \hat{k}_a \rangle|^2 = \frac{1}{2} \sum_{a,b} |f_{ba}|^2 = \frac{1}{2} \sum_{\mu,\mu'} |f_{\mu\mu'}|^2$$

$$\frac{d\vec{\sigma}}{d\Omega} \cdot \vec{p} = \frac{1}{2} \sum_{a,b,c} \langle a | f | b \rangle \langle c | f | b \rangle^* \langle c | \vec{\sigma} | a \rangle = \frac{1}{2} \sum_{a,b,c} f_{ab} f_{cb}^* \langle c | \hat{k}_c | \vec{\sigma} | a \hat{k}_a \rangle$$

If we are going to use helicity amplitudes, it is crucial to notice that spin states a and c in this last equation had to be quantized in the $\hat{k}_f$ direction. Thus, if we then decide to choose this direction for a z axis, we get:

$$\frac{d\vec{\sigma}}{d\Omega} \cdot \hat{k}_f = \frac{d\vec{\sigma}}{d\Omega} \cdot \vec{p}_{\text{long}} = \frac{1}{2} \sum_{a,b,c} f_{ab} f_{cb}^* a \delta_{ac} = \frac{1}{2} \sum_{a,b} a |f_{ab}|^2$$

Comparison of the two normalizations used in density matrix.

The first normalization is that of Goldberger and Watson. They insist that all density matrices have unit trace. Thus:

$$\rho_f = \frac{f g; f^\dagger}{h(f g; f^\dagger)} \quad h(\rho_f) = 1 \quad \langle \sigma \rangle^f = h(\sigma \rho_f)$$

$$\frac{d\sigma}{d\Omega} = h(f g; f^\dagger) = h(\rho_f) \cdot h(f g; f^\dagger)$$

$$\langle \sigma \rangle^f = h(\sigma \rho_f) = \frac{h(\sigma f g; f^\dagger)}{h(f g; f^\dagger)} \Rightarrow \frac{d\sigma}{d\Omega} \langle \sigma \rangle^f = h[\sigma f g; f^\dagger]$$

Everyone seems to agree that at least the initial density matrix should be normalized to unity. It is only the final matrix that is different. The second normalization is that used by Martin and Spearman and Williams. Namely,

$$\tilde{\rho}_f = f g; f^\dagger \quad h(\tilde{\rho}_f) = h(f g; f^\dagger)$$

Of course, the definition of ρ_f is not going to change the final equations:

$$\frac{d\sigma}{d\Omega} = h(\tilde{\rho}_f) = h(f g; f^\dagger)$$

$$\langle \sigma \rangle^f = \frac{h(\sigma \tilde{\rho}_f)}{h(\tilde{\rho}_f)} = \frac{h(\sigma f g; f^\dagger)}{h(f g; f^\dagger)} \Rightarrow \frac{d\sigma}{d\Omega} \langle \sigma \rangle^f = h[\sigma f g; f^\dagger]$$

The Full Formalism of Polarization

This formalism will describe any 2-to-2 scattering process (S_1, S_2) goes to (S_3, S_4) , arbitrary spins. In other words, we select a specific two-body entrance channel, and a specific two-body exit channel. It will be obvious how the formalism could be generalized to describe the general scattering process of N to M particles.

We choose to normalize the final system density matrix to the particular differential cross section, as is done by Martin and Spearman, rather than to unity, as is done by Goldberger and Watson. Also, we shall use the representation of the multipole tensors as used by Martin and Spearman on pages 196 and 197. These tensors are nothing more and nothing less than Clebsh Gordon coefficients (the elements of these tensors are...).

Each statistical particle involved will be called a "beam". Thus, there are two initial beams, and two final beams. In the lab frame, one of the initial beams happens to be at rest. Each of the four beams is characterized by a certain set of numbers called multipole parameters. Basically, these parameters are just the coefficients of the basis matrices that appear in the expansion of the density matrix which describes that beam. It turns out that these basis matrices can be written as "tensors" (objects which transform under rotations in the usual way). If the beam has spin S , then there are $(2S + 1)$ tensors and thus $(2S + 1)$ sets of multipole parameters. The L value of the tensor ranges from 0 to $2S$. As usual, for a given tensor of rank L , there are $(2L + 1)$ actual multipole moments. In all, there are $(2S + 1)^2$ real numbers needed to describe the beam (one of which is unity, so really $(2S + 1)^2 - 1 =$). And there are as many basis matrices, one of which is the unit matrix. Three of the multipole parameters (those in the $L=1$ multipole) always describe the polarization vector of the beam. The higher multipole moments have little physical interpretation that I know of, though I have seen certain "surfaces" drawn to represent them (see Pol. Conference...).

In an "experiment", one is supposed to know all the multipole parameters of both initial beams. By measuring some of the multipole parameters of one (or both) of the final beams, the experimenter produces some numbers which can be compared with theory. It is the purpose of this formalism to show clearly what the predictions are for all these final multipole moments in terms of, say, the helicity amplitudes. It should be noted that one does not gain much information by just measuring the particular differential cross sections (PDCS) (by "particular" is meant from a particular initial state or set of initial multipole parameters, to all final states). In fact, with higher spins, the bulk of the totality of information lies in the final multipole moments. Thus the importance of "polarization experiments." For example, in spin 0/spin 1 scattering, polarization data is necessary to resolve the so-called Minami ambiguity.

The density matrix formalism is of course independent of the nature of the spin states chosen (for example, the direction of quantization chosen.) The spin state choice known as the "helicity formalism" will be used here. It seems to be the most modern and convenient.

The fundamental equations are these:

$$\frac{d\sigma}{d\Omega} = \text{PDCS} = h(p_f) = h(f_f; f^+)$$

$$\langle \sigma_f \rangle = \frac{h(\sigma_f p_f)}{h(p_f)} \Rightarrow \frac{d\sigma}{d\Omega} \langle \sigma_f \rangle = h(\sigma_f p_f) = h(\sigma_f f_f; f^+)$$

In these equations, tr means trace, f is the scattering operator, the ρ are the initial and final density matrices, the final being unnormalized as mentioned above. O_f is some operator describing an observable of the final system (for example, polarization of one of the final beams).

The operator f needs a little discussion. It is an operator in two senses. First and foremost, it is an operator in the quantum mechanical state Hilbert space. It is the S-matrix. If we agree to use cms two body states, then f becomes the operator which Martin and Spearman call $\mathcal{S}_p$. This operator is defined on the "little Hilbert Space" called h . It happens that h is isomorphic to E^3 , the usual euclidean 3-space, and the operators defined on this Hilbert space are rotations, contractions, inversions (if you like), etc. Actually, f is related to $\mathcal{S}_p$ by a constant, see elsewhere.

Secondly, f is an operator in spin space. It is a matrix, possibly not square, which connects the initial two particle spin states to the final ones. For example,

To get a good understanding of these density matrix equations, one might read the appropriate sections in both books already mentioned. What follows is just an explicit representation of these equations.

The density matrix describing any of the four beams is given by,

$$\frac{\rho^{(i)}}{h(\rho^{(i)})} = \frac{1}{(2S_i+1)} \sum_{L,M} (2L+1) t_{LM}^{*(i)} T_{LM}^{(i)}$$

$$t_{LM}^{(i)} = \langle T_{LM} \rangle = \frac{h(\rho^{(i)} T_{LM}^{(i)})}{h(\rho^{(i)})}$$

First we have written the density matrix as a function of the various multipole parameters, then the inversion giving the parameters in terms of the density matrix. The multipole tensors are described by a few equations:

$$T_{L,-M}^{(i)} = (-1)^M T_{LM}^{(i)\dagger} \quad t_{L,-M} = (-1)^M t_{LM}^*$$

$$\text{trace}(T_{LM}^{(i)} T_{LM}^{(i)\dagger}) = \frac{(2S^{(i)}+1)}{(2L+1)} \delta_{LL} \delta_{MM}$$

$$\langle j m | T_{LM} | j m \rangle = \langle j L; m' M | j m \rangle$$

In passing, we note that GW do not include the factor $(2L+1)$ in their expansion, and thus this same factor does not appear in their normalization (268). Highly unimportant.

Now the whole problem is writing down an explicit expression for the PDCS and for the final multipole moments in terms of helicity amplitudes. It is really quite trivial:

$$S_3 S_1 \overrightarrow{b} | f | \overleftarrow{a} S_1 S_2$$

$$\frac{d\sigma}{d\Omega} = \text{PDCS} = h(\rho_f) = h(f \rho_i f^\dagger)$$

$$= \sum_{m_3 m_4} \sum_{m_1 m_2} \sum_{m_1' m_2'} \langle m_3 m_4 | f | m_1 m_2 \rangle \langle m_1 m_2 | \rho_i | m_1' m_2' \rangle \langle m_1' m_2' | f^\dagger | m_3 m_4 \rangle$$

$$\begin{aligned} \frac{d\sigma}{d\Omega} &= \sum_{m_3 m_4} \sum_{m_1 m_2} \sum_{m'_1 m'_2} \langle m_3 m_4 | f | m_1 m_2 \rangle \langle m_3 m_4 | f | m'_1 m'_2 \rangle^* \langle m_1 | \rho_i^{(1)} | m'_1 \rangle \langle m_2 | \rho_i^{(2)} | m'_2 \rangle \\ &= \sum_{m_3 m_4} \sum_{m_1 m_2} \sum_{m'_1 m'_2} f_{m_3 m_4, m_1 m_2} f_{m_3 m_4, m'_1 m'_2}^* \rho_{i m_1 m'_1}^{(1)} \rho_{i m_2 m'_2}^{(2)} \end{aligned}$$

If one wanted to compute a PDCS, he would simply insert the initial density matrices as given on the previous page. For example,

$$\rho_{i m_1 m'_1}^{(1)} = \frac{1}{(2S^{(1)} + 1)} \sum_{L, M} (2L + 1) \langle S^{(1)} m'_1; LM | S^{(1)} m_1 \rangle t_{LM}^{(1)*}$$

The f 's are helicity amplitudes and are functions of energy and angle. Of course one might want to insert partial wave expansions for these "full" helicity amplitudes, but that is another story.

Next, we write an expression for the predicted multipole moments of, say, beam 3. Clearly a similar formula will hold for beam 4.

$$t_{LM}^{(3)} = \frac{h(T_{LM}^{(3)} \rho_f)}{h(\rho_f)}$$

$$\frac{d\sigma}{d\Omega} t_{LM}^{(3)} = h(T_{LM}^{(3)} f \rho_i f^\dagger)$$

$$= \sum_{m_3 m_4} \sum_{m'_3 m'_4} \sum_{m_1 m_2} \sum_{m'_1 m'_2} \langle m_3 m_4 | T_{LM}^{(3)} | m'_3 m'_4 \rangle \langle m'_3 m'_4 | f | m_1 m_2 \rangle \langle m_1 m_2 | \rho_i | m'_1 m'_2 \rangle \langle m'_1 m'_2 | f^\dagger | m_3 m_4 \rangle$$

$$= \sum_{m_3 m_4} \sum_{m'_3} \sum_{m_1 m_2} \sum_{m'_1 m'_2} \underbrace{(T_{LM}^{(3)})_{m_3 m'_3}}_{L = \langle S^{(3)} m'_3 LM | S^{(3)} m_3 \rangle \in \text{note primes}} f_{m'_3 m_4, m_1 m_2} f_{m_3 m_4, m'_1 m'_2}^* \rho_{i m_1 m'_1}^{(1)} \rho_{i m_2 m'_2}^{(2)}$$

That is the full theory. Everything else is just special cases.

One final point. A particular representation of any tensor implies a choice of axes, especially a choice of z axis. On page 2, we wrote a Clebsh Gordon coefficient representation for the various polarization tensors. There we see spin quantum numbers m and m' . Since we have chosen the helicity formalism, we have already decided to have m and m' refer to quantization along the direction of motion of the particle that this tensor is connected with. Thus, if particle (beam) 3 goes off in the $\hat{k}_f$ direction, this direction has to be taken as the z -axis for the tensors relating to beam 3. One cannot "choose" the z axis arbitrarily, once the helicity formalism has been selected. The choice of x and y axes is then arbitrary. Martin and Spearman suggest using $\hat{y} = \hat{k} \times \hat{k}_f$, since this in some sense conforms to the usual Euler conventions.

em. See last sheet.

no

On the Relation between the Cartesian Rotation Matrices of Classical Mechanics,
and the Rotation Matrices of Angular Momentum of Quantum Mechanics.

The definition of a cartesian vector is something that transforms like the position vector under rotations. For ejemplo,

$$A_i = R_{ij}(\alpha, \beta, \gamma) A_j \quad A_{ij} = R_{im} R_{jn} A_{mn}$$

Here we have also written the transformation of a cartesian rank-2 tensor. The explicit form of these rotation matrices is given in the "group" notes.

The definition of a rank-1 spherical tensor is:

$$T_{lm} = D_{m'm}^{(l)} T_{lm'}$$

These are the so-called "irreducible tensors" and it is probably not proper to refer to l and the "rank". One can show that the nine-component cartesian rank-2 tensor A_{ij} can be expressed in terms of $l=0, 1$, and 2 irreducible tensors. Let's come back to this later.

Going immediately to the case $l=1$, we ask how the two rotation matrices R_{ij} and $D_{m'm}^{(1)}$ are related. Rather than go at it that way, let us ask another question. Given the two different matrices, what linear combination of the T_{lm} gives an object that transforms as a cartesian vector.

$$J_i' = \sum_m a_{im} T_{1m} \quad \begin{cases} R J_i' R^\dagger = R_{ij} J_j' = J_i' \\ R T_{1m} R^\dagger = D_{m'm}^{(1)} T_{1m'} = T_{1m} \end{cases}$$

$$\begin{cases} J_i' = \sum_m a_{im} T_{1m} = \sum_m a_{im} \sum_{m'} D_{m'm}^{(1)} T_{1m'} \\ J_i' = R_{ij} J_j = \sum_j R_{ij} \sum_m a_{jm} T_{1m} \end{cases}$$

$$\Rightarrow \sum_{m,m'} a_{im'} D_{m'm}^{(1)} T_{1m} = \sum_j R_{ij} a_{jm} T_{1m}$$

$$\Rightarrow \sum_{m'} a_{im'} D_{m'm}^{(1)} = \sum_j R_{ij} a_{jm} \quad \forall m \quad (m = -1, 0, 1)$$

$$\Rightarrow (A\tilde{D})_{im} = (RA)_{im} \Rightarrow A\tilde{D} = RA \Rightarrow A = R A D^\dagger$$

all just matrices notation.

I am aborting this effort. We end up with 9 equations in 9 unknowns so it could be solved, but something says I am not doing this in the right way.

Constructing Spherical Tensors from two cartesian vectors.

Consider the cartesian vector $\overset{\text{operator}}{J}$ where J represents the angular momentum of some system. To say J is a cartesian vector is to say that it transforms under rotations as follows,

$$J'_i = \sum_{j=1}^3 R_{ij} J_j = R J_i R^{-1}$$

where R_{ij} are the usual cartesian rotation matrices. Notice that if you scale the vector J up or down, you do not change its transformation properties so it remains a cartesian vector.

Corresponding to the cartesian vector $\overset{\text{operator}}{J}$, there is a spherical vector $\overset{\text{operator}}{J}$ which transforms according to the following rule under rotations:

$$R J_m R^{-1} = \sum_{m'} D_{m'm}^{(j)} J_{m'} = J'_m$$

where here the D are the rotation matrices of angular momentum theory. Again, the scale is not set, but the relative sizes of the spherical components are of course determined.

As shown elsewhere, the relation between these two kinds of vectors is given by:

$$J_+ = a \left(-\frac{1}{\sqrt{2}}\right) [J_x + i J_y]$$

$$J_- = a \left(\frac{1}{\sqrt{2}}\right) [J_x - i J_y]$$

$$J_0 = a J_z$$

Again, a is any constant you like. If you want the angular momentum matrix elements of the operator $\text{script-}J$ to be exactly equal to the CG coefficients, then

$$a = \frac{1}{\sqrt{j(j+1)}}$$

But here we will simply take $a = 1$.

Now we claim that starting with any cartesian vector A , we can form an object that transforms as a $J=1$ spherical tensor in a similar way. Strictly this could be proved by relating the R and D rotation matrices, but somehow I have never been able to do that. Nevertheless, I am sure of the answer:

$$[A_x, A_y, A_z] \rightarrow [A_+, A_0, A_-] \quad \text{where} \quad A_+ = -\frac{1}{\sqrt{2}} [A_x + i A_y]$$

$$A_- = \frac{1}{\sqrt{2}} [A_x - i A_y]$$

$$A_0 = A_z$$

Now as a practice problem, given the six numbers $\vec{A}$ and $\vec{B}$, how would one combine them to form an object that transforms as a $J=0$ spherical tensor (ie, as a scalar). We know the answer ahead of time, but lets see how it obtains via the Clebsch Gordon coefficients:

$$|JM\rangle = \sum_{m_a, m_b} |1 m_a 1 m_b\rangle \langle 1 m_a 1 m_b | JM\rangle$$

$$\text{scalar comb} \propto |00\rangle = \sum_{m_a, m_b} A_{m_a} B_{m_b} \langle 1 m_a 1 m_b | 00\rangle$$

Here we are treating the tensors A and B as if they were angular momentum states. I am not completely clear why I am doing this, but it is going to work. So,

$$\begin{aligned}
 \text{scalar comb} \propto |00\rangle &= \sum_{m_a} A_{m_a} B_{-m_a} \langle 1 m_a | -m_a | 00 \rangle \\
 &= A_+ B_- \langle 1 1 | -1 | 00 \rangle + A_0 B_0 \langle 1 0 | 0 | 00 \rangle + A_- B_+ \langle 1 -1 | 1 | 00 \rangle \\
 &= \sqrt{\frac{1}{3}} A_+ B_- - \sqrt{\frac{1}{3}} A_0 B_0 + \sqrt{\frac{1}{3}} A_- B_+ \\
 &= \sqrt{\frac{1}{3}} \left[\left(-\frac{1}{2}\right) (A_x + i A_y)(B_x - i B_y) - A_z B_z + \left(-\frac{1}{2}\right) (A_x - i A_y)(B_x + i B_y) \right] \\
 &= \sqrt{\frac{1}{3}} [-A_x B_x - A_y B_y - A_z B_z] \\
 &= -\sqrt{\frac{1}{3}} \vec{A} \cdot \vec{B}
 \end{aligned}$$

Again, the scale is undetermined.

Next, let's try to form a J=1 object. This will correspond to the cross product, no doubt:

$$|1 m\rangle = \sum_{m_a m_b} A_{m_a} B_{m_b} \langle 1 m_a | m_b | 1 m \rangle = \sum_{m_a} A_{m_a} B_{m-m_a} \langle 1 m_a | m-m_a | 1 m \rangle$$

$$\begin{aligned}
 |1 1\rangle &= G_+ = A_+ B_0 \langle 1 1 | 0 | 1 1 \rangle + A_0 B_+ \langle 1 0 | 1 | 1 1 \rangle \\
 &= \sqrt{\frac{1}{2}} [A_+ B_0 - A_0 B_+] = \sqrt{\frac{1}{2}} \sqrt{\frac{1}{2}} [(A_x + i A_y) B_z - A_z (B_x + i B_y)] \\
 &= -\frac{1}{2} [A_x B_z - A_z B_x + i (A_y B_z - B_y A_z)] \\
 &= -\sqrt{\frac{1}{2}} [C_x + i C_y]
 \end{aligned}$$

$$\begin{aligned}
 |1 0\rangle &= A_+ B_- \langle 1 1 | -1 | 1 0 \rangle + A_0 B_0 \langle 1 0 | 0 | 1 0 \rangle + A_- B_+ \langle 1 -1 | 1 | 1 0 \rangle \\
 &= \sqrt{\frac{1}{2}} [A_+ B_- - A_- B_+] \\
 &= \sqrt{\frac{1}{2}} \left[\left(-\frac{1}{2}\right) [(A_x + i A_y)(B_x - i B_y) - (A_x - i A_y)(B_x + i B_y)] \right] \\
 &= -\frac{1}{2} \sqrt{\frac{1}{2}} [2i (A_y B_x - B_y A_x)] = -\sqrt{\frac{1}{2}} i (A_y B_x - B_y A_x) \\
 &= G_z
 \end{aligned}$$

This leaves us with three equations:

$$\begin{aligned}
 C_x + i C_y &= \frac{\sqrt{2}}{2} [(A_x B_z - A_z B_x) + i (A_y B_z - B_y A_z)] \\
 C_x - i C_y &= \frac{\sqrt{2}}{2} [(A_x B_z - A_z B_x) - i (A_y B_z - B_y A_z)] = -C_x + i C_y \\
 C_z &= -\sqrt{\frac{1}{2}} i (A_y B_x - B_y A_x)
 \end{aligned}$$

Assuming all the C_i are imaginary, we obtain:

$$\begin{aligned} 2iC_y &= \sqrt{2}(A_x B_z - A_z B_x) \Rightarrow C_y = \frac{1}{\sqrt{2}}i(A_z B_x - A_x B_z) \\ 2C_x &= \sqrt{2}i(A_y B_z - B_y A_z) \Rightarrow C_x = \frac{1}{\sqrt{2}}i(A_y B_z - B_y A_z) \\ C_z &= \frac{1}{\sqrt{2}}i(A_x B_y - A_y B_x) \end{aligned}$$

Finally, we ask what is the $J=2$ object like? It has 5 components. We get:

$$|2m\rangle = \sum_{m_a m_b} A_{m_a} B_{m_b} \langle 1m_a 1m_b | 2m \rangle$$

$$\begin{aligned} |22\rangle &= A_+ B_+ = \frac{1}{2}[A_x + iA_y][B_x + iB_y] = \frac{1}{2}[A_x B_x - A_y B_y + i(A_y B_x + A_x B_y)] \\ |2-2\rangle &= A_- B_- = \frac{1}{2}[A_x B_x - A_y B_y - i(A_y B_x + A_x B_y)] \end{aligned}$$

Similarly we could write out the other three linear combinations. Notice that these elements are symmetric in A and B. These 5 components can be related to the 5 independent elements of a traceless ~~antisymmetric~~ symmetric, rank-2 cartesian tensor.

A better way to approach this whole thing is as follows: from the angular momentum idea, we know that with the two vectors A and B we can generate objects corresponding to $J=0$, $J=1$ and $J=2$. From the property of the Clebsch Gordon coefficients:

$$\langle j_1 j_2 m_1 m_2 | J M \rangle = (-1)^{J-j_1-j_2} \langle j_1 j_2 m_2 m_1 | J M \rangle$$

we know that in this case, the $J=0,2$ objects will be symmetric under interchange of A and B, whereas the $J=1$ object will be antisymmetric. We also know the number of elements in each of these spherical tensors. It is easiest to construct them by hand directly in the cartesian space. There, the two vectors combine to form a scalar, a vector, and a rank-2 tensor. However, we know that there are only 5 elements in this rank-2 tensor, so we can set it up to be traceless (it already is ~~antisymmetric~~ symmetric.)

$$S = A_i B_i$$

$$V_k = \epsilon_{kij} A_i B_j$$

$$T_{jk} = (A_j B_k + A_k B_j) - \delta_{jk}(\vec{A} \cdot \vec{B})$$

Relation between J and T_{lm} .

We know from the start that,

$$T_{11} = a_+ J_+$$

$$T_{1-1} = a_- J_-$$

$$T_{10} = a_0 J_z$$

Any tensor operator with a 11 label transforms in the same way as J^+ . There is the possibility of a constant (independent of m , but dependent on j). To get these constants, just play with some matrix elements:

$$\langle jm | T_{10} | jm \rangle = \langle jm | a_0 J_z | jm \rangle$$

$$\langle jm | 10 | jm \rangle = a_0 m = \langle j | m_0 | jm \rangle = \frac{m}{\sqrt{j(j+1)}} \quad \text{GW p 86}$$

$$\Rightarrow a_0 = + [j(j+1)]^{-1/2}$$

$$\langle jm | T_{11} | j, m-1 \rangle = a_+ \langle jm | J_+ | j, m-1 \rangle = a_+ \langle j, m-1 | J_- | jm \rangle \quad \text{since real}$$

$$= a_+ \sqrt{(j+m)(j-m+1)}$$

$$= \langle j, m-1 | 11 | jm \rangle = \langle j | m-1, 1 | jm \rangle = - \sqrt{\frac{(j+m)(j-m+1)}{2j(j+1)}}$$

$$\Rightarrow a_+ = - [2j(j+1)]^{1/2}$$

$$\langle jm | T_{1-1} | j, m+1 \rangle = a_- \langle jm | J_- | j, m+1 \rangle = a_- \langle j, m+1 | J_+ | jm \rangle$$

$$= a_- \sqrt{(j-m)(j+m+1)}$$

$$= \langle j | m+1, -1 | jm \rangle = + \sqrt{\frac{(j-m)(j+m+1)}{2j(j+1)}}$$

$$\Rightarrow a_- = + [2j(j+1)]^{1/2}$$

Thus we see this famous minus sign, that $A_+ = -A_-$ and $A_- = (2j(j+1))^{1/2}$.

$$T_{11} = a_+ J_+$$

$$a_- = -a_+ = \frac{1}{\sqrt{2j(j+1)}}$$

$$J_x = \frac{1}{2} \sqrt{2j(j+1)} [T_{1-1} - T_{11}]$$

$$T_{1-1} = a_- J_-$$

$$a_0 = + \frac{1}{\sqrt{j(j+1)}}$$

$$J_y = \frac{i}{2} \sqrt{2j(j+1)} [T_{1-1} + T_{11}]$$

$$T_{10} = a_0 J_z$$

$$J_z = \sqrt{j(j+1)} T_{10}$$

Special Case: unpolarized input beams.

Here, we use "unpolarized" in the sense that all multipole moments of the incident beams vanish, not just the components of the polarization vector. (Of course for spin- $\frac{1}{2}$ beams these amount to the same thing.)

The equations reduce to:

$$\frac{d\sigma}{d\Omega} = \frac{1}{(2S_1+1)(2S_2+1)} \sum_{1234} |f_{34,12}|^2$$

$$t_{LM}^{(3)} = \frac{\sum_{12343'} \langle S^{(3)}_{m_3}; LM | S^{(3)}_{m_3'} \rangle f_{3'4,12} f_{34,12}^*}{\sum_{1234} |f_{34,12}|^2}$$

In particular, the polarization vector components are:

plus sign diagram with m's.

$$P_z = \frac{1}{S_3} \sum_{1234} m_3 |f_{34,12}|^2 / \sum_{1234} |f_{34,12}|^2$$

$$P_x = -\frac{1}{S_3} \sum_{1234} \sqrt{(S_3+m_3)(S_3-m_3+1)} \operatorname{Re} \left[\tilde{f}_{m_3-1, m_4, m_1, m_2} \tilde{f}_{34,12}^* \right] / \sum_{1234} |f_{34,12}|^2$$

$$\hat{y}=\hat{n} \quad P_y = +\frac{1}{S_3} \sum_{1234} \sqrt{(S_3+m_3)(S_3-m_3+1)} \operatorname{Im} \left[\tilde{f}_{m_3-1, m_4, m_1, m_2} \tilde{f}_{34,12}^* \right] / \sum_{1234} |f_{34,12}|^2$$

Remember that the x and y axes referred to here depend on which type of helicity amplitude is being used. If we use the regular f 's of Jacob and Wick and Martin and Spearman, then $\hat{x}$ and $\hat{y}$ are very inconvenient. If we use the un-pre-rotated convention, then $\hat{y} = \hat{n}$. I have been using twiddles to indicate such amplitudes.

$$\left[\tilde{f}_{m_3-1, m_4, m_1, m_2} \tilde{f}_{34,12}^* \right] = \left[e^{-i\phi} f_{m_3-1, m_4, m_1, m_2} f_{m_3, m_4, 1, 2}^* \right]$$

Of course, if we chose our initial axis so that $\hat{y} = \hat{n}$ to start with, then $\phi = 0$ and no distinction is needed. I will assume these axes from now on.

Special Case: the polarization vector!

We found that,

$$\frac{d\sigma}{d\Omega} t_{LM}^{(3)} = \sum_{\dots} \langle S_{m_3}^{(3)} LM | S_{m_3}^{(3)} \rangle \dots$$

We also know that:

$$\vec{P}^{(3)} = \frac{\langle \vec{S}^{(3)} \rangle}{S^{(3)}}$$

From now on, refer to $S^{(3)}$ simply as J to simplify notation. Then we know:

$$P_i = \frac{\langle J_i \rangle}{J} = ?$$

$$P_x = \frac{\langle J_x \rangle}{J} = \frac{1}{2} \frac{\sqrt{2J(J+1)}}{J} [t_{1-1}^{(3)} - t_{11}^{(3)}]$$

$$P_y = \frac{\langle J_y \rangle}{J} = \frac{i}{2} \frac{\sqrt{2J(J+1)}}{J} [t_{1-1}^{(3)} + t_{11}^{(3)}]$$

$$P_z = \frac{\langle J_z \rangle}{J} = \frac{\sqrt{J(J+1)}}{J} t_{10}^{(3)}$$

First lets treat the longitudinal polarization P_z . We know:

$$\begin{aligned} \frac{d\sigma}{d\Omega} P_z &= \sum_{\dots} \left[\frac{\sqrt{J(J+1)}}{J} \langle S_{m_3}^{(3)} 10 | S_{m_3}^{(3)} \rangle \right] \dots \\ &= \sum_{\dots} \left[\frac{m_3}{J} \delta_{m_3 m_3'} \right] \dots \end{aligned}$$

Thus we can immediately write:

$$\frac{d\sigma}{d\Omega} P_z^{(3)} = \frac{1}{J} \sum_{m_3 m_4} \sum_{m_1 m_2} \sum_{m_1' m_2'} m_3 \cdot f_{m_3 m_4, m_1 m_2}^* f_{m_3 m_4, m_1' m_2'} S_{i m_1 m_1'}^{(1)} S_{i m_2 m_2'}^{(2)}$$

Doing P_x and P_y at the same time we get:

$$\begin{aligned} \frac{d\sigma}{d\Omega} \begin{Bmatrix} P_x \\ i P_y \end{Bmatrix} &= \sum_{\dots} \left[\frac{1}{2} \frac{1}{J} \sqrt{2J(J+1)} \{ \langle S_{m_3}^{(3)} 1-1 | S_{m_3}^{(3)} \rangle \mp \langle S_{m_3}^{(3)} 1+1 | S_{m_3}^{(3)} \rangle \} \right] \dots \\ &= \sum_{\dots} \left(\frac{1}{2J} \sqrt{2J(J+1)} \right) \left[\delta_{m_3', m_3+1} \langle S_{m_3+1}^{(3)} 1-1 | S_{m_3}^{(3)} \rangle \mp \right. \\ &\quad \left. \delta_{m_3', m_3-1} \langle S_{m_3-1}^{(3)} 1+1 | S_{m_3}^{(3)} \rangle \right] \dots \\ &= \sum_{\dots} \left(\frac{1}{2J} \sqrt{2J(J+1)} \right) \left[\delta_{m_3', m_3+1} \frac{\sqrt{(J-m_3)(J+m_3+1)}}{\sqrt{2J(J+1)}} \mp \delta_{m_3', m_3-1} \frac{(-1) \sqrt{(J+m_3)(J-m_3+1)}}{\sqrt{2J(J+1)}} \right] \end{aligned}$$

$$0 = \frac{1}{2J} \sum_{\dots} \left[\sqrt{(J-m_3)(J+m_3+1)} \delta_{m'_3, m_3+1} \pm \sqrt{(J+m_3)(J-m_3+1)} \delta_{m'_3, m_3-1} \right] \dots$$

Thus we can write the x and y polarizations as,

$$\frac{d\sigma}{d\Omega} \begin{Bmatrix} P_x \\ -iP_y \end{Bmatrix} = \frac{1}{2J} \sum_{m_3 m_4} \sum_{m_1 m_2} \sum_{m'_1 m'_2} \sqrt{(J-m_3)(J+m_3+1)} f_{m_3+1 m_4, m_1 m_2} f_{m_3 m_4, m'_1 m'_2}^* \rho_{m_1 m'_1}^{(1)} \rho_{m_2 m'_2}^{(2)} \\ + \frac{1}{2J} \sum_{\text{same}} \sum_{m_3 m_4} \sum_{m'_1 m'_2} \sqrt{(J+m_3)(J-m_3+1)} f_{m_3-1 m_4, m_1 m_2} f_{m_3 m_4, m'_1 m'_2}^* \rho_{m_1 m'_1}^{(1)} \rho_{m_2 m'_2}^{(2)}$$

Now redefine the summation index in the first equation. Note that the range of these m sums is restricted such that the helicity amplitude subscripts make sense, no matter what summation index is chosen.

$$1^{\text{st}} \text{ term} = \frac{1}{2J} \sum_{\tilde{m}_3 m_4} \sum_{i, i'} \sum_{i', i} \sqrt{(J+\tilde{m}_3)(J-\tilde{m}_3+1)} f_{\tilde{m}_3 m_4, m_1 m_2} f_{\tilde{m}_3-1, m'_1 m'_2}^* \rho_{i m_1 m'_1}^{(1)} \rho_{i' m_2 m'_2}^{(2)} \\ \tilde{m}_3 \equiv m_3+1$$

Now drop the twiddle and add the two terms:

$$\frac{d\sigma}{d\Omega} \begin{Bmatrix} P_x \\ -iP_y \end{Bmatrix} = \frac{1}{2J} \sum_{m_1 m_2} \sum_{m'_1 m'_2} \sum_{m_3 m_4} \sqrt{(J+m_3)(J-m_3+1)} \rho_{i m_1 m'_1}^{(1)} \rho_{i' m_2 m'_2}^{(2)} \\ \times \left[f_{m_3 m_4, m_1 m_2} f_{m_3-1 m_4, m'_1 m'_2}^* \pm f_{m_3-1 m_4, m_1 m_2} f_{m_3 m_4, m'_1 m'_2}^* \right]$$

$$\frac{d\sigma}{d\Omega} \{P_z\} = \frac{1}{J} \sum_{m_1 m_2} \sum_{m'_1 m'_2} \sum_{m_3 m_4} m_3 \rho_{i m_1 m'_1}^{(1)} \rho_{i' m_2 m'_2}^{(2)} f_{m_3 m_4, m_1 m_2} f_{m_3 m_4, m'_1 m'_2}^*$$

In general, these initial density matrices are not symmetric, so the Im part simplification cannot be made (but see other sheet for unpolarized case.)

Polarization and Helicity State Phase Choice.

We run into a slight problem of interpretation with our final polarization equations. We found, for example, an explicit expression for " P_z ", the longitudinal polarization of one of the final beams. Here, we knew that " z " meant the direction along the beam because that is the way the helicity spin states are quantized.

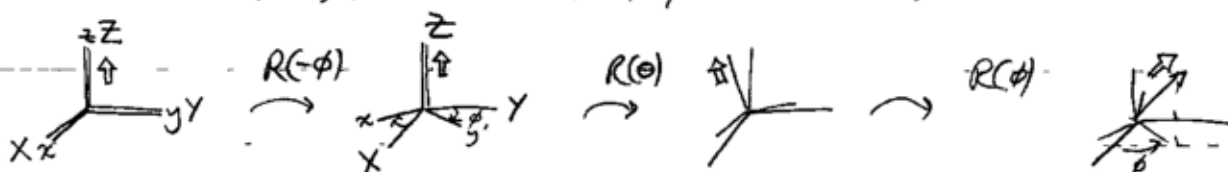
However, we also came up with an equation for " P_x " and now we have a problem: what is the x axis? It seems like a trivial question, but it is not really so trivial. What is the x axis to which all those multipole tensors are referenced! It all boils down to another question: in the helicity amplitudes which we have been using, to what x axis are those spin states m_3 and m_4 referenced?

$$\langle p\theta\phi, m_3 m_4 | f | p\theta\phi, m_1 m_2 \rangle = f_{m_3 m_4, m_1 m_2}$$

At first it might seem that you can put the x axis wherever you want. But that can't be right, because the polarization formula for P_x is for some definite direction! We know that rotating the x axis about the z axis causes the spin states to pick up some phase. It is this phase that makes all the difference.

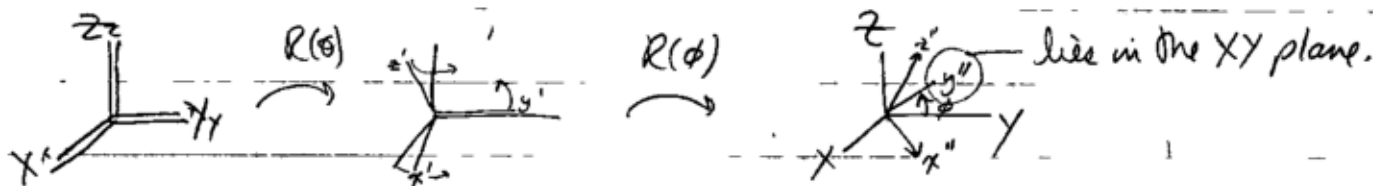
Let's examine how the helicity states were constructed. The single particle helicity state was built like this:

$$| \vec{p}, \mu \rangle = | p\theta\phi, \mu \rangle = R(\theta, \phi) | p_z, \mu \rangle \quad R(\theta, \phi) = e^{-i\phi J_z} e^{-i\theta J_y} e^{+i\phi J_z}$$



The spin states are referenced to a set of xyz axes (body axes) that are "stuck" to the momentum vector as it is rotated. Due to the Jacob-Wick pre-rotation, the final position of the x and y axes is very obscure and inconvenient. (And very hard to draw.)

So, it seems that the best thing to do is to go back and define different helicity states, ones that do not use the prerotation. Then it is easy to locate the body x and y axes:



$$\hat{y}'' = \frac{\hat{z} \times \hat{z}''}{\sin\theta} = \hat{k} \times \hat{k}_z = \hat{n}$$

So the reference y axis is none other than the familiar n axis, perpendicular to the scattering plane. Using a twiddle to denote these new helicity states, (and no twiddle to refer to MS or JW states),

$$| \widetilde{p\theta\phi}, \mu \rangle = \widetilde{R}(\theta, \phi) | p_z, \mu \rangle = R(\theta, \phi) e^{-i\phi J_z} | p_z, \mu \rangle = e^{-i\phi \mu} | p\theta\phi, \mu \rangle$$

For the single particle state, the difference is this simple phase.

Next, let's find out what the new two particle cms helicity state looks like.

$$|p\theta\phi\mu_1\mu_2\rangle = R(\theta,\phi) |p00\mu_1\mu_2\rangle$$

$$|p00\mu_1\mu_2\rangle = |m_1j_1p_z\mu_1\rangle \otimes |m_2j_2-p_z\mu_2\rangle \times (-1)^{j_2-\mu_2}$$

Notice here that u_2 is the component of the spin of particle 2 along the $-z$ axis. The new state will be:

$$|\widetilde{p\theta\phi\mu_1\mu_2}\rangle = \widetilde{R}(\theta,\phi) |p00\mu_1\mu_2\rangle$$

$$= R(\theta,\phi) e^{-i\phi J_z} |m_1j_1p_z\mu_1\rangle \otimes |m_2j_2-p_z\mu_2\rangle \times (-1)^{j_2-\mu_2}$$

$$= e^{-i\phi\mu_1} e^{-i\phi(-\mu_2)} \times |p\theta\phi\mu_1\mu_2\rangle$$

$$= e^{-i\phi(\mu_1-\mu_2)} |p\theta\phi\mu_1\mu_2\rangle$$

Thus, the new helicity amplitude is:

$$\widetilde{f}_{\mu_3\mu_4;\mu_1\mu_2} = \langle \widetilde{p\theta\phi\mu_3\mu_4} | f | p00\mu_1\mu_2 \rangle = e^{+i\phi(\mu_3-\mu_4)} f_{\mu_3\mu_4;\mu_1\mu_2}$$

Effect of Parity Invariance on Results for unpolarized beams.

Parity does not make any of the multipole parameters vanish. It does however cause two of the three components of the polarization vector to vanish. The longitudinal case is easiest:

$$P_z \sim \sum_{1234} (m_3) |f_{34,12}|^2 = \sum_{1234} (-m_3) |f_{3,-4,-1,-2}|^2 = \sum_{1234} (-m_3) |f_{34,12} e^{i\pi}|^2$$

$\uparrow$ relabel $\uparrow$ parity

$$= -P_z \Rightarrow \boxed{P_z = 0}$$

P_x , the component in the scattering plane, is a little harder to show:

$$P_x \sim \sum_{1234} \sqrt{(S_3+m_3)(S_3-m_3+1)} \operatorname{Re} [f_{3,-1,4,1,2} f_{3,4,1,2}^*]$$

$$= \sum_{1234} \sqrt{(S_3+m_3)(S_3-m_3+1)} \operatorname{Re} [f_{3,-4,-1,-2} f_{-(3-1),-4,-1,-2}^*]$$

This step took indices 1, 2, and 4 to minus themselves (a relabelling). It also took m_3 to $-(m_3-1)$, which happens to leave the square root factor invariant. Then,

$$= \sum_{1234} \sqrt{\quad} \operatorname{Re} [f_{34,12} \eta (-1)^{[m_3-m_4]-[m_1-m_2]} f_{3,-1,4,1,2}^* \eta^* (-1)^{[m_3-1-m_4]-[m_1-m_2]}]$$

The η factors are + or - one and depend only on spins and intrinsic parities. Thus, the two η 's combine to give unity, being helicity independent. Then,

$$= \sum_{1234} \sqrt{\quad} \operatorname{Re} [f_{34,12} f_{3,-1,4,1,2}^* (-1)]$$

$$= - \sum_{1234} \sqrt{\quad} \operatorname{Re} [f_{3,-1,4,1,2} f_{34,12}^*] = -P_x \Rightarrow \boxed{P_x = 0}$$

P_y does not vanish because the presence of Im rather than Re adds an extra sign, so you get $P_y = P_y$. (It could vanish, of course.)

Polarization. Want to know "all there is to know" about polarization in all formalisms:

- 1) π -N case with (q, h) , (f_1, f_2) , (f_3, f_4) ...
- 2) N-N case with a, b, c, m, n and $(\phi_1, \phi_2 \dots \phi_5)$
- 3) full helicity formalism analysis.

Present references are:

- 1) Jacob+Wick (1959). This of course contains everything - initial and final density matrices, transverse pol, stat. tensors - but the stuff is very compact. I would prefer to learn it from a monograph which explains all. Someone in the last 15 years must have written such a monograph.
- 2) Collins+Squires (1967). As far as specific regge fits go, these guys do not go into full detail. For example, they have a section on π -N CEX which has the g , but all equations are simply lifted from Höhler (1966). Here, do/dt has that famous dip in t. In their NN discussion, they seem to use those same ϕ_i 's as Rarita et al. Sharp + Wagner, P + Rarita, Singh are all referenced. Also, note ref to Jackson on an "absorptive" model. Later, in section 8.7, there is a very brief section on polarization and a very significant formula purported to be taken from J+W. The normal pol of particle 3 in the usual $\hat{n}$ direction is:

$$P = \frac{\sum_{\lambda_1 \lambda_2 \lambda_3 \lambda_4} [(\sigma_3 + \lambda_3)(\sigma_3 - \lambda_3 + 1)]^{1/2} \operatorname{Im} [\langle \lambda_3 - \lambda_1 | A | \lambda_1 \lambda_2 \rangle \langle \lambda_3 \lambda_4 | A | \lambda_1 \lambda_2 \rangle^*]}{\sum_{\lambda_1 \lambda_2 \lambda_3 \lambda_4} |\langle \lambda_3 \lambda_1 | A | \lambda_1 \lambda_2 \rangle|^2}$$

This thing is completely general.

- 3) Martin+Speiser (1969). In their chap. 4 they do a long section on P, but it seems to be specialized to spin-0 :: spin- π . They do not make clear how this connects to "unpolarized-beams = spin-0".

Density Matrix Approach to Total Cross Section

Let μ = helicity set for initial state
 μ' = helicity set for final state.

We know, eg, that: $\frac{d\sigma}{d\Omega} = \sum_{\mu'} |f_{\mu'\mu}|^2 = \langle \mu | f^\dagger f | \mu \rangle$

This leads to:

$$\left(\frac{d\sigma}{d\Omega} \right)^{\text{static}} = \left\langle \frac{d\sigma}{d\Omega} \right\rangle = \langle f^\dagger f \rangle = \text{tr}(f^\dagger f \rho_i)$$

Similarly, we have

$$\sigma_{\text{tot}} \approx \text{tr } f_{\mu,\mu} (\text{forward}) \approx \text{tr} \langle \mu | f | \mu \rangle$$

Thus, one would expect that:

$$\sigma_{\text{tot}}^{\text{static}} = \langle \sigma_{\text{tot}} \rangle = \langle \text{tr } f \rangle = \text{tr}(\text{tr } f \rho_i)$$

Specifically, then:

$$\langle \sigma_{\text{tot}} \rangle = \text{tr}(\text{tr } f \rho_i) = \sum_{\mu, \mu'} \text{tr} \langle \mu | f | \mu' \rangle \langle \mu' | \rho_i | \mu \rangle$$

But, $\langle \mu | f | \mu' \rangle = \delta_{\mu\mu'} \langle \mu | f | \mu \rangle$ for f at $t=0$, thus

$$\langle \sigma_{\text{tot}} \rangle = \sum_{\mu} \text{tr} \langle \mu | f | \mu \rangle \cdot \langle \mu | \rho_i | \mu \rangle$$

With explicit helicity indices,

$$\langle \sigma_{\text{tot}} \rangle = \sum_{a,b} \text{tr} \langle a,b | f | a,b \rangle \cdot \langle a | \rho^A | a \rangle \cdot \langle b | \rho^B | b \rangle$$

where

$$\langle a | \rho^A | a \rangle = \frac{1}{(2S_a+1)} \sum_{L,M} (2L+1) T_{LM}^{A*} \cdot \langle a | T_{LM}^A | a \rangle$$

where

$$\begin{aligned} \langle a | T_{LM}^A | a \rangle &= \langle j_a; LM | j_a \rangle = \langle S_a m_a; LM | S_a m_a \rangle \\ &= \delta_{M,0} \langle S_a m_a; L0 | S_a m_a \rangle \end{aligned}$$

$$\text{Thus, } \langle a | \rho^A | a \rangle = \frac{1}{(2S_a+1)} \sum_L (2L+1) T_{L0}^{A*} \langle S_a a; L0 | S_a a \rangle$$

If A and B are unpolarized, then

$$\langle a | \rho^A | a \rangle = \frac{1}{(2S_a+1)} \quad \text{so} \quad \langle \sigma_{\text{tot}} \rangle = \frac{1}{(2S_a+1)} \frac{1}{(2S_b+1)} \sum_{a,b} \text{tr} \langle a,b | f | a,b \rangle$$

In case of $\pi-N$, say, we have:

$$\langle a | p^A | a \rangle = \frac{1}{2}$$

$$\langle b | p^B | b \rangle = \frac{1}{2} \sum_L (2L+1) t_{L0}^* \langle \frac{1}{2} b; L0 | \frac{1}{2} b \rangle.$$

Since $L=0$ or 1 only:

$$\langle b | p^B | b \rangle = \frac{1}{2} [1 + 3 t_{10}^* \langle \frac{1}{2} b; 10 | \frac{1}{2} b \rangle] = \frac{1}{2} [1 \pm \sqrt{3} t_{10}^*].$$

$$\left[\langle b | p^B | b \rangle = \frac{1}{2} [1 + 2 \vec{P}_B \cdot \langle b | \vec{S} | b \rangle] \right] \text{ not.}$$

Now, $t_{10}^* = \frac{1}{\sqrt{2(L+1)}}$

ab, only S_z has diagonal, so

$$\langle b | S_z | b \rangle = b_z$$

So, $\langle b | p^B | b \rangle = \frac{1}{2} [1 + 2 b \vec{P}_B \cdot \hat{z}] \quad b = \pm \frac{1}{2}$

Then, for such a polarized incident nuclear beam,

$$\begin{aligned} \langle \sigma_{tot} \rangle &= \sum_{ab} \dots = \sum_b \text{Im} \langle b | f | b \rangle \cdot \frac{1}{2} [1 + 2 b \vec{P}_B \cdot \hat{z}] \\ &= \text{Im} f_{++} \left(\frac{1 + \vec{P}_B \cdot \hat{z}}{2} \right) + \text{Im} f_{--} \left(\frac{1 - \vec{P}_B \cdot \hat{z}}{2} \right) \\ &= \text{Im} f_{++} [1] = \text{Im} f_{++} \quad / \text{parity.} \end{aligned}$$

Well, since both fcs's are the same, the mixture does not matter.

In NN case $\langle \sigma_{tot} \rangle = \sum_{ab} \text{Im} \langle ab | f | ab \rangle \cdot \frac{1}{2} [1 + 2 a \vec{P}_A \cdot \hat{z}] \pm [1 + 2 b \vec{P}_B \cdot \hat{z}]$

$$\langle \sigma_{tot} \rangle = \frac{1}{4} \left\{ (1 + \vec{P}_A \cdot \hat{z})(1 + \vec{P}_B \cdot \hat{z}) \text{Im} f_{++++} + \text{etc} \right\}$$

The main conclusion is this: For unpolarized in,

$$\boxed{\langle \sigma_{tot}^{AB} \rangle_{unp} = \frac{4\pi}{q_i} \langle \text{Im} f_{ab,ab} \rangle}$$

$$= \frac{4\pi}{q_i} \times \frac{1}{(2S_A+1)} \frac{1}{(2S_B+1)} \sum_{ab} \text{Im} [f_{ab,ab}]$$

Problem: what is the density matrix which describes a spin- σ beam which is in a pure spin state? I.e., what are the various multipole parameters?

① The density matrix for such a particle has the form $\langle m | \rho | m' \rangle$. A general "statistical beam" will have diagonal and off-diagonal elements, eg, $\rho_i = \frac{1}{2} [1 + \vec{P}_i \cdot \vec{\sigma}_i]$ in spin- $\frac{1}{2}$ case. However, if the beam is in a pure helicity state m , then: $\langle m | \rho | m' \rangle = \delta_{m',m} \delta_{m,m} \rho_{mm} \sim$ i.e., the density matrix has only one entry somewhere on the diagonal.

② So by "pure spin state" we mean the particle is in one of its angular eigenstates (w.r.t. some chosen axis), then boosted up to $\hat{p}$. Helicity closes $\hat{p}$ as "chosen axis." I will use helicity.

③ Now we can calculate the multipole parameters:

$$t_{LM} = \frac{\text{tr}(\rho T_{LM})}{\text{tr}(\rho)} = \frac{\sum_{m,m'} \langle m | \rho | m' \rangle \langle m' | T_{LM} | m \rangle}{\sum_m \langle m | \rho | m \rangle}$$

Well, $\sum_m \langle m | \rho | m \rangle = \rho_{\lambda\lambda}$, where $\lambda =$ helicity of pure state.

$$\begin{aligned} \text{Then: } \rho_{\lambda\lambda} t_{LM} &= \sum_{m,m'} \delta_{m,m'} \delta_{m,\lambda} \langle \sigma m' | T_{LM} | \sigma m' \rangle \rho_{\lambda\lambda} \\ &= \rho_{\lambda\lambda} \langle \sigma \lambda | L M | \sigma \lambda \rangle \end{aligned}$$

$$\text{So: } t_{LM} = \langle \sigma \lambda | L M | \sigma \lambda \rangle = \delta_{M0} \langle \sigma \lambda | L 0 | \sigma \lambda \rangle$$

④ Comments: clearly, need $M=0$ or else get 0. There will be a t_{L0} for all L such that $\sigma < L < 2\sigma$, i.e., $\sigma \otimes L \otimes \sigma = 1$, i.e., $\forall L \Rightarrow L < \sigma \otimes \sigma = 2\sigma, 2\sigma-1, 2\sigma-2, \dots, 0$. I.e., there will be $(2\sigma+1)$ non vanishing multipole parameters.

$$t_{00}, t_{10}, t_{20}, \dots, t_{(2\sigma),0}$$

⑤ For spin- $\frac{1}{2}$, you will have $t_{00} \neq 0$ and $t_{10} \neq 0$.

Answer: $\chi_2^\dagger \chi_2$ is the total cross section into all final helicities starting with initial state $\chi_1 = (a, b)$, whatever you like.

But why is this = trace ρ_2 ?

$$\rho_2 \equiv \chi_2 \chi_2^\dagger = (f \chi_1) (f \chi_1)^\dagger = f \chi_1 \chi_1^\dagger f^\dagger = f \rho_1 f^\dagger \quad \rho_1 \equiv \chi_1 \chi_1^\dagger$$

$$\begin{aligned} \text{tr}(\rho_2) &= \text{tr}(\chi_2 \chi_2^\dagger) = \sum_i (\chi_2 \chi_2^\dagger)_{ii} = \sum_i (\chi_2)_i (\chi_2^\dagger)_i = \sum_i (\chi_2^\dagger)_i (\chi_2)_i \\ &= \chi_2^\dagger \chi_2 = d\sigma/d\Omega \text{ (totn of } a, b, \theta, \phi) \end{aligned}$$

Problem: given a target polarized $\vec{P}_1$, what do you get for $\vec{P}_2$?

$$\rho_1 = \frac{1}{2} (1 + \vec{P}_1 \cdot \vec{\sigma})$$

$$\rho_2 = \frac{d\sigma}{d\Omega} (1 + \vec{P}_2 \cdot \vec{\sigma})$$

$$\rho_2(\vec{P}_1=0) = \rho_2^0 = f \rho_1^0 f^\dagger = \frac{1}{2} (f f^\dagger)$$

$$\frac{d\sigma}{d\Omega}(\vec{P}_1=0) = \text{tr}(\rho_2^0) = \frac{1}{2} \text{tr}(f f^\dagger) = \frac{1}{2} \sum_{\mu, \mu'} |f_{\mu\mu'}|^2$$

↙ average on initial.

In this case, what is $\vec{P}_2$?

$$\frac{1}{2} (f f^\dagger) = \frac{1}{2} \text{tr}(f f^\dagger) [1 + \vec{P}_2 \cdot \vec{\sigma}] \quad \text{could extract } \vec{P}_2.$$

Parity will force it to be in the $\hat{n}$ direction, no doubt. Now, more generally,

$$\rho_2(\vec{P}_1 \neq 0) = \rho_2(\vec{P}_1) = f \rho_1(\vec{P}_1) f^\dagger = \frac{1}{2} f (1 + \vec{P}_1 \cdot \vec{\sigma}) f^\dagger$$

$$\frac{d\sigma}{d\Omega} = \text{tr}(\rho_2) = \text{tr}(\frac{1}{2} f (1 + \vec{P}_1 \cdot \vec{\sigma}) f^\dagger)$$

Then: $\vec{P}_2$ could be found from:

$$\frac{1}{2} f (1 + \vec{P}_1 \cdot \vec{\sigma}) f^\dagger = \text{tr} \left\{ \frac{1}{2} f (1 + \vec{P}_1 \cdot \vec{\sigma}) f^\dagger \right\} \times (1 + \vec{P}_2 \cdot \vec{\sigma})$$

Rather messy. But suppose $\vec{P}_1 = P_1 \hat{z}$ where $\hat{z} = \hat{n}$. We know already that $\vec{P}_2$ must be in $\hat{z}$ also by parity, so:

$$(1 + P_1 \sigma_z) = \begin{pmatrix} 1+P_1 & 0 \\ 0 & 1-P_1 \end{pmatrix}$$

$$f \begin{pmatrix} 1+P_1 & 0 \\ 0 & 1-P_1 \end{pmatrix} f^\dagger = f \begin{pmatrix} (1+P_1) f_{\frac{1}{2}\frac{1}{2}}^\dagger & (1+P_1) f_{\frac{1}{2}-\frac{1}{2}}^\dagger \\ (1-P_1) f_{-\frac{1}{2}\frac{1}{2}}^\dagger & (1-P_1) f_{-\frac{1}{2}-\frac{1}{2}}^\dagger \end{pmatrix} \dots$$

Problem: In the usual density matrix scheme, exactly what is meant by the $d\sigma/d\Omega$ that appears in $\psi(\theta^{\text{out}}) = d\sigma/d\Omega$?

Suppose you start with a pure state $\chi_1 = \begin{pmatrix} a \\ b \end{pmatrix}$. Here, a is the amplitude that initial particle has $\mu_i = +1/2$. Now we can ask: what is the total $d\sigma/d\Omega$ for this state summed over all final states? First guess is:

$$\frac{d\sigma}{d\Omega} = \sum_{\text{final}} |\text{amp}|^2 = \sum_{\mu_f} |a f_{\frac{1}{2}\mu_f} + b f_{-\frac{1}{2}\mu_f}|^2 \quad (*)$$

Now, defining $\chi_2 = f\chi_1$, what is $\chi_2^\dagger \chi_2$?

$$\begin{aligned} \chi_2^\dagger \chi_2 &= \chi_1^\dagger f^\dagger f \chi_1 = (a^* \ b^*) f^\dagger f \begin{pmatrix} a \\ b \end{pmatrix} \\ &= (a^* \ b^*) \begin{bmatrix} (f^\dagger f)_{11} & (f^\dagger f)_{12} \\ (f^\dagger f)_{21} & (f^\dagger f)_{22} \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = (a^* \ b^*) \begin{bmatrix} a(f^\dagger f)_{11} + b(f^\dagger f)_{12} \\ a(f^\dagger f)_{21} + b(f^\dagger f)_{22} \end{bmatrix} \\ &= |a|^2 (f^\dagger f)_{11} + a^* b (f^\dagger f)_{12} + b^* a (f^\dagger f)_{21} + |b|^2 (f^\dagger f)_{22} \\ &= |a|^2 (f^\dagger f)_{\frac{1}{2}\frac{1}{2}} + a^* b (f^\dagger f)_{\frac{1}{2}-\frac{1}{2}} + b^* a (f^\dagger f)_{-\frac{1}{2}\frac{1}{2}} + |b|^2 (f^\dagger f)_{-\frac{1}{2}-\frac{1}{2}} \quad (**)$$

Now expand formula (*) above:

$$\begin{aligned} &\sum_{\mu_f} [a f_{\frac{1}{2}\mu_f} a^* f_{\frac{1}{2}\mu_f}^* + |b|^2 f_{-\frac{1}{2}\mu_f} f_{-\frac{1}{2}\mu_f}^* + a f_{\frac{1}{2}\mu_f} b^* f_{-\frac{1}{2}\mu_f}^* + b f_{-\frac{1}{2}\mu_f} a^* f_{\frac{1}{2}\mu_f}^*] \\ &= \sum_{\mu_f} \{ |a|^2 f_{\frac{1}{2}\mu_f} f_{\frac{1}{2}\mu_f}^* + |b|^2 f_{-\frac{1}{2}\mu_f} f_{-\frac{1}{2}\mu_f}^* + b^* a f_{\frac{1}{2}\mu_f} f_{\mu_f-\frac{1}{2}}^* + a^* b f_{-\frac{1}{2}\mu_f} f_{\mu_f\frac{1}{2}}^* \} \\ &= |a|^2 (f f^\dagger)_{\frac{1}{2}\frac{1}{2}} + |b|^2 (f f^\dagger)_{-\frac{1}{2}-\frac{1}{2}} + b^* a (f f^\dagger)_{\frac{1}{2}-\frac{1}{2}} + a^* b (f f^\dagger)_{-\frac{1}{2}\frac{1}{2}} \end{aligned}$$

Here I have used: first subscript = initial helicity. If I had used other convention, we'd get:

$$\left. \frac{d\sigma}{d\Omega} \right|_{\text{total from } (a,b)} = \sum_{\mu_f} |a f_{\mu_f\frac{1}{2}} + b f_{\mu_f-\frac{1}{2}}|^2 \quad (*)$$

$$\begin{aligned} &= \sum_{\mu_f} [|a|^2 f_{\mu_f\frac{1}{2}} f_{\mu_f\frac{1}{2}}^* + a f_{\mu_f\frac{1}{2}} b^* f_{\mu_f-\frac{1}{2}}^* + b f_{\mu_f-\frac{1}{2}} a^* f_{\mu_f\frac{1}{2}}^* + |b|^2 f_{\mu_f-\frac{1}{2}} f_{\mu_f-\frac{1}{2}}^*] \\ &= |a|^2 (f^\dagger f)_{\frac{1}{2}\frac{1}{2}} + a b^* (f^\dagger f)_{-\frac{1}{2}\frac{1}{2}} + b a^* (f^\dagger f)_{\frac{1}{2}-\frac{1}{2}} + |b|^2 (f^\dagger f)_{-\frac{1}{2}-\frac{1}{2}} \end{aligned}$$

Why is (*) = (**)? Look at (*) closely:

$$a f_{\mu_f\frac{1}{2}} + b f_{\mu_f-\frac{1}{2}} = \begin{pmatrix} f_{\frac{1}{2}\frac{1}{2}} & f_{\frac{1}{2}-\frac{1}{2}} \\ f_{-\frac{1}{2}\frac{1}{2}} & f_{-\frac{1}{2}-\frac{1}{2}} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \quad \text{one row of.}$$

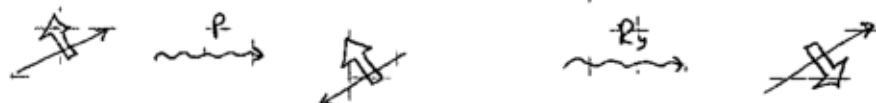
This paper of some five pages describes the polarization experiment at CERN. They use a polarized target to do this.

The experimental set up is shown. Many "hodoscopes" in this apparatus. This paper describes the logic design and use of on-line computer monitoring. There is no physics here. But, we are given a reference for "the experiment." However, this ref. turns out to be a paper submitted at the Vienna 1968 conference, + is not included in those proceedings.

So, we can look at our PL 31B (1970) 405 ref. Earliest ref. with Bonghini, et al. is PL 24B (1967) 77. As time went on, the τ -range improves, but I think the exper. remains the same. But, again here we are referred again to PL 21 (1966) 114. But, this reference is on π -p, not pp. Nevertheless, ... there is one equation of physics here which shows how you can measure P_z , the polarization normal to the scattering plane. Where can I read about pol-target physics? There is a connection here to parity conservation, methinks.



These pictures are supposed to show that a longitudinal polarization is not possible. What about "transverse, in-plane" pole?



Similarly it vanishes. Only a $\hat{n}$ polarization can survive so this must be what all the experiments are about. But where does the target enter?

Denom. is the total (into all hel. states) d.o.s. with an unpolarized beam.

Another way to write it:

$$P_3^z = \left\{ \frac{\sum_{\mu_1, \mu_2, \mu_3, \mu_4} \mu_3 |f_{1234}|^2}{\sum_{\mu_1, \mu_2, \mu_3, \mu_4} |f_{1234}|^2} \right\}$$

Now all we need is a regge form for the helicity amplitude: Should remove kinematic singularities: First, remove direct channel sing.

$$\hat{f}_{1234} = \left(\frac{1-z}{2}\right)^{-\frac{(\lambda-\lambda')}{2}} \left(\frac{1+z}{2}\right)^{-\frac{(\lambda+\lambda')}{2}} f_{1234}$$

$\hat{f}_{1234}$ has no weird z -plane sing.

Next, remove kin sing induced by crossed channel:

$$\hat{\hat{f}}_{1234} = (s - 4M^2)^{\alpha/2} S^{\lambda/2} \hat{f}_{1234}$$

$$\begin{aligned} \lambda &= \mu_1 + \mu_2 \\ \lambda' &= \mu_3 + \mu_4 \end{aligned}$$

$$\alpha = 2 + \text{Max}_i \left\{ \sum_{j=1}^4 \left(\frac{1}{2} - \epsilon_j \right) \right\} - |\lambda - \lambda'| - |\lambda + \lambda'|$$

$$\alpha/2 = \left(1 - \frac{|\lambda - \lambda'|}{2} - \frac{|\lambda + \lambda'|}{2} \right) \quad \beta = \text{Max}_i 1 = 1.$$

Thus, it would seem that:

$$\hat{\hat{f}}_{1234} = [s - 4M^2]^{1 - \frac{|\lambda - \lambda'|}{2} - \frac{|\lambda + \lambda'|}{2}} S^{-1/2} \hat{f}_{1234}$$

At large s , this becomes a complicated power of s , and:

$$1 - z_s = -\frac{t}{2q^2} = -\frac{2t}{s - 4M^2} \approx -2t/s \quad \frac{1-z_s}{2} = -t/s.$$

Thus, it would seem that:

$$\begin{aligned} \hat{\hat{f}}_{1234} &= \text{Regge poles} = (-t/s)^{-\frac{\lambda-\lambda'}{2}} \cdot s^{1 - \frac{|\lambda-\lambda'|}{2} - \frac{|\lambda+\lambda'|}{2} - \frac{1}{2}} f_{1234} \\ &= t^a s^b f_{1234}. \end{aligned}$$

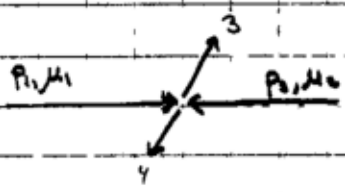
These powers are helicity dependent. Next, use a regge form:

$$f_{1234} \text{ or } \hat{\hat{f}}_{1234} = \sum_i \frac{(2\alpha_i + 1)}{\sin(\pi(\alpha_i + \pi/2))} \beta_{1234}^{\alpha_i \pm} d_{\lambda\lambda'}^{\pm}(\alpha_i, -z) z^{\alpha_i} s^{\alpha_i}$$

Zowie, this gets complicated very fast indeed. Need some papers on polarization.

More general experiment.

5.



We can still look at particle 3.

What happens to:

$$\chi_3 = \mathcal{M} \chi_1 \dots \quad \frac{d\sigma}{d\Omega} |_{\mu_1 \mu_2 \mu_3 \mu_4} = |f_{\mu_1 \mu_2 \mu_3 \mu_4}|^2$$

But, what now is the density matrix? We need a combined spin state...

$$\chi_{34} = \chi_3 \otimes \chi_4 \quad |p\phi \mu_3 \mu_4\rangle = \chi_{34}$$

$$|p\phi \mu_3 \mu_4\rangle = \langle p\phi \mu_3 \mu_4 | A | p_0 0 \mu_1 \mu_2 \rangle \times \dots$$

This state $\uparrow$ does not norm to $d\sigma/d\Omega$. We could define:

$$|\mu_3 \mu_4\rangle = f_{\mu_1 \mu_2 \mu_3 \mu_4} |\mu_1 \mu_2\rangle \quad \chi_{34} = f_{\mu_1 \mu_2 \mu_3 \mu_4} \chi_{12}$$

Then f again is a matrix with $(\mu_3 \mu_4)$ together labelling a row...

$$\rho_f = |\mu_3 \mu_4\rangle \langle \mu_3 \mu_4| = \chi_{34} \chi_{34}^\dagger = (f \chi_{12})(\chi_{12}^\dagger f^\dagger) = f \rho_i f^\dagger$$

$$\rho_i = |\mu_1 \mu_2\rangle \langle \mu_1 \mu_2| \quad \langle \mu_1 \mu_2 | \mu_1 \mu_2 \rangle = \chi_{12}^\dagger \chi_{12} = 1$$

$$\langle \mu_3 \mu_4 | \mu_3 \mu_4 \rangle = |f_{\mu_1 \mu_2 \mu_3 \mu_4}|^2 = \text{specific d.c.s.} = \frac{d\sigma}{d\Omega} |_{\mu_3 \mu_4}$$

So in some sense all the old equations follow, and:

$$\vec{P}_3 = \frac{\langle \vec{S}_3 \rangle}{S_3} = \frac{1}{S_3} \frac{\text{Tr}(\vec{S}_3 \rho_{\text{out}})}{\text{Tr}(\rho_{\text{out}})} \quad \rho_{\text{out}} = f \rho_i f^\dagger$$

$$\begin{aligned} \text{Tr}(\rho_{\text{out}}) &= \text{Tr}(\chi_{34} \chi_{34}^\dagger) = \text{Tr} [|\mu_3 \mu_4\rangle \langle \mu_3 \mu_4|] = \sum_{\mu_0 \mu_0} \langle \mu_0 \mu_0 | \mu_3 \mu_4 \rangle \langle \mu_3 \mu_4 | \mu_0 \mu_0 \rangle \\ &= \sum_{\mu_0 \mu_0} | \langle \mu_0 \mu_0 | \mu_3 \mu_4 \rangle |^2 = \sum_{\mu_0 \mu_0} | \langle \mu_3 \mu_4 | \mu_0 \mu_0 \rangle |^2 = \sum_{\mu_1 \mu_2} |f_{\mu_1 \mu_2 \mu_3 \mu_4}|^2 \end{aligned}$$

Note that here, $[\mu_0 \mu_0]$ are some basis states, normed to 1.

Lets defer discussion here and find a solid reference. Now do case of longitudinal pol:

$$\begin{aligned} P_3(\text{long}) &= \frac{1}{(2S_1+1)(2S_2+1)} \sum_{S_1 S_2} \sum_{S_4} \left[\sum_{\mu_3} |f_{\mu_1 \mu_2 \mu_3 \mu_4}|^2 \mu_3 \right] \\ &= \frac{1}{(2S_1+1)(2S_2+1)} \sum_{\mu_1 \mu_2} \sum_{\mu_3 \mu_4} |f_{\mu_1 \mu_2 \mu_3 \mu_4}|^2 \end{aligned}$$

The polarization final is:

$$I(\theta, \phi) \vec{P}(\theta, \phi) = \frac{1}{s} \text{Tr}(\vec{S} f f^\dagger) \frac{1}{2s+1} = \frac{1}{s(2s+1)} \text{Tr}(f f^\dagger \vec{S})$$

On the 2x2 case, $\vec{S} = \hat{z}$, but more generally we know J_x, J_y, J_z in the usual rep. See "group" section: the x and y have nonzero el's one off the diagonal. We can now explicitly compute:

$\vec{P}_0 \cdot \hat{p}' =$ final long. pole. with initial unpol.

$$= \frac{1}{s(2s+1)} \cdot \frac{1}{I_0} \cdot \text{Tr}(f f^\dagger \vec{S} \cdot \hat{p}')$$

But in this x'y'z' system, $\hat{p}' = \hat{z}'$ so this is just $S_z = \mu \delta_{\mu\mu'}$.

$$\vec{P}_0 \cdot \hat{p}' = \frac{1}{s(2s+1) I_0} \sum_{\mu'' \mu'''} f_{\mu'' \mu} f_{\mu'''}^\dagger \mu'' \delta_{\mu'' \mu'''} = \frac{1}{s(2s+1) I_0} \sum_{\mu, \mu''} f_{\mu \mu''} f_{\mu'' \mu}^\dagger \mu$$

$$\vec{P}_0 \cdot \hat{p}' = \frac{1}{s(2s+1) I_0} \sum_{\mu} \mu |f_{\mu \mu}|^2$$

Thus, a knowledge of matrix f tells you the long pole. Next, do the $\perp$ p x p' component:

$$\begin{aligned} \vec{S} \cdot \hat{n} = S_y &\Rightarrow \sum_{\mu \mu'} (f f^\dagger)_{\mu \mu'} (S_y)_{\mu' \mu} = -\frac{1}{2} i \sum_{\mu \mu'} (f f^\dagger)_{\mu \mu'} \sqrt{(s-\mu)(s+\mu+1)} \delta_{\mu', \mu+1} \\ &= -\frac{1}{2} i \sum_{\mu} (f f^\dagger)_{\mu, \mu+1} \sqrt{(s-\mu-1)(s+\mu+2)} \\ &= -\frac{1}{2} i \sum_{\mu} (f f^\dagger)_{\mu-1, \mu} \sqrt{(s-\mu)(s+\mu+1)} \dots \end{aligned}$$

Examine this piece:

$$(f f^\dagger)_{\mu-1, \mu} = \sum_{\mu''} f_{\mu-1, \mu''} f_{\mu'' \mu}^\dagger = \sum_{\mu''} f_{\mu-1, \mu''} f_{\mu, \mu''}^*$$

But, had I included the other term, $\delta_{\mu', \mu-1}$, we'd get the purely real answer demanded, since $\vec{P}_0 \cdot \hat{n}$ is real.

Thus, (4.196 a, b, c) are all reasonable + computable in terms of f .

Question: so far we can handle O-J scattering. What about more general J-J'?

The answer is a multipole expansion, viz.,

$$g = (2j+1)^{-1} \sum_{L,M} (2L+1) t_{LM}^* T_{LM} \quad t_{LM} = \langle T_{LM} \rangle$$

These matrices T_{LM} are normalized so that:

$$\langle j m | T_{LM} | j m' \rangle = \langle j L m' M | j m \rangle = \text{simple Clebsch-Gordan coeff.}$$

Although we know these matrices explicitly:

$$T_{j m, j m'}^{LM} = \langle j L m' M | j m \rangle$$

it is only their average values that we need to use:

$$\langle T^{LM} \rangle = t_{LM}$$

In the special case of $j = 1/2$, we had:

$$(t_{11}, t_{10}, t_{1-1}) = (P_x + i P_y, P_z, P_x - i P_y) \quad \vec{r} \rightarrow \vec{p}$$

Now write out the first terms in the g sum:

$$g = (2j+1)^{-1} \left[I + 3 \sum_M t_{1M}^* T_{1M} + 5 \sum_M t_{2M}^* T_{2M} + \dots \right]$$

Examine the dipole term:

$$\sum_M t_{1M}^* T_{1M} = \sum_M t_{1M} T_{1M} = [j(j+1)]^{-1} \langle \vec{J} \rangle \cdot \vec{J}$$

What do these things mean in the general case? What operators lie beyond $\vec{J}$? I guess there is no physical meaning beyond the dipole piece. They are just terms in a finite expansion — each term is a "moment".
So:

$$g = (2j+1)^{-1} \left[I + 3 \underset{\text{dipole}}{\langle T_1 \rangle \cdot T_1} + 5 \underset{L=2}{\langle T_2 \rangle \cdot T_2} + 7 \underset{L=3}{\langle T_3 \rangle \cdot T_3} + \dots \right]$$

Now, return to that unpolarized example. All moments are zero, so:

$$g^{in} = (2j+1)^{-1} I$$

Thus, at once many results obtain:

$$I(\theta, \phi) = \frac{d\sigma}{d\Omega} = h (f g^{in} f^\dagger) = (2j+1)^{-1} h (f f^\dagger) = \frac{1}{2s+1} \sum_{j m} |f_{j m}|^2$$

$$h(\rho^{\text{out}}) = d\sigma/d\Omega \equiv I(\theta, \phi) = \text{intensity or cross section}$$

$$\vec{P} = \frac{1}{\sigma} h(\vec{S} f \rho^{\text{in}} f^*) / I(\theta, \phi)$$

- Recall this analysis is for $a+b \rightarrow a+b$ where a has spin- s , b has no spin. Now do some examples: first, unpolarized incoming: Timeout. Must go back + read 7.1 on multipole parameters. But this in turn leads us back to page 59 on density matrix.

Hermiticity of ρ :

$$\textcircled{1} \rho_{kj} = \sum_i \langle k | X_i \rangle \rho_i \langle X_i | j \rangle$$

$$\rho_{ik}^* = \rho_{kj}^* = \sum_i \langle j | X_i \rangle^* \rho_i^* \langle X_i | k \rangle^* = \sum_i \langle k | X_i \rangle \rho_i \langle X_i | j \rangle = \rho_{kj} \quad \text{Q.E.D.}$$

$$\textcircled{2} \sum_{j,k} a_{jk} \rho_{kj} = \langle A \rangle = \langle A \rangle^* = \sum_{j,k} a_{jk}^* \rho_{kj}^* = \sum_{j,k} a_{kj}^* \rho_{jk}^* = \sum_{j,k} a_{jk} \rho_{kj} \quad \checkmark$$

Now, produce a set of orthogonalized operators, one is $c \times$ unity:

$$Tr(T_i T_k^\dagger) = \delta_{ik} \quad \{T_i\} = \text{set of matrices} \quad T_0 = cI$$

- This set $\{T_i\}$ should be complete in the sense that:

$$A = \sum_i h(AT_i^\dagger) T_i = \text{full expansion:}$$

$$h(A) = \sum_i h(AT_i^\dagger) h(T_i) = h(AT_0^\dagger) h(T_0) = c h(A) \cdot c h(I) = c^2 N h(A) \quad \checkmark$$

$$h(AT_j^\dagger) = \sum_i h(AT_i^\dagger) h(T_i T_j^\dagger) = h(AT_j^\dagger) \quad \checkmark$$

In particular, expand the special matrix operator ρ :

$$\rho = \sum_i h(\rho T_i^\dagger) T_i \equiv \sum_i t_i^* T_i = \sum_i t_i T_i^\dagger = \rho^\dagger = \rho \quad (2.27)$$

$$\langle T_j \rangle = h(T_j \rho) = \sum_i t_i^* h(T_j T_i) = \dots = \sum_i t_i h(T_j T_i^\dagger) = t_{ij}$$

$$t_{ij} = \langle T_j \rangle$$

- In the 2×2 case, these T_i matrices are N^2 in number + are in fact just the Pauli matrices + identity.

The problem is this: how do you extend this idea from $i = \frac{1}{2}$ to any s :

$$\rho = \frac{1}{2} I + \vec{P} \cdot \vec{\sigma}$$

7.2 Polarization Formulae.

Think of X_1 as a state vector whose components $\langle X_1 | i \rangle$ are the projections onto the base states. X_1 is an initial state, X_2 the final state. Both have the same "dimension" so that:

$$X_2 = M X_1$$

where M is a matrix telling how much goes where and so forth. Now, the density matrix associated with pure state X_1 is:

$$\rho_{ij}^{X_1} = \langle i | X_1 \rangle \langle X_1 | j \rangle$$

Assume these normalizations:

$$X_1^\dagger X_1 = 1 \quad h(\rho_1) = \sum_i \langle i | X_1 \rangle \langle X_1 | i \rangle = \sum_i |\langle X_1 | i \rangle|^2 = \|X_1\|^2 = 1$$

$$X_2^\dagger X_2 = \frac{d\sigma}{d\Omega} \quad h(\rho_2) = \|X_2\|^2 = d\sigma/d\Omega$$

Now we can relate ρ_2 and ρ_1 :

$$\rho_2 = X_2^\dagger X_2 = (M X_1)^\dagger (M X_1) = X_1^\dagger M^\dagger M X_1 \quad \dots \text{error}$$

$$\rho_2 = X_2 X_2^\dagger = |X_2\rangle \langle X_2| = \text{a matrix} = (M X_1) (M X_1)^\dagger = M X_1 X_1^\dagger M^\dagger = M \rho_1 M^\dagger$$

Hence, the famous results:

$$\rho_2 = M \rho_1 M^\dagger \quad \frac{d\sigma}{d\Omega} = X_2^\dagger X_2 = h(\rho_2) = h(M \rho_1 M^\dagger)$$

Thus we support (4.190). The elements of matrix M could be labelled:

$$M_{\mu\mu'} \quad \mu = \text{incoming helicity} \quad \mu' = \text{outgoing helicity}$$

$\{i\}$ = helicity basis

In the density matrix formalism we can express the "polarization" as:

$$\begin{aligned} \vec{P} &\equiv \frac{\langle \vec{S} \rangle}{S} & \vec{S} &= \text{spin vector of outgoing in "out" frame } x'y'z' \\ &= \frac{1}{S} h(\vec{S} \rho^{\text{out}}) / h(\rho^{\text{out}}) \end{aligned}$$

But recall that: