

QED / Radiation

Bjorken & Drell (1964) Chapt 8: Higher Order Corrections

"Insertions" and Ward Identity

Magnetic Moment

Triangle Anomaly

Positronium

Lamb Shift

QED Processes

Radiation Theory

Recent QED Tests - Strauch

1964

Bjorken and Drell, CHAPTER 8, "Higher Order Corrections"

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Overall Comments on BD chapter 8: "Higher Order Corrections": ---

1. This was my third "shot" at this chapter. The first time I looked at it was during 230. I absorbed very little because I failed to ignore the integration details. Also, I was unhappy with the feynman rules at the time and so did not believe the various expressions. The second reading must have been during 230C. Again, I got very little out.

This time, I got the whole picture. The reason I suppose is that I have more general knowledge of what is going on, but also, this time I took careful notes, section by section, to keep myself honest. I made an decided effort to find the "results" and to ignore the details of integration. Maybe some other time I will look at all the tricks for doing these integrals.

Usually it is the job of the author to cleanly separate the "results" from the messy details, not the job of the reader. That is one of the major faults of Bjorken and Drell, that you cannot go in there and read the results without trampling over all calculational detail.

For example, I wanted to look up the result for the Lamb shift. The dialogue necessary to locate this result was interstrewn with calculation, so I had to read the whole damn chapter to get the result. In retrospect, of course, it was good to have read it, but I would have preferred to make that decision independently.

2. Many ideas are contained in this chapter, so I will just list some key words. Many of these were on my "list", so this was a profitable digression:

relative signs of feynman graphs and generalized Pauli
photon symmetrization can generate disconnected graphs
the cutoff method of doing integrals
subtractions
gauge condition
 Z_3 and vac pol charge renormalization
photon vac pol propagator correction
vac pol Uehling potential and -27 mc
order by order unitarity
external wavefunction renormalization for photons
electron propagator modification
mass shift δm , mass renormalization
 Z_2 charge renormalization
external electron wavefunction renormalization
mass counterterms
Ward's Identity
anomalous magnetic moment
the infrared problem and Brem.
Lamb shift and vacuum fluctuations

3. This chapter should have been required reading before we went into our renormalization discussion in 230C. It took me maybe 3 days to do this.

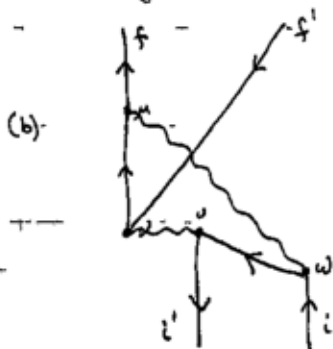
Bjorken + Drell: Section 8.1: "e⁻e⁺ elastic, 4th order"

m B + P.

① Comments: I never "got along" with BD vol. 1 because they did all their calcs based on propagator theory instead of field theory. I certainly am stubborn! It is clear that any of their volume 1 perturbation calcs can be thought of either way. Just use the damn rules, you slobbing fool!

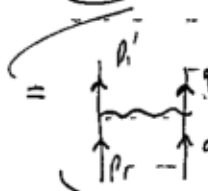
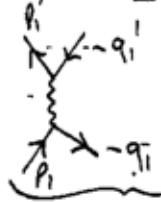
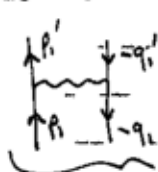
In particular, $\Psi_i^{(+)} = \text{initial electron external wavefunction}$
 $= \Omega^{-1/2} u(p_i, s_i) e^{-ip_i \cdot x}$

② The game here is "the relative signs".



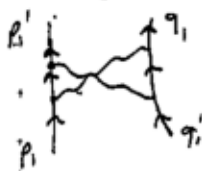
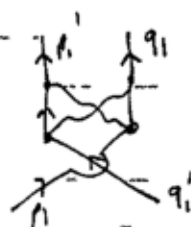
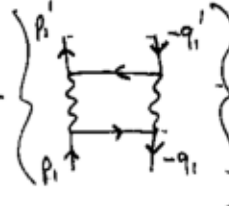
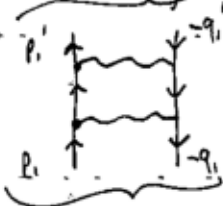
Maybe this idea does not apply to (a) vs. (b).

Go back to



After you do crossing, you can understand why the sign.

Similarly:



(These have exchanged electrons in initial, hence the minus.)

The "Dirac structure" "knows" about crossing.

relative signs

③ So, BD open this section by writing in full, with signs, 5 of the 18 graphs. In order 4th order. I dig it.

Other graphs?

1 like c

3 like d

1 like a

1 like b.

6

+ 5 = 11 so far, 7 missing

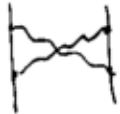
How about:



4



2

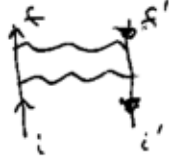


1

That's it! No, already sym'd.

$$(\text{diagram 1} + \text{diagram 2}) + (\text{diagram 3} + \text{diagram 4}) ? \times$$

No, I am doing this all wrong.

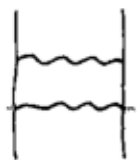


is the only (a) type.

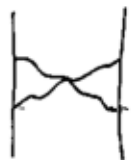
We do get 6 from 1, so 6+5=11, 7 left to find.

1 like c? $\Rightarrow$ 3 left to find. Perhaps the stuff?
3 like d

④ I'm not sure of these 18 graphs. No reference given. Not in IR, Sak, M, Sch, Cas. Each graph is already a no, you should do explicit photon sym.



+



These are two separate graphs.

a) 2

b) 2

c) 2

d) 4

e) 1

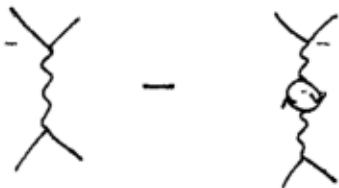
f) 6

g) 1



here are your 18, at last.

⑤ Why do fermion loops get minus signs?



Well, consider:

$$\int d^4x d^4y \langle a_\mu(p) : \bar{\psi} \gamma_\mu \psi A_\mu : : \bar{\psi} \gamma_\nu \psi A_\nu : | a_\nu(p) \rangle$$

$$\sim \int d^4x d^4y C_\mu C_\nu \epsilon^{\mu\nu} \epsilon^{\alpha\beta} \langle 0 | : \bar{\psi} \gamma_\mu \psi : \bar{\psi} \gamma_\nu \psi : | 0 \rangle$$

$$\bar{\psi}_a^{(x)} (\gamma_\mu)_{\alpha\beta} \psi_\beta^{(x)} \bar{\psi}_a^{(y)} (\gamma_\nu)_{\alpha\beta} \psi_\beta^{(y)}$$

$$= (\gamma_\mu)_{\alpha\beta} (\gamma_\nu)_{\alpha\beta} \langle \bar{\psi}_a^{(x)}, \bar{\psi}_a^{(y)} | 0 \rangle \langle \psi_\beta^{(x)}, \psi_\beta^{(y)} | 0 \rangle$$

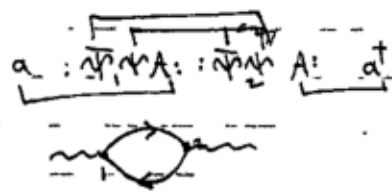
$$= C_\mu C_\nu (\gamma_\mu)_{\alpha\beta} (\gamma_\nu)_{\alpha\beta} \bar{\psi} i S_F(x-y)_{\alpha\beta} i S_F(x-y)_{\beta\alpha}$$

$$\equiv \epsilon^{\mu\nu} (-1) \text{trace} (i S_F(x-y) \gamma_\mu i S_F(x-y) \gamma_\nu)$$

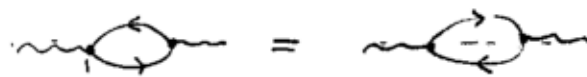
→ The minus is because you have to reorder one pair from $\bar{\psi}\psi$ to $\psi\bar{\psi}$ in order to get an SF.

⑥ How does the counting work?

- x 1 / two ways to hook up photon
- x 2 / two ways to contract $\bar{\psi}\psi$
- ÷ 2! since second order



I seem to end up with extra rule of ~~X~~ X1 ✓



I guess you do not really add these, but I would like to see it "come out" directly. Ok, done!

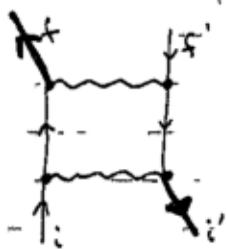
→ trace, and (-1), and no loop correction.

minus sign
in fermion
loop.

trace.



⑦ Once more look at (a) and (b):



No rule.

We are interchanging final electron with initial position, so - sign.

⑧ Now, how do you do the integrals? We can only start by doing the various "insertions".

⑨ This section at last rings loud and clear!



○ Bjorken & Drell, Section 8.2 : "Vacuum Polarization"

① They define: $\left[\text{loop} = I_{\mu\nu} \right]$

Then: $\text{loop} = q_\mu(1) a_\nu(2) I^{\mu\nu}(1,2)$

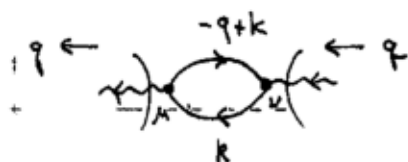
If $q_\mu(1)$ is a physical photon, then $q_\mu \rightarrow q_\mu + 2X$ makes no diff.
In momentum space:

$$q_\mu I^{\mu\nu}(1,2) = 0 \quad \text{if "1" physical}$$

In mass-space:

$$\left[q_\mu I^{\mu\nu} = 0 \right]$$

② Set up $I_{\mu\nu}$ specifically:



○ Note convention: the $\frac{-i}{\not{p}-m}$, p is momentum along arrow, so label carefully.

$$\begin{aligned} I_{\mu\nu}(q) &= \int \frac{d^4k}{(2\pi)^4} (-1) \cdot \text{trace} \left(\frac{+i}{\not{k}-m} [-ie\gamma_\nu] \frac{+i}{\not{k}-\not{q}-m} [-ie\gamma_\mu] \right) \\ &= -(e^2) \int \frac{d^4k}{(2\pi)^4} \text{trace} \left(\frac{1}{\not{k}-m} \gamma_\nu \frac{1}{\not{k}-\not{q}-m} \gamma_\mu \right) \end{aligned}$$

Note that it is symmetric in $\mu \leftrightarrow \nu$. Now:

$$q^\mu I_{\mu\nu} = 0 = -(e^2) \int \frac{d^4k}{(2\pi)^4} \text{trace} \left[\not{q} \frac{1}{\not{k}-m} \gamma_\nu \frac{1}{\not{k}-\not{q}-m} \right]$$

③ Now comes that old quinnick:

$$\frac{1}{\not{k}-\not{q}-m} \not{q} \frac{1}{\not{k}-m} = \frac{1}{\not{k}-\not{q}-m} - \frac{1}{\not{k}-m}$$

Hence:

$$0 = -(e^2) \int \frac{d^4k}{(2\pi)^4} \text{trace} \left[\left(\frac{1}{\not{k}-\not{q}-m} - \frac{1}{\not{k}-m} \right) \gamma_\nu \right]$$

④ So what?!

⑤ We are introducing the idea of "cutoffs" and "subtractions".
 Eg, define:

$$\bar{I}_{\mu\nu}(q) = (I_{\mu\nu}(q, m^2) - I_{\mu\nu}(q, M^2)) \leftarrow \text{seems this is done differently in Vol 2}$$

Clearly $\lim_{M \rightarrow \infty} I_{\mu\nu}(q, M^2) = 0$ so $\lim_{M \rightarrow \infty} \bar{I}_{\mu\nu}(q) = I_{\mu\nu}(q, m^2)$.

But do not take $M \rightarrow \infty$ until the very end! M is a "subtraction point".

Notice that: $q_\mu \bar{I}_{\mu\nu}(q) = 0$! $\leftarrow$ convenience.

⑥ Now it is just business to show that:

$$\bar{I}_{\mu\nu}(q) = \frac{i\alpha}{3\pi} (q_\mu q_\nu - g_{\mu\nu} q^2) \cdot \left[\log \frac{M^2}{m^2} - 6 \int_0^1 dz z(1-z) \log \left(1 - \frac{q^2}{m^2} z(1-z) \right) \right]$$

It is of course the gauge invariance which tells us ahead of time that $\bar{I}_{\mu\nu}$ has this form.

Note: I would have appendicized these calculations!

⑦ In $\pi\pi$ the $q_\mu q_\nu$ term goes away, so:

$$\begin{aligned} \frac{\text{diagram}}{\text{diagram}} &= -\frac{i g_{\mu\nu}}{q^2} + \frac{i g_{\mu\nu}}{q^2} \left(\frac{\alpha}{3\pi} \ln \left(\frac{M^2}{m^2} \right) \right) \quad \text{at } q^2 = 0 \\ &= -\frac{i g_{\mu\nu}}{q^2} \left(1 - \frac{\alpha}{3\pi} \ln \left(\frac{M^2}{m^2} \right) \right) \end{aligned}$$

$$\text{Thus: } \boxed{\frac{Z}{e_{\text{bare}}} \left(1 - \frac{\alpha}{3\pi} \ln \frac{M^2}{m^2} \right) = e_{\text{phys.}}}$$

Z_3

$$\text{So: } \boxed{e_R^2 = Z_3 e_0^2}$$

⑧ So, this is a specific calculation of Z_3 to second order.
 Note $\alpha' = 137$, period.

⑨ What is the "propagator correction"?

$$\frac{2\alpha}{\pi} \int_0^1 dz z(1-z) \left[-\frac{q^2 z(1-z)}{m^2} \right] = -\frac{2\alpha q^2}{\pi m^2} \int_0^1 x^2(1-x)^2 dx$$

This definite integral is:

$$\int_0^1 x^2(1-2x+x^2) dx = \frac{1}{3}x^3 - 2\frac{1}{4}x^4 + \frac{1}{3}x^5 = \frac{1}{3} - \frac{1}{2} + \frac{1}{3} = \frac{6-15+10}{30} = \frac{1}{30}$$

So $= -\frac{\alpha q^2}{\pi m^2 \cdot 15}$ This means:

$$e^2(-\frac{i q \omega}{q^2}) \rightarrow \frac{-i q \omega}{-q^2} \left[\left(1 - \frac{\alpha}{3\pi} \log \frac{M^2}{m^2} \right) - \left(\frac{\alpha q^2}{15\pi m^2} \right) \right] e_B^2$$

$$e_R^2 = Z_3 e_B^2, \text{ so we have: } \left[Z_3 - \frac{Z_3 \alpha q^2}{15\pi m^2} \right] e_B^2$$

$$= Z_3 e_B^2 \left[1 - \frac{\alpha q^2}{Z_3 \cdot 15\pi m^2} \right] \text{ This seems wrong...}$$

Certainly it is $\alpha_R^{-1} = 137$, not α_B^{-1} . And this certainly is α_B .

⑩ Repeat: $e_B^2 \left[1 - \frac{\alpha_B}{3\pi} \ln \frac{M^2}{m^2} - \frac{\alpha_B}{15\pi} \left(\frac{q^2}{m^2} \right) \right] \leftarrow$

$$= e_B^2 \left[Z_3(\text{to first order in } \alpha_B) - \frac{\alpha_B}{15\pi} \frac{q^2}{m^2} \right]$$

In order that $Z_3 \approx 1 - \frac{\alpha_B}{3\pi} \ln \frac{M^2}{m^2}$ to be reasonable approx, we need correction small. Then $\alpha_B \approx \alpha_R$, so

$$\alpha_B = Z_3^{-1} \alpha_R \approx \alpha_R \text{ to lowest order.}$$

So $\left[\left(1 - \frac{\alpha}{15\pi} \frac{q^2}{m^2} \right) \right]$ is the result for small q^2

⑪ Recall that $\frac{1}{q^2} \rightarrow -\frac{1}{4\pi R}$ under F.T., $q^0 \approx 0$.

Well,

$$-\frac{\alpha_R}{15\pi m^2} \rightarrow ? = -\frac{\alpha_R}{15\pi m^2} \delta^2(x) !$$

$$\int \frac{d^3 q}{(2\pi)^3} \frac{1}{|\vec{q}|^2} e^{+i\vec{q} \cdot \vec{x}} = \frac{1}{4\pi |\vec{x}|}$$

V.P.
correction
to propagator.

$$S_0 \rightarrow \frac{1}{-|q|^2} [1] - \frac{\alpha R}{15\pi m^2} \rightarrow \left[\underbrace{\frac{1}{4\pi |x|^2}}_{\text{old potential}} + \underbrace{\frac{\alpha R}{15\pi m^2} \delta^3(\vec{x})}_{\text{new term}} \right]$$

⑫ How does this affect SE?

$$H = H_0 + V \quad \text{where } V = \frac{(Ze) \cdot e}{4\pi \epsilon_0 r}$$

strengthens potential,
⇒ energy drops.

$$\lambda = \frac{h}{c}$$

The shift is always

energy shift

$$\Delta E = \langle \psi | H' | \psi \rangle = \int d^3x \psi^*(x) \left(\frac{Ze^2 \alpha}{15\pi m^2} \delta^3(\vec{x}) \right) \psi(x)$$

$$= -\frac{Ze^2 \alpha}{15\pi m^2} |\psi(0)|^2$$

What is $|\psi(0)|^2$ for hydrogen atom?

$$\psi = u = R_{nl}(r) Y_{lm}(\theta, \phi)$$

all $l=0$ states in hydrogenlike are lowered in this way.

For H $\frac{2}{s} \rightarrow \frac{3}{p} \rightarrow \frac{4}{d}$

⑬ Laguerres as $r \rightarrow 0$?

$$L_{n+l}^{2l+1}(s) \propto (-1)^{2l+1} \frac{[(n+l)!]^2}{(n-l-1)!(2l+1)!}$$

$$= \text{const.} \quad \Delta E = \frac{4 d^5 \cdot m_e}{15\pi (2^3)}$$

Now R_{nl} all $\rightarrow 0$ unless $l=0$, so:

$$L_n(s) = (-1)^l \frac{(n!)^2}{(n-l)!}$$

$$= -1.12 \times 10^{-13} \text{ MeV}$$

$$f = \frac{1}{t} = \frac{E}{h} = \frac{1}{6.58 \times 10^{-22}} = -27 \text{ mHz}$$

Then:

$$R_{n0}(r) = - \left\{ \left(\frac{2Z}{n a_0} \right)^3 \frac{(n-1)!}{2n (n-1)!^2} \right\}^{1/2} \cdot (-1)^l \frac{(n!)^2}{(n-l)!}$$

27.1 result

And, $Y_{00} = \frac{1}{\sqrt{4\pi}}$. Thus:

$$|\psi|^2 = |u|^2 = \frac{1}{4\pi} |R_{n0}|^2 = \frac{1}{4\pi} \left(\frac{2Z}{n a_0} \right)^3 \frac{(n-1)!}{2n (n-1)!^2} \frac{(n!)^4}{((n-1)!)^2}$$

$$= \frac{1}{4\pi} \frac{8Z^3}{n^3 a_0^3} \frac{n!^2}{2n^2 (n-1)!^2} = \frac{1}{4\pi} \frac{4Z^3}{n^3 a_0^3} = |\psi_{0n}(0)|^2$$

Thus,

$$\Delta E = -\frac{Ze^2 \alpha}{15\pi m^2} \cdot \frac{1}{\pi} \frac{Z^3}{n^3 a_0^3}$$

$$= -\frac{Z^4 \cdot 4 \alpha^5 \cdot m}{15\pi n^3} = \Delta E$$

$$= -\frac{Z^4 \cdot 4\pi \alpha^5}{15\pi^2 m^2 n^3} \cdot \frac{(m a_0)^3}{\pi n^3} = \frac{Z^3 (m a_0)^3}{\pi n^3} = \frac{1}{\pi} \left(\frac{Z m a_0}{n} \right)^3$$

Veiling formula

(14) To summarize: the vac. pol. correction to the propagator causes an observable effect. By changing the propagator, it changes the NR potential in coord space. The corrected potential is pointlike $\delta^3(\vec{x})$; this is the Uehling potential, 1935.

$$eV = -\left(\frac{e^2}{4\pi r} + \frac{2e^2}{15\pi m^2} \delta^3(\vec{x})\right)$$

$$e = H.L.$$

In gaussian units

$$e_{HL} V_{HL} = -\frac{e_{HL}^2}{4\pi r} = e_G V_G = -\frac{e_G^2}{r}$$

So:

$$eV = -\left[\frac{e^2}{r} + \frac{4}{15} \alpha \frac{e^2}{m^2} \delta^3(\vec{x})\right]$$

Uehling correction

$$\hbar = c = 1, \quad e = e_G$$

Causes $\Delta E = -2.7 \text{ mHz}$ in classic case:

$$[\text{pole} + \text{pole} = \text{pole}_{\text{Uehling}}]$$

This is not the Lamb shift's major component.

(15) I presume there should also be a muon correction here

$$\Delta E(\text{muon}) = \left(\frac{m_\mu}{m_e}\right) \Delta E(\text{electron}) \approx 210 \times (2.7 \text{ MeV})?$$

□? Where is this baby?

(16) Next, they examine the large q^2 limit, second order, of

$$\text{pole} + \text{pole}$$

Very unclear what happens. Maybe $e_R \rightarrow e_{\text{bare}}$; if $e_{\text{bare}} = 0$, then maybe $e_R \rightarrow 0$ (asym. freedom?!).

(17) If $q^2 > 4m^2$, I_{uv} acquires an imag part, as you'd expect:

$$| \text{pole} |^2 = \text{pole} ?$$

But:

$$\text{pole}^{(2)} \sim \text{Im} \left[\text{pole}^{(1)} \right]$$

I have not worked on "order by order field theory unitarity."

Bjorkel Drell on "Vacuum Polarization"

1. Any time you have an electron photon situation, it is possible that the photon can "spend some of its time" being an e^+e^- pair. When we correct the photon propagator for this possibility of having the photon "polarize the vacuum", two things result:
 - a) the charge² gets renormalized by a $Z_3 = 1 - (a/3\pi)\ln(M/m)^2$, where M is a cutoff.
 - b) the momentum space propagator has added to it a small constant piece $-(a/15\pi m^2)$. This constant piece in configuration space appears as a origin point correction to a Coulomb potential, a strengthening of the negative potential. As a result, energy levels for $L=0$ atomic states are stabilized by 10^{-5} ev. For hydrogen, it is -27 mc.
2. This atomic correction is only a small part of the Lamb shift which is more like $+1000$ mc total. Our small piece is known as the Uehling correction and was found by him in 1935.
3. We can always consider an "external photon" to be just a propagator very close to mass shell, so vac pol has the same effect. The self interactions on the propagator "increase" the size of the propagator by Z_3 . We can associate a $\text{SQRT}(Z_3)$ with each end, and then simply forget about the source end. Hence, the charge where an external photon attaches is renormalized in the same way as the internal changes.
4. In the LSZ picture we allow an external photon do all its self interactions, just as if it were any other photon. Then, however, we amputate it as if it were a noninteracting photon propagator. Thus, that photon has donated a factor $\text{SQRT}(Z_3)$ to our amplitude of interest which we do not want, so in the LSZ rules we divide by this SQRT . In other words, we should completely remove the full, selfinteracting external line to get just the Greens functions.

Bjorken Drell, Section 8.4 "Self-Mass of the Electron"

① "electron proper self-energy part to order e^2 " =



② $\int \frac{d^4 k}{k^2 \cdot k} \sim \int dk \sim$ linear divergence

$$\textcircled{3} \int \frac{d^4 k}{(2\pi)^4} \left[\frac{-i g_{\mu\nu}}{k^2 - \lambda^2 + i\epsilon} \right] \gamma^\mu \left[\frac{i}{\not{p} - \not{k} - m + i\epsilon} \right] \gamma^\nu (-ie)^2 = -i \Sigma(\not{p})$$

Of course $\Sigma(\not{p})$ is a Dirac matrix object, and if we know it, we'd say:

$$\begin{aligned} \text{---} + \text{---} &= i \not{S}_F(p) + i \not{S}_F(p) (-i \Sigma(\not{p})) i \not{S}_F(p) \\ &= i \not{S}_F(p) + i \not{S}_F(p) \Sigma(\not{p}) \not{S}_F(p) = \text{---} = \not{S}_F(p; m_{\text{phys}}) \end{aligned}$$

④ After a few steps, we get:

$$\Sigma(\not{p}) = \frac{\alpha}{2\pi} \int_0^1 dz [2m - \not{p}(1-z)] \mathcal{J}(\not{p}, m, \lambda)$$

This section merely computes $\Sigma(\not{p})$: makes no interpretations.

⑤ $\lambda =$ photon mass (regulator).

⑥ $\mathcal{J}(\not{p}, m, \lambda)$ blows as $\ln$ at low λ end, so instead compute

$$\bar{\Sigma}(\not{p}) = \dots [\mathcal{J}(\not{p}, m, \lambda) - \mathcal{J}(\not{p}, m, \lambda_0)] \quad \begin{array}{l} \uparrow \\ \text{in photon mass} \\ \text{role; a cutoff.} \end{array}$$

This difference of $\mathcal{J}$'s converges + we get:

$$\begin{aligned} \bar{\Sigma}(\not{p}) &= \frac{3\alpha m}{4\pi} \ln\left(\frac{\Lambda^2}{m^2}\right) - \frac{\alpha}{4\pi} (\not{p} - m) \ln\left(\frac{\Lambda^2}{m^2}\right) \\ &\quad + \frac{\alpha}{2\pi} \int_0^1 dz [2m - \not{p}(1-z)] \ln \left\{ \frac{m^2 z^2 + \lambda^2 (1-z)}{m^2 z + \lambda^2 (1-z) - \not{p}^2 z(1-z)} \right\} \end{aligned}$$

⑦ If $\lambda \ll$ everything, then:

result $\rightarrow \bar{\Sigma}(\not{p}) \approx \frac{3\alpha}{4\pi} m \ln\left(\frac{\Lambda^2}{m^2}\right) - \frac{\alpha}{4\pi} (\not{p} - m) \left(\ln\left(\frac{\Lambda^2}{m^2}\right) + 4 \ln\left(\frac{m^2 - \not{p}^2}{m^2}\right) \right)$ $p^2 - m^2 \gg m\lambda$

⑧ As let of p^2 :

$$\bar{\Sigma}(\not{p}) \sim \begin{array}{|c|} \hline L^2 \\ \hline \end{array} \quad \text{OR} \quad \log\left(\frac{\lambda}{m}\right) \quad \begin{array}{l} \leftarrow p^2 - m^2 \ll m\lambda \\ \leftarrow \text{cut starts at } p^2 = m^2 \\ \leftarrow \text{so gets in part.} \end{array}$$

Broken Shell Section 8.5: "Renorm. of Electron Propagator"

① As already noted:

$$\begin{aligned} \Rightarrow \text{new prop} &= \frac{i}{\not{p}-m} + \frac{i}{\not{p}-m} (-i \Sigma(p)) \frac{i}{\not{p}-m} \\ &= \frac{i}{\not{p}-m} \left[1 + \Sigma(p) \frac{1}{\not{p}-m} \right] = \frac{i}{\not{p}-m} \dots \end{aligned}$$

Well, if ∞ series were there, we'd have

$$\begin{aligned} A + ABA + ABAB + \dots &= C \Rightarrow A = C - ABC \\ ABA + ABAB + \dots &= ABC \Rightarrow C = (1-AB)C \\ \therefore &= (A^{-1} - B)C \end{aligned}$$

So: $C = \frac{A}{1-AB} = (1-AB)^{-1} A$

$C = (A^{-1} - B)^{-1} = ?$ $A = \frac{i}{\not{p}-m}$ $A^{-1} = (-i)(\not{p}-m)$

$B = (-i \Sigma(p))$ so $C = ((-i)(\not{p}-m) + (-i) \Sigma(p))^{-1} = i(\not{p}-m - \Sigma(p))^{-1}$

So: $\left[\text{line} + \text{loop} \right] \approx \frac{i}{\not{p}-m - \Sigma(p)}$ to order α^2

② However, we have already computed $\Sigma(p)$:

$$\Sigma(p) = \left[\underbrace{\frac{3\alpha}{4\pi} m \ln \left(\frac{\Lambda^2}{m^2} \right)}_{\equiv \delta m_0} \right] - \underbrace{\frac{\alpha}{4\pi} (\not{p}-m) \left(\dots \right)}_{\equiv \bar{Z}_2^{-1} - 1 + c(p)}$$

So $\Sigma(p) = \delta m_0 - [\bar{Z}_2^{-1} - 1 + c(p)](\not{p}-m)$

③ Which piece of $\frac{\alpha}{4\pi} \left(\ln \frac{\Lambda^2}{m^2} + 4 \ln \frac{m^2 - p^2}{m^2} \right)$ shall we call " $c(p)$ "?

We'd like $c(0) = 0$. Then $\left[\bar{Z}_2^{-1} - 1 + 0 = \frac{\alpha}{4\pi} \left(\ln \frac{\Lambda^2}{m^2} + 4 \ln \frac{m^2}{m^2} \right) \right]$
 $\leftarrow$ depends on Λ and λ !

④ So far, then

$$\frac{i}{\not{p}-m - \Sigma(p)} = \frac{\bar{Z}_2 i}{\bar{Z}_2 (\not{p}-m) - \bar{Z}_2 [\delta m_0 - (\bar{Z}_2^{-1} - 1 + c(p))(\not{p}-m)]}$$

$$= \frac{Z_2 i}{Z_2 p - Z_2 m - Z_2 \delta m + (1 - Z_2 + Z_2 c(p))(p - m)}$$

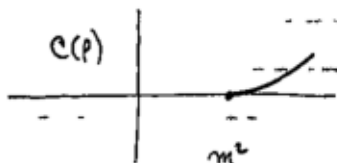
$$= \frac{i Z_2}{(p - m)(1 + Z_2 c(p)) - Z_2 \delta m} \approx \left[\frac{i Z_2}{(p - m - \delta m)(1 + c(p))} \right]$$

Now if we "pretend" that $Z_2 \approx 1$, this is $\uparrow$ also assumes $c(p)$ small.

⑤ Thus, $m_{\text{phys}} = m + \delta m$ and it is not clear exactly how $c(p)$ depends on p .

$$\frac{dc(p)}{dp^2} = \frac{\alpha}{4\pi} \cdot 4 \cdot \frac{m^2}{m^2 - p^2} \cdot (-1)$$

Now let $p^2 \rightarrow m^2$, slope is infinite — but no, actually slope is constant.



Need exact result, not two approximations.

⑥ Classically, $m_{\text{ph}} = m \pm m_{\text{em field}}$. "Inertia of electron self field."

⑦ How do you deal with δm ? Use:

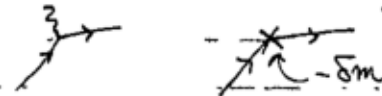
$$\frac{(i\not{\partial} - m_0)\psi}{-\delta m \psi} = \frac{e\not{A}\psi}{-\delta m \psi}$$

$$\Rightarrow [(i\not{\partial} - m_{\text{ph}})\psi = e\not{A}\psi - \delta m \psi]$$

So, the real Lagrangian has m_0 , but we'll make it look like it has m_{ph} by using m_{ph} and a "mass counterterm."

Lagrangian: $[\bar{\psi}(i\not{\partial} - m_{\text{ph}})\psi - e\bar{\psi}\not{A}\psi - \delta m \bar{\psi}\psi]$

So prior to correction, consider $m = m_{\text{ph}}$ everywhere.



⑧ At $p \approx m$, $\left[\text{prop} = \frac{i Z_2}{p - m} \right]$ analogous to Z_3 .
 $\uparrow$ physical.

Comments on Electron Propagator Renormalization:

1. When we add the second order self energy part term to the bare electron propagator, we of course get a corrected propagator, just as the second order photon self energy part caused a correction to the photon propagator.
2. Thus, we first have the problem of computing the self energy part, $\Sigma(p)$. As in the photon case, we do this via a subtraction, and the result is given in 8.40.
3. In this self energy part calculation, you need both a cutoff (subtraction) and a photon mass (regulator). In other words, both the IR and UV catastrophes are present.
4. So, what happens to the electron propagator as a result of this correction?
 - a) the mass is shifted by a (computable) amount.
 - b) the propagator is multiplied by Z_2
 - c) no other effects for p^2 very close to m^2 where $C(p) = 0$.

5. What about a p^2 modification? In the photon case we had a q^2 change in the photon propagator which led to an observable effect, namely vacuum polarization caused shifts in energy levels because the static potentials were modified (Uehling). Of course "potentials" involved photons, not electrons, so we get no corresponding analysis here. Still, I feel there should be some observable effect due to this $C(p)$. We ignored this in the course, BD ignore it in their text.

6. What about the scale factor Z_2 ? An electron propagator (as with the photon) always occurs between two "vertices". Thus, we could take a $\text{SQRT}(Z_2)$ to each vertex and renorm the charge. Then, we have $\text{SQRT}(Z_3 Z_2)$ so far renorming the charge. Later Z_2 's effect will be exactly cancelled via Ward's identity, but wait on that.

7. External electrons? I retract my earlier paragraph on this. In computing a Greens Function, you do want to include all self interactions on those external lines, because those self interactions are part of the physical external particles. When we apply the amputator operators, we do pick up the Z_2 sitting on each external electron propagator.

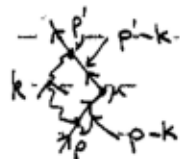
However, somehow this is too much. It is not consistent to keep the entire Z_2 because the "source" should get half of it. This is a poor argument, but there are other proofs that you only want to pick up a $\text{SQRT}(Z_2)$ for an external electron, not a full Z_2 . In perturbation theory, then, we can include all self interactions by using bare charge and then multiplying by $\text{SQRT}(Z_2)$. Or, we can just use the Z_2 renormed charge and do nothing else. OR, in LSZ, we will gain a full Z_2 by treating the external line as internal. Then on amputation we have to divide by $\text{SQRT}(Z_2)$ to correct.

8. Mass counterterms: if in the full free lagrangian you decide to go with the physical mass, then you must correct with an interaction-mass counterterm. This is because lagrangians are really only supposed to have bare quantities in them. Of course, if we neglect self energy graphs, we can sometimes use renormalized charge, eg, but then we would never get the observable corrections!

See vol II
vacuum
of m^2 and
which
does contribute

Biaken+Drell: Section 8.6: "THE VERTEX CORRECTION"

① "The second order vertex part":



$$\int \frac{d^4 k}{(2\pi)^4} \frac{(-ie)}{k^2 - \lambda^2} \gamma_\mu \frac{i}{\not{p}' - \not{k} - m} \gamma_\mu \frac{i}{\not{p} - \not{k} - m} \gamma^\nu (-ie)^2 \equiv \Lambda_\mu(p, p')$$

We leave off the charge at $e\mu$, so one would compare Λ_μ to just γ_μ itself.

② Think now as p, p' both on-shell. Then also, let $p' \rightarrow p$. In this case

define Z_1

$$\left[\bar{u}(p) \Lambda_\mu(p, p) u(p) = 4u\bar{u} = \text{const} \times \bar{u}(p) \gamma_\mu u(p) \right]$$

We could have used $\not{p}$ in there, but (the Gordon thing) shows that in this limit $\not{p} \sim \gamma_\mu$. So write

$$\text{const} = \underline{Z_1 - 1}$$

③ Now compute:

$$\frac{\partial}{\partial \not{p}_\mu} \left[\frac{i}{\not{p} - \not{k} - m + i\epsilon} \right] = -\frac{\partial}{\partial q_\mu} \left[\frac{i}{\not{q} - m} \right] = \frac{\partial}{\partial q_\mu} \left(\frac{\not{q} + m}{q^2 - m^2} \right) i$$

$$= \left(\frac{1}{q^2 - m^2} \right) \gamma_\mu + (\not{q} + m) (-1) \left(\frac{1}{q^2 - m^2} \right)^2 2q_\mu i$$

$$= \left(\frac{1}{q^2 - m^2} \right)^2 \left[(q^2 - m^2) \gamma_\mu - 2q_\mu (\not{q} + m) \right] i$$

Now start with: $[\gamma_\mu, \not{q}] = 2q_\mu$

$$\begin{aligned} (\not{q} + m) \gamma_\mu (\not{q} + m) &= \gamma_\mu (-\not{q} + m)(\not{q} + m) + \{\not{q}, \gamma_\mu\} (\not{q} + m) \\ &= \gamma_\mu (-q^2 + m^2) + 2q_\mu (\not{q} + m) \\ &= -[] \end{aligned}$$

Ward identity

Thus, $\lim_{m \rightarrow 0} \left\{ \frac{\partial}{\partial \not{p}_\mu} \left[\frac{1}{\not{p} - m} \right] = -\frac{1}{\not{p} - m} \gamma_\mu \frac{1}{\not{p} - m} \right.$

On, $\left. \frac{\partial}{\partial \not{p}_\mu} \left[\frac{i}{\not{p} - m} \right] = i \left[\frac{i}{\not{p} - m} \right] \gamma_\mu \left[\frac{i}{\not{p} - m} \right] \right.$

Ward.

Thus: $\left[\frac{\partial}{\partial \not{p}_\mu} \Sigma(p) = +i(-ie)^2 \int \dots \gamma_\nu \left\{ i \frac{i}{\not{p} - m} \gamma_\mu \frac{i}{\not{p} - m} \right\} \gamma^\nu = -\Lambda_\mu(p, p) \right]$

④ So, let's compute $\Delta_\mu(p,p)$ from the Ward + find out what Z_1 is:

compute $\Delta_\mu(p,p)$

$$\Delta_\mu(p,p) = -\partial_\mu^{(p)} \Sigma(p) = -\partial_\mu^{(p)} [\delta m - (\bar{Z}_2^{-1} - 1 + 0)(\not{p} - m)]$$

$$= +\partial_\mu^{(p)} (\not{p}) \cdot (\bar{Z}_2^{-1} - 1) = (\bar{Z}_2^{-1} - 1) \gamma_\mu$$

Therefore, $\boxed{Z_1 = \bar{Z}_2}$ ← also called "Ward"

⑤ So, $\gamma_\mu \rightarrow \gamma_\mu + (\bar{Z}_2^{-1} - 1) \gamma_\mu = \bar{Z}_2^{-1} \gamma_\mu$

Corrected vertex is "weaker" than γ_μ by $\bar{Z}_2^{-1}$. So, renorm the charge again



$$e^2 \rightarrow e^2 \cdot (\sqrt{Z_3}) \cdot \underbrace{\sqrt{Z_2} \cdot \sqrt{Z_2} \cdot \bar{Z}_1^{-1}}_{\rightarrow 1}$$

⑥ Look again at

p. 169
addition of
corrections



$$\gamma_\mu \rightarrow \gamma_\mu (-i \Sigma(p)) \frac{i}{\not{p} - m} \gamma_\mu$$

walk backwards from
end of fermion line.

$$= \gamma_\mu (-i) [\delta m - (\bar{Z}_2^{-1} - 1)(\not{p} - m)] \frac{i}{\not{p} - m} \gamma_\mu$$

$$\gamma_\mu \rightarrow \gamma_\mu + \underbrace{\gamma_\mu \cdot \frac{1}{\not{p} - m} \delta m - (\bar{Z}_2^{-1} - 1) \gamma_\mu}_{\text{vertex correction} \quad [-ie \text{ kept out}]}$$

⑦ Here we have used the physical mass propagator, so we must use a counterterm correction:



$$= \frac{i}{\not{p} - m} \cdot (+i \delta m) \cdot \gamma_\mu = -\delta m \frac{1}{\not{p} - m} \gamma_\mu$$

The rule is:

$$e \bar{\psi} \psi - \delta m \psi \rightarrow -ie \bar{\psi} \psi + \underbrace{i \delta m \psi}_{+ i \delta m \text{ at vertex}}$$

⑧ In (8.57) are listed off all the corrections which we have considered so far. As expected, the total effect is to renormalize the charge by $\sqrt{Z_3}$, period.

⑨ So far we have dealt with $\Lambda_\mu(p, p')$ only in the case that $p' = p$, because charge can be dealt with in this $q \rightarrow 0$ limit. For $q \neq 0$, we cannot use Ward, so we proceed as usual. For small q , we find:

$p \neq p'$
direct calc

$$\gamma_\mu + \Lambda_\mu(p, p) \approx \gamma_\mu \left[1 + \frac{d}{3\pi} \frac{q^2}{m^2} \left(\ln \frac{m}{\lambda} - \frac{3}{8} \right) \right] + \frac{\alpha}{8\pi m} [q, \gamma_\mu]$$

Note: we think of $Z_1 \approx 1$, so $\Lambda_\mu \approx \Lambda_\mu^c$. The fact that $Z_1 \neq 1$ is already accounted for:

$$\gamma_\mu + \Lambda_\mu = \gamma_\mu + (Z_1^{-1} - 1) \gamma_\mu + \Lambda_\mu^c = (Z_1^{-1} \gamma_\mu) + \Lambda_\mu^c$$

↑ forget for now.

⑩ Hey! That q^2 term should cancel a propagator + give a term just like vac. pol.

vertex: $-\frac{3}{8} \cdot \frac{\alpha}{3\pi} \cdot \frac{q^2}{m^2}$

vac. pol: $-\frac{3}{15} \frac{\alpha}{3\pi} \frac{q^2}{m^2}$ ✓

So at this point we just "add on" our old vac. pol. result.

⑪ Anomalous magnetic moment:

$$\gamma_\mu \rightarrow \gamma_\mu + \frac{\alpha}{8\pi m} [q, \gamma_\mu]$$

$$= q^\nu \frac{2}{i} \sigma_{\nu\mu} = 2i \sigma_{\nu\mu} q^\nu$$

$$= \boxed{\frac{\alpha}{2\pi}} \cdot \frac{1}{2m} \cdot i \sigma_{\nu\mu} q^\nu$$

usually called F_2

and $\mu_{anom} \equiv \mu_B \cdot 2m F_2$

$$= \mu_B \cdot \frac{\alpha}{2\pi}$$

So what appears here is always,
the anomalous m.m. relative to unity μ_B .

Therefore: $mm \rightarrow mm \left(1 + \frac{\alpha}{2\pi} \right)$

↑

$$\log \frac{m}{\lambda}$$

⑫ Now we start in on an analysis of the ~~third~~ term in (8.62). So far we have gotten rid of all divergences, but this one still remains.

⑬ Resolution: if ΔE is the uncertainty in the measurement of your final electron, then $\sim \Delta E$ worth of soft photons could have been emitted without your knowledge. In fact, what you are actually measuring is the (incoherent) sum of two processes:

$$|\text{hard}|^2 + |\text{soft} + \text{hard}|^2$$

where we integrate the soft photon up to something like ΔE .

Of course with a perfect photon sensor, you could rule out events where a photon is produced. But, then you would not have any events left to measure.

$$\left. \frac{d\sigma}{d\Omega} \right|_{\text{with soft}} = \left. \frac{d\sigma}{d\Omega} \right|_{\text{elastic}} \times \frac{2\alpha}{\pi} \left[\ln \left(\frac{k_{\max}}{k_{\min}} \right) \right] \frac{4}{3} \beta^2 \sin^2 \frac{\theta}{2} \quad \text{NR}$$

I presume we have integrated all soft photons from $k_{\min}$ to $k_{\max}$. As $k_{\min} \rightarrow 0$, the soft photon emission has infinite probability as compared to the pure elastic!

⑭ Keep in mind that $q^2 = (p' - p)^2 = 2m_e^2 - 2(E'E - p'p \cos \theta)$. If q is small, then

$$= 2m_e^2 - 2E^2(1 - \cos \theta) = 2m_e^2 - 2m_e^2 - 2p^2(1 - \cos \theta) = 2m_e^2 - 2E^2 \cdot 2 \sin^2 \frac{\theta}{2}$$

So $\beta = p/E \propto p/m \Rightarrow p = \beta m$, so $\left. \frac{d\sigma}{d\Omega} \right|_{\text{elastic}} = -4p^2 \sin^2 \frac{\theta}{2} = -4m^2 \beta^2 \sin^2 \frac{\theta}{2}$

Thus: $q^2 = -4m^2 \beta^2 \sin^2 \frac{\theta}{2}$ and $\left(-\frac{q^2}{m^2} \right) = 4\beta^2 \sin^2 \frac{\theta}{2}$

Thus: $\left. \frac{d\sigma}{d\Omega} \right|_{\text{soft}} = \left. \frac{d\sigma}{d\Omega} \right|_{\text{elas}} \times \frac{2\alpha}{\pi} \ln \left(\frac{k_{\max}}{k_{\min}} \right) \left(-\frac{q^2}{m^2} \right) = X(q^2)$

⑮ So much for Bremsstrahlung review. Now we want to compute the elastic DCS correction due to $\gamma_\mu \rightarrow \gamma_\mu + \gamma$. We know that

$$\gamma_\mu \rightarrow \gamma_\mu(1 + \epsilon), \quad \text{so } |\gamma_\mu|^2 \rightarrow |\gamma_\mu|^2(1 + 2\epsilon).$$

Thus, from (8.62) we would say: -

$$\left. \begin{aligned} \frac{d\sigma}{d\Omega} \Big|_{\text{vertex correction}} &= 2 \times \frac{d\sigma}{d\Omega} \Big|_{\text{elastic}} \cdot \frac{\alpha}{3\pi} \frac{q^2}{m^2} \log \frac{m}{\lambda} \\ &= \left(\frac{d\sigma}{d\Omega} \right)_0 \times \left[-\frac{2\alpha}{\pi} \log \frac{m}{\lambda} \left(-\frac{1}{3} \frac{q^2}{m^2} \right) \right] \end{aligned} \right\} \text{prior to photon modification}$$

$$\frac{d\sigma}{d\Omega} \Big|_{\text{brems. correct}} = \left(\frac{d\sigma}{d\Omega} \right)_0 \times \left[+\frac{2\alpha}{\pi} \log \frac{k_{\max}}{k_{\min}} \left(-\frac{1}{3} \frac{q^2}{m^2} \right) \right]$$

We can see that these are trying to cancel $\sim$ at least hopefully as $\lambda \rightarrow 0$, $k_{\min} \rightarrow 0$, the divergent pieces will cancel. Notice that $(d\sigma/d\Omega)$ from the vertex correction is negative.

plan (16) Now, to be consistent, we are going to suppress that portion of the vertex correction where photon has energy less than $k_{\min}$. Keep $\lambda \ll k_{\min}$. Then compare!

(17) So, on p174 we calculate $\delta\Lambda_\mu(p, p)$, the change in Λ_μ due to our restriction. I.e.,

$$\Lambda_\mu(p, p, \lambda) = \Lambda_\mu(p, p, k_{\min}) + \delta\Lambda_\mu$$

$\uparrow$ $\uparrow$
 the thing we what we
 got earlier want

The temp. result is (8.72)

(18) However, the vertex is also affected by changes in $\Sigma(p)$ caused by changing the photon mass. It is, after all, the sum of all corrections that we want to compare with the Brem. formula.

So, on p175 they compute $\Sigma(p)$. Apparently, other pictures are not seriously affected $\sim$ there are no other photon loops. Also, another reason you had to change Σ was that if you did not, Ward would be violated, $Z_1 \neq Z_2$, and renorm. is fouled up.

As usual, we state the small q^2 result:

$$\delta\Lambda_\mu = \Lambda_\mu \cdot \frac{\alpha}{3\pi} \frac{q^2}{m^2} \left(\ln \frac{2k_{\min}}{\lambda} - \frac{5}{6} \right)$$

(19) Now, let's add this directly to 8.62:

(8.62) says: $\Lambda_\mu^e \approx \gamma_\mu \cdot \frac{\alpha}{3\pi} \frac{q^2}{m^2} \left(\log \frac{m}{\lambda} - \frac{3}{8} \right) + \frac{\alpha}{8\pi m} [\not{q}, \gamma_\mu]$

(8.76) says: $-\delta \Lambda_\mu^e \approx -\gamma_\mu \cdot \frac{\alpha}{3\pi} \frac{q^2}{m^2} \left(\log \frac{2k_{min}}{\lambda} - \frac{5}{6} \right)$

$\Lambda_\mu^e(\text{tot}, k_{min}) = \gamma_\mu \frac{\alpha}{3\pi} \frac{q^2}{m^2} \left(\log \frac{m}{2k_{min}} + \frac{5}{6} - \frac{3}{8} \right) +$

$\frac{\alpha i \sigma_{\mu\nu} q^\nu}{4\pi m}$

(20) Aside:

$[\not{q}, \gamma_\mu] = \not{q} \gamma_\mu - \gamma_\mu \not{q} + \gamma_\mu \not{q} - \gamma_\mu \not{q} = \{\not{q}, \gamma_\mu\} - 2\gamma_\mu \not{q}$

Gordon says: $2m\gamma_\mu = (\not{p} + \not{p}')_\mu + i\sigma_{\mu\nu} q^\nu$

inside $\bar{u}(p')$ $u(p)$

Also: $\bar{u}(p') [\not{p}' - \not{p}, \gamma_\mu] u(p) =$

$$\begin{aligned} [\not{p}' - \not{p}, \gamma_\mu] &= \not{p}' \gamma_\mu - \gamma_\mu \not{p}' - \gamma_\mu \not{p} + \not{p} \gamma_\mu = (m - \not{p}) \gamma_\mu - \gamma_\mu (\not{p}' - m) \\ &= 2m\gamma_\mu - (\not{p}' \gamma_\mu + \gamma_\mu \not{p}') \\ &= 2m\gamma_\mu + \gamma_\mu m - 2\not{p}'_\mu + m\gamma_\mu - 2\not{p}_\mu \\ &= 4m\gamma_\mu - 2(\not{p} + \not{p}')_\mu \end{aligned}$$

Then use $4m\gamma_\mu - 2(\not{p} + \not{p}')_\mu = 2i\sigma_{\mu\nu} q^\nu$

Thus

$$[\not{q}, \gamma_\mu] = 2i\sigma_{\mu\nu} q^\nu$$

Of course! $[\not{q}, \gamma_\mu] = q^\nu [\gamma_\nu, \gamma_\mu] \frac{1}{2} \cdot \frac{2}{i} = q^\nu \cdot \sigma_{\mu\nu} \cdot \frac{2}{i}$
 $= 2i q^\nu \sigma_{\mu\nu}$

No need for Gordon

Note: $\mathcal{M}_{anom} = \mu_B \cdot 2m \times \dots$ (coeff of $i\sigma_{\mu\nu} q^\nu$)

So here, $\mathcal{M}_{anom} = \mu_B \cdot \frac{\alpha}{2\pi} \frac{q^2}{m^2} = \boxed{\frac{\alpha}{2\pi} \mu_B}$

(21) So, let's summarize:

$$\gamma_\mu \rightarrow \gamma_\mu + \frac{\alpha}{2\pi} \left[\frac{i\sigma_{\mu\nu} q^\nu}{2m} + \frac{2}{3} \frac{q^2}{m^2} \gamma_\mu \left(\ln \frac{m}{2k_{min}} + \frac{5}{6} - \frac{3}{8} - \frac{1}{5} \right) \right]$$

complete corrections

base.

vertex corr.

photon k_{min}

vac. pol. vertex corr.

Now we can use $|X_{nl}|^2 \rightarrow |X_{nl}|^2 (1+2\epsilon)$ to say:

$$\left. \frac{d\sigma}{dR} \right|_{\text{div. piece}}^{\text{radiative correction}} = \left. \frac{d\sigma}{dR} \right|_0 \cdot \left[\frac{\alpha}{\pi} \frac{2}{3} \frac{q^2}{m^2} \ln \frac{m}{2k_{\min}} \right]$$

In essence, $\left(\frac{m}{\lambda}\right)$ has been replaced by $\left(\frac{m}{2k_{\min}}\right)$ in the vertex piece due to all our fiddling with the photon. And

$$\left. \frac{d\sigma}{dR} \right|_{\text{Beam}} = \left. \frac{d\sigma}{dR} \right|_0 \cdot \left[-\frac{\alpha}{\pi} \frac{2}{3} \frac{q^2}{m^2} \ln \frac{k_{\max}}{k_{\min}} \right]$$

If we add these two we get

$$\left[\left. \frac{d\sigma}{dR} \right|_{\text{up to } k_{\max}} \right] = \left. \frac{d\sigma}{dR} \right|_0 \left[1 + \left(-\frac{2}{3} \frac{\alpha}{\pi} \frac{q^2}{m^2} \ln \frac{k_{\max}}{(m/2)} \right) \right]$$

In an exper,  $E \neq m$

② So, if your resolution is such that photons up to $k_{\max}$ can get away, then the theory predicts this in your measured $d\sigma/dR$. This correction is really negative, subtracts from the one. I would have guessed addition.

So, we compute this thing in semiclassical or whatever (Dirac on electron, field is just a pert.)

Aside on Pert. theory

$$E = E_0 + \underbrace{\langle n | H' | n \rangle}_{\text{first order}} + \sum_m' \frac{|\langle m | H' | n \rangle|^2}{E_m - E_n} + \dots$$

second order

⑤ The result of Bethe's 1947 "old-fash p. theory" is (8.86). After a comment, it is (8.87) top. Now there will be some number $\bar{E}$ that makes the equality shown there, you can in principle calculate it from (8.87) top from a knowledge of the states of the atom, i.e., $\sum_m$.

Result:

$$\Delta E^< = -\frac{4\alpha Z\alpha}{3 m^2} \ln \frac{k_{\text{min}}}{\bar{E}} |\psi(0)|^2$$

Now add this to (8.84).

$$\Delta E = \Delta E^> + \Delta E^< = \frac{4\alpha}{3} \cdot \frac{Z\alpha}{m^2} |\psi|^2 \left(\log \frac{m}{2\bar{E}} + \frac{11}{24} - \frac{1}{5} \right)$$

⑥ Now recall that

$$|\psi|^2 = \frac{1}{\pi} \left(\frac{Zm\alpha}{n} \right)^3 = |\psi_{\text{non}}(0)|^2, \text{ hydrogenlike}$$

$$\text{So: } \Delta E_{\text{LAMB}} = \frac{4\alpha}{3} \cdot \frac{Z\alpha}{m^2} \cdot \frac{1}{\pi} \cdot \left(\frac{Z\alpha m}{n} \right)^3 \left(\log \frac{m}{2\bar{E}} + \frac{11}{24} - \frac{1}{5} \right)$$

$$\Delta E_L = \frac{4\alpha (Z\alpha)^4 m}{3\pi n^3} \left(\log \frac{m}{2\bar{E}} + \frac{11}{24} - \frac{1}{5} \right)$$

$$= \frac{+5.8}{8.6} - \frac{3.6}{8.6} = \frac{40-18}{48}$$

⑦ The Bethe calc shows that $\bar{E} \approx 8.9 \alpha^2 m$ in hydrogen.

⑧ But wait; we forget the magnetic moment correction.

$$\Delta E = -\mu_{\text{Bohr}} B_{\text{reflective}}?$$

I estimate B_{eff} as ... wait.

mag moment
attempt

① Anom. moment vs. Lamb shift:

$$u^\dagger(\rho) \left[-\frac{Ze^2}{|\vec{q}|^2} - \frac{\alpha}{4\pi m} \vec{\sigma} \cdot \vec{q} \right] u(\rho)$$

Well, $u^\dagger u = 1$ in $C_0 = 2m$. So first term gives

$$-\frac{Ze^2}{|\vec{q}|^2} \rightarrow -\frac{Ze^2}{4\pi|\vec{q}|} = -\frac{Z\alpha}{r} \rightarrow \text{potential}$$

Second term gives:

$$u^\dagger \vec{\sigma} u = \begin{pmatrix} u^\dagger & 0 \end{pmatrix} \begin{pmatrix} 0 & \sigma \\ -\sigma & 0 \end{pmatrix} \begin{pmatrix} u \\ 0 \end{pmatrix} = \begin{pmatrix} u^\dagger & 0 \end{pmatrix} \begin{pmatrix} 0 \\ -\sigma u \end{pmatrix} \approx 0.$$

As $\vec{p} \rightarrow 0$, this really does vanish. More precisely:

$$\begin{pmatrix} u^\dagger & u^\dagger \frac{\vec{\sigma} \cdot \vec{p}}{2m} \end{pmatrix} \begin{pmatrix} 0 & \sigma \\ -\sigma & 0 \end{pmatrix} \begin{pmatrix} -u \\ \frac{\vec{\sigma} \cdot \vec{p} u}{2m} \end{pmatrix}$$

$$\begin{pmatrix} u^\dagger & u^\dagger \frac{\vec{\sigma} \cdot \vec{p}}{2m} \end{pmatrix} \begin{pmatrix} \sigma (\frac{\vec{\sigma} \cdot \vec{p}}{2m} u) \\ -\sigma u \end{pmatrix}$$

$$= u^\dagger \sigma \frac{(\vec{\sigma} \cdot \vec{p})}{2m} u - u^\dagger \frac{(\vec{\sigma} \cdot \vec{p})}{2m} \sigma u$$

$$= \frac{1}{2m} u^\dagger [\sigma_i, (\vec{\sigma} \cdot \vec{p})] u$$

$$\text{Now } [\sigma^i, (\vec{\sigma} \cdot \vec{p})] = p_j [\sigma^i, \sigma^j] = p_j 2i \epsilon_{ijk} \sigma_k$$

$$= 2i \epsilon_{ijk} p_j \sigma_k = 2i \vec{p} \times \vec{\sigma}$$

$$\text{So, } u^\dagger \vec{\sigma} u = \frac{i}{m} \vec{p} \times u^\dagger \vec{\sigma} u$$

$$\text{So, } \vec{q} \cdot u^\dagger \vec{\sigma} u = \frac{i}{m} \vec{q} \cdot (\vec{p} \times u^\dagger \vec{\sigma} u)$$

$$= \frac{i}{m} \vec{q} \times \vec{p} \cdot u^\dagger \vec{\sigma} u$$

$$= \frac{i}{m} \vec{q} \times \vec{p} \cdot u^\dagger \vec{\sigma} u$$

But, $\vec{L} = 0$ for the $2s$. But the $2p$ does shift. This is $\vec{L} \cdot \vec{S}$.

⑩ ~~Meson~~ magnetic moment shift. So far I have obtained:

$$\Delta E = - \frac{\alpha}{2\pi} \cdot \frac{1}{2m} \cdot \cancel{\vec{L}} \cdot \frac{\vec{u} \vec{\sigma} u}{2} \cdot 2$$

$$= - \frac{\alpha}{2\pi} \cdot \frac{1}{2m} \cdot 2 \cdot \cancel{\vec{L} \cdot \vec{S}}$$

If we put in ~~Q~~

$$\cancel{\vec{L} \cdot \vec{S}} = \frac{1}{2} (j(j+1) - l(l+1) - 3/4)$$

Then we have:

$$\left[\Delta E = - \frac{\alpha}{2\pi} \cdot \frac{1}{2m} \cdot \left[\cancel{\vec{S}(\vec{S}+1)} - L(L+1) - 3/4 \right] \right]$$

units wrong.

all dumb.

$\vec{q} \neq$ coordinate.

⑪ Go back to:

$$- \frac{\alpha}{4\pi m} \cdot \frac{i}{m} \vec{q} \times \vec{p} \cdot \vec{u} \vec{\sigma} u$$

Fourier transform this, $\vec{p} = \text{constant, small}$.

$$\text{F.T.} = \int \frac{d^3 q}{(2\pi)^3} \left[- \frac{\alpha}{4\pi m} \frac{i \vec{q} \times \vec{p}}{m} \cdot \vec{u} \vec{\sigma} u \right] e^{+i \vec{q} \cdot \vec{x}}$$

$$= - \frac{\alpha}{4\pi m^2} \vec{u} \vec{\sigma} u \cdot (-i \vec{p} \times \int \frac{d^3 q}{(2\pi)^3} \vec{q} \cdot e^{+i \vec{q} \cdot \vec{x}})$$

$$= - \frac{\alpha}{4\pi m^2} \vec{u} \vec{\sigma} u \cdot (-i \vec{p} \times i \vec{\nabla}_x \delta^3(\vec{x})) \quad \leftarrow i \vec{\nabla}_x \delta^3(\vec{x})$$

So, we get a correction to the potential that is:

$$\Delta V = - \frac{\alpha}{2\pi m^2} \vec{S} \cdot \vec{p} \times \vec{\nabla}_x \delta^3(\vec{x})$$

⑫ ~~Q~~ So,

$$\int \psi^*(\vec{x}) \vec{\nabla}_x \delta^3(\vec{x}) \psi(\vec{x}) = \vec{\nabla}_x |\psi(\vec{x})|^2 \Big|_0$$

Forget it!

⑬ Neglecting the anomalous moment correction, what is the Lamb shift? We know that:

$$\left(\log \frac{2\pi}{2 \times 8.9 \times 10^{-18} \alpha^2} + \frac{11}{24} - \frac{1}{5} \right)$$

$$= \left(5 \log \left[\frac{d^2}{2 \times (8.9)} \right] + \frac{55}{24} - 1 \right)$$

$$= (34.8 + 2.3 - 1) = +36.1$$

The -1 gives -27 mc, so we get

$$36.1 \times 27 = \boxed{974.7 \text{ mc.}}$$

← bulk from $\log \left(\frac{m}{E} \right)$

⑭ Discussion to follow.

Comments on Bjorken and Drell:

$$\bar{u}(p) \Lambda_\mu(p,p) u(p) + \bar{u}(p) \gamma_\mu u(p) = Z_1^{-1} \bar{u}(p) \gamma_\mu u(p)$$

1. The vertex Correction. In this section we first deal with the Dirac matrix element of the "vertex" at $q=0$. We show that when $q=0$, this matrix element must be proportional to γ_μ so this defines constant Z_1 .
2. However, from the Ward Identity which relates the second order vertex part at $q=0$ to the second order electron self energy, we can show that Z_1 just defined is equal to Z_2 . The result is that the charge renormalization induced by the electron self energy is exactly cancelled by the charge renormalization induced by the vertex correction.
3. Specifically, BD add up all the vertex corrections (6 terms) including the mass counterterms. The result of adding all these things (at $q=0$) is that only the $\text{SQRT}(Z_3)$ charge renormalization survives. This was induced by the second order photon self energy (vacuum polarization).
4. Note on mass counterterms: the electron "propagator" should really have the bare mass. But we like using the physical mass in this propagator. In order to be able to use the physical mass, we have to modify the Dirac equation and the Lagrangian (from whence the propagator derives) with a mass counterterm. Then we have to include all separate graphs with counterterms. Of course this counterterm "X" already is equivalent to two vertices in "order" because as we computed, it is proportional to α .
5. The Feynman rule for a mass counterterm is (idem) at vertex.
6. Now BD begin to compute the vertex correction for $q \neq 0$. This requires once again both IR and UV cutoffs in the photon mass position, but the small q^2 result depends only on the infrared cutoff. This result (page 17) shows three terms:
 - a) an infrared divergent term $\sim \ln(m/\lambda)$
 - b) a $-3/8$ term which is exactly like the vac pol which is $-1/5$, not yet included into this result.
 - c) for the first time, the anomalous magnetic moment term appears.

All three terms come from the one Feynman graph in which a photon bridges the vertex.

7. Now begins a very long analysis of that infrared divergent term. This term modifies elastic Coulomb scattering. If we were to remove the infrared cutoff, this modification would be infinite and Coulomb scattering would not make sense.

We ran into a similar problem with Bremsstrahlung. There, unless we put an infrared cutoff on the photon energies emittable, the Brem cross section again became an infinite modification.

Here it is argued that when the differential cross section modifications of these two processes are added, the infrared cutoff disappears.

$$|\mathcal{M}_{\text{Brem}}|^2 + |\mathcal{M}_{\text{Vertex}}|^2 \text{ --- cancels --- } |\mathcal{M}_{\text{Coulomb}}|^2$$

However, we cannot directly add our results to show this cancellation because the Brem was computed using massless photons and excluded energies up to k_{min} . On the other hand, the vertex correction was done with massive photons and included all lower energies. We have to get these onto the same footing somehow.

8. In order to go to this "same footing", BD in effect recompute the vertex correction by removing photon phase space up to $k_{\min}$. They assume that $k_{\min}$ is much larger than the photon mass. So, after computing the change in the vertex due to this photon phase space modification, they discover that they also have to include an effective change in the vertex caused by a change in the electron self energy part, because it too has a photon loop. The sum of these corrections is illustrated on top of page 176.

9. What results from this photon phase space correction?

- a) the infrared term which was $\ln(m/\lambda)$ in 8.62 is now replaced with $\ln(m/2k_{\min})$. Thus, we can add to the Brem result in a second.
- b) a $+5/6$ vac pol type term appears out of thin air.

10. Now we can add the Brem correction to the Vertex correction, and also add in the zeroth order elastic scattering. On page 21 we see that the sum is no longer infrared divergent, but depends only on $k_{\max}$. This in turn is determined by your experimental resolution. Although this equation does not seem to say so, I would assume that as your resolution gets more and more perfect, the modification should vanish.

Note here that we only added in that infrared piece of the vertex. From now on we do not simply ignore this piece. We have explained how we experimentally avoid the infrared catastrophe, but that "piece" is still there in the vertex correction and will lead to still more observable effects (Lamb).

11. The Lamb Shift. The vertex graph is the primary contributor to the Lamb shift. We learned that the vertex graph contributes that $k_{\min}$ term, the $-3/8$ term; and the $+5/6$ term. It turns out that $k_{\min}$ has a natural value of about nine inverse Bohr radii. We certainly expected it to have something to do with the scale of the problem (atomic radius). So, if the vacuum polarization term (the $-1/5$) were -1 , then the $k(\min)$ term would be about $+35$, and the other two fractions contribute $+2.3$.

Thus, when these three effects are added, I get $+975$ mc for the energy shift. I have left out the anomalous magnetic moment correction, and maybe some other corrections I don't know about. But the point is clear: the main reason for the Lamb shift is the fact that the electron in the hydrogen atom can emit a photon and then catch it again on the other side of its Coulomb interaction vertex. This is called "fluctuations." This "vertex" correction is about 35 times larger than the Uehling vacuum potential correction we discussed earlier.

12. In order to evaluate $k_{\min}$, BD go to the Bethe nonrel calculation of the energy shift for photons of energy below $k_{\min}$. This is done via a standard old fashioned second order perturbation calculation using as a perturbation the $\mathbf{j} \cdot \mathbf{A}$, where $\mathbf{j}$ is the electron's Dirac current, and $\mathbf{A}$ is just $\hat{\mathbf{e}} e^{-ikr}$. It is as if the VACUUM caused the electron to emit a photon, and then the electron catches the photon again. The vacuum tickles the electron and makes it do this, hence the name "vacuum fluctuations". We know that even the vacuum has a certain nonzero VEV for electric field, and it is this electric field which is inducing this transition, loosely speaking.

Having done the Bethe calculation, BD add its result to their former result. The $k(\min)$'s cancel, and you get a specific result, (8.89).

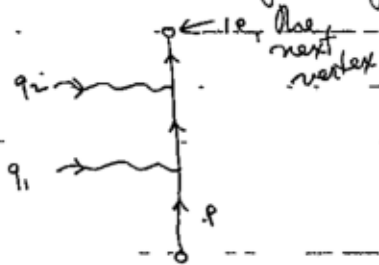
Main
Lamb Source:



"Insertions" + Ward Identity

The concept of "insertions" + Ward Identity

- ① Suppose you have a piece of a Feynman graph which looks like this:



At each vertex we usually have $-ie\gamma_\mu$, or $\not{\epsilon}_\mu$, but for generality we can imagine an arbitrary 4×4 Dirac matrix at each vertex. Call this a_1 and a_2 .

Thus, the above picture is:

$$\frac{1}{\not{p} + \not{q}_1 + \not{q}_2 - m} a_2 \frac{1}{\not{p} + \not{q}_1 - m} a_1 \frac{1}{\not{p} - m}$$

- ② Now let us define what we mean by "an insertion". For a couple in a photon ~~set~~ of momentum q such that the coupling is specifically $\not{q}$. An insertion thus generates a new spin structure. For example

$$= \frac{1}{\not{p} + \not{q}_1 + \not{q}_2 + \not{q} - m} a_2 \left\{ \frac{i}{\not{p} + \not{q} + \not{q}_2 - m} \not{q} \frac{1}{\not{p} + \not{q}_1 - m} \right\} a_1 \frac{1}{\not{p} - m}$$

We have bracketed the characteristic triple product. But this identity:

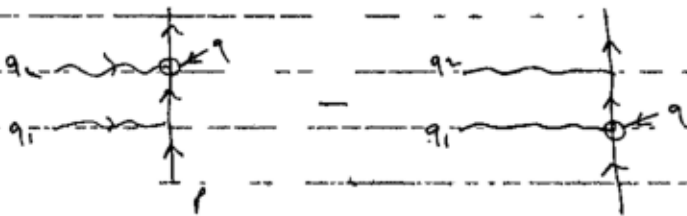
$$\frac{1}{\not{K} + \not{q} - m} \not{q} \frac{1}{\not{K} - m} = \frac{1}{\not{K} - m} - \frac{1}{\not{K} + \not{q} - m}$$

allows us to expand each "insertion" into two terms, pretty obvious. In each of these terms, the momentum q suddenly appears out of nowhere. For example, in the above specific case, the two terms are

$$+ \left[\frac{1}{p+q_1+q_2+q-m} a_2 \frac{1}{p+q_1-m} a_1 \frac{1}{p-m} \right]$$

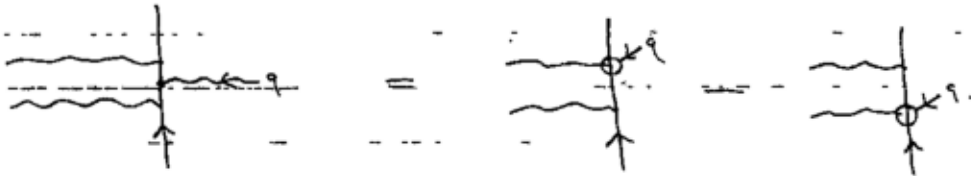
$$- \left[\frac{1}{p+q_1+q_2+q-m} a_2 \frac{1}{p+q_1+q-m} a_1 \frac{1}{p-m} \right]$$

Here is my graphical notation for these terms:

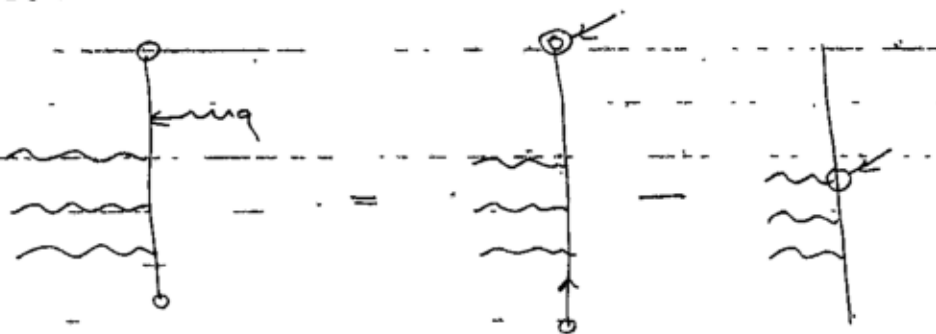


The little arrow $\leftarrow q$ merely tells you where the momentum q starts appearing as you move up the chain.

② Thus, here is our Theorem:



A special case occurs if you insert at the "end" of a chain:



But the meaning of is clear. Same as .

I.e., the added q has no effect on this part of the diagram. But does make a difference.

④ Now using this concise graphical language, we can prove some easy theorems. First, an example: ---

$$\left(\begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} \begin{array}{c} \circ \\ \text{---} \\ \text{---} \\ \text{---} \\ \circ \end{array} \right) + \left(\begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \circ \end{array} \right) + \left(\begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \circ \end{array} \right) + \left(\begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \circ \end{array} \right) = \text{"sum of insertions on a line"} \\ = \left(\begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} \begin{array}{c} \circ \\ \text{---} \\ \text{---} \\ \text{---} \\ \circ \end{array} \right) - \left(\begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \circ \end{array} \right) + \left(\begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \circ \end{array} \right) - \left(\begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \circ \end{array} \right) + \left(\begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \circ \end{array} \right) - \left(\begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \circ \end{array} \right) + \left(\begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \circ \end{array} \right) - \left(\begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \circ \end{array} \right)$$

Here you can "see" the pairwise cancellation, and the complete sum is:

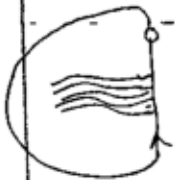
$$= \left(\begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} \begin{array}{c} \circ \\ \text{---} \\ \text{---} \\ \text{---} \\ \circ \end{array} \right) - \left(\begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \circ \end{array} \right)$$

⑤ Clearly, this works for any number of photons on the chain, so here is the theorem: ---

$$\sum_q \left(\begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} \begin{array}{c} \circ \\ \text{---} \\ \text{---} \\ \text{---} \\ \circ \end{array} \right) = \left(\begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} \begin{array}{c} \circ \\ \text{---} \\ \text{---} \\ \text{---} \\ \circ \end{array} \right) - \left(\begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \circ \end{array} \right)$$

Theorem 1

⑥ Now, suppose this were a closed loop then the top & bottom are the same place, i.e.

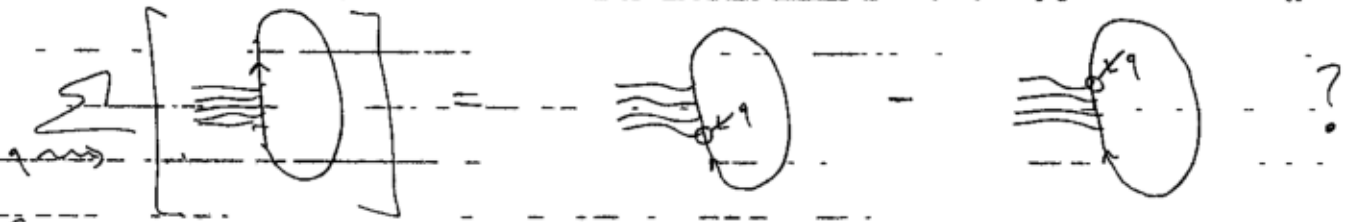


Thus, the right hand side of the above theorem vanishes! We then get result that "sum of insertions on a closed loop = 0"



$$\sum_q \left(\begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} \begin{array}{c} \circ \\ \text{---} \\ \text{---} \\ \text{---} \\ \circ \end{array} \right)$$

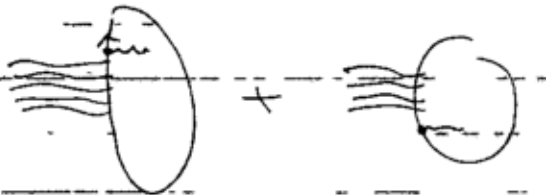
⑥ Now suppose we have a closed loop. Then: ---



But look how things are labelled:

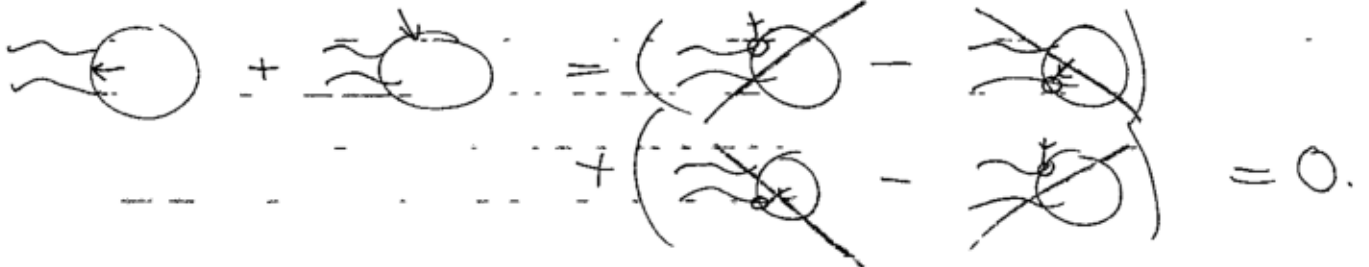


But, we have over counted because we included both these insertions in our sum:

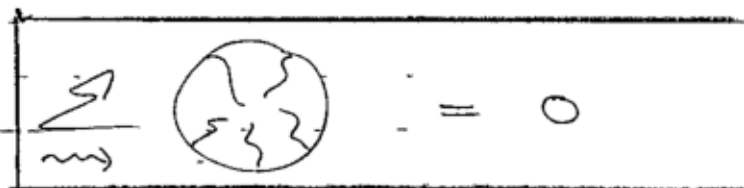


So, subtract away one of these to get right answer.
But this is RHS, so get 0.

⑦ Alternatively, it is clear that things are just cancelling in pairs:



⑧ So our second theorem:



Theorem 2.

- ⑨ Approaching the Ward Identity, we want to define certain objects. —
First, the "proper self-energy part" for the electron is this:

$$\text{Diagram: } \text{electron line with a shaded self-energy loop} = \text{Diagram: } \text{electron line with a photon loop} + \text{Diagram: } \text{electron line with a fermion loop} + \dots$$

Note that this does not contain any on any types. —
Of course these are in the full propagator, which is

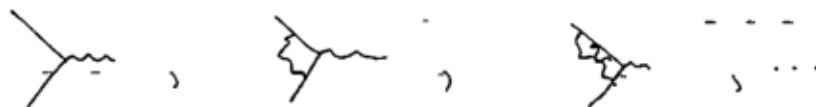
$$iS_F' = iS_F + iS_F [\text{self-energy}] iS_F + \dots$$

Thus, $iS_F' = \frac{1}{\not{p} - m - \Sigma}$

Now, by the way: $\text{Diagram: } \text{shaded loop} = \Sigma$, no external propagators.

" $\text{Diagram: } \text{electron line with shaded loop}$ " = $iS_F'(p)$, again no "external propagators."

- ⑩ Now, what do we mean by a "vertex", ie, a "proper vertex part." —
We want to include:



We want to exclude:



because these are not "proper", ie, they become disconnected if you remove just one line (of either kind).

- ⑪ Claim: you can generate the proper vertex part by attaching photons to the proper self energy part. We have this expansion:

$$\Sigma = \text{Diagram: } \text{shaded loop} = \text{Diagram: } \text{photon line attached to shaded loop} + \text{Diagram: } \text{photon line attached to shaded loop} + \dots$$

⑫ Now, let me write another version of Theorem 1. It is

$$\Sigma \left(\begin{array}{c} \text{diagram} \end{array} \right) = \begin{array}{c} \text{diagram} \end{array} - \begin{array}{c} \text{diagram} \end{array}$$

Theorem 1

This time we have photons at both ends which are shown. This does not change the theorem. Example

$$\Sigma \left(\begin{array}{c} \text{diagram} \end{array} \right) = \begin{array}{c} \text{diagram} \end{array} = a_2 \frac{1}{p+q-m} - \frac{1}{p-m} a_1$$

$$= -a_2 \frac{1}{p-m} a_1 - a_2 \frac{1}{p+q-m} a_1$$

$$= \begin{array}{c} \text{diagram} \end{array} - \begin{array}{c} \text{diagram} \end{array}$$

⑬ Now, claim:

$$q_\mu \left(\begin{array}{c} \text{diagram} \end{array} \right) = (\text{proper vertex part})^\mu q_\mu = \sum_{q \rightarrow} \left(\begin{array}{c} \text{diagram} \end{array} \right)$$

But by the above theorem, we get:

$$= \begin{array}{c} \text{diagram} \end{array} - \begin{array}{c} \text{diagram} \end{array}$$

⑭ Better to draw it this way

$$q_\mu \left[\begin{array}{c} \text{diagram} \end{array} \right] = \sum_{q \rightarrow} \left(\begin{array}{c} \text{diagram} \end{array} \right) = \begin{array}{c} \text{diagram} \end{array} - \begin{array}{c} \text{diagram} \end{array} = \Sigma(p) - \Sigma(p+q)$$

OK, but this is how they would define Λ^μ , not Γ^μ .

$$\text{One: } q_\mu \Lambda^\mu = \Sigma(p) - \Sigma(p+q)$$

(15) Thus, we have Ward's identity:

$$\Lambda^\mu(p', p) \equiv \text{diagram} = \text{diagram} + \text{diagram} + \text{etc.}$$

Notice that photon lines must always be attached to both circles, else this would not be proper.

Ward Says: →

$$\begin{aligned} q_\mu \text{diagram} &= q_\mu \Lambda^\mu(p', p) \\ &= \text{diagram} - \text{diagram} \\ &= \Sigma(p) - \Sigma(p') \end{aligned}$$

∴ $q_\mu \Lambda^\mu(p', p) = \Sigma(p) - \Sigma(p')$

(16) The actual object used by BD is:

$$\text{line coming out} \rightarrow \text{diagram} = \text{diagram} + \text{diagram}$$

$$p_\mu = \gamma_\mu + \Lambda_\mu$$

This enables you to write $q_\mu \Lambda^\mu(p', p) = \Sigma(p') - \Sigma(p)$
But this adds nothing new.

(17) There is order-by-order statement of Ward:

zeroth order: $q_\mu \left(\text{diagram} \right) = \left(\text{diagram} \right) - \left(\text{diagram} \right) = \text{diagram}$

verify: $\text{diagram} = \frac{1}{p+m} - \frac{1}{p-m} = \frac{1}{p-m} - \frac{1}{p'+m}$

This works, but is not really contained in $q_\mu \Lambda^\mu = \Sigma - \Sigma'$ for reasons that should be clear. This is the thing you add to get the second form of Ward.

order α : $\sum \left(\text{diagram} \right) = \text{diagram} = \text{diagram} - \text{diagram}$

OK After all, diagram is diagram to order α . Also, diagram is Λ^N to order α .

More familiar notation:

$\text{diagram} = \text{diagram} = \text{diagram} - \text{diagram}$

order α^2 : Dis means two photons:

$\text{diagram} \text{ order } \alpha^2 = \text{diagram} + \text{diagram} + \text{diagram}$

I do not bother with loops since insertions $\neq 0$. So

$\left(\text{diagram} \right) + \left(\text{diagram} \right) + \left(\text{diagram} \right) = \text{diagram} \text{ to order } \alpha^2.$

Write this out:

$\left(\text{diagram} + \text{diagram} + \text{diagram} \right) + \left(\text{diagram} + \text{diagram} + \text{diagram} \right) + \text{diagram} \left(\text{diagram} + \text{diagram} \right) = \text{RHS}.$

OK, so much for the Ward Identity.

⑧ Relation between gauge invariance, current conservation:

$$A^\mu \rightarrow A^\mu + \partial^\mu \Lambda \quad \text{should make no difference}$$

$$\int d^4x \, J_\mu A^\mu \quad \text{picks up term} \quad \int d^4x \, \partial_\mu J^\mu \Lambda$$

which vanishes if current is conserved.

⑨ "Conserved current" in simple case:



$$\partial^\mu J_\mu = 0$$

I.e, extra term is: $\sim \partial^\mu \langle p' | J_\mu(x) | p \rangle = i q^\mu \langle p' | J_\mu(x) | p \rangle$
by translation.

Verify: $\bar{u}(p') \gamma_\mu u(p) q^\mu = 0 \checkmark$

⑩ Conserved current after renormalization of vertex.

Must show that $\bar{u}(p') \gamma_\mu u(p) q^\mu = 0$.
I.e, $J_\mu^{\text{renorm}} = \bar{u}(p') \gamma_\mu u(p)$.

From Ward identity,

$$q_\mu J_\mu^{\text{renorm}} = \bar{u}(p') q_\mu \Gamma^\mu u(p) = \bar{u}(p') [S_F^{-1}(p) - S_F^{-1}(p')] u(p)$$

Will consider $\bar{u}(p') q_\mu \Gamma^\mu u(p) = \bar{u}(p') [\Sigma(p') - \Sigma(p)] u(p)$.

Typical $\Sigma(p)$ term is

$$= \int d^4k_1 \int d^4k_2 \frac{1}{\not{p} + \not{k} - m} \gamma^\mu \frac{1}{\not{k}_1} \gamma^\nu \frac{1}{\not{k}_2} \dots$$

Can you show: $\bar{u}(p') [\not{\epsilon}(p) - \not{\epsilon}(p')] u(p) = 0$.

Then: Ward Identity $\longleftrightarrow$ conservation of renormalized current.

$\longleftrightarrow$ gauge invariance

$\longleftrightarrow$ you must

sum over all corrections of a given order in order to get a gauge covariant result.

② By the way, generally you have to add all possible exchange situations to get a gauge invariant amplitude, eg. Compton case. Here, gauge invariance means $\epsilon \rightarrow \epsilon + q$. In any case here, $\cancel{p_\mu \epsilon^\mu} \rightarrow p_\mu \epsilon^\mu + p_\mu q^\mu$

and you need $p_\mu q^\mu$ to vanish when in $\bar{u}(p') \dots u(p)$.

Magnetic Moment

Anomalous Magnetic Moment of Electron.

But insert "renormalized current" to get measured moment. This is what form factors do.
 $\cancel{J_0} \rightarrow J_0$ renorm.
like J_0

1. The term "magnetic moment" seems to have a "nominal" meaning and an "operational" meaning. Nominally, we write down an operator for the magnetic moment by just cribbing from classical E&M. This operator is a certain spatial integral of the current J . If you put in the well-known Dirac electron current, you get $M = e/2m_e$. This is exactly what you get, there are no "corrections" when you define magnetic moment in this way. Similarly, if you insert fancier currents (with form factors, etc), you again find the nominal magnetic moment for that particle.

2. The "operational" magnetic moment is the magnetic moment you would operationally measure. The only way you can measure it is via an interaction between the electron and a classical magnetic field. In "external potential field theory" the "source" of magnetism is represented by an "x". This "x" is exact in that if you made a field theory model for the external potential (ie, you make a proton go in a circle and have a current which generates a vector potential etc), you would have to include corrections "on the proton side" to all orders to get "x". (In passing, we note that this is why there are no corrections to the Mott formula; you cannot talk about "double Coulomb scattering" etc because the entire effect of the source proton has been included in the Coulomb Potential).

However, you can still do corrections on the "electron side" of the interaction. This means that the operational magnetic moment can only be exactly computed from an infinite sum of feynman graphs. The lowest order graph is the nominal magnetic moment which you cannot measure because you cannot "turn-off" the corrections.

In this picture, the operational magnetic moment is that magnetic moment you would have to use with no corrections to get the same $\bar{S}$ -matrix.

3. The first order correction is $1.16 \times 10^{-3} = .00116$ relative to 1. Namely, $a/2\pi$. It was first computed by Schwinger in 1948. The second order correction was done in 1950 by Karplus and Kroll to be $-2.973(a/\pi)^2$. They goofed, however, and in 1957-8 Sommerfeld and Petermann redid the calc to get $-.328(a/\pi)^2$. This new result ¹⁵ (quoted by BD and GAS). As of 1963, exper says $.328 \pm .005$, see BD).

Anomalous Moment Numerics

① The 1973 tables give: $\bar{\alpha}^1 = 137.03602(21)$

I read this as:

$$\bar{\alpha}^1 = 137.0360 \pm .0002$$

Therefore, $\alpha = .00729735 \pm 1$

② Thus, the first order correction is:

$$\frac{\delta\mu}{\mu} = .001161410 \pm 1 \rightarrow 1161.410 \pm 1$$

So the uncertainty in $\frac{\alpha}{2\pi}$ is in the "seventh place"

$$1161.410 \pm 1$$

③ Sommerfeld + Pottermann [1957-8] independently got

$$\frac{\delta\mu}{\mu}^{(1)} = -.328 \left(\frac{\alpha^2}{\pi^2} \right) = -1.7692 \times 10^{-6} = -1.77 \times 10^{-6}$$

The limit here is that .328 is only 3 places; hence, 3 places

$$= -.00000177$$

So, relative to the 1161, this is $-.000177 \pm 1$, say

this:

$$1159.64 \pm 1 \leftarrow \text{best theoretical.}$$

④ Conclusion: The limiting factor is the knowledge of the .328

$$+.001161410 \pm 1$$

order α

$$-.000001770 \pm 10$$

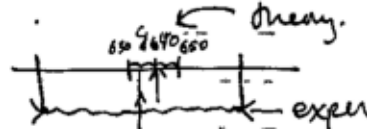
order α^2

$$+.001159640 \pm 10$$

BD and Sakurai

⑤ Exper [1963, Wilkinson + Crane]: say 1764 ± 27 instead of 1790 ± 10 . So they say

$$+.001159634 \pm 27$$



⑥ In 1961, Schupp, Pidd + Crane (Gas p 203) said (exper)

$$.001160900 \pm 2400$$

⑦ In 1956, Franken + Lieber (Mandl p 150 said)

$$.001165000 \pm 11,000$$

⑧ So, most recent quote is 1973 p.d.t.

$$.0011596577 \pm 35$$

$$\equiv .001159,658 \pm 4$$

So consider

1956	.00	11	65	000	$\pm$	11,000
1961	.00	11	60	900	$\pm$	2400
1963	.00	11	59	634	$\pm$	27
1973	.00	11	59	658	$\pm$	4

Who says experimental physics is not making progress?

and Dirac Magnetic Moment of Electron. (see p5)

Attempt calculation of "magnetic" scattering with Golden Rule.

① For Coulomb scatter we used:

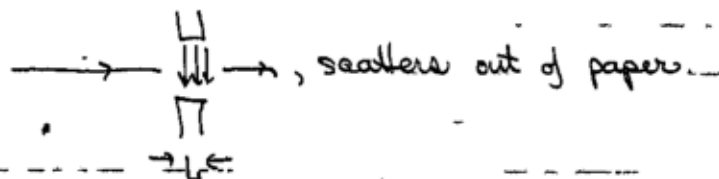
$$A_0 = \frac{e}{4\pi|\mathbf{r}|} \quad \vec{A} = 0 \quad \rightarrow A_\mu(\mathbf{x})$$

then we fourier'd to get

$$\vec{A}(\vec{p}) = \int d^3x \vec{A}(\mathbf{x}) \cdot e^{-i\vec{p}\cdot\vec{x}}$$

Then: $A_0(\vec{q}) = \frac{e}{|\vec{q}|^2}$ and $\vec{A}(\vec{q}) = 0$.

② Now what is a respectable magnetic potential



uniform $\vec{B}$

Let's build a uniform field in there.

$$\vec{B} = \vec{\nabla} \times \vec{A} \quad \text{What vector has a constant curl?}$$

Try: $\boxed{\vec{A} = \frac{1}{2}(\vec{B} \times \vec{r})}$ for uniform $\vec{B}$.

$$\vec{\nabla} \times \vec{A} = \frac{1}{2} \vec{\nabla} \times (\vec{B} \times \vec{r}) = \frac{1}{2} [\vec{B} \cdot (\vec{\nabla} \cdot \vec{r}) + (\vec{r} \cdot \vec{\nabla}) \vec{B} - (\vec{B} \cdot \vec{\nabla}) \vec{r}]$$

But $\vec{\nabla} \cdot \vec{r} = 3$, $(\vec{r} \cdot \vec{\nabla}) \vec{B} = 0$, $(\vec{B} \cdot \vec{\nabla}) \vec{r} = B_x \hat{x} + \dots = \vec{B}$

So $= \frac{1}{2} [3\vec{B} - \vec{B}] = \vec{B} \quad \checkmark \quad A_0(\mathbf{x}) = 0$

③ Now do your fourier:

$$\int d^3x \cdot \frac{1}{2} (\vec{B} \times \vec{r}) \cdot e^{-i\vec{p}\cdot\vec{x}} = \frac{1}{2} \vec{B} \times \int d^3x \cdot \vec{x} \cdot e^{-i\vec{p}\cdot\vec{x}}$$

Now if we were to fill all space with this field, this integral would become:

$$\int d^3x x^i e^{-i\vec{p}\cdot\vec{x}} = \int d^3x x^i e^{-i\vec{p}\cdot\vec{x}}$$

Now, $\frac{d}{dp^i}(e^{-i\vec{p}\cdot\vec{x}}) = -ix^i = -\frac{d}{dp^i}(e^{-i\vec{p}\cdot\vec{x}}) = -\partial_{p^i}(e^{-i\vec{p}\cdot\vec{x}})$

So: $x^i = -i \partial_{p^i}(e^{-i\vec{p}\cdot\vec{x}})$ and:

$$\int d^3x x^i e^{-i\vec{p}\cdot\vec{x}} = -i \partial_{p^i} \cdot (2\pi)^3 \delta^3(\vec{p})$$

Then we'd have:

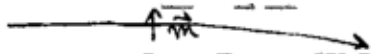
$$\vec{A}(\vec{p}) = -\frac{i}{2} \vec{B} \times \vec{\nabla}_p \delta^3(\vec{p}) \times (2\pi)^3$$

So, we would obtain:

$$\begin{aligned} \langle p_2 | H' | p_1 \rangle &= C_1 C_2 \bar{u}(2) [-ie\gamma_\mu] u(1) \cdot \vec{A}^*(\vec{p}) \\ &= C_1 C_2 \bar{u}(2) [-ie\gamma_\mu] u(1) \cdot [-\frac{i}{2}(2\pi)^3] \vec{B} \times \vec{\nabla}_p \delta^3(\vec{p}_1 - \vec{p}_2) \end{aligned}$$

We ain't going nowhere fast with this singular amplitude. It makes no physical sense either. (Might be OK with packets, however)...

④ Suppose we try to scatter off a static magnetic dipole:



⑤ Normalization of dipole moment: In gaussian units, Dave says:

$$m = \frac{I}{c} \times \text{area}, \text{ i.e., } m_G = \frac{I_G}{c} \cdot \text{area.}$$

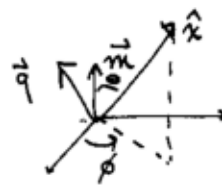
Now to go to Heaviside, $q_H = \sqrt{4\pi} q_G$, so $i_H = \sqrt{4\pi} i_G$

OK, so A, m, I, e, B all "ride" on the same scale. In any scale,

$$\left[\vec{A} = \frac{\vec{m} \times \hat{x}}{|\vec{x}|^2} \right] \leftarrow \text{for mag. dipole}$$

⑥ So, compute the Fourier: $\int d^3x \vec{A}(\vec{x}) e^{-iq\cdot\vec{x}}$

$$\vec{A}(\vec{q}) = \vec{m} \times \int d^3x \frac{\hat{x}}{|\vec{x}|^2} e^{-i\vec{q} \cdot \vec{x}}$$



Well

$$\vec{x} \cdot e^{-i\vec{q} \cdot \vec{x}} = \dots \text{stop}$$

$$\vec{A}(\vec{q}) = \int d^3x \cdot \frac{\vec{m} \times \hat{x}}{|\vec{x}|^2} e^{-i\vec{q} \cdot \vec{x}}$$

$$\vec{m} \times \hat{x} = m \hat{\phi} \sin \theta$$

$$= \int d^3x \cdot \frac{m \hat{\phi}(\vec{x}) \sin \theta}{|\vec{x}|^2} e^{-i\vec{q} \cdot \vec{x}} \quad (\text{looks messy})$$

$$\vec{A} = \vec{m} \times \left[\left(\int d^3x \frac{x}{r^2} e^{-i\vec{q} \cdot \vec{x}} \right) \hat{x} + \left(\dots \right) \hat{y} + \left(\dots \right) \hat{z} \right]$$

Idea:

$$\vec{x} \cdot e^{-i\vec{q} \cdot \vec{x}} = i \nabla_q \cdot e^{-i\vec{q} \cdot \vec{x}} = i \frac{\partial}{\partial q_i} e^{-i\vec{q} \cdot \vec{x}} \quad \checkmark$$

So:

$$\vec{A} = \vec{m} \times (i \nabla_q) \cdot \underbrace{\int d^3x \frac{1}{|\vec{x}|^3} e^{-i\vec{q} \cdot \vec{x}}}_{2\pi \int_0^\infty \frac{r dr}{r^3} \int_{-1}^1 dz e^{-iqrz}}$$

Do the z integral:

$$\begin{aligned} \int_{-1}^1 dz e^{-iqrz} &= \frac{1}{-iqr} [e^{-iqr} - e^{iqr}] = \frac{2}{2iqr} (e^{iqr} - e^{-iqr}) \\ &= \frac{2 \sin(qr)}{qr} \end{aligned}$$

So we now need:

$$\int_0^\infty \frac{dr}{r^2} \cdot \left[\frac{2 \sin(qr)}{qr} \right] = 2q \cdot \int_0^\infty \frac{\sin x dx}{x^3}$$

$$\begin{aligned} \int_0^\infty \frac{\sin x dx}{x^3} &= -\frac{\sin x}{2x^2} \Big|_0^\infty + \frac{1}{2} \int_0^\infty \frac{\cos x dx}{x^2} = -\frac{\sin x}{2x^2} \Big|_0^\infty + \frac{1}{2} \left[-\frac{\cos x}{x} \Big|_0^\infty - \int_0^\infty \frac{\sin x dx}{x} \right] \\ &= \left(-\frac{\sin x}{2x^2} - \frac{\cos x}{2x} \right) \Big|_0^\infty - \frac{1}{2} \cdot \frac{\pi}{2} \end{aligned}$$

→ Clearly this thing diverges ~ of course; the dipole "potential" is not valid "close in" and a F.T. requires full spatial information

no go.

Still, you can see the interesting features:

- (a) only the piece $\vec{u} \cdot \vec{x} \cdot \vec{u}$ will be involved, and in soft limit this means $\vec{u} \cdot \vec{x} \cdot \vec{u}$, and this will be related to μ

- ⑦ Note: go back and assume that our magnetic "source" was a magnetized sphere of radius a . Then we can separate our $\int d^3x$ integral!

The "outside piece" gives:

$$\vec{A} = \vec{m} \times i \vec{\nabla}_q \cdot \left(2\pi \cdot 2q \cdot \left[+ \frac{\sin(qa)}{2(qa)^2} + \frac{\cos(qa)}{2(qa)} - \frac{\pi}{4} \right] \right)$$

Well, this remains messy, but I could do it if I wanted: ✓

- ⑧ Suppose we go back to the "plate of uniform field" case. There we had:

$$\vec{A}(\vec{q}) = \int_{\text{plate}} d^3x \frac{1}{2} (\vec{B} \times \vec{x}) e^{-i\vec{q} \cdot \vec{x}} = \frac{1}{2} \vec{B} \times \int_{\text{plate}} i \vec{\nabla}_q \cdot d^3x e^{-i\vec{q} \cdot \vec{x}}$$

Well $\int_{\text{plate}} d^3x e^{-i\vec{q} \cdot \vec{x}} = \left(\int_0^{L_x} dx e^{-iq_x x} \right) \left(\int_0^{L_y} dy e^{-iq_y y} \right) \left(\int_0^{L_z} dz e^{-iq_z z} \right)$

$$= \frac{1}{-iq_x} \left(e^{-iq_x L_x} - 1 \right) \cdot \frac{1}{-iq_y} \left(e^{-iq_y L_y} - 1 \right) \cdot \frac{1}{-iq_z} \left(e^{-iq_z L_z} - 1 \right)$$

Better, let's do $\int_{-L_x/2}^{+L_x/2} dx e^{-iq_x x}$

$$= \frac{1}{-iq_x} \left[e^{-iq_x L_x/2} - e^{+iq_x L_x/2} \right]$$

$$= \frac{2}{q_x} \sin(q_x L_x/2)$$

Then:

$$\int_{\text{plate}} d^3x e^{-i\vec{q} \cdot \vec{x}} = \frac{8}{q_x q_y q_z} \sin(\dots) \sin(\dots) \sin(\dots)$$

However, if $\vec{q}$ is small we can say:

$$\int_{\text{plate}} d^3x [1 - i\vec{q} \cdot \vec{x}] = \text{volume} - i\vec{q} \cdot \int_{\text{plate}} d^3x \cdot \vec{x}$$

Well, $\int_{\text{plate}} d^3x \cdot \vec{x} = 0$ if origin at center of cube

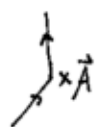


This must be the coordinate of the cube center, so

$$\int_{\text{plate}} d^3x e^{-i\vec{q} \cdot \vec{x}} \approx V - i\vec{q} \cdot \vec{x}_0$$

Then, $i \vec{\nabla}_q \cdot [V - i q \cdot \vec{x}_c] = + \chi_c$, so

$\vec{A}(\vec{q}) = \frac{1}{2} \vec{B} \times \vec{x}_c$, which I might have guessed.



How does Feynman rule amp. will be, for small $\vec{q}$:

$$M_f = \bar{u}(2) [-ie \not{\epsilon}] u(1) \cdot \frac{1}{2} \vec{B} \times \vec{x}_c$$

This says the magnitude of the scatt amp. depends on where the mag cube is... very unpleasant.

Drop this since no such thing as a "magnetic cube".

compute $\langle \vec{m} \rangle$

⑨ New approach: you have some current flowing and this current has a magnetic moment:

$$\vec{m} = \frac{1}{2} \int \vec{x} \times \vec{j}(\vec{x}) d^3x$$

This is the way to find magnetic moment!

Suppose we just calculate: (for an electron).

$$\frac{\langle p' | \vec{m} | p \rangle}{\langle p' | p \rangle} = -\frac{1}{2} \cdot \frac{1}{\langle p' | p \rangle} \int d^3x \cdot \vec{x} \times \langle p' | : e \cdot \vec{\Psi}(x) \not{\epsilon} \Psi(x) : | p \rangle$$

$$= -\frac{1}{2} \cdot \frac{1}{\langle p' | p \rangle} \cdot C_4 \cdot C_4 \cdot \bar{u}(p') \not{\epsilon} u(p) \times \int d^3x \cdot \vec{x} \cdot e^{-ix \cdot p} e^{ix \cdot p'} = -\frac{1}{2} \cdot \frac{1}{\langle p' | p \rangle} \cdot C_4 \cdot C_4 \cdot \bar{u}(p') \not{\epsilon} u(p) \times \int d^3x \cdot \vec{x} \cdot e^{-ix \cdot (p-p')}$$

Now replace: $[\vec{x} = -i \vec{\nabla}_p]$ and extract it leaving:

$$-i \vec{\nabla}_p \left[e^{-it(E-E')} \cdot (2\pi)^3 \delta^3(\vec{p}-\vec{p}') \right]$$

Well, $-i \vec{\nabla}_p (e^{-it(E-E')}) = e^{-it(E-E')} \cdot (-it) \cdot (-i \vec{\nabla}_p E)$. This term vanishes at $t=0$, so lets work at $t=0$. Then:

$$= -\frac{1}{2} \langle p' | p \rangle \cdot C_4 \cdot C_4 \cdot [\bar{u}(p') \not{\epsilon} u(p)] \times [-i \vec{\nabla}_p \delta^3(\vec{p}-\vec{p}')] \cdot (2\pi)^3$$

What does this mean? It means that the magnetic moment of a plane wave electron is highly singular + we would do better if we made some wave packets.

Nevertheless, we know that $\vec{j} \rightarrow 0$ as $p \rightarrow p'$, so we can

consequence of doing parts to get:

$$= -\frac{1}{2} \langle p' | p \rangle \cdot C_4 C_4 \cdot (+i \vec{\nabla}_p [\bar{u}(p') \vec{\gamma} u(p)]) \cdot (2\pi)^3 \delta^3(\vec{p} - \vec{p}')$$

Then we can use $\langle p' | p \rangle = 2E C_2 \delta^3(\vec{p} - \vec{p}')$ to get

$$= -\frac{1}{2} \cdot \frac{1}{2EC_2} \cdot C_4^2 \cdot (2\pi)^3 \cdot (+i \vec{\nabla}_p [\cdot])$$

But, $\frac{C_4^2}{C_2} = \delta$ and $E \rightarrow m$, so $= -\frac{1}{2} \cdot \frac{1}{2m} \cdot (2\pi)^3 \delta \cdot (+i \vec{\nabla}_p [\cdot])$

We had:

$$\begin{aligned} \int d^3p \vec{\gamma} \times \vec{\nabla}_p \delta^3 &= \int d^3p \epsilon_{ijk} \gamma^i (\partial_p)_k \delta^3 \\ &= \epsilon_{ijk} \int d^3p \gamma^i \partial_k \delta^3 \rightarrow -\epsilon_{ijk} \int d^3p \partial_k \gamma^i \cdot \delta \\ &= + \int d^3p \vec{\nabla} \times \vec{\gamma} \cdot \delta = + \vec{\nabla} \times \vec{\gamma} \end{aligned}$$

So in this "parts", we have two cancelling minus signs, so

$$\langle \vec{m} \rangle = -\frac{1}{2} \cdot \frac{1}{2m} \cdot (2\pi)^3 \delta \cdot [-i \vec{\nabla}_p \times [\bar{u}(p') \vec{\gamma} u(p)]]$$

Choose Eden norm, $C_6 = 1$; then, $\delta = (2\pi)^3$ and

$$\langle \vec{m} \rangle = \frac{e}{4M} \cdot i \vec{\nabla}_p \times (\bar{u}(p') \vec{\gamma} u(p)) \quad \text{as } p' \rightarrow p.$$

So, in momentum space, $\langle \vec{m} \rangle$ is basically the curl of the current.

So: $\langle m^i \rangle = \frac{e}{4M} \cdot i \epsilon^{ijk} \frac{d}{dp^j} [\bar{u}(p') \gamma^k u(p)] = \frac{e}{4M} \cdot i \epsilon^{ijk} \bar{u}(p') \gamma^k \left[\frac{du(p)}{dp^j} \right]$

Meanwhile;

$$u = \sqrt{E+m} \begin{bmatrix} u \\ \frac{\vec{\sigma} \cdot \vec{p}}{E+m} u \end{bmatrix}$$

Now maybe we can get away with doing $E \rightarrow m$ limit before deriv:

$$u = \sqrt{2m} \begin{bmatrix} u \\ \frac{\vec{\sigma} \cdot \vec{p}}{2m} u \end{bmatrix} \quad \text{OK.}$$

Then, $\frac{d}{dp^j} u(p) = \sqrt{2m} \begin{bmatrix} 0 \\ \frac{\sigma^j u}{2m} \end{bmatrix} = \frac{1}{\sqrt{2m}} \begin{bmatrix} 0 \\ \sigma^j u \end{bmatrix} \leftarrow$

In this case, we can now compute:

$$\bar{u}(p) \gamma^k \frac{du(p)}{dp^j} = \sqrt{2m} \left[u^\dagger \quad -u^\dagger \frac{\vec{\sigma} \cdot \vec{p}}{2m} \right] \underbrace{\begin{bmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{bmatrix}}_{\left[\sigma^k \sigma^j u \right]} \frac{1}{\sqrt{2m}} u$$

$$= u^\dagger \sigma^k \sigma^j u \Rightarrow$$

$$\langle m^i \rangle = \frac{e}{4M} \cdot i \epsilon^{ijk} [u^\dagger \sigma^k \sigma^j u]$$

But:

$$\epsilon^{ijk} \sigma^j \sigma^k \text{ stop.}$$

$$\epsilon^{ijk} \sigma^k \sigma^j = \frac{1}{2} \epsilon^{ijk} [\sigma^k \sigma^j - \sigma^j \sigma^k] = \frac{1}{2} \epsilon^{ijk} [\sigma^i \sigma^j]$$

$$\text{So: } \langle m^i \rangle = \frac{e}{8M} \cdot i \epsilon^{ijk} [u^\dagger [\sigma^k, \sigma^j] u]$$

Anyway, $\epsilon^{ijk} \sigma^k \sigma^j = \epsilon^{ijk} (i \epsilon_{kjl} \sigma^l) = ? = -2i \delta^i_l \sigma^l = -2i \sigma^i$

$$\epsilon^{ijk} \epsilon_{kjl} = -\epsilon^{ijk} \epsilon_{ljk} = -[\delta^i_l \cdot 2]$$

$$\text{So } \langle m^i \rangle = \frac{e}{4M} \cdot (i) \cdot u^\dagger \underbrace{[\epsilon^{ijk} \sigma^k \sigma^j]}_{-2i \sigma^i} u = \frac{e}{2M} u^\dagger \sigma^i u$$

Results.

$$\text{So: } \left[\langle \vec{m} \rangle = \frac{e}{2M} u^\dagger \vec{\sigma} u \right]$$

Suppose electron is spins up along z. Then $u^\dagger \sigma^z u = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1$

$$\text{So } \boxed{m_z = \frac{e}{2M}} \leftarrow \text{magnetic moment of electron (lowest order).}$$

This is the electrons "intrinsic moment", more precisely.

$$m = \frac{e}{2M} \cdot \frac{\hbar}{c} = .93 \times 10^{-20} \text{ erg/gauss} \equiv \text{Bohr Magnetron} = \mu_B$$

Note: a spinless electron with $l=1$ orbit would have just this m , classically; this is Bohr Atom, hence Bohr Magnetron.

Triangle Anamoly

Axial-Vector Vertex in Spinor Electrodynamics

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Working within the framework of perturbation theory, we show that the axial-vector vertex in spinor electrodynamics has anomalous properties which disagree with those found by the formal manipulation of field equations. Specifically, because of the presence of closed-loop "triangle diagrams," the divergence of axial-vector current is not the usual expression calculated from the field equations, and the axial-vector current does not satisfy the usual Ward identity. One consequence is that, even after the external-line wave-function renormalizations are made, the axial-vector vertex is still divergent in fourth- (and higher-) order perturbation theory. A corollary is that the radiative corrections to $\nu\bar{\nu}$ elastic scattering in the local current-current theory diverge in fourth (and higher) order. A second consequence is that, in massless electrodynamics, despite the fact that the theory is invariant under γ_5 transformations, the axial-vector current is not conserved. In an Appendix we demonstrate the uniqueness of the triangle diagrams, and discuss a possible connection between our results and the $\pi^0 \rightarrow 2\gamma$ and $\eta \rightarrow 2\gamma$ decays. In particular, we argue that as a result of triangle diagrams, the equations expressing partial conservation of axial-vector current (PCAC) for the neutral members of the axial-vector-current octet must be modified in a well-defined manner, which completely alters the PCAC predictions for the π^0 and the η two-photon decays.

INTRODUCTION

THE axial-vector vertex in spinor electrodynamics is of interest because of its connections (i) with radiative corrections to $\nu\bar{\nu}$ scattering and (ii) with the γ_5 invariance of massless electrodynamics. We will show in this paper, within the framework of perturbation theory, that the axial-vector vertex has anomalous properties which disagree with those found by the formal manipulation of field equations. In particular, because of the presence of closed-loop "triangle diagrams," the divergence of the axial-vector current is not the usual expression calculated from the field equations, and the axial-vector current does not satisfy the usual Ward identity. One consequence is that, even after external-line wave-function renormalizations are made, the axial-vector vertex is still divergent in fourth- (and higher-) order perturbation theory. A corollary is that the radiative corrections to $\nu\bar{\nu}$ elastic scattering in the local current-current theory diverge in fourth (and higher) order. A second consequence is that, in massless electrodynamics, despite the fact that the theory is invariant under γ_5 transformations, the axial-vector current is not conserved.

In Sec. I we derive the usual formulas for the axial-vector divergence and Ward identity, and then show how they are modified by the presence of triangle diagrams. In Sec. II we discuss various consequences of the additional term found in Sec. I. In the Appendix we show that it is not possible to redefine the triangle diagram in a physically acceptable way so as to eliminate the anomalous behavior discussed in Secs. I and II. We also discuss in the Appendix a possible connection between our results and the $\pi^0 \rightarrow 2\gamma$ and $\eta \rightarrow 2\gamma$ decays. In particular, we argue that as a result of triangle diagrams, the equations expressing partial conservation of axial-vector current (PCAC) for the neutral members of the axial-vector current octet must be modified in a

well-defined manner, which completely alters the PCAC predictions for the π^0 and the η two-photon decays.

I. AXIAL CURRENT DIVERGENCE AND WARD IDENTITY

We work in the usual spinor electrodynamics described by the Lagrangian density¹

$$\mathcal{L}(x) = \bar{\psi}(x)(i\gamma \cdot \square - m_0)\psi(x) - \frac{1}{2}F_{\mu\nu}(x)F^{\mu\nu}(x) - :e_0\bar{\psi}(x)\gamma_\mu\psi(x)A^\mu(x):, \quad (1)$$

$$F_{\mu\nu}(x) = \frac{\partial A_\nu(x)}{\partial x^\mu} - \frac{\partial A_\mu(x)}{\partial x^\nu}, \quad \gamma \cdot \square \equiv \gamma^\mu \frac{\partial}{\partial x^\mu}.$$

We define the axial-vector current $j_\mu^5(x)$ and the pseudoscalar density $j^5(x)$ by

$$j_\mu^5(x) = :\bar{\psi}(x)\gamma_\mu\gamma_5\psi(x):, \quad (2)$$

$$j^5(x) = :\bar{\psi}(x)\gamma_5\psi(x):;$$

the corresponding vertex parts $\Gamma_\mu^5(p, p')$ and $\Gamma^5(p, p')$ are defined by

$$S_{F'}(p)\Gamma_\mu^5(p, p')S_F(p') = - \int d^4x d^4y e^{ip \cdot x} e^{-ip' \cdot y} \langle T(\psi(x) j_\mu^5(0) \bar{\psi}(y)) \rangle_0, \quad (3)$$

$$S_{F'}(p)\Gamma^5(p, p')S_F(p') = - \int d^4x d^4y e^{ip \cdot x} e^{-ip' \cdot y} \langle T(\psi(x) j^5(0) \bar{\psi}(y)) \rangle_0.$$

Using the equations of motion which follow from Eq. (1), the divergence of the axial-vector current may

¹ We use the notation and metric conventions of J. D. Bjorken and S. D. Drell, *Relativistic Quantum Fields* (McGraw-Hill Book Co., New York, 1965), pp. 377-390. Note that $\epsilon_{0123} = -\epsilon^{0123} = 1$.

easily be calcu

From Eqs. (3) Ward identity

$$(\not{p} - \not{p}')^\mu \Gamma_\mu^5(p, p')$$

Our task in th (5), which w equations, act

To this end theory. It is c

where the vert self-energy pa the free propa in the free pro renormalizatio and Σ , Eq. (5)

$$(\not{p} - \not{p}')^\mu \Lambda_\mu^5(p, p')$$

In order to de contributing t in which the a fermion line b γ' and ending diagrams in attached to an 1(b), respect has the form

$$\sum_{n=1}^{\infty} \prod_{i=1}^{n-1} \left[\gamma^{(i)} - \not{p}^{(i)} \right]$$

where we have the $\gamma_\mu\gamma_5$ vert remainder of th by $(\not{p} - \not{p}')^\mu$ and

$$\frac{1}{\not{p} + \not{p}_k - m_0} (\not{p} - \not{p}') \times \frac{1}{\not{p}' + \not{p}_k}$$

Stephcn Adler

"Axial Vector Vertex in Spinor Electrodynamics"

PR177, 2426 (1969)

1. First recall some basic QED facts. The Ward Identity relates the full vertex to the full spinor propagator. An alternative form relates the vertex correction to the spinor proper self-energy part. In lowest order the Ward identity may be trivially proven by using the little Dirac spinor space identity.
2. The axial version of this identity appears in (9) of Adlers paper. There are three terms instead of two on the right side. This leads to the three term axial Ward identity shown in (4) and again in (5). This Ward identity is formally "proved" in the usual way by considering all possible "insertions" of the "axial photon" onto the axial vertex blob. Unlike the vector Ward case, the closed loop contributions do not vanish. However, when one adds the loop and line contributions, the formal result (7) obtains.
4. But there is a catch. Part of the proof requires translation of the origin d^4k in a divergent integral. Such a translation is only justifiable if the integral is only logarithmically divergent. It turns out that the triangle closed loop is actually linearly divergent, as opposed to the higher loops, so you are not really sure that the Ward holds. It turns out in fact that the naive Ward is wrong, and that there is an extra term as shown in (32).
5. You can directly derive the Ward identity from current conservation and axiomatic stuff (reduction stuff). See equations (3) and (4). The fact that the actual perturbation theory Ward has that extra term means that current conservation (axial) also has an extra term as shown in (30).
6. The effect described above is called "the triangle anomaly". It is directly related to another triangle effect which is this: according to Weinberg, you can obtain the "asymptotic behavior" in a certain sense of a Feynman graphs by doing a certain power count. For axial stuff, the actual power is one lower than the power count gives because the leading power has vanishing coefficient. BUT, for the triangle graph, Weinbergs power is correct.

7. What are some of the implications of this triangle anomaly? First, recall the theorem proved by Preparata and Weisberger: if the current is conserved apart from usual mass terms, then you can multiplicatively renormalize the vertex: ie, the theory can be renormalized. Adler shows that his extra term in the current conservation ruins this theorem and causes the 4th order vertex correction to be divergent even after the standard wavefunction renormalizations. OUCH.

8. Next, he shows that the 4th order radiative corrections to $e\nu_e$ elastic scattering diverge for the same reason. (However, to understand this you have to understand that Fierz transformation which takes you to the charge retention ordering; I do not know about this)

9. The third unusual fact is that although the axial current is not "conserved", the lagrangian still has perfect gamma-5 symmetry.

10. Finally, on page 2438 Adler gives the form for a coefficient in the broken current conservation. If you can make your axial charges add up right, you can evade this problem altogether. Harari in another paper claims that the condition is actually that the hadronic quark + lepton charges must add up to 0. This fact obtains in the GIM theory with a fourth quark.

Positronium

1. reference cards: Pend-Dicke, Deutsche, Green-Lee, Ore Powell. — — — — —
other refs: Heitler book, Segree book, — — — — —
2. General description. Positronium has the same general energy level structure as hydrogen, except spacings are all scaled down by factor 2 due to smaller reduced mass. Thus, ground state binding energy is -6 ev relative to total mass of about 1000 ev. This is why potential concept still works. — — — — —
3. Ground state hyperfine: due to the obvious spin situation, you get a split ground state with the triplet 3S_1 lying about 10^{-5} eV above the singlet 1S_0 . The singlet (also called "para" although spins are not "parallel") decays into two photons in ~~XXX~~ 10^{-10} seconds. This is a standard QED calculation of annihilation, you have to multiply by wavefunction at origin etc.

The triplet (also called "ortho" meaning straight in Greek, so spins aligned) state is metastable in that it can only decay into three photons in 10^{-7} sec. I.e., this triplet state is long lived. Of course there are three such states. I have not seen a "modern" calculation of three photon decay in QED, but the graphs are pretty clear. Old calculation is Ore Powell above. You would expect three photon decay to be down by order-alpha. Actually, factor is more like 1000.
4. You can also radiatively go from triplet down to singlet, but easy to show that only M1 transition can do this. (No parity change). The spontaneous M1 lifetime is on the order of a month, so you can forget about this. But, when positronium is in a paramagnetic gas whose molecules like O_2 have magnetic moments, I gather that M1 radiation floats around and can stimulate this triplet to singlet transition. This effect is called quenching. In fact, this can happen very fast so that quench time is 10^{-10} seconds. Depends on gas pressure.
5. Selection rules: you can show that singlet must go into even number of photons merely my charge conjugation arguments. The catch is that $(-1)^{L+S}$ tells you the C of a spectroscopic state. Similarly, this shows that triplet cannot go into two photons. Analysis is analogous to π^0 decay into two photons. By the way, ~~XXX~~ π^0 has never been seen going into three photons! (Will 1970)

a note on parity conservation and $\pi^0 \rightarrow \gamma\gamma$; also, positronium $\rightarrow \gamma\gamma$.

- ① There are in the cms only three vectors floating around: $\vec{E}_1, \vec{E}_2$ and $\vec{k} = (\vec{p}_1 - \vec{p}_2)$. In general, the decay matrix element will have this form:

$$M = A \cdot \vec{E}_1 \cdot \vec{E}_2 + B (\vec{E}_1 \times \vec{E}_2) \cdot \vec{k}$$

Under parity x form we have $[\vec{k} \rightarrow -\vec{k}]$.

$$M \rightarrow M' = A \vec{E}_1 \cdot \vec{E}_2 - B (\vec{E}_1 \times \vec{E}_2) \cdot \vec{k}$$

Clearly if parity is conserved, both terms cannot be present.

- ② Final state has $P = +1 \Rightarrow M = A \vec{E}_1 \cdot \vec{E}_2$
 $P = -1 \Rightarrow M = B (\vec{E}_1 \times \vec{E}_2) \cdot \vec{k}$

③ How would you tell which one was true?

$P = +1 \Rightarrow$ photon linear pols. tend to be parallel
 $P = -1 \Rightarrow$ " " " " " " perpendicular.

In fact the second case is observed, so π^0 has $P = -1$.

- ③ Positronium decay. You know positronium is 1S_0 or 3S_1 . Both singlet and triplet have same parity of $-1 (-1)^L = -1$. Thus, you expect to see $\vec{E}_1 \perp \vec{E}_2$. In circular analysis, expect one photon L, one R, says Segré p69.

- ④ Contradiction: if one is L, the other is R, then $\Rightarrow \leftarrow \rightarrow$
 $J_3 = +2$ in final state. How is this possible if largest initial $J = 1$? More work needed on this. (see $\rightarrow$)

- ⑤ References: Bethe Hoffman "Mesons + Fields" v3 ~ p 20
 Sakurai ~ p 45
 Segré ~ p 69

⑥ I think Segie has made one error. Both photons must always have the same circular polarization when 1S_0 positronium decays, regardless of parity, just from angular momentum conservation. So circular does not provide a correlation test on parity.

⑦ Suppose initial state has $P=1$. Then you expect

$$\begin{aligned} M &\sim \vec{E}_1 \cdot \vec{E}_2 = E_{1x}E_{2x} + E_{1y}E_{2y} \\ &= (E_{1x} + iE_{1y})(E_{2x} - iE_{2y}) + (E_{1x} - iE_{1y})(E_{2x} + iE_{2y}) \\ &= E_1^+ E_2^- + E_1^- E_2^+ \\ &= E_1^R E_2^R + E_1^L E_2^L \quad \text{going over to helicity.} \end{aligned}$$

$$\begin{aligned} \text{So } \vec{E}_1 \cdot \vec{E}_2 &\sim RR + LL \\ E_1 \times E_2 &\sim RR - LL \end{aligned} \quad \left\{ \begin{array}{l} \text{ie, you never get } RL \text{ or } LR \\ \text{possibility.} \end{array} \right.$$

⑧ The argument is Hoffman is this: They want to guess whether π^0 has $J=0$ or $J=1$. The final photon state has $M_J = 0, \pm 2$, rule out ± 2 , this $M_J = 0$. If you do $R_x(\pi)$ on this state, it goes into + itself for either parity.

	$R_x(\pi)$
$RR + LL$	+
$RR - LL$	+

However, $J=1, M_J=0$ initial state is $\sim P_1(\cos\theta) = z$ into itself. Thus, rule out $1, 1, 0$ + conclude that initial state is spin 0.

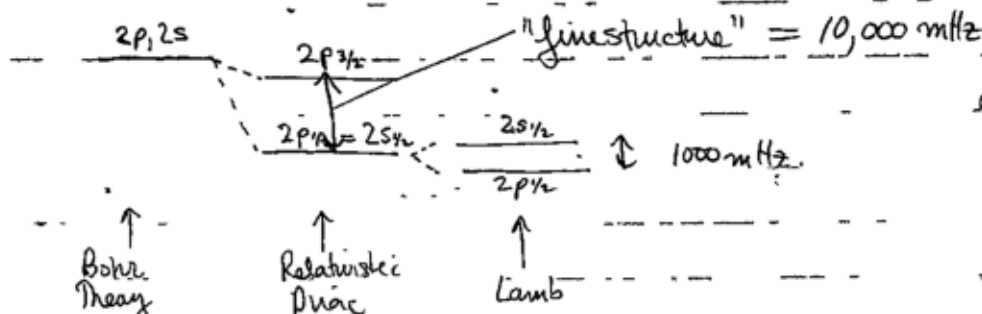
Then look at $\vec{E}_1, \vec{E}_2$ correlation to see what initial parity is

⑨ Positronium: choice of 3S_1 or 1S_0 . Rule out 3S_1 because $J=1$ not allowed as shown above in ⑧. Expect $\uparrow$ correlation on final photons since parity of positronium state is -1 .

⑩ Charge conjugation says $(-1)^{L+S} = (-1)^n \Rightarrow ^3S_1$ cannot go into $\gamma\gamma$!
This is easier way

Lamb Shift

1. First, let's get Uehling out of the way. Even before that, let's say what the Lamb shift is: it is the energy difference between the $2S_{1/2}$ and $2P_{1/2}$ levels in hydrogen. In basic theory, these are degenerate. (i.e., Schrodinger theory). In the relativistic Dirac theory of hydrogen where you use the Dirac Hamiltonian in the SE, you get some "fine structure" namely:



Gottfried page 369

Dirac theory automatically includes spin effects (perturbation theory calls this spin-orbit interaction) and thus yields the fine structure. But I think states with the same n and J are still degenerate. For sure, the Dirac theory of hydrogen does not give the Lamb shift. You really need QED. By the way; the Lamb line is what interstellar hydrogen does all the time.

2. Now for Uehling. The Uehling correction accounts for 27 mc of the Lamb shift about 1/40 th of the full shift. It is simply the vacuum polarization correction to the photon propagator. This correction causes a constant term to appear in the renormalized (to lowest order) propagator. This constant term in the momentum space photon propagator appears as a delta function term in the potential in configuration space, a simple Fourier fact. This is the Uehling potential, also known as a contact potential. This potential clearly only affects S-waves which have some amplitude at the origin. It tends to stabilize S-orbitals, although as shown above the total Lamb shift is a raising of the S-state. For details on Uehling, see page 8 of BD notes chapter 8.

In the messy formulas which are obtained in BD and below, this Uehling term always appears as a $-1/5$ term in the vertex correction (vertex in the non-"proper" sense). I.e., vacuum polarization can be considered as one term in a full vertex correction.

3. So, where does that bulk 1000 mc of the Lamb shift come from? To get the answer, we can think in terms of the Uehling effect described above. In the above, we obtained the Uehling correction as a correction to the photon propagator. But you could also think of this as a correction to the vertex. Then this Uehling correction appears in this way: *an improper*

$$\gamma_\mu \rightarrow \gamma_\mu + \Delta\gamma_\mu \quad \Delta\gamma_\mu = \frac{-i}{2\pi} \cdot \frac{2}{3} \cdot \frac{q^2}{m^2} \cdot \gamma_\mu \cdot \left(-\frac{1}{5}\right)$$

From this point of view, you can see the q^2 factor which cancels the q^2 of the photon propagator coming from your ~~max~~ external potential x to this corrected vertex. Thus, the size of this $q^2 \gamma_\mu$ term in the vertex correction will always determine the size of the contact potential, which in turn will determine the Lamb shift.

We shall now show that the $(-1/5)$ is only a small part of the total.

← from the VP

one of these is the proper vertex correction

4. Now imagine that you calculate the actual ~~vertex~~ ^{radiative} corrections in lowest order. As BD show, this means you add together 6 graphs. Three have simple photon lines, one is the VP, and two are mass counterterm graphs which ~~recall are also in this order~~. When you add all this stuff up, you get three terms for your vertex correction.

4. The vertex correction is only 3 terms

- a) the anomalous magnetic moment term appears with usual α/π . This does not concern the Lamb Shift however because not a q^2 term.
- b) a $(-3/8)$ term appears like the Uehling $(-1/5)$.
- c) an infrared divergent term $\ln(m/\Lambda)$ appears because you have to regulate the vertex loop integral.

5. Clarification on Uehling versus vertex. We are arguing that the vertex corrections give terms which are effectively like the Uehling term. BUT you still have to include the Uehling along with the vertex correction. Uehling is a correction to the photon propagator, vertex correction is something else. ~~One and two graphs do nothing Lambwise.~~

6. OK, so now what do you do with that infinite tending infrared correction??? This is where things get complicated. The first thing that BD do is make a correction to the vertex correction so that loop photons with k below some $k(\min)$ are not counted in the integrations. When this is done, the photon mass vanishes from the new correction and the new log term is $\ln(m/k_{\min})$. Also, in the process of this correction to the vertex correction, a q^2 term of $(+5/6)$ appears. So, the "vertex correction without soft photons below $k_{\min}$ is now given by:-

$$\Lambda_\mu = \frac{i\sigma_{\mu\nu}q}{2\pi} \cdot \frac{\alpha}{2\pi} + \frac{\alpha}{2\pi} \cdot \frac{2}{3} \frac{q^2}{m^2} \delta_\mu \left[\ln \frac{m}{2k_{\min}} + \frac{5}{6} - \frac{3}{8} - \frac{1}{5} \right]$$

Actually, I have included the Uehling here even though it is not really a part of the ^{proper} vertex correction itself. The $(-3/8)$ came from the original vertex calculation, the $(+5/6)$ comes when you modify the vertex correction for soft photons.

7. So, we have gotten rid of the photon mass, but now we have this arbitrary soft photon cutoff $k(\min)$ appearing. What do we do with that?? This part is very subtle. There is still another "vertex correction" that we have not yet done, and that is the possibility of soft physical photons being radiated from the vertex. (called Bremsstrahlung). Well, actually BD do not work it this way. They only show that. They ~~claim~~ claim that that $k(\min)$ really is present in the vertex correction. It must be there to cancel another $k_{\min}$ which occurs in the Bremsstrahlung cross section. Then when you calculate elastic scattering (Coulomb say) with Brem and with vertex correction containing $k_{\min}$, the two $k_{\min}$ s cancel and your actual Coulomb scattering is independent of $k(\min)$ as well as photon mass.

8. So, we still have $k_{\min}$ present in the vertex correction. Now comes a stupendous gimmick. This is the "hard photons" vertex correction. We showed above that if you include the soft photons, the vertex correction depends on photon mass. However, somehow if you handle the soft photons in a Fermi nonrel calculation to get the soft photon energy shift, again the $k_{\min}$ s cancel. This extra calculation in effect gives you a vertex correction which cancels the $k_{\min}$ term. No terms like $(-1/5)$ are added here.

9. Total Lamb shift then becomes:

$$\Delta E = \frac{4\alpha(Z\alpha)^4 m}{3\pi n^3} \left[+ \ln \left(\frac{m}{2E} \right) + \frac{5}{6} - \frac{3}{8} - \frac{1}{5} \right]$$

$E = 9\alpha^2 m$ in hydrogen.

Here is the relative size of these terms:

$$\Delta E = \text{const.} \left[5 \ln \left(\frac{m}{2E} \right) + \frac{25}{6} - \frac{15}{8} - 1 \right]$$

She 1 gives -27 mc

$$= \text{const.} [34.8 + 2.3 - 1] = 36.1 \times 27 \text{ mc} = 974.7 \text{ mc.}$$

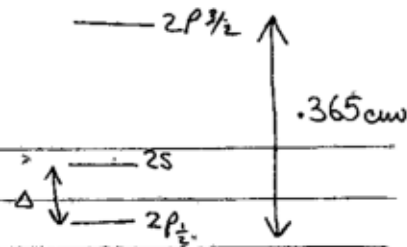
10. Some numbers from Schweber. If I take out the Uehling, my actual number is 1002 mc. Schweber says this should be 1010. Now, the anomalous magnetic moment increases the shift by 68 mc, Uehling pulls it down by 27 mc. This takes you to 1052. Experiment says 1057.8. The remaining few mHz come from higher order stuff plus finite size effects (I read this as: proton or nucleus is not really a point but has a radius. Not clear how this changes calculation which was based on an external potential anyway. Perhaps finite proton mass also enters in this category).
11. Conclusion and comments: We have seen that the Lamb shift is a direct consequence of higher order QED corrections. The vacuum polarization correction is relatively unimportant coming in at 27 mc. (However, the VP correction is proportional to electron mass so would be about 200 times larger for muonic hydrogen. Also, VP is proportional to Z^4 for hydrogenlike system. This at least suggests that VP is more significant in heavy atoms, but I have not thought that through.) Except for about 5eV due to higher order and finite size effects, the entire rest of the 1057mc Lamb shift comes from the simple lowest order vertex correction. This vertex correction generates among other things the anomalous magnetic moment term which accounts only for 68 mc + . The real problem is how to handle the soft photon problem in actually calculating the vertex correction graph. Somehow BD end up not using QED for a part of their calculation which is supposed to test QED. I think a bound state theory is a little better at doing this Lamb shift. Roughly, the pictures you might draw for Lamb shift are these: -



Roughly speaking, virtual photons from the nucleus allow the electron to emit and recapture photons. This process shifts the energy of the electrons state.

[H.A. Bethe [Cornell]
 "The E.M. Shift of energy levels."
 PR 72, 339 (1947)]

(3 pages)



Dirac's theory says that $E(2s) = E(2p_{1/2})$ in H. But Rutherford and Lamb (PR 72, 241, 1947) have shown that $\Delta = .033 \text{ cm}^{-1} = 1000 \text{ mHz}$. This was suspected even back in 1937. Several explanations have been considered

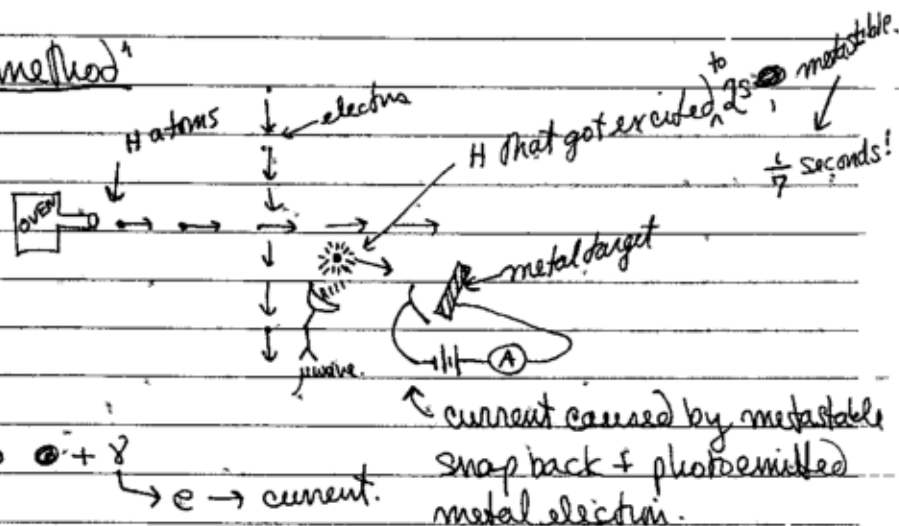
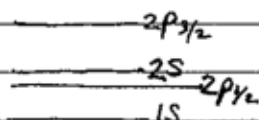
- Kemble — its not a nuclear interaction (1932)
- Uehling — its not vac. pole (wrong sign) (1935)
- Schwinger, Weisskopf, Oppenheimer — radiative corrections?

Bethe suggests that the divergence encountered by (c.) above can be "subtracted out" leaving

$$W_{\text{is}} = 136 \ln \left[\frac{mc^2}{E_n - E_{n-1}} \right] = 1040 \text{ mc}! \quad (\text{now } 1059.8)$$

"This paper grew out of extensive discussions at the 1947 Shelter Island Conference"

[Lamb + Rutherford
 "Fine Structure of H by microwave method"
 PR 72, 241 (1947)]



$\text{H} \rightarrow \text{H}^+ + e^-$
 $e^- \rightarrow \text{current}$

If you perturb the metastable beam so that 2s and 2p_{1/2} mix, then excited atoms decay quick + current (A) goes away.

So, zap the metastable beam with $\approx 1000 \text{ mc}$ microwaves
 $[\lambda = c/\nu = 3 \times 10^{10} / 10^9 \approx 30 \text{ cm}]$ Farinik!

QED

Processes

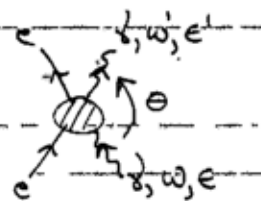
QED Processes

#	process	formula name	historic names
1	$e\gamma \rightarrow e\gamma$	Klein-Nishina (1929) Formula	Compton Scattering ; Thomson Cross Section
2	$e p \rightarrow e p$	Mott Cross Section	Rutherford Cross Section ("Coulomb scattering")
3	$e e^+ \rightarrow e e^+$	Bhabha	
4	$e e^+ \rightarrow \mu^+ \mu^-$	muon pair production	
5	$e e \rightarrow e e$	Møller (1932)	/full row calculation without /using Rules; LSZ etc.
6	$e p \rightarrow e p \gamma$	Bethe-Heitler (1934)	Bremsstrahlung

The major QED cross section formulas:

① Compton Process The DCS is simplest in the lab system where the exact (electron spin averaged.) result is:

$$\frac{d\sigma}{d\Omega} = \frac{r_0^2}{4} \left(\frac{\omega'}{\omega}\right)^2 \left[\frac{\omega}{\omega'} + \frac{\omega'}{\omega} - 2 + 4 |\epsilon \cdot \epsilon'|^2 \right]$$



where

$$\frac{\omega'}{\omega} = \left[1 + \frac{\omega}{m} (1 - \cos \theta) \right]^{-1}$$

This is Klein-Nishina (1929). If we do polarization averaging we get:

$$\frac{d\bar{\sigma}}{d\Omega} = \frac{r_0^2}{2} \left(\frac{\omega'}{\omega}\right)^2 \left[\frac{\omega}{\omega'} + \frac{\omega'}{\omega} - \sin^2 \theta \right]$$

The low energy limit for both these formulas [$\omega \ll m$] is:

$$\frac{d\sigma}{d\Omega} = r_0^2 \cos^2 \alpha$$

$$\cos \alpha = \vec{\epsilon} \cdot \vec{\epsilon}'$$

$$\frac{d\bar{\sigma}}{d\Omega} = r_0^2 \cdot \frac{1}{2} (1 + \cos^2 \theta)$$

$$\sigma_{\text{tot}} = \frac{8\pi}{3} r_0^2$$

These limits were known to Thompson in 1933. The general total cross section is:

$$\sigma(\omega) = 2\pi r_0^2 \left\{ \frac{1+\omega}{\omega^3} \left[\frac{2\omega(1+\omega)}{1+2\omega} - \ln(1+2\omega) \right] + \frac{\ln(1+2\omega)}{2\omega} - \frac{(1+3\omega)}{(1+2\omega)^2} \right\}$$

where ω is really ω/m here. The high energy limits are

$$\frac{d\bar{\sigma}}{d\Omega} = \frac{r_0^2}{4} \cdot \left(\frac{m}{\omega}\right) \cdot \csc^2 \theta/2$$

$$\bar{\sigma} = \pi r_0^2 \left(\frac{m}{\omega}\right) \cdot \left(\ln \frac{2\omega}{m} + \frac{1}{2} \right)$$

But need $\theta^2 \gg 2m/\omega$ here. Thus, the total cross section decreases as $(\ln \omega)/\omega$ at high energy.

Note:

$$r_0 = 2.8 \times 10^{-13} \text{ cm} = 2.8 \text{ fermi}$$

$$r_0^2 = 7.8 \text{ f}^2 = 7.8 \times 10^{-26} \text{ cm}^2 = 78 \times 10^{-27} \text{ cm}^2 = 78 \text{ mb}$$

$$\text{So } \sigma_{\text{Thomson}} \cong 640 \text{ mb.}$$

Compton Scatt.

-14-

Compton.

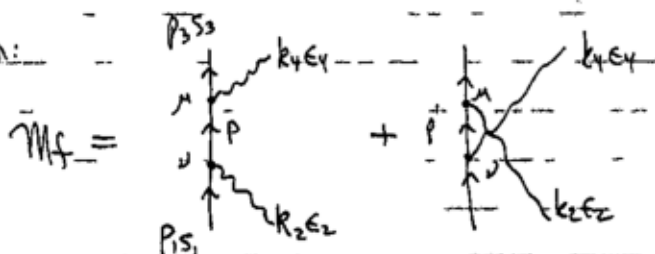
1

(see end of "Matter" for first page of this)

Norms are same as before, and now:

$$M_f = (-ie)^2 \cdot \epsilon_1^\mu \cdot \bar{u}(3) \gamma_\mu \left[\frac{+i}{\not{p}-m+i\epsilon} \right] \gamma_\nu u(1) \epsilon_2^\nu \\ + (-ie)^2 \cdot \epsilon_2^\mu \cdot \bar{u}(3) \gamma_\mu \left[\frac{+i}{\not{p}'-m+i\epsilon} \right] \gamma_\nu u(1) \epsilon_1^\nu$$

Picture:



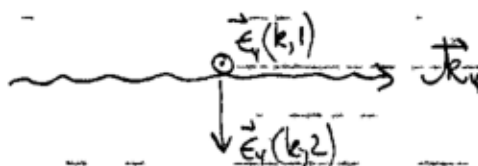
In shorthand: $M_f = (-ie)^2 \cdot \left[\bar{u}(3) \not{\epsilon}_1 \frac{-i}{\not{p}-m} \not{\epsilon}_2 u(1) + \bar{u}(3) \not{\epsilon}_2 \frac{-i}{\not{p}'-m} \not{\epsilon}_1 u(1) \right]$

This is the classic Compton amplitude.

So new rule:

- for each ext. photon, put ϵ^μ . It will "lie" onto the $(-ie \gamma_\mu)$ at some vertex. **polarization.**

Next, what about spin sums here?! This is a fussy detail. An external photon can have only two spin states.



Of course it is up to the "operator" to choose a plane here. But $\vec{\epsilon}(1) \times \vec{\epsilon}(2) = \vec{k}$. Define:

$$\epsilon^\mu = (0, \vec{\epsilon}).$$

Now, compute M_f^R . Make sure ϵ_1 and ϵ_2 are real

$$M_f^* = (+ie)^2 \left[\bar{u}(1) \not{\epsilon}_2^* \left(\frac{+i}{\not{p}-m+i\epsilon} \right)^* \not{\epsilon}_1 u(3) + \bar{u}(1) \not{\epsilon}_1 \left(\frac{+i}{\not{p}'-m+i\epsilon} \right)^* \not{\epsilon}_2 u(3) \right]$$

Now again we take these things together. Replace:

$$\frac{+i}{\not{p}-m} = \frac{+i(\not{p}+m)}{p^2-m^2}$$

Then,

$$M_f = (-ie)^2 \left(\frac{+i}{p^2 - m^2} \right) \bar{u}(3) \not{\epsilon}_r (\not{p} + m) \not{\epsilon}_\mu u(1)$$

$$+ (-ie)^2 \left(\frac{+i}{p'^2 - m^2} \right) \bar{u}(3) \not{\epsilon}_\mu (\not{p}' + m) \not{\epsilon}_r u(1)$$

$$M_s^* = (-ie)^2 \left(\frac{+i}{p^2 - m^2} \right)^* \bar{u}(1) \not{\epsilon}_\mu (\not{p} + m) \not{\epsilon}_r u(3)$$

$$+ (-ie)^2 \left(\frac{+i}{p'^2 - m^2} \right)^* \bar{u}(1) \not{\epsilon}_r (\not{p}' + m) \not{\epsilon}_\mu u(3)$$

Then, clearly

$$|M|^2 = (-ie)^4 [A + B + C + D]$$

$$A = \left(\frac{+i}{p^2 - m^2} \right) \left(\frac{+i}{p'^2 - m^2} \right)^* \bar{u}(3) \not{\epsilon}_r (\not{p} + m) \not{\epsilon}_\mu u(1) \bar{u}(1) \not{\epsilon}_\mu (\not{p} + m) \not{\epsilon}_r u(3)$$

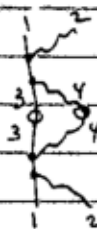
$$= \left| \frac{i}{p^2 - m^2} \right|^2 \text{tr} (\not{\Lambda}_3 \not{\epsilon}_r (\not{p} + m) \not{\epsilon}_\mu \not{\Lambda}_1 \not{\epsilon}_\mu (\not{p} + m) \not{\epsilon}_r)$$

$$B = \left(\frac{+i}{p'^2 - m^2} \right) \left(\frac{+i}{p^2 - m^2} \right)^* \bar{u}(3) \not{\epsilon}_\mu (\not{p}' + m) \not{\epsilon}_r u(1) \bar{u}(1) \not{\epsilon}_\mu (\not{p} + m) \not{\epsilon}_r u(3)$$

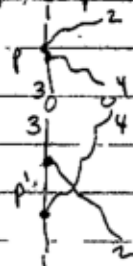
$$= \left(\frac{+i}{p'^2 - m^2} \right) \left(\frac{+i}{p^2 - m^2} \right)^* \text{tr} (\not{\Lambda}_3 \not{\epsilon}_\mu (\not{p}' + m) \not{\epsilon}_r \not{\Lambda}_1 \not{\epsilon}_r (\not{p} + m) \not{\epsilon}_\mu) \checkmark$$

Pictorially:

A =



B =



C = ...

D = ...

It is just like before; in Møller, we put a $\not{\epsilon}$ at each vertex, here we put $\not{\epsilon}_i$ instead.

Now, electron spin sum goes as before $\sim \rightarrow \frac{1}{2}$ in front and $1 \rightarrow (\not{p} + m) / 2m$. We are left with the photon spin sums: about the end.

$$\sum_{\text{spin}} \sum_{\text{final photon}} A = \sim \sum_{\epsilon_\mu} \text{tr} (\not{\Lambda}_3 \not{\epsilon}_\mu (\not{p}' + m) \not{\epsilon}_r \not{\Lambda}_1 \not{\epsilon}_r (\not{p} + m) \not{\epsilon}_\mu)$$

Extract to get: $\sum_{\epsilon_\mu} \epsilon_\mu^\nu \epsilon_\mu^\mu$

I think this is $-g^{\nu\mu}$ if we use gauge terms.

$$\frac{1}{4} \sum_{\epsilon_2, \epsilon_4} (\epsilon_2 \cdot \epsilon_4)^2 = \frac{1}{4} \sum_{\epsilon_2, \epsilon_4} (g_{\alpha\beta}^{\mu\nu} \epsilon_2^\alpha \epsilon_4^\beta)^2$$

$$= \frac{1}{4} g_{\alpha\beta}^{\mu\nu} (g^{\alpha\beta}) = \frac{1}{4} \cdot 4 = 1$$

$$= \frac{1}{4} \sum_{\epsilon_2, \epsilon_4} (g_{\alpha\beta}^{\mu\nu} \epsilon_2^\alpha \epsilon_4^\beta) (g_{\mu\nu}^{\rho\sigma} \epsilon_2^\rho \epsilon_4^\sigma)$$

$$= \frac{1}{4} g_{\alpha\beta}^{\mu\nu} g_{\mu\nu}^{\rho\sigma} \left(\sum_{\epsilon_2} \epsilon_2^\alpha \epsilon_2^\rho \right) \left(\sum_{\epsilon_4} \epsilon_4^\beta \epsilon_4^\sigma \right)$$

$$= \frac{1}{4} g_{\alpha\beta}^{\mu\nu} g^{\alpha\mu} g_{\mu\nu}^{\rho\sigma} g^{\beta\rho} = \frac{1}{4} g_{\alpha\beta}^{\mu\nu} g_{\mu\nu}^{\rho\sigma} g^{\alpha\mu} g^{\beta\rho}$$

$$= \frac{1}{4} \text{tr}(g g g g) \quad \text{With } g^2 = \begin{pmatrix} 1 & & \\ & -1 & \\ & & -1 \end{pmatrix} \begin{pmatrix} 1 & & \\ & -1 & \\ & & -1 \end{pmatrix} = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} = I$$

$$= \frac{1}{4} \text{tr}(I \cdot I) = \frac{1}{4} \cdot \text{tr}(I) = \frac{1}{4} \cdot 4 = 1$$

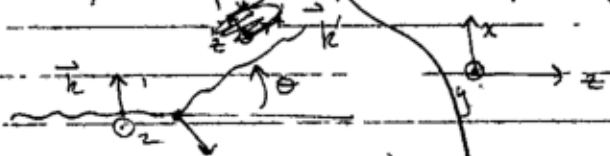
$$\text{So } \sum_{\epsilon \epsilon'} \left[\frac{k'}{h} + \frac{k}{h} + 4(\epsilon \cdot \epsilon')^2 - 2 \right]$$

$$= \frac{1}{2} \left[\frac{k'}{h} + \frac{k}{h} + 4 \cdot (4) - 2 \right] = \frac{1}{2} \left[\frac{k'}{h} + \frac{k}{h} + 16 - 2 \right]$$

Something is wrong here! Perhaps we can only

$$\text{For: } \sum_{\epsilon_2, \epsilon_4} (\epsilon_2 \cdot \epsilon_4)^2 = \sum_{\epsilon_2, \epsilon_4} \{ \vec{\epsilon}_2 \cdot \vec{\epsilon}_4 \}^2 = \sum_{\epsilon_2, \epsilon_4} (\vec{\epsilon}_2 \cdot \vec{\epsilon}_4)^2$$

How to set this up! In lab frame,



First deal with final sum: $\vec{\epsilon}_1 = \hat{z}$, $\vec{\epsilon}_2 = \hat{y}$

No prob.

$$\vec{\epsilon}(1) = \hat{x}$$

$$\vec{\epsilon}'(1) = \hat{x} \cos \theta - \sin \theta \hat{z}$$

$$\vec{\epsilon}(2) = \hat{y}$$

$$\vec{\epsilon}'(2) = \hat{y}$$

$$\text{So: } \sum_{\epsilon \epsilon'} (\vec{\epsilon} \cdot \vec{\epsilon}')^2 = (\vec{\epsilon}(1) \cdot \vec{\epsilon}'(1))^2 + (\vec{\epsilon}(2) \cdot \vec{\epsilon}'(1))^2 + (\vec{\epsilon}(1) \cdot \vec{\epsilon}'(2))^2 + (\vec{\epsilon}(2) \cdot \vec{\epsilon}'(2))^2$$

$$= \cos^2 \theta + 0 + 0 + 1 = 1 + \cos^2 \theta$$

$$6 - 4 \sin^2 \theta$$

$$\text{So we get: } +4(1 + \cos^2 \theta) - 2 = 2 + 4(1 - \sin^2 \theta)$$

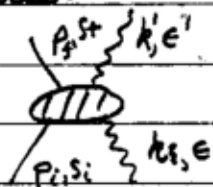
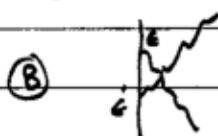
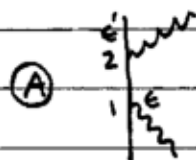
THE COMPLETE COMPTON CALCULATION FROM M TO do:

1. the invariant matrix element
2. general rules for M of the form $\bar{u}Ou$.
3. explicit trace calculations
4. M^2
5. The cross section Klein Nishina formula

from 225 notes.

Trace techniques. I have gone through getting σ in terms of $|M|^2$. Now let's work out $|M|^2$ for several cases.

Compton Scattering: There are two diagrams of course:



When you add them the matrix element is

$$M = e^2 \bar{u}(p_f, s_f) \left[\not{\epsilon}' \frac{1}{\not{p}_i + \not{k} - m} \not{\epsilon} + \not{\epsilon} \frac{1}{\not{p}_i - \not{k}' - m} \not{\epsilon}' \right] u(p_i, s_i)$$

Is this Lorentz invariant? Let's consider $\bar{u} \not{\epsilon} u$. This is really $p_\mu \cdot \bar{u} \gamma^\mu u$. Both pieces transform as a vector so yes. But what about, eg,

$$M' = \bar{u}(p_f, s_f) \not{\epsilon}' \frac{1}{(\not{p}_i - \not{k} + m)} \not{\epsilon} u(p_i, s_i)$$

So what can we say about $\bar{u}_1 \gamma^\mu \gamma^\nu \gamma^\rho u_2$ $\not{\epsilon}'_\mu \not{p}'_\nu \not{\epsilon}_\rho$. Those are different states in there, $\bar{u}_1$ and u_2 . Let's examine (p III)

$$[\bar{u}_2 \gamma_\mu u_1]$$

$$u'_i = ?$$

done. ★
see N13

I had better learn how spinors transform in general. (as opposed to spinor fields).

$$\psi'_a(x) = U(a, \Lambda) \psi_a(x) U^\dagger(a, \Lambda) = (S(\Lambda^{-1}))_{a\bar{q}} \psi_{\bar{q}}(\Lambda x + a)$$

says Halpern. This is the definition, and

$$S(\Lambda) = e^{\frac{i}{2} \sigma_{\mu\nu} \omega^{\mu\nu}}$$

Now, returning to the trace stuff:

$$M = e^2 \bar{u}_s \left[\frac{\not{\epsilon}' (\not{p}_i + \not{k} + m) \not{\epsilon}}{(p_i + k)^2 - m^2} + \frac{\not{\epsilon} (\not{p}_i - \not{k}' + m) \not{\epsilon}'}{(p_i - k')^2 - m^2} \right] u_i$$

At once we go to lab frame and use transverse photons:

$$\begin{aligned} p_i &= (m, 0) & p_i \cdot \epsilon &= 0 & k \cdot \epsilon &= 0 \\ \epsilon &= (0, \underline{\epsilon}) & p_i \cdot \epsilon' &= 0 & k' \cdot \epsilon' &= 0 \\ \epsilon' &= (0, \underline{\epsilon}') \end{aligned}$$

$$D_1 = 2p_i \cdot k \quad D_2 = -2p_i \cdot k'$$

$$\begin{aligned} \not{p}_i \not{\epsilon} &= -\not{\epsilon} \not{p}_i & \not{p}_i u_i &= m u_i & \not{\epsilon}' \not{p}_i \not{\epsilon} u_i &= -\not{\epsilon}' \not{\epsilon} m u_i \\ \not{\epsilon} \not{p}_i \not{\epsilon}' u_i &= -\not{\epsilon} \not{\epsilon}' m u_i \end{aligned}$$

Both p_i terms vanish + we are left w/

$$\begin{aligned} M &= e^2 \bar{u}_s \left(\frac{\not{\epsilon}' \not{k} \not{\epsilon}}{2p_i \cdot k} + \frac{-\not{\epsilon} \not{k}' \not{\epsilon}'}{-2p_i \cdot k'} \right) u_i \\ &= \frac{e^2}{2m} \bar{u}_s \left[\frac{\not{\epsilon}' \not{k} \not{\epsilon}}{\omega} + \frac{\not{\epsilon} \not{k}' \not{\epsilon}'}{\omega'} \right] u_i \end{aligned}$$

Now quote yr old theorem:

$$\bar{\sigma} = \sigma_0 \sigma^\dagger \sigma_0$$

$$M = \bar{u}_s \sigma u_i \quad M^{*+} = \bar{u}_i \bar{\sigma}^\dagger u_s$$

$$|M|^2 = M M^{*+} = \overbrace{(\bar{u}_s)_\lambda (\sigma)_{\lambda\kappa} (u_i)_\kappa (\bar{u}_i)_\mu (\sigma^\dagger)_{\mu\nu} (u_s)_\nu}$$

Now we look up our projectors:

$$u_\kappa \bar{u}_\mu = u(p, s)_\kappa \bar{u}(p, s)_\mu = \left(\frac{\not{p} + m}{2m} \right)_\kappa \left(\frac{1 + \gamma_5 \not{s}}{2} \right)_{\lambda\mu}$$

How does this work? (topic of another note!)

So we may in general say

$$M = \frac{1}{2} \text{Tr} \left[\left(\frac{\not{p}_f + m}{2m} \right) \sigma \left(\frac{\not{p}_i + m}{2m} \right) \sigma^\dagger \right]$$

This is for a particular initial + final spin. At this stage we can sum-average to get

$$\overline{M} = \frac{1}{2} \text{Tr} \left[\left(\frac{\not{p}_f + m}{2m} \right) \sigma \left(\frac{\not{p}_i + m}{2m} \right) \sigma^\dagger \right]$$

OK, lets go ahead + do this where

$$\sigma = \frac{e^2}{2m} \left[\frac{\not{\epsilon}' \not{k}}{\omega} + \frac{\not{\epsilon} \not{k}'}{\omega'} \right]$$

Then as total thing ... well:

$$\overline{\sigma} = \frac{e^2}{2m} \left[\dots \right]$$

To keep it looking like Null, use $\begin{cases} \not{k} \not{\epsilon} = -\not{\epsilon} \not{k} \\ \not{k}' \not{\epsilon}' = -\not{\epsilon}' \not{k}' \end{cases}$ to get

$$M = -e^2 \bar{u}_f \left(\frac{\not{\epsilon}' \not{\epsilon} \not{k}}{2p_i \cdot k} + \frac{\not{\epsilon} \not{\epsilon}' \not{k}'}{2p_i \cdot k'} \right) u_i$$

Then

$$\sigma = -\frac{e^2}{2m} \left[\frac{\not{\epsilon}' \not{\epsilon} \not{k}}{\omega} + \frac{\not{\epsilon} \not{\epsilon}' \not{k}'}{\omega'} \right]$$

$$\bar{\sigma} = -\frac{e^2}{2m} \left[\frac{\not{k} \not{\epsilon} \not{\epsilon}'}{\omega} + \frac{\not{k}' \not{\epsilon}' \not{\epsilon}}{\omega'} \right]$$

Note the implicit theorem:

$$\overline{\not{a} \not{b} \not{c} \dots} = \dots \not{c} \not{b} \not{a}$$

So we are left with:

$$|\overline{M}|^2 = \frac{1}{8m^2} \frac{Q^4}{4m^2} \text{Tr} \left[(\not{p}_5 + m) \left\{ \frac{\not{\epsilon}' \not{\epsilon} \not{k}}{\omega} + \frac{\not{\epsilon}' \not{k}'}{\omega'} \right\} (\not{p}_i + m) \left\{ \frac{\not{k} \not{\epsilon}'}{\omega} + \frac{\not{k}' \not{\epsilon}' \not{\epsilon}}{\omega'} \right\} \right]$$

We need essentially 4 traces:

$$T_1 = \text{Tr} (\not{p}_5 + m) \not{\epsilon}' \not{\epsilon} \not{k} (\not{p}_i + m) \not{k} \not{\epsilon} \not{\epsilon}'$$

$$T_2 = \text{Tr} (\not{p}_5 + m) \not{\epsilon}' \not{\epsilon}' \not{k}' (\not{p}_i + m) \not{k} \not{\epsilon} \not{\epsilon}$$

$$T_3 = \text{Tr} (\not{p}_5 + m) \not{\epsilon}' \not{\epsilon} \not{k} (\not{p}_i + m) \not{k}' \not{\epsilon}' \not{\epsilon}$$

$$T_4 = \text{Tr} (\not{p}_5 + m) \not{\epsilon}' \not{\epsilon}' \not{k}' (\not{p}_i + m) \not{k} \not{\epsilon} \not{\epsilon}'$$

Now always use tricks instead of brute force. For T_2 , write T_2 backward:

$$T_2 = \text{Tr} \not{\epsilon}' \not{\epsilon}' \not{k}' (\not{p}_i + m) \not{k} \not{\epsilon} \not{\epsilon} (\not{p}_5 + m)$$

Just T_2 again.

Now recall that $\not{p}_5 = \not{p}_i + \not{k} - \not{k}'$. Thus, T_1 and T_2 are:

$$T_1 = \text{Tr} (\not{p}_i + \not{k} - \not{k}' + m) \not{\epsilon}' \not{\epsilon} \not{k} (\not{p}_i + m) \not{k} \not{\epsilon} \not{\epsilon}'$$

$$T_2 = \text{Tr} (\not{p}_i + \not{k} - \not{k}' + m) \not{\epsilon}' \not{\epsilon} \not{k} (\not{p}_i + m) \not{k}' \not{\epsilon}' \not{\epsilon}$$

Clearly, $k \leftrightarrow -k'$ and $\epsilon \leftrightarrow -\epsilon'$. So get T_1 :

$$T_1 = \text{Tr} (\not{p}_5 \not{\epsilon}' \not{\epsilon} \not{k} \not{p}_i \not{k} \not{\epsilon} \not{\epsilon}') + m^2 \text{Tr} (\not{\epsilon}' \not{\epsilon} \not{k} \not{\epsilon} \not{\epsilon}')$$

$$= +2(k \cdot p_i) \text{Tr} (\not{p}_5 \not{\epsilon}' \not{\epsilon} \not{k} \not{\epsilon} \not{\epsilon}') = +2(k \cdot p_i) \text{Tr} (\not{p}_5 \not{\epsilon}' \not{k} \not{\epsilon}')$$

$$= 2(k \cdot p_i) \left\{ \cancel{\text{Tr} (\not{p}_5 \not{p}_i \not{k} \not{\epsilon}')} + 2(k \cdot p_i) \right\} + 2(k \cdot \epsilon') \left\{ \cancel{\text{Tr} (\not{p}_5 \not{\epsilon}') \not{k} \not{\epsilon}'} \right\}$$

$p_i \cdot k' \qquad (k \cdot \epsilon')$

$$= 8(k \cdot p_i)(k' \cdot p_i) + 16(k \cdot p_i)(k \cdot \epsilon')^2$$

$$T_1 = 8(k \cdot p_i) [(k' \cdot p_i) + 2(k \cdot \epsilon')^2] /$$

So at once $T_2 = 8(k'p_i) [(k'p_i) - 2(k'\epsilon)^2] \checkmark$

Now $T_3 = T_4$ by reversal so get T_3 . Use $p_4 = p_i + k - k'$

$$T_3 = T_1(p_i + m) \epsilon' \epsilon' k(p_i + m) k' \epsilon' \epsilon + T_1(k - k') \epsilon' \epsilon k p_i k' \epsilon' \epsilon$$

① run $\epsilon' \epsilon$ around to the end in first term.

② $k(p_i + m) = -(p_i + m)k + 2h \cdot p$

③ $(p_i + m)(-p_i + m) = 0$

The first term yields: $2(k'p_i) T_1(p k' \epsilon' \epsilon \epsilon' \epsilon)$

$$= 2(k'p_i) [-(4p \cdot k') + 2(\epsilon \cdot \epsilon') h(p k' \epsilon' \epsilon)]$$

$$\rightarrow [p \cdot k' \epsilon' \epsilon - \cancel{p \epsilon' k' \epsilon}] 4$$

$$= -8(p \cdot k)(p \cdot k') + 16(k'p_i)(\epsilon \cdot \epsilon') [(\epsilon \cdot \epsilon')(p \cdot k') - \cancel{(p \epsilon' k' \epsilon)}]$$

$$= -8(p \cdot k)(p \cdot k') + 16(p \cdot k)(p \cdot k')(\epsilon \cdot \epsilon')^2 - 16(p \cdot k) \cancel{(p \epsilon' k' \epsilon)} (\epsilon \cdot \epsilon') (\epsilon \cdot k')$$

$$= -8(p \cdot k)(p \cdot k') [1 - 2(\epsilon \cdot \epsilon')^2] - 16(p \cdot k) \cancel{(p \epsilon' k' \epsilon)} (\epsilon \cdot \epsilon') (\epsilon \cdot k')$$

The second term is quite tricky. We take 8's to 4's on next page.

reduce $T_n(\mathbf{R}-\mathbf{R}') \epsilon' \epsilon \mathbf{R} \rho \mathbf{R}' \epsilon' \epsilon$

$$= T_n [\mathbf{R} \rho \mathbf{R}' \epsilon' \epsilon (\mathbf{R}-\mathbf{R}') \epsilon' \epsilon]$$

use identity:

$$(\mathbf{R}-\mathbf{R}') \epsilon' \epsilon = \epsilon' \epsilon (\mathbf{R}-\mathbf{R}') + 2(\epsilon \cdot \mathbf{R}') \epsilon' + 2(\epsilon' \cdot \mathbf{R}) \epsilon$$

① ② ③

three terms: first is:

$$T_n(\mathbf{R} \rho \mathbf{R}' \epsilon' \epsilon \epsilon' \epsilon (\mathbf{R}-\mathbf{R}')) = -h(\mathbf{R} \rho \mathbf{R}' \epsilon' \epsilon \epsilon' \epsilon \mathbf{R}')$$

On this use: $\epsilon' \epsilon \epsilon' \epsilon = 2(\epsilon \cdot \epsilon') \epsilon' \epsilon - 1$

Two terms. first is $-2(\epsilon \cdot \epsilon') h(\mathbf{R} \rho \mathbf{R}' \epsilon' \epsilon \mathbf{R}')$
second is 0. ($\mathbf{R} \mathbf{R}'$). Now do this ↑

$$\begin{aligned} ① &= 2(\epsilon \cdot \epsilon') h(\mathbf{R} \rho \epsilon' \mathbf{R}' \epsilon' \mathbf{R}') = 2(\epsilon \cdot \epsilon') 2(\epsilon \cdot \mathbf{R}') h(\mathbf{R} \rho \epsilon' \mathbf{R}') \\ &= \underline{-4(\epsilon \cdot \epsilon')(\epsilon \cdot \mathbf{R}') h(\mathbf{R} \rho \mathbf{R}' \epsilon')} \end{aligned}$$

Second term is:

$$\begin{aligned} &2(\epsilon \cdot \mathbf{R}') h(\mathbf{R} \rho \mathbf{R}' \epsilon' \epsilon \epsilon') \\ &\quad = h(\mathbf{R} \rho \mathbf{R}' \epsilon) + 2\epsilon \cdot \epsilon' h(\mathbf{R} \rho \mathbf{R}' \epsilon') \\ &= 2(\epsilon \cdot \mathbf{R}') h(\mathbf{R} \rho \mathbf{R}' \epsilon) + \underline{4(\epsilon \cdot \epsilon')(\epsilon \cdot \mathbf{R}') h(\mathbf{R} \rho \mathbf{R}' \epsilon')} \end{aligned}$$

Third term is:

$$2(\epsilon' \cdot \mathbf{R}) h(\mathbf{R} \rho \mathbf{R}' \epsilon' \epsilon \epsilon) = -2(\epsilon' \cdot \mathbf{R}) h(\mathbf{R} \rho \mathbf{R}' \epsilon')$$

Add em all up: $= \boxed{2(\epsilon \cdot \mathbf{R}') h(\mathbf{R} \rho \mathbf{R}' \epsilon) - 2(\epsilon' \cdot \mathbf{R}) h(\mathbf{R} \rho \mathbf{R}' \epsilon')}$

Now reduce these:

$$p_i + k = p_f + k'$$

$$h(k \cdot p, k' \cdot p) = 4(k \cdot \epsilon')(p \cdot k')$$

$$h(k' \cdot p, k \cdot p) = 4(k' \cdot \epsilon)(p \cdot k)$$

So "second term" is:

$$8(\epsilon \cdot k')(\epsilon' \cdot k)(p \cdot k) - 8(\epsilon' \cdot k)(\epsilon \cdot k')(p \cdot k')$$

$$= 8(\epsilon \cdot k')(\epsilon' \cdot k) p_i (k'_i - k_i)$$

$$p_i (k'_i - k_i) = p_i (p_i - p_{fi}) \\ = m^2 - (p_i \cdot p_{fi})$$

$$= \frac{8m^2(\epsilon \cdot k')(\epsilon' \cdot k)}{8p_i}$$

$$= 8(\epsilon \cdot k')(\epsilon' \cdot k) [m^2 - (p_i \cdot p_{fi})]$$

So total result is, for T_3 :

$$T_3 = 8(p_i \cdot k)(p_i \cdot k') [2(\epsilon \cdot \epsilon')^2 - 1] + 8(\epsilon \cdot k')^2 (p \cdot k) - 8(\epsilon' \cdot k)^2 (p \cdot k')$$

Some now want $T_1 + T_2 + 2T_3 = \text{total}$:

$$\begin{aligned} & 8(k \cdot p)(k' \cdot p) + 16(k \cdot p)(k' \cdot \epsilon')^2 \\ & + 8(k' \cdot p)(k \cdot p) - 16(k \cdot p)(k' \cdot \epsilon')^2 \\ & + 16(k \cdot p)(k' \cdot p) [2(\epsilon \cdot \epsilon')^2 - 1] + 16(\epsilon \cdot k')^2 (p \cdot k) - 16(\epsilon' \cdot k)^2 (p \cdot k) \\ & = 16(p \cdot k)(p \cdot k') [2(\epsilon \cdot \epsilon')^2] \end{aligned}$$

We now seek: $\frac{T_1}{\omega^2} + \frac{T_2}{\omega'^2} + \frac{2T_3}{\omega \omega'}$

$$\begin{aligned}
&= \frac{8}{\omega^2} (p \cdot k)(p \cdot k') + \frac{16}{\omega^2} (p \cdot k)(k \cdot \epsilon')^2 \\
&+ \frac{8}{\omega'^2} (p \cdot k)(p \cdot k') - \frac{16}{\omega'^2} (p \cdot k')(k' \cdot \epsilon)^2 \\
&+ \frac{16(p \cdot k)(p \cdot k')}{\omega \omega'} [2(\epsilon \cdot \epsilon')^2 - 1] + \frac{16}{\omega \omega'} (\epsilon \cdot k')^2 (p \cdot k) - \frac{16(\epsilon' \cdot k)^2 p \cdot k'}{\omega \omega'}
\end{aligned}$$

Now $p = m$ etc:

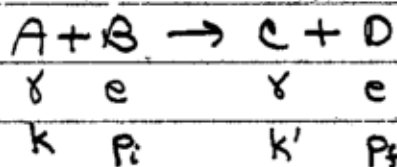
$$\begin{aligned}
&= m^2 8 \frac{\omega'}{\omega} + \frac{16}{\omega} (k \cdot \epsilon')^2 m \\
&+ m^2 8 \frac{\omega}{\omega'} - \frac{16}{\omega'} (k' \cdot \epsilon)^2 m \\
&+ 16 m^2 [2(\epsilon \cdot \epsilon')^2 - 1] + \frac{16}{\omega} m (\epsilon \cdot k')^2 - \frac{16}{\omega'} m (\epsilon' \cdot k)^2 \\
&= 8 m^2 \left(\frac{\omega'}{\omega} + \frac{\omega}{\omega'} \right) + 16 m^2 [2(\epsilon \cdot \epsilon')^2 - 1] \\
&+ 16 (k \cdot \epsilon')^2 m \left(\cancel{\frac{1}{\omega}} - \cancel{\frac{1}{\omega'}} \right) + 16 (k' \cdot \epsilon)^2 m \left(\cancel{\frac{1}{\omega'}} - \cancel{\frac{1}{\omega}} \right) \\
&= \boxed{8 m^2 \left[\frac{\omega'}{\omega} + \frac{\omega}{\omega'} + 4(\epsilon \cdot \epsilon')^2 - 2 \right]} \\
&+ \cancel{16 m \left(\frac{\omega' - \omega}{\omega \omega'} \right) [(k \cdot \epsilon')^2 + (k' \cdot \epsilon)^2]}
\end{aligned}$$

So at last,

$$\sim \text{Tr} \left(\frac{A_1 + m}{2m} \right) \dots =$$

$$\begin{aligned}
&= \frac{1}{2} \text{Tr} \left(\frac{A_1 + m}{2m} \right) \dots \\
&= \frac{1}{2} \text{Tr} \left(\frac{A_1 + m}{2m} \right) \dots
\end{aligned}$$

Now let's go for cross section! Reaction was



$$\begin{array}{ll}
 p_A = k & p_C = k' \\
 p_B = p_i & p_D = p_f
 \end{array}$$

We will integrate over electron p_D & quote my own general all frame result: (see [NII])

$$d\sigma = \frac{N |M|^2}{(8\pi)^2} \frac{k' d\Omega_{k'}}{m k \left(E_f - k' \left(\frac{p_f}{k'} \right) \hat{k}' \cdot \hat{p}_f \right)}$$

However, this thing simplifies since $E_C = p_i$:

$$\begin{aligned}
 E_f - \frac{1}{k'} \vec{k}' \cdot \vec{p}_f &= \frac{1}{k'} (k' E_f - \vec{k}' \cdot \vec{p}_f) = \frac{1}{k'} (\vec{p}_f \cdot \vec{k}') \\
 &= \frac{1}{k'} (p_i \cdot k) \\
 &= m \frac{k}{k'}
 \end{aligned}$$

$$\begin{aligned}
 p_i - k' &= p_f - k \\
 p_i + k &= p_f + k'
 \end{aligned}$$

Thus, in our case,

$$d\sigma = \frac{N |M|^2}{(8\pi)^2} \frac{1}{m^2} \left(\frac{k'}{k} \right)^2$$

$$N = 4m^2$$

$$\alpha = \frac{e^2}{4\pi}$$

$$\boxed{\frac{m^2}{m^2} \frac{k^2}{k^2}}$$

And we found that $|M|^2 = \frac{e^4}{4m^2} []$, thus:

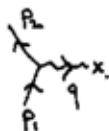
$$\boxed{\frac{e^4}{4m^2} []}$$

At last, I have gone completely from M to $d\sigma$.

Coulomb Scatter (Mott, Rulhufnd, $2\pi\delta(\dots)$ phase space, etc.)

This is to practice using the Feynman rule.

One graph:



$$q = p_1 - p_3 = \text{mom. transferred into Coulomb.}$$

$$M_f = \bar{u}(2) [-ie\gamma^\mu] u(1) \cdot (-2e) g_{\mu 0} \cdot \frac{2M_p}{c^6} \cdot \frac{1}{|q|^2}$$

Notice that we are elastic so $\nu = q^0 = 0$. So, using $e_h^2 = 4\pi\alpha$, we have:

$$M_f = i \cdot 4\pi\alpha \cdot 2 \cdot \frac{2M_p}{c^6} \cdot \bar{u}(2) \gamma^0 u(1) \cdot \frac{1}{|q|^2}$$

$|M|^2$

Meanwhile, $|M_f|^2 = (4\pi\alpha)^2 2^2 \left(\frac{2M_p}{c^6}\right)^2 \frac{1}{|q|^4} \cdot \text{tr}(\Lambda_1 \gamma^0 \Lambda_2 \gamma^0)$

What happens to phase space? The proton is really present, so if we go to a daggar room (say, Eden), then

$\frac{d\sigma}{d\Omega}$ stop x. don't want cms.

$$\text{Nelly, } d\sigma = \frac{1}{4 \cdot \text{flux}} |T_{fi}|^2 \cdot (2\pi)^4 \delta^4(\text{ext}) \cdot (2\pi)^6 d^4 p_2 \delta^4(1) d^4 p_3 \delta^4(\dots)$$

Now go to lab where $\text{flux} = M_p \cdot p_1$

$$\text{So: } d\sigma = \frac{1}{4 M_p p_1} |T_{fi}|^2 \cdot \frac{1}{(2\pi)^2} \cdot \delta^4(\text{ext}) d^4 p_2 d^4 p_3 \delta^4(\dots) \delta^4(\dots)$$

Now, for heavy proton, lab $\approx$ cms, but we then need a final recoil momentum. Maybe set $p_2 = -p_1$, so

$$\omega = \frac{p_1}{4 M_p} d\Omega$$

$$\text{Then we've got: } d\sigma = \frac{1}{4 M_p p_1} |T_{fi}|^2 \cdot \frac{1}{(2\pi)^2} \cdot \frac{p_1}{4 M_p} \cdot d\Omega$$

$$\frac{d\sigma}{d\Omega} = \frac{|T_{fi}|^2}{64\pi^2 M_p^2}$$

/could have read this off the chart.

Meanwhile, $|q|^2 = |p_1 - p_3|^2 = p_1^2 + p_3^2 - 2p_1 p_3 \cos\theta = 2p^2 - 2p^2 \cos\theta = 2p^2(1 - \cos\theta) = 4p^2 \sin^2 \theta/2$

Now Eden uses $C_6 = 1$, so I have:

$$|T|^2 = \frac{16\pi^2 \alpha^2 Z^2 \cdot 4m_p^2}{(4p_i^2 \sin^2 \theta/2)} h(1, \delta^0, 1, \delta^0)$$

$$= \frac{4\pi^2 \alpha^2 Z^2 m_p^2}{p_i^4 \sin^4 \theta/2} h(1, \delta^0, 1, \delta^0) \leftarrow |T_{Eden}|^2$$

Then: $\frac{d\sigma}{d\Omega} = \frac{4\pi^2 \alpha^2 Z^2 m_p^2}{16 p_i^4 \sin^4 \theta/2} h(1, \delta^0, 1, \delta^0) = \frac{(\alpha Z)^2}{16 p_i^4 \sin^4 \theta/2} h(1, \delta^0, 1, \delta^0)$

Now, last thing, we can do the trace.

$$\frac{1}{2} h((\vec{p}_1 + m)\delta^0 (\vec{p}_2 + m)\delta^0) = \frac{1}{2} \underbrace{h(\vec{p}_1 \delta^0 \vec{p}_2 \delta^0)}_A + \frac{1}{2} m^2 \underbrace{h(\delta^0 \delta^0)}_B$$

Well, $A = 4(E_1 E_2 + E_1 E_2 - \vec{p}_1 \cdot \vec{p}_2 g^{00})$
 $= 4(2E_1 E_2 - [E_1 E_2 - p_1 p_2 \cos \theta]) = 4(E_1 E_2 + p_1 p_2 \cos \theta)$

$B = h(\delta^0 \delta^0) = h(1) = 4$, so

$$\rightarrow \frac{1}{2} \cdot 4(E_1 E_2 + p_1 p_2 \cos \theta) + \frac{1}{2} m^2 \cdot 4 = 2E_1 E_2 + 2p_1 p_2 \cos \theta + 2m^2$$

$$= 2(m^2 + E_1 E_2 + p_1 p_2 \cos \theta)$$

But elastic means $= 2(m^2 + E^2 + p^2 \cos \theta) = 2(m^2 + m^2 + p^2(1 + \cos \theta))$
 $= 2(2m^2 + p^2 2 \cos^2 \theta/2)$
 $= 4(m^2 + p^2 \cos^2 \theta/2)$

I seem to get: $\frac{d\sigma}{d\Omega} = \frac{(\alpha Z)^2 (m^2 + p^2 \cos^2 \theta/2)}{4 p^4 \sin^4 \theta/2}$

But $m^2(1 + (\frac{p}{m})^2 \cos^2 \frac{\theta}{2}) = m^2(1 + \beta^2 \cos^2 \frac{\theta}{2})$

$$\frac{m}{p} = \frac{1}{\beta}$$

So: $\frac{d\sigma}{d\Omega} = \frac{(\alpha Z)^2 (m^2 + p^2 \cos^2 \theta/2)}{4 p^4 \sin^4 \theta/2}$

$$\beta = p/E$$

Rewrite: $(m^2 + p^2 \cos^2 \theta/2) = (E^2 - p^2 + p^2(1 - \sin^2 \theta/2)) = (E^2 - p^2 \sin^2 \theta/2)$
 $= E^2(1 - \beta^2 \sin^2 \theta/2)$

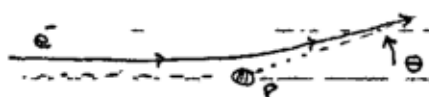
Thus,
$$\frac{d\sigma}{d\Omega} = \frac{(\alpha Z)^2 \cdot E^2 (1 - \beta^2 \sin^2 \theta/2)}{4 p^4 \sin^4(\theta/2)} = \frac{(\alpha Z)^2 \cdot (1 - \beta^2 \sin^2 \theta/2)}{4 \beta^4 p^2 \sin^4(\theta/2)}$$

This is the correct Mott result. At very low energy, $\beta \ll 1$, we get

$$\frac{d\sigma}{d\Omega} = \frac{(\alpha Z)^2 \cdot m_e^2}{4 p^4 \sin^4(\theta/2)} \quad \text{Rutherford} \quad p = \text{lab.}$$

Notice that this blows up at small angle or small momentum. The total cross section is infinite.

Q: why aren't p-p total cross sections infinite on account of this effect? Better, e-p, since that is our case. Maybe they are! But



I guess the answer is that at any reasonable angle, this effect is very small compared to a hadronic event. This question requires more work:

- 1) how large is the Mott in mb? ($d\sigma \sim 20 \text{ mb} \times d\Omega$)
- 2) how does this compare to a typical hadronic?
- 3) what happens to the optical theorem when you include this long range force?

still
alive till
checked!

Now, is there a more conventional way to deal with the phase space?

$$(2\pi)^4 \cdot \frac{d^3 p_2}{(2\pi)^3 2E} \cdot \frac{d^3 p_2}{(2\pi)^3 2E_2} \cdot \delta^4(\text{ext}) = \frac{d^3 p_2}{(2\pi)^3 2E_2} \cdot \frac{\delta(E_i - E_f)}{(2\pi)^3 2M_P} \cdot (2\pi)^4$$

$$= \frac{d^3 p_2}{(2\pi)^3 2E_2} \cdot \frac{1}{2M_P} \cdot 2\pi \delta(E_i - E_f)$$

← This becomes the rest of your phase space.
← This is what you see.

Repeat this in all-norm: (well, use daggers only):

$$dQ_f = (2\pi)^4 \delta^4(\text{ext}) \cdot \left(\frac{\delta_A d^3 p_A}{2E_A} \right) \cdot \prod_{\text{rest}} \frac{\delta_f d^3 p_f}{2E_f} = \frac{(2\pi)^4 \delta_A}{2M_A} \cdot \delta(\text{energy}) \cdot \prod_{\text{rest}} \left(\frac{\delta_f d^3 p_f}{2E_f} \right)$$

↑
the x-

phasespace to use
with $\sim x$

-4-

2

Repeat:
$$\left[dQ_f = \left[\frac{(2\pi)^3 \delta_A}{2M_A} \right] \cdot 2\pi \delta(E_f - E_i) \cdot \prod_{\text{out}} \left(\frac{4 d^3 p_f}{2E_f} \right) \right]$$

In BD, $\delta_A = \frac{C_6}{(2\pi)^3} = \frac{2M_A}{(2\pi)^3}$, so $[] = 1$. Notice that no energy is transferred to fat particle A.

check.

Now repeat that to check this: (Eden norm):

$$\begin{aligned} d\sigma &= \frac{1}{4 \cdot M_p \cdot p} |T_{fi}|^2 dQ_f \\ &= \frac{1}{4 \cdot M_p \cdot p} \cdot \frac{1}{2M_p} \cdot 2\pi \delta(E_f - E_i) \cdot \frac{(2\pi)^3 d^3 p}{2E} \cdot \frac{4 \cdot \pi^2 (\alpha Z)^2 M_p^2}{p^4 \sin^4 \theta/2} \cdot h(1, \delta^0, 1, \delta^0) \\ &= \frac{h \delta(E_f - E_i) d^3 p \cdot \pi^2 (\alpha Z)^2}{p \cdot 2E \cdot 8\pi^3 \cdot p^4 \sin^4 \theta/2} \text{ here.} \end{aligned}$$

Now $d^3 p = p^2 dp d\Omega = p E dE d\Omega$ so

$$d\sigma = \frac{p E d\Omega (\alpha Z)^2}{16 p E p^4 \sin^4 \theta/2} \cdot h(\dots) \Rightarrow d\sigma = \frac{(\alpha Z)^2 d\Omega}{16 p^4 \sin^4(\theta/2)} \cdot h(1, \delta^0, 1, \delta^0).$$

Now replace:

$$\begin{aligned} h(\dots) &\rightarrow \frac{1}{2} \cdot h(\dots) = 2(m^2 + E^2 + p^2 \cos \Theta) \\ &= 4(m^2 + p^2 \cos^2 \theta/2) \\ &= 4E^2(1 - \beta^2 \sin^2 \theta/2) \end{aligned}$$

So: $h(1, \delta^0, 1, \delta^0) \rightarrow 4E^2(1 - \beta^2 \sin^2 \frac{\theta}{2})$ on spin sum.

Then:
$$\frac{d\sigma}{d\Omega} = \frac{(\alpha Z)^2 (1 - \beta^2 \sin^2 \theta/2)}{4 \beta^2 p^2 \sin^4(\theta/2)} \quad \checkmark$$

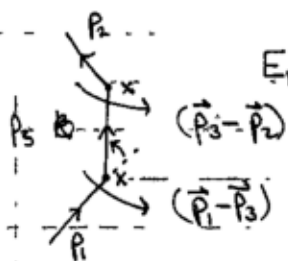
"double
Brems
scatter"

? For double scattering off the same proton: my guess is:

- use same phase space
- use $-2e \cdot \vec{q}_0 / |q|^2$ at each vertex.
- use an electron propagator.



So, for double Coulomb scatter:



$$E_1 = E_2 = E_3 = E$$

$$M_f = \bar{u}(2) [-ie\gamma^\mu] \frac{+i}{\not{p}_3 - m + i\epsilon} [-ie\gamma^\nu] u(1) \times \frac{-(Ze)\delta_{\mu 0}}{|\vec{p}_2 - \vec{p}_1|^2} \times \frac{-(Ze)\delta_{\nu 0}}{|\vec{p}_3 - \vec{p}_2|^2}$$

$$= \frac{(-i)^2 Z^2 e^4}{|\vec{p}_2 - \vec{p}_1|^2 |\vec{p}_3 - \vec{p}_2|^2} \bar{u}(2) \gamma^0 (\not{p}_3 + m) \gamma^0 u(1) (p_3 p_2 - m_e^2)$$



diverges!

$$p_1^\mu = (E, 0, 0, p) \\ p_2^\mu = (E, p \cos \theta_1, 0, p \sin \theta_1) \\ p_3^\mu = (E, p \cos \theta_2, 0, p \sin \theta_2)$$

Why are there no corrections to Rutherford, yet these higher terms seem to diverge? See T+R for discussion. The problem was solved by Dalitz, 1951.

Size: Recall that $\alpha = e^2 = 10^{-2} m_e$, so $\alpha^2 = 10^{-4} m_e^2$, and:

$$\left. \frac{d\sigma}{d\Omega} \right|_{\text{Mott}} = \frac{10^{-4} m_e^2 Z^2 E^2}{4 p^4 \sin^4(\theta/2)}$$

$$\frac{m_e^2 E^2}{p^4} = \frac{m_e^2 E^2}{\beta^4 E^4} = \frac{m_e^2}{\beta^4 E^2}$$

Use $\beta = \frac{p}{E}$ $\gamma = \frac{E}{m_e}$ $\gamma\beta = p/m_e =$

$$= \frac{1}{\beta^4 \gamma^2}$$

So:

$$\left. \begin{array}{l} \text{Mott.} \\ \text{Ruth.} \end{array} \right\} \left\{ \begin{array}{l} \left. \frac{d\sigma}{d\Omega} \right|_{\text{Mott}} = \frac{r_0^2 Z^2}{4 \sin^4(\theta/2)} \cdot \frac{1}{\beta^4 \gamma^2} \cdot (1 - \beta^2 \sin^2 \frac{\theta}{2}) \\ \left. \frac{d\sigma}{d\Omega} \right|_{\text{Ruth}} = \frac{r_0^2 Z^2}{4 \beta^4 \sin^4(\theta/2)} \end{array} \right. \quad \left. \begin{array}{l} \text{Rutherford is obvious} \\ \frac{r_0^2}{4} = 20 \text{ mb} \end{array} \right\}$$

How to Apply the Golden Rule to Coulomb Scatter

① What density of states should you use with F.G.R.?

$$\frac{d^3p \cdot V}{(2\pi)^3} \quad \text{comes to mind}$$

$$\text{So: } dW = 2\pi \delta(E_2 - E_1) \frac{|\langle p_2 | H' | p_1 \rangle|^2}{\langle p_1 | p_1 \rangle \langle p_2 | p_2 \rangle} \cdot \frac{d^3p_2 \cdot V}{(2\pi)^3} \quad \checkmark$$

To avoid $\delta^3(0)$ stuff... well, write this as:

$$dW = 2\pi \delta(E_2 - E_1) \frac{|\langle p_2 | H' | p_1 \rangle|^2}{2E_1 C_1 \delta^3(0) \cdot 2E_2 C_2 \delta^3(0)} \times d^3p_2 \cdot \delta^3(0)$$

These states like $|p_i\rangle$ are the $b^\dagger(p_i)|0\rangle$ of field theory. We shall use $\langle p_1 | p_2 \rangle = 2E_1 C_1 \delta^3(p_1 - p_2)$ as our state norm. H' is fixed in scales, namely:

$$H' = \int d^3x : e \bar{\psi}_\mu(x) A^\mu(x) :$$

On another sheet, in a dagger norm, we know that using a classical Coulomb potential, we get

$$\langle p_2 | H' | p_1 \rangle = e^2 C_1 C_2 \bar{u}(2) \gamma_0 u(1) \cdot 2\pi \delta(E_1 - E_2) \cdot \frac{1}{|q|^2} \quad \times$$

Thus,

$$|\langle p_2 | H' | p_1 \rangle|^2 = e^4 C_1^2 C_2^2 [\bar{u}(2) \gamma_0 u(1)]^2 \cdot \frac{1}{|q|^4} \cdot 2\pi \delta(E_1 - E_2) \cdot 2\pi \delta(0) \quad \times$$

When this goes into our rate formula, we get extra $\delta(E_1 - E_2)$ floating around. Ah yes, I guessed:

$$H' = \int d^3x : e \bar{\psi}_\mu(x) A^\mu(x) :$$

This "undoes" the $2\pi \delta(E_1 - E_2)$, so we have:

$$\langle p_2 | H' | p_1 \rangle = e^2 C_1 C_2 \bar{u}(2) \gamma_0 u(1) \cdot \frac{1}{|q|^2}$$

then:

$$dW_1 = 2\pi \delta(E_2 - E_1) \frac{[e^4 C_2^2 C_4^2 |\bar{u}(2) \gamma_0 u(1)|^2 \cdot \frac{1}{|q|^4}]}{2E_2 C_2 \delta^3(0) \cdot 2E_1 C_2 \delta^3(0)} \cdot d^3 p_2 \cdot \delta^3(0)$$

This should be the "rate with one of everything". Of course, rate is volume dependent ~ a small box $\Rightarrow$ hi densities $\Rightarrow$ fast rate

② The corresponding entity in the 4-particle theory is obtainable from WP, C, B - p24.

$$dW_2 = \frac{\text{Prob}}{2\pi \delta(0)} = \frac{(2\pi)^4 \cdot \delta^4(\text{ext}) \prod_{\text{all}} C_i \cdot \frac{d^3 p_2 d^3 p_4}{2E_2 C_2 2E_4 C_2} \cdot (2\pi)^3 \delta^3(0) \cdot |T_{fi}|^2}{2E_1 C_2 \delta^3(0) \cdot 2E_3 C_2 \delta^3(0)}$$

Recall that $C_3 = (2\pi)^4$ in a F.T. system. Now, I have shown elsewhere that in the soft limit:

$$|T_{fi}| = |M_{fi}| = e^2 \cdot \frac{2m_p}{C_6} \cdot |\bar{u}(2) \gamma_0 u(1)| \cdot \frac{1}{|q|^2}$$

③ No I feel that my two dW 's should be equal. Do:

$$\delta^4(\text{ext}) \cdot d^3 p_4 = \delta(\text{en})$$

then here is what we want to show:

$$\begin{aligned} & 2\pi \delta(\text{ext}) \cdot e^4 \cdot C_1^2 C_2^2 \cdot \frac{1}{|q|^4} \cdot d^3 p_2 \cdot \delta^3(0) \\ &= 2\pi \delta(\text{ext}) \cdot (2\pi)^6 \cdot \delta^3(0) \cdot C_1^2 C_2^2 \cdot e^4 \cdot \left(\frac{2m_p}{C_6}\right)^2 \cdot \frac{1}{|q|^4} \cdot \frac{1}{2E_3 C_2 \cdot 2E_4 C_2} \cdot d^3 p_2 \end{aligned}$$

This boils down to:

$$1 = (2\pi)^6 \cdot C_1^2 \cdot C_2^2 \cdot \left[\frac{2m_p}{C_6}\right]^2 \cdot \frac{1}{2E_3 C_2} \cdot \frac{1}{2E_4 C_2}$$

But we have agreed that

$$C_4^2 = \frac{C_2 C_6}{(2\pi)^3} \text{ so } 1 = (2\pi)^6 \cdot \frac{C_2 C_6}{(2\pi)^3} \cdot \frac{C_2 C_6}{(2\pi)^3} \cdot \frac{4m_p^2}{C_6^2} \cdot \frac{1}{2m_p C_2} \cdot \frac{1}{2m_p C_2} = 1 \checkmark \text{ beautiful!}$$

So, I have now established that the Golden Rule is a clean way to handle external field problems. It is of course still perturbation theory + thus corresponds only to a lowest-order field theory calculation. However, we know that lowest-order pert theory with an external potential somehow "is all there is" in the soft limit. It is inconsistent to think of multiple scatterings.

① The relation between dW and cross section in "box" is:

$$dW = \frac{\text{Prob.}}{2\pi\delta(0)} = \frac{d\sigma \cdot V_B \cdot (2\pi)^3}{\delta^3(0)} = \frac{d\sigma \cdot V_B}{V} = d\sigma \cdot \text{flux}$$

Here, $\text{flux} = V_B \cdot g = V_B \cdot 1/V$

So, we record here the Golden rules:

$$\begin{cases} dW = \frac{2\pi\delta(E_2-E_1) |\langle P_2 | H' | P_1 \rangle|^2}{2E_1 C_1 \delta^3(0) \cdot 2E_2 C_2 \delta^3(0)} \cdot d^3 p_2 \delta^3(0) \\ d\sigma = (2\pi)^3 \delta^3(0) \cdot \frac{dW}{V_B} \\ = (2\pi)^3 \cdot \frac{2\pi\delta(E_2-E_1) |\langle P_2 | H' | P_1 \rangle|^2}{V_B \cdot 2E_1 C_1 \cdot 2E_2 C_2} \cdot d^3 p_2 \end{cases}$$

the actual interaction Hamiltonian, eg, $e \cdot \vec{p}_\mu \cdot \vec{A}^\mu$

Note of course that $d\sigma$ is volume independent. $\langle A P \rangle = \text{usual}$, $H' = \delta^3 x \cdot \mathcal{H}$, standard.

[Comment: The Golden rule exists in a first quantized (single particle) theoretical framework only. You cannot expect it to do a general process.]

Note: in any dagger norm,

$$\rightarrow \langle P_2 | H' | P_1 \rangle = C_1 \cdot C_2 \cdot M_f$$

Eq, in Coulomb case, $M_f = \bar{u}(2) [-ie\gamma_\mu] u(1) \cdot \frac{-iq_{\mu 0} e}{|q|^2}$

ie, just in this situation we have a "rule":

$$x = \frac{-iq_{\mu 0} e}{|q|^2}$$

Bhabha Scattering (as Calibrator of $\mathcal{L}$)

① When $E \gg m_e$, we get this spin-averaged result (7.87 of BD):

$$\frac{d\bar{\sigma}}{d\Omega} = \frac{\alpha^2}{8E^2} f(\theta) = \frac{\alpha^2}{8s} f(\theta)$$

$$f_H(\theta) = \left[\frac{1 + C_V^4}{S_V^4} + \frac{2}{S_V^2 C_V^2} + \frac{1 + S_V^4}{C_V^4} \right] \quad \text{Møller} \quad \alpha \equiv \theta/2$$

$$f_B(\theta) = \left[\frac{1 + C_V^4}{S_V^4} - \frac{2C_V^4}{S_V^2} + \frac{1 + C_V^4}{2} \right] \quad \text{Bhabha}$$

② For very small angles,

$$f_B(\theta \rightarrow 0) \approx \left[\frac{1}{\alpha^4} - \frac{2}{\alpha^2} + \frac{1}{2} \right]$$

If we integrate over small angles, $\int d\Omega$ contains $\int \sin\theta d\theta \approx \int d\theta$
 $\sim \int d\alpha$. First two terms diverge, so $\sigma_{\text{tot}}^{\text{BH}} = \infty$.

③ What makes this thing infinite? It's the exchange graph being active up to such a long range, same as in Coulomb scatt. How do the exper's deal with this infinity? Maybe they just look at 90° or something...? Or maybe they subtract out this QED stuff?

$$\textcircled{4} \text{ At } 90^\circ, \quad f_B(90^\circ) = \left[\frac{1 + \cos^4 45^\circ}{\sin^4 45^\circ} - \frac{2 \cos^4 45^\circ}{\sin^2 45^\circ} + \frac{1 + \cos^2 90^\circ}{2} \right]$$

$$\cos 45^\circ = \sin 45^\circ = 1/\sqrt{2}, \text{ so}$$

$$f_B(90^\circ) = \left[\frac{1 + \frac{1}{4}}{\frac{1}{4}} - \frac{2 \cdot \frac{1}{4}}{\frac{1}{2}} + \frac{1 + 0}{2} \right] = \left[5 - 1 + \frac{1}{2} \right] = \frac{9}{2}$$

$$\text{Then: } \frac{d\bar{\sigma}}{d\Omega}(90^\circ) = \frac{9}{2} \cdot \frac{\alpha^2}{8s}$$

At 3.105 GeV, this is:

$$\left[\frac{9}{2} \cdot \left(\frac{1}{137} \right)^2 \cdot \frac{1}{8} \cdot \frac{1}{(3.105 \text{ GeV})^2} \times 389 \times 10^6 \right] \text{ nb} = [1.21 \text{ nb}]$$

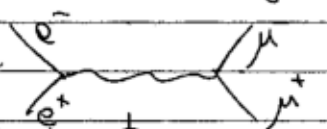
$$\text{So, } d\bar{\sigma} = 1.21 d\phi \sin\theta d\theta \Rightarrow$$

$$\boxed{\left. \frac{d\bar{\sigma}}{d\theta} \right|_{90^\circ}^{\text{Bhabha}} = 7.60 \text{ nb/radian}}$$

⑤ I conclude that there is no such thing as $\sigma_{\text{tot}}^{B_h}$. They measure $\sigma = \int_0^{\theta_0} + \text{compare to QEP to calibrate their luminosity}$.

⑥ However, we can get a σ_{tot} for $e^+e^- \rightarrow \mu^+\mu^-$ since here there is no exchange graph.

WIDTH of the ϕ



$$\frac{d\sigma}{d(\cos\theta)} = \frac{\pi\alpha^2}{8s} \left(1 + \cos^2\theta + \frac{m_\mu^2}{E^2} \sin^2\theta \right) \beta_\mu = \sigma_0$$

This is from p61 of 1971 Int Symp on Electron + Photon Int, QC721, I596 in the room. Thus:

$$\begin{aligned} \sigma_{\text{tot}} &= \int_{-1}^1 dz \left[\frac{\pi\alpha^2}{8s} \left(1 + z^2 + \frac{m_\mu^2}{s} (1 - z^2) \right) \beta_\mu \right] \\ &= \frac{\pi\alpha^2}{8s} \beta_\mu \left[\int_{-1}^1 dz \cdot 1 + \int_{-1}^1 dz z^2 + \frac{m_\mu^2}{s} \int_{-1}^1 dz (1 - z^2) \right] \\ &= \frac{\pi\alpha^2 \beta_\mu}{8s} \left[2 + 2 \cdot \frac{1}{3} + \frac{m_\mu^2}{s} \left(2 - \frac{2}{3} \right) \right] \end{aligned}$$

$$\sigma_{\text{tot}} = \frac{\pi\alpha^2 \beta_\mu}{8s} \left[\frac{8}{3} + \frac{4}{3} \frac{m_\mu^2}{s} \right]$$

At $\sqrt{s} = 3.1 \text{ GeV} = 3105 \text{ MeV}$, we have $\frac{m_\mu}{\sqrt{s}} = \text{small}$, $\beta_\mu \approx 1$, so

$$\begin{aligned} \sigma_{\text{tot}} &\approx \frac{\pi\alpha^2}{8s} \left[\frac{8}{3} \right] = \frac{\pi}{3} \cdot \frac{\alpha^2}{s} \approx \left(\frac{4}{137} \right)^2 \cdot \frac{1}{(3.1)^2} \cdot (1.389) \times 10^6 \\ &\approx \frac{1000}{137} \cdot \frac{1000}{137} \cdot \frac{1.389}{9} \approx 7.7 \cdot \frac{4}{90} \approx \frac{50.4}{90} \approx \frac{45}{9} \approx \underline{\underline{5 \text{ nb}}} \end{aligned}$$

⑦ For interference with a resonance (at that time it was the ϕ ,

which has $\sigma = \frac{1}{(1+\cos\theta)}$

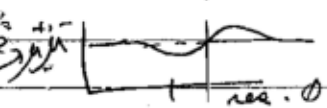
$$\sigma = \sigma_0 \left[1 + \frac{8\sqrt{B_1 B_2} m\Gamma}{\alpha + 4E^2 - m^2 + i m\Gamma} \right]^2$$

where $B_{1,2}$ are dimensionless branching ratios for $\phi \rightarrow e^+e^-$, $\phi \rightarrow \mu^+\mu^-$. Here, $m = m_\phi$. Γ also has mass dim.

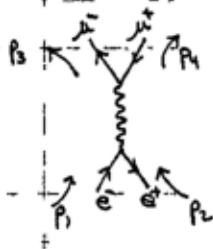
This could cause dips.

By fitting to σ_{obs} , they deduce $\sqrt{B_1 B_2} = 2.6 \times 10^{-4}$. Used

$\Gamma = 4.0 \text{ MeV}$; Cross section $\frac{d\sigma}{d\cos\theta}$



Calculation Hit Parade



$$T_{fi} = \bar{u}(3) [-i\sqrt{4\pi\alpha} \gamma_\mu] u(4) \left[\frac{-i}{s} \right] \bar{v}(2) [-i\sqrt{4\pi\alpha} \gamma^\mu] v(1)$$

$$|T_{fi}|^2 = \frac{(4\pi\alpha)^2}{4s^2} \underbrace{\text{tr}([p_3 + m_\mu] \gamma_\mu [p_4 - m_\mu] \gamma_\nu)}_{A_{\mu\nu}} \underbrace{\text{tr}([p_2 - m_e] \gamma^\mu [p_1 + m_e] \gamma^\nu)}_{B^{\mu\nu}}$$

$$\begin{aligned} \textcircled{2} \quad A_{\mu\nu} &= \text{tr}(p_3 \gamma_\mu p_4 \gamma_\nu) - m_\mu^2 \text{tr}(\gamma_\mu \gamma_\nu) \\ &= 4 [-p_{3\mu} p_{4\nu} + p_{3\nu} p_{4\mu} - g_{\mu\nu} (p_3 \cdot p_4 + m_\mu^2)] \end{aligned}$$

But $p_3 \cdot p_4 = \frac{1}{2}(s - m_\mu^2 - m_\mu^2) = \frac{1}{2}s - m_\mu^2$, so

$$A_{\mu\nu} = 4 [p_{3\mu} p_{4\nu} + p_{3\nu} p_{4\mu} - \frac{1}{2}s g_{\mu\nu}]$$

$$B^{\mu\nu} = 4 [p_1^\mu p_2^\nu + p_1^\nu p_2^\mu - \frac{1}{2}s g^{\mu\nu}]$$

$$\begin{aligned} \textcircled{3} \quad \text{So } A_{\mu\nu} B^{\mu\nu} &= 16 [2 p_{3\mu} p_{4\nu} - \frac{1}{2}s g_{\mu\nu}] [p_1^\mu p_2^\nu + p_1^\nu p_2^\mu - \frac{1}{2}s g^{\mu\nu}] \\ &= 16 [2((p_3 \cdot p_1)(p_4 \cdot p_2) + (p_3 \cdot p_2)(p_4 \cdot p_1)) - s((p_3 \cdot p_4) + (p_1 \cdot p_2)) + s^2] \end{aligned}$$

$$\textcircled{4} \quad \text{But, } \begin{aligned} (p_1 \cdot p_3) &= (p_2 \cdot p_4) = E_1 E_3 - p_1 p_3 \cos \theta_{13} & \hat{p}_1 \cdot \hat{p}_3 &= \cos \theta_{13} \\ (p_1 \cdot p_4) &= (p_2 \cdot p_3) = E_1 E_3 + p_1 p_3 \cos \theta_{13} \end{aligned}$$

$$(p_1 \cdot p_2) = \frac{1}{2}(s - 2m_e^2) \quad (p_3 \cdot p_4) = \frac{1}{2}(s - 2m_\mu^2)$$

Thus:

$$A_{\mu\nu} B^{\mu\nu} = 16 \left\{ 4(E_1^2 E_3^2 + p_1^2 p_3^2 \cos^2 \theta) + s(m_\mu^2 + m_e^2) \right\}$$

$$\textcircled{5} \quad |T_{fi}|^2 = \frac{(4\pi\alpha)^2}{4s^2} A_{\mu\nu} B^{\mu\nu}$$

$$= \frac{16\pi^2 \alpha^2}{4s^2} \cdot 16 \left\{ 4E_1^2 E_3^2 + s(m_\mu^2 + m_e^2) + 4p_1^2 p_3^2 \cos^2 \theta \right\}$$

$$= 64\pi^2 \alpha^2 s^2 \{ a + b \cos^2 \theta \} \quad \begin{aligned} a &= 4E_1^2 E_3^2 + s(m_\mu^2 + m_e^2) \\ b &= 4p_1^2 p_3^2 \end{aligned}$$

⑥ Now, cross section:

$$d\sigma = \frac{|\overline{T}_{fi}|^2 d\Omega_f}{4\sqrt{s} p_i} = \frac{|\overline{T}_{fi}|^2}{4\sqrt{s} p_i} \times$$

$$d\sigma = \frac{1}{64\pi^2 s} \left(\frac{p_3}{p_i}\right) |\overline{T}_{fi}|^2 d\Omega_3$$

So, integrate to get:

$$\sigma = \frac{1}{64\pi^2 s} \cdot \left(\frac{p_3}{p_i}\right) \cdot \frac{64\pi^2 \alpha^2}{s^2} \cdot 4\pi \left\{ a + \frac{1}{3} b \right\}$$

$$\sigma(s) = \left(\frac{p_3}{p_i}\right) \cdot \frac{4\pi \alpha^2}{s^3} \left\{ a + \frac{1}{3} b \right\}$$

⑦ More detail:

$$a = 4 \cdot \left(\frac{\sqrt{s}}{2}\right)^2 \left(\frac{\sqrt{s}}{2}\right)^2 + s(m_\mu^2 + m_e^2) = \frac{1}{4} s^2 + s(m_\mu^2 + m_e^2)$$

$$= s(m_\mu^2 + m_e^2 + s/4) = \frac{s^2}{4} \left(1 + \frac{4}{s}(m_\mu^2 + m_e^2)\right)$$

$$b = 4 \cdot \frac{s}{4} \left[1 - \frac{4m_e^2}{s}\right] \cdot \frac{s}{4} \left[1 - \frac{4m_\mu^2}{s}\right]$$

$$= \frac{s^2}{4} \left(1 - \frac{4m_e^2}{s}\right) \left(1 - \frac{4m_\mu^2}{s}\right)$$

$$\text{So: } \left\{ a + \frac{1}{3} b \right\} = \frac{s^2}{4} \left\{ 1 + \frac{4}{s}(m_\mu^2 + m_e^2) + \frac{1}{3} \left(1 - \frac{4m_e^2}{s}\right) \left(1 - \frac{4m_\mu^2}{s}\right) \right\}$$

$$= \frac{s^2}{3} \left\{ 1 + \frac{2}{s}(m_\mu^2 + m_e^2) + \frac{4}{s^2} m_e^2 m_\mu^2 \right\}$$

Then:

$$\frac{d\sigma}{d\Omega} = \frac{p_3}{p_i} \frac{64\pi^2 \alpha^2}{64\pi^2 s} \frac{s^2}{4} \left[\left(1 + \frac{4}{s}(m_\mu^2 + m_e^2)\right) + \left(1 - \frac{4m_e^2}{s}\right) \left(1 - \frac{4m_\mu^2}{s}\right) \cos^2 \Theta \right]$$

$$\left[\frac{d\sigma}{d\Omega} = \left(\frac{p_3}{p_i}\right) \cdot \frac{\alpha^2}{4s} \left[\left\{ 1 + \frac{4}{s}(m_\mu^2 + m_e^2) \right\} + \left\{ \left(1 - \frac{4m_e^2}{s}\right) \left(1 - \frac{4m_\mu^2}{s}\right) \right\} \cos^2 \Theta \right] \right]$$

⑧ And,

$$\left[\sigma(s) = \frac{4\pi \alpha^2}{3s} \cdot \left(\frac{p_3}{p_i}\right) \cdot \left\{ 1 + \frac{2}{s}(m_\mu^2 + m_e^2) + \frac{4}{s^2} m_e^2 m_\mu^2 \right\} \right]$$

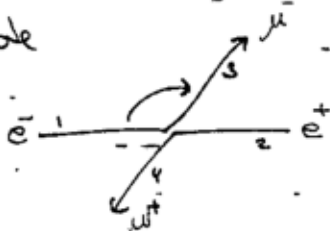
⑨ For $\frac{2m_\mu^2}{s} \ll 1$, we get this H.E. result:

$$\left\{ \frac{d\bar{\sigma}}{d\Omega} = \frac{\alpha^2}{4s} [1 + \cos^2\Theta] \right\}$$

$$\bar{\sigma} = \frac{4\pi}{3} \frac{\alpha^2}{s}$$

see F.p/63

⑩ Note



Θ = angle between leptons of same charge!

⑪ If we neglect m_e , but not m_μ , then result of ⑨ is:

$$\left[\frac{d\bar{\sigma}}{d\Omega} = \frac{p_2}{p_1} \cdot \frac{\alpha^2}{4s} \left[1 + \cos^2\Theta + \frac{4m_\mu^2}{s} \sin^2\Theta \right] \right]$$

Also, note that $\frac{p_3}{A} = \frac{E_3 \beta_3}{E_1 \beta_1} \approx (\beta_\mu / \beta_e) \approx \beta_\mu$

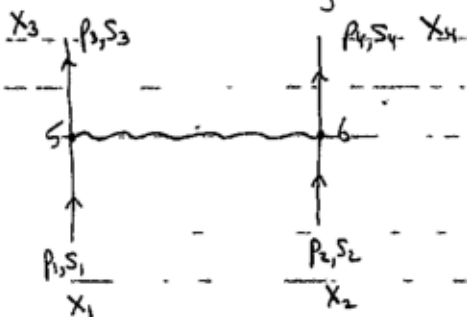
So $\left\{ \frac{p_2}{p_1} \approx \beta_\mu \right\}$

Moller, using
"reduction method"

-2-

ee setup LSZ.
5

⑥ Now we are ready to do a sample calculation.



$$H_I = -\mathcal{L}_I = +e \bar{\psi} \gamma^\mu \psi A_\mu$$

$$= e J^\mu A_\mu$$

First, we need a reduction formula:

$$\langle p_3, s_3; p_4, s_4 | p_1, s_1; p_2, s_2 \rangle_{in} = \langle f | S | i \rangle$$

$$= (-i)^4 \int d^4x_1 d^4x_2 d^4x_3 d^4x_4 \cdot \exp(i[-p_1 \cdot x_1 - p_2 \cdot x_2 + p_3 \cdot x_3 + p_4 \cdot x_4])$$

$$\times \prod_{all} C_i \times \quad \quad \quad \vec{D} = (i\vec{\partial} - m)$$

$$\bar{u}_r(p_3, s_3) \vec{D}_{rs}^{(x_3)} \cdot \bar{u}_l(p_4, s_4) \vec{D}_{lm}^{(x_4)}$$

$$\langle 0 | T(\psi_s(x_3) \psi_m(x_4) \bar{\psi}_n(x_1) \bar{\psi}_a(x_2)) | 0 \rangle$$

$$\bar{D}_{nc}^{*(x_1)} u_c(p_1, s_1) \bar{D}_{ab}^{*(x_2)} u_b(p_2, s_2)$$

For the first time, we are in "all norm" notation. Everyone norms their fields the same way. All other norms are incorporated into the C_i factors. The size of the one particle states, the size of u and $\bar{u}$, these are all taken care of.

We pause and analyze the T -function:

$$\langle 0 | T(\psi_s(x_3) \psi_m(x_4) \bar{\psi}_n(x_1) \bar{\psi}_a(x_2)) | 0 \rangle$$

$$= \langle 0 | T(\psi_s^{in}(x_3) \psi_m^{in}(x_4) \bar{\psi}_n^{in}(x_1) \bar{\psi}_a^{in}(x_2)) | 0 \rangle \times \frac{(-i)^4}{i^4}$$

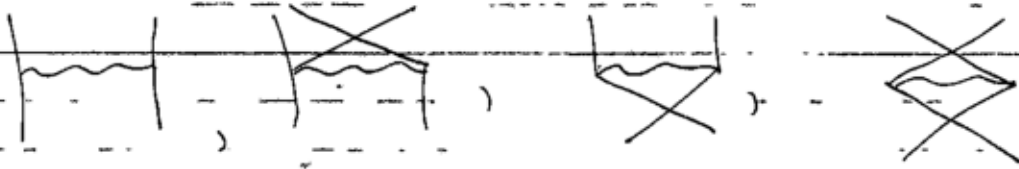
$$\times \int d^4x_5 d^4x_6 : H'(x_5) : : H'(x_6) : | 0 \rangle$$

All other lower terms vanish. Higher terms we ignore (i.e., other skeleton graphs). Self-energies we also ignore & we use renormed coupling e . So, keeping $(-i/2) \int d^4x_5 d^4x_6 \bar{e}$ in mind,

$$\langle 0 | T(\psi_s^{in}(x_3) \psi_m^{in}(x_4) \bar{\psi}_n^{in}(x_1) \bar{\psi}_a^{in}(x_2) : \bar{\psi}_d(x_5) \psi_c(x_5) A_\mu(x_5) : : \bar{\psi}_f(x_6) \psi_g(x_6) A_\nu(x_6) : | 0 \rangle$$

$$= \mathcal{E}_{3mn} a d e \mu f g \nu (x_1 \dots x_6)$$

So there are really 4 appropriate contractions, namely:



But, these are pairwise equal, so we pick up

$$= 2 \left(\text{diagram 1} - \text{diagram 2} \right)$$

Specifically, we can go back now & write our S-matrix element:

Now comes Wick's Theorem. We have 10 fields inside, but the normal ordering via Wick's stuff blocks strange graphs (contractions). We had better pause at this point to consider this idea:

$$T(\phi(x_1)\phi(x_2)) = : \phi(x_1)\phi(x_2) : + \langle 0 | T(\phi(x_1)\phi(x_2)) | 0 \rangle \quad (*)$$

This is the lowest claim of Wick. How do you prove this?

$$\phi(x_1) = \phi^{(+)}(x_1) + \phi^{(-)}(x_1) \quad \text{where } \phi^{(+)} \text{ is the } a^+ \text{ piece.}$$

$$\text{Well, } T(\phi(x_1)\phi(x_2)) = \Theta(t_1 - t_2) \phi(x_1)\phi(x_2) + \Theta(t_2 - t_1) \phi(x_2)\phi(x_1) \\ = \text{one or the other.}$$

$$\text{E.g., } \phi(x_1)\phi(x_2) = (\phi_1^+ + \phi_1^-)(\phi_2^+ + \phi_2^-) \\ = \phi_1^+\phi_2^+ + \phi_1^-\phi_2^+ + \phi_1^+\phi_2^- + \phi_1^-\phi_2^-$$

whereas $: \phi(x_1)\phi(x_2) :$ = same but reads second term.

But, each time you "read" a term, you pick up a commutator:

$$[\phi_1^-, \phi_2^+] \sim [a, a^+] \sim \text{c-number.}$$

In the $\Theta(t_2 - t_1)$ case, again you have to read all terms to "get to" $: \phi(x_1)\phi(x_2) :$. You pick up only c numbers so:

$$T(\phi(x_1)\phi(x_2)) = : \phi(x_1)\phi(x_2) : + c$$

Well, apply $\langle 0 | \dots | 0 \rangle$ to determine c. QED for (*).

Now we must do the next higher case:

$$T(\phi(x_1)\phi(x_2)\phi(x_3)) = : \phi(x_1)\phi(x_2)\phi(x_3) : \\ + \langle 0 | T(\phi(x_1)\phi(x_2)) | 0 \rangle : \phi(x_3) : + \text{perms.}$$

Again, pick a time ordering, say $t_1 > t_2 > t_3$. Then

$$T(\phi(x_1)\phi(x_2)\phi(x_3)) = (\phi_1^+ + \phi_1^-)(\phi_2^+ + \phi_2^-)(\phi_3^+ + \phi_3^-)$$

$$= \phi_1^+\phi_2^+\phi_3^+ + \phi_1^+\phi_2^+\phi_3^- + \phi_1^+\phi_2^-\phi_3^+ + \phi_1^+\phi_2^-\phi_3^- \\ + \phi_1^-\phi_2^+\phi_3^+ + \phi_1^-\phi_2^+\phi_3^- + \phi_1^-\phi_2^-\phi_3^+ + \phi_1^-\phi_2^-\phi_3^-$$

Now lets return to our special case:

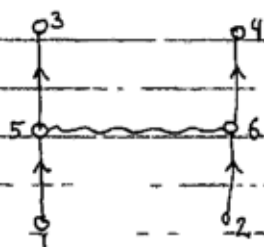
$$\langle 0 | T \left(\psi_s(3) \psi_m(4) \bar{\psi}_n(1) \bar{\psi}_a(2) : \psi_d(5) \bar{\psi}_e(5) A_\mu(5) : : \psi_f(6) \bar{\psi}_g(6) A_\nu(6) : \right) | 0 \rangle$$

We rewrite in fast shorthand:

$$\begin{array}{ccccccc} 3 & 4 & \bar{1} & \bar{2} & : & \bar{5} & 5 & 5_A & : & : & \bar{6} & 6 & 6_A \end{array}$$

→ do get photon prop.

This "contraction" is this picture:



$$= i S_{F_{sd}}(x_3 - x_5) i S_{F_{mf}}(x_4 - x_6) i S_F(\quad) i S_F(\quad) \times i D_F(x_5 - x_6)$$

We could not do:

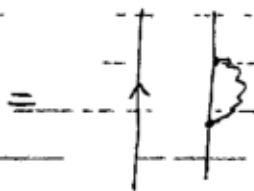
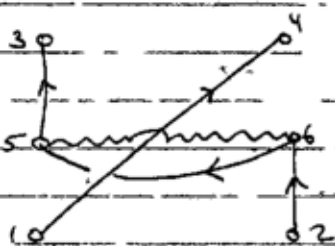
$$\begin{array}{cccc} 3 & 4 & \bar{1} & \bar{2} \end{array}$$

Because $\psi(3)$ commutes with $\bar{\psi}(4)$!

If we did:

$$\begin{array}{ccccccc} 3 & 4 & \bar{1} & \bar{2} & : & \bar{5} & 5 & 5_A & : & : & \bar{6} & 6 & 6_A \end{array}$$

This is:



ee self energy graphs + contractions.

You cannot contract two pairs like so

$$: \sim \psi \sim : : \sim \bar{\psi} \sim :$$

because that is a self energy graph.

Now each time we reorder, we pick up, again a comm. times one field. I.e.,

$$T(\phi(x_1)\phi(x_2)\phi(x_3)) = :\phi(x_1)\phi(x_2)\phi(x_3): \\ + \phi_1^+ [\phi_2^-, \phi_3^+] + \phi_3^- [\phi_1^-, \phi_2^+] + \phi_2^- [\phi_1^-, \phi_3^+] + \phi_1^- [\phi_2^-, \phi_3^+] \\ + \phi_3^+ [\phi_1^-, \phi_2^+] + \phi_2^+ [\phi_1^-, \phi_3^+]$$

This last piece is:

$$\phi_1^- \phi_2^- \phi_3^+ = \phi_1^- \phi_3^+ \phi_2^- + \phi_1^- [\phi_2^-, \phi_3^+] \\ = \phi_3^+ \phi_1^- \phi_2^- + \underbrace{\phi_2^- [\phi_1^-, \phi_3^+] + \phi_1^- [\phi_2^-, \phi_3^+]}$$

And,

$$\phi_1^- \phi_2^+ \phi_3^+ = \phi_2^+ \phi_3^+ \phi_1^- + \phi_2^+ [\phi_1^-, \phi_3^+] + \phi_3^+ [\phi_1^-, \phi_2^+]$$

Each time a field passes through another, on its way to the end, you pick up a comm. of the two involved, times all the other fields.

Now, regroup the results:

$$T(\phi(x_1)\phi(x_2)\phi(x_3)) = :\phi(x_1)\phi(x_2)\phi(x_3): \\ + \phi(x_1) [\phi_2^-, \phi_3^+] + \phi(x_2) [\phi_1^-, \phi_3^+] + \phi(x_3) [\phi_1^-, \phi_2^+]$$

Hey, what can we say about:

$$[\phi_2^-, \phi_3^+] = \left[\int \frac{d^3p}{2E} c_5^- a(p) e^{-ip \cdot x_2}, \int \frac{d^3p'}{2E} c_5^+ a^\dagger(p') e^{ip' \cdot x_3} \right] \\ = \int d^3p d^3p' \left(\frac{c_5^-}{2E} \right) \left(\frac{c_5^+}{2E'} \right) e^{i[-p \cdot x_2 + p' \cdot x_3]} [a(p), a^\dagger(p')] \\ = \int \left(\frac{c_5^-}{2E} \right)^2 e^{i[-p \cdot x_2 + p \cdot x_3]} \cdot 2E c_2 \frac{d^3p}{(2\pi)^3} = \int \frac{1}{(2\pi)^3} \frac{2E c_2}{2E \cdot 2E} e^{-ip \cdot (x_3 - x_2)} d^3p \\ = \int \frac{1}{(2\pi)^3 2E} e^{ip \cdot (x_3 - x_2)} d^3p = i \Delta_F(x_2 - x_3) \text{ when } t_2 > t_3$$

If we had chosen $t_1 > t_3 > t_2$ at the start, we would have had:

$$T(123) = \phi_1 \phi_3 \phi_2 = \dots + \phi_1 \phi_3^+ \phi_2^- + \dots \quad \phi_1^- \phi_3^+ \phi_2^- \\ \rightarrow \phi_2^- [\phi_1^-, \phi_3^+] \quad \rightarrow \phi_2^- [\phi_1^-, \phi_3^+] \\ \rightarrow \phi_1^- [\phi_3^+, \phi_2^-]$$

Wick

Now, with $t_1 > t_3 > t_2$, we pick up $\phi(x_1) [\phi_3^-, \phi_2^+]$ instead of $\phi(x_1) [\phi_2^-, \phi_3^+]$. This of course causes "the other side" of $i\delta$ to appear.

Thus, we have proven that:

$$T(\phi(x_1)\phi(x_2)\phi(x_3)) = : \phi(x_1)\phi(x_2)\phi(x_3) :$$

$$+ \phi(x_1) \langle 0 | T(\phi(x_2)\phi(x_3)) | 0 \rangle + \text{perms.}$$

$\phi = : \phi :$
when alone

The idea is this: once you choose a time ordering, in order to get each term in the LHS to normal order, you have to pick up products of commutators.

$$\text{Eg. } \phi^-(1)\phi^-(2)\phi^+(3)\phi^+(4)$$

$$\hookrightarrow \phi^-(1)\phi^-(2) [\phi^+(3), \phi^+(4)] + \text{perms.}$$

$$\phi^-(1)\phi^-(2)\phi^+(3)\phi^+(4)$$

$$\hookrightarrow \phi^-(1) [\phi^-(2), \phi^+(3)] \phi^+(4) + \phi_1 \phi_3^+ \phi_2^- \phi_4^+$$

$$\hookrightarrow [,] \times [,]$$

More clearly:

$$\phi_1^+ \phi_2^+ \phi_3^+ \phi_4^+ = \phi_3^+ \phi_1^+ \phi_2^+ \phi_4^+ + \phi_1^+ \phi_2^+ [\phi_3^+, \phi_4^+] \times 2$$

$$= \phi_3^+ \phi_4^+ \phi_1^+ \phi_2^+ + 4 \cdot \phi_1^+ \phi_2^+ [\phi_3^+, \phi_4^+]$$

More clearly still:

$$\begin{aligned} (- - + +) &= (- + [-, +]) + (+ + [-, +]) + (- + [+ , -]) + (+ + + -) \\ &= (+ - [-, +]) + (- + [-, +]) + (+ + [-, +]) \end{aligned}$$

A rigorous proof would be tedious, but I have done enough to see the desired point: in

$$\langle 0 | T(\phi\phi\phi : \phi\phi\phi\phi : \phi\phi\phi) | 0 \rangle$$

you can never "pick up commutators" in a past as shown, eg, $\phi_1 \phi_2 \phi_3 \phi_4$ are already in normal order!

$$\langle p_3, s_3; p_4, s_4 | S | p_1, s_1; p_2, s_2 \rangle = \langle f | S | i \rangle = S_{fi}$$

$$= (-i)^4 \int dx_1 dx_2 dx_3 dx_4 dx_5 dx_6 \times \frac{(-i)^2}{2!} \times e^2$$

$$\times \prod_{all} C_F \times e^{i(-p_1 \cdot x_1 - p_2 \cdot x_2 + p_3 \cdot x_3 + p_4 \cdot x_4)}$$

$$\times [A + B] \times 2$$

where,

$$A = \text{diagram} = \bar{u}(p_3, s_3) (i\not{\partial}^2 - m)_{rs} i S_F(x_3 - x_5) (\not{\partial}^M)_{dc} i S_F(x_5 - x_1)$$

$$\times \frac{1}{(i\not{\partial}^2 - m)_{nc}} u_c(p_1, s_1)$$

$$\times [\text{similarly, } \bar{u}(p_4) (i\not{\partial}^2 - m) i S_F(x_4 - x_6) \not{\partial}^N i S_F(x_6 - x_2)$$

$$(i\not{\partial}^2 - m)^M u(z)]$$

$$\times i D_F(x_5 - x_6)$$

The term B is just:

$$B = \text{diagram}$$

At this point we can perform the "amputations", for example:

$$(i\not{\partial}^2 - m) i S_F(x_3 - x_5) = i \not{\partial}^N(x_3 - x_5) \cdot 1$$

By the way, if $\underline{A} \underline{B} = c \cdot \underline{1}$, then $\underline{B}^+ \underline{A}^+ = c^* \cdot \underline{1}$.

Is S_F hermitian? Use:

$$S_F(z) = (i\not{\partial} - m) \Delta_F(z)$$

$$\Rightarrow S_F^+(z) = (i\not{\partial} - m)^+ \Delta_F(z)$$

I have reached a technical impasse here, but no doubt we can also say:

$$i S_F(x_5 - x_1) (i\not{\partial}^2 - m)_{nc}^* = i \not{\partial}^N(x_5 - x_1) \cdot 1_{cc}$$

Then we can remove $\int dx_4 dx_3$ and replace x_1, x_3 with x_5 .

With this in mind, we get:

norm.

So, using the "second technique" I get a contradictory answer, namely:

$$S_i = (2\pi)^4 \delta^4(\text{ext}) \prod_{\text{all}} \underbrace{\frac{C_i}{(2\pi)^3}}_{\text{norm.}} \cdot M_{\text{Feyn.}}$$

Let's rethink norms a little. First, one chooses C_6 and C_2 however one likes. Let $C_3 = (2\pi)^4$. If one then agrees to choose C_4 such that:

$$C_4 = \sqrt{C_2 C_6} / (2\pi)^3$$

Then, $\{v_i, b_i\} = C_4 e^{i\pi} u$

and then both "methods" work without the need for extra factors.

This relation holds for these norms:

Sakurai
Commins
Bjorken & Drell
Jackson
Horn & Zach
Eden
CS
Williams

but not Kallen
not Gurtman
not Gas.
not MS

for these guys, if you want to use the "second method", must add various factors in each ext leg.

I think the second technique is faster if you want, say, to derive Feynman rules, etc. Of course once you know the rules, you are independent of method.

One remaining problem (at the "operational" level) is how to deal with spin. For spin $\frac{1}{2}$ I know that:

$$u\bar{u} = \frac{\not{p} + m}{C_6} \cdot \frac{1 + \gamma_5 \not{s}}{2}$$

$$-v\bar{v} = \frac{-\not{p} + m}{C_6} \cdot \frac{1 + \gamma_5 \not{s}}{2}$$

$$u\bar{v} = \frac{\not{p} + m}{C_6} \cdot \frac{1 + \gamma_5 \not{s}}{2}$$

$$v\bar{u} = \frac{-\not{p} + m}{C_6} \cdot \frac{1 + \gamma_5 \not{s}}{2}$$

$$S_{fi} = (-i)^4 \int dx_5 dx_6 \times \frac{(-i)^2}{2!} \times e^2 \times \Pi C_i \times e^{i(-p_1 x_5 - p_2 x_6 + p_3 x_5 + p_4 x_6)} \\ \times \cancel{2} [A+B]$$

where: $A = \bar{u} \cdot i \cdot \gamma^\mu \cdot i \cdot u \cdot i D_{F\mu\nu}(x_5 - x_6) \cdot (\bar{u} \cdot i \cdot \gamma^\nu \cdot i \cdot u)$

$$= (i)^4 [\bar{u}(3) \gamma^\mu u(1)] \cdot i D_{F\mu\nu} \cdot [\bar{u}(4) \gamma^\nu u(2)]$$

Now, cancel the 2's as shown: Pack $(-i)^4 \times (i)^4 = (1)^4 = 1$.
Group $(-ie\gamma^\mu)$ together. Hence:

$$S_{fi} = \int dx_5 dx_6 \cdot \Pi C_i \cdot e^{i(p_3 x_5 + p_4 x_6)} \cdot [A+B]$$

where $\tilde{A} = \bar{u}(3) \gamma^\mu u(1) \cdot i D_{F\mu\nu} \cdot \bar{u}(4) \gamma^\nu u(2)$

Now, insert Sn D:

$$(i D_F)_{\nu\mu} = + g_{\nu\mu} D_F = - g_{\nu\mu} \int \frac{d^4 k}{(2\pi)^4} e^{-ik(x)} \cdot \frac{+i}{k^2 + i\epsilon}$$

$$= g_{\nu\mu} \int \frac{d^4 k}{(2\pi)^4} e^{-ik(x)} \cdot \left(\frac{-i g_{\nu\mu}}{k^2 + i\epsilon} \right)$$

This gives us:

$$S_{fi} = \Pi C_i \cdot \int d^4 x_5 d^4 x_6 e^{i(-p_1 x_5 - p_2 x_6 + p_3 x_5 + p_4 x_6 - k x_5 + k x_6)} \\ \times \left[\bar{u} \gamma^\mu u \cdot \left(\frac{-i g_{\mu\nu}}{k^2 + i\epsilon} \right) \cdot \bar{u} \gamma^\nu u + \text{second} \right]$$

Why does this look like two δ^4 's?!

$$\int \frac{d^4 k}{(2\pi)^4}$$

OK, do the $\int d^4 x_6$ + pick up $(2\pi)^4 \delta^4(-p_2 + p_4 + k)$
do the $\int d^4 x_5$ + pick up $(2\pi)^4 \delta^4(-p_1 + p_3 - k)$

Then, do the $\int \frac{d^4 k}{(2\pi)^4}$ to get: $(2\pi)^4 \delta^4(p_3 + p_4 - p_1 - p_2)$

Then:

$$S_{fi} = \Pi C_i \cdot (2\pi)^4 \delta^4(\text{ext}) \cdot \left[\bar{u} \gamma^\mu u \cdot \left(\frac{-i g_{\mu\nu}}{k^2 + i\epsilon} \right) \cdot \bar{u} \gamma^\nu u \right]$$

"rules" drop out

4 []

norms.

Now, choose ~ compare to my notation:

$$S_{fi} = \delta_{fi} + i C_1 C_3 \delta^4(\text{ext}) \cdot \prod_{\text{all}} C_i \cdot T_{fi}$$

$$S_{fi} = \delta_{fi} + (2\pi)^4 \delta^4(\text{ext}) \cdot \prod_{\text{all}} C_i \cdot [\bar{u} \gamma \dots]$$

Thus, conclude that:

$$i C_1 C_3 T_{fi}^{me} = (2\pi)^4 \cdot M_{\text{Seymour}}$$

So we observe that $C_3 = (2\pi)^4$ comes out very naturally.

And, it seems that $C_1 = 1$ is appropriate? Or, $C_1 = -1$.

Let's use $C_1 = -1$, then:

$$T_{fi}^{(me)} = +i M_{\text{Seymour}}$$

I have used:

$$C_3 = (2\pi)^4 \quad C_1 = -1$$

But all other C 's are any way you like!

Summary: I think that now I know how to calculate some graphs using the "rules". This business of using all these propagators, then of amputating them ~ that seems a bit of extra work. How can we reformulate the theory?

One way not used by BD but taught us by sub + sm is this:

$$\langle p_3, s_3; p_4, s_4 | S | p_1, s_1; p_2, s_2 \rangle_{in} = \langle f | S | i \rangle$$

Here, we need a way to express the operator S . I do not know how to derive the method, off hand, but I think this is the result.

$$= \frac{i!}{2^i} \langle 0 | b(p_3, s_3) b(p_4, s_4) \int d^4x d^4x_0 : \bar{\psi} \psi A : : \bar{\psi} \psi A : b^\dagger(p_1, s_1) b^\dagger(p_2, s_2) | 0 \rangle$$

Now contact at once: Let's look more closely

-9
Second method: "Moller."

5

My suspicion is that:

$$\langle 0 | b(3) b(4) : \frac{(-ie)^2}{2!} \int dx_1 dx_2 : \bar{\psi}(x_1) \gamma_\mu \psi(x_1) A^\mu(x_1) : : \bar{\psi}(x_2) \gamma_\nu \psi(x_2) A^\nu(x_2) : | 1(1) b^\dagger(2) | 0 \rangle$$

Now we do:

$$\begin{array}{c} \begin{array}{cc} \begin{array}{c} 3 \\ \uparrow \\ x_1 \end{array} & \begin{array}{c} 4 \\ \uparrow \\ x_2 \end{array} \\ \begin{array}{c} 1 \\ \uparrow \\ x_1 \end{array} & \begin{array}{c} 2 \\ \uparrow \\ x_2 \end{array} \end{array} = b(3) b(4) \overbrace{\bar{\psi} \gamma_\mu \psi A^\mu}^{b \leftrightarrow d^\dagger} \overbrace{\bar{\psi} \gamma_\nu \psi A^\nu}^{b \leftrightarrow d^\dagger} b^\dagger(1) b^\dagger(2) \end{array}$$

So, we sort of ignore the T-product + conclude:

$$= \frac{(-ie)^2}{2!} \int dx_1 dx_2 \langle 0 | \{ b(3), \bar{\psi}(x_1) \} \{ b(4), \bar{\psi}(x_2) \} \dots$$

Clearly if you "move the b's in"

$$= \frac{(-ie)^2}{2!} \int dx_1 dx_2 \langle 0 | \{ b(3), \bar{\psi}(x_1) \} \delta_\mu \{ \psi(x_1), b^\dagger(1) \} T(A^\mu(x_1) A_\nu(x_2)) \cdot \{ b(4), \bar{\psi}(x_2) \} \delta_\nu \{ \psi(x_2), b^\dagger(2) \} | 0 \rangle$$

At this point, enter normalization:

$$\begin{aligned} \{ \psi, b^\dagger \} &= \sqrt{\frac{c_2 c_6}{(2\pi)^3}} u \cdot e^\dots & \{ b, \bar{\psi} \} &= \sqrt{\frac{c_2 c_6}{(2\pi)^3}} \bar{u} \cdot e^\dots \\ \{ b, \psi^\dagger \} &= \sqrt{\frac{c_2 c_6}{(2\pi)^3}} u^\dagger \cdot e^\dots \end{aligned}$$

So, executing all these commutators, we arrive at:

$$= \frac{(-ie)^2}{2!} \int dx_1 dx_2 \underbrace{\prod_{\text{all}} \sqrt{\frac{c_2 c_6}{(2\pi)^3}}}_{\text{same}} \bar{u}(3) \gamma_\mu u(1) \cdot \langle 0 | T(A^\mu(x_1) A_\nu(x_2)) | 0 \rangle \bar{u}(4) \gamma_\nu u(2)$$

Now we use:

$$\langle 0 | T(A^\mu(x_1) A^\nu(x_2)) | 0 \rangle = i D_F^{\mu\nu} = \int \frac{d^4 k}{(2\pi)^4} e^{-i k \cdot (x_1 - x_2)} \frac{-i g_{\mu\nu}}{k^2 + i\epsilon}$$

Then we do up the two $d^4 k$ to get

$$= \frac{(-ie)^2}{2!} \prod_{\text{all}} \sqrt{\frac{c_2 c_6}{(2\pi)^3}} (2\pi)^4 \delta^4(\text{ext}) \cdot \text{Meyman}$$

↑
dump

Dirac spin sums - 11 -

Then we can sum over spins very easily.

$$\sum_s \frac{1 + \gamma_5 \not{s}}{2} = \frac{1 + \gamma_5 \not{s}}{2} + \frac{1 - \gamma_5 \not{s}}{2} = 1.$$

Let's do our example.

$$M_f = (-ie)^2 \left[\bar{u}(3) \gamma^\mu u(1) \cdot \left(\frac{-i g_{\mu\nu}}{k^2 + i\epsilon} \right) \cdot \bar{u}(4) \gamma^\nu u(2) \right] \\ \left[-\bar{u}(4) \gamma^\mu u(1) \cdot \left(\frac{-i g_{\mu\nu}}{k^2 + i\epsilon} \right) \cdot \bar{u}(3) \gamma^\nu u(2) \right].$$

Now we want M_f^* :

$$M_f^* = (+ie)^2 \left[\bar{u}^*(3) \gamma^{\mu*} u^*(1) \left(\frac{+i g_{\mu\nu}}{k^2 - i\epsilon} \right) \bar{u}^*(4) \gamma^{\nu*} u^*(2) - \dots \right]$$

Well, consider:

$$[\bar{u}(3) \gamma^\mu u(1)]^* = \left\{ \dots \right\}^* \left(\cancel{\gamma^\mu} \right) \left(\begin{matrix} \vdots \\ \vdots \end{matrix} \right)^* \\ = ?$$

Wait, can we say: $(V_2^\dagger \cdot A \cdot V_1) = \text{number}$.

$$(V_2^\dagger \cdot A \cdot V_1)^\dagger = (\text{number})^\dagger = (\text{number})^* \\ = V_1^\dagger \cdot A^\dagger \cdot V_2 \quad \checkmark$$

$$\text{So } (V_2^\dagger A V_1)^* = (V_2^\dagger A V_1)^\dagger = V_1^\dagger A^\dagger V_2.$$

$$\text{So: } (\bar{u}(3) \gamma^\mu u(1))^* = u^\dagger(1) \gamma^{\mu\dagger} \gamma^{0\dagger} u(3) = \bar{u}(1) \gamma^0 \gamma^{\mu\dagger} \gamma^0 u(3) \\ = \bar{u}(1) \gamma^\mu u(3)$$

there are general theorems on this somewhere. In general:

$$\left[\bar{u}(2) \tilde{\sigma} u(1) \right]^* = \bar{u}(1) \cdot \tilde{\sigma} \cdot u(2)$$

So

We now have:

$$M_{fi} = (-ie)^2 \left[\bar{u}(3) \gamma^\mu u(1) \cdot \left(\frac{-ig_{\mu\nu}}{k^2 + i\epsilon} \right) \cdot \bar{u}(4) \gamma^\nu u(2) \right. \\ \left. - \bar{u}(4) \gamma^\mu u(1) \cdot \left(\frac{-ig_{\mu\nu}}{k^2 + i\epsilon} \right) \cdot \bar{u}(3) \gamma^\nu u(2) \right]$$

$$M_{fi}^* = (+ie)^2 \left[\bar{u}(1) \gamma^\mu u(3) \cdot \left(\frac{+ig_{\mu\nu}}{k^2 - i\epsilon} \right) \cdot \bar{u}(2) \gamma^\nu u(4) \right. \\ \left. - \bar{u}(1) \gamma^\mu u(4) \cdot \left(\frac{+ig_{\mu\nu}}{k^2 - i\epsilon} \right) \cdot \bar{u}(2) \gamma^\nu u(3) \right]$$

This allows us to say:

$$|M_{fi}|^2 = e^4 \cdot \frac{1}{(k^2 - i\epsilon)(k^2 + i\epsilon)} \cdot \bar{u}(3) \gamma^\mu u(1) \cdot \bar{u}(4) \gamma_\mu u(2) \cdot \bar{u}(1) \gamma^\nu u(3) \cdot \bar{u}(2) \gamma_\nu u(4) \\ + e^4 \cdot \frac{1}{(k^2 + i\epsilon)(k^2 - i\epsilon)} \cdot \bar{u}(4) \gamma^\mu u(1) \cdot \bar{u}(3) \gamma_\mu u(2) \cdot \bar{u}(1) \gamma^\nu u(4) \cdot \bar{u}(2) \gamma_\nu u(3) \\ - 4 e^4 \cdot \frac{1}{(k^2 + i\epsilon)(k^2 - i\epsilon)} \cdot \bar{u}(3) \gamma^\mu u(1) \cdot \bar{u}(4) \gamma_\mu u(2) \cdot \bar{u}(1) \gamma^\nu u(4) \cdot \bar{u}(2) \gamma_\nu u(3) \\ - e^4 \cdot \frac{1}{(k^2 + i\epsilon)(k^2 - i\epsilon)} \cdot \bar{u}(4) \gamma^\mu u(1) \cdot \bar{u}(3) \gamma_\mu u(2) \cdot \bar{u}(1) \gamma^\nu u(3) \cdot \bar{u}(2) \gamma_\nu u(4)$$

Now rewrite the first term.

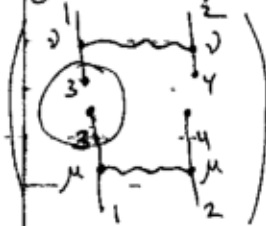
$$e^4 \cdot \frac{(-i)}{k^2 + i\epsilon} \cdot \frac{(+i)}{k^2 - i\epsilon} \cdot \underbrace{\bar{u}(3) \gamma^\mu u(1) \bar{u}(1) \gamma^\nu u(3)}_{= i (\Lambda_3 \gamma^\mu \Lambda_1 \gamma^\nu)} \cdot \underbrace{\bar{u}(4) \gamma_\mu u(2) \bar{u}(2) \gamma_\nu u(4)}_{= i (\Lambda_4 \gamma_\mu \Lambda_2 \gamma_\nu)}$$

where $\Lambda_3 = \frac{\not{p}_3 + m}{C_6} \cdot \frac{1 + \gamma_5 \not{\epsilon}_3}{2}$ etc.

Now prior to spin sums we can say:

$$1^{st} \text{ term} = \left| \frac{(-ie)^2 (-i)}{k^2 + i\epsilon} \right|^2 \cdot i (\bar{u}_3 \gamma^\mu u_1 \bar{u}_1 \gamma^\nu u_3) \cdot i (\bar{u}_4 \gamma_\mu u_2 \bar{u}_2 \gamma_\nu u_4)$$

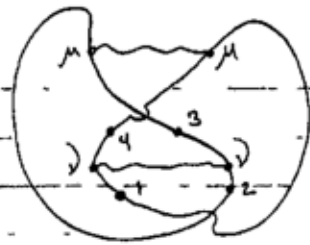
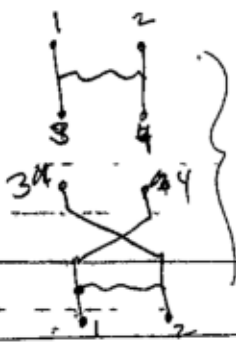
Is there a pictorial way to have this?



This picture tells you how to write $|M_{fi}|^2$ first term.

hace:

Second term:



$\Rightarrow -(-ie)^4 \left\{ \begin{array}{c} \text{diagram 1} \\ \text{diagram 2} \end{array} \right\}$

Clearer if you say:

$$= (-ie)^4 \cdot \frac{(+i)}{k^2 - i\epsilon} \cdot \frac{(-i)}{k^2 + i\epsilon} \cdot \text{tr}(\gamma_\mu \gamma_\nu \gamma_\rho \gamma_\sigma)$$

So, we can now summarize:

$$|M|^2 = \left\{ \begin{array}{c} \text{diagram 1} \\ \text{diagram 2} \\ \text{diagram 3} \\ \text{diagram 4} \end{array} \right\}$$

This graphical notation is very nice. Now write it out.

hace technique

$$\begin{aligned}
 \frac{|M|^2}{(-ie)^4} &= \left| \frac{(-i)}{k^2 + i\epsilon} \right|^2 \text{tr}(\gamma_\mu \gamma_\nu \gamma_\rho \gamma_\sigma) \text{tr}(\gamma_\mu \gamma_\nu \gamma_\rho \gamma_\sigma) \\
 &= \left(\frac{(-i)}{k^2 + i\epsilon} \right)^* \left(\frac{(-i)}{k^2 + i\epsilon} \right) \text{tr}(\gamma_\mu \gamma_\nu \gamma_\rho \gamma_\sigma \gamma_\mu \gamma_\nu \gamma_\rho \gamma_\sigma) \\
 &= \left(\frac{(-i)}{k^2 + i\epsilon} \right) \left(\frac{(-i)}{k^2 + i\epsilon} \right)^* \text{tr}(\gamma_\mu \gamma_\nu \gamma_\rho \gamma_\sigma \gamma_\mu \gamma_\nu \gamma_\rho \gamma_\sigma) \\
 &+ \left| \frac{(-i)}{k^2 + i\epsilon} \right|^2 \text{tr}(\gamma_\mu \gamma_\nu \gamma_\rho \gamma_\sigma) \text{tr}(\gamma_\mu \gamma_\nu \gamma_\rho \gamma_\sigma)
 \end{aligned}$$

Now, if we do "summar" on spins, this means:

$$\begin{aligned}
 &\frac{1}{2s_i + 1} \frac{1}{2s_f + 1} \sum_{s_i, s_f} \sum_{s_i, s_f} \\
 &= \frac{1}{2} \cdot \frac{1}{2} \cdot \sum_{s_1, s_2, s_3, s_4}
 \end{aligned}$$

But, eq. $\sum_i \rightarrow 1$ from each term, $1 \rightarrow (\frac{p+m}{E})$. (Moller)

Thus, add $\frac{1}{4}$ out front + replace.

After doing this, you have to work out all the traces.
Then, use M_F just as rule says:

$$T_{fi}^{me} = +i M_F \quad \text{with } C_1 = -1, C_3 = (2\pi)^4$$

To get cross section, I would say to use:
if you like.

$$d\sigma = \frac{(2\pi)^6}{4 \cdot \text{flux}} \cdot |i M_F|^2 dQ_F$$

In Eden norm, eq. will get:

$$\begin{aligned} d\sigma &= \frac{(2\pi)^6}{4 \cdot \lambda(s, m_a^2, m_b^2)/2} \cdot (2\pi)^6 \cdot |i M_F|^2 \\ &\quad \times (2\pi)^4 \delta^4(\text{ext}) \cdot (2\pi)^6 \cdot d^4 p_1 \delta^4(\dots) d^4 p_2 \delta^4(\dots) \\ &= \frac{1}{(2\pi)^2} \cdot \frac{1}{4 \cdot \frac{1}{2} \lambda} \cdot |i M_F|^2 \cdot \underbrace{\delta^4(\text{ext}) d^4 p_1 \delta^4(\dots) d^4 p_2 \delta^4(\dots)}_{\rightarrow \frac{P_F^*}{4\sqrt{s}} \cdot d\Omega} \end{aligned}$$

OK, OK, well get:

$$\left. \frac{d\sigma}{d\Omega} \right|^* = \frac{(2\pi)^3 \cdot (2\pi)^8 \cdot (2\pi)^{12} \cdot |i M_F|^2}{16s} = \frac{1}{64\pi^2 s} |i M_F|^2$$

$$\text{Then: } \sigma_{\text{tot}} = \frac{1}{2!} \int d\Omega \cdot \left. \frac{d\sigma}{d\Omega} \right|^*$$

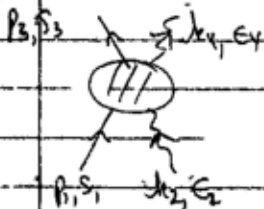
"Nothing can stop us". The exact result for $\frac{d\sigma}{d\Omega}$ is the
"Moller Formula", 1932. Look at Jauch + Palsch

So, in principle I feel I know how to operate the
QED rules for at least $ff \rightarrow ff$ in lowest order

(What about external photons? And photon sums?)

start Compton:

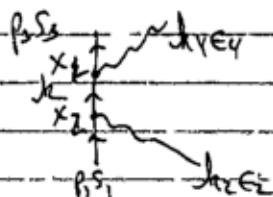
Let's take another example:



Using our "second method," we would set this up:

$$\int d^4x d^4x_2 \frac{(-ie)^2}{(2!)^2} \langle 0 | a(k_1, \epsilon_1) b(p_1, s_1) : \bar{\psi}(x_1) \psi(x_1) : : \bar{\psi}(x_2) \psi(x_2) : a^\dagger(k_2, \epsilon_2) b^\dagger(p_1, s_1) | 0 \rangle$$

One contraction is



$$a b : \bar{\psi}(x_1) \psi(x_1) : : \bar{\psi}(x_2) \psi(x_2) : a^\dagger b^\dagger$$

$$= \frac{(-ie)^2}{2!} \int d^4x d^4x_2 : [a(k_1, \epsilon_1), A^\dagger(x_1)] \{ b(p_1, s_1), \bar{\psi}(x_1) \} \gamma_\mu \langle 0 | T(\bar{\psi}(x_1) \bar{\psi}(x_2)) | 0 \rangle : \\ \cdot \gamma_\nu \{ \psi(x_2), b^\dagger(p_1, s_1) \} [A(x_2), a^\dagger(k_2, \epsilon_2)]$$

Now I would assume:

$$[A^\dagger, a^\dagger] = \sqrt{\frac{c_2}{(2\pi)^3}} e^{i p \cdot x} \epsilon^\mu$$

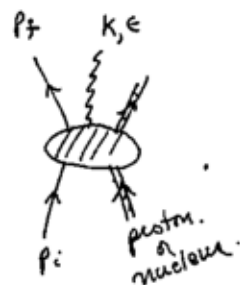
Now, for electron we could have said $S^\mu = S^\mu(p, \lambda)$ where $\lambda = 1, 2$ for up, down + p because $S \cdot p = 0$. So I'd write $\epsilon^\mu(k, \lambda)$, but no big deal, yet.

So, do all this and use

$$\langle 0 | T(\bar{\psi}(x_1) \bar{\psi}(x_2)) | 0 \rangle = i S_F(x_1, x_2)$$

$$= \int \frac{d^4k}{(2\pi)^4} e^{i k \cdot (x_1 - x_2)} \frac{(\not{p} + \not{m})}{(p^2 - m^2 + i\epsilon)} = \left\{ \frac{+i}{p - m + i\epsilon} \right\}$$

$$\text{Hence: } \frac{(-ie)^2}{2!} (2\pi)^4 \delta^4(x_1 - x_2) \prod_{\text{all}} \sqrt{\frac{c_2 c_6}{(2\pi)^3}} \\ \times \epsilon_4^\mu \bar{u}(3) \gamma_\mu \left[\frac{+i}{p - m + i\epsilon} \right] \cdot \gamma_\nu u(1) \cdot \epsilon_2^\nu$$

Bremsstrahlung

① Process is

Usually done in LAB frame.

② The full $d\sigma$ is called Bethe-Heitler, see p 244 of Heitler's book. (1931)③ In the limit $k \rightarrow 0$ the sum of obvious two Feynman graphs is:

$$S_{fi} = \left\{ \bar{u}(p_f, s_f) [-i\gamma_0] u(p_i, s_i) \right\} \times \left(\frac{\epsilon \cdot p_f}{k \cdot p_f} - \frac{\epsilon \cdot p_i}{k \cdot p_i} \right)$$

But this is essentially the Mott elastic amplitude given by BD (7.11), = 

So you find:

$$\left(\frac{d\sigma}{d\Omega_f} \right)^{\text{Brem}} = \left(\frac{d\sigma}{d\Omega_f} \right)^{\text{Mott}} \left[\frac{e^2 k dk d\Omega_k \Theta(E_i - m - k)}{16\pi^3} \cdot \frac{(\epsilon \cdot p_f - \epsilon \cdot p_i)^2}{k \cdot p_f k \cdot p_i} \right]$$

↑ cause peaks when $\vec{k} \parallel \vec{p}_i$ or $\vec{k} \parallel \vec{p}_f$.

But $k \cdot p_i = k[E_i - p_i \cos\theta_i]$ etc, so $\left(\frac{d\sigma}{d\Omega_f} \right)^{\text{Brem}} \propto \frac{dk}{k}$.
Blows up as photon gets very soft.

④ Suppose your exper catches soft photons from $k_{\min}$ to $k_{\max}$. Then you get

$$\left(\frac{d\sigma}{d\Omega_f} \right)^{\text{Brem}} = \left(\frac{d\sigma}{d\Omega_f} \right)^{\text{Mott}} \cdot \frac{2\alpha}{\pi} \ln\left(\frac{k_{\max}}{k_{\min}}\right) \cdot \frac{4}{3} \beta^2 \sin^2 \frac{\theta}{2}, \text{NR.}$$

⑤ Exper cross does not blow up because as $k_{\min} \rightarrow 0$ you also catch pure elastic events with their order α^2 corrections! Hence you must include  and . These cancel the IR divergence.

Radiation Theory

Review of Radiation Theory

1. First, a quick summary of Jackson's chapter on classical radiation by moving charges. Probably this is unrelated to atomic radiations, but maybe it will be relevant. The basic idea is simply to say: E and B fields may be computed from A_μ which in turn may be obtained (at retarded time) from the current density J_μ . But the current density of a single particle is trivial. Thus, you obtain the trivial Lienard Wiechart potentials on page 465. The only fancy aspect is the little k factor which causes potential fields to compress into a beam in the direction of the particles motion.

Now, given these LW potentials, you can calculate E and B . But it turns out that $B = n \times E$ so you only need calculate E . This is done in 14.14 where we get a fast falloff "velocity field" which we shall neglect, and the standard radiation field which is proportional to particle acceleration. This is probably the fundamental formula of the chapter.

Next, JDJ goes to a nonrel frame and computed the radiated energy into solid angle. You get the standard result that $dP/d\Omega = \alpha \times \text{accel}^2 \times \sin(\text{angle})^2$. This then integrates over solid angle to give the Larmor formula.

This Larmor formula is non-rel but is easily relativized to give result 14.24 or 14.26. The main differences from the nonrel Larmor are the presence of a γ^6 factor and a second velocity dependent term. JDJ then goes on to calculate radiation from linear and circular accelerators. This gives the famous γ^4 radiated power dependence which so much limits energy of electrons in circular accelerators. It is noted that radiation losses in a linear accelerator are totally negligible, hence why slac was built.

Next comes the "angular distribution of radiated power" stuff. You get the searchlight effect with a width in angle of γ^{-1} . From an estimate of the "see time" you can guess that the frequency spectrum will run up to γ^3 times the fundamental frequency of the rotating particle. Another rule of thumb is that it is always the transverse acceleration that causes radiation.

Next, he mentions that one simple way to make a particle accelerate is to apply a plane wave. This leads to the Thomson formula for the cross section for radiation upon a free particle (or effectively free). The result is the standard $(1 + \cos^2)$ angular distribution, with Thomson total cross section of $8\pi/3$ times r_0^2 , the classical electron radius. At high energy this process is called Compton scattering and the Klein Nishina formula must be used which shows peaking in the forward direction.

~~Jackson closes out the chapter with comments on Cerenkov effect.~~

The main idea is that when a particle travels at a speed greater than the speed of light in the medium, the particle radiates only in a specific direction whose cosine is the ratio c/v . By measuring this radiation direction, you can figure out the exact speed of the particle. Of course it has to be going faster than the local speed of light!

2. Having reviewed this business on radiation of single particles in motion, we now

③ turn to the chapter on multipole fields, hoping this will have more relevance to atomic and charmonium transitions. We are still completely classical, no quantum mechanics yet.

We start on page 543 which shows that there are two ways to make the wave equation describe electromagnetism in the absence of sources. One way has B satisfying the wave equation. B must then have no divergence (transversality condition on B), and you get E from B by taking a curl. This is the "electric" mode like for $E1$ transition to be described later.

But once you know that B satisfies the wave equation (by the way, the time dependence is always fixed at a certain frequency Ω . Thus, $(\nabla^2 + k^2)$ is the wave operator), you can expand this thing onto spherical harmonics as in 16.35. I.e., the angular stuff is described by spherical harmonics because these solve the angular equation. The radial equation is then solved by some version of spherical Bessel functions. You usually choose hankels because these have distant plane wave behavior that you want for radiation theory. Of course in this expansion there are certain undetermined vector coefficients. These may be partially determined by observing that B must be "transverse". Hence you end up with $B = L Y$ where L is the angular momentum vector operator and Y is spherical harmonic. Then there is only a scalar coefficient $f_l(kr)$. See 16.42.

These fields $B_{lm} = f_l(kr) L Y_{lm}$ and $E_{lm} = \text{curl } B_{lm}$ are the famous "multipole fields". Of course these are only half of a complete set of expansion fields because you also have to include the "magnetic possibilities" as shown in 16.44.

3. Selection rules: Jackson claims that angular momentum selection rules are simply given by vector addition. The multipole field carries off L . If your initial state is J , final state is J' , then J' must be contained in $J \oplus L$ to get action. Obviously this is just Wigner-Eckart and the multipole field is a tensor operator.

The parity selection rule is more interesting. The parity of the radiated "multipole" field is actually given by the parity of B_{lm} which is $(-1)^l$

for electrics, and $(-1)^{l+one}$ for magnetics. Thus, for example, in E1 transition (electric dipole), the field has parity -1 to the initial and final atomic states must have opposite parity. If the atomic states have same parity, then E1 is forbidden and you can then have only M1 or E2 which are both two photon operations. I think. I.e., they are at the same level of weakness relative to E1.

In this discussion of selection rules there is mention only of parity and total J. No mention of orbital L as in Russel Saunders stuff. I will have to look elsewhere for discussion of more detailed atomic selection rules and rules of thumb.

~~By the way, it seems rather clear xxx.~~

OK, now we have set up the multipole field analysis. The next question is how do you generate a particular multipole field. All fields are generated by moving charges, so it all boils down to looking at the J of the radiating source. If you integrate your charge density against a spherical harmonic, you get the electric multipole moment Q_{lm} . Roughly, this is the scalar coefficient in the radiation expansion. Thus, just from the physical shape of the radiating charge distribution you can figure out which electric multipoles will go. For example, a linear antenna has $a_E(l,m)$ only when $m=0$ and $l = 1, 3, 5, \dots$. A circular antenna probably has no electric moments. It certainly has a strong M1 moment though. You get the magnetic moments M_{lm} by looking at $M = \text{curl } J$. I do not want to make any inferences yet about the relation between the shape of quantum wave functions and the radiation. Wait until we incorporate quantum mechanics later.

Now here are some rules of thumb for atomic and nuclear radiation without any quantum mechanics. : the relative contribution of multipoles is set by size of $(ka)^2$ where k = photon energy and a = characteristic distance. For atoms, a = Bohr like distance and k = 1 eV more or less. One gets that ka = order of $Z \times \text{ALPHA}$ so generally small except for heavy atoms. Thus, usually you need only deal with the leading multipole. Thus, you have a high quality perturbation expansion. By making a simple model, you can show that M is down from E at the same l by same ratio of $(Z \alpha)^2$. That is why E1 sits by itself, then M1 and E2 are same strength in atomic radiations.

In nuclear stuff you take a = few fermis, but k varies over very large range from 10 keV to 10 MeV. Still, it is generally true that electrics are stronger than magnetics by maybe factor 25 to 100. (of the same order, i.e., E1 stronger than M2 M1,). But what about relation between E2 and M1? In atoms these were the same size, but in nuclear M1 tends to be 20 times larger than E2. So the order would be E1, then M1, then E2, and so on if all allowed.

What do Blatt and Weiskopf have to say?

1. First, they clarify the business of "intrinsic magnetization". Recall how this magnetization found its way into Jackson's multipole scalar coefficients. In the wave equations you put a source. Wherever the current J appears, you also want to include the bound current $\text{curl}(M)$. Thus, this M occurs in the various moments Q_{lm} and M_{lm} . For example, suppose you have a neutron irradiated by a plane wave. Since there is no charge density, there will be no electric multipole radiation. (Except possibly in higher order due to Q'_{lm}). There will be magnetic radiation however because you will have an oscillating magnetic moment. Perhaps a plane wave xx . Obviously, this has something to do with spin S of particle because spin is always tied to intrinsic magnetic moment hence to intrinsic "magnetization" which causes radiation.

2. Also in there pre-quantum comments. Blatt Weiskopf observe that the angular distribution of radiation for order (l,m) is the same for either electric or magnetic. See 16.72 of Jackson.

3. In first part of "transition to QM" BW note that radiation actually occurs in quanta which carry $h\nu$. This is a "photon". However, this photon ~~xxx~~ carries angular momentum of l where l is the multipole order. Roughly, this must be the total angular momentum carried off by the photon, which is the vector combination of the photon spin (1 unit) with the orbital angular momentum. This idea comes across in the Heitler type theory below where the exiting photon is put in a plane wave $e^{ik \cdot r}$ which we know is a superposition of orbital angular momentum states. This also supports the idea that a photon cannot carry off angular momentum 0 which you might think possible because $S=1$ and $L=1$ could combine to give 0. The reason must be that $M_J = 0$ is not possible for a photon due to masslessness. Thus, you never get multipole radiation with $l=0$. Always starts with dipole.

So, I will drop my erroneous idea that multipole radiation of order l involves an l -photon process. In fact, only one photon is involved in multipole radiation.

4. Now finally comes the transition where you must somehow replace the classical charge and current distributions which occur in the multipole coefficients. The idea is to replace charge density ρ with $u_f^* u_i$. I.e., a charge density becomes the actual physical overlap of two wavefunctions (orbitals).

Similarly, the current goes over into $u_f^* p u_i$ / where p = momentum operator.

The reason that these quantum mechanical analogues appear is of course that in quantum mechanics, the radiation rate (transition probability) is obtained from perturbation matrix elements $(u_f/H'/u_i)$ where $H' = v \cdot A$ roughly, where A is the vector potential. Now, you might also include in your perturbation Hamiltonian some kind of spin term like $s \cdot B$ and this will then cause the magnetization style terms to appear when you calculate the multipole moments.

Blatt and W then go on to discuss some of the nuclear photo processes in their chapter on nuclear physics and radiation. I will read instead the abbreviated stuff in Segre.

What about Schiff?

He presents some of the Heitler style semiclassical theory of radiation in which the atomic system is quantized but the radiation field is classical. When you do this, you find a matrix element which looks like $\int e^{ik \cdot r} A_{\mu} p_{\mu} u_i$. The exponential comes from the plane wave of the outgoing photon, the $A_{\mu} p_{\mu}$ comes from the interaction hamiltonian. If you replace the exponential by unity in the long wavelength limit (as you integrate over the small atomic orbitals), you clearly see that only electric dipole radiation survives. ie, you can replace the matrix element of p with the matrix element of r and this you at once recognize as dipole electric radiation: $\int r^L u_f^* u_i Y_{LM}$ is general electric multipole.

If you go on to expand the exponential, you next obtain E2 and M1 radiation multipoles, and so on. ie, you expand the photon plane wave in spherical harmonics.

One also sees from this treatment that the transition rate is proportional to the em-field present $(\propto A^2)$. This is usual stimulated emission idea. The idea of spontaneous emission I believe requires full quantization of the radiation field. (ie, "second quantization"). This is the magic business of n going to $n+1$ and ground state energy (quantum fluctuation). Although you really cannot explain this in semi classical, you can fake it. This is what Schiff does in his chapter 11 which has a section on spont emission. Then when he does second quantization at the end of the book, he gets spont to be part of the straightforward QED calculation.

Actually, Schiff is probably a good place to read how to relate the various levels of radiation theory: classical, semiclassical, full QED.

Jackson's Chapter 17

1. He opens by discussing the fact that we have been neglecting reactive effects so far in the book. When an electron changes configuration in a time that is very short, on the order of the characteristic time $\tau = 10^{-24}$ sec for electron, then you must consider reactive effects. For example, an electron that is accelerated by external fields radiates and then responds to its own radiated field!
2. Section 17.2 gives a quick Larmor-based derivation of the Abraham-Lorentz equation of motion, equation 17.9. What is new here is a "damping term" which is proportional to $\dot{v}$. Recall that usual classical damping terms are proportional just to v . The effect of such a $\dot{v}$ term is similar to a standard damping term, as shown later.
3. I now skip to section 17.7 where the Abraham-Lorentz equation is transcribed into a more newtonian-looking form 17.51. By using this form, you get rid of undesired blow-up solutions of the other equation. This "integrodifferential equation" is clearly acausal, as illustrated by the preacceleration example. But you could never make observations on such a small time-scale anyway, and QM uncertainty would swamp such an effect anyway, so no problem. Referred to as : classical microscopica causality violation.
4. Section 17.8 gives the first application: linewidth. A radiating atom has a damping term given by the AL equation and you calculate a universal linewidth called Γ . Also, you find a shift from the natural frequency which is much smaller than the width in this model. However, when you go over into QED, you get a much larger shift than this classical model predicts due to photon loops and vertex corrections etc ("vacuum fluctuations" so-called). This makes the Lamb shift much larger.
5. Section 17.9 deals with atomic scattering of light. You assume an external electric field and put in the AL damping term into the equation of motion. Here, in addition, Davè puts in a standard v -type damping term to account for other forms of damping like thermal effects, vibrational decays, etc. You solve for the electrons motion $x(t)$. From this (ie, from $\dot{x}$) you calculate the radiated field, at the same frequency of course. You then get the cross section for such radiation into Ω and at energy ω . Then 17.67 shows the total radiative cross section (elastic). If you are above the resonance assumed in the model here, but still at reasonably low energy, you recover the Thomson cross section result. (Remember that at high energy you have to use QED to get Klein-Nishina result)(ie, Compton scattering). Dipole sum rules are then mentioned. For notes on skipped sections, see other sheet.

Section 17.1. A criterion for when "reactive" forces are important is given. It states that you can neglect reactive radiation etc. when the mechanical time τ is long compared to $\tau \approx e^2/m$ for particles of interest. In this case, problems can be solved by one of these methods:

a) given charges, what are the fields

b) given fields, how do the charges move

section 17.2. I will model the reactive effect as a force in the usual force equation, called $\vec{F}_{\text{rad}}$. By insisting that the work done by this "force" plus the energy radiated vanish, I find that:

$$\vec{F}_{\text{rad}} = m\tau \ddot{\vec{v}} \quad \text{where } \tau = \text{characteristic time} \quad \tau = \frac{2}{3} \frac{e^2}{mc^3}$$

Then one finds: $m(\dot{\vec{v}} - \tau \ddot{\vec{v}}) = \vec{F}_{\text{ext}}$

Al...

Then, radiation appears as a classical damping of the motion. Because particle does work to radiate, it must slow down, + here it is: the Abraham-Lorentz equation.

As a crude example of application, I suppose a central potential $V(r)$. Easy to compute $\dot{\vec{v}}$ and hence dE/dt . Also, the radiation forces causes an angular loss:

$$\frac{dL}{dt} = \text{torque} = \text{radiative reaction torque} = \vec{r} \times [m\tau \ddot{\vec{v}}] \text{ etc.}$$

This example could explain radiation from a muonic atom cascading down.

section 17.3. In 1903 Abraham + Lorentz concocted a theory to explain their equation. This theory appears in Lorentz 1915 Dover book "Theory of Electrons". I review their argument: let P = total momentum of an electron, assume it has no mechanical part. Then, it is "all fields". A + L argue that: $F_{\text{ext}} + F_{\text{self}} = 0$ so $F_{\text{ext}} = \frac{dp}{dt} = -F_{\text{self}}$ I do not see the basis of this guess, but take as axioms.

Now, $F_{\text{self}} = \int (\rho \vec{E} + \vec{J} \times \vec{B}) d^3x$. Make a model for electron's charge, then compute $\vec{E}$, let $\vec{B} = 0$. Series expand [I get, eventually get results shown in 17.29. The zeroth term is roughly "m" a, where "m" = electrostatic self energy, modulo 4/3. The first term

is the same reaction term we found earlier, $-\frac{2}{3}e^2 \ddot{\vec{a}}$. All higher terms vanish as $a \rightarrow 0$, size of electron. So:

$$(\frac{4}{3}m_0)\ddot{\vec{V}} - \frac{2}{3}\frac{e^2}{c^3}\ddot{\vec{V}} = \vec{F}_{ext}$$

section 17.4. comments on this model? - Dave lists of deficiencies

- a) non-rel.
- b) what holds charge together
- c) if $a=0$, $\ddot{V}=\infty$. If $a \neq 0$, then higher terms must count....
- d) something is wrong? with the mass.

section 17.5 stress problem. Recall that $P_{\mu} = \int d^3x T_{\mu 4}$. J. shows that if there is stress in the rest frame of an electron $T_{33}^{(0)} \neq 0$, then P_{μ} does not transform right under LT. The reason that $T_{33} \neq 0$ in Ab model is clear - a ball of charge does have stress & is unstable.

In 1906 Poincaré remedied both problems by postulating an extra non-EM stress tensor. (hadronic?) such that $S_{\mu\nu} = eT_{\mu\nu} + P_{\mu\nu} =$ full animal. Then $m_e = \int d^3x S_{44}$ and $S_{33} = 0$. Of course Poincaré could not explain this added tensor $P_{\mu\nu}$.

J points out that the mass diverges also in QFT where electron is taken as point so $a=\infty$. Though divergence is not quite so bad. Mentions that renorm procedure does work. Some people have added new vector fields to make things work better.

section 17.6. Earlier we found that EM + Poincaré = covariant, but not the pieces separately. Here J fixes that up so EM now appears covariant. Big deal.

section 17.7. Rewrites ALE as "integrodifferential" equation. Notes violation of microscopic causality, giving "pre-acceleration" as example.

$$m\ddot{\vec{a}} = \int_{-\infty}^{\infty} \ddot{\vec{a}}(t+\tau) F(\tau) d\tau$$

sections 17.8, 17.9 - atomic applications of the ALE: linewidth, res. fluo, etc.

Reduces to usual $\vec{F} = m\ddot{\vec{a}}$ when $\tau = 0$.

contents of these pages.

① Stationary state perturbation theory ($H' \neq H'(t)$)

- first order ΔE shift
- second order ΔE shift
- Zeeman where $H' = \mu_B \vec{B} \cdot \vec{L} = \mu_B B L_z$
- Stark where $H' = e \vec{E} \cdot \vec{r} = e E z$

② Time dependent perturbation theory ($H' = H'(t)$) ("Diac method of varying parameters")

- the S.E. as $\dot{a}_k^{(1)}(t) = (i\hbar)^{-1} \sum_n \langle k | H'(t) | n \rangle a_n^{(0)} e^{-i(E_k - E_n)t/\hbar}$
- start with $\delta_{mn} = a_n^{(0)}$, so $a_k^{(1)}(t) = (i\hbar)^{-1} \int_0^t dt' e^{i(E_k - E_m)t'/\hbar} \langle k | H'(t') | m \rangle$
- get rate $W = (2\pi/\hbar) \delta(k) |\langle k | H'(0) | m \rangle|^2$ "the golden rule"

③ Semiclassical radiation theory: transitions

- use $H' = \frac{q}{c} \vec{A} \cdot \vec{v}$ where $\vec{v}$ = appropriate operator
 - $\vec{v} = \frac{1}{m} \vec{p} = \frac{i\hbar}{m} \vec{\nabla}$ in S.E.
 - $\vec{v} = c \vec{\alpha}$ in Dirac equation
- use radiation = coulomb = transverse gauge $\vec{\nabla} \cdot \vec{A} = 0$ $\phi = 0$ so that
- $\vec{E} = -\frac{1}{c} \dot{\vec{A}}$ $\vec{B} = \text{curl } \vec{A}$
- $\vec{A} = (\text{norm factor}) \cdot \vec{e} \cdot e^{i(\omega t - \vec{k} \cdot \vec{r})}$ for plane wave.
 - $\vec{E} \sim \vec{e}$
 - $\vec{B} \sim \hat{n} \times \vec{e}$ $\hat{n} = \hat{k}$
- for absorption or induced emission, get #. quanta in box from Poissoning / tw to get rate $\propto I(\omega) \int d^3x u_{\vec{k}}^* e^{i\vec{k} \cdot \vec{r}} \cdot \vec{e} \cdot \vec{v} \cdot u_{\vec{k}}(\vec{r})^2$

④ Semiclassical spontaneous emission theory

- in the above, replace $\vec{A}_{\text{ext}}$ with $\vec{A}$ of spontaneously emitted photon. Get same result for rate, except $\langle n \rangle = 1$, not $I(\omega_0)/\hbar\omega_0$.
- alternatively, calculate classical energy flux then rate from classical current $\vec{J}$. Identify this radiation with spont. emission, identify $\vec{J}$ with $\sum_j u_j^*(\vec{r}) \vec{v} \cdot u_j(\vec{r}) d^3r$. Get same result.

⑤ How to get classical radiation

- in coulomb gauge, $\nabla^2 \vec{A} = \frac{4\pi}{c} \vec{J}$, solve to get $\vec{A} = \int \frac{[\vec{J}]}{r} d^3r$
- then get E, B , hence S , hence $dP/d\omega d\Omega$.

Quick Review of Perturbation Theory in Quantum Mechanics and Radiation Theory.

- ① First, "stationary" perturbation theory. Idea is break up $H = H_0 + H'$ where H' is time independent, λ = small dimensionless parameter. Then expand wavefunctions + energies

$$E^{(m)} = E_0 + \lambda E_1 + \lambda^2 E_2 + \dots \quad \text{ie} \quad \langle \psi_0 | H_0 | \psi_0 \rangle = E_0$$
$$\psi = \psi_0 + \lambda \psi_1 + \lambda^2 \psi_2 + \dots$$

- ② When you play around, you find that

$$(\lambda E_1) = \langle \psi_0 | \lambda H' | \psi_0 \rangle$$

Better notation

$$E_m = E_m^{(0)} + \lambda E_m^{(1)} + \lambda^2 E_m^{(2)} + \dots$$
$$\psi_m = \psi_m^{(0)} + \lambda \psi_m^{(1)} + \lambda^2 \psi_m^{(2)} + \dots$$

Then: $\lambda E_m^{(1)} = \langle \psi_m^{(0)} | \lambda H' | \psi_m^{(0)} \rangle$ and $\psi_m^{(1)} \sim \sum_k \frac{\langle \psi_k^{(0)} | H' | \psi_m^{(0)} \rangle}{E_m - E_k} \psi_k^{(0)}$

first order shift

It's always very easy to compute first order shift in an energy level knowing only H' and $\psi_m^{(0)}$.

- ③ The second order energy shift is:

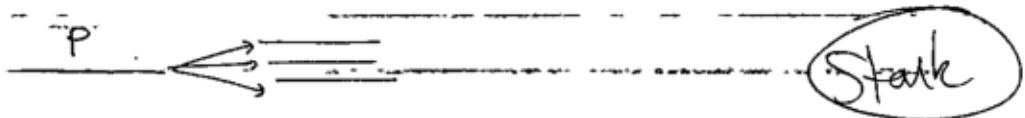
$$\lambda^2 E_m^{(2)} = \sum_n \frac{\langle m | H' | n \rangle \langle n | H' | m \rangle}{E_m - E_n}$$

- ① If two states are degenerate under H_0 , correction $\psi_m^{(1)}$ will blow up unless $\langle m' | H' | m \rangle = 0$. Thus, you have to choose degenerate states which diagonalize H' before applying pert. theory. If your two levels have same value of $\langle m | H' | m \rangle$, then they are still degenerate thru first order. Will only be removed in second order.

- ② Example: put hydrogen atom in B-field. Then $H' = \mu_B \cdot B \cdot L$.
 $= \mu_B B L_z$. Then

$$\langle m | H' | m \rangle = \mu_B \cdot m B \quad \text{(Zeeman)}$$

This splits levels according to their magnetic quantum numbers.



⑥ Example: electric field on hydrogen. $H' = eEz$ has odd parity. Thus, this H' has no diagonal matrix elements in hydrogen since Ψ_{lm} have definite parity. Thus, Stark is second order only! This is only true for a nondegenerate state. The $n=2$ level of hydrogen does get split in first order.

Schiff spends ~ 30 pages on stationary pert theory. On page 279 he turns to

time dependent perturbation theory

① Let $H = H_0 + H'(t)$. The fact that H depends on t means that energy is not constant of motion $\Rightarrow$ no true stationary states. But H_0 has stat. states! Three separate ways to treat this

A) sudden approx

B) adiabatic approx

C) "time dep. pert. theory" ✓

② Examine the third. Let $\Psi = \sum a_n e^{iE_n t/\hbar} \phi_n(H_0)$

③ Then Schrodinger equation says

$$\frac{da_k(t)}{dt} = \frac{1}{i\hbar} \sum_n \langle k | H' | n \rangle a_n e^{i\omega_{kn}t}$$

↑ inject dependence.
↓
time dep of a_n , so
Schrodinger Picture

④ Now we expand these a^s :

$$a_m(t) = a_m^{(0)} + \lambda a_m^{(1)} + \dots$$

↑ no time dep., i.e., these are coeffs if you turn H' off.

Thus, get

$$\dot{a}_k^{(1)}(t) = \frac{1}{i\hbar} \sum_n \langle k | H' | n \rangle a_n^{(0)} e^{i\omega_{kn}t}$$

The SE.

⑤ Now start off with $\psi = \text{some } a_m$ so $a_m^{(0)} = \delta_{m,m}$. Then integrate to get

$$a_k^{(1)}(t) = (i\hbar)^{-1} \int_{-\infty}^t \langle k | H'(t') | m \rangle e^{i\omega_{km}t'} dt'$$

At $t = -\infty$, $a_k^{(1)}(t) = 0$, but starts to gain occupation as time proceeds. Not linear in time, in general.

⑥ But, suppose you run $H' = \sim \sin(\omega t)$ from $t = -t_0$ to t_0 , then deactivate. Then after $t = t_0$, $a_k^{(1)}$ takes on this constant value

$$a_k^{(1)}(\text{after pulse over}) = \left(\frac{1}{\omega_{km} + \omega} - \frac{1}{\omega_{km} - \omega} \right)$$

Thus, $a_k^{(1)}$ can be large if perturbation was tuned to level separation between m and k . Either emission or absorption is allowed hence interpretation



Assume $\omega \approx \omega_{km}$, then square:

$$|a_k^{(1)}(\text{after pulse of length } 2t_0)|^2 \approx \frac{\sin^2 \frac{1}{2}(\omega_{km} - \omega)t_0}{(\omega_{km} - \omega)^2}$$

probability into some final state.

⑦ Transition tends to occur if ω is "near ω_{km} ". Suppose we are thinking emission. Then "space" will accept any ω in proportion to phase space. So sum over all emissions to get

total prob = $\sum_{\omega \neq k} |a_k^{(1)}|^2 g(\omega) d\omega \approx \pi \cdot t_0$

not $\approx |H'|^2 \cdot g(\omega_{km}) \cdot \left[\int d\omega \cdot \frac{\sin^2}{\dots} \right]$

- ③ Here we get idea that prob. total $\propto t$, so get rate and golden rule

$$w = \text{rate} = \frac{2\pi}{\hbar} g(k) |\langle k | H' | m \rangle|^2$$

- ④ For states in a box, Schiff says $g(k) = \frac{\mu L^3}{8\pi^3 \hbar^2} k \cdot d\Omega$ where $\mu \equiv$ mass of electron ejected into box when H is ionized by a perturbation $H' = e z E_0 \sin \omega t$ where $\omega =$ large enough to i.e. $\hbar \omega > 13.6 \text{ eV}$. OK, but. This is not a photon emission example.

- ⑤ Second order perturbation theory gives $2w = w_{km}$
These are two-quanta processes like two photon absorption - a Compton.

- ⑥ We now skip to page 400 of Schiff where we get into radiation theory.

Idea is this: you have a particle in em field characterized by A^μ . Take $p \rightarrow p - eA$ in Hamiltonian to obtain this fact:

$$H' = \frac{ie\hbar}{mc} \vec{A} \cdot \vec{\nabla}$$

- ① In the radiation = coulomb = transverse gauge $\vec{\nabla} \cdot \vec{A} = 0$, $\phi = 0$, so only $\vec{A}$ is present. Recall that even $\vec{E}$ field can be given by $\vec{A}$:

$$\vec{E} = -\frac{1}{c} \frac{\partial \vec{A}}{\partial t} \quad \vec{B} = \vec{\nabla} \times \vec{A}$$

So $H' = \frac{ie\hbar}{mc} \vec{A} \cdot \vec{\nabla} = \frac{e}{mc} \vec{A} \cdot \vec{p} = \frac{e}{c} \vec{A} \cdot \vec{v}$ velocity

is the standard perturbation Hamiltonian.

- ② We take $\vec{A}$ to be a plane wave

$$\left(\vec{A} \equiv \vec{A}_0 \cdot e^{i(\vec{k} \cdot \vec{r} - \omega t)} + \text{c.c.} \right)$$

Therefore $\vec{E}, \vec{B}$ are plane waves, and $\langle |S| \rangle = \frac{\omega^2}{2\pi c} |\vec{A}_0|^2 =$ energy flux pointing.

③ Now the relevant $\langle f | H' | i \rangle$ in t.d. pert. theory is

$$\langle k | H' | m \rangle = \int d^3x \cdot u_k^*(x) \cdot \left[\frac{ie\hbar}{mc} \{ \vec{A}_0 e^{i\vec{k}\cdot\vec{r}} \} \cdot \vec{\nabla} \right] u_m(x)$$

sign

The sign depends on whether absorption & emission.

④ We now use the rule:

$$\text{prob} = |a_k^{(1)}|^2 = 4 \cdot |\langle k | H' | m \rangle|^2 \cdot \frac{\sin^2 \frac{1}{2}(\omega_{km} - \omega)t_0}{\hbar^2 (\omega_{km} - \omega)^2}$$

↑
residual from $H' = H_1 - H_2$

again, we assume some kind of constant spectrum of radiation present in external, spectrum is $I(\omega)$. Then integrate etc to find

$$\text{transition rate} = \frac{4\pi^2 e^2}{m^2 c \omega_{kn}^2} I(\omega_{kn}) \left| \sum_k u_k^* e^{\pm i\vec{k}\cdot\vec{r}} \nabla_A u_n \right|^2$$

- spectrum external.

↑ for up
↓ for down

where $\nabla_A = \frac{\partial}{\partial x_n}$ where x_n is in direction of external $\vec{A}_0$.

⑤ In long wavelength limit, we see how $|\langle k | p_A | n \rangle|^2$ will appear, the dipole moment along $\vec{A}$ direction. This is electric dipole radiation, i.e., $|\langle k | p_A | n \rangle|^2$.

⑥ Detail: what really appears in dipole is $|\langle k | p_A | n \rangle|^2$, which is $|\langle k | z | n \rangle|^2$ if you take $\vec{A}_0$ along z . But, k, n are isotropic ~ well, somehow if you average over all directions of $\vec{A}_0$, then $\langle |z|^2 \rangle = \langle |y|^2 \rangle = \langle |x|^2 \rangle$ so

$$|\langle k | p_A | n \rangle|^2 \rightarrow \frac{1}{3} |\langle k | \vec{r} | n \rangle|^2$$

but only for average situations.

Now we come to spontaneous emission

① We know that a classical $\vec{J}$ will radiate if $\vec{J} \sim e^{i\omega t}$.
Rule says $\vec{J} \rightarrow \sum_n \vec{J}_n e^{i\omega_n t} =$ matrix element
of current operator.

② The classical radiated flux is:

$$S = \text{power} = \text{flux} = \frac{k^2}{2\pi r^2 c} \left| \int \vec{J}_{\perp k}(\vec{r}) e^{-ik \cdot \vec{r}} d^3 r \right|^2$$

This is flux radiated by a classical current,

③ Aside on this matter: in the Lorenz gauge where $\partial_\mu A^\mu = 0$, you
get $\square \vec{A} = \vec{J}$. In section 6.6, Jackson shows
how you solve this equation to get

$$\vec{A}(\vec{r}, t) = \int \frac{d^3 r'}{|\vec{r} - \vec{r}'|} [\vec{J}(\vec{r}', t')]_{\text{ret}}$$

Te, you always get causal retardation. $t' = t - r/c$.
Back to p 269 J. Assume $\vec{J} \sim \sin \omega t$. Then

$$\vec{A} = \vec{A}(\vec{r}) = \int \frac{\vec{J}(\vec{r}') e^{ik|\vec{r} - \vec{r}'|}}{|\vec{r} - \vec{r}'|} d^3 r'$$

This gives the vector potential due to charge. Te, think
of radiation. Far away, you tend to get

$$\vec{A}(\vec{r}) \approx \frac{1}{r} \int \vec{J}(\vec{r}') e^{ik(\vec{r} - \vec{r}') \cdot \hat{n}} d^3 r'$$

Then if you take the 1, $\vec{A}$ corresponds to dipole
electric emission because

$$\begin{aligned} \int \vec{J} d^3 r &= - \int \vec{r} (\vec{\nabla} \cdot \vec{J}) d^3 r = - \int \vec{r} \rho d^3 r = \vec{p} \\ &= -i\omega \vec{p} \end{aligned}$$

↑ dipole moment

- ④ Here we see classically why a radiating current $\vec{J} = \vec{J}_0 e^{i\omega t}$ is electric dipole radiation:

$$\vec{A} \sim \frac{1}{r} \int \vec{J} d^3r \sim -i\omega \left(\frac{e}{\pi} \right) \cdot \vec{p}$$

The vector potential is roughly equal to the dipole moment of your charge distribution which one obtains from $\vec{J}$ by continuity, of course $\vec{p}$ could vanish.

⑤ So, $\vec{A} \approx \frac{1}{rc} \int d^3x \vec{J}(\vec{x}) \cdot e^{-ik|\vec{r}-\vec{x}|}$
 ← note $\frac{1}{r}$. ↑ a phase due to distance from point in source.

This means that

$$\begin{aligned} \vec{E} &\sim \omega \vec{A} \\ \vec{B} &\sim \omega \hat{k} \times \vec{A} \end{aligned} \quad k|\vec{r}-\vec{x}|$$

So $|\vec{S}_{\text{into } k}| \sim \frac{c}{4\pi} \cdot \omega^2 \cdot |\vec{A}|^2$

Wait: $\vec{E} \times \vec{B} \sim \omega^2 \cdot \vec{A} \times [\hat{k} \times \vec{A}]$



90° cause $\vec{S}$ is $\perp$ to $\vec{A}$ $\sim \vec{J}$

$$\vec{S} \neq \vec{A} \cdot \vec{B}$$

⑥ Thus, I have verified that:

energy flux out of $\vec{J}$ into $\hat{k} = \frac{k^2}{r^2} \cdot \frac{1}{2\pi c} \left| \int \vec{J}(\vec{r}') \cdot e^{ik\hat{k} \cdot \vec{r}'} d^3r' \right|^2$

The significance of this is: let $\vec{J} \rightarrow \vec{J}_{\text{em}} = \sum_k \vec{J}_k^* \cdot \vec{\nabla} \cdot \vec{u}_k$.
 Then

transition rate = $\frac{\text{energy flux}}{\hbar\omega} = \frac{k^2}{r^2} \cdot \frac{1}{2\pi c} \left| \int d^3r' u_k^*(\vec{r}') \cdot \vec{\nabla} u_k(\vec{r}') \right|^2$
 ("spont. em")

⑦ Then you can take dipole limit to get the dipole spont. emission rate.

→ in effect, you add a 1 to n in induced formulae.

Comments on the spin-flip synchrotron radiation

1. First some remarks on classical theory. If you take some charge and make it accelerate, you get radiation. There are several ways to analyze this. One way is to compute the vector potential A directly from the current J , then get the fields E and B from A , then get radiated power spectrum $dI(\omega)/d\Omega d\omega$ and so on. This is how you compute radiation pattern from an antenna (see Jackson chapter 9) of radiation from a charge (see Jackson chapter 14). This might be called "charge radiation". However, suppose you have a magnetic moment which accelerates, but you have no free current as such. Well obviously that magnetic moment can be simulated by a current according to $\mathbf{J} = \text{curl}(\mathbf{M})$, and this current is being accelerated and hence radiates. This radiation will tend to be $M\omega$, or "magnetic radiation" whereas the charge radiation tends to be $E\omega$. In either case, you can get the relativistic radiation pattern by doing a simple time integral of the path then squaring (this is for a particle) where the particle velocity appears (for charge radiation) and its magnetic moment appears for magnetic radiation. Generally speaking, if a particle has both charge and magnetic moment, the charge radiation dominates.

2. Jackson reviews the notion of regular synchrotron radiation as spontaneous emission, as if the accelerator were a huge atom. You start by pretending there is an external A field and take $H' = A \cdot v$. I.e., as if A were an external perturbation (time dependent) which is causing a transition. But then you let A be the potential of the spontaneously emitted photon itself. This results in a transition rate and radiation spectrum shown in (33), i.e., the QM matrix element appears $(\mathbf{r}/c) \cdot \sqrt{\omega} e^{i\mathbf{k} \cdot \mathbf{r}}$ inside a time integral, all done in Heisenberg picture. But this can be taken at once to classical limit and QM disappears and you are left with the classical velocity (instead of a matrix element of the Heisenberg velocity operator). This is of course justified because the accelerator orbit has such a huge $n = 10^{17}$. Thus, we recover via theory of spontaneous emission the classical "charge radiation" of synchrotrons. The spin of the radiating electron is irrelevant to this business of charge radiation. Notice how the QM matrix element of velocity was replaced with the classical velocity.

3. Jackson's text does on page 481 give a classical formula for radiation by a fast moving dipole moment. However, the magnetic dipole moment of the moving charge seems to be always in an eigenstate of mag moment. I.e., you see μ as a classical object in the equation. In QM this is like $(\uparrow/\mu/\uparrow)$. The mag moment is always "up" because classical physics does not know about spin flips.

4. To arrive at something reasonable for the magnetic moment radiation, you need something with more QM in it. Jackson takes over his previous spontaneous emission argument but goes with $H' = g\mu_B \vec{S} \cdot \vec{B}$. As before, you first imagine that B is an external time dependent perturbing magnetic field (like from a plane wave) which is inducing transitions between unperturbed levels. But then you let B = field of spontaneously radiated photon. Then in direct analogy you can compute the radiation pattern and lifetime of the "upper" level using perturbation theory. You get nearly the right answer for lifetime, but predicts 100% polarization of course. The reason is that the spin is not at rest really in your comoving frame and one must include Thomas precession terms. This yields the effective interaction Hamiltonian given in equation (46). Note that $a=0$ is for $g=2$. So all you need do is grind through your perturbation theory with this hamiltonian. The net effect is shown in (46) where you replace the velocity operator of charge radiation with a spin operator. The electric polarization $\vec{e}$ is of course more or less replaced with the magnetic polarization $\vec{A} \times \vec{e}$. The result is schematically given in (47) and completely stated in (55) with all Thomas terms.

5. Now, this "spin radiation" has both spin-flip and spin-nonflip pieces. The important point is that only the spin flip pieces can affect the beam polarization; non-flip transitions leave the beam where it was. So you simply compute the total rate of transition from up to down, and down to up (Jackson calls this $\tilde{\epsilon}$). These rates are not equal, as equation (57) shows, and the obvious result is a polarization. (59). Since the rate of these spin radiations is rather low, you have to wait a while so that enough of these transitions occur to get the beam to a steady state for up versus down transitions.

Recent QED Tests
Strauch
(1973)

Strauch

2 Quantum electrodynamic processes

Strauch's Bonn 1973 talk.
— recent QED tests.

The three reactions of order α^2 for the study of the validity of QED with e^+e^- colliding beams are:

$$e^+ + e^- \rightarrow e^+ + e^-, \quad (1)$$

$$e^+ + e^- \rightarrow \gamma + \gamma, \quad (2)$$

$$e^+ + e^- \rightarrow \mu^+ + \mu^-. \quad (3)$$

The QED calculations for these processes are based on the assumption that (1) leptons behave like point-like Dirac particles; (2) Maxwell's equations are valid, the photon propagator is simply given by $1/q^2$; (3) diagrams of the order α^2 dominate in the calculations, and contributions from higher order diagrams can be taken care of by the usual methods of radiative corrections. The validity of these assumptions has been tested with increasing accuracy and over an increasingly large energy region in experiments reported at this Symposium from Frascati and CEA: let me say right away that ~~no deviations have been found~~. I will not discuss reaction (3) since no new results have become available since last year's Chicago Conference.

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Reactions (2) and (3) contain no e^+ or e^- in the final state and are thus "annihilation reactions"; they do not occur in e^-e^- collisions. As will be discussed below, the Bhabha scattering reaction (1) involves an additional annihilation diagram not present in the e^-e^- elastic scattering. Reaction (3) is the only one of the three to uniquely involve a time-like photon.

Interesting results on the validity of QED and on the search for heavy leptons are reported at this symposium from studies of the α^3 order Bremsstrahlung reaction:

$$e^+ + e^- \rightarrow e^+ + e^- + \gamma, \quad (4)$$

Three QED reactions of order α^4 have been studied in e^+e^- colliding beam systems:

$$e^+ + e^- \rightarrow e^+ + e^- + \gamma + \gamma, \quad (5)$$

$$e^+ + e^- \rightarrow e^+ + e^- + e^+ + e^-, \quad (6)$$

$$e^+ + e^- \rightarrow e^+ + e^- + \mu^+ + \mu^-. \quad (7)$$

The double Bremsstrahlung reaction (5) was first studied at Novosibirsk [5] and has since served as a convenient luminosity monitor for medium luminosity experiments. New interesting results are reported at this symposium for reactions (6) and (7) from experiments at Frascati in which the inelastically scattered electron and positron are detected.

In reactions (4)–(7) the final state contains the initial state leptons plus one or more particles or photons. These reactions occur both in e^+e^- and in e^-e^- collisions. The name "electroproduction" is traditionally used to describe the production of particles by electrons (or positrons) inelastically scattered from a target; reactions (4)–(7) are examples of electroproduction in contrast to reactions (2) and (3) which are annihilation reactions and only occur in e^+e^- collisions.

2.1 Bhabha scattering $e^+e^- \rightarrow e^+e^-$

The elastic scattering process $e^+e^- \rightarrow e^+e^-$ is described by two Feynman diagrams:

$$\begin{array}{cc}
 \begin{array}{c} e^- \\ \diagdown \\ \text{---} \text{---} \text{---} \\ \diagup \\ e^- \end{array} & \begin{array}{c} e^+ \\ \diagup \\ \text{---} \text{---} \text{---} \\ \diagdown \\ e^+ \end{array} \\
 q^2 = -s \sin^2 \frac{\theta}{2} & q^2 = s
 \end{array}$$

For experiments which do not measure the charge of the lepton (the only ones reported to date) the annihilation diagram (on the right) plays its most significant role at the scattering angle of $\theta = 90^\circ$. ~~All the experiments detect electrons and positrons scattered into a solid angle centered around $\theta = 90^\circ$.~~

At Frascati the luminosity is measured using small angle Bhabha scattering; at CEA the double Bremsstrahlung process is used for this purpose. Both monitors detect events dominated by low q^2 space-like processes where it is reasonable to assume that QED is valid [6].

The most accurate experiment spanning the region of $\sqrt{s} = 1.2\text{--}3.0$ GeV has been carried out at Frascati by the BCF group [7]. Their results are based on 12,827 events scattered into the region $45^\circ < \theta < 135^\circ$ and distributed among 11 values of $\sqrt{s}$. The authors fit their beautiful results shown in fig. 2 with the expression: