

UIRs of $SO(3,1) = SL(2, \mathbb{C})$

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DECOMPOSITION OF $SL(2, \mathbb{C})$ WITH RESPECT
TO $SU(1,1)$

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University of Cambridge,
England.

October 1968.

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ABSTRACT

We perform the decomposition of $SL(2, C)$ with respect to $SU(1, 1)$ and evaluate the matrix elements in the implied basis of a boost not contained in the subgroup. These functions contain the singularities associated with the $SU(2) - SU(1, 1)$ overlap functions; we use them to give a decomposition of a Lorentz pole in a natural way.

Group properties of the New $\tilde{d}$ -functions for $SU(2)$.

D. Definition of these functions:

$$\tilde{d}_{m'm}^j(z) = \Gamma(j+1+m)\Gamma(j+1-m') \tilde{P}_{m'm}^j(z)$$

1. Addition Theorem:

$$\tilde{d}_{m'm}^j(g) = \sum_{n=-j}^j \tilde{d}_{m'n}^j(g_1) \tilde{d}_{n'm'}^j(g_2) / H_{nm}^j$$

2. Completeness on the interval:

$$2\delta(z_1 - z_2) = \sum_{j=\max}^{\infty} (2j+1) \tilde{d}_{mn}^j(z_1) \tilde{d}_{nm}^j(z_2) / H_{nm}^j$$

3. Completeness on the group with 4PI azimuths:

$$\delta(g_1 - g_2) = \sum_{jmn} (2j+1) \tilde{d}_{mn}^j(g_1) \tilde{d}_{mn}^j(g_2) / H_{mn}^j$$

4. Meaning of inverse group element:

$$g_2^{-1} = (\pi - \phi_2, \Theta_2, \pi - \phi_2)$$

$$\tilde{d}_{mn}^j(g_2^{-1}) = \tilde{d}_{-m-n}^j(g)$$

5. Orthogonality of the UIR's:

$$\int dg \tilde{d}_{mn}^j(g) \tilde{d}_{m'n'}^{j'}(g^{-1}) = \delta_{jj'} \delta_{m'm} \delta_{n'n} \cdot \frac{H_{mn}^j}{(2j+1)}$$

6. Basic data: same as before, only new stuff is:

$$\tilde{d}_{mn}^j(z) = (-1)^{m-n} \tilde{d}_{nm}^j(z) = \tilde{d}_{-n-m}^j(z)$$

7. Standard expansion on the group:

$$f(g) = \sum_{jmn} (2j+1) \tilde{d}_{mn}^j(g) F_{mn}^j$$

$$F_{mn}^j = \int dg f(g) \tilde{d}_{mn}^j(g) / H_{mn}^j$$

8. Integral of product theorem:

$$\int dg f(g) h(g^{-1}) = \sum_{jmn} (2j+1) f_{mn}^j h_{mn}^j \cdot H_{mn}^j$$

Group Properties in terms of the $\tilde{P}$ -functions for $SU(2)$.1. Addition Theorem:

$$\tilde{P}_{mn}^j(g) = \sum_{n=-j}^j \tilde{P}_{mn}^j(g_1) \tilde{P}_{nm}^j(g_2)$$

2. Completeness on the interval:

$$2\delta(z_1 - z_2) = \sum_{j=\max}^{\infty} (2j+1) \tilde{P}_{mn}^j(z_1) \tilde{P}_{mn}^j(z_2)$$

3. Completeness on the group with 4PI azimuthal intervals:

$$\delta(g_1 - g_2) = \sum_{j=0, \frac{1}{2}, 1, \dots} (2j+1) \cdot \sum_{m, n=-j}^j G_{mn}^j \tilde{P}_{mn}^j(g_1) \tilde{P}_{mn}^j(g_2^{-1})$$

4. The meaning of the inverse group element:

$$g_2 = (\phi_2, \Theta_2, \phi_2') \Rightarrow g_2^{-1} = (\pi - \phi_2', \Theta_2, \pi - \phi_2)$$

$$\Rightarrow \tilde{P}_{mn}^j(g_2^{-1}) = G_{mn}^j \tilde{P}_{-m, -n}^j(g_2)$$

5. Orthogonality of UIR's:

$$\int dg \tilde{P}_{mn}^j(g) \tilde{P}_{m'n'}^{j'}(g^{-1}) = \delta_{jj'} \delta_{mm'} \delta_{nn'} \frac{G_{mn}^j}{(2j+1)}$$

6. Standardization and basic data:

$$\delta(g_1 - g_2) = 4\pi \delta(\phi_1 - \phi_2) \cdot 2\delta(z_1 - z_2) \cdot 4\pi \delta(\phi_1' - \phi_2')$$

$$\int dg = \int_0^{4\pi} \frac{d\phi_1}{4\pi} \cdot \int_{-1}^1 \frac{dz_1}{2} \cdot \int_0^{4\pi} \frac{d\phi_1'}{4\pi}$$

$$\tilde{P}_{mn}^j(g) = e^{-im\phi} \tilde{P}_{mn}^j(z) e^{-in\phi'}$$

$$g = (\phi, \Theta, \phi'), z = \cos\Theta$$

$$\tilde{P}_{mn}^j(z) = (-1)^{m-n} G_{mn}^j \tilde{P}_{nm}^j(z)$$

$$\tilde{P}_{mn}^j(z) = \tilde{P}_{-n-m}^j(z)$$

7. Expansion on the group:

$$f(g) = \sum_{j=0, \frac{1}{2}, \dots} (2j+1) \sum_{m, n=-j}^j G_{mn}^j F_{mn}^j \tilde{P}_{mn}^j(g_2^{-1})$$

$$F_{mn}^j = \int dg f(g) \tilde{P}_{mn}^j(g)$$

8. Product integral theorem:

$$\int dg f(g) h(\bar{g}) = \sum_{j=0, \frac{1}{2}, \dots} (2j+1) \sum_{m, n=-j}^j G_{mn}^j F_{mn}^j H_{mn}^j$$

Completeness Relation for Rotation Group.

1. First, we know that this is true:

$$\sum_{m=0, \pm 1, \pm 2}^{\pm \infty} e^{im\phi} = 2\pi \delta(\phi) \quad 0 < \phi < 2\pi$$

If I had to prove this, I would do the greens function thing for simple ODE on this interval, but dont bother because everybody knows it is true.

2. Now change variables slightly:

$$n \equiv \frac{1}{2} m \quad \psi \equiv 2\phi \quad \Rightarrow \quad m\phi = n\psi$$

$$\Rightarrow \sum_{n=0, \pm \frac{1}{2}, \pm 1, \dots}^{\pm \infty} e^{in\psi} = 4\pi \delta(\psi) \quad 0 < \psi < 4\pi$$

This is the correct completeness relation for functions of azimuth in this 4PI range. I think I screwed up in my LBL by doing either/or, rather than both at once.

3. Consider this change of variables:

$$\begin{aligned} & \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \sum_{j=\max(|m|, |n|)}^{\infty} F_{mn}^j \\ & \quad \uparrow \quad \uparrow \\ & \quad \text{in the above full sense} \\ & = \sum_{j=0, \frac{1}{2}, 1, \frac{3}{2}, \dots}^{\infty} \sum_{n=-j}^j \sum_{m=-j}^j F_{mn}^j \\ & = \sum_{\epsilon=0, \frac{1}{2}}^{\infty} \sum_{j=\epsilon}^{\infty} \sum_{n=-j}^j \sum_{m=-j}^j F_{mn}^j \end{aligned}$$

4. Next, consider the completeness relation for the P functions:

$$2\delta(z_1 - z_2) = \sum_{j=\max}^{\infty} (2j+1) \tilde{P}_{mn}^j(z_1) \tilde{P}_{-m-n}^j(z_2)$$

$\uparrow$ $e^{-im(\phi_1 - \phi_2)} e^{-in(\phi_1' - \phi_2')}$ $\uparrow$ $e^{-im\phi_1}$ $e^{-in\phi_1'}$ $e^{+im\phi_2}$ $e^{+in\phi_2'}$

Add the exponential factors on both sides, then do the full sums on both sides as described above. Result:

$$\delta(g_1 - g_2) = \sum_{j=0, \frac{1}{2}, 1, \dots}^{\infty} (2j+1) \tilde{P}_{mn}^j(g_1) \tilde{P}_{-m-n}^j(g_2)$$

where $\delta(g) = \frac{\delta(\phi)}{4\pi} \delta(z)$

$$\sum_{m=-j}^j \sum_{n=-j}^j$$

$$\delta(g_1 - g_2) = \frac{\delta(\phi_1 - \phi_2)}{4\pi} \cdot 2\delta(z_1 - z_2) \cdot 4\pi \delta(\phi'_1 - \phi'_2)$$

On:

$$\delta(g_1 - g_2) = 4\pi \delta(\phi_1 - \phi_2) \cdot 2\delta(z_1 - z_2) \cdot 4\pi \delta(\phi'_1 - \phi'_2)$$

5. Now make these changes:

$$\tilde{P}_{-m, -n}^j(g_2) = (-1)^{m-n} \tilde{P}_{nm}^j(g_2^{-1}) = G_{nm}^j \tilde{P}_{mn}^j(g_2^{-1})$$

↑ arises from $g_2^{-1} = (\pi - \phi'_2, \theta_2, \pi - \phi_2)$

(-1) goes away in $\tilde{P}$ switch!

So $\tilde{P}_{-m, -n}^j(g_2) = G_{nm}^j \tilde{P}_{mn}^j(g_2^{-1})$

$$\Rightarrow \delta(g_1 - g_2) = \sum_{j=0, \frac{1}{2}, \dots}^{\infty} (2j+1) \sum_{m=-j}^j \sum_{n=-j}^j G_{nm}^j \tilde{P}_{mn}^j(g_1) \tilde{P}_{mn}^j(g_2^{-1})$$

6. Now we can find the desired orthogonality relation:

$$\begin{aligned} \int dg_1 \tilde{P}_{m'n'}^{j'}(g_1^{-1}) \delta(g_1 - g_2) &= \tilde{P}_{m'n'}^{j'}(g_2^{-1}) \\ &= \sum_j (2j+1) \sum_{m,n} G_{nm}^j \tilde{P}_{mn}^j(g_2^{-1}) \left[\int dg_1 \tilde{P}_{m'n'}^{j'}(g_1^{-1}) \tilde{P}_{mn}^j(g_1) \right] \\ &= \frac{\delta_{jj'} \delta_{mm'} \delta_{nn'}}{(2j+1) G_{nm}^j} \end{aligned}$$

$$\Rightarrow \int dg \tilde{P}_{mn}^j(g) \tilde{P}_{m'n'}^{j'}(g^{-1}) = \frac{\delta_{jj'} \delta_{mm'} \delta_{nn'}}{(2j+1)} G_{nm}^j$$

$$dg = \frac{d\phi}{4\pi} \cdot \frac{dz}{z} \cdot \frac{d\phi'}{4\pi}$$

7. Now consider:

$$\begin{aligned}
 I &= \int dg f(g) h(\bar{g}^{-1}) \\
 &= \int dg \left[\sum_{jmn} (2j+1) G_{mn}^j f_{mn}^j \tilde{P}_{mn}^j(\bar{g}^{-1}) \right] \left[\sum_{j'm'n'} (2j'+1) \right. \\
 &\quad \left. \times G_{m'n'}^{j'} h_{m'n'}^{j'} \tilde{P}_{m'n'}^{j'}(g) \right] \\
 &= \sum_{\substack{jmn \\ j'm'n'}} (2j+1) G_{mn}^j (2j'+1) G_{m'n'}^{j'} f_{mn}^j h_{m'n'}^{j'} \\
 &\quad \times \left[\int dg \tilde{P}_{m'n'}^{j'}(g) \tilde{P}_{mn}^j(\bar{g}^{-1}) \right] \\
 &= \frac{G_{mn}^j}{(2j+1)} \delta\delta\delta
 \end{aligned}$$

$$\Rightarrow \boxed{\int dg f(g) h(\bar{g}^{-1}) = \sum_{jmn} (2j+1) G_{mn}^j f_{mn}^j h_{mn}^j}$$

second-kind completeness relation in P, Q notation:

① Start with:

$$\delta(ch v_1 - ch v_2) = \frac{1}{2\pi i} \oint_{|z_j| > 1} dz_j (z_j + 1) \underbrace{P_{\mu, \mu'}^j(ch v_1)}_{= P_{-\mu, -\mu'}^j(ch v_1)} \underbrace{Q_{\mu, \mu'}^j(ch v_2)}_{= Q_{-\mu, -\mu'}^j(ch v_2)}$$

Apply $e^{+\mu \xi_1} e^{+\mu' \xi_1'} e^{-\mu \xi_2} e^{-\mu' \xi_2'}$

Then:

$$e^{+\mu(\xi_1 - \xi_2)} e^{+\mu'(\xi_1' - \xi_2')} \delta(ch v_1 - ch v_2) = \frac{1}{2\pi i} \oint dz_j (z_j + 1) \left[e^{+\mu \xi_1} P_{-\mu, -\mu'}^j(ch v_1) e^{+\mu' \xi_1'} \right] \times \left[e^{-\mu \xi_2} Q_{\mu, \mu'}^j(ch v_2) e^{-\mu' \xi_2'} \right]$$

② Now define these two functions over S_0^+ ; $g = (\xi, v, \xi')$

$$P_{\mu, \mu'}^j(g) \equiv e^{-\mu \xi} P_{\mu, \mu'}^j(ch v) e^{-\mu' \xi'}$$

$$Q_{\mu, \mu'}^j(g) \equiv e^{-\mu \xi} Q_{\mu, \mu'}^j(ch v) e^{-\mu' \xi'}$$

$$\text{So } \left[\begin{array}{c} \vdots \\ \vdots \end{array} \right] \left[\begin{array}{c} \vdots \\ \vdots \end{array} \right] = P_{-\mu, -\mu'}^j(g_1) Q_{\mu, \mu'}^j(g_2)$$

③ Next apply $(-i) \oint du$ $(-i) \oint du'$ to get deltas. So:

$$\delta(g_1 - g_2) = (-i) \oint du (-i) \oint du' \frac{1}{2\pi i} \oint dz_j (z_j + 1) P_{-\mu, -\mu'}^j(g_1) Q_{\mu, \mu'}^j(g_2)$$

where

$$\delta(g_1 - g_2) \equiv 2\pi \delta(\xi_1 - \xi_2) \delta(ch v_1 - ch v_2) 2\pi \delta(\xi_1' - \xi_2')$$

$$dg = \frac{d\xi}{2\pi} d(ch v) \frac{d\xi'}{2\pi}$$

$$\int_{S_0^+} dg = \int_{-\infty}^{\infty} \frac{d\xi}{2\pi} \int_0^{2\pi} dv / \sinh v \int_{-\infty}^{\infty} \frac{d\xi'}{2\pi}$$

② This implies

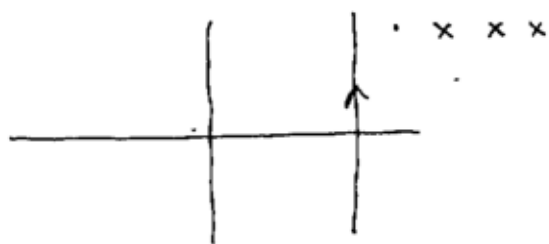
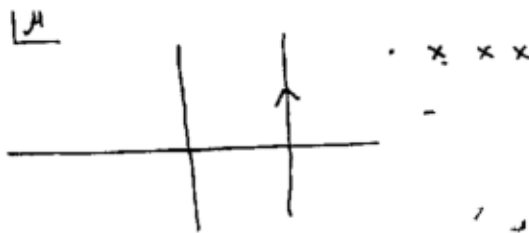
$$f_{\mu\nu}^j = \int_{S_0^+} dg_2 f(g_2) \Phi_{\mu\nu}^j(g_2)$$

$$f(g_1) = \frac{1}{2\pi i} \oint dj(2j+1) \text{ "trace" } P_{-\mu-\mu'}^j(g_1) f_{\mu\nu}^j$$

where

$$\text{"trace"} = (-i) \oint du (-i) \oint du'$$

④ Contours: move j -contour to the right, then shift helicity contours to the right to end up with:



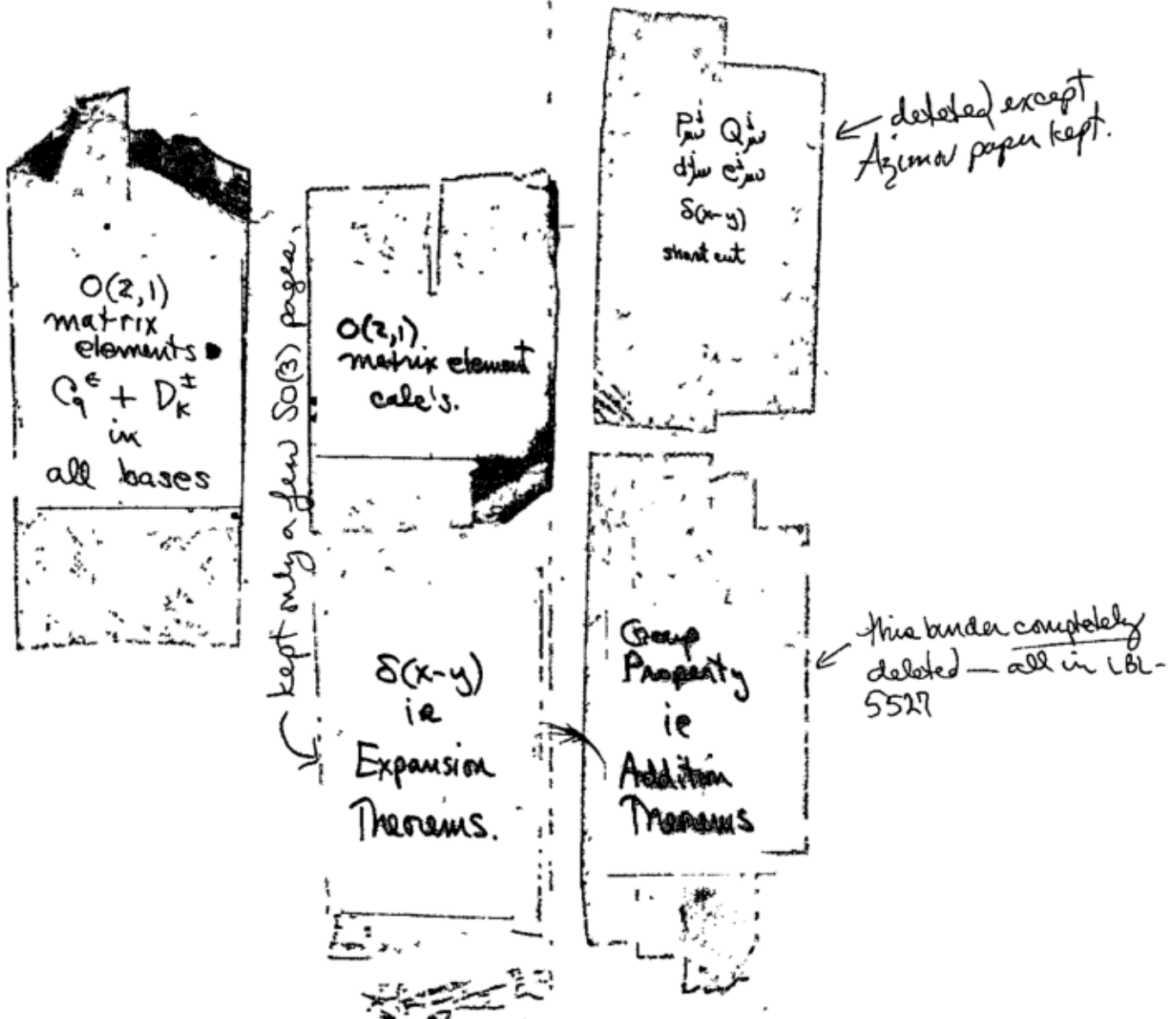
⑤ To interchange order of integration between "trace" and $\oint dj$, you need uniform convergence. Now:

I have decided that there is no point in retaining all this stuff. At the time I thought I would write a paper on how to get all the $SU(1,1)$ matrix elements in a uniform way, but I never did it. Much of the results did appear in LBL-5527, however.

This work was a sort of composite of the work of Vilenkin, Mukunda, and all these people. I have retained their papers and my notes on them.

As noted on the next page, I did write a "paper" that contained an outline of my efforts to get all the matrix elements. However if I ever go back to this subject again I will probably have to start from scratch. Thus, no reason to keep a sotrage file of related integrals and representation forms, which is what most of these two binders was.

Deleted: about 3 inches of notes, mostly written on boths sides.



July 7, 1976

ORIGINAL

Contents of this Binder:

Paper:

^{kept} shows how to compute matrix elements for C_q^e and D_k^e in all three bases of interest. (discrete, continuous, mixed). The rest of this binder is supportive material for this paper.

Data 1: data taken from Muk 67 concerning the continuous basis.

Data 2: important support material for above paper

Data 3: data for discrete basis calculations. All new. Parallels earlier work on discrete basis which may be found in another binder.

History: now useless notes on how I worked out the idea of computing all matrix elements in the same way.

Data 4: improvements to the paper, more details, see index there (July 14)

Diff Gens: how to calculate differential generators on the group manifold. Was used in paper.

How to Compute the Matrix Elements of $SU(1,1)$ in All Bases for all UIR series which occur in the Plancherel Theorem.

There are four classes of UIR's of interest. They are the C_q^0 , the $C_q^{\frac{1}{2}}$ which we group together as the $C_q^{\frac{1}{2}}$; and there are the D_k^+ and the D_k^- which we will also tend to treat together.

There are three types of bases of interest. The discrete basis has matrix elements of the form $\langle m/.../n \rangle$ where ~~maxim~~ m and n are both integers or both half-integers. The continuous basis matrix elements are of the form $\langle p/.../p' \rangle$ where p and p' are real. Finally, there is the mixed basis whose matrix elements have the form $\langle m/.../p \rangle$, sort of half way in between the other bases. There are other combinations possible if one includes the lightlike basis, but we do not concern ourselves with this possibility.

If we group together the continuous series, and group together the discrete series, we find that we have $2 \times 3 = 6$ types of matrix elements to compute. Here is a chart:

	$C_q^{\frac{1}{2}}$	$D_k^{\pm}$	trivial gens.
discrete	$^1 \langle m/U(g)/n \rangle$	$^2 \langle m/U(g)/n \rangle$	$K_2, (K_1)$
continuous	$^3 \langle p/U(g)/p' \rangle$ (4)	$^4 \langle p/U(g)/p' \rangle$	$K_2, (J_3)$
mixed	$^5 \langle p/U(g)/m \rangle$ (2)	$^6 \langle p/U(g)/m \rangle$	K_2

One notices that wherever the continuous series and the discrete series meet, there is a multiplicity index which we call r or r' in the chart. Thus, for example, in the continuous basis, continuous series, there will be four matrix elements to compute, for each group element, rather than just one matrix element.

Since $SU(1,1)$ is a three parameter Lie group, there are three generators which we call J_3 , K_1 and K_2 . The generator J_3 is diagonal in the discrete basis $/m$, whereas the generator K_1 is diagonal in the basis $/p$. Thus, a large number of the single-parameter subgroup matrix elements will be trivial. The non-trivial ones we indicate in the following chart: (see last column). A generator in parentheses means that matrix element can be trivially obtained from the previous generators group matrix element, so becomes trivial in this way.

We see then that instead of considering the general $SU(1,1)$ group element, one need only consider elements of the form $e^{i\nu K_2}$. All other matrix elements can be trivially obtained from this one.

So, counting up, we find that we shall have to compute $(1+4+2)^2 + (1+1+1)^2$ or $14 + 6 = 20$ matrix elements. However, it will turn out that certain elements can be computed simultaneously so we won't have all that much work to do.

We shall treat the matrix elements in the order that they appear in the chart.

External References:

Certain of these matrix elements have been computed by various authors and exist in the literature in some form. We make the following chart to indicate references:

	C_q^0	$C_q^{\frac{1}{2}}$	D_K^I
discrete	JLY, 70 Bayman Vilenkin AG	Bayman Vilenkin	Bayman Vilenkin AG (implicit)
	← BF →		
continuous	MR, PR MukSZ (Muk 6?) CDM, AM	MR, P MukSZ	MR, PR MukS
mixed	CDM, but $m=0$ only. GW JLY ($m=0$) Kahnine		X GW ($m=0$) X X

MacFayden?

Clearly, practically all the matrix elements we are after have been computed somewhere by somebody. We hope to present the results in a uniform notation.

Generalized Legendre functions:

It turns out that the various matrix elements we seek satisfy certain simple differential equations. In fact, they all satisfy a certain form of the generalized Legendre equation introduced by Azimov. For each basis, the equation is slightly different. We record the results here:

Define:
$$L(j, \mu, \nu, z) = (1-z^2) \frac{d^2}{dz^2} - 2z \frac{d}{dz} + j(j+1) - \frac{(\mu^2 - 2\mu\nu z + \nu^2)}{1-z^2}$$

Then:

discrete $L(j, m, n, \nu) \langle m | e^{-i\nu k_z} | n \rangle = 0$

continuous $L(j, ip, ip', \nu) \langle p | e^{-i\nu k_z} | p' \rangle = 0$

mixed $L(j, ip, m, -i\nu) \langle p | e^{-i\nu k_z} | m \rangle = 0 \checkmark$

These equations apply to matrix elements in all UIR classes. Each equation is merely a statement of the fact that a matrix element of an UIR labelled by j must be an eigenfunction of the Casimir operator J^2 expressed as a differential operator on the group manifold. Thus we know right from the start that any matrix element must be a linear combination of first and second kind generalized Legendre functions. For example, one must be able to conclude that:

$$\langle j, p, + | e^{-i\nu k_z} | j, p', + \rangle \Big|_Q^e = A \cdot Q_{ip, ip'}^j(\nu) + B \cdot Q_{ip, ip'}^{j-1}(\nu)$$

where A and B do not depend on ν . I.e., Q^j and Q^{j-1} are independent solutions of the Legendre equation. Thus, we need only calculate the coefficients. Sometimes it is very easy to do this by looking at $\nu = 0$ or $\nu = \infty$. We shall use this trick below.

General Method:

In order to compute matrix elements in various UIR's, one needs several ingredients:

- a group representation (group action on some Hilbert Space)
- a scalar product
- basis functions.

There are many ways to choose a group representation. We shall choose a representation that looks the same for all classes of UIR's. However, we shall find that the scalar product must vary slightly (or more than slightly) as one shifts from one UIR to the next. It is a characteristic of the multiplier representations (the type we shall use) that this is what happens. The advantage of using the multiplier method is that the group rep stays roughly the same, the differential generators are always first order differential operators, but the scalar product changes and sometimes becomes non-local. Sometimes, the scalar product appears as a double integral.

An alternative approach is to use a non-local representation (ie, not a multiplier representation). The advantage here is that the scalar product may be made uniform over all UIR classes. However, the group representation changes from one class to the next. Moreover, the eigenfunction equations for the basis functions becomes second order, so the basis functions become Special Functions. (Laguerres, eg). Each approach has its good points and its bad. See Mukunda Radhakrishnana.

We shall now list off the group representation and scalar products and eigenfunctions (basis functions) for the various cases: the group rep we take from Vilenkin, replacing B with -B to conform to Muk 67 notation:

$$[U(g)f](z) = z^{k+\epsilon} A^{-k-\epsilon} B^{-k+\epsilon} f(A/B)$$

where

$$z = e^{i\theta}$$

$$k = j+1$$

$$A = (\alpha z - \bar{\beta})$$

$$B = (\bar{\alpha} - \beta z)$$

$$\epsilon = 0 \text{ or } \frac{1}{2}$$

$$f = \text{function defined on unit circle}$$

$$g = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} = \text{SU}(1,1) \text{ element.}$$

The group $SU(1,1)$ is here represented by operators $U(g)$ acting on a space of functions defined on the unit circle. These representations are irreducible in general and are labelled by (j, ϵ) . For certain values of j , the representations are also unitary. For certain integral values of $j+\epsilon$ the representation is slightly reducible in that it contains both sides of the discrete series labelled by j . One can easily see that the representations with $\epsilon = \frac{1}{2}$ have the property that $U(-\text{identity}) = -1$, whereas the $\epsilon = 0$ reps have $U(-e) = 1$. That is the significance of the index ϵ . The label j is angular momentum (or its continuation), or simply the value taken by the Casimir operator of $SU(1,1)$:

$$j(j+1) = J^2 = J_3^2 - K_1^2 - K_2^2 = -\frac{1}{2}$$

From this representation, one may easily compute the following differential generators:

$$J_3 = \frac{1}{i} \frac{\partial}{\partial \theta} - \epsilon$$

$$K_1 = -J_2 = ik \cos \theta - \sin \theta J_3$$

$$K_2 = J_1 = i k \sin \theta + \cos \theta J_3$$

$$K_{\pm} \equiv K_1 \pm i K_2 = i e^{\pm i \theta} [k \pm J_3]$$

The basis functions which diagonalize J_3 are clearly proportional to this function:

$$\phi_m(z) \equiv z^{m+\epsilon} = e^{i(m+\epsilon)\theta}$$

where we are using the variables z and θ , always understanding that $z = e^{i\theta}$.

Given a function on the circle, we may define the following Fourier projection:

$$\begin{aligned} f_m &\equiv \frac{1}{2\pi} \int_0^{2\pi} d\theta \phi_m(z) f(z) \\ &= \frac{1}{2\pi} \int_0^{2\pi} d\theta e^{-i(m+\epsilon)\theta} f(e^{i\theta}) \end{aligned}$$

Here, m can be integer or half-integer, but $m+\epsilon$ is always integer.

For the series of UIR's C_q^ϵ , the scalar product is:

$$(f, g)_\epsilon = \sum_{m=-\infty}^{\infty} \overline{f_m} g_m$$

where the sum is over integers or half integers depending on ϵ . This scalar product may be trivially rewritten as:

$$(f, g)_\epsilon = \frac{1}{2\pi} \int_0^{2\pi} d\theta \overline{f(e^{i\theta})} g(e^{i\theta})$$

This scalar product is not just any scalar product, it is a scalar product that is invariant under the group action (see appendix _____). This invariance is needed if you want the group elements to be represented by unitary operators, which is to say, you want the generators to be hermitian. The normalized J_3 basis functions for the C_q^ϵ UIR are clearly:

$$|m\rangle \rightarrow \phi_m(z)$$

$$\langle m|m'\rangle = \delta_{m,m'}$$

For the UIR D_k^+ the scalar product given in is not invariant because j does not have the magic value $-\frac{1}{2} + i\epsilon$. A slight modification is needed to get a viable invariant scalar product * which is:

$$(f, g)_k^+ = \sum_{m=k}^{\infty} \frac{\Gamma(i-k+m)}{\Gamma(k+m)} \bar{f}_m g_m.$$

By the way, the space of functions which is the representation space for the UIR D_k^+ is simply a subspace of the space of functions on the circle, which space characterizes the C_q representations. A function in the D_k^+ Hilbert Space has the form:

$$f(z) = \sum_{m=k}^{\infty} \bar{f}_m g_m(z).$$

where, as usual, m is integral or halfintegral, depending on k . The usual way people have viewed these functions is to say:

$$f(z) = z^{k+\epsilon} \cdot \left(\sum_{n=0}^{\infty} \bar{f}_{n+k} z^n \right).$$

In other words, $f(z)$ is the boundary value of the product of $z^{k+\epsilon}$ times a function analytic in the open unit disc. It is easy to see from the representation that functions of this form do form an invariant subspace of the larger space of functions on the unit circle. Since we restrict our interest to $k = \frac{1}{2}, 1, \dots$ and ϵ takes the appropriate value 0 or $\frac{1}{2}$ depending on k , it happens that the functions $f(z)$ is itself analytic in the unit disc. However, $f(z)$ is not arbitrary in that it has a zero of order $(k+\epsilon)$ at the center of the disc. Some authors map this open disc unto an upper half plane and thereby characterize the D_k^+ by functions which are boundary values of functions analytic in an upper half plane. We do not need this construction, however.

By substituting the expansion above for $f(z)$ into the following expression; one regenerates the scalar product above. Therefore, an equivalent form for the scalar product for the D_k^+ is:

$$(f, g)_k^+ = \frac{1}{\pi \Gamma(2k-1)} \int_{\text{unit disc}} d^2z (1-|z|^2)^{2k-2} |z|^{-2k} \bar{f}(z) g(z).$$

Here, $d^2z = dx dy = r dr d\theta$, ie, this is an integral of a weighted product of the two functions over the unit disc. One may show directly — ie, without going through and — that this is an invariant scalar product by realizing that this weight function is an invariant density much like the Haar measure. We refer the reader to page of Bargmann.

This version of the D_k^+ scalar product, while rather messy, is at least "local" in that both functions are evaluated at the same point. Vilenkin gives another

version of this scalar product which is non-local. It has the form:

$$(f, g)_k^+ = \int_0^{2\pi} d\theta \int_0^{2\pi} d\theta' W(\theta - \theta') f(e^{i\theta}) g(e^{i\theta'})$$

We shall have no use for this form, nor for the disc version, and will therefore use only that form shown in . The normalized J_3 eigenfunctions are clearly:

$$|m\rangle \rightarrow \frac{\sqrt{\Gamma(k+m)}}{\sqrt{\Gamma(1-k+m)}} \phi_m(z) \quad m = k, k+1, \dots$$

The setup for D_k^- is very similar. Functions in the associated Hilbert Space have the form:

$$f(z) = \sum_{m=-\infty}^{-k} f_m \phi_m(z) \\ = z^{-k+\epsilon} \left(\sum_{n=-\infty}^0 f_{n-k} z^n \right)$$

So here we are dealing with functions analytic outside the unit circle. The various forms of the scalar product are similar to those already noted, and we take just the following form, which is the one we shall use:

$$(f, g)_k^- = \sum_{m=-\infty}^{-k} \frac{\Gamma(1-k-m)}{\Gamma(k-m)} f_m g_m$$

The basis functions are:

$$|m\rangle \rightarrow \frac{\sqrt{\Gamma(k-m)}}{\sqrt{\Gamma(1-k-m)}} \phi_m(z) \quad m = -k, -k-1, \dots$$

Discrete Basis Matrix Elements.

We have at this point assembled enough information to compute all the matrix elements corresponding to the first row of our chart on page 1. As noted, we are only interested in $U(g) = e^{-i\nu K_2}$ so from our rep on page 4 we may gather that:

$$U(g) = e^{-i\nu K_2} \quad \alpha = \sinh \frac{\nu}{2} \quad \beta = -i \cosh \frac{\nu}{2}$$

$$[U(g) \phi_m](z) = z^{k+\epsilon} A^{-k+n} B^{k-n}$$

$$[e^{-i\nu K_2} \phi_n](z) = z^{k+\epsilon} \left(\cosh \frac{\nu}{2} z - i \sinh \frac{\nu}{2} \right)^{-k+n} \left(\cosh \frac{\nu}{2} + i \sinh \frac{\nu}{2} z \right)^{-k-n}.$$

For the C_q^ϵ series, our matrix elements are:

$$\langle m | U(g) | n \rangle_\epsilon = \frac{1}{2\pi} \int_0^{2\pi} d\theta \cdot \bar{\phi}_m(e^{i\theta}) [U(g) \phi_n](e^{i\theta})$$

Ergo,

$$\langle m | e^{-i\nu K_z} | n \rangle = \frac{1}{2\pi} \int_0^{2\pi} d\theta \cdot e^{i\theta(k-m)} \left(\cosh \frac{\nu}{2} e^{i\theta} - i \sinh \frac{\nu}{2} \right) \left(\cosh \frac{\nu}{2} + i \sinh \frac{\nu}{2} e^{i\theta} \right)^{-k+n}$$

By expanding the last two factors into appropriate convergent ^{binomial} power series and making use of the following elementary fact:

$$\sum_{n=0}^{\infty} \binom{A}{n} \binom{B}{n+C} z^n = \frac{\Gamma(B+1)}{\Gamma(B+1-C)} \cdot \frac{F(-A, C-B; C+1; z)}{\Gamma(C+1)}$$

we may reduce the integral to a hypergeometric function form which we discover without much surprise is a Legendre function:

$$\langle m | e^{-i\nu K_z} | n \rangle_\epsilon = \frac{\Gamma(j+1-m)}{\Gamma(j+1-n)} P_{mn}^j(\cosh \nu)$$

This is the result, say, for $\nu > 0$. For $\nu < 0$ one must add $(-1)^{m-n}$. In these calculations, we always drop irrelevant phase factors since these have no significance. The reader might be warned that sometimes these factors take strange forms, for example:

$$\sqrt{\frac{\Gamma(j+1+m)\Gamma(j+1-m)}{\Gamma(j+1+n)\Gamma(j+1-n)}} = \text{phase}$$

when $j = -\frac{1}{2} + iS$. We may at once replace the P function with the d-function of Andrews and Gurdon (and defined in appendix ---). Again, dropping phases:

$$\langle m | e^{-i\nu K_z} | n \rangle_\epsilon = d_{mn}^j(\cosh \nu + i\epsilon) \quad \nu > 0$$

Here we have arbitrarily evaluated the d-function above its cut. (The P function is uncut on the right; one of its advantages here!).

$$\left[\times (-1)^{m-n} \quad \nu < 0 \right]$$

We note that the discrete basis matrix elements for the C_q^e which we have just calculated do not explicitly depend on e . It is just that m and n are either both integers or both half-integers depending on whether $e = 0$ or $\frac{1}{2}$. Of course, $j = -\frac{1}{2} + is$, which is to say $k = \frac{1}{2} + is$.

The matrix elements in the D_k^+ and D_k^- UIR's may be obtained directly from what we have already done. We use the same group action, the slightly changed scalar product and basis functions given in to get:

$$\begin{aligned} \langle m | U(g) | n \rangle_{k^+} &= \sum_{m'=k}^{\infty} \frac{\Gamma(1-k+m')}{\Gamma(k+m')} \cdot \left[\frac{1}{2\pi} \int_0^{2\pi} d\theta \overline{\phi_{m'}(z)} \cdot \left(\sqrt{\frac{\Gamma(k+m)}{\Gamma(1-k+m)}} \phi_m(z) \right) \right]^* \\ &\quad \times \left[\frac{1}{2\pi} \int_0^{2\pi} d\theta \overline{\phi_m(z)} \cdot [U(g) \phi_n](z) \cdot \sqrt{\frac{\Gamma(k+n)}{\Gamma(1-k+n)}} \right] \\ &= \left[\frac{\Gamma(1-k+n)}{\Gamma(k+n)} \cdot \frac{\Gamma(k+n)}{\Gamma(1-k+n)} \right]^{\frac{1}{2}} \cdot \frac{1}{2\pi} \int_0^{2\pi} d\theta \overline{\phi_m(z)} [U(g) \phi_n](z) \\ &= \left[\frac{\Gamma(k+n)}{\Gamma(k-n)} \cdot \frac{\Gamma(k-m)}{\Gamma(k+m)} \right]^{\frac{1}{2}} \cdot P_{mn}^j(\chi) \\ &= \sqrt{G_{mn}^j} P_{mn}^j(\chi) = \tilde{d}_{mn}^j(\chi + i\epsilon) \end{aligned}$$

When the dust settles, we find that the D_k^+ matrix elements are the same functions as the C_q matrix elements. The symbol $\doteq$ indicates equality up to a phase. We have made continual use of the property of gamma functions,

$$\frac{\Gamma(a)}{\Gamma(b)} = \frac{\Gamma(1-b)}{\Gamma(1-a)} \frac{\sin \pi b}{\sin \pi a}$$

Of course for the D_k^+ we have $k = j+1$, and $k = \frac{1}{2}, 1, 3/2, \dots$ and $m = k, k+1, \dots$ and $n = k, k+1, \dots$

One might imagine that the D_k^- matrix elements are once again the same functions. They are.

We have arrived, therefore, at the following conclusion: all of the UIR matrix elements of $e^{-i\nu K_2}$ in the discrete basis are given by the same d-functions. One must keep in mind the rule that when the boost parameter changes sign, a factor $(-1)^{m-n}$ must be added (as one finds from evaluating the integral, or which one sees must be true to preserve unitarity, given the d-function symmetry relations).

SU(2) matrix elements:

In passing, we should point out that our same representation also gives the SU(2) matrix elements. We ~~xxxxx~~ take our representation and replace $\bar{\beta}$ with $-\bar{\beta}$ throughout (but we do not change β !). This is the change that takes you from $su(1,1)$ back to $su(2)$. The group action is then:

$$[U(g)f](z) = z^{k+\epsilon} A^{-k-\epsilon} B^{-k+\epsilon} f(A/B)$$

$$A = (\alpha z + \bar{\beta}) \quad B = (\bar{\alpha} - \beta z) \quad g = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} \in SU(2).$$

In particular, we have:

$$[e^{-i\phi J_2} f_n](z) = z^{k+\epsilon} \left(\cos \frac{\phi}{2} z - \sin \frac{\phi}{2} \right)^{-k+n} \left(\cos \frac{\phi}{2} + \sin \frac{\phi}{2} z \right)^{-k-n}$$

$$\text{Since } \alpha = \cos \frac{\phi}{2} \quad \beta = -\sin \frac{\phi}{2}$$

A comparison between these equations and shows that we are essentially examining what in $su(1,1)$ would be an imaginary boost. I.e, the above group rep would be obtained if we were to replace v by $-i\phi$ in equations. This fact becomes completely transparent when one considers the actual 2×2 SL(2,c) generator matrices:

$$J_2 = \frac{1}{2} \sigma_2 \quad K_2 = \frac{i}{2} \sigma_2 \quad \text{so} \quad K_2 = iJ_2$$

$$\Rightarrow -i\phi J_2 = -i(-i\phi) K_2$$

We found earlier the invariant scalar product for the UIR's of SU(1,1) by insisting that K_1 and K_2 be hermitian. We now require that J_1 and J_2 be hermitian which means that K_1 and K_2 are antihermitian, because we are looking for UIR's of SU(2). One easily finds that the ^{invariant} ~~correct~~ scalar product for SU(2) is:

$$(f, g) = \sum_{m=-j}^j \frac{1}{\Gamma(j+1+m)\Gamma(j+1-m)} \bar{f}_m g_m$$

Normalized basis functions are:

$$|m\rangle \rightarrow \frac{1}{\sqrt{\Gamma(j+1+m)\Gamma(j+1-m)}} f_m(z).$$

Having the basis functions, the scalar product, and the group action, we may compute the SU(2) matrix elements. We encounter exactly the same integral and things reduce to the same d-functions.

The result is certainly not earthshaking:

$$\langle m | e^{-i\phi J_z} | n \rangle = d_{mn}^j(\cos\phi)$$

This business of relating $SU(2)$ and $SU(1,1)$ in this way is loosely referred to as "Weyl's Unitary Trick". The usual unitary irreducible-representations of $su(2)$ may be identified with the non-unitary finite dimensional representations of $SU(1,1)$ which nobody uses for anything.

We have gone into so much detail here in order to be able to make the statement: the matrix elements of the UIR's of $SU(1,1)$ in the discrete basis are simply the analytic continuations to $z = x + i\epsilon$, $x > 1$, of the matrix elements of the UIR's of $SU(2)$, $d_{mn}^j(z)$. In sharp contrast, we shall show that the $su(1,1)$ UIR matrix elements in the other bases are not d-functions. In fact, they will turn out to be mostly second-kind Legendre functions (Q functions, or e functions).

Continuous Basis Matrix Elements:

We now move down to the second row of our chart back on page 1, and attempt calculation of the $SU(1,1)$ matrix elements in the continuous basis. As ~~usual~~ usual, we shall need a group representation, scalar products, and basis functions. We follow basically the approach of Mukunda 1967.

We have already stated our fundamental representation and given the differential generators. We found, for example,

$$K_1 = ik \cos \theta - \sin \theta \left[\frac{1}{i} \frac{\partial}{\partial \theta} - \epsilon \right]$$

We wish now to go to a space of functions of the real variable q such that the differential generator K_1 has the following form:

$$K_1 = -\frac{1}{i} \frac{d}{dq}$$

We shall relate every function in this new space to functions on the circle by some as yet unknown function multiplier:

$$f(z) \rightarrow F(q) = h(q) f(z(q))$$

We have two unknown functions, $h(q)$ and $z(q)$. By insisting that K_1 have the simple form shown when it acts on the q -space, we may solve for both these unknown functions:

$$F(q) = h(q) f(z) \Rightarrow [K_1 F](q) = h(q) [K_1 f](z)$$

$$\Rightarrow -\frac{1}{i} \frac{d}{dq} [h(q) f(z)] = h(q) [ik \cos \theta - \sin \theta (-\epsilon + \frac{1}{i} \frac{\partial}{\partial \theta})] f(z)$$

$$\Rightarrow + \frac{d\theta}{dq} = \sin \theta \quad \text{and} \quad -\frac{1}{i} \frac{dh}{dq} = h(ik \cos \theta + \epsilon \sin \theta)$$

$$\Rightarrow q(\theta) = \ln(\tan \frac{\theta}{2}) \quad \text{and} \quad h(q) = (\cosh q)^{-k} \left(\frac{1+i e^q}{1-i e^q} \right)^{-\epsilon}$$

$$\Rightarrow z(q) = \left[\frac{1+i e^q}{1-i e^q} \right]$$

The new variable q must be related to the old variable z (or θ) as shown, and the function multiplier $h(q)$ must be given by above.

We have arrived at the continuous basis, which for short we shall refer to as q -space. The eigenfunctions which diagonalize K_1 are obviously simple exponentials:

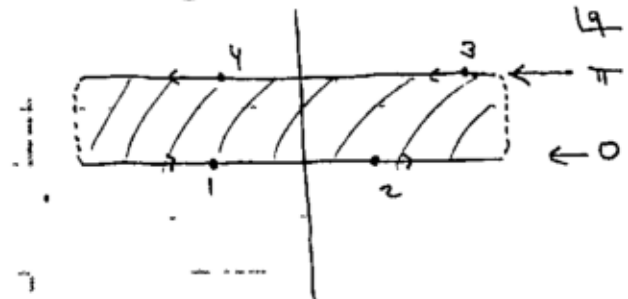
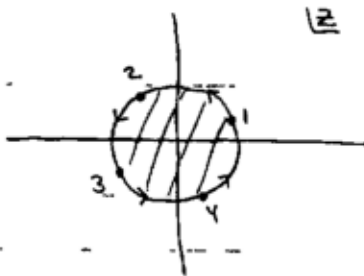
$$|p\rangle \rightarrow e^{-ipq}$$

$$K_1 |p\rangle = p |p\rangle$$

Let us now examine the change of variable more closely. We have:

$$q = \ln(\tan \frac{\theta}{2}) = \ln \left[-i \left(\frac{z-1}{z+1} \right) \right]$$

If we choose the $+i\pi$ branch of the log when $\tan \frac{\theta}{2}$ goes negative (which happens on the lower half circle) we find that as z traces the unit circle in z -space, q traces a certain rectangle in q -space. We refer to this rectangle sometimes as the "upper" rectangle. It is shown in the sketch below. In fact, by looking at $q(z)$ above, it is easy to see that the inside of the unit circle in z -space maps into this upper rectangle.



Of course the sides of this "rectangle" are at infinity and correspond to the points $z = \mp 1$. It is perhaps better to refer to this rectangle as two real lines, as Mukunda does. The fact that there are two real lines and not one, corresponding to the circle, is the source of the infamous multiplicity index which arises in the continuous basis. In order to get $K_1 = i\dot{q}/dq$, we were forced to make the change of variable we did, and this forced the circle to map only two real lines. There is no way out.

We pause here to mention that this multiplicity is a group theoretic fact and will of course arise in any "realization" of $su(1,1)$ in the continuous basis. For example, Kalnins calculates certain matrix elements by considering the group of motions on the light cone (the simplest $O(2,1)$ surface). In his analysis, the multiplicity index magically arises because the light cone has two halves. Group theoretically, MAR have shown that the origin of this multiplicity is the fact that when the full $su(1,1)$ algebra $[J_3, K_+, K_-]$ is reduced onto the subalgebra whose generators are $[J_3, K_+]$, this subalgebra occurs twice in the reduction. The present author, however, feels most comfortable with the above Mukunda construction as showing the origin of this index.

We recall that the space of functions for the UIR's C_q^e was the set of all (reasonable) functions on the circle. In the q -space, this maps into the set of arbitrary (but fourier projectable) functions on the two real lines. That is, the elements of the space $\mathcal{K}^e$ are function pairs $(f_+(q), f_-(q))$ which we can consider to be both functions of real q . Since in C^q there was no correlation

← wrong! generators!

between the function $f(z)$ on the two halves of the circle, the two functions $f_{\pm}(q)$ are independent. We shall choose as our two basis functions the following:

$$|P_1+\rangle \rightarrow \begin{pmatrix} e^{-ipq} \\ 0 \end{pmatrix} \quad |P_1-\rangle \rightarrow \begin{pmatrix} 0 \\ -e^{-ipq} \end{pmatrix}$$

where column vectors refer to:

$$\begin{pmatrix} f_+(q) \\ f_-(q) \end{pmatrix} \quad \text{or sometimes} \quad \begin{pmatrix} f_1(q) \\ f_2(q) \end{pmatrix}$$

In what follows, we shall usually use the label r to have values $+$ and $-$. We can then write:

$$\phi_{r'}^{(pr)}(q) = e^{-ipq} \delta_{rr'}$$

$$K_1 |P_1 r\rangle = p |P_1 r\rangle \quad r = \pm$$

Since K_1 is supposed to be a hermitian generator, p is of course real. One usually phrases the last equation above in this way: in the continuous basis, the spectrum of the diagonal generator for the C_q^e UIR's is the real line twiced.

In contrast, we recall that the D_k^+ UIR's had to do with functions which were analytic in the unit disc. In q -space, these will map into functions which are analytic in the upper box of figure . Thus, for the D_k^+ (and also for the D_k^-), one cannot separately set the functions on the upper and lower "lines" because they are analytically connected. I.e, they will be roughly ~~opposite~~ boundary values of the same analytic function on opposite sides of the box. The function on the upper line is determined completely by the function on the lower line. This is why there is no multiplicity index for the discrete series UIRs.

Our next task is to take our group representation over into q -space. First, we shall define a certain function that will arise frequently:

$$f_m(q) \equiv (chq)^{-k} \cdot \left[\frac{1+ieq}{1-ieq} \right]^m = (chq)^{-k} (z(q))^m$$

This happens to be ~~an~~ eigenfunction of J_3 in q -space, but ~~however~~ we shall return to that fact later in our consideration of the mixed basis. Right now we use this function to whorten our expression for the multiplier $h(q)$ which is:

$$h(q) = f_{-e}(q)$$

The q-space version of our group representation first,

may be obtained as follows:

$$\begin{aligned} [U(g)F](q) &= f_{-e}(q) [U(g)f](z) \\ &= f_{-e}(q) z^{k+e} A^{-k-e} B^{k+e} f(z') \end{aligned}$$

where recall $A = (\alpha z - \beta)$, $B = (\bar{z} - \beta z)$, $z' = A/B$. The variable z' will be related to a variable q' in the same way that z is related to q . To keep things clear, we note the sequence of events:

$$z \equiv e^{i\theta} \longrightarrow q \equiv \ln \left(\tan \frac{\theta}{2} \right) = \ln \left[-i \left(\frac{z-1}{z+1} \right) \right]$$

$$z' \equiv \frac{(\alpha z - \beta)}{(\bar{z} - \beta z)} = A/B$$

$$q' \equiv \ln \left(-i \left(\frac{z'-1}{z'+1} \right) \right) = \ln \left(\tan \frac{\theta'}{2} \right) \text{ where } z' \equiv e^{i\theta'}$$

$$z = \frac{1+ie^q}{1-ie^q} = -\operatorname{th} q + i \operatorname{sech} q = \left(\frac{i - \operatorname{sh} q}{\operatorname{ch} q} \right)$$

$$z' = \frac{1+ie^{q'}}{1-ie^{q'}} = \text{etc.}$$

Obviously one can deduce the direct connection between q' and q , but we shall not do that yet. Except where we use θ and θ' , we mean all these equations to be true in the complex sense. The original Vilenkin multiplier representation takes a point on (or inside) the circle z and maps that point onto a new point z' which lies on (or inside) the circle. In q space, our representation will take a point q on the rectangle and map it into some other point q' on the rectangle. In general, q and q' will not lie on the same "line".

We now finish the conversion of the group rep to q space which we began above. We replace:

$$f(z') = \frac{1}{f_{-e}(q')} F(q')$$

to get

$$\begin{aligned} [U(g)F](q) &= \left[\frac{f_{-e}(q)}{f_{-e}(q')} \right] \cdot z(q)^{k+e} A^{-k-e} B^{k+e} F(q') \\ &= \left(\frac{\operatorname{ch} q}{\operatorname{ch} q'} \right)^{-k} \cdot (z')^e \cdot z^k A^{-k-e} B^{-k+e} F(q') \end{aligned}$$

However, $z' = A/B$, so we get:

$$[U(g)F](q) = \left[\frac{\text{ch } q \cdot A \cdot B}{\text{ch } q' \cdot z} \right]^{-k} F(q')$$

This is just a standard multiplier representation: the function $F(q)$ here defined on the rectangle, is mapped via a point transformation into some new function $\tilde{F}(q)$ whose value is given by a multiplier factor times the original functions evaluated at the new point q' .

In order to express the multiplier entirely in terms of q , and for purposes to be seen below, we define some new quantities and observe some new relations:

$$z' = \frac{A}{B} = \frac{(\alpha z - \bar{\beta})}{(\bar{\alpha} - \beta z)} = \frac{(\alpha z - \bar{\beta})(\alpha - \bar{\beta} \bar{z})}{|\alpha - \beta z|^2} = \frac{N}{D}$$

where

$$N = -\text{Im } q (\alpha^2 + \bar{\beta}^2) - 2\alpha\bar{\beta} + i(\alpha^2 - \bar{\beta}^2)/\text{ch } q$$

$$D = |\alpha|^2 + |\beta|^2 + 2\text{Im } q \text{Re}(\alpha\bar{\beta}) + 2\text{Im}(\alpha\beta)/\text{ch } q > 0$$

Now,

$$z' = e^{i\theta'} \Rightarrow \begin{aligned} \sin \theta' &= \text{Im } N / D \\ \cos \theta' &= \text{Re } N / D \end{aligned}$$

One can also show that

$$\begin{aligned} \sin \theta &= (\text{ch } q)^{-1} & \sin \theta' &= (\text{ch } q')^{-1} \\ \cos \theta &= -\text{th } q & \cos \theta' &= -\text{th } q' \\ \tan \theta &= (-\text{sh } q)^{-1} & \tan \theta' &= (-\text{sh } q')^{-1} \end{aligned}$$

Finally, we note that

$$z = \frac{AB}{D} \Rightarrow \left[\frac{\text{ch } q AB}{\text{ch } q' z'} \right] = (\text{ch } q \text{Im } N)$$

Therefore, we arrive at this form for our group representation:

$$[U(g)F](q) = (\text{ch } q \text{Im } N)^{-k} \tilde{F}(q')$$

Several comments are in order here. First, notice that the label ϵ has completely vanished from the representation. Nor does it occur in our basis functions. Nor will it appear in the scalar product to be derived below. As might have been expected, the quantity ϵ which had to do with whether m was integral or half-integral in the discrete basis, plays no role whatsoever in the continuous basis.

Secondly, we wish to remind the reader that all the q 's which appear on the previous page lie on the rectangle, which is to say, they are either real or real + $i\pi$. Below we shall use the index q as a purely real variable.

In fact, our next task is to re-express the group representation in terms of function-pairs of a real variable, rather than "functions on a rectangle". I.e., we simply want to "lower" the function defined on the upper line. Before doing this, we have to handle a slight phase detail which is this: if we imagine that our region of interest in q -space is the inner boundary of the upper box (ie, as if we were dealing with boundary values of functions analytic in the upper box), then one can show that the phase of the function $ch(q)$ is:

$$\arg ch q = \begin{cases} 0 & \text{lower edge} \\ \pi \operatorname{sign}(Re q) & \text{upper edge.} \end{cases}$$

Therefore, we find,

$$(ch q)^{-k} = (ch[Re q])^{-k} \quad \text{lower line} \quad q = Re q + i\epsilon$$

$$(ch q)^{-k} = (ch[Re q])^{-k} e^{-i\pi k s(Re q)} \quad \text{upper line} \quad q = Re q + i(\pi - \epsilon).$$

If we temporarily use the symbol x as the real part of q , then we may convert our function $F(q)$ defined on the rectangle into two functions of x as follows:

$$f_+(x) \equiv F(x) = f_\epsilon(x) f(z) = (ch x)^{-k} \left[\frac{1+ie^x}{1-ie^x} \right]^{-\epsilon} f(e^{i\theta})$$

$$f_-(x) \equiv e^{+i\pi k s(x)} F(x+i\pi) = (ch x)^{-k} \left[\frac{1+ie^x}{1-ie^x} \right]^{\epsilon} f(e^{i\theta})$$

Therefore:

$$\begin{aligned} f_r(q) &= (ch q)^{-k} \left(\frac{1+ie^q}{1-ie^q} \right)^{-r\epsilon} f(z) \\ &= f_{-r\epsilon}(q) f(z). \end{aligned} \quad r = \pm$$

where we now take q to be real only.

Using this mapping of the function on the rectangle to function-pairs, we may with a slight amount of work convert the representation to this form:

$$[U(q)f]_r(q) = (\ln |\partial_m N_r|)^{-1} \sum_{r'=\pm} \Theta(r' \partial_m N_r) f_{r'}(q')$$

where the N_r ^{are} ~~were given back in below~~. The theta functions are very important and arise in the simple way to be described below. We hasten to point out that the q and q' which appear in this form of the representation are pure real. That is:

$$q = \ln |\tan \frac{\theta}{2}|$$

$$q' = \ln |\tan \frac{\theta'}{2}| = \ln \left| \frac{z' - 1}{z' + 1} \right|$$

$$= \ln \left| \frac{1 - \cos \theta'}{\sin \theta'} \right| = \ln \left[\frac{(D_r - R_e N_r)}{|\partial_m N_r|} \right]$$

where $z_r = \left(\frac{1 + i e^q}{1 - i e^q} \right)^{1/2} = \frac{1 + i e^q}{1 - i e^q}$

~~$z_r = \left(\frac{1 + i e^{q'}}{1 - i e^{q'}} \right)^{1/2}$~~

The reason these $z(q)$ ~~now~~ ^{carry} an r label is that we are now treating q as real. The r now carries the information of upper or lower line. That is:

$$e^q = + e^q$$

$$e^{q+i\pi} = - e^q$$

The theta functions in the above representation arise because the upper half circle is mapped into the lower line ~~and function f_+~~ , while the lower half circle is mapped into the ~~upper line, and function $f_-(q)$~~ . Therefore:

~~$f_r(q) = f_{-re}(q) f(z_r(q))$~~

~~$f(z) = \sum_{r=\pm} \Theta(r \sin \theta) \left(\frac{f_r(q)}{f_{-re}(q)} \right)$~~

The theta function selects out the proper q -space function based on the location of z .

Similarly we have,

$$f(z') = \sum_{r'=\pm} \Theta(r' \sin \theta') : f_{r'}(q') / f_{r'e}(q')$$

But $\sin \theta' = \frac{\text{Im } N_r}{D_r}$ so $\Theta(r' \sin \theta') = \Theta(r' \text{Im } N_r)$

By the way, $z' = z'(z)$ In the present notation, we are interested in z' and q' and θ' all as functions of q . To get back to q , we have to go through z , and $z = z_r(q)$ as given in above, where r indicates whether this z lay on the upper or lower half circle; or, whether q lay on the upper or or upper "line". Thus, the variables q' , z' and θ' implicitly depend on this index r , as well as the real variable q . In the representation, this index r is the index which appears on the left side of the equation.

To summarize, we have converted our representation to q space where the representation space consists of function pairs, where each function is a function of the real variable q . That representation is:

$$[U(q)f]_r(q) = (\text{ch } q |\text{Im } N_r|)^{-k} \sum_{r'} \Theta(r' \text{Im } N_r) f_{r'}(q')$$

where

$$N_r = -\text{th } q (\alpha^2 + \beta^2) - 2\alpha\beta + i r (\alpha^2 - \beta^2) / \text{ch } q$$

$$D_r = |\alpha|^2 + |\beta|^2 + 2\text{th } q \text{Re}(\alpha\beta) + 2r \text{Im}(\alpha\beta) / \text{ch } q$$

$$q' = \ln \left[\frac{D_r - \text{Re } N_r}{|\text{Im } N_r|} \right]$$

For example, $U(q) = e^{+i\nu K_2}$ $\alpha = \text{ch } \frac{\nu}{2}$ $\beta = i \text{sh } \frac{\nu}{2}$

$$\text{ch } q \text{Im } N_r = \text{sh } \nu \text{ch } q + r \text{ch } \nu$$

$$\text{ch } q \text{Re } N_r = -\text{sh } q$$

$$\text{ch } q D_r = \text{ch } \nu \text{ch } q + r \text{sh } \nu$$

$$e^{q'} = \left| \frac{\text{ch } \nu \text{ch } q + r \text{sh } \nu + \text{sh } q}{\text{sh } \nu \text{ch } q + r \text{ch } \nu} \right| = \left| \frac{\text{th } \frac{\nu}{2} + r e^q}{1 + r \text{th } \frac{\nu}{2} e^q} \right|$$

Thus,

$$[e^{+i\omega K_2}]_r(q) = [sh\omega chq + r ch\omega]^{-k} \sum_{r'} \Theta[r'(sh\omega chq + r ch\omega)] \bar{f}_{r'} \left(\ln \left| \frac{sh\omega + re^q}{1 + rth\omega e^q} \right| \right).$$

We shall use this form below to calculate matrix elements. We have chosen the plus sign in the exponential to conform to other authors who have calculated these matrix elements and whose results we shall use below.

Scalar Product for the C_q :

Starting with our original scalar product on the unit circle, we may convert easily to a form involving integration on our rectangle:

$$(f, g)_C = \frac{1}{2\pi} \oint_{\Gamma} dq \bar{F}(q) G(q)$$

Here we have made use of the important fact that, for the principle UIR series, $-k + \bar{k} = 1$, and also:

$$\frac{dz}{dq} = \frac{iz}{chq}$$

Using , the scalar product may be at once reinterpreted in terms of the function-
pairs introduced above, so.

$$(f, g)_C = \frac{1}{2\pi} \int_{-\infty}^{\infty} dq \cdot \sum_r \bar{f}_r(q) g_r(q)$$

From this scalar product, one sees at once that the normalized basis functions which diagonalize the generator K_1 are:

$$|p, r\rangle \rightarrow g^{(p, r)}(q) = \text{function vector given in } \underline{\hspace{2cm}}$$

$$\langle p', r' | p, r \rangle = \delta(p - p') \delta_{r', r}$$

Scalar product for the D_k :

The scalar product for the discrete series is slightly more complicated. It may be obtained by starting from the disc form in z -space, converting to an integral over the upper rectangle, and then converting to the form which follows. Details are given in . The result is:

$$(f, g)_K = \frac{1}{4\pi} \int_{-\infty}^{\infty} dp \, w(p) \, \bar{f}(p) \, g(p).$$

where:

$$g(p) = \int_{-\infty}^{\infty} dq \, e^{+ipq} G(q) \quad \text{correct & L.} \quad \rightarrow \text{same for } f.$$

$$w(p) = e^{-\pi p} |\Gamma(k + \frac{1}{2}p)|^{-2}$$

We wish to emphasize that this is the same scalar product that we used earlier for the D_K^+ . The normalization has not been changed. The properly normalized basis function for K_1 is :

$$|p\rangle \rightarrow \frac{1}{\sqrt{\pi w(p)}} \cdot e^{-ipq}$$

$$\langle p' | p \rangle_K = \delta(p' - p)$$

If we wish, we can view this basis function as a function pair. The form given is the correct function for the lower line. For the upper line, we use our equation to get:

$$|p\rangle \rightarrow \begin{bmatrix} \frac{1}{\sqrt{\pi w(p)}} e^{-ipq} \\ \frac{1}{\sqrt{\pi w(p)}} e^{-\pi p} e^{-i\pi k(s(q))} e^{-ipq} \end{bmatrix}$$

As already pointed out, the upper line function is completely determined by the function on the lower line.

The group representation adapted to the D_K^+ is, from

$$[U(g)f]_+(q) = (dq |dmN+|)^{-k} \left[\Theta(dmN+) f_+(q') + \Theta(-dmN+) e^{-i\pi k s(q')} f_+(q' + i\pi) \right]$$

where q and q' are both real. Thus, we have a representation which maps functions on the ~~lower~~ line to functions on the lower line. Whenever the point transformation lands on the upper line, the function is "lowered" across the analytic rectangle so that everything is given in terms of the lower line.

One can see immediately that if the function on the left side of the scalar product given in is a K_1 basis function, the p integration goes away. Thus:

$$\begin{aligned} \langle p | g \rangle_k^+ &= \left(\frac{1}{\sqrt{\pi w(p)}} e^{-i p q}, -g \right)_k^+ \\ &= \frac{1}{4\pi} \sqrt{\frac{w(p)}{\pi}} \int_{-\infty}^{\infty} dq e^{+i p q} g_+(q) \end{aligned}$$

We shall use this form below to calculate the D_k^+ matrix elements.

For the D_k^- there are, as usual, just a few changes. The scalar product starts out in z -space as an integral over the outside of the unit circle. In the q -space, this appears as an integral over the lower box going from $\text{Im}(q) = 0$ down to $-i\pi$. As with the D_k^+ everything may be expressed in terms of functions on the real line (actually, $q - i\epsilon$ in this case). When the point q' lands on the line $\text{re}(q') - i\pi$, it is ~~raised~~ raised through the lower analytic box and placed back on the real axis. The results are all the same except for a few sign changes arising from $-\pi$ in place of $+\pi$.

$$f_+(q) = F(q - i\epsilon) = (dq)^k \left[\frac{1+i\epsilon q}{1-i\epsilon q} \right]^{-\epsilon} f(e^{i\theta})$$

$$\begin{aligned} f_-(q) &= F(q - i\pi + i\epsilon) \cdot e^{-i\pi k \arg(q)} \\ &= (dq)^k \left[\frac{1+i\epsilon q}{1-i\epsilon q} \right]^{\epsilon} f(e^{i\theta}) \end{aligned}$$

$$\Rightarrow f_r(q) = f_{re}(q) f(z)$$

! need correction.

These equations show how a function defined on the perimeter of the lower rectangle are converted into function-pairs on the two lines. Here, q is real.

The scalar product, basis functions, and group action for D_k^- are as follows:

$$(f, g)_k = \frac{1}{4\pi} \int_{-\infty}^{\infty} dq \tilde{w}(p) \bar{f}(p) g(p)$$

$$\begin{aligned} \text{where} \quad g(p) &= \int_{-\infty}^{\infty} dq e^{+i p q} g_+(q) \\ \tilde{w}(p) &= e^{+\pi p} |p(k+i p)|^{-2} \end{aligned}$$

$$|p\rangle \rightarrow \frac{1}{\sqrt{\pi \tilde{w}(p)}} \begin{bmatrix} e^{-i p q} \\ e^{+\pi p} e^{+i\pi k s(q)} e^{-i p q} \end{bmatrix}$$

$$[U(g)f]_+(q) = (dq | \partial_m N_+ |)^{-k} \left[\theta(\partial_m N_+) f_+(q') + \theta(-\partial_m N_+) e^{+i\pi h_3(q')} f_+(q' + i\pi) \right]$$

and

$$\langle p | g \rangle_{\bar{e}} = \frac{1}{4\pi} \sqrt{\frac{w(p)}{\pi}} \int_{-\infty}^{\infty} dq e^{+ipq} g_+(q).$$

At this point, let us summarize the forms of matrix elements in all four classes of UIR's which have a normalized K_1 eigenfunction on the left. For the $D_k^{\pm}$ we have from and ,

$$\langle p | g \rangle_{k^{\pm}} = \frac{1}{4\pi} \sqrt{\frac{w_{\pm}(p)}{\pi}} \int_{-\infty}^{\infty} dq e^{+ipq} g_{\pm}(q)$$

From , and , we get for the $C_q^{\pm}$ series:

$$\langle p | g \rangle_e = -\frac{1}{2\pi} \int_{-\infty}^{\infty} dq e^{+ipq} g_r(q)$$

~~This shows that~~ If we now consider the function g to be the group action on a normalized K_1 eigenstate on the right, we find:

$$\begin{aligned} \langle p | U(g) | p' \rangle_{k^{\pm}} &= \frac{1}{4\pi^2} \sqrt{\frac{w_{\pm}(p)}{w_{\pm}(p')}} \int_{-\infty}^{\infty} dq e^{+ipq} [U(g) e^{-ip'q}] \\ &= \frac{1}{4\pi^2} \sqrt{\frac{w(p)}{w(p')}} \int_{-\infty}^{\infty} dq e^{ipq} (dq | \partial_m N_+ |)^{-k} [\theta(\partial_m N_+) + \theta(-\partial_m N_+) e^{+i\pi p}] e^{-ip'q} \end{aligned}$$

This tells us all $SU(1,1)$ matrix elements in the continuous basis for both discrete series classes D_k^+ and D_k^- . For the classes $C_q^{\pm}$ we find:

$$\begin{aligned} \langle p | U(g) | p' \rangle_e &= \frac{1}{2\pi} \int_{-\infty}^{\infty} dq e^{+ipq} [U(g) g_r(q)] \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} dq e^{+ipq} (dq | \partial_m N_r |)^{-k} [e^{+i\pi p} g_r(q)] e^{-ip'q} \end{aligned}$$

From these expressions, one notices that the $D_k^{\pm}$ matrix elements can be obtained directly as linear combinations of the C_q matrix elements as follows:

$$\langle p | U(g) | p' \rangle_{k^{\pm}} = \frac{1}{4\pi^2} \sqrt{\frac{w(p)}{w(p')}} \left[\langle p | U(g) | p' \rangle_e + e^{+i\pi p} \langle p | U(g) | p' \rangle_e \right]$$

Therefore, to get all matrix elements in the continuous basis, we need only calculate those in the C_q for the generator K_2 . These are precisely the functions calculated by CDM, for example, in their appendix. They used the same basis that we use and gave these results:

$$\begin{aligned} \langle p', r' | e^{+iVK_2} | p, r \rangle &= \frac{1}{2\pi} \int_{-\infty}^{\infty} dq e^{+ipq} |sh\sqrt{q} + rch\sqrt{q}|^{-k} \\ &\quad \times \Theta(r' [sh\sqrt{q} + rch\sqrt{q}]) \cdot \left| \frac{r\sqrt{k} + re^q}{1 + r\sqrt{k}e^q} \right|^{-ip'} \\ &= d_{-j(r'), -j(r)}^{k-1}(v') \text{ of CDM.} \end{aligned}$$

where $u = ip$ and $u' = ip'$. Using their results (which we have checked) and converting to the P and Q function notation, we find for $v > 0$:

$$\langle p', + | e^{+iVK_2} | p, + \rangle = \frac{\Gamma(k + ip')}{\Gamma(k + ip)} \cdot \frac{1}{\pi} Q_{ip, ip'}^{k-1}(ch v)$$

$$\langle p', - | e^{+iVK_2} | p, - \rangle = \frac{\Gamma(-k + 1 + ip)}{\Gamma(-k + 1 + ip')} \cdot \frac{1}{\pi} Q_{ip', ip}^{-k}(ch v)$$

$$\langle p', - | e^{+iVK_2} | p, + \rangle = \text{mess.}$$

$$\langle p', + | e^{+iVK_2} | p, - \rangle = 0$$

Using , we find at once for the $D_k^{\pm}$ (thanks to the vanishing of the last element):

$$\langle p | e^{+iVK_2} | p' \rangle_{k^{\pm}} = \frac{1}{4\pi^2} \cdot e^{\mp \pi(p-p')/2} \cdot \frac{1}{\pi} Q_{ip, ip'}^{k-1}(ch v)$$

where as usual we have dropped a phase factor.

Mixed basis:

Finally, we move down to the last row of our chart on page 1 and discuss calculation of the various matrix elements in the mixed basis. We may do this calculation either in z-space or q-space. To work in q-space, we will need the eigenfunctions of J_3 which are:

$$g_m(z) = e^{i\theta(m+\epsilon)} = z\text{-space eigenfunction of } J_3$$

$$f_r(q) = f_{re}(q) f(z) = \text{conversion.}$$

Therefore:

$$g_r^{(m)}(q) = (dq)^{-k} \left(\frac{1+ie^q}{1-ie^q} \right)^{-re} \left(\frac{1+ie^q}{1-ie^q} \right)^{r(m+\epsilon)}$$

$$= f_{rm}(q)$$

This is only normalized for the C_q^ϵ . For the discrete series we must add an appropriate factor, as shown in or

Using this J_3 eigenfunction, the matrix elements in the mixed basis in the q-space formalism are:

$$\langle p, r^+ | U(q) | m \rangle_\epsilon = \frac{1}{2\pi} \int_{-\infty}^{\infty} dq e^{+ipq} [U(q) | \Phi^{(m)}]_r(q)$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} dq e^{+ipq} (dq | \partial_m N_r |)^{-k} \sum_{r'} \Theta(r' \partial_m N_r) f_{r'm}(q')$$

$$\langle p | U(q) | m \rangle_{k^\pm} = \frac{1}{4\pi} \cdot \sqrt{\frac{W_\pm(p)}{\pi}} \cdot \sqrt{\frac{P(k \pm m)}{P(1 \mp k \pm m)}}$$

$$\times \int_{-\infty}^{\infty} dq \cdot e^{+ipq} \cdot (dq | \partial_m N_+ |)^{-k} \sum_{r'} \Theta(r' \partial_m N_+) f_{r'm}(q')$$

Alternatively, the mixed basis matrix elements may be stated in z-space. In this case, we need the K_1 eigenfunctions converted to z-space. These are:

$$f^{(p,s)}(z) = e^{i\theta\epsilon} | \sin \theta |^{-k} \left| \tan \frac{\theta}{2} \right|^{-ip} \Theta(s \cdot \sin \theta) \text{ on } C_q^\epsilon$$

This follows from the rules.

$$f(z) = \sum_r \theta(r \sin \theta) [f_e(q)] f_r(q)$$

$$e^q = |\tan \frac{\theta}{2}|.$$

$$dq = (\sin \theta)^{-1}.$$

For the $P_k^\pm$, the corresponding functions are:

$$f^{(P)}(z) = \frac{1}{\sqrt{\pi W_\pm(P)}} \cdot e^{-ipq} \cdot [f_e(q) + f_{-e}(q) e^{-\pi p} e^{-i\pi k s(q)}]$$

$$= \frac{1}{\sqrt{\pi W_\pm(P)}} \cdot |\tan \frac{\theta}{2}|^{-p} \left[e^{i\theta e} + e^{-i\theta e} e^{\mp \pi p} e^{\mp i\pi k s(-\sin \theta)} \right] |\sin \theta|^{-k}.$$

These functions are so unpleasant, that I think I will just forget about doing these mixed basis calculations in the z-space and stick with the q-space.

History of the Search for a $C^q = D_k^+$ connection.

- ① First, I typed up sort of a "proposal" for the idea of searching for this connection. Goal was to compute both C^q and D_k^+ matrix elements in the same manner using the same group action etc.
- ② Next, I looked at the variable change from z to q and discovered that yes indeed, the function $f(q) = (chq)^{-k} f(z)$ is analytic in the box if $f(z)$ is one of those subspace functions in z -space appropriate to D_k^+ , ie, $f(z) = z^k f_{an}(z)$.
- ③ Then I came up with a nice group action for D_k^+ , namely, I just used $UF(q)$ on the lower line and shifted down the result using a little phase. The down shift is OK because $F(q)$ is analytic in the box. This group action, then mapped a function on the lower line into another function on the lower line. It looked good. In particular, I computed the group action on e^{ipq} .
- ④ Next, I simply assumed that the Muk p -space local scalar product was the one to use. I used this scalar product for D_k^+ and began calculating D_k^+ matrix elements. Sure enough, I found that the integrals occurring in the D_k^+ calculations were the same ones which appear in the C^q . This was the desired goal!
- ⑤ Next, I showed that the various integrals $I_{rr'}$ were exactly the integrals used by CDM. (At least for one of the generators which CDM call K_x). I then related the CDM functions to my Azimov PQ functions and filed this under PQ.
- ⑥ Feeling exuberant, and excited to boot, I started at this point to consider the possibility of computing mixed basis matrix elements. I stuck to the Muk p -space scalar product and merely replaced the right-side e^{ipq} with good old $f_m(q)$. This then gave a nice single integral rep for the mixed basis matrix elements. I gave this integral rep the identity test, and the trivial K_1 test. It worked!
- ⑦ Then came the first sign of trouble when I tried out the J_3 test. There are of course two terms and this is the case where the second term does not vanish from its theta function. The J_3 test failed. You are supposed to just get a phase times the cross matrix, but it just didn't work. I began to wonder if my generators were wrong.
- ⑧ I then took a look at the generators, they looked OK, but then it hit me: my use of the Muk p -space scalar product was not justified. Here, I stopped worrying about the J_3 failure, and began thinking of scalar products.

9. So, I then started into the problem of finding a nice scalar product for D_k^+ consistent with my idea of doing C^q and D_k^+ at the same time. My first idea was to take the m -sum form and replace the two Fourier coefficients with box integrals (using the usual function multiplier). This form looked promising for mixed basis matrix elements (though I did not pursue the idea), but was clearly no good for fully continuous basis matrix elements.
10. So, I then started with the disc form and began converting it to a box area integral form. Discovered the nice Jacobian idea for area. I then realized for the first time that you need to get a function of $\text{Im}(q)$ only inside your integrand if you expect to get a local scalar product in p -space. Without thoroughly checking, I just looked at my integrand and concluded that it was not simply a function of $\text{Im}(q)$ alone, so I concluded that a different multiplier had to be used.
11. The factor $(1 - |z|^2)$ was clearly a problem. I looked at how Mukunda handled this going to w -space, and then to q -space. I saw how he made it work, and I concluded that he did need a different multiplier. Had I looked closer, I would have seen roughly the form $(\text{ch}q)^{-k}$ times z^k as this new multiplier.
12. However, Mukunda's version of D_k^+ led to several confusions:
- a) he seemed to have β replaced with $i\beta$ so picture was rotated 90° . He did this, I guessed, to have top half circle map into upper w -plane.
 - b) also, however, his z differs from the z I am used to. This I learned after being unable to "undo" that 90° rotation.
- So, finding all these unpleasant differences, I sort of changed the subject for a while.
13. At this time I worked on showing why the m -sum form of the D_k^+ scalar product really was invariant. This led to an attempt to find the non-local scalar product in z -space by executing the "super-sum". This was a huge digression during which I learned some more ideas about contour integrals, but which has no direct relevance here, so I have pulled this section and put elsewhere.
14. Next, I began to consider the type of functions which belong in the D_k^+ subspace. I know this in z -space, and I converted it to q -space. The functions do have a certain well defined form. The functions $F(q)$ are analytic in the box, but the related ("upper and lower" functions are not exactly the boundary values of this function). There is a little $(-1)^k$ phase factor for the upper function. This could later be relevant to the J_3 problem mentioned above.

15. Now I define a special meaning for $f(z)$. This means a D_k^+ function, and it therefore has the form $z^k f_{an}(z)$. So $f(z)$ is a power series starting at k , not 0. I then wrote down the disc form of the scalar product in terms of functions of the type $f(z)$. A little different from the usual forms.

16. Having done this above, I then converted this disc integral to a box integral and searched for a viable multiplier such that the p -space scalar product would be local. After first goofing this up, I later found that the same old multiplier $(chq)^{-k}$ does the job. That is, $F(q) = (chq)^{-k} f(z)$. We use the old representation for $f(z)$ that we have always used, because after all, it is a function defined on the circle.

17. Thinking I had gotten the function multiplier $(chq)^{+k}$, I was giving up on the program of relating our two series of reps, and I began consideration of the Bargmann representation for the D_k^+ in terms of analytic functions. This led me to a very complicated multiplier for the group action. But now, correcting for the error, I see that this approach gives the same q -space group action as in the C^q . There is no need to go to this Bargmann thing. You lose the idea of subspace, so skip this. All equivalent.

18. Next, after making many corrections, I have worked on the conversion of Vilenkins subspace representation $[U[z^k f_{an}(z)]]$ to q -space. I end up with what I think is a viable and simple D_k^+ representation in q -space. As far as I am concerned, this thing maps the real line $F(q)$ onto the real line again. I do not yet know how this thing connects to the C^q form, nor have I checked the generators to see if they are right. I bet they are right.

19. I will now continue from this point.

Index of this support material for paper:

①. Miscellany:

- a) some relations between q, q', x, x' etc
- b) how do you make K_2 diagonal instead of K_1 : what happens.
- c) getting the $\exp(-i\phi J_3)$ elements from the $\exp(-ivK_2)$ elements
- d) group action directly in m -space (un-used)
- e) the lightlike basis functions.

②. The Group Representation

- a) derivation of the z -space form from first principles
- b) conversion to general $\tilde{f}(q)$ form
- c) conversion to D_K^S form
- d) conversion to C_q^L form

③. The D_K^S Scalar Products

- a) evaluation of $(1-z\bar{z})$, check
- b) derivation of the D_K^+ in ~~maximal~~ disc, box, p -space, check
- c) derivation of the D_K^- in disc ext, lower box, p -space, check

④. Summary of scalar products and basis functions

- a) list of various scalar products
- b) list of standard basis functions
- c) closer derivation of two basis functions
- d) the lightlike basis functions (see misc above)

⑤. Matrix Elements

- a) group action on basis functions
- b) matrix element setups
- c) integral representations for all elements
- d) computation of explicit mixed basis elements

6. JUNK

Summary of scalar products and basis functions

① For the C_q^S : $(f, g)_C = \frac{1}{2\pi} \int_0^{2\pi} d\theta \overline{f(z)} g(z) = \sum_{m=-\infty}^{\infty} \overline{f_m} g_m$

$$= \frac{1}{2\pi} \sum_r \int_{-\infty}^{\infty} dq \overline{f_r(q)} g_r(q) = \sum_r \int_{-\infty}^{\infty} dq \overline{f_{r,p}} g_{r,p} \cdot \frac{1}{4\pi^2}$$

where $f_{r,p} = \int_{-\infty}^{\infty} dq e^{+ipq} f_r(q)$.

② For the D_k^S : $(f, g)_S = \sum_{m=-\infty}^{\infty} \frac{\Gamma(1-k+sm)}{\Gamma(k+sm)} \overline{f_m} g_m$

$$= \frac{1}{\pi \Gamma(2k-1)} \int_{|z|>s} d^2z [s(1-|z|^2)]^{2k-2} |z|^{-2k-2\epsilon} \overline{f(z)} g(z)$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} dp \frac{e^{+ipq}}{|k+iq|^2} \overline{f_p} g_p$$

where $f_p = \int_{-\infty}^{\infty} dq e^{+ipq} \tilde{f}(q)$, and $f_m = \frac{1}{2\pi} \int_0^{2\pi} d\theta (\tilde{z})^{m+\epsilon} f(z)$

③ J_2 basis fctns: for C_q^S $\tilde{f}(q) = (dq)^{-k} \cdot \tilde{z}^{-\epsilon} f(z) = \frac{1}{2\pi i} \oint dz \cdot \tilde{z}^{-m-\epsilon+1} f(z)$

$|m, \epsilon\rangle \rightarrow (\tilde{z})^{m+\epsilon} = e^{i\theta(m+\epsilon)} = \phi^{(m, \epsilon)}(z)$

$\tilde{\phi}^{(m, \epsilon)}(q) = (dq)^{-k} \cdot (\tilde{z}(q))^m = (dq)^{-k} \cdot \left[\frac{1+ie^q}{1-ie^q} \right]^m$

$(dq)^k \tilde{\phi}^{(m, \epsilon)}(q) = (dx)^k \cdot \phi_r^{(m, \epsilon)}(x) = \left[\frac{1+ie^x}{1-ie^x} \right]^{rm}$

So: $\phi_r^{(m, \epsilon)}(x) = (dx)^{-k} \cdot \left[\frac{1+ie^x}{1-ie^x} \right]^{rm}$

for D_k^S :

$|m, s\rangle \rightarrow \sqrt{\frac{\Gamma(k+sm)}{\Gamma(1-k+sm)}} \cdot (\tilde{z})^{m+\epsilon} = \phi^{(m, s)}(\tilde{z})$

$\tilde{\phi}^{(m, s)}(q) = \sqrt{\frac{\Gamma(k+sm)}{\Gamma(1-k+sm)}} \cdot (dq)^{-k} \cdot \left[\frac{1+ie^q}{1-ie^q} \right]^m$

① The k basis fctns: for C_q^ϵ

$$\phi_{r'}^{(p,\epsilon)}(x) = \delta_{r,r'} e^{-ipx}$$

$$\phi_{r'}^{(p,\epsilon)}(x) = \delta_{r,r'} e^{-ipx}$$

$$\sum_{r'} \Theta(r' dq) \left(\frac{dx}{dq} \right)^k \phi_{r'}^{(p,\epsilon)}(x) = \left(\frac{dx}{dq} \right)^k \tilde{\phi}^{(p,\epsilon)}(q)$$

$$= \left(\frac{dx}{dq} \right)^k \sum_{r'} \Theta(r' dq) \delta_{r,r'} e^{-ipx}$$

$$= \left(\frac{dx}{dq} \right)^k \Theta(r dq) e^{-ipx}$$

$$\text{So } \tilde{\phi}^{(p,\epsilon)}(q) = \left(\frac{dx}{dq} \right)^k e^{-ipx} \Theta(r dq)$$

$$= \begin{cases} e^{-ipx} \Theta(r) & q = x \\ e^{-ipx} \left[\frac{dx}{dq} \right]^k \Theta(r) & q = x + i\pi \end{cases}$$

$$\phi_{r'}^{(p,\epsilon)}(z) = \left(\frac{dz}{dq(z)} \right)^k z^\epsilon \tilde{\phi}^{(p,\epsilon)}(q)$$

$$= \left(\frac{dz}{dq} \right)^k z^\epsilon \left(\frac{dx}{dq} \right)^k e^{-ipx} \Theta(r dq)$$

$$= z^\epsilon \left(\frac{dx}{dq} \right)^k e^{-ipx} \Theta(r dq) \quad \checkmark$$

Now, $dx = \left| \frac{2z}{z^2-1} \right|$ $e^x = \left| \frac{z-1}{z+1} \right|$ $\Theta(r dq) = \Theta(r \partial_m z)$

$$\text{So } \phi_{r'}^{(p,\epsilon)}(z) = z^\epsilon \cdot \left| \frac{2z}{z^2-1} \right|^k \cdot \left| \frac{z-1}{z+1} \right|^{-ip} \Theta(r \partial_m z)$$

Here, z on circle, so

$$\phi_{r'}^{(p,\epsilon)}(z) = z^\epsilon \cdot |z-1|^{-k-ip} \cdot |z+1|^{-k+ip} \cdot \Theta(r \partial_m z) \cdot 2^k \quad \checkmark$$

For k :

$$\tilde{\phi}^{(p,s)}(x) = \sqrt{\frac{|\Gamma(k+ip)|^2}{2\pi e^{\pi p}}} \cdot e^{-ipx} \quad \checkmark$$

Also here $x \rightarrow q$, q in correct box.

Now $\phi^{(p,s)}(z) = \frac{|\Gamma(k+ip)|^2}{2\pi e^{s\pi p}} e^{-ipx} (chq)^{-k} z^{\epsilon}$

see correct sheet.

$$= \int \left[-i \left(\frac{z-1}{z+1} \right) \right]^{-ip} \left(\frac{2iz}{z^2-1} \right)^k z^{\epsilon}$$

So: $\phi^{(p,s)}(z) = \frac{|\Gamma(k+ip)|^2}{2\pi e^{s\pi p}} 2^k (i)^{k+ip} (z+1)^{-k-ip} (z+1)^{-k+ip} z^{k+\epsilon}$ ✓

⑤ Some shorthand:

$$\frac{|\Gamma(k+sm)|}{|\Gamma(1-k+sm)|} \equiv {}^{(ks)}A_m$$

$$\frac{|\Gamma(k+ip)|^2}{2\pi e^{s\pi p}} \equiv {}^{(ks)}B_p$$

Then we have:

$$(f, g)_s = \sum_{m=-\infty}^{\infty} {}^sA_m^{-2} \bar{f}_m g_m$$

$$= \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} dp {}^sB_p^{-2} \bar{f}_p g_p$$

$$\left[\begin{array}{l} (1) \quad \phi_3^{(m,s)}(z) = {}^sA_m z^{m+\epsilon} \\ (2) \quad \tilde{\phi}_3^{(m,s)}(q) = {}^sA_m (chq)^{ik} \left[\frac{1+ie^q}{1-ie^q} \right]^m \end{array} \right]$$

And

$$\phi_p^{(m,s)} = \int_{-\infty}^{\infty} dx e^{+ipx} \tilde{\phi}^{(m,s)}(x) = {}^sA_m \int_{-\infty}^{\infty} dx e^{+ipx} (chx)^{-k} \left(\frac{1+ie^x}{1-ie^x} \right)^m$$

So

$$\phi_p^{(m,s)} = {}^sA_m \cdot 2^k (is)^{ip+k} \frac{|\Gamma(k+ip)|^2}{\Gamma(2k)} F(k-sm, k+ip; 2k; 2)$$

$$\phi_m^{(m,s)} =$$

Also, [

$$(3) \quad \tilde{\phi}^{(p,s)}(q) = {}^s B_p e^{-ipq} \checkmark$$

$$(4) \quad \phi^{(p,s)}(z) = {}^s B_p \cdot 2^k (-i)^{k+ip} (z+1)^{-k-ip} (z+1)^{-k+ip} z^{k+\epsilon} \checkmark$$

⑥ Detail, now summarize Q^ϵ :

$$(5) \quad \phi^{(n,\epsilon)}(z) = z^{n+\epsilon}$$

$$(6) \quad \phi_r^{(n,\epsilon)}(x) = (ix)^k \left[\frac{1+ie^x}{1-ie^x} \right]^{rm}$$

$$(7) \quad \phi_r^{(p,\epsilon)}(x) = \sum_{a,r} e^{-ipx} \checkmark$$

$$(8) \quad \phi^{(p,\epsilon)}(z) = z^\epsilon \cdot |z-1|^{-k-ip} \cdot |z+1|^{-k+ip} \cdot \Theta(ad_m z) \cdot 2^k \checkmark$$

derivation of the $\phi^{(pr, \epsilon)}(z)$ basis fctn.

① Start with

$$\phi_{r'}^{(pr, \epsilon)}(x) = \delta_{r, r'} e^{-ipx} \quad \text{then:}$$

$$\tilde{\phi}^{(pr, \epsilon)}(z) = \delta_{r, +} e^{-ipx} \quad \text{when } q = x$$

$$\tilde{\phi}^{(pr, \epsilon)}(q) = e^{-i\pi sk S(x)} \delta_{r, -} e^{-ipx} \quad \text{when } q = x + i\pi s.$$

So you could combine to say

$$\tilde{\phi}^{(pr, \epsilon)}(q) = \theta(chq) \phi_+^{(pr, \epsilon)}(x) + \theta(-chq) e^{-i\pi sk S(x)} \phi_-^{(pr, \epsilon)}(x)$$

② Now, add in $z^\epsilon (chq)^k$ and you have

$$\phi^{(pr, \epsilon)}(z) = z^\epsilon \left[\theta(chq) \phi_+^{(pr, \epsilon)}(x) + \theta(-chq) (chq)^k e^{-i\pi sk S(x)} \phi_-^{(pr, \epsilon)}(x) \right]$$

$$= z^\epsilon (chx)^k \left[\theta(chq) \phi_+ + \theta(-chq) \phi_- \right]$$

$$= z^\epsilon (chx)^k \sum_{r'} \theta(r' chq) \phi_{r'}^{(pr, \epsilon)}(x) = \theta(r chq) \phi e^{-ipx}$$

$$= z^\epsilon (chx)^k \cdot \theta(r chq) \cdot e^{-ipx}$$

③ Now $chx = |chq| = \left| \frac{2iz}{z^2 - 1} \right| = 2|z^2 - 1|^{-1} = 2|1 - z^2|^{-1}$

$$e^x = |e^q| = \left| i \left(\frac{1-z}{1+z} \right) \right| = \left| \frac{1-z}{1+z} \right|$$

$$S_0 = z^\epsilon \cdot 2^k \cdot |1-z|^{-k} \cdot |1+z|^{-k} \cdot |1-z|^{-ip} \cdot |1+z|^{+ip} \cdot \theta(r chq)$$

$\Rightarrow$

$$\phi^{(pr, \epsilon)}(z) = 2^k \theta(r \operatorname{Im} z) \cdot z^\epsilon \cdot |1-z|^{-k-ip} \cdot |1+z|^{-k+ip}$$

derivation of the $\phi^{(p,s)}(z)$ basis function.

① Our scalar product reads:

$$(f, g)_s = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} dp \left({}^s B_p \right)^{-2} \bar{f}_p g_p.$$

Claim: $\boxed{\tilde{\phi}^{(p,s)}(q) = {}^s B_p \cdot e^{-ipq}} \checkmark$

Proof: $g_{p'} = \phi_{p'}^{(p,s)} = \int_{-\infty}^{\infty} dx \cdot e^{+ip'x} {}^s B_p e^{-ipx} = 2\pi {}^s B_p \delta(p-p')$

So $|g_{p'}|^2 = 4\pi^2 ({}^s B_p)^2 \delta(p-p') \delta(p-p')$

$\Rightarrow \langle p|p' \rangle = \delta(p-p') \checkmark$

② Now convert this:

$$\begin{aligned} \phi^{(p,s)}(z) &= z^e \cdot (dq)^{+k} \cdot \tilde{\phi}^{(p,s)}(q) \checkmark \\ &= {}^s B_p \cdot (e^q)^{-ip} z^e \cdot (dq)^{+k} \checkmark \end{aligned}$$

③ Ansatz: $e^q = -i \left(\frac{z-1}{z+1} \right) = i \left(\frac{1-z}{1+z} \right) \checkmark$

$$dq = \frac{2iz}{z^2-1} = \frac{-2iz}{1-z^2} = \frac{(-2i)z}{(1-z)(1+z)} \checkmark$$

Then:

$$\begin{aligned} &= {}^s B_p \cdot (i)^{-ip} (1-z)^{-ip} (1+z)^{+ip} \cdot z^e \cdot (2)^k \cdot (-i)^k \cdot z^k \cdot (1-z)^{-k} (1+z)^{-k} \checkmark \\ &= {}^s B_p (i)^{-k-ip} 2^k \cdot z^{k+e} (1-z)^{-k-ip} (1+z)^{-k+ip} \checkmark \end{aligned}$$

So $\boxed{\phi^{(p,s)}(z) = {}^s B_p 2^k (i)^{-k-ip} \cdot z^{k+e} (1-z)^{-k-ip} (1+z)^{-k+ip}} \checkmark$

It is much more useful to deal with $(1-z)$ rather than $(z-1)$ when you are planning to be dealing with hypergeometric and gamma interpretations !!!

group action on basis functions

$$\textcircled{1} \quad [U \phi^{(m,s)}](x) = |\beta dx|^{-k} \left\{ \Theta(\beta) \phi^{(m,s)}(x') + \Theta(-\beta) e^{+i\pi k \Im(x')} \tilde{\phi}^{(m,s)}(x' + i\pi) \right\}$$

But $\tilde{\phi}^{(m,s)}(x') = {}^s A_m (dx')^{-k} \left[\frac{1+ie^{x'}}{1-ie^{x'}} \right]^m$. In first term, though,

$x' = q'$ and $\left(\frac{1+ie^{x'}}{1-ie^{x'}} \right)^m = \text{here } (z')^m = \left(\frac{N(q)}{D(q)} \right)^m$. Now $q = \text{real}$, so $= \left(\frac{N_+(x)}{D_+(x)} \right)^m$.

Next, $\tilde{\phi}^{(m,s)}(x' + i\pi) = \tilde{\phi}^{(m,s)}(q') = {}^s A_m (dx')^{-k} \left[\frac{1+ie^{q'}}{1-ie^{q'}} \right]^m$, but

now $q' = x' + i\pi$. Still, this is $(z')^m = \left(\frac{N}{D} \right)^m = \left(\frac{N_+(x)}{D_+(x)} \right)^m$. So

both terms are the same, phase is cancelled, $\Theta(\beta) + \Theta(-\beta) = 1$.

$$[U \phi^{(m,s)}](x) = |dm N_+(x) dx|^{-k} (dx')^{-k} \left(\frac{N_+(x)}{D_+(x)} \right)^m$$

But $(|dm N_+(x) dx dx'|) = (D_+(x) dx)$.

$$\Rightarrow [U \phi^{(m,s)}](x) = (D_+(x) dx)^{-k} {}^s A_m \left[\frac{N_+(x)}{D_+(x)} \right]^m$$

$$\textcircled{2} \quad [U \phi^{(m,s)}]_r(x) = |\beta_r dx|^{-k} \sum_{r'} \Theta(r' \beta_r) \phi_{r'}^{(m,s)}(x')$$

Again, in $\Theta(\beta_r)$ term, $[] \rightarrow (z')$

$= \left(\frac{N(q)}{D(q)} \right)^m = \left(\frac{N_r(x)}{D_r(x)} \right)^m$ since r now locates q .

$(dx')^{-k} \left[\frac{1+ie^{x'}}{1-ie^{x'}} \right]^{r'm}$

In $\Theta(-\beta_r)$, have $\left[\frac{1+ie^{x'}}{1-ie^{x'}} \right]^{-m} = \left[\frac{1+ie^{q'}}{1-ie^{q'}} \right]^{+m} = (z')^m = \left(\frac{N_r(x)}{D_r(x)} \right)^m$.

Again, combine them: So:

$$[U \phi^{(m,s)}]_r(x) = (D_r(x) dx)^{-k} \left[\frac{N_r(x)}{D_r(x)} \right]^m$$

④

$$[U\phi^{(p,s)}]^\sim(x) = |\beta \cdot dx|^{-k} \left\{ \Theta(\beta) \tilde{\phi}^{(p)}(x') + \Theta(-\beta) e^{+i\pi s k \delta(x') - (p,s)} \tilde{\phi}(x' + i s \pi) \right\}$$

$$[U\phi^{(p,s)}]^\sim(x) = s\beta_r \cdot e^{-ipx'} \cdot |\beta_r dx|^{-k} \left\{ \Theta(\beta) + \Theta(-\beta) e^{+i\pi s k \delta(x) - \pi s p} \right\}$$

$$\begin{aligned} \textcircled{5} \quad [U\phi^{(p,s)}]_r(x) &= |\beta_r dx|^{-k} \sum_{r'} \Theta(\beta_r) \underbrace{\phi_{r'}^{(p,s)}(x')}_{\delta_{0,r'} e^{-ipx'}} \\ &= |\beta_r dx|^{-k} \Theta(\alpha\beta_r) e^{-ipx'} \end{aligned}$$

So:

$$[U\phi^{(p,s)}]_r(x) = e^{-ipx'} |\beta_r dx|^{-k} \cdot \Theta(\alpha\beta_r)$$

⑥ Now go to z -space, same order.

$$\begin{aligned} [U\phi^{(m,s)}](z) &= z^{k+\epsilon} A^{-k-\epsilon} B^{-k+\epsilon} \underbrace{\phi^{(m,s)}(z')}_{s A_m(z')^{m+\epsilon}} \\ &= z^{k+\epsilon} A^{-k-\epsilon} B^{-k+\epsilon} s A_m \cdot A^{-m+\epsilon} B^{-m-\epsilon} \\ &= z^{k+\epsilon} A^{-m-k} B^{-m-k} s A_m \end{aligned}$$

So

$$[U\phi^{(m,s)}](z) = s A_m \cdot z^{k+\epsilon} A^{-k+m} B^{-k-m}$$

⑦ Next,

$$[U\phi^{(m,e)}](z) = z^{k+\epsilon} A^{-k-\epsilon} B^{-k+\epsilon} \underbrace{\phi^{(m,e)}(z')}_{(z')^{m+\epsilon}} \rightarrow (z')^{m+\epsilon}$$

$$\Rightarrow [U\phi^{(m,e)}](z) = z^{k+\epsilon} A^{-k+m} B^{-k-m}$$

$$\begin{aligned}
 \textcircled{8} \quad [U \phi^{(p,q)}](z) &= z^{k+\epsilon} A^{-k-\epsilon} B^{-k+\epsilon} \underbrace{\phi^{(p,q)}(z')}_{s B_p (-i)^{k+i} 2^k (z'+1)^{-k-i} (z'+1)^{-k+i} (z')^{k+\epsilon}} \\
 &= (z^{k+\epsilon} s B_p (-i)^{k+i} 2^k) \cdot \cancel{A^{-k-\epsilon}} \cdot \cancel{B^{-k+\epsilon}} \cdot (A+B)^{-k-i} \cdot \cancel{B^{-k+i}} \\
 &\quad \times (A+B)^{-k+i} \cdot \cancel{B^{-k-i}} \cdot \cancel{A^{k+\epsilon}} \cdot (B)^{-k-\epsilon}
 \end{aligned}$$

$$\Rightarrow [U \phi^{(p,q)}](z) = s B_p \cdot 2^k (-i)^{k+i} z^{k+\epsilon} (A+B)^{-k-i} (A+B)^{-k-i}$$

$$\begin{aligned}
 \textcircled{9} \quad [U \phi^{(p,q,\epsilon)}](z) &= z^{k+\epsilon} A^{-k-\epsilon} B^{-k+\epsilon} \underbrace{\phi^{(p,q,\epsilon)}(z')}_{2^k (z')^\epsilon |z'+1|^{-k+i} |z'-1|^{-k-i} \Theta(a \partial_m z')} \\
 &= (z^{k+\epsilon} 2^k) \cdot \cancel{A^{-k-\epsilon}} \cdot \cancel{B^{-k+\epsilon}} \cdot \cancel{A^{-k-i}} \cdot \cancel{B^{k+i}} \cdot |A+B|^{-k+i} \cdot |B|^{k-i} \cdot |A-B|^{-k-i} \cdot |B|^{k+i} \Theta(a \partial_m z') \\
 &= 2^k z^{k+\epsilon} \cdot \underbrace{A^{-k} B^{-k} |B|^{2k}}_{= z^{-k}} \cdot |A+B|^{-k+i} \cdot |A-B|^{-k-i} \Theta(a \partial_m z')
 \end{aligned}$$

$$\text{So} \quad [U \phi^{(p,q,\epsilon)}](z) = 2^k z^{k+\epsilon} |A+B|^{-k+i} |A-B|^{-k-i} \Theta(a \partial_m z')$$

matrix element setups

① First, do special form setups:

$$\langle m, g \rangle_s = \langle \phi^{(m,s)}, g \rangle_s = \sum_{m' \in \mathbb{Z}} s A_{m'}^{-2} \overline{\phi_{m'}^{(m,s)}} g_{m'}$$

But,

$$\phi_{m'}^{(m,s)} = \frac{1}{2\pi} \int_0^{2\pi} d\theta e^{-i\theta(m'+\epsilon)s} A_m e^{i\theta(m+\epsilon)s} = s A_m \delta_{m,m'}$$

So:

$$\begin{aligned} \langle \phi^{(m,s)}, g \rangle_s &= s A_m^{-1} g_m \\ &= s A_m^{-1} \cdot \frac{1}{2\pi i} \oint_{|z|=1} dz \cdot z^{-m-\epsilon-1} g(z) \end{aligned}$$

② Second, consider:

$$\langle \phi^{(p,s)}, g \rangle_s = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} dp \ s B_p^{-2} \overline{\phi_p^{(p,s)}} g_p$$

But

$$\begin{aligned} \phi_p^{(p,s)} &= \int_{-\infty}^{\infty} dx e^{ipx} \tilde{\phi}^{(p,s)}(x) = \int_{-\infty}^{\infty} dx e^{ipx} s B_p e^{-ip'x} \\ &= 2\pi \delta(p-p') s B_p \end{aligned}$$

So:

$$\begin{aligned} \langle \phi^{(p,s)}, g \rangle_s &= \frac{1}{2\pi} s B_p^{-1} g_p \\ &= s B_p^{-1} \cdot \frac{1}{2\pi} \int_{-\infty}^{\infty} dx e^{ipx} \tilde{g}(x) \end{aligned}$$

③ Repeat for C_q :

$$\langle \phi^{(n,\epsilon)}, g \rangle_\epsilon = \sum_{m' \in \mathbb{Z}} \overline{\phi_{m'}^{(n,\epsilon)}} g_{m'}$$

But $\phi_{m'}^{(n,\epsilon)} = \delta_{m,m'}$ so

$$\langle \phi^{(n,\epsilon)}, g \rangle_\epsilon = g_n = \frac{1}{2\pi i} \oint dz \cdot z^{-n-\epsilon-1} g(z)$$

④ Now:

$$(\phi^{(p,q,\epsilon)}, g)_\epsilon = \frac{1}{4\pi^2} \sum_r \int_{-\infty}^{\infty} dp \cdot \overline{\phi_{r,p}^{(p,q,\epsilon)}} g_{r,p}$$

$$\text{But } \phi_{r,p}^{(p',q,\epsilon)} = \int_{-\infty}^{\infty} dx e^{ipx} \cdot \underbrace{\phi_r^{(p',q)}(x)}_{\delta_{a,r} e^{-ip'x}} = \delta_{a,r} \cdot 2\pi \delta(p-p')$$

So

$$\begin{aligned} (\phi^{(p,q,\epsilon)}, g)_\epsilon &= \frac{1}{2\pi} g_{a,p} \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} dx e^{+ipx} g_a(x) \end{aligned}$$

⑤ Comparisons:

$$(\phi^{(p,s)}, g)_s = S_p^{-1} \cdot (\phi^{(p+, \epsilon)}, g)_\epsilon$$

This seems true provided that $g_+(x) = \tilde{g}(x)$. Look in more detail later. Also:

$$(\phi^{(m,s)}, g)_s = A_m^{-1} \cdot (\phi^{(m, \epsilon)}, g)_\epsilon$$

⑥ In particular:

$$(\phi^{(p,s)}, u(g) \phi^{(p',s)})_s = S_p^{-1} (\phi^{(p+, \epsilon)}, ?)$$

$$= S_p^{-1} (\phi^{(p+, \epsilon)}, u(g) \phi^{(p', \epsilon)})_\epsilon$$

↑
This is where confusion
needs alleviation.

integral representations for all matrix elements.

$$\textcircled{1} \quad (\phi^{(m,s)}, U \phi^{(m',s)})_s = s^{-1} A_m^{-1} \frac{1}{2\pi i} \oint dz \cdot z^{-m-e-1} [U \phi^{(m',s)}](z)$$

$$s A_m z^{k+e} A^{-k+m'} B^{-k-m'}$$

$$\Rightarrow \boxed{\langle m | U | m' \rangle_s = \frac{s A_m'}{s A_m} \frac{1}{2\pi i} \oint dz (z)^{k-m-1} A^{-k+m'} B^{-k-m'}} \quad \text{done}$$

$$\textcircled{2} \quad (\phi^{(m,e)}, U \phi^{(m',e)})_e = \frac{1}{2\pi i} \oint dz z^{-m-e-1} [U \phi^{(m',e)}](z)$$

$$z^{k+e} A^{-k+m'} B^{-k-m'}$$

$$\Rightarrow \boxed{\langle m | U | m' \rangle_e = \frac{1}{2\pi i} \oint dz (z)^{k-m-1} A^{-k+m'} B^{-k-m'}} \quad \text{done}$$

$$\textcircled{3} \quad (\phi^{(m,s)}, U \phi^{(p,s)})_s = s^{-1} A_m^{-1} \frac{1}{2\pi i} \oint dz z^{-m-e-1} [U \phi^{(p,s)}](z)$$

$$s B_p 2^k (i)^{k+i} z^{k+e} (A+B)^{-k+i} (A-B)^{-k-i}$$

$$\Rightarrow \boxed{\langle m | U | p \rangle_s = \frac{s B_p}{s A_m} 2^k (i)^{k+i} \frac{1}{2\pi i} \oint dz z^{k-m-1} (A+B)^{-k+i} (A+B)^{-k-i}} \quad \text{done by 5, not nice}$$

by ⑤

$$\textcircled{4} \quad (\phi^{(m,e)}, U \phi^{(p,e)})_e = \frac{1}{2\pi i} \oint dz z^{-m-e-1} [U \phi^{(p,e)}](z)$$

$$2^k z^e |A+B|^{-k+i} |A-B|^{-k-i} O(\partial \theta \epsilon).$$

$$\Rightarrow \boxed{\langle m | U | p \rangle_e = 2^k \frac{1}{2\pi i} \oint dz O(\partial \theta \epsilon) \cdot z^{-m-1} |A+B|^{-k+i} |A-B|^{-k-i}}$$

A/B

ugly, use ⑥

Aside: a) $A+B = (\alpha-\beta)z + (2-\bar{\beta})$
 $A-B = (\alpha+\beta)z - (2+\bar{\beta})$

b) $z' = \frac{1+ie^{x'}}{1-ie^{x'}}$ in context of ⑤ and ⑥ [i.e., not A/B].

⑤

$$(\phi^{(p,s)}, u \phi^{(m,s)})_s = {}^s B_p^{-1} \frac{1}{2\pi} \int_{-\infty}^{\infty} dx e^{ipx} \underbrace{[u \phi^{(m,s)}]_s(x)}_{\substack{\left(\frac{N_+(x)}{D_+(x)} \right)^m}} \\ {}^s A_m (D_+(x) dx)^{-k} \left(\frac{N_+(x)}{D_+(x)} \right)^m \quad \text{done by 4a}$$

$$\Rightarrow \langle p | u | m \rangle_s = \frac{{}^s A_m}{{}^s B_p} \cdot \frac{1}{2\pi} \int_{-\infty}^{\infty} dx' e^{ipx'} (D_+(x) dx)^{-k} \left(\frac{N_+(x)}{D_+(x)} \right)^m \\ = \frac{{}^s A_m}{{}^s B_p} \frac{1}{2\pi} \int_{-\infty}^{\infty} dx' e^{ipx'} (D_+(x) dx)^{-k-m} (N_+(x) dx)^m$$

⑥

$$(\phi^{(p,e)}, u \phi^{(m,e)})_e = \frac{1}{2\pi} \int_{-\infty}^{\infty} dx' e^{+ipx'} \underbrace{[u \phi^{(m,e)}]_r(x)}_{\substack{\left(\frac{N_r(x)}{D_r(x)} \right)^m}} \\ (D_r(x) dx)^{-k} \left(\frac{N_r(x)}{D_r(x)} \right)^m \quad \text{done by 4a}$$

$$\Rightarrow \langle p_r | u | m \rangle_e = \frac{1}{2\pi} \int_{-\infty}^{\infty} dx e^{+ipx} (D_r(x) dx)^{-k} \left(\frac{N_r(x)}{D_r(x)} \right)^m \\ = \frac{1}{2\pi} \int_{-\infty}^{\infty} dx e^{+ipx} (D_r(x) dx)^{-k-m} (N_r(x) dx)^m$$

⑦

$$(\phi^{(p,s)}, u \phi^{(p',s)})_s = {}^s B_p^{-1} \frac{1}{2\pi} \int dx e^{ipx} \underbrace{[u \phi^{(p',s)}]_s(x)}_{\substack{{}^s B_{p'} e^{-ip'x'} |\beta dx|^{-k} \{ \Theta(\beta) + \alpha(-\beta) \exp(+i\pi s k \mathcal{S}(x')) \} e^{+i\pi s p}}}}$$

$$\Rightarrow \langle p | u | p' \rangle_s = \frac{{}^s B_{p'}}{{}^s B_p} \cdot \frac{1}{2\pi} \int_{-\infty}^{\infty} dx e^{ipx} e^{-ip'x'} |\beta dx|^{-k} \left\{ \Theta(\beta) + \alpha(-\beta) e^{+i\pi s k \mathcal{S}(x')} e^{+i\pi s p} \right\} \quad \text{relate to below}$$

⑧

$$(\phi^{(p,a,e)}, u \phi^{(p',a,e)})_e = \frac{1}{2\pi} \int dx e^{ipx} \underbrace{[u \phi^{(p',a,e)}]_a(x)}_{\substack{e^{-ip'x'} |\beta_a dx|^{-k} \Theta(a' \beta_a)}} \\ e^{-ip'x'} |\beta_a dx|^{-k} \Theta(a' \beta_a)$$

$$\Rightarrow \langle p_r | u | p'_r \rangle_e = \frac{1}{2\pi} \int_{-\infty}^{\infty} dx e^{ipx} e^{-ip'x'} |\beta_r dx|^{-k} \Theta(r' \beta_r) \quad \text{done by 4a}$$

deal with $\langle p | U | m \rangle$ AND $\langle p | U | m \rangle_s$

$-i\nu k_2$

$$① = \frac{1}{2\pi} \int_{-\infty}^{\infty} dx e^{ipx} (dx)^{-k} [D_r(x)]^{-k} \left[\frac{N_r(x)}{D_r(x)} \right]^m$$

② First, look at $\nu \rightarrow +\infty$. Then $N_r(x) = i \operatorname{chv} \left(\frac{r}{\operatorname{chx}} - 1 \right)$

$$D_r(x) \approx -\operatorname{chv} \left(\frac{r}{\operatorname{chx}} - 1 \right)$$

$$\frac{N_r(x)}{D_r(x)} = (-i)$$

$$(\operatorname{chx} D_r(x))^{-k} = \operatorname{chx} \operatorname{chv} \left(1 - \frac{r}{\operatorname{chx}} \right) = \operatorname{chv} (\operatorname{chx} - r)$$

So $I = (-i)^m \cdot \frac{1}{2\pi} \left(\frac{1}{2} e^\nu \right)^{-k} \cdot \int_{-\infty}^{\infty} dx e^{+ipx} (\operatorname{chx} - r)^{-k} \rightarrow \underline{\underline{0}}$

③ Next, set $\nu = 0$ or $\pm i\epsilon$. Then $z' = z$. I.e.,

$$\frac{N_r(x)}{D_r(x)} = N_r(x) = \frac{(-\operatorname{chx} + ir/\operatorname{chx})}{1} = z = \frac{1 + i e^x}{1 - i e^x}$$

$$D_r(x) \approx 1 - re \rightarrow 1$$

Then $= \frac{1}{2\pi} \int_{-\infty}^{\infty} dx e^{ipx} (dx)^{-k} \left[\frac{1 + i e^x}{1 - i e^x} \right]^m$

④ Now, $k = j+1 = \frac{1}{2} + is$ so $\operatorname{Re} k = \frac{1}{2}$. If $(rm) > \frac{1}{2}$, we get answer ①. If $(-rm) > \frac{1}{2}$, other answer. i.e. $rm < -\frac{1}{2}$. So:

$$= \frac{1}{2\pi} \cdot \frac{\Gamma(k+ip)\Gamma(k-ip)}{\Gamma(2k)} 2^k \left[\begin{array}{l} \Theta(rm - \operatorname{Re} k) \cdot \frac{1}{\Theta(rm - \frac{1}{2})!} (i)^{ip+k} F(k-rm, k+ip; 2k; 2) \\ + \Theta(-rm - \frac{1}{2}) (-i)^{ip+k} F(k+rm, k+ip; 2k; 2) \end{array} \right]$$

Well, combine. Use $\Theta(rm)$ so

$$= \frac{1}{2\pi} \frac{\Gamma(k+ip)\Gamma(k-ip)}{\Gamma(2k)} 2^k (\operatorname{sign}(rm) i)^{ip+k} F(k-|rm|, k+ip; 2k; 2)$$

⑤ From the ODE, we know we are dealing with

$$\cancel{Q}_{\mu m}^{k-1}(-ishv)$$

$$\searrow \approx e^{-vk}$$

and

$$\cancel{Q}_{\mu m}^{-k}(-ishv)$$

$$\nearrow \searrow e^{-v(1-k)}$$

Rule out.

So answer must be:

$$C_1 \cancel{Q}_{\mu m}^{k-1}(-ishv)$$

⑥ Now take to $v \rightarrow 0$. Get: Let $v \rightarrow \pm i\epsilon$. Then

$$C_1 \cancel{Q}_{\mu m}^{k-1}(\mp i\epsilon) = C_1 e^{\pm i\pi[k + \frac{1}{2}(\mu+m)]} 2^{k-1}$$

$$\frac{\Gamma(k+\mu)\Gamma(k-m)}{\Gamma(2k)} F(k+i\epsilon, k-m; 2k; 2)$$

$$\text{Try } C_1 \cancel{Q}_{\mu, |rm|}^{k-1}(\mp i\epsilon) = C_1 e^{\pm i\pi[k + \frac{1}{2}(\mu+|rm|)]} 2^{k-1}$$

$$\frac{\Gamma(k+\mu)\Gamma(k-|rm|)}{\Gamma(2k)} F(k-|rm|, k+i\epsilon; 2k; 2)$$

⑦ Compare

(but this no sense w/ ODE!)

$$\cancel{\frac{1}{2\pi}} \frac{\cancel{\Gamma(k+\mu)}\Gamma(k-\mu)}{\cancel{\Gamma(2k)}} \cancel{2^k} (\text{sign}(rm)i)^{ip+k} \cancel{A}$$

$$= C_1 \frac{\cancel{\Gamma(k+\mu)}\Gamma(k-|rm|)}{\cancel{\Gamma(2k)}} e^{\pm i\pi[k + \frac{1}{2}(\mu+|rm|)]} \cancel{2^k} \cancel{\frac{1}{2}} \cancel{A}$$

$$\Rightarrow \frac{\Gamma(k-\mu)}{\pi} e^{\text{sign}(rm)i\frac{\pi}{2}(ip+k)} = C_1 e^{\pm i\pi[k + \frac{1}{2}(\mu+|rm|)]}$$

$$\Rightarrow C_1 (for v \gtrless 0) = e^{\pm i\pi k} e^{\pm i\frac{\pi}{2}(ip+|rm|)} e^{\text{sign}(rm)i\frac{\pi}{2}k}$$

$$\times \frac{\Gamma(k-ip)}{\pi} e^{-\frac{\pi}{2}p \text{sign}(rm)}$$

⑤

Now break into two cases:

I. $m > 0$
II. $m < 0$

$$\frac{1}{2\pi} \frac{\Gamma(k+\mu)\Gamma(k-\mu)}{\Gamma(2k)} 2^k \begin{cases} (-i)^{ip+k} F(k-\mu, k+\mu; 2k; z) \\ (-i)^{ip+k} F(k+\mu, k+\mu; 2k; z) \end{cases}$$

In case I, we take more cases:

I. $+$ $r > 0$
I. $-$ $r < 0$ etc.

Case I. $+$ $(+i)^{ip+k} F(k-\mu, \dots)$

$$\Rightarrow \Delta_{\mu, m}^{k-1} (+ishv)$$

Case I. $-$ $(+i)^{ip+k} F(k+\mu, \dots)$

$$\Rightarrow \Delta_{\mu, +m}^{k-1} (+ishv)$$

Case II. $+$ $F(k+\mu) \Rightarrow \Delta_{\mu, +m}^{k-1} (+ishv)$

II. $-$ $F(k-\mu) \Rightarrow \Delta_{\mu, m}^{k-1} (+ishv)$

Continue: Let $\sigma = \text{sign}(m)$. Then general answer is

$$\Delta_{\mu, \sigma m}^{k-1} (+ishv) = \Delta_{\mu, |m|}^{k-1} (+ishv)$$

⑥ Now we shall assume $C_1 \Delta_{\mu, |m|}^{k-1} (+ishv)$. Look up limit as $v \rightarrow \pm 0$.

$$C_1 \Delta_{\mu, |m|}^{k-1} (\pm ish) = C_1 \cdot e^{\pm i\pi [k + \frac{1}{2}(\mu \mp |m|)]} 2^{k-1}$$

$$\times \frac{\Gamma(k+ip)\Gamma(k+|m|)}{\Gamma(2k)} F(k \mp |m|, k+ip; 2k; z)$$

$$\text{Compare: } C_1 e^{\pm i\pi [k + \frac{1}{2}(\mu \mp |m|)]} \frac{1}{2} \Gamma(k+|m|) = \frac{1}{2\pi} \Gamma(k-ip) \cdot (r\sigma i)^{ip+k}$$

$$\text{So } C_1(v \geq 0) = \frac{1}{\pi} \frac{\Gamma(k-ip)}{\Gamma(k+|m|)} \cdot (r\sigma i)^{ip+k} \left(\frac{+i}{\eta} z\right)^{ip-|m|} \cdot e^{\pm i\pi k}$$

$$\langle p | e^{-i\nu K_z} | m \rangle_{\epsilon} = C_1 \cdot \bigoplus_{\mu, -\sigma m}^{k-1} (+i\sigma \text{sh} \nu)$$

$$\text{where } \sigma = \text{sign}(m)$$

$$\begin{aligned} \langle p | e^{-i\nu K_z} | m \rangle_{\epsilon} &= C_1 \bigoplus_{\mu, -|m|}^{k-1} (+i \text{sign}(m) \text{sh} \nu) \\ C_1 &= \frac{1}{\pi} \frac{\Gamma(k-ip)}{\Gamma(k+|m|)} (r\sigma i)^{ip+k} (\eta i)^{ip-|m|} e^{\eta i\pi k} \\ \text{where } \sigma &= \text{sign}(m); \quad \eta = \text{sign}(\nu) \end{aligned}$$

$$\textcircled{1} \text{ Now look at } \langle p | e^{-i\nu K_z} | m \rangle_s = \frac{{}^s A_m}{{}^s B_p} \left(\text{above with } r = + \right)_{\sigma = s}$$

$$\text{So } -|m| = -\sigma m$$

$$C_1 = \frac{1}{\pi} \frac{\Gamma(k-ip)}{\Gamma(k+|m|)} (is)^{ip+k} (\eta i)^{ip-|m|} e^{i\pi k \eta}$$

$$\langle p | e^{-i\nu K_z} | m \rangle_s = \frac{{}^s A_m}{{}^s B_p} C_1 \bigoplus_{\mu, -|m|}^{k-1} (+is \text{sh} \nu) \quad \eta = \text{sign}(\nu).$$

$$\text{Now } {}^s A_m = \frac{\Gamma(k+|m|)}{\sqrt{\Gamma(1-k+|m|)}} \quad {}^s B_p = \frac{|\Gamma(k+ip)|}{\sqrt{2\pi} e^{\frac{\pi p}{2}}}$$

$$\begin{aligned} \text{So } \langle p | e^{-i\nu K_z} | m \rangle_s &= \frac{(is)^{ip+k} (\eta i)^{ip-|m|} e^{i\pi k \eta}}{\pi \sqrt{\Gamma(k+|m|)\Gamma(1-k+|m|)}} \cdot \frac{\Gamma(k-ip)}{|\Gamma(k-ip)|} \sqrt{2\pi} \cdot e^{\frac{\pi p}{2}} \\ &\quad \times \bigoplus_{\mu, -|m|}^{k-1} (+is \text{sh} \nu) \end{aligned}$$

$$\text{Group: } (is)^{ip} (\eta i)^{ip} e^{\frac{\pi p}{2}} = e^{-\frac{\pi p}{2}}$$

So dropping phases I get:

$$\langle p | e^{-i\omega k_2} | m \rangle_s = \sqrt{\frac{2}{\pi}} \cdot e^{-\frac{E p^2}{2}} \cdot \frac{1}{\sqrt{\Gamma(k+|m|)\Gamma(1-k+|m|)}} \cdot Q_{ip_1-|m|}^{k-1} (+is\hbar\nu)$$

where $\kappa = \text{sign}(\nu)$.

June 17, 1976

A review of recent activities in the world of mixed-basis D^{k+} .

Several long pieces of work led up to this present work, so I will first mention some of the older things. I have long been interested in calculating the mixed basis matrix elements for $SO(2,1)$. Kainins succeeded in doing this using a realization of motions on a light cone. ~~XXX~~ He obtained the pleasant functions $Q_{m,p}$ and put me onto Azimov's paper. He noted that the discrete series stuff could not be calculated by his method. Now I was wondering how you might calculate these things. One particular realization in which I had invested much time was the MR 1971 paper. I had used this, in fact, to calculate the continuous basis D^k matrix elements and this led me to make the connection with Azimov's addition theorem and the group property of these functions.

Now, with MR71 I had for amusement one time computed the eigenfunctions of J_3 . These had come out being Laguerre polynomials and I spent lots of time solving the related problem of the radial differential equation for the hydrogen atom in 2 and 3 dimensions. After I calculated these things, they just sat around for a long time. I had once tried computing the cross matrix but left it in integral representation only, but got suggestion of a $F(\dots, -1)$ type thing. (IN agreement with LN).

So my recent surge of activity began with calculation of the specific basis functions in the two bases, in particular I wanted to get the normalizer factors. This attempt met with success. I had tried this earlier in the Bargmann realization of the D^{k+} and was very frustrated to end up with a triple sum of 15 gamma functions just for the normalizer calculation! I had given up on that "disc" representation.

So, once I had the explicit basis functions, I thought I might calculate the cross matrix. This was very easy, I used a standard Laguerre integral to get this. At this point I began one of the hardest calculations I have ever done, and it turned out to be all for nothing. I computed the matrix element for e^{iJ_3} in the mixed basis. Of course this is just a constant times the cross matrix, but I was not thinking very clearly at the time and I failed to realize this obvious fact. Thus, I entered the calculation blindly.

Actually, I first worked this out in the discrete basis. The e^{iJ_3} operator was just a multiplier representation, so the matrix element ended up being the product of two Laguerres. Luckily this was a standard form, and I got the answer as a HGF. After a few shifts, I finally arrived at the P^J_{mm} form with certain normalizer factors. I here encountered the problem of the "switch" symmetry relation of my functions not working in the D^{k+} sector.

At this point I said: "if you know the ME in the discrete basis, and if you know the explicit cross matrix, then you ought to be able to find the mixed basis ME using the cross matrix as the unitary basis change."

So, I started out writing the mixed bases ME as a helicity sum. This was then SWT'd into a helicity integral, the integrand being the $F(\dots, 2)$ and P_{μ} . Then I had to scramble to find a way to do this integral. I found that all integral reps were failing so I had to switch indicies on P_{μ} using the just-discovered "switch" relation. This left me with a helicity integral of: the product of two f 's, some gammas, and an expo. I then chose some $(0,1)$ range integral reps and quickly ran into theta function troubles. Then I chose the better integral reps and was able to elevate away everything and get some delta function action out of the helicity contour. This left us with our ME as a single s integral. But eventually I was able to do this integral, and the end product was --- much to my surprise --- just the cross matrix times ϵ^{ij} . Actually, I do not really claim that all factors are correct, but I tried to get them right here.

So, this was my first successful mixed basis calculation and the fact that I got a good answer lent support to my general procedure. The next obvious idea was to get a non-trivial matrix element. To do this, I used the hyperbolic boost theorems to get the new ME. In fact, this merely added a phase to the helicity integral integrand, which was surprising. However, that phase was just what was needed to destroy the convergence of my previous method, after elevation of the f 's into their integral reps. To remedy this situation, I changed the P into Q - Q and used or considered using my old integral rep for the Q . This method looked useable, but I did not finish it. The first Q integral was basically my old byproduct theorem, but the second integral was entirely new, so I just stopped at that point.

Then I was about to quit when I thought: why not just for fun write down this mixed basis matrix element as the double integral which MR say it must be. I did that and got double infinite integral of Laguerre times Bessel. I didn't think I could do this integral fast, but it turned out that the first integral was just the Bessel and that was in GR. The remaining integral was Laguerre times Whittaker. Then I converted Laguerre to ϕ to Whittaker so had product of two Whittakers. Then much to my amazement, I noticed that you can integrate any number of whittakers to get a new animal, the multivariable HGF's. I did this and got answer in terms of an F_2 Appells function of 7 arguments. Then, after studying up on these functions a bit, I noticed that this F_2 reduced to an F_1 and that low and behold that F was exactly $Q_{m,ip}$ (some argument). I never got the argument right but lots of fish 's were popping up around then so I assume that is the argument. (As I know based on the Casimir DE).

It then occurred to me that Bessel and Laguerre and Whittaker are all confluent HGF's and I noticed the direct formula in Bateman for the integration of two matched confluent to give an F . This reestablished the fact that $Q_{m,ip}$ really

was the right answer.

So, knowing the Matrix Elements for D^{k+} in mixed and continuous basis, the next obvious thing was to work up the mixed basis addition theorem which I have been thinking about so long. I considered the "type 1" relation which puts the continuous basis on the left side and puts an azimuth in the "middle". This effort resulted in more or less the Hobson sum rule, except for a factor of π^2 which I cannot explain yet. I reconsidered the convergence of this sum rule and concluded that, as before, it does not converge, even where it is supposed to, unless I have not been careful enough with my limits.

Maimark

1. Mairmark: "Linear Reps. of Lorentz Group" (1958 Moscow, 1964 New York)
QA171 N157

$\gamma_{\mu\nu} \geq 1 \Rightarrow$ "complete" Lorentz group $\mathfrak{so}_{1,3}$
 $\wedge \det = 1 \Rightarrow$ "proper" Lorentz group $\mathfrak{so}_{1,3}^+$ } $\mathfrak{so}_{1,3}$ and $\mathfrak{so}_{1,3}^+$ connected by $S = \text{parity}$.

$$G_0 = SO(3, R)$$

$$U = SU(2) = \text{"the unitary group"}$$

reps of $SU(2)$ are à la Valentin, on $f(z_1, z_2) \rightarrow \phi(z)$.

Contents:

- I. Rotation + Lorentz Groups
- II. Reps. of $SU(2)$
- III. The irreducible reps of $\mathfrak{so}_{1,3}$ and $\mathfrak{so}_{1,3}^+$.
- IV. Invariant Equations.

Note: This book is 450 pages; says nothing about Poincaré group.

Chapter III

Paragraph 7: Define the inf. operators.

Section 7.1: The 4×4 matrices
exponentiation of generators $a = \text{rotation}$
 $g = u \cdot b(t) \cdot v$ $b = \text{boost}$
(central decomposition)

Section 7.2: Lie algebra of $a_i, b_i = 4 \times 4$ generators

Section 7.3: infinitesimal operators A_i, B_i

Section 7.4: Lie of A_i, B_j .

Paragraph 8: What are general A_i, B_i ?

$$A \rightarrow H_{\pm}, H_3$$

Section 8.1: define H, F .

$$B \rightarrow F_{\pm}, F_3$$

Section 8.2: weight vectors... get H 's

Section 8.3: get F 's.

Theorem 2, p. 116: every irred. rep of G_+ is set by (k_0, c) where $k_0 = \text{int or half-int}$, $c = \text{complex}$. The actual action of $H_+, H_-, H_3, F_+, F_-, F_3$ on a basis f_{λ}^k is explicitly stated, analogous to $SU(2)$. If $c^2 = (k_0 + n)^2$ where $n = 1, 2, \dots$, then rep is finite dim and then $k = k_0, k_0 + 1, \dots, k_1$ and $\lambda = k, k-1, \dots, -k$. Otherwise, $k = k_0 \dots \infty$ and rep is inf dim.

Section 8.4: unitary reps?

Theorem 3: (1) (k_0, c) if $2k_0 = 1, 2, 3, \dots$ and $c = \text{imaginary}$
(2) $0 \leq c \leq 1$ and $k_0 = 0$

Paragraph 9: The finite-dim reps of $\mathcal{L}_+$

Paragraph:

Section 9.1: $SL(2, \mathbb{C})$ and $q' = aqa^*$ $U \equiv SL(2, \mathbb{C})$ now

Section 9.2: compare G_+ and U .

Section 9.3: reps of $SL(2, \mathbb{C})$ set up on $f(z, \bar{z})$.

Basis is $\begin{matrix} z^p \bar{z}^q \\ \uparrow \quad \uparrow \\ i's \text{ (dotted)} \quad j's \text{ (undotted)} \end{matrix}$ of rep χ_{pq} in Liebnicki notation

Section 9.4: the inf. ops in terms of $\partial/\partial z, \partial/\partial \bar{z}$.

Section 9.5: irreducibility

Section 9.6: standard basis for reps, to get Theorem 2 exactly.

Paragraph 10: The Principle Series Reps of U . (∞ dim).

Section 10.1: subgroup $K, \mathbb{Z}$

$$k = \begin{pmatrix} \lambda^{-1} & \mu \\ 0 & \lambda \end{pmatrix} \quad z = \begin{pmatrix} 0 \\ z \end{pmatrix}$$

Section 10.2: canonical decomp

$$a = k\mathbb{Z}. \quad (\text{trivial})$$

Section 10.3: "residue class" = a coset, labelled by z .
 $\mathbb{Z} \equiv$ space of cosets U/K $U = SL(2, \mathbb{C})$.

Section 10.4: $z' = \frac{az + b}{c\bar{z} + d} = \text{transform of } z$.

Section 10.5: invariant measure $dz = dx dy$.

Section 10.6: define the Principle Series Reps:

Works on $L^2(\mathbb{Z})$ with $(f_1, f_2) = \int dz f_1 f_2$, simple!

Sets up a simple multiplier rep $\forall a f(z) = \dots$, then
- hunts for multiplier χ . This leads to

$$\chi(\lambda) = |\lambda|^{is-m} \lambda^m$$

which is the inducing rep. of Schwarz/Toller, but here it is just a multiplier.

$m = \text{integer}$. $s = \text{real}$

Theorem 5: for (s, m) as above, $\forall a f(z) = d(za) f(za)$
- give you the principle series reps, unitary.

Section 10.7: irreducibility.

Paragraph 11:

Section 11.1: connect the Principle reps with $SU(2)$.

$$a = k \cdot u$$

Section 11.2: $L_2^m(u) =$ space of functions on $SU(2)$, which are left-invariant under z -rotation.

Section 11.3: proof that $\int |\phi(u)|^2 du = \int |f(z)|^2 dz$

Theorem 7: you can realize the Principle Rep in $L(u)$ as
 $\forall a \phi(u) = \frac{\alpha(ua)}{\alpha(u)} \phi(ua)$; similar to S/Toller.

Section 11.4: The rep S_k of $SU(2)$ [where $k = \text{usual } j$] is contained at most once in $\Gamma_{g,m}$. This happens when $m = 2k, \dots, -2k$.

Section 11.5: spherical functions. Pertains to $\Gamma_{g,m}$ when $m=0$, so
 $\chi(au) = \chi(ua) = \chi(a)$, $u \in SU(2)$ $a \in SL(2, \mathbb{C})$. There seems to be one specific form which does this.

Section 11.6: standard basis for Principle Reps.

$$\begin{aligned} \Delta = \dots & \quad \Delta = \text{fn of } (g, m) \\ \Delta' = \dots & \quad \Delta' = -mg. \end{aligned} \quad \left\{ \begin{array}{l} \text{casimirs} \end{array} \right.$$

Conclusion:

$$k = |m/2| \text{ and } c = -i(\text{sign } m) \pm g$$

$$\text{Also: } \Gamma_{m,g} \approx \Gamma_{-m,-g}.$$

Section 11.7: X

Paragraph 12: Complementary Series of $SL(2, \mathbb{C})$. These are the second ones listed in Theorem 3.

Paragraph 13: Trace of a Rep. ; group rings

Paragraph 14: Plancherel Theorem for $SL(2, \mathbb{C})$. (Proved 1949 by GN). See page 225. Only one principle rep appears!

Paragraph 15: details. $(g, m) =$ all irred. reps $m = \text{int}$, $g = \text{complex}$ (ie, not nec. unitary).

Paragraph 16: adding parity to get reps of complete L.G called G_0 .

Comments: This book is very similar to Gelfand, Minlos, Shapiro, also written in 1958. Similar stuff also in Vol 5 of, AGV; also see Liubarski;

books

{ Mairmark (1958)
Gelfand, Minlos, Shapiro (1958)
Gelfand, Graev, Vilenkin
Liubarski
Rühl (1970)

Gelfand

I.M. M.I. N.Ya.

Gelfand, Graev, Vilenkin

Volume 5 of "Generalized Functions"

AP, 1966, Russian 1962
NY

see p. 86 of H.

The series:

- 1) Properties + Operations
- 2) Applied Harmonic Analysis
- 5) Integral Geometry + Rep. Theory
- 2) Function Spaces
- 3) Differential Equations

This volume: separates cleanly into two pieces

A) Integral geometry

I. Radon Transforms of Test Functions

II. Integral Transforms

III. Spaces of constant curvature, horospheres, etc.

B) $SL(2, \mathbb{C})$

→ III. Reps of $SL(2, \mathbb{C})$ [Use this]

IV. Harmonic Analysis on $SL(2, \mathbb{C})$

V. HA on Lorentz group

→ VII. Reps of $SL(2, \mathbb{R})$

Some comments on Gelfand et al on $SL(2, \mathbb{C})$.

I have been reading their chapter on the reps. of $sl(2, \mathbb{C})$ and am a little confused about exactly what they are up to. This is an attempt to clarify.

There are several questions to which I would like the answers. First, how do you find all the irreducible representations of G . Then of these, which ones are unitary? How do they answer these questions?

Other questions are: what is the "monomial method" and does it work to find all representations or not, if not why.

Usually when one treats the Lorentz group, one mentions generators, Lie algebra, realizations, Casimirs, and all that. For some reason, GGV have totally avoided all Lie algebraic concepts. They deal only with Group.

Consider their starting point: they set up the usual shift representation on a space of two complex variables, actually a space of functions of two complex variables z_1 and z_2 . The group element g is "represented" by an operator which takes one such function into another in such a way that the group property is maintained. This is the standard "regular" representation.

Then they show that if you restrict your interest to functions which are "homogeneous of degree n_1-1 and n_2-1 ", then these functions form an invariant subspace. This subspace is called $D(n_1, n_2)$.

Except at certain integer points, a particular subspace $D(n_1, n_2)$ is subspace and operator irreducible. They proved operator irreducibility using the theory of intertwining operators with the Schur concept. At these special integer points, the space $D(n_1, n_2)$ does actually have an invariant subspace called E , so the rep can be further reduced; I am not sure whether D is also operator irreducible in this case.

So, using the technique of restriction of the regular representation to the space $D(n_1, n_2)$, they obtain all the irreducible representations of the group $sl(2, \mathbb{C})$. Usually, due to homogeneity, the $f(z_1, z_2)$ is replaced with a single function $\phi(z)$ and the group representation is presented on these functions in a uniform way for all representations.

What is the significance of complex numbers n_1 and n_2 ? Somehow, these are representation labels. No doubt if Lie Algebra were introduced, these would be Casimir eigenvalues, so the group must have two complex Casimirs! This agrees with the earlier idea of separating the 6 generator algebra into two disjoint pieces to get representations $T(P, Q)$ as in Liubarski. I recall that a vector particle corresponds to $(\frac{1}{2}, \frac{1}{2})$ but I forget the significance of having two Casimirs. (not much for LG; real sign. is in Poincaré)

There is a huge amount of discussion of proving the existence of bilinear functionals connecting the representation spaces of two representations X_1 and X_2 . Why?! Well, in particular they search for bilinear functionals which are invariant under respective group operators $T(x_1)$ and $T(x_2)$. They show that if you really want an invariant bilinear functional (BF) for all group elements, then the form is very restricted. In fact, you simply cannot form invariant BF's unless the labels (n_1, n_2) and (m_1, m_2) take special values! In fact, except at those integer points, there is no such invariant BF unless $x_1 = x_2$ or $x_1 = -x_2$ where $x_1 = (n_1, n_2)$. In these cases, they actually give the explicit form of the invariant BF. It is an integral over $dz_1 dz_2 dz_3 dz_4$. In the integral cases, there are other conditions for existence and these are stated I think in Theorem 2. Criterion for integer points is s_1 and $s_2 =$ positive integers or something.

By the way, I have tended to confuse the conditions for existence of invariant BFs with the conditions for equivalence of two reps x_1 and x_2 . Clear this up later.

So, repeat: why are they interested in invariant BFs? I think the big reason is that ultimately you want to find unitary representations. The notion of "unitary" is that $S^+S = I$ and requires the notion of a hermitean adjoint operator S^+ . This thing is defined, I think, as $(Ax, y) = (x, A^+y)$. I.e., it is the operator which makes both sides equal. Of course A acts in space E_1 whereas A^+ must act on a different space E_2 . Recall that in general the two sides of a BF are different spaces.

However, ultimately we want to have a scalar product, so at this stage we want the BF to be hermitian, meaning $(x, y)^* = (y, x)$ roughly. In this case, to get such a hermitian thing, you must choose $x_1 = X$ and $x_2 = X^*$ so your two spaces are actually complex conjugates in the usual sense of quantum mechanics, say.

So, having made this condition of Hermiticity, they show that a hermitian invariant BF only exists for rep X when a regular invariant BF exists for $x_1 = X$ and $x_2 = X^*$. Aside from positive definiteness, this added condition gives you a scalar product.

So, unitary reps only exist when $X = X^*$ or $X = -X^*$, because these were the conditions found earlier for the existence of invariant BD. The first case causes one of the integer point situations where $s_1 = s_2 = 0$, and the appropriate HIBF is stated. The second case gives $n_1 = n_2 = \text{real number}$ etc.

So the point is that to get unitary reps, only very special values of $x = (n_1, n_2)$ are acceptable. Somehow I should be able to reduce these results to either $SU(2)$ or $SU(1, 1)$ and see what the hell is really going on here. This formalism must be obtaining the reps of both groups at once.

What about "basis functions". This very concept implies the existence of a scalar product, or at least an ~~in~~ reasonable BF. I.e., you really want an orthogonal basis and orthogonality implies scalar product. I.e., a basis function exists in space $D(x)$ and its image exists in $D(x^*)$ so you are really talking about a hermitian BF.

Off hand I do not see why the scalar product has to be "invariant" under group action in order to have a set of basis vectors. Also, I don't quite see where positive definiteness is needed. Nevertheless, we shall later have some basis functions in space $D(x)$.

But, what is a basis function? Usually it is chosen to be an eigenfunction of a generator, but GGV have not even mentioned generators at this point, so we cannot yet have basis functions. Quite obvious how you make generators via tangent matrices, of course, so no sweat. I just suspect that GGV are going to do all their harmonic analysis in operator notation so there will be no need for a particular basis. Maybe this will be advantageous. pay attention. Nice to get results that are valid in all bases!!

Now let's get back to this business of the "monomial method". By this I refer to a certain method of obtaining matrix elements, and thus we are speaking in terms of a basis. Usually, one defines a set of generators corresponding to the group parameters. On the space $f(z_1, z_2)$ each generator may be realized as a certain differential operator. It is sometimes possible to find a simple function which is eigenfunction of one of these differential generators. This is then a basis function $f(z_1, z_2; m)$ where $m = \text{eigenvalue or set of same}$. We know exactly what $T_x(g)$ does to this function which, I forget to say, is also chosen to be in space D_x . Thus: $f(z_1, z_2; X, M)$. By simply applying T and using binomial theorem, you can easily get the matrix elements. This could conceivably work in any basis! Also, can do in mixed basis as long as you always know the basis functions.

but sums are too hard to perform!

1
One fact about this "monomial method" of obtaining matrix elements is that you never need your scalar product. You don't need the other half of the bilinear functional, that is to say. For $sl(2, \mathbb{C})$ you can have your representation for any complex pair (n_1, n_2) , so you can ergo obtain matrix ~~rep~~ elements for any irreducible representation X . Then if you like you can subsequently restrict to unitary rep values of X .

GGV

Vol. 1: Chapter III "Representations of $SL(2, \mathbb{C})$ "

intro: This chapter is a 66 page data base for the next chapter on harmonic analysis on $SL(2, \mathbb{C})$. We shall generalize $\mathcal{O}^{1,1}$ from fourier group to $SL(2, \mathbb{C})$. The intersection of this chapter with GMS is small, so good complements.

Outline of the chapter is

1. $SL(2, \mathbb{C})$ + some realizations
 - 1.1 connect to $SO(3, 1)$, $SO(4)$
 - 1.2 Lobachevski
2. Reps acting on $f(z_1, z_2)$
 - 2.1 group reps
 - 2.2 the D_x space
 - 2.3 two useful realizations of D_x : $\phi(z)$ and $f(w_1, w_2)$
 - 2.4 rep. of $SL(2, \mathbb{C})$ on D_x
 - 2.5 $T_x(g)$ in the $\phi(z)$ and $f(w_1, w_2)$ realizations
 - 2.6 dual reps
3. Summary of Basic Results of reps on D_x
 - 3.1 irreducibility of reps; integer points
 - 3.2 equivalence of reps; integer points
 - 3.3 equivalence at integer points
 - 3.4 unitary reps
4. Invariant Bilinear Functionals (IBF)
 - 4.1 stating the problem and main results
 - 4.2 conditions needed for PT and D invariance
 - 4.3 invariance
 - 4.4 sufficiency to show IBF's exist (non-singular case)
 - 4.5 " " " " (singular case)
 - 4.6 degeneracy of IBF
 - 4.7 conditional invariance of IBF
5. Equivalence of Representations of G
 - 5.1 intertwining operators
 - 5.2 equivalence of two reps
 - 5.3 partial equivalence
6. (over)

6. Unitary Reps of G

6.1 invariant hermitean functionals on D_X

6.2 positive definite

6.3 IHF for $|g| \geq 1, g \neq \text{integer}$

6.4 IHF for $n_1 = n_2 = \text{integer}$

6.5 unitary reps of G by operators on HS

6.6 subspace irreducibility of UR's.

1. $SL(2, \mathbb{C})$ + realizations

1.1. Proper Lorentz Group

$$g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \quad \alpha\delta - \beta\gamma = 1.$$

Let $X = \begin{pmatrix} x_0 - ix_3 & x_2 - ix_1 \\ -x_2 + ix_1 & x_0 + ix_3 \end{pmatrix}$. Let $X' = g^* X g$.

↑
herm!

This defines a map $A_g: \mathbb{R}^4 \rightarrow \mathbb{R}^4$, where $x^\mu \in \mathbb{R}^4$. This map can be restricted to $\det A_g = +1$ and $x_\mu x^\mu = \text{invariant}$.

The set $\{A_g\}$ represent the group! The relation $SO(3,1) : SL(2, \mathbb{C})$ is 1:2, the usual sign. All explicit forms are stated.

1.2. Lobachevski. Every LT describes motion on the two-sheeted $x^\mu x_\mu > 0$ surface, the single sheeted $x^\mu x_\mu < 0$ surface, or one of the lightcones, depending on value of $x^\mu x_\mu$ + sign of x_0 .



This upper sheet = model of "Lob. Space"
space is transitive



There exists some $g \in G$
connecting 2 points.

from now on: $SL(2, \mathbb{C}) = \text{"Lorentz Group"}$

1/11 Test seeing how well
this paper takes this ink
when you write fast. It
is the paper and not the pen!

2. Reps. of LG on $f(z_1, z_2)$

2.1 Reps of Groups

$E = \text{linear, topological space}$

$T(g) = \text{operator acting on } E$

$G = \text{top group} = \{G, m, i\}$... open set structure of G .
 $\uparrow$ mult. table

"Rep" $\equiv$ group property plus $T(e) = 1$.
 all with $T(g)$ on E .

Note: you do not need a scalar product to have a group rep!

"Unitary Rep" $\equiv \exists$ scalar product s.t. $(T(g)\xi, T(g)\chi) = (\xi, \chi)$.

So, you must have a Hilbert Space to do a UR.

Example:

$$T(g) P(\bar{z}_1, z_1) = P(\bar{z}_1, z_1) \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = P(\alpha z_1 + \gamma z_2, \beta z_1 + \delta z_2)$$

"Nucleus" — page 141

D_X will essentially give all irreducible reps.

2.2 The D_X Spaces of homogeneous functions

homogeneous: $f(x_1, x_2) = a_0 x_1^n + a_1 x_1^{n-1} x_2 + a_2 x_1^{n-2} x_2^2 \dots$
 $\uparrow$ all pairs of exp's add to $n \equiv \text{degree}$

$$f(rx_1, rx_2) = a_0 r^n x_1^n + a_1 r^n x_1^{n-1} x_2 + \dots = r^n f(x_1, x_2)$$

$x_1, x_2 = \text{real so far.}$

complex: $f(z_1, z_2; \bar{z}_1, \bar{z}_2) = z_1^{n_1} \bar{z}_1^{m_1} z_2^{n_2} \bar{z}_2^{m_2} + \dots$

if correct, then: $f(z_1, z_2)$

$$\begin{aligned} f(az_1, az_2) &= (az_1)^{n_1} (a\bar{z}_1)^{m_1} (az_2)^{n_2} (a\bar{z}_2)^{m_2} + \dots \\ &= a^{n_1+m_1+n_2+m_2} \bar{a}^{m_1+m_2} \dots \\ &= a^n \bar{a}^m f(z_1, z_2). \end{aligned}$$

if $z_1^{n_1} \bar{z}_1^{m_1} + z_1^{n_2} \bar{z}_1^{m_2}$, then no go.

Point: there are two degrees!

Let $\chi \equiv (n_1, n_2)$ and $(n_1-1, n_2-1) \equiv \text{degree of } f(z_1, z_2)$.

Let n_1, n_2 be complex, but $n_1 - n_2 = \text{integer}$. Reason:

$$a^\lambda \bar{a}^\mu = |a| e^{i\lambda \arg a} |a|^\mu e^{-i\mu \arg a} = |a|^{\lambda+\mu} e^{i(\lambda-\mu)\arg a} = e^{iN \arg a} = \text{s.v. of } (a)^N$$

$\leftarrow \lambda - \mu \text{ integer so this has no cts in } \arg.$

Now set up D_x :

1. $f(z_1, z_2) = \text{homo of degree } (n_1-1, n_2-1)$
2. all derivatives exist except at $(0,0) = (z_1, z_2)$

Topology: $|f_m(z_1, z_2) - f(z_1, z_2)| \rightarrow 0$
 $\hookrightarrow$ i.e., uniform convergence.

space is complete in this topology.

contour remark.

2.3 Two useful Realizations of the D_x .

$$A) f(az_1, az_2) = a^\lambda \bar{a}^\mu f(z_1, z_2) \Rightarrow f(az_1, a) = a^\lambda \bar{a}^\mu f(z_1, 1)$$

$$\Rightarrow f(z_2 \bar{z}_1, z_2) = z_2^\lambda \bar{z}_2^\mu f(z_1, 1) \quad \forall z_1 \text{ let } z_2 \bar{z}_1 = z_1$$

$$\Rightarrow f(z_1, z_2) = z_2^\lambda \bar{z}_2^\mu f(z_1/z_2, 1)$$

$$\text{Let } f(\bar{z}, 1) \equiv \phi(z). \text{ Then } f(z_1, z_2) = z_2^\lambda \bar{z}_2^\mu \phi\left(\frac{z_1}{z_2}\right).$$

What do we know about $\phi(z)$?

1) differentiable $^\infty$ in $z \cap \bar{z}$.

$$\text{Define: } \hat{\phi}(z) = z^{n_1-1} \bar{z}^{n_2-1} \phi(-\bar{z}^{-1})$$

$$f(1, z) = z^{n_1-1} \bar{z}^{n_2-1} \phi(z^{-1})$$

Claim: $\hat{\phi}(z) \in C^\infty$ (inf. diff.) $\hat{\phi}$ = "inversion" of ϕ .

$$\text{asym: } \phi(\bar{z}) \rightarrow z^{n_1-1} (\bar{z})^{n_2-1} f(1, 0)$$

$$B) \text{ Homo. says } f(e^{i\theta} w_1, e^{i\theta} w_2) = (e^{i\theta})^{n_1-1} (\overline{e^{i\theta}})^{n_2-1} f(w_1, w_2) \\ = e^{i\theta(n_1-n_2)} f(w_1, w_2).$$

Idea is that (w_1, w_2) lie on unit sphere $|w_1|^2 + |w_2|^2 = 1$ in 4-space. If you know $f(z_1, z_2)$ on such a sphere, then you can find it anywhere by scaling. I.e.

$$f(z_1, z_2) = r^{n_1+n_2-2} f\left(\frac{z_1}{r}, \frac{z_2}{r}\right) \quad r = \sqrt{|z_1|^2 + |z_2|^2}$$

$\Omega = 4 \text{ sphere} = \text{compact space!}$

whereas $\phi(z), z \in \mathbb{C}$, not compact.

2.4 Representations of $\bar{G}$ on D_X

- ① Claims that $T_X(g) f(z_1, z_2) = f\left((z_1, z_2) \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\right)$ is a rep in D_X . Verifies continuity requirements etc. So, $X = (n_1, n_2)$ $n_1, n_2 = \text{integer}$ $X = \text{weight of representation}$. For each X , you get a rep.

2.5 Other realizations

$$f'(z_1, z_2) = T_X(g) f(z_1, z_2) = f(\alpha z_1 + \gamma \bar{z}_2, \beta z_1 + \delta \bar{z}_2)$$

note: $f(z_1, z_2) \in D_X$, forgot to point this out.

$$f'(a z_1, a z_2) = f(\alpha a z_1 + \gamma a \bar{z}_2, \beta a z_1 + \delta a \bar{z}_2) = f(a z_1', a z_2')$$

$$= a^{\lambda} \bar{a}^{\mu} f(z_1', z_2') \dots$$

$$= (\beta z_1 + \delta \bar{z}_2)^{n_1-1} (\bar{\beta} \bar{z}_1 + \bar{\delta} \bar{z}_2)^{n_2-1} \phi\left(\frac{\alpha z_1 + \gamma \bar{z}_2}{\beta z_1 + \delta \bar{z}_2}\right)$$

$$f'(z_1, z_2) = z_1^{n_1-1} \bar{z}_1^{n_2-1} \phi\left(\frac{z_1}{\bar{z}_1}\right) \Rightarrow f'(z_1, 1) = \phi(z_1)$$

$$\Rightarrow \phi(z) = (\beta z + \delta)^{n_1-1} (\bar{\beta} \bar{z} + \bar{\delta})^{n_2-1} \phi\left(\frac{\alpha z + \gamma}{\beta z + \delta}\right) = T_X(g) \phi(z) \quad \checkmark$$

Redoes this for $(w_1, w_2) \in \Omega = \text{some "contour" crossing all "lines"}$ (like surface ①). Here, instead of using a real scalar $r = \sqrt{|z_1|^2 + |z_2|^2}$, she uses $r = g(z_1, z_2)$ defined by shape of surface. For sphere, boils down to

$$T_X(g) f(w_1, w_2) = f(\alpha w_1 + \gamma w_2, \beta w_1 + \delta w_2)$$

$$= r_g^{n_1+n_2-2} f\left(\frac{\alpha w_1 + \gamma w_2}{r_g}, \frac{\beta w_1 + \delta w_2}{r_g}\right) \quad \text{shift.}$$

$$\text{where } r_g = \sqrt{|\alpha w_1 + \gamma w_2|^2 + |\beta w_1 + \delta w_2|^2} = r_g = r_g \text{ in general}$$

If you want to shift your result back down to the surface of interest

$$\text{i.e., } \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = g(z_1, z_2) \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

← says how to get from $w_1 \in \text{surface}$ to z_1, z_2 on line.

2.6 Dual Reps

① $D_X =$ space whose functions are f :

$D'_X \equiv$ space of functionals on D_X .

$F \in D'_X \Rightarrow (F, \phi)$ exists as bilinear form, $\phi \in D_X$.

Define $\tilde{T}(g)$: $(\tilde{T}_X(g)F, \phi) = (F, T_X(g)\phi)$
 $\nwarrow$ note inverse.

Thus: $\tilde{T}_X(g) =$ "dual to $T_X(g)$ ".

Thus: let $\phi = T_X(g)$: $(\tilde{T}_X(g)F, T_X(g)\phi) = (F, \phi)$ always.
 by definition

② Now let $\phi \in D_{X_1}$ } two spaces.
 $\psi \in D_{X_2}$

$B: D_{X_1}, D_{X_2} \rightarrow \mathbb{C} =$ bilinear form. $B(\phi, \psi)$.

Suppose $B(T_{X_1}(g)\phi, T_{X_2}(g)\psi) = B(\phi, \psi)$ "invariant BF"

$B(\phi, \psi)$ "nondegenerate" $\Rightarrow$ if $B(\phi, \psi) = 0 \forall \psi$, then $\phi = 0$. etc.

Claim: you can 1-1 map $D_{X_1} \leftrightarrow D'_{X_2}$.

Proof: $B(\phi, \psi)$, $\phi \in D_{X_1} =$ functional on $D_{X_2} =$ element of D'_{X_2}
 $\uparrow$ think of $\phi \in D_{X_1}$ as parametrizing your functional, i.e. $(B\phi, \psi)$. $B\phi \in D'_{X_2}$

OK, redo this piece if needed later.

$\tilde{T}_{X_2}(g) = T_{X_1}(g)$ in some sense

3. Summary of Basic Results on "reps on D_X "

Can skip from end of 3. to next chapter.

not both ways for ∞ dim reps.

3.1 irreducibility

{absence of invariant subspaces} $\Leftarrow$ irreducible "truly"
 $\equiv$ "subspace irreducibility"

"operator irreducibility" $\equiv \left\{ \begin{array}{l} E = \text{space} \quad A = \text{closed operator} \\ [A, T_X(q)] = 0 \Rightarrow A = \lambda \cdot I \end{array} \right\}$

$$X = (n_1, n_2)$$

$\forall q$

$\nwarrow$ the Schur business.

op. irred. $\not\Rightarrow$ ss. irred. (if $X = (2, 2)$ eg.)

$X = \text{integer point} \Rightarrow (n_1, n_2) = \text{integers of same sign, } \neq 0$. - double spots.

$(X = \text{noninteger}) \Rightarrow$ all notions of irreducibility coincide

When $(n_1, n_2) = \text{positive integers}$, D_X does have an invariant subspace - namely $E_X = \text{polynomials}$. Thus E_X is of finite dimension n_1, n_2 . Reason is that when $n_i = \text{integer}$, 'derivatives stop doing anything after n_i applications!'. I.e., n_i is like j .

You can represent $\mathfrak{g}$ on $D_X/E_X = \text{functions up to a polynomial}$.

$$F = D_{n_1, n_2} / E_{n_1, n_2} \quad (n_1, n_2) = (2, 5), \text{ eg.}$$

$\nwarrow$ This factor space is also irreducible, ∞ dim.

If n_1, n_2 are negative, then $F_{n_1, n_2} = \text{an } \infty \text{ dim inv. subspace (here, derivatives do keep on going)}$ but $D_{n_1, n_2} / F_{n_1, n_2} = \text{finite dim factor space called } E_{-n_1, -n_2}$.

Let n_1, n_2 be positive. Then D_X has ∞ dim inv. ss F_X .

Let $f(z) \in F_X$. OK, a little clearer now.

3.2 Equivalence of Reps.

$T_{\lambda_1}(g)$ equiv $T_{\lambda_2}(g)$ is $\exists$ intertwining operator A .

Results. λ_1 eq. λ_2 if $\begin{cases} \lambda_1 = \lambda_2 & \text{I} \\ \lambda_1 = -\lambda_2 \text{ but non-integer.} & \text{II} \end{cases}$

For case II, One explicit A is stated.

3.3 Equivalence at integer points.

A is usually supposed to be an ITO.

$B: D_{-\lambda} \rightarrow D_{\lambda}$ fails to be onto for $\lambda = \text{integer point}$.

In fact $B: D_{-\lambda} \rightarrow E_{\lambda} = \text{polys } n_1, n_2$

Also: $B: E_{-\lambda} \subseteq D_{-\lambda} \rightarrow 0$
 $\uparrow$ nullspace of B .

$A: D_{\lambda} \rightarrow D_{-\lambda}$, but $A: E_{\lambda} \rightarrow 0$.
 $A: D_{\lambda} \rightarrow E_{-\lambda}$, actually.

OK, at integer points, there are more equivalences

$$D_{n_1, n_2} \sim D_{-n_1, n_2} \sim D_{n_1, -n_2} \sim D_{-n_1, -n_2}$$

3.4 Unitary Reps.

unitary rep $\Rightarrow \exists$ posdef, invariant, Hermitian, BLF i.e. invariant scalar product.

When does $\overline{VIR}$ exist

1) $n_1 = -\overline{n_2}$.

$$(q, \psi) = \frac{i}{2} \int q(z) \overline{\psi(z)} dz d\bar{z}$$

2) $n_1 = n_2 = g \neq 0$ but real $-1 < g < 1$

$$(q, \psi) = \left(\frac{i}{2}\right)^2 \int |z_1 - z_2|^{-2g-2} q(z_1) \overline{\psi(z_2)} \dots$$

where are discrete series?

"supplementary"

4. Invariant Bilinear Functionals

4.1 stating the problem. The results are: WR 's exist when $n_1 + \bar{n}_2 = 0$ or when $n_1 = n_2 = g$ $-1 < g < 1$. In these cases we have the explicit scalar products. Now we shall see how that is derived.

Decompose group elements into 3 classes + consider each one separately.

The idea is to find a $B(\phi, \psi)$ where $\phi \in E_1$, $\psi \in E_2$. I.e., these will be two different rep. spaces. You could consider this

$B[Tx_1(g)\phi, Tx_2(g)\psi]$, eg. Now $X_1 = (n_1, m_1)$ $X_2 = (m_1, m_2)$.

Let $S_1 = -\frac{1}{2}(n_1 + m_1)$ $S_2 = -\frac{1}{2}(m_1 + m_2)$. Then cases depend on nature of S_1, S_2 : I will not copy results here.

4.2. two of three cases of forms of $g \in G$.

① generalized function: sort of a functional, eg: an integral operator.

$$(B_0, w) = B_0^{(z)} w(z) = \int z^{n-1} w(z) dz.$$

② Theorem: if $B(\phi, \psi)$ is a BIF , then you can write it as (B_0, w) where $w(z) = \int \phi(z') \psi(z+z') dz'$. You can check that the result is bilinear and is translation invariant: i.e.

$$B(\phi(z), \psi(z)) = B(\phi(z+z_0), \psi(z+z_0))$$

check: $(B_0, w(z+z_0)) = \int \phi(z') \psi(z+z_0+z') dz' = \text{stop. Seems wrong.}$
It must mean:

$$B(\phi_t, \psi_t) = B(\phi, \psi) \quad \text{or} \quad (B_0, w_t) = (B_0, w)$$

$$\text{now } w_t(z) = \frac{1}{2} \int dz' \phi_t(z') \psi_t(z+z') = \frac{1}{2} \int dz' \phi(z+\Delta z') \psi(z+z'+\Delta z') = w(z) \checkmark$$

$\rightarrow$ So, $w(z) = \frac{1}{2} \int \phi(z') \psi(z+z') dz' d\bar{z}'$ is forced upon us, but B_0 is not yet specified.

③ What info. do dilatations add?

$$w_D(z) = \int \phi_d(z) \psi_d(z+z_1) dz_1 = ?$$

$$\phi_d(z) = T_X(D) \phi(z) = \alpha^{-n_1+1} (\bar{\alpha})^{-n_2+1} \phi(\alpha^2 z).$$

$$\begin{aligned} w_d(z) &= \alpha^{-n_1+1} (\bar{\alpha})^{-n_2+1} \int \phi(\alpha^2 z_1) \psi(\alpha^2 z + \alpha^2 z_1) d^2 z_1 \\ &= |\alpha|^{-4} \int \phi(z_1) \psi(z + z_1) d^2 z_1 \\ &= |\alpha|^{-4} w(\alpha^2 z). \end{aligned}$$

$$\text{Thus: } w_d(z) = (\alpha)^{-n_1-m_1+2} (\bar{\alpha})^{-n_2-m_2+2} |\alpha|^{-4} w(\alpha^2 z).$$

$$\begin{aligned} \text{So: } B(\phi_d, \psi_d) &= (B_0, w_d) = \alpha^{-n_1} \bar{\alpha}^{-n_2} |\alpha|^{-4} (B_0, w(\alpha^2 z)) \\ &= (B_0, w) \end{aligned}$$

$$\text{So: } \text{cancel } |\alpha|^4 \left[(B_0, w) = (\alpha)^{-n_1-m_1} (\bar{\alpha})^{-n_2-m_2} (B_0, w(\alpha^2 z)) \right]$$

What does this mean? Well,

$$f(\alpha z) = \alpha^\lambda \bar{\alpha}^\mu f(z) \Rightarrow f(z) = \text{homogeneous of degree } \lambda, \mu.$$

Thus, $w(z)$ must be homo. of degree $\lambda = s_1, \mu = s_2$.

$$w_d(z) = w \dots$$

$$(B_0, w(\alpha^2 z)) = \alpha^{\frac{(n_1+m_1)}{2}} \bar{\alpha}^{\frac{(n_2+m_2)}{2}} (B_0, w(z)).$$

What does homogeneous mean here?

④ def: " (B_0, w) is homogeneous of degree μ, λ " means

$$(B_0, w) = \int (z)^\mu (\bar{z})^\lambda w(z) dz d\bar{z}.$$

$$\text{This then implies: } (B_0, w(\alpha^2 z)) = (\alpha)^{-\mu-1} (\bar{\alpha})^{-\lambda-1} (B_0, w(z)).$$

(4.2 cont'd 1)

⑤ So, we have $(B_0, w) = \frac{i}{2} \int (z)^{s-1} (\bar{z})^{s-1} w(z) dz d\bar{z}$

where $dz d\bar{z} = -2i dx dy = -2i r dr d\theta$. What happens when you try to area integrate a pole?

$$\int_0^\infty r dr \int_0^{2\pi} d\theta \cdot \frac{1}{re^{i\theta}} = \int dz d\bar{z} \cdot \frac{1}{(z-\theta)} \Rightarrow \text{OK at origin.}$$

But: $\int dz d\bar{z} \frac{1}{z^n} \rightarrow \int_0^\infty r dr \cdot r^{-n} = \int_0^\infty r^{-n+1} dr \Rightarrow \text{OK if } n=0,$
but $n = \text{negative} \Rightarrow \text{ouch.}$

$$\int dz d\bar{z} \bar{z}^n \sim \int_0^\infty r^{-n+1} dr \quad \text{Let } x = \frac{1}{r} \quad dx = -\frac{1}{r^2} dr = -x dx$$

Then $\sim \int_2^\infty x^{n-1} \frac{dx}{x} = \int_2^\infty x^{n-2} dx \sim \frac{1}{n-2}$ by A.C.

Parts: consider $\int_1^\infty x^n dx = f(n)$. For $n < -1$, OK.

Parts says:

$$= - \int_1^\infty \left(\frac{1}{n+1} \frac{d}{dx} x^{n+1} \right) \cdot 1 dx = ?$$

Apparently, $\int dz d\bar{z} \frac{1}{z^n} \rightarrow \Gamma(n)$. Aha!

Consider: $\int_0^\infty e^{-\alpha x} x^n dx = \frac{\Gamma(n+1)}{(\alpha)^{n+1}} \quad (\alpha > 0)$

If $\text{Re } \alpha > 0 \rightarrow 0$, converges and $\int_0^\infty x^n dx = \Gamma(n+1) \cdot x$

$$\int_0^\infty x^n dx = \frac{1}{-n+1} x^{-n+1} \Big|_0^\infty = \frac{1}{n-1} \leftarrow \text{This is anal. ant!! not } \Gamma(n).$$

How about: $\int_0^\infty x^n f(x) dx = \int_0^\infty \frac{1}{-n+1} \frac{d}{dx} x^{-n+1} f(x) \sim \int_0^\infty x^{-n+1} \frac{df}{dx}$
 $\sim \int_0^\infty x^{-n+2} \frac{2x}{2x^2} dx$. If $f, f', f'' \dots$ all $\rightarrow$ const at ∞ , then get $\Gamma(-n)$.

OK. Claim: $\int \int dz d\bar{z} \cdot \phi(z) \cdot \frac{1}{(z-a)^{n+1}}$
 $\rightarrow$ behaves as $\Gamma(-n)$ near integers in n .

"regularization" — look up.

⑦ OK, so use (3) in (7) to get:

$$\begin{aligned} (B_0, w) &= B(\phi, \psi) = \frac{i}{2} \int (z)^{s_1-1} (\bar{z})^{s_2-1} \left[\frac{i}{2} \int \phi(z_1) \psi(z_2) dz_1 d\bar{z}_2 \right] dz d\bar{z} \\ &= \left(\frac{i}{2}\right)^2 \int dz d\bar{z} dz_1 d\bar{z}_1 (z_1 - z)^{s_1-1} (\bar{z}_1 - \bar{z})^{s_2-1} \phi(z_1) \psi(z) \end{aligned}$$

where $s_1, s_2 \neq (\text{negint}, \text{negint})$. If they are, then

$$(B_0, w) = \frac{i}{2} \int \frac{z^{s_1-1} \bar{z}^{s_2-1}}{2 z^{s_1} \bar{z}^{s_2}} \phi(z) \psi(z) dz d\bar{z}.$$

Finally, he writes (ϕ, ψ) using the two variable method.

4.3 Inversion invariance

Here, we do the nonsingular case only. The added requirement of inversion invariance restricts s_1, s_2 , i.e. of $(n_1, n_2; m_1, m_2)$. In fact, (ϕ, ψ) exists only if

$$\begin{aligned} n_1 &= m_1 & \text{i.e., } \chi_1 &= \chi_2 & \text{i.e. } s_1 &= -\frac{1}{2}(n_1 + m_1) = -n_1 \\ n_2 &= m_2 & \text{so } s_1 &= -n_1 \\ & & & s_2 &= -n_2 \end{aligned}$$

So both spaces must be same rep space. So result is same as earlier, but $s_1 \rightarrow -n_1, s_2 \rightarrow -n_2$ $\chi_1 = \chi_2$ only.

4.4 sufficiency — We have shown that

if (ϕ, ψ) exists $\Rightarrow$ certain form is necessary

We have not shown that (ϕ, ψ) really exists yet. I ignore this part and go right to statement of "Theorem 1" which I have already given above.

4.5 singular case Here the question is: for what values m_1, m_2, n_1, n_2 is the singular BLF inversion invariant? Answer:

singular case $\Rightarrow S_1$ and S_2 are both $\odot$ negative integers including 0 but not nec. the same.

Results: yes, BLF exists if

- ① ~~$m_1 = m_2$~~ $(n_1, n_2) = (+m_1, +m_2)$ i.e. $S_1 = +m_1$ $S_2 = -m_2$
- ② $(n_1, n_2) = (-m_1, -m_2)$ i.e. $S_1 = 0$ $S_2 = 0$
- ③ $(n_1, n_2) = (m_1, -m_2)$ i.e. $S_1 = -m_1$ $S_2 = 0$
- ④ $(n_1, n_2) = (-m_1, m_2)$ i.e. $S_1 = 0$ $S_2 = -m_2$

Case ② is simplest since $S_1 = S_2 = 0$. It boils down to:

$$B(\phi, \psi) = \frac{i}{2\pi} \int \phi(z) \psi(z) dz^2.$$

So all this becomes stated as Theorem 2.

4.6 degeneracy of IBF

concept: $B(\phi, \psi)$ is "degenerate on $E_1 \subset D_1$ " if $B(\phi, \psi) = 0 \quad \forall \phi \in E_1$

i.e. $B(\phi_1, \psi) = B(\phi_2, \psi)$ even if $\phi_1 \neq \phi_2$ but $\phi_1 - \phi_2 \in E_1$

Look at Case 1: $B(\phi, \psi) = \int g^{(n_1, n_2)}(z) \psi(z) dz^2$

obviously, $g(z) = \text{low polynomial} \Rightarrow B(\phi, \psi) = 0$

OK, this correlates with idea of $F = D/E$.

4.7 conditional invariance For $(n_1, n_2) = (-2, -3)$, say, then you can actually find another IBF called $B_2(\phi, \psi)$. This is shown in (6).

5. Equivalence of Reps. of G

Recall: $ATx_1 = Tx_2A \Leftrightarrow$ "A intertwines Tx_1 and Tx_2 "
to be used: equiv., partial equiv.

5.1 intertwining operators.

define: $(Tx_1 \text{ "equivalent" } Tx_2) \equiv (\exists \text{ a bicontinuous and 1-1 intertwiner})$

consider D_{x_1} and D_{x_2} . invariance of BLI says:

$$B(\phi, \psi) = B(Tx_2(g)\phi, Tx_1(g)\psi)$$

question: show that A intertwines $Tx_1, Tx_2 \Leftrightarrow B(\phi, \psi) \equiv (\psi, A\phi)$
is invariant under $Tx_1, Tx_2 \Leftrightarrow x_1 \text{ equiv. } x_2$

$$B[Tx_1\phi, Tx_2\psi] = (Tx_2\psi, ATx_1\phi) = (Tx_2\psi, Tx_2A\phi)$$

$$B[\phi, \psi] = (\psi, A\phi) \stackrel{\uparrow}{=} (Tx_2\psi, Tx_2A\phi) \quad \text{because } \psi \in D_{x_2}, \phi \in D_{x_1}.$$

so:

we assume that (ψ, ϕ) is invariant $= (Tx_2\psi, Tx_2\phi)$.

we ~~assume~~ then:

$$A \text{ intertwines } x_1, x_2 \Leftrightarrow B(\phi, \psi) = B(Tx_1\phi, Tx_2\psi).$$

Note that $B(\phi, \psi)$ is a new BLF.

② Corollary: They have shown that A exists in certain cases 1 $\rightarrow$ 5. One of these cases is $x_1 = x_2$. Thus A exists $\Leftrightarrow [Tx_1, A] = 0$.
But, A exists also $\Rightarrow A\phi = c\phi$. Thus we have proven operator irreducibility of Tx for any non-int x .

$$\text{If } \exists A \ni [Tx, A] = 0 \quad \forall g, \text{ then } A = cI.$$

5.2 Equivalence of Two Reps. Proof of claim that: T_{x_1}, T_{x_2} are equivalent
($x_1 \neq x_2$) only when $x_2 = -x_1$, as long as x not integer point.

5.3 Partial Equivalence. They say to omit.

6. Unitary Reps on G

6.1 Invariant Hermitian Functionals on D_X

① Hermitian $\Rightarrow B(\phi, \psi) = B(T_X(g)\phi, T_{\bar{X}}(g)\psi) \quad \bar{X} = (\bar{n}_1, \bar{n}_2)$

$$B(\phi, \psi) = (\phi, \bar{\psi})$$

← a bilinear
so defined

← a herm F.

$$\text{so let } \begin{aligned} X_1 &= X \\ X_2 &= \bar{X} \end{aligned}$$

When do such BLI exist?

$$X_1 = X_2 \Rightarrow X = \bar{X} = X = \text{real}$$

$$X_1 = -X_2 \Rightarrow X = -\bar{X}$$

$$X = \bar{X} \Rightarrow \begin{aligned} n_1 &= \bar{n}_1 \\ n_2 &= \bar{n}_2 \end{aligned} \quad \text{stop!}$$

② Reexamine $B(\phi, \psi) \equiv (\phi, \bar{\psi})$. If $\psi \in D_X$, then where is $\bar{\psi}$?

$$T_X(g)\psi(z) = (\beta z + \delta)^{n_1-1} (\bar{\beta} \bar{z} + \bar{\delta})^{n_2-1} \psi\left(\frac{\alpha z + \gamma}{\beta z + \delta}\right) \Rightarrow \psi \in D_X$$

$$\overline{T_X(g)\psi(z)} = (\bar{\beta} \bar{z} + \bar{\delta})^{\bar{n}_1-1} (\beta z + \delta)^{\bar{n}_2-1} \bar{\psi}\left(\frac{\alpha z + \gamma}{\beta z + \delta}\right) \Rightarrow$$

$$\text{thus } [D_{\bar{X}} = (\bar{n}_2, \bar{n}_1) \text{ if } D_X = (n_1, n_2)]$$

③ Now herm. BLF exists if $B(\phi, \psi)$ exists, i.e., if $X = \bar{X}$
or $X = -\bar{X}$

$$\text{i.e., } \begin{cases} \textcircled{1} n_1 = \bar{n}_2 \\ \textcircled{2} n_1 = -\bar{n}_2 \end{cases} \leftarrow \text{only cases!}$$

Now consider ①: $n_1 = \bar{n}_2$ and $n_1, n_2 = \text{integer}$
i.e., we always know $\text{Im } n_1 = \text{Im } n_2$
 $\text{Re } n_1 = \text{Re } n_2 + 1 \text{ or } 0$

Now we add:

$$\text{Im } n_1 = + \text{Im } n_2 \text{ and } \text{Re}(n_1) = \text{Re}(n_2)$$

So, at once, $\text{Im } n_1 = \text{Im } n_2 = 0$ and hence real. Then $n_1 = n_2$, but I don't see why $n_1 = n_2$

So nothing new on imag, but $\text{Re } n_1 - \text{Re } n_2 = \text{int} \Rightarrow \text{Re } n_1 = \text{int} \Rightarrow \text{Re } n_2 = \text{int}$

So, when $n_1 = -\bar{n}_2$, why does he use the integer form?

Well think of S_1 and S_2 :

$$S_2 = -\frac{1}{2}(n_2 + \bar{n}_2).$$

$$\begin{aligned} \text{Now } X_1 = X &= (n_1, n_2) \\ X_2 = \bar{X} &= (\bar{n}_1, \bar{n}_2) = (-\bar{n}_2, \bar{n}_1) \Rightarrow \begin{aligned} n_1 &= \bar{n}_2 \\ n_2 &= \bar{n}_1 \end{aligned} \end{aligned}$$

$$\text{Thus } \begin{aligned} S_1 &= -\frac{1}{2}(n_1 + \bar{n}_2) \\ S_2 &= -\frac{1}{2}(n_2 + \bar{n}_1) = \bar{S}_1. \end{aligned}$$

So $n_1 = -\bar{n}_2 \Rightarrow S_1 = S_2 = 0 \Rightarrow$ singular case

Note: $\begin{aligned} n_1 &= -m_1 \\ n_2 &= -m_2 \end{aligned} \Rightarrow \text{Case 2 of theorem 2} \Rightarrow \int \phi(z) \bar{\psi}(z) dz^2 = \text{principle series.}$

$$\textcircled{4} \text{ Next, } X_1 = +X_2 \Rightarrow n_1 = \bar{n}_2 \Rightarrow \begin{aligned} S_1 &= -\frac{1}{2}(n_1 + \bar{n}_2) \\ S_2 &= -\frac{1}{2}(n_2 + \bar{n}_1) \text{ as before} \end{aligned}$$

$$\text{but now } \begin{aligned} S_1 &= -n_1 \\ S_2 &= -\bar{n}_1 \end{aligned}$$

$$\text{Also, } n_1 = \bar{n}_2 \Rightarrow \begin{aligned} \text{Re } n_1 &= \text{Re } n_2 \\ \text{Re } n_1 &= \text{Re } n_2 + \text{int} \Rightarrow \underline{\text{int}} = 0 \quad h = 0. \end{aligned}$$

$$\text{So } n_1 = \bar{n}_2 = g = \text{real \#}.$$

$$\text{Avoid } g = 1 \Rightarrow S_1, S_2 = \text{neg int's.}$$

Three cases given:

$$\text{A. } n_1 = -\bar{n}_2 \Rightarrow S_1 = S_2 = 0 \Rightarrow (\phi, \psi) = \frac{i}{2} \int \phi(z) \bar{\psi}(z) dz^2$$

$$\text{B. } n_1 = n_2 = g \text{ real } g \neq 0, 1, \dots \quad (\phi, \psi) = \dots$$

$$\text{C. } g = q = 0, 1, 2, \dots \Rightarrow (\phi, \psi) = \dots$$

Chapter 3 "Reps. of $SL(2, \mathbb{C})$ " GG+V.

introduction: $SL(2, \mathbb{C})$ is simpler than $SL(2, \mathbb{R})$. Want to generalize the idea of fourier integral. For rep theory see Gel'fand, Minlos + Shapiro which has a lot on $SL(2, \mathbb{C})$ also, small Λ with this book. Also, see Naimark.

1. $SL(2, \mathbb{C})$ + realizations

1.1. Connection with Proper Lorentz Group. Let R_4 be ^{note: hermitian} space of 4-vectors $x = (x_0, x_1, x_2, x_3)$. Then if $a = \begin{pmatrix} x_0 - x_3 & x_2 - ix_1 \\ x_2 + ix_1 & x_0 + x_3 \end{pmatrix}$, then $a' = g^* a g$ defines A_g acting on $V \in R_4$. ($* = \dagger$). Properties are: A_g leaves $x_0^2 - x^2$ invariant, $\det A_g = 1$, positive cone $x_0 - x^2 = 0$ is invariant, $x_0 > 0$. These linear x-forms make the "proper L.T.". The $SL(2, \mathbb{C})$ to L.G. is 2:1. It writes x_0 in terms of $x_1, \dots, x_3$; $\alpha \beta \delta \gamma$, p 137.

1.2 Connection with Lobachevskian + Other Motions. Lorentz x-forms induce "motions" of the "surfaces": cones, two sheeted hyperboloid, etc. One upper sheet + motion = "Lob. space". So $SL(2, \mathbb{C}) =$ [locally] isomorphic group of Lob. motions. These surfaces are examples of "transitive" ones, all points are connected by $g \in G$.

2. Reps of Lorentz Group Acting on Homogeneous Fctns of 2 Complex Var's.

2.1 Reps of Groups. [Define a rep: if $G =$ "topological group" [ie, parameter space is a top space, $G =$ abstract group, so $\{G, m, i, \tau\}$], and if $E =$ a linear topological space, then a "rep. of G on E " is a set of linear operators $T(g)$ acting on E such that "invariant h.b.f. + adj" $T(g_1 g_2) = T(g_1) T(g_2)$. A rep is unitary if you can introduce a scalar product on E with $(T(g) \xi, T(g) \eta) = (\xi, \eta) \forall \xi, \eta \in E, g \in G$. Note: "unitary w.r.t. some scalar product." ! As examples of finite dim. spaces, G itself ≈ 2 dim. Or, hom. poly of degree $n-1 =$ space of dim n . x-forms will be $n_1 \times n_1$ matrix acting on coefficients:

$$P(z_1, z_2) = a_0 z_1^{n-1} + a_1 z_1 z_2^{n-2} + \dots + a_{n-1} z_2^{n-1}$$

Here a rep is $T(g) \cdot P(z_1, z_2) = P(\alpha z_1 + \beta z_2, \gamma z_1 + \delta z_2)$. Below, however, we shall deal with Dx + infinite dim reps.

define
a rep.

2.2 The D_x space of homogeneous functions. $D_x = \{f(z_1, z_2, \bar{z}_1, \bar{z}_2)\}$
For short $f(z_1, z_2)$. Homogeneous of degree (λ, μ) means: —

$$f(az_1, az_2) = a^\lambda \bar{a}^\mu f(z_1, z_2)$$

Let $X = (n_1, n_2)$ such that $n_2 - n_1 = \text{integer}$. Now refine the space D_x s.t.:

- 1) $f(z_1, z_2)$ is homo. of degree $(n_1 - 1, n_2 - 1)$
- 2) f is ∞ differentiable except at $z_1, z_2 = (0, 0)$.

2.3 Two Useful Realizations of the D_x . Note: $\phi(z) \equiv f(z, 1)$ and $f(z_1, z_2) = z_1^{n_1-1} \bar{z}_2^{n_2-1} g(z_1/z_2)$. Also, define: $\hat{\phi}(z) = z^{n_1-1} \bar{z}^{n_2-1} \phi(-\frac{1}{z})$. Also, $f(1, z) = z^{n_1-1} \bar{z}^{n_2-1} g(\frac{1}{z})$. Thus, $\phi(z) \xrightarrow{z \rightarrow \infty} C z^{n_1-1} \bar{z}^{n_2-1}$.

2.4 Reps of G on D_x . $T_x(g)f(\frac{z_1}{z_2}) = f(\tilde{g}(\frac{z_1}{z_2})) = f \dots$
Well

$$\{T_x(g) f(z_1, z_2) = f(z_1, z_2) \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\}$$

Easy to show this is a rep. $X = \text{"weight" of rep.}$

2.5 The $T_x(g)$ operators + other realizations of D_x

$$\rightarrow \{T_x(g)\phi(z) = (\beta z + \delta)^{n_1-1} (\bar{\beta} \bar{z} + \bar{\delta})^{n_2-1} g\left(\frac{\alpha z + \gamma}{\beta z + \delta}\right)\}$$

Note that $\phi(az) = f(az, 1)$ not related to $\phi(z)$. I do not see any relation to $X = (l, e)$ except maybe $n_1 - 1 = l + e$ $n_2 - 1 = l - e$ so $n_1 - n_2 = 2e$, $n_1 + n_2 = 2l + 2$.

2.6 The Dual Reps. "Dual space to D_x " = space of linear functionals on D_x . Eg, $F \in D_x \rightarrow (F, \phi) = \text{something } \phi \in D_x$. You can make a rep on D_x in this way.

$$(\tilde{T}_x(g)F, \phi) = (F, T_x(\bar{g})\phi)$$

So $\tilde{T}_x$ is "dual to T_x ". In particular this means

$$(\tilde{T}_x(g)F, T_x(g)\phi) = (F, \phi)$$

(F, ϕ) is called a "bilinear functional". There is a 1-to-1 map from D_x to D_x' . T in D_x is called an "extension" of T in D_x .

3. Summary of Basic Results concerning Reps on D_x . These things will all be proved in sections 4, 5, 6 of this chapter, but you can skip proofs + go on to H.A. if you want.

3.1 Irreducibility of D_x reps; Role of Integer Points. Define a particular form of irred: "subspace irreducibility" means no closed, invariant subspaces of E , OK for finite reps. Also need "operator irreducibility" for ∞ dim reps: every closed operator on E which commutes with all "operators of the rep" is multiple of unit operator. Note: A closed $\Rightarrow [x, Ax]$ = closed set in $E \oplus E$. This $[x, Ax] \forall x$ = "graph of A ".

It is possible to have O.I. but not S.I. There is another form of I.R. having to do with $E \otimes F$ tensor products. Skip. An "integer point" $x = (n_1, n_2)$ if n_1, n_2 are integers of the same sign. Then D_x has invariant subspaces. Clearly, these subspaces are E_x . One finite dim polynomials already noted. Polys. of degree (n_1-1, n_2-1) , hence $\dim = n_1 n_2$. So D_x/E_x can be defined as factor space, ∞ dim. It turns out that $D_{n_1, n_2}/E_{n_1, n_2} \sim D_{n_1, n_2} \sim D_{n_1, -n_2}$, n_1, n_2 both ≥ 0 . However, $D_{-x} \approx D_{-n_1, -n_2}$ has $D_{-n_1, -n_2}/F_{-n_1, -n_2} \sim E_{n_1, n_2}$, so factor space is finite dim. Note: $F_{-n_1, -n_2} = D_{n_1, n_2}/E_{n_1, n_2}$.

3.2 Equivalence of Reps on D_x ; The Role of Integer Points. When are two reps "equivalent"? Answer: when $ATx_1 = Tx_2A$. Here, A is called an "intertwining operator" for Tx_1 and Tx_2 reps. It will turn out that $Tx_1 \sim Tx_2$ if $x_1 = x_2$, or $x_1 = -x_2$, i.e., $(n_1, n_2) = \pm(n_1, n_2)$. We will also find $A = I_e$, $A: D_x \rightarrow D_{-x}$, $A\phi(z) = \bar{\phi}(\bar{z})$... see p 152. Think of each rep as pair of complex numbers (n_1, n_2) , think of as a Riemann surface.

3.3 The problem of Equivalence at Integer Points. We are using $A: D_x \rightarrow D_{-x}$ and $B: D_{-x} \rightarrow D_x$. When n_1, n_2 are integers, you have troubles. I.e., $B\phi(z) = a(n_1-1, n_2-1)$ poly. in $z, \bar{z}$.

Thus, $B: D_x \rightarrow E_x$, only a subspace of D_x . If the "moments" of $\phi(z)$ all vanish, -

$$b_{jk} \equiv \frac{1}{2} \int z^j \bar{z}^k dz d\bar{z} = 0$$

$$j = 0, \dots, n_1-1 \\ k = 0, \dots, n_2-1$$

$$\text{Then } B: \mathcal{F} \rightarrow 0 \in E_x.$$

This "kernel" of B is called F_x , a piece of D_x . In notation, then, we say $D_x / F_x \cong E_x$.

Meanwhile, consider $A: D_x \rightarrow D_x$. It turns out that in fact, $A: D_x \rightarrow F_x$, and $A: \phi(z) \rightarrow 0$ if $\phi = \text{poly. in } E_x$. So kernel is E_x , and $D_x / E_x = F_x$.

So, when n_1, n_2 are pos. integers, there are various isomorphisms between reps, eg, $D_{n_1, n_2} / E_{n_1, n_2} \cong D_{-n_1, n_2}$ etc. "Closest relatives" are mentioned here:

3.4 Unitary Reps. There may exist a "hermitean bilinear functional" (ϕ, ψ) in D_x . If it is pos. def., then this h.b.f. defines a scalar product. If it is group "invariant", i.e., if $(T_x(g)\phi, T_x(g)\psi) = (\phi, \psi)$, then the rep $T_x(g)$ is unitary wrt this scalar product. $\Leftarrow$ [p.d. i. h. b.f.]

When can you introduce such a scalar product? See section 6. Results:

$$1. n_1 = -n_2; \quad (\phi, \psi) = (i/2) \int dz d\bar{z} \phi(z) \bar{\psi}(\bar{z})$$

This is "principle series"

$$2. n_1 = n_2 = g \neq 0 = \text{real}, -1 < g < 1. \text{ Then}$$

$$(\phi, \psi) = (i/2)^2 \int dz_1 d\bar{z}_1 dz_2 d\bar{z}_2 |z_1 - z_2|^{2g-2} \phi(z_1) \bar{\psi}(\bar{z}_2)$$

This is "supplementary series".

4. Invariant Bilinear Functionals

4.1 Statement of The Problem; Basic Results Consider two topo. spaces E_1, E_2 ; have $T_1(g), T_2(g)$; let $\phi \in E_1, \psi \in E_2$. Then define: "continuous, bilinear, functional"

$$B(\phi, \psi) \geq : \quad \begin{aligned} - B(a\phi_1 + b\phi_2, \psi) &= a B(\phi_1, \psi) + b B(\phi_2, \psi) \\ - B(\phi, a\psi_1 + b\psi_2) &= a B(\phi, \psi_1) + b B(\phi, \psi_2) \end{aligned}$$

"Continuous" means continuous set of ϕ, ψ in $E_1 \oplus E_2$. Now B is "invariant under seps T_1, T_2 " if $\forall g \in G, B(\phi, \psi) = B(T_1(g)\phi, T_2(g)\psi)$.

Now, our group $G = SL(2, \mathbb{C})$, so $T_{X_1}(g)\phi(z) = \dots n_1, n_2$
 $T_{X_2}(g)\psi(z) = \dots m_1, m_2$
 These act on D_{X_1}, D_{X_2} .

Can we find an invariant B ?

Define: $\begin{cases} s_1 = -\frac{1}{2}(n_1 + m_1) \\ s_2 = -\frac{1}{2}(n_2 + m_2) \end{cases}$

If s_1, s_2 are not neg. ints. or 0, then $\exists$ an i.b.f. iff $n_1 = n_2$ and $m_2 = n_2$. In this case, (Theorem 1).

$$B(\phi, \psi) = (i/2) \int (z_1 - \bar{z}_2)^{-n_1-1} (\bar{z}_1 - z_2)^{-n_2-1} \phi(z_1) \psi(z_2) dz_1 d\bar{z}_1 dz_2 d\bar{z}_2$$

If s_1, s_2 are neg. ints. or 0, then 4 special cases.

[1] $m_1 = n_1, m_2 = n_2$ as above. (note: n_1, n_2 will be pos. ints here).

$$B(\phi, \psi) = (i/2) \int \phi^{(n_1, n_2)}(z) \psi(z) dz d\bar{z}$$

where $\phi^{(n_1, n_2)}(z)$ means $\frac{\partial^{n_1+n_2} \phi(z)}{\partial z^{n_1} \partial \bar{z}^{n_2}}$

[2] $m_1 = -n_1, m_2 = -n_2$

$$B(\phi, \psi) = (i/2) \int \phi(z) \psi(z) dz d\bar{z}$$

[3] $m_1 = n_1, m_2 = -n_2, n_1 = 0, 1, \dots$

$$B(\phi, \psi) = (i/2) \int \phi^{(n_1, 0)}(z) \psi(z) dz d\bar{z}$$

[4] $m_1 = -n_1, m_2 = n_2, n_2 = 0, 1, \dots$

$$B(\phi, \psi) = (i/2) \int \phi^{(0, n_2)}(z) \psi(z) dz d\bar{z}$$

How does one prove all this stuff? First, identify g with group of linear + fractional x-forms: $w = (\alpha z + \beta)/(\gamma z + \delta)$. Then, we can generate any group element via:

$w = z + z_0$	$\begin{pmatrix} 1 & z_0 \\ 0 & 1 \end{pmatrix}$	"parallel translation"
$w = \lambda z$	$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$	"dilatation"
$w = -1/\bar{z}$	$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$	"inversion"

4.2. Translation + Dilator: seek invariant... b.f. The idea here is to specialize general $T_x(g)$ to these cases, + search for i.b.f.'s. They draw here on material from earlier volume.

4.3. Inversions OK, shows how "basic results" in 4.1 are found.

4.4. Sufficiency. Previous sections were only necessity; this is only for the "nonsingular case." Ends with "Theorem 1" which was already stated: need $\chi_1 = \chi_2$, $B(\phi, \psi) = \dots$, valid when s_1, s_2 are not both neg. wts or zero.

4.5. Singular Case. Here, "Theorem 2" lists off the 4 cases above.

4.6. "Degeneracy" of IBF's; "Conditional Invariance" Skip.

5. Equivalence of Reps of G. skip

6. Unitary Reps of G.

6.1 Invariant Hermitian Functionals on D_x . Here, ~~"Hermitian"~~ We ^{want} ~~positive~~ positive definite, so scalar product is definable. We already know what "invariant" means, + we have listed off all invariant b.f.'s on G , see section 4.
So what does "Hermitian" mean?

$$(\phi, \psi) = B(\phi, \bar{\psi}) = \text{hermitean functional}$$

$$\text{Or: } B(\phi, \psi) = (\phi, \bar{\psi}) \quad \phi \in D_x - \quad \psi \in D_{\bar{x}}$$

So (ϕ, ψ) invariant under $T_x \Leftrightarrow B(\phi, \psi)$ invariant on $T_x, T_{\bar{x}}$

The conclusion of this section is: "An invariant herm. functional exists on D_x in two cases

$$\left\{ \begin{array}{l} 1) \quad \eta_1 = -\bar{\eta}_2, \quad (\phi, \psi) = (i/2) \int \phi(z) \bar{\psi}(z) dz d\bar{z} \\ 2) \quad \eta_1 = \bar{\eta}_2 = g \text{ real } (\eta_1, \eta_2 = \text{int.}) \quad (\phi, \psi) = \text{see 3.4 notes.} \\ \quad \quad (\phi, \psi) = c(i/2)^g \int |z_1 - z_2|^{-2g-2} \phi(z_1) \bar{\psi}(z_2) dz_1 d\bar{z}_1 dz_2 d\bar{z}_2, \quad g \neq 0, 1, \dots \\ \quad \quad (\phi, \psi) = c(i/2) \int \phi^{(g,0)}(z) \bar{\psi}(z) dz d\bar{z} \quad g = 0, 1, 2, \dots \end{array} \right.$$

6.2. When is this i.h.f. pos def? We just showed: ---

Case 1: $n_1 = -\bar{n}_2$ so $n_1 = \frac{1}{2}(n+ig)$, due to $n_1 - n_2 = n$, int.
 $n_2 = \frac{1}{2}(-n+ig)$ (single valuedness).

$X = (\frac{1}{2}[n+ig], \frac{1}{2}[-n+ig])$, principle series.

$(\phi, \psi) = (i/2) \int \phi(z) \bar{\psi}(\bar{z}) dz d\bar{z}$, auto. mathematically pos def.

Case 2: $n_1 = n_2 = g = \text{real}$. In this case he shows that $-1 < g < 0 \rightarrow$ pos def h.b.f.i., not totally trivial. Next, if $g = \text{int}$, considers $-g-1 < g < -g$, gets (ϕ, ϕ) , claims not pos def. If $g = -g-1$, again, not def sign.

So, when $g < 0$, only $-1 < g < 0$ works. By equivalence, $0 < g < 1$ also "works" (i.e., gives a pos def h.b.f.).

$X = (g, g)$, supp. series.

$(\phi, \psi) = 1/P(-g) (i/2)^2 \int |z_1 - z_2|^{2g-2} \phi(z_1) \bar{\psi}(\bar{z}_2) dz_1 d\bar{z}_1 dz_2 d\bar{z}_2$.

6.3 Special study: $-g-1 < g < -g$. Shows why not pos def. Valid also for $g < g < g+1$, hence $|g| \geq 1$, $g \neq \text{integer case}$.
 $\phi_X =$ two pieces; in one piece $(\phi, \phi) > 0$, in the other < 0 .

6.4 Special study $g = g$. Again, no go.

6.5 Unitary Reps. of G by Operators on a Hilbert Space. Once you have a viable scalar product [i.e., pos def...], then you have a Hilbert space. $T_X(g)$ acting on this H.S. is now a unitary operator. So "To every rep. $T_X(g)$ with an invariant, pos def. hermitian, b.f., there exists a rep by unitary operators on a Hilbert space."

6.6 Proof of Subspace Irreducibility of T_X . Skip.

Ruhl 1970

Comments on relation between UIR's and Spinor Reps of $SL(2,C)$.

1. First think about $SU(1,1)$. The reps are usually set up with a homogeneity condition and a parameter j . For certain unitary values of j , you can find a scalar product and can produce the UIR matrix elements. The clearest view of what is going on is provided by the Lie Algebra. For integer $2j$ values, recall that things truncate and you can get finite-dimensional invariant subspaces. However, the standard "norm" of the generators will be negative, which is to say, the generators cannot be hermitian in this representation. Nevertheless, in $SU(1,1)$ both the UIR's and the FDR's are provided by the same group action (homo polys) and the same lie algebraic operations. One of the FDR's of $SU(1,1)$ is, of course, the nominal $SU(1,1)$ matrix set.

Note that for $SU(1,1)$ the actual matrix elements of the UIR's and the FDR's are given by the same function, namely, $d_{mm}^j(z)$.

2. How does this work out for $SL(2,C)$? I think that when (n_1, n_2) take integer values, somehow the same group action that you use for the UIR's, and the same Lie algebra, does have invariant subspaces of finite dimension. So the FDR's evolve from the same group action, but the functions do not look the same. I.e., and FDR would be $D^{(j_1, j_2)}_{m_1 m_1', m_2 m_2'}(a)$. Note that we now have 4 helicity labels instead of two jm pairs. Somehow a shuffle has taken place. Here, if we set $j_1 = \frac{1}{2}$ and $j_2 = 0$, we should get the usual 2×2 matrices as we let a run through the 6 OP subgroups. Also, $j_1 = j_2 = \frac{1}{2}$ will give us the usual 4×4 matrices. It does not appear that Ruhl or Gelfand or Vilenkin have clearly stated the connection. I wonder what V has to say on this.

Some details from Puhl.

① Follows GGT. Homogeneous functions $F(\alpha z_1, \alpha z_2) = \alpha^\lambda \bar{\alpha}^\mu F(z_1, z_2)$ are carrier space. Single value means $\mu - \lambda = m = \text{integer}$ ✓.

$\Rightarrow \Delta m = \Delta \lambda \equiv \frac{1}{2} s$, so can use (m, s) . Also, (n_1, n_2) .

② $SL(2, \mathbb{C})/K = \text{cosets}$, typical coset is aK .

③ Group action is $T_a^\lambda F(z_1, z_2) = \dots$. Former notation is:

$$T_a^\lambda F\left[\begin{pmatrix} \vdots \\ z_1, z_2 \end{pmatrix}\right] = F\left[\begin{pmatrix} \vdots \\ z_1', z_2' \end{pmatrix}\right] = F\left[\begin{pmatrix} \vdots \\ z_1, z_2 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\right].$$

④ $a = k \cdot z$. $a = k'z$ so z labels cosets

$$z = \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix}.$$

⑤ Anyway, $T_a^\lambda F(z) = \alpha(z, a) F(za)$ is a new rep with one z -variable, regard as $z \in \text{coset space}$, so $M \hat{=} \text{base of bundle} = \text{coset space } SL(2, \mathbb{C})/K$.

⑥ Now go to $u \in SU(2)$, Section 3-3. $u = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix}$.

Let $g(u) = F(u_{21}, u_{12})$. Then

$$g(\circlearrowleft R_z(\omega)u) = e^{i\omega(n_2 - n_1)} g(u) \quad \leftarrow \text{let } m = n_2 - n_1$$

Measure Dz replaced with $du(u)$, as in S.T.

$$(g_1, g_2) = \int g_1^* g_2 du.$$

$$g_q^j(u) = D_{\frac{1}{2}m, q}^j(u) \quad \text{Then, satisfy the covariance, where } m = n_2 - n_1$$

That is (why m must be here),

canonical basis.

$f_q^j(z) = \text{basis functions in } z\text{-space!}$

$$\text{The } g_q^j(u) = D_{\frac{1}{2}m, q}^j(u) \text{ span } D_x = \mathcal{L}_m^2(U)$$

⑦ What is u_a ? $u_a = (k^{-1} u a)$

Note that $a = ?$ Well, $T_a^X \phi(u) = \alpha(u, a) \cdot \phi(u_a)$

I presume $u_a \in O(3)$.

⑧ For $a = u$, $T_u^X \phi(u) = \phi(u, u)$, no multiplier,

$$\begin{aligned} D_{j_1 q_1, j_2 q_2}^X(u) &= (\phi_{q_1}^{j_1}(u'), \phi_{q_2}^{j_2}(u'u)) \\ &= \int du' \cdot D_{q_1, m_1/2}^{j_1}(u') D_{q_2, m_2/2}^{j_2}(u'u) \\ &= \delta_{j_1 j_2} D_{q_1 q_2}^j(u). \end{aligned}$$

So:

$$D_{j_1 q_1, j_2 q_2}^X(a) = \langle j_1 q_1 | T_a^X | j_2 q_2 \rangle = (\phi_{q_1}^{j_1}, T_a^X \phi_{q_2}^{j_2})$$

$$D_{j_1 q_1, j_2 q_2}^X(u) = \delta_{j_1 j_2} D_{q_1 q_2}^{j_1}(u)$$

$$\mathcal{L}_m^2(u) = \text{space spanned by } \phi_q^j(u) \text{ where } -j < q < j \text{ and } j \geq \frac{1}{2}|m|.$$

$$\phi_q^j(u) = \sqrt{2j+1} \cdot D_{\frac{m}{2}, q}^j(u) \quad \text{note } j \geq \frac{1}{2}|m|$$

$$\mathcal{L}_m^2(u) = \sum_j \oplus \mathcal{H}^j \quad j \Rightarrow \frac{1}{2}|m| \dots +\infty$$

This says: The $SL(2, \mathbb{C})$ OIR rep space labelled by (m, s) may be written as a sum of spaces in this way.

$$D_{j_1 q_1, j_2 q_2}^X(a) = D_{j_1 q_1, j_2 q_2}^X(u_1 d u_2)$$

$$d = \begin{pmatrix} e^{\eta/2} & 0 \\ 0 & e^{-\eta/2} \end{pmatrix} = B_Z(\eta)$$

$$D_{j_1 q_1, j_2 q_2}^X(d) = \int \phi_{q_1}^{j_1}(u') \phi_{q_2}^{j_2}(u'd) du' \quad \text{turns out diagonal.}$$

$$d_{j_1 j_2}^x(\tau) \equiv D_{j_1 j_2}^x(d)$$

Note: The group action T_a^x , hence all matrix elements, also depend on g .

$$\begin{aligned} n_1 + n_2 &= i g \\ n_2 - n_1 &= m \end{aligned}$$

Also:

$$d_{00}^{(0,g)}(\tau) = 2 \sin(\frac{1}{2} \tau g) / g \operatorname{sh}(\tau)$$

IR classification:

1) $n_1 = n_2 = \text{positive integer}$

/ suppl. series

2) $\bar{n}_1 = -n_2 \Rightarrow g \text{ real,}$

/ main series

The projection as operator:

$$T_x^x \equiv \int d\mu(a) x(a) T_a^x$$

some integrable fctn on $SL(2, \mathbb{C})$

This operator-projection must have some action on elements of space of fctns of u , i.e. the rep space of our $SL(2, \mathbb{C})$ rep. This action is characterized by a kernel

$$T_x^x \phi(u) \equiv \int d\mu(u') K_x^x(u, u') \phi(u')$$

The usual "projections" of $x(a)$ are:

projection

$$\langle j_1 q_1 | T_x^x | j_2 q_2 \rangle \equiv \cancel{X_{j_1 q_1 j_2 q_2}^x} = \int d\mu(a) x(a) D_{j_1 q_1 j_2 q_2}^x(a)$$

Notion of trace:

$$\text{trace}(T_x^x) = \int d\mu K_x^x(u, u) = \sum_{j, q} X_{j, q, j, q}^x \quad \text{where } j = \frac{1}{2}|m|, \dots, |m| \leq j$$

Plancherel: (expansion theorem)

$$X_{j_1 q_1, j_2 q_2}^{(s, m)} \equiv \int du(a) x(a) D_{j_1 q_1, j_2 q_2}^{(s, m)}(a) \\ = \text{the projection.}$$

$$x(a) = \frac{1}{2} \int_{-\infty}^{+\infty} dg \sum_{m=-\infty}^{\infty} (m^2 + g^2)$$

$$\times \sum_{\substack{j_1 \\ j_2 = \frac{1}{2}|m|}}^{\infty} \sum_{q_1 = -j_1}^{j_1} \sum_{q_2 = -j_2}^{j_2} \cdot D_{j_1 q_1, j_2 q_2}^{(s, m)*}(a) X_{j_1 q_1, j_2 q_2}^{(s, m)}$$

Aside:

$$D_{m_1 m_2}^j(u)^* = D^* = D^{\sim T} = D^{\sim -1} = \tilde{D}(\tilde{u}^{-1}) \\ = D_{m_2 m_1}^j(\tilde{u}^{-1})$$

Now, $D_{j_1 q_1, j_2 q_2}^{(s, m)}(a) = \sum_q D_{q q_1}^{j_1}(u_1) d_{j_1 j_2 q}^{(s, m)}(\tilde{r}) D_{q q_2}^{j_2}(u_2)$

$$\Rightarrow D_{j_1 q_1, j_2 q_2}^{(s, m)*}(a) = \sum_q D_{q q_1}^{j_1}(\tilde{u}_1^{-1}) d_{j_2 j_1 q}^{(s, m)}(\tilde{r}^{-1}) D_{q q_2}^{j_2}(u_2^{-1})$$

since $d_{j_1 j_2 q}^{(s, m)*}(\tilde{r}) = d_{j_2 j_1 q}^{(s, m)}(-\tilde{r})$

$$\Rightarrow = \sum_q D_{q q_2}^{j_2}(\tilde{u}_2^{-1}) d_{j_2 j_1 q}^{(s, m)}(\tilde{r}^{-1}) D_{q q_1}^{j_1}(\tilde{u}_1^{-1})$$

$$= D_{j_2 q_2, j_1 q_1}^{(s, m)}(\tilde{u}_2^{-1} \tilde{r}^{-1} \tilde{u}_1^{-1}) = D_{21}((u_1 \tilde{r} u_2)^{-1})$$

$$\Rightarrow D_{j_1 q_1, j_2 q_2}^{(s, m)*}(a) = D_{j_2 q_2, j_1 q_1}^{(s, m)}(\tilde{a}^{-1})$$

Same as for SU_2 , statement of unitarity.

So I can restate Plancherel as:

$$x(a) = \frac{1}{2} \int_{-\infty}^{\infty} dg \sum_{m=-\infty}^{\infty} (m^2 + g^2) \cdot \text{trace}^{(m)} \left[D^{(g,m)}(\bar{a}') X^{(g,m)} \right]$$

where D and X are obvious operators, and

$$\text{trace}(AB) = \sum_i (AB)_{ii} = \sum_{n,n'} A_{n'n} B_{nn'}$$

and

$$\text{trace}^{(m)} = \left(\sum_{j=\frac{1}{2}|m|}^{\infty} \sum_{q=-j}^j \right)$$

Finally, with:

$$X^{(g,m)} = \int da' x(a') D^{(g,m)}(a')$$

We can say:

$$\int d\Lambda = \frac{1}{2} \int_{-\infty}^{\infty} dg \sum_{m=-\infty}^{\infty} (m^2 + g^2)$$

Then:

$$\begin{aligned} x(a) &= \int d\Lambda \cdot \text{trace}^{(m)} \left[D(\bar{a}') \int da' x(a') D(a') \right] \\ &= \int da' x(a') \cdot \underbrace{\left[\int d\Lambda \cdot \text{trace}^{(m)} [D(\bar{a}') D(a')] \right]}_{= \delta(a'-a)} \end{aligned}$$

So final version is:

$$\delta(a-a') = \int d\Lambda^{(g,m)} \cdot \text{trace}^{(m)} \left[D^{(g,m)}(\bar{a}') D^{(g,m)}(a') \right]$$

These results all follow from p 102 (4-32) of Ruhl.

Other symmetries:

$$d_{j_1 j_2 q}^{(s, m)}(\eta) = d_{j_1 j_2 -q}^{(s, -m)}(\eta)$$

$$d_{j_1 j_2 q}^{(s, m)}(\eta) = (-1)^{j_1 - j_2} d_{j_2 j_1 q}^{(-s, -m)}(\eta)$$

Also:

$$d_{j_1 j_2 q}^{(s, m)}(\eta)^* = d_{j_1 j_2 q}^{(-s, m)}(\eta)$$

1/12/77

W. Ruehl: "The Lorentz Group and Harmonic Analysis" Benjamin 1970.

(298 pages, written Sept. 1968)

Chapter 1: Definitions, Notations, Rep. Theory.

- 1.1 Poincare group: Mentions $SL(2, \mathbb{C})$, complex rotations, the various subgroups. Pieces of IG identified. Connection to $SL(2, \mathbb{C})$ is credited to Joos in 1962. No mention yet of representations.
- 1.2 Functions on groups. The Shift Representations (regular reps) are defined. Invariant measures. Functions form a Banach algebra called the group algebra. Referenced to Naimark rings book on this. He shows existence of invariant meas for s.l.c groups.
- 1.3 Unitary Reps. Defines usual c.u. rep. Irreducibility. Type I groups are defined (factor reps). Semisimple Lie, and certain direct product groups are all of type I. Central decomposition simply means the type-I decomposition theorem, and that each continuous component may contain ~~some~~ several irreducibles (eg, like those labelled by helicity indices).
- 1.5 Induced Representations. An Excellent section to read later. Incorporates multiplier reps, ratios of measures, the direct product-semi analysis, and shows Wigner angle idea.

Chapter 2: Unitary Reps of Poincare Group.

- 2.1 The orbits and their invariant measures.
- 2.2 The little groups. $E(2)$ is so simple it will be excluded.
- 2.3 Boosts for Timelike Orbits. Here the usual "canonical" and "helicity" boosts are defined. Things are done mostly in $SL(2, \mathbb{C})$ matrices. The Moussa-Stora like covariance wavefunctions are set up.
- 2.4 Boosts for other orbits. Basically, how do you want to move around on the spacelike surface, or on lightcone.
- 2.5 Finite-dim IR's of $SL(2, \mathbb{C})$ (called spinor reps). The matrix elements $D^{AA'}(a)$ are obtained by monomial method - just as in $SU(2)$ but are extended to $SL(2, \mathbb{C})$. The non-equivalence of D and D^* causes there to be two kinds of reps, dotted and undotted. Thus, the general spinor rep is (S_1, S_2) showing how much of each kind. Odd that this subject is injected at this point. (ref: Baade Jehle)
- 2.6 Classical spinor fields. Interesting! Start with your Moussa Stora wavefunction $\phi_m(x)$ and process it by applying $D(a^{-1})$, or D^* . This gives objects having the trivial covariance property like Toller's $M(a...)$, rather than a D-function covariance. If you then FT these guys to x-space, you get objects which transform as usual classical free fields!

Chapter 3: Representations of $SL(2, \mathbb{C})$.

This chapter seems like a useful conglomerate of Sciarrino Toller plus Naimark plus Gelfand et al. Reps are labelled by $\chi = (n_1, n_2, 0) = (m, q) = (M, \lambda)$ in Toller notation. Homogeneous functions, canonical basis. On page 63 the D functions are defined, the "central d-function" is explicitly written only as an integral of elementary functions. Ruhl uses index q for helicity. Bilinear functionals, equivalence, integer case, Lie Algebra realization in z and $\bar{z}$. Covariant operators are discussed in last section.

Chapter 4: Harmonic Analysis of nice functions on $SL(2, \mathbb{C})$.

We have here discussion of representations for the group algebra, then mainly the Plancherel Theorem for $SL(2, \mathbb{C})$. Shows how this decomposes the regular representation into irreducible components. Generalized functions, distributions.

Right in the middle of this chapter is a section on second-kind functions. Ruhl says: "there is no direct group theoretical meaning of the functions of the second kind". However, he is referring to the central d and e functions of $SL(2, \mathbb{C})$, not to the $SU(1, 1)$ functions. Lots of properties of these things are given (called a by Toller). E analogous to D are also defined. Clearly Ruhl will need these things later in his Lorentz pole analysis (but no).

Chapter 5: Representations of $SL(2, \mathbb{R}) = SU(1, 1)$.

This is more of Gelfand's. Bilinear functionals and so on. Connection to $SU(1, 1)$ is shown. Reps labelled by $\chi = (\sigma, \epsilon)$. The matrix elements are given on page 144-145 and are called c and C, rather than d and D. Lie generators are given. The B^k series are discussed as induced reps.

No continuous basis. No second-kind functions. (yet)

Chapter 6: Harmonic Analysis on $SL(2, \mathbb{R})$

Plancherel is derived a la Gelfand again.

Section now on e and d functions. The c-functions used above are redefined in the more standard d-functions. Then the e-functions are defined. All is in accord with Andrews and Gunson.

Chapter 7: Reps of $SL(2, \mathbb{C})$ on the subgroup $SU(1, 1)$.

Basically this is the same thing as done by Sciarrino and Toller. The idea is to use this for Lorentz pole reduction.

Chapter 8: Regge and Toller Poles

Toller pole = what Toller calls a Lorentz pole. Ruhl just defines these objects in obvious way, but does not show the family reduction idea. I think the book ended sooner than he had planned, since the Sciarrino Toller analysis was presented in last chapter.

Ruhl
 "Lorentz Group + Harmonic Analysis"
 1970, Benjamin

Preface — Naimark wrote the classic text in 1964 on Lorentz group representations. The problem was to construct covariant operators (generalized Dirac matrices). Ruhl will take a different point of view — "bilinear invariant functionals". A Chap on Regge + Toller poles. This guy, at CERN, is caught in that area between physics + math — I like it.

Outline of the book:

	page
1. Defs, notations, sep. theory	1
2. Unitary Reps on Inhom. $SL(2, \mathbb{C})$	30
3. Reps of $SL(2, \mathbb{C})$	51
4. Harmonic Anal of Polybounded fctns on $SL(2, \mathbb{C})$	93
5. Reps of $SL(2, \mathbb{R})$	135
6. Harm An of Polybound fctn on $SL(2, \mathbb{R})$	179
7. Same as 6. but on $SU(1, 1) \subset SL(2, \mathbb{C})$	221
8. Regge + Toller Poles	247

Appendix

References	Bergmann	Toller
	Gelfand	Weinberg
	Naimark	Wigner

Chapter One

1. Minkowski space: $xy = g_{\mu\nu} x^\mu y^\nu = x^0 y^0 - x^1 y^1 - x^2 y^2 - x^3 y^3$

2. linear transform in M-space: $x \rightarrow x' = \Lambda x + y$ $x'^\mu = \Lambda^\mu_\nu x^\nu + y^\mu$
 use: $(\Lambda, y) \uparrow$ where $g_{\mu\nu} \Lambda^\mu_\alpha \Lambda^\nu_\beta = g_{\alpha\beta}$

Show: $x'y' = g_{\mu\nu} x'^\mu y'^\nu = g_{\mu\nu} [\Lambda^\mu_\alpha x^\alpha + y^\mu] [\Lambda^\nu_\beta y^\beta + y^\nu] = g_{\alpha\beta} x^\alpha y^\beta = xy$

So $(\Lambda, 0)$ transforms leave xy invariant.

multiplicative law: $(\Lambda_1, y_1)(\Lambda_2, y_2)^x = (\Lambda_1, y_1)(\Lambda_2 x + y_2)$
 $= \Lambda_1 [\Lambda_2 x + y_2] + y_1$
 So, $(\Lambda_1, y_1)(\Lambda_2, y_2) = (\Lambda_1 \Lambda_2, y_1 + \Lambda_1 y_2)$
 $= (\Lambda_1 \Lambda_2) x + (\Lambda_1 y_2 + y_1)$
 $= (\Lambda_1 \Lambda_2, y_1 + \Lambda_1 y_2) x$ ✓

$(\Lambda, 0) =$ hom transforms

The set $\{(\Lambda, 0)\} = \text{group } L \text{ [Lorentz]}$. $T_4 = \text{transformations}$

Then $P = L \times T_4 = \text{Poincaré group}$.

"semidirect product" $G = A \times B \Rightarrow A = \text{subgroup}$
 $B = \text{invariant subgroup}$.

All groups will be Lie groups.

There are 4 disconnected pieces of L :

$$g_{\mu\nu} \Lambda_0^\mu \Lambda_0^\nu = g_{\mu\nu} = 1 \Rightarrow \Lambda_0^\alpha \Lambda_0^\alpha = (\Lambda_0^0)^2 = 1$$

$$(\Lambda_0^0)^2 = 1 + \sum_k (\Lambda_0^k)^2 \geq 1$$

By the way, the 4 vectors Λ_0^μ are all real.

$$\text{Thus } (\Lambda_0^0)^2 \geq 1 \Rightarrow \Lambda_0^0 \geq 1 \text{ or } \Lambda_0^0 \leq -1$$

Also notice that: $x'^\mu = \Lambda_0^\mu x^\nu$ (backwards)

The homo. Lorentz group has four disconnected manifolds:

$\Lambda_0^0 =$	$\det = +1$	$\det = -1$
≥ 1	$L_+^\uparrow \supset I$	$L_-^\uparrow \supset I_3$
≤ -1	$L_+^\downarrow \supset I_4$	$L_-^\downarrow \supset I_4$

Note that only $L_+^\uparrow$ is a group since it alone contains the identity

$$T_3 = \begin{pmatrix} 1 & & \\ & -1 & \\ & & -1 \end{pmatrix} = \text{space inversion element etc.}$$

$$\text{Now group these: } L_+^\uparrow \cup L_-^\uparrow = \text{all with } \Lambda_0^0 \geq 1 \\ \equiv \text{"orthochronous" LG} = L_+^\uparrow$$

$$L_+^\uparrow \cup L_+^\downarrow = \text{"proper" LG} = L_+$$

$$L_+^\uparrow = \text{proper, orthochronous LG}$$

(I would always take $\begin{pmatrix} m \\ 0 \\ 0 \\ 0 \end{pmatrix}$ into something with $E > 0$.)

$$\text{Thus, we hope: } L_+^\uparrow = \text{proper, orthochronous} = SO(3,1)$$

$$SL(2, \mathbb{C}) \equiv \left\{ \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \right\} \quad a_{ij} \text{ complex, but } \det(a) = 1.$$

$$a = a_0 e + \vec{a} \cdot \vec{\sigma}$$

4. Look at $\underline{x} \equiv x^0 e + x^k \sigma_k =$ a 2×2 complex matrix
 $=$ element of $SL(2, \mathbb{C})$.

Note: $\det(\underline{x}) = x^0 \det(e) + x^k \det(\sigma_k) =$ stop: $\det(A+B) \neq \det A + \det B$.

$$\det(\underline{x}) = \det \begin{bmatrix} x^0 + x^3 & x^1 - i x^2 \\ x^1 + i x^2 & x^0 - x^3 \end{bmatrix} = (x^0)^2 - (x^3)^2 - (x^1)^2 - (x^2)^2 = x^0$$

So: $\det(\underline{x}) = x^0$.

Consider: $\underline{x}' = a \underline{x} a^\dagger \Rightarrow \det \underline{x}' = \det \underline{x}$

Such a transform preserves xy so

$$\underline{x}' = a \underline{x} a^\dagger \iff x' = \Lambda(a) x.$$

Here $a \in SL(2, \mathbb{C})$. a is not necessarily unitary, but $SL \Rightarrow \det a^\dagger = (\det a)^* = 1$.
Thus, it seems that

$$L_+ = SL(2, \mathbb{C}) \text{ in some sense}$$

Of course $a = a_0 e + a_k \sigma_k$ so each a is 4 numbers (not 8).

$\det a = 1 \Rightarrow$ 1 equation, but 2 equations (Re, Im): $8 - 2 = 6$ parameters

$$(x^0)' e + (x^k)' \sigma_k = [a_0 e + a_k \sigma_k] [x^0 e + x^k \sigma_k] [a_0^* e + a_k^* \sigma_k] \\ = \Lambda_{\mu}^{\nu} x^{\nu}$$

Now: Look at upper left element: $(x^0)' + (x^3)' =$

RHS = $a_0^* x^0 e +$ 6 terms:

1. $a_0 a_0^* x^0 e e e = |a_0|^2 x^0 e$
2. $a_0 a_k^* x^j e \sigma_j e = |a_0|^2 x^j \sigma_j$
3. $a_i e x^0 e a_k^* \sigma_k = a_0 a_k^* x^0 \sigma_k$
4. $a_0 e x^j \sigma_j a_k^* \sigma_k = a_0 a_k^* x^j \sigma_j \sigma_k$

etc. etc. Turns out: $\Lambda_0^0 = |a_0|^2 + |a_1|^2 + |a_2|^2 + |a_3|^2$

Thus, we are dealing with $\Lambda_0^0 > 0$, hence ≥ 1 .

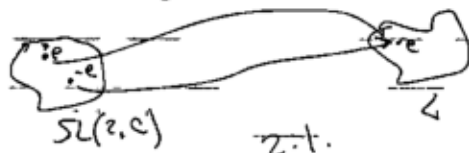
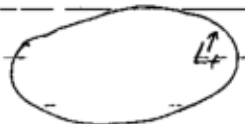
$$\begin{bmatrix} a_0 + a_3 & a_1 - ia_2 \\ a_1 + ia_2 & a_0 - a_3 \end{bmatrix}$$

5. $\Delta_v = h(\Lambda) = |\text{Tr } a|^2 = 4|a_d|^2$

$h(a) = \text{Tr } a = a_{11} + a_{22} = ?$

Maybe it's a trace $\sum a_{ii} = a_0 + a_1 + a_2 + a_3$

But



So for each $a \in SL(2, \mathbb{C})$, we have found a unique $\Lambda \in L_+^{\uparrow}$.
 This — well, we have not used all of $SL(2, \mathbb{C})$ — yes.

$$a \rightarrow \Lambda \quad \begin{array}{ccc} f: SL(2, \mathbb{C}) & \rightarrow & L_+^{\uparrow} \\ \uparrow a & & \uparrow \Lambda \end{array}$$

The image of $SL(2, \mathbb{C})$ is contained in $L_+^{\uparrow}$ and is a group.

6. "center" = set of central elements of a group

"central element" = ~~one~~ commutes with all group els

[eg, in $\sigma(3)$, 1 and $\bar{J}$ are central.]

Turns out that $SL(2, \mathbb{C})$ has two central els, called e_+ , e_- .
 Both map into

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = e_+ \rightarrow I \text{ of } L_+^{\uparrow}$$

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = e_- \rightarrow I \text{ of } L_+^{\uparrow} \text{ also.}$$

Let $\Sigma_2 =$ group whose els are $(\pm e, -e)$. Then:

$$\Sigma_2 \subset SL(2, \mathbb{C}) \text{ but } \Sigma_2 \not\subset L_+^{\uparrow}.$$

$$SL(2, \mathbb{C}) = L_+^{\uparrow} \times \Sigma_2$$

$$L_+^{\uparrow} \cong SL(2, \mathbb{C}) / \Sigma_2$$

↑ isomorphism

covers for 2:1.

This A/B means take el of $SL(2, \mathbb{C})$.

Sciarrino '67

Decomposition of the Unitary Irreducible Representations of the Group $SL(2C)$ Restricted to the Subgroup $SU(1, 1)$

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(Received 31 October 1966)

The unitary irreducible representations of the group $SL(2C)$ belonging to the principal series restricted to the subgroup $SU(1, 1)$ are decomposed into a direct integral of unitary irreducible representations of $SU(1, 1)$. The matrix elements of the unitary operator which performs the decomposition are given explicitly and used to obtain a relation between the matrix elements of the unitary irreducible representations of the groups $SL(2C)$ and $SU(1, 1)$. Similar identities between the matrix elements of nonunitary representations of these groups are obtained by means of analytic continuation. The relevance of these results to the theory of complex angular momentum and of high energy nearly forwardscattering is pointed out.

1. INTRODUCTION

RECENTLY various authors^{1,2} have suggested and investigated a connection between complex angular momentum and the (not necessarily unitary) irreducible representations of the Poincaré group corresponding to a spacelike four-momentum (the momentum transfer). These representations are strictly connected³ to the representations of the little group corresponding to a spacelike four-momentum, i.e., the three-dimensional Lorentz group, or the corresponding spinor group, the group $SU(1, 1)$ which is homomorphic to it. The scattering amplitude at fixed momentum transfer can be expanded in terms of the matrix elements of these representations. This expansion is strictly connected with the expansion obtained by means of the Sommerfeld-Watson transform.⁴ It can be considered as a generalization of the partial wave analysis, which is an expansion of the scattering amplitude at fixed energy in terms of the matrix elements of the representations of the group $SU(2)$.

It has also been suggested⁵⁻⁷ that, when the momentum transfer vanishes, it is more natural to expand the scattering amplitude in terms of the matrix elements of the representations of the little group

corresponding to a vanishing four-momentum, i.e., the homogeneous Lorentz group or the corresponding spinor group $SL(2C)$ which is homomorphic to it. In other words, many considerations lead us to think that, at vanishing momentum transfer, the expansion in terms of the matrix elements of the representations of $SL(2C)$ permits a simpler description of the high-energy scattering amplitude. ✓

As the scattering amplitude is an analytic, and therefore continuous, function of the momentum transfer, a connection must exist between the $SL(2C)$ expansion at vanishing momentum transfer and the $SU(1, 1)$ expansion for very small momentum transfer. This connection takes a more suggestive form if we make the assumption (supported by recent research⁸) that the scattering amplitude is dominated in the very-high-energy region by Regge pole contributions. As shown in the above-mentioned papers, each Regge pole contribution can be described in terms of the matrix elements of an irreducible nonunitary representation of $SU(1, 1)$. Gribov and Volkov^{9,10} have pointed out that, when the momentum transfer vanishes, the various Regge pole contributions can no longer be independent from each other, but must be arranged in families of poles which are displaced from one another by integral numbers. However, there is a large ambiguity in determining the structure of these families, so further hypotheses are needed. According to the ideas sketched above, we suggest¹¹ that at vanishing momentum transfer the sum of the Regge pole contributions belonging to a family gives rise to a contribution which can be described in terms of an

¹ H. J. Lipkin, *Lectures in Theoretical Physics*, W. E. Brittin and A. O. Barut, Eds. (University of Colorado Press, Boulder, Colo., 1965), Vol. VIIA, p. 132; L. Sertorio and M. Toller, *Nuovo Cimento* **33**, 413 (1964); F. T. Hadjoannou, *Nuovo Cimento* **44**, 185 (1966).

² J. F. Boyce, *J. Math. Phys.* **8**, 675 (1967).

³ E. P. Wigner, *Ann. Math.* **40**, 149 (1939).

⁴ E. J. Squires, *Complex Angular Momentum and Particle Physics* (W. A. Benjamin, Inc., New York, 1963). This book contains the references to the original papers.

⁵ M. Toller, *Nuovo Cimento* **37**, 631 (1965).

⁶ M. Toller, "The Laplace Transform on the Lorentz Group and a Generalization of the Regge Pole Hypothesis," Istituto di Fisica dell'Università di Roma, Report No. 76 (1965).

⁷ M. Toller, "Some Consequences of a Generalization of the Regge-Pole Hypothesis," Istituto di Fisica dell'Università di Roma, Report No. 84 (1965).

⁸ R. J. N. Phillips and W. Rarita, *Phys. Rev.* **139**, B1336 (1965).

⁹ D. V. Volkov and V. N. Gribov, *Zh. Eksperim. i Teor. Fiz.* **44**, 1068 (1963) [English transl.: *Soviet Phys.—JETP* **17**, 720 (1963)].

¹⁰ V. N. Gribov, *Zh. Eksperim. i Teor. Fiz.* **43**, 1529 (1962) [English transl.: *Soviet Phys.—JETP* **16**, 1080 (1963)].

¹¹ Volkov and Gribov use a different assumption.

What is Tollers d-function phase convention?

- Sci + Toller say:
- 1) d is a function of z , not of chz
 - 2) but they do restrict range to $z > 0$.

$$\frac{z+1}{2} = \left(ch \frac{z}{2}\right)^2 \quad \frac{z-1}{2} = \left(sh \frac{z}{2}\right)^2$$

So

$$\begin{aligned} d_{mm'}^l(z) \Big|_{\text{Toller}} &= \frac{\Gamma(l+1+m)}{\Gamma(l+1+m')} \left(\frac{z+1}{2}\right)^{\frac{m+m'}{2}} \left(\frac{z-1}{2}\right)^{\frac{m-m'}{2}} \\ &\quad \times F(-l+m, l+1+m, m'-m'+1; \frac{1-z}{2}) / \Gamma(m-m'+1) \\ &= \frac{\Gamma(l+1+m)}{\Gamma(l+1+m')} P_{m'm}^l(z) \end{aligned}$$

Again:

$$\textcircled{1} \quad d_{mm'}^l(z) \Big|_{\text{Toller}} = \frac{\Gamma(l+1+m)}{\Gamma(l+1+m')} P_{m'm}^l(chz)$$

- ② Therefore, what are Tollers symmetric?

$$d_{-m-m'}^l(z) = \frac{\Gamma(l+1-m)}{\Gamma(l+1-m')} P_{mm'}^l(chz) = \frac{\Gamma(l+1-m)}{\Gamma(l+1-m')} G_{mm'}^l P_{m'm}^l$$

So:

$$\textcircled{2} \quad d_{-m-m'}^l(z) = d_{mm'}^l(z)$$

$$\frac{\Gamma(l+1-m)}{\Gamma(l+1-m')} \frac{\Gamma(l+1+m)}{\Gamma(l+1+m)} \frac{\Gamma(l+1-m')}{\Gamma(l+1+m')}$$

③ Next,

$$d_{m'm}^l = \frac{\Gamma(l+1+m')}{\Gamma(l+1+m)} P_{mm'}^l = \frac{\Gamma(l+1+m')}{\Gamma(l+1+m)} G_{mm'}^l P_{m'm}^l$$

$$= \frac{\Gamma(l+1+m')}{\Gamma(l+1+m)} \frac{\Gamma(l+1+m)}{\Gamma(l+1-m)} \frac{\Gamma(l+1-m')}{\Gamma(l+1+m')}$$

$$= \frac{\Gamma(l+1+m') \Gamma(l+1-m')}{\Gamma(l+1+m) \Gamma(l+1-m)} d_{mm'}^l$$

So

$$\textcircled{3} \quad d_{m'm}^l(z) = \frac{\Gamma(l+1+m') \Gamma(l+1-m')}{\Gamma(l+1+m) \Gamma(l+1-m)} d_{mm'}^l(z)$$

Next: $d_{mm'}^{-l-1} = \frac{\Gamma(-l+m)}{\Gamma(-l+m')} P_{m'm}^l = \frac{\Gamma(l+1+m)}{\Gamma(l+1+m')} P_{m'm}^l$

$\times \frac{\Gamma(-l+m)}{\Gamma(l+1+m)} \left(\frac{\Gamma(l+1+m')}{\Gamma(-l+m')} \right)$
 $\equiv U_m^l$

So

$$d_{mm'}^{-l-1}(s) = U_m^{-l-1} \cdot d_{mm'}^l \cdot U_{m'}^l$$

$$U_m^l \equiv \frac{\Gamma(l+1+m)}{\Gamma(-l+m)}$$

④

For $\Re(l) = -\frac{1}{2}$, this shows equivalence.

③ How related to AG d ?

$$d_{mm'}^l(s) \Big|_{\text{Toller}} = \frac{\Gamma(l+1+m)}{\Gamma(l+1+m')} \underbrace{P_{m'm}^l(ch\zeta)}_{\rightarrow e^{-i\pi(m'-m)/2} \tilde{P}_{m'm}^l(ch\zeta \pm i\epsilon)}$$

$\sqrt{G_{m'm}^j} \cdot d_{mm'}^l(z \pm i\epsilon)$

So $d_{mm'}^l(s) \Big|_{\text{Toller}} = \left[\frac{\Gamma(l+1+m)}{\Gamma(l+1+m')} \cdot \frac{\Gamma(l+1+m') \Gamma(l+1-m)}{\Gamma(l+1-m') \Gamma(l+1+m)} \right]^{\frac{1}{2}} (\mp i)^{m'-m} d_{mm'}^l(z \pm i\epsilon).$

So

$$d_{mm'}^l(s) \Big|_{\text{Toller}} = (\mp i)^{m'-m} \sqrt{\frac{\Gamma(l+1+m) \Gamma(l+1+m')}{\Gamma(l+1+m') \Gamma(l+1+m)}} \cdot d_{mm'}^l(ch\zeta \pm i\epsilon) \Big|_{\text{AG}}$$

⑤

When j is on the principle series, we see that these two functions really do just differ by a phase factor, so they are both unitary representations!

Combine ③, ④ to get:

$$d_{mm'}^{-l-1}(s) = (-1)^{m-m'} d_{m'm}^l(s)$$

⑥

Comments on Toller's functions:

1. First, the functions of Andrews and Gunson are really best adapted for $SU(2)$ and the continuation out of $SU(2)$. They have the square roots of gamma functions, and the wrong cut structure for $SU(1,1)$ to be convenient. Toller's functions have all these advantages:

- a) good cut structure of my P-functions.
- b) lack of ambiguous square roots of gamma functions
- c) simple negative symmetry relation
- d) they are unitary $SU(1,1)$ matrix elements for principle series.

Only bad aspects might be taken to be:

- a) not defined for complex helicity per se
- b) do not simply reduce to Legendre functions, which have lengthy history.

2. Once Toller has selected a phase convention for his principle series functions, he must stick to it as he goes on to define the effects of the discrete symmetries. I.e., he has a specific algebra to satisfy and he must produce some functions to do it. He has done all this work for us in his signature paper.

3. It is very important that these d-functions are unitary.

4. If I were to try to redo Toller's work using AG functions, I would have to change the way that s' is represented, for example. So be wary of this. Probably best to stick very closely to Toller's functions.

The Toller d-function for negative argument:

Assume that j is on the unitary line for principle series. Then $j^* = -j-1$ as usual. Then you can show that:

$$d_{mm}^j(s) = (-1)^{m-j} d_{mm}^j(1/s); \text{ when } s < 0$$

I think the correct philosophy for going to complex j should be: establish the above result for j on the special line, and then continue to general j . If you try to continue first, you might end up with $f(j^*)$ which is not even an analytic function.

Toller's second-kind function.

$$z = \operatorname{ch} g \quad \left(\operatorname{sh} \frac{g}{2}\right)^2 = \frac{\operatorname{ch} g - 1}{2} = \frac{z-1}{2} \quad \left(\operatorname{ch} \frac{g}{2}\right)^2 = \frac{z+1}{2-1}$$

$$a_{mm'}^l(g) = \frac{(-1)^{m-m'}}{\Gamma(-l-m)\Gamma(-l+m)} \underbrace{\left(\frac{z-1}{2}\right)^{-l-1} \left(\frac{z+1}{2}\right)^{\frac{m+m'}{2}} F(l+1+m', l+1+m; 2l+2; \frac{z}{1-z})}_{= 2\Gamma(2l+3) \Phi_{mm'}^l(\operatorname{ch} g)}$$

Now we have:

$$\frac{\Gamma(-2l-1)}{\Gamma(-l-m)\Gamma(-l+m)} = \frac{\sin \pi(-l-m)\Gamma(l+1+m)}{\sin \pi(-2l-1)\Gamma(2l+2)} \frac{\sin \pi(-l+m)\Gamma(l+1-m)}{\pi}$$

Now result:

$$a_{mm'}^l(g) = (-1)^{m-m'} \frac{2}{\pi} \Phi_{mm'}^l(\operatorname{ch} g) \Gamma(l+1+m)\Gamma(l+1-m) \underbrace{\left[\frac{\sin \pi(l+m)\sin \pi(l-m)}{\sin \pi(2l)} \right]}_{(S_{mm}^l)^{-1}}$$

$$\Rightarrow a_{mm'}^l(g) = (-1)^{m-m'} \frac{\tan \pi(l+\epsilon)}{\pi} \Gamma(l+1+m)\Gamma(l+1-m) \underbrace{\Phi_{mm'}^l(\operatorname{ch} g)}_{\frac{\Phi_{mm'}^l(\operatorname{ch} g)}{\Gamma(l+1+m)\Gamma(l+1-m)}} = \frac{(S_{mm}^l)^{-1}}{[2 \cot \pi(l+\epsilon)]^{-1}}$$

$$\text{Then: } a_{mm'}^l(g) = (-1)^{m-m'} \frac{\tan \pi(l+\epsilon)}{\pi} \frac{\Gamma(l+1-m)}{\Gamma(l+1-m')} \Phi_{mm'}^l(\operatorname{ch} g)$$

$$\text{Another idea: } \sin \pi(l+m) = \sin \pi(l+m'-m'+m) = (-1)^{m-m'} \sin \pi(l+m').$$

Then:

$$\Phi_{mm'}^l(g) = \Phi_{mm'}^l(g)$$

$$a_{mm'}^l(g) = \frac{2}{\pi} \Gamma(l+1+m)\Gamma(l+1-m) \Phi_{mm'}^l(\operatorname{ch} g) \cdot (S_{mm}^l)^{-1}$$

$$a_{mm'}^l(g) = \frac{2}{\pi} \frac{\Gamma(l+1-m)}{\Gamma(l+1-m')} \left(\frac{\Phi_{mm'}^l(\operatorname{ch} g)}{S_{mm}^l} \right)$$

Finally:

$$a_{mm'}^l(g) = \frac{\Gamma(l+1-m)}{\Gamma(l+1-m')} \left[\frac{2}{\pi} \frac{\Phi_{mm'}^l(\operatorname{ch} g)}{S_{mm}^l} \right]$$

From my formula (H.27) I would conclude:

$$\begin{aligned}
 \left\{ \frac{\Gamma(l+1-m')}{\Gamma(l+1-m)} a_{mm'}^l(\xi) \right\} + \left\{ l \leftrightarrow -l-1 \right\} &= P_{mm'}^l \\
 &= \frac{\Gamma(l+1+m)}{\Gamma(l+1+m')} d_{mm'}^l(\xi) / \Gamma \\
 &= \frac{\cancel{\Gamma(l+1+m)} \Gamma(l+1+m') \Gamma(l+1-m')}{\cancel{\Gamma(l+1+m')} \cancel{\Gamma(l+1+m)} \Gamma(l+1-m)} d_{mm'}^l \\
 &= \frac{\Gamma(l+1-m')}{\Gamma(l+1-m)} d_{mm'}^l
 \end{aligned}$$

Therefore:

$$a_{mm'}^l(\xi) = d_{mm'}^l(\xi) - \frac{\Gamma(l+1-m)}{\Gamma(l+1-m')} - \frac{\Gamma(-l-m')}{\Gamma(-l-m)} a_{mm'}^{-l-1}(\xi)$$

$\Rightarrow$

$$d_{mm'}^l(\xi) = a_{mm'}^l(\xi) + \frac{\Gamma(l+1-m)}{\Gamma(-l-m)} a_{mm'}^{-l-1}(\xi) \cdot \frac{\Gamma(-l-m')}{\Gamma(l+1-m')}$$

Pause: From earlier form:

$$\begin{aligned}
 a_{-m, -m'}^l(\xi) &= a_{mm'}^l(\xi) \\
 d_{-m, -m'}^l(\xi) &= d_{mm'}^l(\xi)
 \end{aligned}$$

← negate relations.

Then:

$$d_{mm'}^l(\xi) = a_{mm'}^l(\xi) + \underbrace{\frac{\Gamma(l+1+m)}{\Gamma(-l+m)}}_{u_m^l} \cdot a_{mm'}^{-l-1}(\xi) \cdot \underbrace{\frac{\Gamma(-l+m')}{\Gamma(l+1+m')}}_{u_{m'}^{-l-1}}$$

Now use: $d_{mm'}^l(z) = U_m^l d_{mm'}^{l-1}(z) U_{m'}^{-l-1}$

$$= U_m^l \left[a_{mm'}^{l-1}(z) + U_m^{-l-1} a_{mm'}^l(z) U_{m'}^{-l-1} \right] U_{m'}^{-l-1}$$

$$= U_m^l a_{mm'}^{l-1}(z) U_{m'}^{-l-1} + a_{mm'}^l(z) \quad \text{same!}$$

So:

$$d_{mm'}^l(z) = U_m^l U_{m'}^{-l-1} a_{mm'}^{l-1}(z) + a_{mm'}^l(z)$$

(want to keep this:

$$= \frac{\Gamma(l+1+m')}{\Gamma(-l+m')} \frac{\Gamma(-l+m')}{\Gamma(l+1+m')} \frac{\Gamma(-l-m')}{\Gamma(-l-m')} \cdot \frac{2}{\pi} \cdot \frac{\phi_{mm'}^{l-1}(z)}{S_{mm'}^{l-1}}$$

$$= \frac{\pi}{\sin \pi(l-m)} \cdot \frac{\sin \pi(-l-m')}{\pi} \cdot \frac{\Gamma(-l+m')}{\Gamma(-l+m')} \cdot \frac{2}{\pi} \cdot \frac{\phi_{mm'}^{l-1}(z)}{S_{mm'}^{l-1}}$$

$$= \frac{\sin \pi(l+m')}{\sin \pi(l+m)} \cdot \frac{\Gamma(l+1+m')}{\Gamma(l+1+m')} \cdot \frac{2}{\pi} \cdot \frac{\phi_{mm'}^{l-1}(z)}{S_{mm'}^{l-1}}$$

$$= (-1)^{m-m'} \cdot \frac{2}{\pi} \cdot \frac{\Gamma(-l+m')}{\Gamma(-l+m)} \cdot \frac{\phi_{mm'}^{l-1}(z)}{S_{mm'}^{l-1}}$$

$$= \frac{(-1)^{m-m'} (-1)}{2(-1)^{m-m'} \sin \pi(l+m)} = -\frac{1}{2} \tan \pi(l+m)$$

So first term is

$$d_{mm'}^l(z) = -\frac{1}{\pi} \frac{\Gamma(l+1+m')}{\Gamma(-l+m')} \phi_{mm'}^{l-1}(z) + \text{other}$$

But $\phi_{mm'}^{l-1} = \Gamma(-l+m) \Gamma(-l-m') \phi_{mm'}^{l-1}(z)$

$$\Rightarrow d_{mm'}^l(z) = -\frac{1}{\pi} \Gamma(-l+m) \Gamma(-l-m') \phi_{mm'}^{l-1}(z) + \text{other}$$

Sciarrino + Toller

28 columns

"VIR's of $SL(2, \mathbb{C})$ on subgroup $SU(1, 1)$ "

JMP 8, 1252 (1967)

1. Introduction. Reread and annotate at the end.

2. VIR's of $SL(2, \mathbb{C})$. The original work is Gel'fand + Maimark, book 1964. Mackey in 1955 invented a generalization: the theory of induced reps. The way this works always involves choosing a subgroup. Use

$$K, k = \begin{pmatrix} \bar{p} & q \\ 0 & p \end{pmatrix}, p \neq 0$$

This subgroup has a one-dim unitary rep: $L^{M\lambda}(k) = |p|^{2(\lambda+M)} (p)^{-2M}$. This rep "induces" a rep of $SL(2, \mathbb{C})$ in this way: it defines a special class of functions on $SL(2, \mathbb{C})$, $a \in SL(2, \mathbb{C})$:

$$f \Rightarrow f(ka) = L^{M\lambda}(k) f(a) \quad f \in F, \text{ say.}$$

The actual reps are: $[D^{M\lambda}(a) f](a') = (2.19)$. We need some basic functions in F . Suggest using $\Phi_{jm}^M = D_{jm}^M(u)$. Write $a = u_1 a_2(s) u_2$ where $u_i \in SU(2)$. Then $a \in SL(2, \mathbb{C})$. There are some special cases:

$$D_{jm'j'm'}^{M\lambda}(u) = \langle \Phi_{jm}^M | D^{M\lambda}(u) | \Phi_{j'm'}^M \rangle = \delta_{jj'} \delta_{mm'} D_{mm'}^j(u) \quad u \in SU(2).$$

$$D_{jm'j'm'}^{M\lambda}(a_2(s)) = \delta_{mm'} d_{mm'}^{M\lambda}(s)$$

So generally: $D_{jm'j'm'}^{M\lambda}(u_1 a_2(s) u_2) = \sum_{m''} \bar{D}_{mm''}^j(u_1) d_{mm''}^{M\lambda}(s) D_{m''m'}^j(u_2)$. We break into subgroups, just as in the $SU(2)$ analysis. Here, an entire $SU(2)$ is "diagonal" on both sides.

3. VIR's of $SU(1, 1)$ Sets up μ, ν .

$$D_{mm'}^{\lambda}(\mu_2(\mu) a_2(s) \mu_2(\nu)) = e^{i\mu\mu - i\nu\nu} d_{mm'}^{\lambda}(s)$$

States classes of reps, Plancherel measure including discrete!

All d-functions are stated, always in traditional $\alpha(2)$ basis.

4. Decomposition Process. I think the essential result here is (4.27) which relates $D_{jmjm'}^{M\lambda}(\nu)$ of $SL(2, \mathbb{C})$ to $D_{mm'}^{M\lambda}(\nu)$ of $SU(1,1)$ by means of auxiliary functions $K_m^{M\lambda}(1, g, j)$

5. Properties of $K_m^{M\lambda}(1, g, j)$ functions. Technical.

6. NonUnitary Reps of $SL(2, \mathbb{C})$

7. "Lorentz poles" ~ how they decompose into series of regge or $SU(1,1)$ poles.

Appendix

A. The explicit $d_{mj}^{M\lambda}(\xi)$ functions

B. The explicit $K_m^{M\lambda}(g, l, +ij)$ functions

Comments: this paper is really not too relevant to my purposes right now. However, it does show that there exists the general problem of finding the representations of more complicated Lie groups. A sensible way is needed, and I guess the way is the method of induced representations invented by Mackey. Also, it was nice to see how things are analogous to finding the reps of say $SU(2)$. I could get more out of this paper by a careful reading, but will shelve it instead.

Sciarrino and Toller

"Decomposition of UIR's of $SL(2, \mathbb{C})$ restricted to $SU(1,1)$ "

JMP 8,1252 (1967)

Section 2: Here the UIR's of $SL(2, \mathbb{C})$ are found as an exercise in induced representation theory. The subgroup is called K ; its rep is a one-dim rep called L having labels M and λ . The induced rep is called D^M_λ and as usual appears as a regular rep with a little multiplier factor (Radon-Nikodyn derivative). The inducing rep is called a covariance condition as usual, see 2.17. The norm for the induced representation is an integral over cosets G/K which are labelled by z , but as shown in Naimark or here, this norm can also be written as an integral over $SU(2)$ as in (2.9). Thus, (2.17) through (2.20) tells the story. The reps of $SL(2, \mathbb{C})$ so obtained in this way are the Principle Series ones; recall that these alone occur in the Plancherel Theorem.

By choosing as basis functions the rotation matrices themselves, one finds that the $SL(2, \mathbb{C})$ matrix elements are, when restricted to $SU(2)$, simply these matrix elements again. Since you can parametrize any 6 parameter $SL(2, \mathbb{C})$ member by $u_1 B_z u_2$, it follows that the full matrix element appears as in 2.28 with a single helicity sum.

These matrix elements $D(a)$ are first-kind functions in the central variable, which leads them to define new object $\mathcal{K}(a)$ which is second-kind function. Keeping only to the central variable, the corresponding functions are $d(\xi)$ and $a(\xi)$.

Section 3: Here the usual $SU(1,1)$ theory is reviewed. Here again we have first kind functions $D(v)$ and $d(\xi)$, and second kind functions $A(v)$ and $a(\xi)$. In order to tell what function you have, look at its rep label which is some λ for $SU(1,1)$ but is $\lambda\lambda$ for $SL(2, \mathbb{C})$.

Section 4: We open by showing that all of $SL(2, \mathbb{C})$ may be divided into two "double cosets" of the form $K \cdot b \cdot SU(1,1)$ so that $a = kbv$. For each of these cosets a function is defined $f^+(v)$, eg, on $SU(1,1)$. These functions are restrictions of the original $f(a)$ of 2.17 and thus maintain some covariance. In 4.9 it is then shown that the old norm for a function in the space of the induced representation may be rewritten as a sum over the double coset label, and an integration over $SU(1,1)$ instead of $SU(2)$.

However, by defining projections on $SU(1,1)$ of these coset functions $F^\pm$, you can further convert this norm to an Plancherel integral as in (4.17). Since norms always tell you how a space is decomposing, we get the decomposition shown in (4.19).

The point is this: when you restrict your representation to $SU(1,1)$ and ignore the $SU(2)$ generator parts, the UIR's of $SL(2,C)$ have subspaces which are invariant with respect to $SU(1,1)$ group action, ie, are reducible with respect to $SU(1,1)$. Of course these UIR's are irreducible with respect to all of $SL(2,C)$. Equation (4.21) shows the reducibility of the $SL(2,C)$ UIR's when they are restricted only to $SU(1,1)$.

Equation (4.26) shows the norm decomposition very directly.

Now, recall that we had basis functions $R(u)$. In the decomposed space these appear, under the projection (4.16), as the new functions K . Thus, the K are simply images of the original ϕ basis functions in the decomposed space.

Section 5: Properties of these K functions. The K functions contain a label l which indicates the rep of $SU(1,1)$ onto which you projected, and a label j which was your original basis function label ϕ^j . Also, contains a label λ from the original $SL(2,C)$ rep labels.

As usual, K is written in terms of a second-kind-like function E which has better l behavior in right-half l -plane. Both functions K and E are very messy finite sums of G -functions, Meier. The functions K have two pole-chains just like my Q functions in variables l or λ .

Section 6: The main result of section 4 was that the actual $SL(2,C)$ matrix elements, when restricted to $SU(1,1)$, may be written in terms of the usual $SU(1,1)$ matrix elements. See either (4.210) or (4.27). Roughly:

$$D^{M\lambda}(v) = \sum_l \int dl K \cdot D^l(v) \cdot K.$$

This equation is the starting point of section 6. If you dump the discrete series terms (later justified), the above integral is essentially a vertical l -plane integration. The parameter λ is usually imaginary, but if you want λ to wander around, you can continue (6.3) by deforming contour around poles as needed. There are 6 poles chains, as shown in figure 1.

But being able to continue in λ is not really the main point here. With the contour still at $\text{Re}(l) = -\frac{1}{2}$, they replace the firstkind $SU(1,1)$ function D with the second-kind function they call A^{-l-1} . Then our formula for $D^{M\lambda}(v)$ appears as in (6.13).

Now the idea is to shift the λ -contour to the left, picking up the three kinds of residues on the three pole chains shown in figure 1. Thus, you get formula (6.15) which gives an asymptotic expansion of the $SL(2,C)$ group function restricted to $SU(1,1)$, in terms of second-kind $SU(1,1)$ group functions A . Finally, you break the left side into $D = \mathcal{K} + U^{-1}\mathcal{A}U$, ie, you break the restricted $SL(2,C)$ function into its two natural pieces of second kind. The result is an asymptotic expansion for $\mathcal{K}$ in terms of A . See (6.18).

An interesting side note is that the discrete series terms are exactly cancelled by the kinematic $SU(1,1)$ Plancherel poles. I am used to seeing this happen and this is no surprise (see (6.17)).

Section 7: Now comes the *raison d'etre* of this paper. At $t=0$ you know that a regge pole will be a Lorentz-pole in the complex λ plane, where λ is the label of the $SL(2,C)$ IR's, analogous to λ for the case $t < 0$. The contribution of this Lorentz pole to the scattering amplitude will be as in (7.2), ie $\text{amp}(a) \sim \mathcal{K}^{-n, -\lambda}(a)$ where now λ is the location of the Lorentz pole, ie, some complex number.

By inserting the magic formula (6.18) for $\mathcal{K}$ in terms of a series of A functions, you get (7.3) which shows, evidently, a whole family of Regge poles associated with this one Lorentz-pole.

An important point is that the regge-residues so obtained factorize, assuming that the Lorentz pole factorizes. Toller notes that this result is non-trivial.

Section 1: Introduction Toller in his Rome house reports 76 and 84 (1965) — neither of which are in the vault because it does not go back that far — has proposed the notion of Lorentz poles at $t=0$. But Volkov and Gribov pointed out a similar notion in 1963, that at $t=0$ regge poles must be somehow correlated to get analyticity.

I would assume Toller has more to say about Lorentz poles in other papers.

To show (2.26):

$$\begin{aligned}
 (\phi_{jm}^M, D^{M\lambda}(u) \phi_{jm'}^M) &= \int_{SU_2} du' \cdot R_{Mm}^j(u') \underbrace{[D^{M\lambda}(u) R_{Mm'}^{j'}(u')]}_{\sum_{m''} R_{Mm''}^{j'}(uu')} \\
 &= \sum_{m''} R_{Mm''}^{j'}(u) \underbrace{\int du' R_{Mm}^j(u') R_{Mm''}^{j'}(u')}_{\delta_{j,j'} \delta_{Mm''} \delta_{mm'}} = \sum_{m''} R_{Mm''}^{j'}(u) R_{m''m'}^{j'}(u) \\
 &= R_{mm'}^j(u) \delta_{jj'} \quad \text{roughly.}
 \end{aligned}$$

Sciarrino + Toller

1967 - 14 pages.

Introduction: There must be some connection between the $t=0$. $SL(2, \mathbb{C})$ analysis, and the $t=\epsilon < 0$, $SU(1,1)$ analysis. We expect a family of regge poles = a Lorentz Pole. This idea is credited to Gribov + Volkov, 1963. Donnay + Suranyi also treated this in 1964 using O4 analysis with Wick rotation.

outline:

2. VIR's of $SL(2, \mathbb{C})$
3. VIR's of $SU(1,1)$
4. Decomposition Process
5. The $K_m^{MA}(1, g; j)$.
6. Non-unitary regge.
7. Decompose a Lorentz-Pole.

A: The $d_{mij}^{MA}(\xi)$,

B: The $K_m^{MA}(\epsilon, l, t; j)$

C: an asymptotic limit

notes on $SL(2, \mathbb{C})$ UTR's:

Section 2: of Scrammo + Toller.

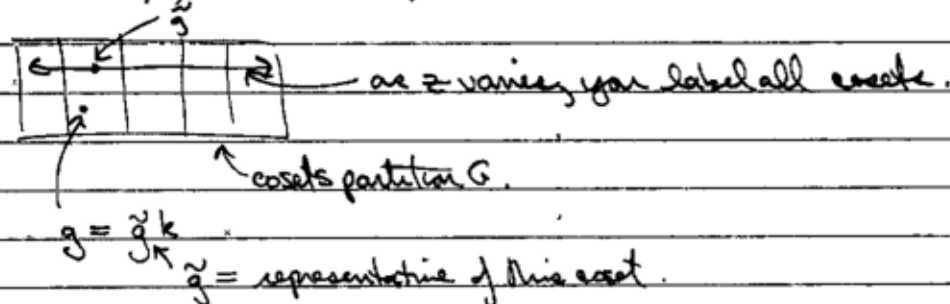
① $k = \begin{pmatrix} \bar{p} & q \\ 0 & p \end{pmatrix} \quad p \neq 0 \quad k \in K \subset SL(2, \mathbb{C})$

$$\begin{pmatrix} \bar{p}_1 & q_1 \\ 0 & p_1 \end{pmatrix} \begin{pmatrix} \bar{p}_2 & q_2 \\ 0 & p_2 \end{pmatrix} = \begin{pmatrix} \bar{p}_1 \bar{p}_2 & \bar{p}_1 q_2 + q_1 \bar{p}_2 \\ 0 & p_1 p_2 \end{pmatrix} = \begin{pmatrix} (\bar{p}_1 \bar{p}_2)^{-1} & [\bar{p}_1 q_2 + q_1 \bar{p}_2] \\ 0 & (p_1 p_2) \end{pmatrix}$$

Thus $K = \text{closed}$. $k^{-1} = \begin{pmatrix} p & -q \\ 0 & \bar{p}^{-1} \end{pmatrix} / 1 \quad \det k = 1$

So $K =$ a subgroup of $SL(2, \mathbb{C})$ unimodular. 4 parameters.

② Let $G = SL(2, \mathbb{C})$. Then $G/K = \text{cosets}$.



$\tilde{g} = \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix}$ so call this z .

Then $a \in G \rightarrow \exists z \in \mathbb{C}$ and $k \in K$ s.t. $a = kz$.

Choose a z . Then $a = \begin{pmatrix} \bar{p} & q \\ 0 & p \end{pmatrix} \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix} = \begin{pmatrix} \bar{p} + qz & q \\ pz & p \end{pmatrix} = 6 \text{ parameters.}$

$\det a = (\bar{p} + qz)p - pzq = 1 \checkmark$

③ For z -fixed, a will be unitary when $\bar{p} = \bar{p} + qz$ and $-pz = \bar{q}$.

$$\left\{ \begin{array}{l} \bar{p} = \bar{p} + qz \\ -pz = \bar{q} \end{array} \right\} \Rightarrow \begin{array}{l} (\bar{p} - \bar{p}) = qz \\ -\bar{q} = pz \end{array}$$

∞ ways to solve for p, q
 so "each coset (each z) contains ∞ unitary".

④ Consider two of these $SU(2)$'s: $[SU(2; \mathbb{C}) \text{ of course}]$ u, u' . When are they in same coset?

$$\begin{array}{l} u = kz \\ u' = k'z \end{array} \Rightarrow \text{at } k^{-1}u = k^{-1}u' \Rightarrow u' = (k'k^{-1})u = k''u.$$

but k'' must be $\in SU_2$, and entry $= 0 \Rightarrow k'' = R_2(\phi)$.

$a = kz \leftarrow$ if $a \in \text{coset } z$, write it as kz .

$u = k'z \leftarrow$ let $u =$ some chosen unitary $\in$ of this coset z .

Then $a = kz = k \cdot k'^{-1}u = k''u$.

Ergo: $a = k \cdot u$ = general way to write $a \in G$; $k \in K$, $u \in SU_2$.

⑤ Since cosets labelled by $z \in \mathbb{C}$, the measure on $G/K = \text{coset space}$ should be expressible in terms of z .

Let $\psi(a) = \psi(ka) =$ left-covariant under K .

But why choose $du(z)$ as in (2.9)? Is it invariant?

$$\int \psi(z'z) du(z) = \int \psi(z'u) du \quad z \in K \quad z \notin SU_2.$$

OK, skip part (2.11) for now.

⑥ Consider: $a: z' \rightarrow z''$ as: $z'' = kz'a$.

Why is this even a " z "? $[z, k] \neq 0$, by the way.

Take some $a \in G$. Then $(z'a) \in G$. This must be in some coset,

so $z'a = k'z''$ for some k and some z'' .

$\rightarrow$ So z'' is the new coset which contains $(z'a)$.

Turns out $z'' =$ usual projective transformation (2.13)

⑦ Let a, a_0 in same coset, so $a = kz \Rightarrow a^{-1} = z^{-1}k^{-1}$
 $a_0 = k'z \Rightarrow a_0^{-1} = z^{-1}k'^{-1}$
 $\Rightarrow a a_0^{-1} = k z z^{-1} k'^{-1} = k k'^{-1} \in K$.

⑧ Show that (2.6) is a UIR:

$$\begin{aligned} \text{Unitary? } \left\{ \begin{aligned} [L^M(k)]^{-1} &= |p(k)|^{-2M-2N} (p(k))^{2M} \\ [L^M(k)]^\dagger &= [L^M(k)]^* = |p(k)|^{-2M+2N} \overline{p(k)}^{-2M} \end{aligned} \right. \\ \text{yes, unitary.} &= |p(k)|^{2N+2M} \frac{|p(k)|^{+4M}}{p^{2M} \bar{p}^{2M}} = \checkmark \end{aligned}$$

Now show really a rep:

$$L(k_1) L(k_2) = \frac{1}{|p_1|} \frac{2(A+M)}{(p_1)}^{-2M} \cdot \frac{1}{|p_2|} \frac{2(A+M)}{(p_2)}^{-2M} \\ = \frac{1}{|p_1 p_2|} \frac{2(A+M)}{(p_1 p_2)}^{-2M} = L(k_1 k_2)$$

because see ①. So yes, (2.6) really is a VIR of K .

Note: K is not abelian, but of course a one-dim rep is allowable.

⑨ Stand by to induce! Covariance condition:

$$L^{M\lambda}(k) f(a) = f(ka)$$

For a scalar product of (f, f') , $f, f' \in \mathcal{H}^L$, Teller says to integrate only over SU_2 . But from (2.9), (2.10), this = integration over coset space! Proof:

$$\overline{f(a)} f'(a) = (f(u), f'(u))^L \text{ in space of rep } L \text{ (on a fiber).}$$

$$\text{Try: } \mathcal{V}(u) = (f(u), f'(u))^L$$

$$\text{Then } \mathcal{V}(a) = (f(a), f'(a)) = (\overline{L^{M\lambda}(k) f(k^{-1}a)}, L^{M\lambda}(k) f'(k^{-1}a)) \\ = (f(k^{-1}a), f'(k^{-1}a)) = \mathcal{V}(k^{-1}a).$$

Thus, $(f, f') = \int f(z) f'(z) d\mu(z)$ as in (2.9) = a true coset or orbit or base integral characteristic of an induced rep.

⑩ The actual induced rep = (2.19), shows Radon-Nikodym.

⑪ What if $a' = u \in SU_2$.

$$\frac{p(u u_0^{-1})}{p(u u_0 (u u_0)^{-1})} \quad a' = u \quad \text{why not choose } (a')_0 = u \text{ because } \dots$$

$$\text{No, } a' = u \Rightarrow a'_0 = u_2(\mu) u \Rightarrow a'(a')_0^{-1} = u \cdot u^{-1} u_2(-\mu) = u_2(-\mu).$$

$$\text{But } |p(u_2(-\mu))| = |e^{i\mu/\epsilon}| = 1, \text{ so no numeration.}$$

Next, $ua(ua)_0^{-1} \in K$, so use covariance to get next line.

$$k\tilde{a} = ua \Rightarrow k = (ua)(ua)_0^{-1} \checkmark$$

$$\tilde{a} = (ua)_0$$

- (12) For Covariance always relates two elements in same coset. If $a=u$, then $u' = \mu \cdot u =$ in same coset, and

$$f(\mu u) = \underbrace{D^M(\mu)}_{!} f(u) = e^{i\frac{1}{2}\mu(-2M)} = e^{-iM\mu}$$

This (2.21) is covariance for $f(u)$ defined on SL_2

Note: since (2.18) involves only $!u$, we can work entirely within SL_2 , hence (2.21) is only covariance condition!

- (13) Obviously the SL_2 matrix elements are so-covariant. (lose).

- (14) Partial decomposition (Naimark!) $a = u_1 B_z(\xi) u_2$
 $0 < \xi < \infty$
 $u_i \in SU_2$.

Now:

$$[D^M(u) \Phi_{jm}^M](u') = ?$$

First, set $u \rightarrow u'$, then $a \rightarrow u$ in (2.20):

$$[D^M(u) f](u') = |p[u'u(u'u)_0^{-1}]|^M (p[u'u(u'u)_0^{-1}])^{-2M} f[(u'u)_0]$$

Let $u'u = u''$, some new rotation! Then

$$(u'')(u'')_0^{-1} = u_2(\mu) \text{ for some } \mu \Rightarrow p = e^{i\frac{1}{2}\mu} \text{ so } |p|=1.$$

So $[D^M(u) f](u') = e^{i\frac{1}{2}\mu(-2M)} \underbrace{f(u'_0)}_{!} = f(u'u)_0$

Wait: just use (2.19).

- (15) Look at (2.25): ok, on SL_2 , the $D_{jm,jm}^M(u) =$ usual SL_2 .

Section 4

① $a = ku$ exist $\forall a$. But $k \cdot u_{\pm}(\mu) = \begin{pmatrix} p & q \\ 0 & p \end{pmatrix} \begin{pmatrix} e^{-i\mu/2} & 0 \\ 0 & e^{i\mu/2} \end{pmatrix} = \begin{pmatrix} p e^{-i\mu/2} & q e^{i\mu/2} \\ 0 & p e^{i\mu/2} \end{pmatrix} = \text{some } k'.$

Thus:

$$a = k \cdot R(0, \theta, \phi) = \text{general decomp } \forall a. \quad (4.1) \checkmark$$

Any $a \in \text{SL}(2, \mathbb{C})$ may be written as one of (4.5), where the $p(k)$ in k is $1/\sqrt{2} \text{deg}$ and $g = \text{parameter in } \alpha_x(g)$.

So, $\exists 3$ double-cosets labelled by: b^+ , b^- , $u_y(\pi/2)$.

② show (4.7): $f^{\pm}(hv) = f(b^{\pm}hv)$ / by 4.6

$$= f(ka) \text{ hopefully}$$

$$= L^{M_x}(k) f(a).$$

Want $a = b^{\pm}v$, so $k b^{\pm}v = b^{\pm}hv \Rightarrow h = (b^{\pm})^{-1}k(b^{\pm})$.

So, if h is of the form $h = (b^{\pm})^{-1}k(b^{\pm})$, then you may conclude that

$$f^{\pm}(hv) = L^{M_x} [b^{\pm} \cdot h \cdot (b^{\pm})^{-1}] \cdot f^{\pm}(v).$$

But also, we want all args to be in $\text{SU}(1,1)$. Thus:

$$h \in H^{\pm} \equiv \text{SU}(1,1) \cap [(b^{\pm})^{-1}K(b^{\pm})].$$

Claim is that $H^{\pm} = \mathbb{Z}$ -rotations, so (4.8).

③ To show (4.9):

$$\int_{\text{SU}_2} |f(u)|^2 d\mu = (4\pi)^{-2} \int_0^{4\pi} d\mu \int_0^{2\pi} d\theta \int_0^{\pi} d\phi \sin \theta |f(\mu, \theta, \phi)|^2$$

But use covariance (2.21) to get rid of μ ($|e^{-iM\mu}| = 1$)

$$= (4\pi)^{-1} \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin \theta |f(0, \theta, \phi)|^2$$

Change variables via (4.2) from θ to g ! Two regions to consider

$$f[u_y(\theta)u_z(\phi)] = f[b^+ \alpha_x(g)u_z(\phi)] = f^+(\alpha_x(g)u_z(\phi))$$

End up with $g = +$ part of (4.9). Also:

$$\tanh \frac{\theta}{2} = \tan \frac{\theta}{2} \quad \text{so} \quad \text{sech}^2 \frac{\theta}{2} \cdot \frac{1}{2} \cdot d\theta = \sec^2 \theta \cdot \frac{1}{2} \cdot d\theta.$$

So $-i \tan(\frac{i\theta}{2}) = \tan(\frac{\theta}{2}) \dots$ OK, I accept middle line of (4.9).

Then, use (4.8) to put $u_2(\mu)$ back in! This, one converts an $SU(2)$ integral to an $SU(1,1)$ one for this special type of function

(i) left covariant

④ Consider separately functions $F^+(\nu)$ and $F^-(\nu)$ on $SU(1,1)$ with norms as in (4.9).

$$\mathcal{H}^+ \quad \mathcal{H}^-$$

But we began with $f(a)$ in (4.6) being $\in \mathcal{H}$ which carries the rep of G shown in (2.19).

⑤ I thought that $\mathcal{H}$ was irreducible? $\mathcal{H}^{\pm M}$ must not be invariant subspaces of $\mathcal{H}$!

⑥ Project $F^{\pm}(\nu)$ into $SU(1,1)$ UIR's (4.16). Covariance of $F^{\pm}(\nu) = (4.6)$ forces helicity lock as in 4.16.

⑦ Next trick: use Plancherel-on-norms idea to show $\mathcal{H}^+ = \bigoplus \mathcal{H}^{\Lambda+}$ d.h.

$\Omega_m = \Lambda$ such that m would be a sensible helicity index.

$\mathcal{H}^{\pm M} =$ therefore obvious.

$F_m^{\Lambda\pm} =$ projection of $\phi_{j,m}^M$ onto $SU(1,1)$ UIR's

$$= \delta_{mm'} K_m^{M\lambda}(\Lambda, \pm; j).$$

λ appears because $f^{\pm}, F^{\pm}$ covariant wrt $L^{M\lambda}$.

So, project the usual $SU(2)$ basis fcn of $SU(2, \mathbb{C})$ onto $SU(1,1)$, get K .

- ⑧ A much line coming up: what we really want to see are the $SL(2, \mathbb{C})$ matrix elements when $a \in SU(1, 1)$. We already know the other case where $a \in SU(2)$:

$$D_{jmjm'}^{M\lambda}(u) = \sum_{jj'} R_{jmjm'}^j(u)$$

In the harder case we have:

$$D_{jmjm'}^{M\lambda}(v) = \int du \cdot \overline{\phi_{jm}^M(u)} \cdot [D_{jmjm'}^{M\lambda}(v) \phi_{jm'}^M(u)]$$

Now, the idea here is to do this in the reduced spaces.

$$f_2(u) = [D_{jmjm'}^{M\lambda}(v) \phi_{jm'}^M(u)](u)$$

$$\rightarrow F_{(2)m}^{\lambda g} = \int_{SU(1,1)} dv' D_{m, gM}^{\lambda}(v') (dv'_g)^{-1} f(b^{\pm} v')$$

$$= \int_{SU_{1,1}} dv' D_{m, gM}^{\lambda}(v') (dv'_g)^{-1} \cdot [D_{jmjm'}^{M\lambda}(v) \phi_{jm'}^M(u)](b^{\pm} v')$$

I have too many D 's?

- ⑨ At any rate, (4.29) says: to get your desired $SL(2, \mathbb{C})$ matrix element when $a = v' \in SU_{1,1}$, integrate the corresponding $SU_{1,1}$ matrix element over certain $SU_{1,1}$ reps with weights as shown in (4.29)

Section 6.

- ① Ignore discrete terms and study:

$$D_{jm, j'm'}^{M\lambda}(\nu) = \frac{1}{2i} \oint_{\gamma} dl \quad K \cdot D_{mm'}^{el}(\nu) \cdot K$$

The factor $K \cdot K$ has poles in the l -plane at $l = 2 - n - 1$. Residues of these poles are given in (6.5), note left-right factorization. So you can reset the contour to $\text{Re}(l) = -1/2$ and explicitly show the poles, as in (6.7). Note that for $\text{Re}(l) < -1/2$, you get a different set of poles; this why (6.8).

- ② Claim: The background + discrete decay as $e^{-\frac{1}{2}l}$, so only the pole terms count asymptotically.

- ③ The P -function D is replaced somehow with the Q -function A^{l+1} so as to yield (6.13):

$$D_{jm, j'm'}^{M\lambda}(\nu) = \oint_{\gamma} dl \cdot K \cdot A^{l+1} \cdot K + \text{junk}$$

- ④ Now that P is replaced by Q , you can shift contour to the left (Mandelstamish idea). Get two series of poles from $K \cdot K$, one series from $\tan l$.

- ⑤ Specialize to $\nu = (\mu, \xi, \eta) = (0, \xi, 0)$. Then break D into two pieces shown in (6.18), (6.19).

- ⑥ Now assume that you could expand scat amp on $SL(2, \mathbb{C})$ and a "Frotyz pole" would then tell you that:

$$\text{pole} \sim A_{jm, j'm'}^{-M, -\lambda}(a)$$

Rewrite (7.2) as (7.3) to get main result.

Appendix A

$$\textcircled{1} D_{j'm, j'm'}^{M\lambda} [a_2(\xi)] = \langle \Phi_{j'm}^M, \bar{D}^{M\lambda} [a_2(\xi)] \Phi_{j'm'}^M \rangle$$

$$= \int_{SU_2} du. \overline{\Phi_{j'm}^M(u)} \{ D^{M\lambda} [a_2(\xi)] \} \Phi_{j'm'}^M(u)$$

So you need $[D(\xi) \cdot \Phi](u)$ as in (2.20).

$(u B_2(\xi))_0 = \text{any element in coset with } u B_2(\xi). \quad (A1) \text{ gives}$

a certain SU_2 element which fills the bill

So out comes (A.5). OK, but functions are really a mess!

- $\textcircled{2}$ New idea here: when you write a function like P^j in terms of Q^j and Q^{-j-1} if it were true that Q_j and Q_{-j-1} were equivalent representations, then they would be linked by a unitary operator. Then you could say something like:

$$P_j = Q_j + U_j^{-1} Q_{-j-1} U_j$$

See (A.15) for example:

$\textcircled{3}$ Summary:

(A5): $d = \text{single integral of two rotation matrices}$

(A10): $d = \text{double sum} \times \text{single integral of exponents}$

$$(A11): d = a_{j,j'}^{M\lambda} + (-1)^{j-j'} a_{j,j'}^{-M,-\lambda}$$

(A12): $a = 4\text{-fold sum.}$

$$(A15): d_{j,j'} = a_{j,j'} + U a_{j,j'}^{-M,-\lambda} U$$

Only reasonable result is the large ξ $a(\xi)$.

$$a_{m,j,j'}^{M\lambda}(\xi) \approx e^{-\xi(\lambda+1+M \cdot m)}$$

$\textcircled{4}$ the $\delta_{m,m'}$ = very obvious.

misc .

{ M K F Wong
 "unitary reps of $SO(n,1)$ "
 JMP 15, 25 (1974)

①	$SO(3,1)$	Poincaré spacelike little group	Bagmann
	$SO(3,1)$	Lorentz group	Gelfand, Naimark
	$SO(4,1)$	De Sitter group	Thomas
	$SO(5,1)$		Kuriyan

② Does all $SO(n,1)$ groups at once. When $n = \text{odd}$, $SO(n,1)$ has no discrete series. He classifies all UIR's of general $SO(n,1)$ group.

{ F. Schwarz
 "reps of $SO(n,1)$ "
 JMP 12, 131 (1971).

Mentions that, for continuous series,
 for suppl

m_{31}
 $\swarrow$
 $j \geq |M|$

$j \geq 0$

m_{21}
 $\swarrow$
 $|j, m\rangle$

Bander + Itzykson
 "Group Theory + H Adm II"
 RMP 38, 346 (1969)

Ref'd by Abner + Saunders

13 pages

I. Introduction We open by defining some entities:

① S_p = unit sphere in p -dim. Euclidean space. The measure on this sphere is:

$$\int_{S_p} d\Omega = \int_{S_p} d^{p-1}\Omega = \frac{2\pi^{p/2}}{\Gamma(p/2)} \equiv \omega_p$$

So ω_p is area of this hypersphere. This is a standard result, see notes. On this sphere also define:

$S_{\text{sph}}(n_i, \vec{n})$

n_i = angles denoting $\times$

$\vec{n}$ = unit vector in p -dimensions
 / locates a point on sphere.

② T_p = unit hyperboloid in p -dim Euc. space. A point on this entity is $\vec{u}$ such that:

$$u_0^2 - \sum_{i=1}^{p-1} u_i^2 = 1$$

$$\Rightarrow u_0 = \pm \sqrt{\sum_{i=1}^{p-1} u_i^2 + 1}$$

Two sheets, so T_p^+ and T_p^- . Try this:

$$u_0 = \cosh \Theta \quad u_i = \sinh(\Theta) \cdot n_i \quad \text{where } \vec{n} = \text{point on } S_{p-1}$$

Eg, for T_3 , $\vec{n}$ = point on a circle. This locates you azimuthally on the hyp. Also:

$$\begin{aligned} \sum_{i=1}^{p-1} u_i^2 &= \sinh^2(\Theta) n_1^2 + \sinh^2(\Theta) n_2^2 \\ &= -\frac{1}{4}(e^{\Theta n_1} - e^{-\Theta n_1})^2 - \frac{1}{4}(e^{\Theta n_2} - e^{-\Theta n_2})^2 \quad \text{no.} \end{aligned}$$

$$u_i = n_i \sinh(\Theta)$$

$$\sum_{i=1}^{p-1} u_i^2 = \sinh^2 \Theta, \text{ then } u_0 = \cosh \Theta \text{ works } \checkmark$$

③ On this surface, measure is: $d^p \mu = (\sinh \Theta)^{p-2} d\Theta d^{p-1}\Omega$

Not clear why we are doing all this fancy stuff

II. The Symmetry Group:

A. Infinitesimal Method

① Here, the usual $O(1,3)$ $[L]$ generators are:

$$L = \frac{1}{2}(\mathbf{r} \times \mathbf{p} - \mathbf{p} \times \mathbf{r})$$

$$M = \left(\frac{1}{2\mu}\right)(\mathbf{p} \times \mathbf{L} - \mathbf{L} \times \mathbf{p}) - \hbar \hat{\mathbf{n}} \quad \text{"Runge-Lenz vector"}$$

We are doing the hydrogen atom, so

$$H = \mathbf{p}^2/2\mu - \hbar^2/|\mathbf{r}|$$

It is simply a fact that these operators, when correctly normalized, represent the Lorentz generators!

② The reps. of L (unitary reps since $H, L, \tilde{M}$ all hermitian) are:

$$\left. \begin{array}{l} l_0 = + \text{integer} \\ c = i\epsilon, \text{ pure imag} \end{array} \right\} \text{"The principle series"}$$

$$\left. \begin{array}{l} l_0 = 0 \\ 0 \leq c \leq 1 \end{array} \right\} \text{"The supplemental series"}$$

③ Rewrite $H = \dots [L^2 - \tilde{M}^2 + \hbar^2]^{-1}$. Note that $(L^2 - \tilde{M}^2)$ is a Casimir of value $l_0^2 + c^2 - 1$, where (l_0, c) label the rep. It turns out that only $l_0 = 0$ is allowed, so:

$$E \equiv (L^2 - \tilde{M}^2 + 1)^{-1} \equiv -1/c^2$$

For scattering we want $E > 0$, so $c = i\epsilon$ and we need only consider the "principle series" with $l = 0$.

B. Global Method - Fock Transformation

① Here, they write the S.E. in f -dim p -space. This p space is then mapped onto an $f+1$ dim hyper T_{f+1}^+ in clear way. This is the Fock x-form. So $\vec{p} \rightarrow u$. Both sheets get used.

Then:

$$d^f p = \frac{d^f u(u)}{|1 + u_0|^f} \quad \text{This is the measure on } T_{f+1}^+.$$

Etc.

③ Now the S.E. is an integral on the hyperboloid:

$$\hat{\phi}(u) = \dots \int_{T_{S+1}} d^f \mu(v) \hat{\phi}(v) / |u-v|^{f/2(f-1)}$$

So v moves over the hyperboloid.

③ Why do all this contortin?!? Recall in $O(3)$ X. If you look at this equation, apparently you can do an $O(1, f)$ transform on the equation & get no change. I do not yet see why, but $d^f \mu$ is suggestive.

④ First, we need some math:

C. Defining the "Spherical Functions" on Hyperboloids.

① Think of $O(1, f)$. If we pick a point in the hyp, you can do an $O(f)$ & have that point remain fixed. Accept this. Presumably, $d^f \mu$ is like dg for $G = O(1, f)$, so $d^f \mu = \text{invariant}$.

② Let g be a fctn on T_{S+1} . The Hilbert space consisting of square-integrable (on T_{S+1}) g is called the [carrier space]. We can of course define an operator which acts on g , to give a new H_{S+1} function:

$$g' = \mathcal{U}g \quad g'(u) = g(\Lambda^{-1}u)$$

$\uparrow$ operator in H_{S+1} $\uparrow$ point in T_{S+1}

Eg, the set of integrable fctns of (α, β, δ) forms a Hilbert space which is the carrier space for $O(3)$.

③ An operator like $\mathcal{U}_N$ is in general reducible: $\mathcal{U}_N = \int_0^\infty dN \mathcal{U}_N^N$, here N is the (continuous) group labels.

④ The problem is to define a set of functions which "span" the T_{S+1} in the same way that Y_m "spans" S_2 . Such fctns will be:

$$H_{N, \alpha, \beta}^{(S+1)}(u) = \sum_{N, \alpha}^{(S)} (\otimes) \prod_{\alpha, \beta}^{(f)} (\vec{m})$$

$\uparrow$ label rep of $O(1, f)$ $\uparrow$ label like l $\uparrow$ label like m $\uparrow$ new fctn to be found $\uparrow$ generalised sph. harmonic

- ⑤ The problem is find these $\sum_{N,\alpha}^{(f)}(\theta)$ Sctns. An usual 3d we might say:

$$\nabla^2 [R^2 Y_{lm}(\theta, \phi)] = 0$$

We generalize this to get eigenctns of L^2 . But now $H_{N,\alpha,\beta}(u)$ must be an eigenctn of G , the $O(5,1)$ casimir.

- ⑥ What results are explicit forms for $\sum_{N,\alpha}^{(f)}(\theta)$. These involve N, α , $\sinh$, $(N\theta)$, and a form of the Legendre (conical).

- ⑦ These H Sctns now satisfy orthogonality over $\int d^5u$, completeness when $\int d^5u$, and a third fancy integral relation.

D. Proof of the Three Relations on the H -Sctns. (Cont turning)

E. More of same,

Note: $\sum_{N,\alpha}^{(f)}(\theta) = \dots$ Lowest "harmonic" in principle series is:

$$\sum_{N,\alpha=0}^{(f-1)}(\theta) = \left\{ \frac{1}{2} \pi N^2 (N^2 + 1^2) (N^2 + 2^2) \right\}^{-\frac{1}{2}} \frac{d}{d \cosh \theta} (\cos N\theta)$$

$$\frac{d}{d(\cosh \theta)} (\cos N\theta) = \frac{d\theta}{d(\cosh \theta)} \cdot \frac{d}{d\theta} \cos(N\theta) = \sim \frac{\sin(N\theta)}{\sinh \theta}$$

I guess this is like $Y_{00}(\theta)$ for $d(3)$.

Zmuidzinas
 "Unitary Reps. of LG on 4-vector Manifolds"
 JMP 7, 764 (1966)

① Lorentz Group has two Casimirs

Maximal abelian subalgebra
 is J^2 and J_3 , which Z.
 calls M^2 and M_3 .

$$\begin{cases} F = -\frac{1}{2} M_{\mu\nu} M^{\mu\nu} \\ G = \frac{1}{4} \epsilon^{\mu\nu\rho\sigma} M_{\mu\nu} M_{\rho\sigma} \end{cases}$$

So in general, you can diagonalize 4 things:

F	f	! ! ! ! !
G	g	
M^2	$j(j+1)$	
M_3	μ	
generator	name of eigenvalue	

basic vectors: $|f, g, j, \mu\rangle$.

② If $G=0$ (ie, the $g=0$ sector), you have "spinless" case, and the 6 LG generators can be realized as diff. operators on the space $f(x)$ where $x = 4$ -vector.

$$M_{\mu\nu} = i(x_\mu \partial_\nu - x_\nu \partial_\mu)$$

[check that $G=0 \dots$]

Can also define operators $X_\mu = x_\mu$ and $P_\mu = i\partial_\mu$. It turns out that, when $g=0$, you can also diag. P^2 with F . Add.

F	f	/ "mass"
P^2	m^2	
M^2	$j(j+1)$	
M_3	μ	

③ Now seek some explicit functions $\Psi_{mfju}(x)$ which diagonalize all four operators

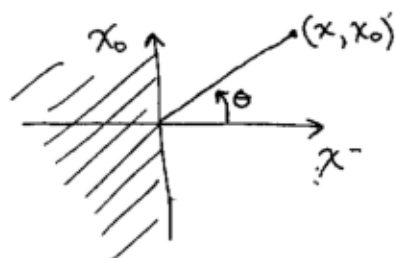
Let $x = (x_0, x \hat{n})$

ie, $\psi_\mu^j(\hat{n})$, usual spherical harmonic

Then $\Psi_{mfju}(x) = \phi_{mf}(x_0, x) a_{ju}(\hat{n})$

(because E, P^2 do not depend on $\hat{n} = (\theta, \phi)$.)

④ So, problem is reduced to finding $\phi_{mf}^j(x, x_0)$ which diagonalize F and P^2 as given above. (3.1).



$$\tan \theta = x_0/x \equiv "s"$$

$$x_0^2 - x^2 \equiv "r^2" \text{ can be } + \text{ or } -.$$

If you change from (x, x_0) to (s, r) , then

$$\phi_{mf}^j(x, x_0) = \underset{\substack{\uparrow \\ \text{solves (3.3)}}}{b_f^j(s)} \times \underset{\substack{\uparrow \\ \text{solves (3.4)}}}{c_m^f(r)} \quad \text{separates}$$

⑤ Timelike-Case: $r^2 = x_0^2 - x^2 = \text{mass}^2 > 0$, $s^2 > 1$.

We know the the F -eigenfunctions " $b_f^j(s)$ " are Legendres like $P_j^\alpha(s)$ with $\alpha = \sqrt{1-f}$.

Turn out: $k=0 \Rightarrow g=kv=0$ so one possibility

$$\Rightarrow \alpha = \sqrt{1-f} = \pm i\nu.$$

So

$$b_f^j(s) \equiv Q_j^{(i\nu)}(s) \text{ as in (3.10).}$$

Another from via Whipple.

⑥ Several Tossian "completeness" relations for $[a_{jm}(\hat{n}) b_v^j(s)]$ are given. Doesn't seem to care about $c_m^f(r)$.

Expansion theorem is misplid, sort of at "fixed mass". In my notation:

$$K_\nu(p) = \text{trace} [D(g) D(g')] \dots = \sum_{j, \mu} \dots$$

$$\text{where } D(g) = D_{jm}(g) = a_{jm}(\hat{n}) b_v^j(s)$$

Then, $\int_0^\infty d\tau K_V(p) = \int_0^\infty d\tau \cdot \hbar [D(g)D(g')] = \delta(x-x').$

where g 's role played by X^M . Here $\uparrow$ is your expansion theorem.
I think this is the $M=0$ sector of Naimark result, $i\omega = \lambda$.

① Spacelike case: $b_j^k(s) \approx p_j^k(s)$

Remarks: I wish the connection to Naimark were made stronger.

More detail from Z.

$|k\nu j\mu\rangle =$ abstract basis function.

Realized in X^4 , called $\psi_{mfj\mu}(x) = \phi_{mf}^j(x_0, |\vec{x}|) Y_{\mu}^j(\theta, \phi)$
 $= b_f^j(s) Y_{\mu}^j(\hat{n}) \cdot C_m^f(r)$

where $s = \sqrt{t}$ and $t = x \cdot x = x_0^2 - \vec{x} \cdot \vec{x}$, $r^2 = t$.

So r labels your hyperboloid.

Reps are labelled by:

$$\begin{aligned} f &= 1 + v^2 - k^2 & F \\ g &= 2kv & G \\ m^2 &\text{ when } g=0 & P^2 \end{aligned}$$

What may α be? $\alpha = \sqrt{1-f} = \sqrt{k^2 - v^2}$.

If $k=0$, then $\alpha = \pm iv$ and $f = 1 + v^2$.

If $v=0$, then $\alpha = \pm k$ and $f = 1 - k^2$.

Answer: $k=0$ only.

$|0\nu j\mu\rangle \rightarrow b_{\nu}^j(s) Y_{\mu}^j(\hat{n})$

where $b_{\nu}^j(s) = (i)^j \left[\frac{\Gamma(j+1-i\nu)}{\Gamma(j+1+i\nu)} \right]^{\frac{1}{2}} \frac{2}{\pi} e^{i\pi\nu} \sqrt{\nu \sinh \pi\nu} (s^2-1)^{\frac{1}{2}} Q_j^{i\nu}(s)$

where usually $s = \frac{x_0}{|\vec{x}|}$

How does projection work?

Key: compare to (m, s) of Puhl
 (M, λ) of ST:

$k=M=\frac{1}{2}m = 0, \frac{1}{2}, 1, \frac{3}{2}, \dots$

$\nu = \bar{s} = i\lambda$

Expansion Theorem looks like:

$$\delta(\hat{Q} - \hat{Q}') = \int_0^\infty ds' \cdot s' \delta^{(4)}(x - x') \quad s' = x'^2$$

= delta in hyperboloid variables

$$= \int_0^\infty dv \cdot \sum_{j\mu} [D_{j\mu}^v(\hat{Q}) D_{j\mu}^v(\hat{Q}')^*]$$

where $D_{j\mu}^v(\hat{Q}) \equiv \left. \begin{matrix} m \\ a_{j\mu}(\hat{n}) b_{j\mu}^v(s) \end{matrix} \right\} \quad \hat{Q} = (s, \hat{n})$

I can see how this appears to be a part of the full Plancherel for $SL(2, \mathbb{C})$.

Projection: $\int dx (x^2)^{-1} \delta(x^2 - r^2) \Theta(x_0) \cdot \bar{\phi}(x) \equiv \tilde{\phi}_{j\mu}(r^2)$
 $\times D_{j\mu}^v(\hat{Q})^*$

Recovery: $\phi(x) = \int_0^\infty dv \cdot \sum_{j\mu} \tilde{\phi}_{j\mu}(r^2) D_{j\mu}^v(\hat{Q})$

Refuses to write... well

$$\int dx \Theta(x_0) \delta(x^2 - r^2) (r^2)^{-1} = \int_1^\infty dg \frac{1}{2} (g^2 - 1)^{-2} \int d\hat{n}$$

where $g = x_0/x = \cosh \eta$ in this case...

$$dg (g^2 - 1)^{-2} =$$

Spacelike Case: now $|\beta| < 1$ so $g = \cosh \eta$ I would say.

Now, $k=0$ case gives

$$b_j^{\pm}(s) = \frac{1}{\sqrt{2}} \cdot (i)^j \sqrt{\frac{\gamma}{\sinh \pi \gamma}} \cdot \sqrt{\frac{\Gamma(j+1-i\gamma)}{\Gamma(j+1+i\gamma)}} \cdot \sqrt{1-s^2}$$

$$[P_j^{iv}(s) \pm (-1)^j P_j^{iv}(-s)]$$

The spacelike Plancherel is messy: it seems that

$$\bar{\delta}(\hat{Q}-\hat{Q}') = \int_0^\infty dv \left(\frac{v}{\sinh v} \right) \cdot D_j^{iv}(\hat{Q}) D_j^{iv}(\hat{Q}')^* \\ + \sum_{k=1}^\infty k \frac{(j-k)!}{(j+k)!} D_j^k(\hat{Q}) D_j^k(\hat{Q}')^*$$

where $= (1-g^2) \delta(g-g') \delta(\hat{n} \cdot \hat{n}' - 1)$.

You seem to have some discrete series stuff here; obviously, what is happening is this:

$$\int_0^\infty dv \cdot \sum_{k=-\infty}^\infty = \text{full measure}$$

+ we are picking out pieces.

Discrete stuff is $\sim P_j^k(g)$.

projection: $\tilde{\phi}_{v,jn} = \int_{-1}^1 d\hat{n} \int dg (g^2-1)^{-2} \cdot P_j^{iv}(g) Y_n^j(\hat{n}) \phi(x)$

note: if $\phi(x) = \phi(x_0, |\vec{x}|)$, then $\int d\hat{n}$ forces $j=0$ and then there are no "discrete" terms.

Then: $\tilde{\phi}_{v,00} = \int dg (g^2-1)^{-2} P_0^{iv}(g) \phi(x)$.

Note: $P_0^\lambda(z) = \frac{1}{\Gamma(1-\lambda)} \left(\coth \frac{\theta}{2} \right)^\lambda$, where $z = \cosh \theta$.

If $z = g = \cosh \theta$, then: $\rightarrow$ must be $(\cosh \theta)^\lambda$
 $\uparrow$ $\theta \rightarrow 0$ to -BST.

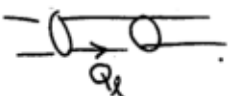
Recovery: $\bar{\phi}(x) = \int_0^\infty dv \cdot \left(\frac{v}{\sinh v} \right) \cdot P_0^{iv}(\cosh v) \cdot \tilde{\phi}_{v,00}$

OK, very rough. See AS for correct theorem.

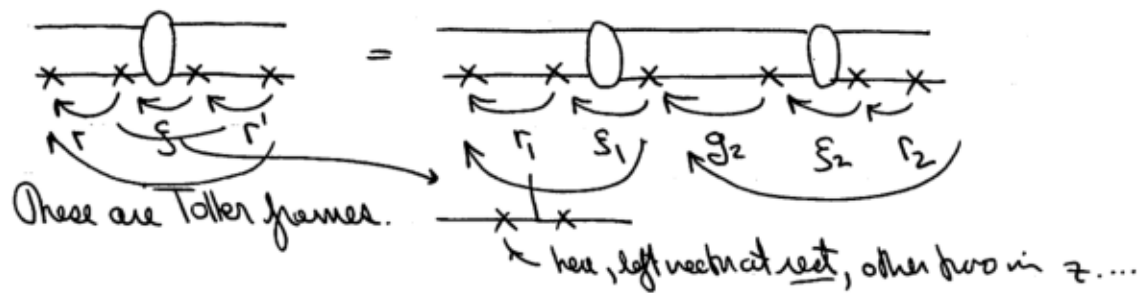
Comments on the $t=0$ diagonalization of the MPE.

1. The way I know how to understand diagonalization is via the addition theorem/invariant measure/ convolution concept. The idea is to write each amplitude as a function of its little group, and this then gives the overall convolution form. For $t < 0$ the properframes are the CDM frames. Naively you end up with a TST configuration, but I have changed that to SSS in order to be able to diagonalize via the second-kind addition theorem.
2. For the $t=0$ case, a similar result follows. The correct frames may now be taken as the lower Toller or BCP frames. Again, as shown on next page, you would like the $O(3) \cdot O(2,1)$, $O(3)$ configuration, which is analogous to the TST. As before, this will require mixed-basis $SL(2,C)$ matrix elements on one side. Perhaps the situation can be cured by going to a full $SO(2,1)$ side-basis throughout. Then the game would be to find the addition theorem for these second-kind functions.
3. The ASF equation is a very symmetric and special base, so it can be diagonalized without knowing the full theory. If you can guess the right projection, then you can prove the diagonalization, just as in the $Q_j(\chi\xi)$ case. Do to the lack of dependences, the correct projection function will be the central harmonic which I might call $a_{000}^{\text{oh}}(\xi) = e^{-\lambda\xi}/\text{sh}\xi$. This is the symmetric limit of the second-kind function; note that $d_{000}^{\text{oh}}(\xi) = \text{sh}(\lambda\xi)/\text{sh}\xi$ and thus does not allow continuation off into the right half λ -plane. This d function is given in Ruhl.
4. Where does the concept of "surface" enter? When the MPE d^4q 4-momentum is written in terms of group variables, we may visualize q^u or part of it (the versor) as moving on a surface. The points on this surface are then labelled by variables which are group variables, but not all of them, as in $d\Omega$, or $d(\lambda\nu)d\xi$, or $d\Omega d\xi$ for $SL(2,C)$. This surface may be used as a homogeneous space for group representations; then its "harmonics" will serve as basis functions, and its measure will give a scalar product.
5. Recall that you can always do your group reps on a homogeneous space, and these are always G/H in form; you do not have to involved induced representations to talk about such spaces. For general $O(n,1)$ groups, I gather it is convenient to always work on hyperboloids as the homo spaces.

Relation Between The MPE and hyperboloid.

- ① We know that d^4Q occurs in mpe.  Also, $Q_L^2 = t_L < 0$.
If overall $t=0$, how to do "frames"?

All frames should keep one $Q^\mu = \text{same}$. If $Q^\mu = (0,0,0,0)$, then any L.T. is OK.



$\left. \begin{array}{l} r_i \in SO(3) \\ g_i \in SO(2,1) \end{array} \right\}$ looks as though you will need some mixed basis
 $SL(2, \mathbb{C})$ matrix elements.

Similar to the $t < 0$ analysis.

$$a = a_1, a_2$$

$$a = r \cdot g \cdot r'$$

$$a_1 = r_1 \cdot g_1$$

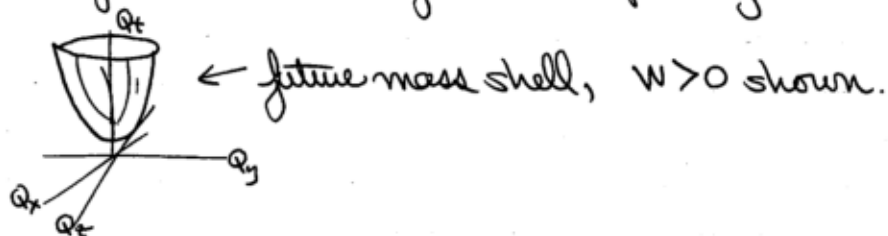
$$a_2 = g_2 \cdot g_2 \cdot r_2$$

$$Q_L^\mu(\text{left}) = L(r_1 \cdot g_1) \widehat{Q_L^\mu(0)} \quad (0,0,0, \sqrt{-t_L}).$$

$$\Rightarrow Q_L = Q_L(\underbrace{\phi_1, \theta_1, g_1}_{\uparrow \text{hyperboloid}}, t_L)$$

The mp integral equation via manifolds.

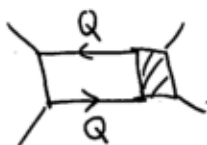
- ① Usually $g =$ parameters of a L.T. like (μ, ξ, ν) . But you can also try to use a manifold as space of motion.



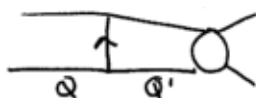
- ② You can replace a full spacetime integral by summing over manifolds:

$$\int d^4 Q = \int_{-\infty}^{\infty} dW \int d\mu_W(Q)$$

where $d\mu_W(Q) = d^4 Q \delta(Q^2 - W)$. In the application of BST, $Q^M \neq 0$



But I think $Q =$ spacelike.



- ③ Where is the "symmetry"?

$$A(Q) = \int d^4 Q' A(Q') \Delta(Q^2) \Theta(Q^0 - Q'^0) \delta[(Q' - Q)^2 - M^2]$$

We could replace $d^4 Q' = d^4 K$ where $K = Q - Q'$. Then

$$= \int d^4 K \delta(K^2 - M^2) \Delta(Q - K)^2 \Theta(K^0) A(Q - K)$$

$$= \int d\mu_K^+(K) \Delta[(Q - K)^2] A(Q - K)$$

where we now have integral over future mass shell.

- ④ But I think idea is to stick with Q' . Assume spacelike. Then

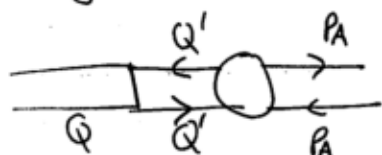
$$\int d^4 Q' = \int_{-\infty}^{\infty} du' \int d\mu(Q')$$

integral over  hyperboloid.

Now produce some variables to replace Q'^M . I.e., if

$$Q'^M = (-\sqrt{u} \sinh \sigma, -\sqrt{u} \cosh \sigma \hat{n}) = 4 \text{ variables.}$$

⑤ Why is $A(Q, P_A)$ spherically symmetric?



Is A a scalar? Yes, so $\tilde{A}(Q^2, P_A^2, P_A \cdot Q)$ ie $A''(u, s; t=0)$.

But in frame where $P_A = (m, 0, 0, 0)$, $\tilde{A}(Q) = \text{independent of } \hat{Q}$ ✓

$P_A \cdot Q = mQ^0$, so A is rotation invariant.

So is $\Delta(Q'^2)$. $\delta[Q'^2 + Q^2 - 2QQ' - M^2]$.

→ In chosen frame, integrand

$(Q' - Q)^2 = \text{Lorentz scalar, so certainly a rotational scalar.}$

→ integrand is rotational scalar

$$\int d^4 Q' \cdot f(Q^0, \vec{Q} \cdot \vec{Q})$$

⑥ Where is our group? Consider Σ measure: has rotation in there somewhere.

$$d^4 Q = dQ^0 d^3 Q = dQ^0 \cdot Q^2 dQ \cdot d\Omega \text{ in usual sense.}$$

$$\text{If } \begin{matrix} Q^0 = \int S \cdot d\sigma \\ Q = \int S \cdot \sigma \hat{r} \end{matrix} \text{ then } \equiv ds \cdot d(\sigma \hat{r}) \cdot d\Omega \text{ or something.}$$

⑦ How about $\int d^4 Q' =$

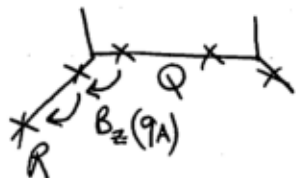
$$\int d^4 Q' f(Q^0, |\vec{Q}'|^2) = \int_0^\infty du' \int d\mu_-(Q') f(Q^0, |\vec{Q}'|^2)$$

⑧ If $Q' = \text{spacelike}$, assume $d^4 Q' = du' d\sigma d\Omega \hat{n}$. So

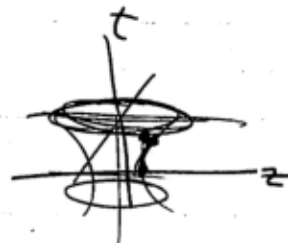
$$A(Q) = \int du' d\sigma \cdot \int d\Omega \hat{n} A(Q') \Delta(Q'^2) \delta[(Q-Q')^2 - M^2] \theta.$$

So what? Where is L.G.? Group of motions on hypersphere!

⑨ Measures + groups



$$Q^{\mu(0)} = (0, 0, 0, \sqrt{-t})$$



$$R_z(\phi) R_y(\theta) R_z(\phi') B_z(q) Q^{\mu(0)} =$$

$$R_z(\phi) R_y(\theta) R_z(\phi') (\text{sh } \sqrt{-t}, 0, 0, \text{ch } \sqrt{-t}).$$



$$(\text{sh } \sqrt{-t}, \text{ch } \sqrt{-t} \hat{n}). \quad \text{Looks like (A.1).}$$

$$\Rightarrow Q^\mu = (\sqrt{-t}) (\text{sh } q, \text{ch } q \hat{n}).$$

$$\hat{n} = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta).$$

You could then deduce $d^4 Q$:

$$Q^2 = -t$$

$$\begin{aligned} Q^0 &= \sqrt{-t} \text{sh } q \\ Q^1 &= \sqrt{-t} \text{ch } q \text{ So } \phi \\ Q^2 &= \sqrt{-t} \text{ch } q \text{ So } \phi \text{ So } \phi \\ Q^3 &= \sqrt{-t} \text{ch } q \text{ Co } \end{aligned} \rightarrow \begin{vmatrix} \text{sh } q & \text{ch } q & 0 & 0 \\ \text{ch } q \text{ So } \phi & \text{sh } q \text{ So } \phi & \text{ch } q \text{ Co } \phi & -\text{ch } q \text{ So } \phi \\ \text{ch } q \text{ So } \phi & \text{sh } q \text{ So } \phi & \text{ch } q \text{ Co } \phi & \text{ch } q \text{ So } \phi \\ \text{ch } q \text{ Co } & \text{sh } q \text{ Co } & -\text{ch } q \text{ So } & 0 \end{vmatrix} \left(-\frac{1}{2\sqrt{-t}} \right)^3$$

$$\frac{d}{dt} \sqrt{-t} = \frac{1}{2} (-t)^{-\frac{1}{2}} = -\frac{1}{2\sqrt{-t}}$$

$$= -\frac{1}{2} \cdot (-t) (\text{ch } q)^2 \text{ So } \begin{vmatrix} \text{sh } q & \text{ch } q & 0 & 0 \\ \text{ch } q \text{ So } \phi & \text{sh } q \text{ So } \phi & \text{Co } \phi & -\text{So } \phi \\ \text{ch } q \text{ So } \phi & \text{sh } q \text{ So } \phi & \text{Co } \phi & \text{Co } \phi \\ \text{ch } q \text{ Co } & \text{sh } q \text{ Co } & -\text{So } & 0 \end{vmatrix}$$

$$- \text{Co } \phi \begin{vmatrix} \text{sh } q & \text{ch } q & 0 \\ \text{ch } q \text{ So } \phi & \text{sh } q \text{ So } \phi & \text{Co } \phi \end{vmatrix} \text{ etc etc.}$$

Get something like: $d^4 Q = J \cdot dt \, dq \, d\theta \, d\phi.$

⑩ Full L.G. measure is

$$a = u_1 a_2(\xi) u_2 \rightarrow \underbrace{du_1 d(\cos \theta_1) dv_1 (\text{sh } \xi)^2 d\xi}_{\text{This piece.}} d(\cos \theta_2) dv_2$$

$$da =$$

So I bet that $d^4 Q = J \cdot dt \cdot da$

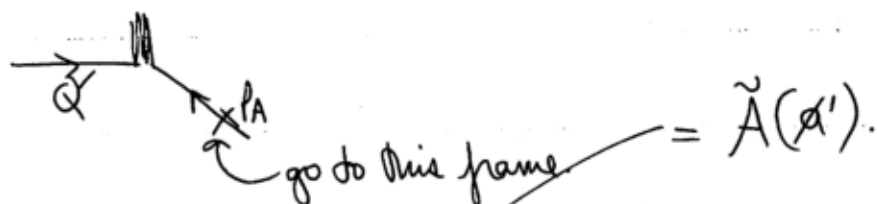
$$\text{Then: } Q' = L(a) Q^0.$$

~~A(Q) =~~

$$A(Q, P_A) = \int d^4 Q' \delta[(Q-Q')^2 - M^2] \Theta(Q-Q') \Delta[Q'] A(Q', P_A).$$

But write $d^4 Q' = dt' d\phi'$.

Also $A(Q', P_A) = A(L(a)Q', L(a)P_A) \Rightarrow$ Lorentz Invariance.



$$A = A(L(a)Q', P_A) \text{ where } \phi' = R(\theta, \phi) B_2(q).$$

Also, $\Delta(Q'^2) = \Delta(Q'^2)$ since scalar. $= \Delta(t')$.

$$(Q' - Q)^2 =$$

$$A(Q, P_A) = \int dt' \int d\phi' \tilde{A}(\phi', t') \Delta(t') \times \Theta(Q' - Q) \delta[(Q - Q')^2 - M^2].$$

What is Q ?

$$Q = L(a)Q^0.$$

$$\text{So } \tilde{A}(a, t) = \int dt' \int d\phi' \tilde{A}(\phi', t') \Delta(t')$$

$$\begin{aligned} (Q - Q')^2 &= (L(a)Q^{(0)} - L(a')Q'^{(0)})^2 \\ &= [L(a')^{-1}L(a)Q^{(0)} - Q'^{(0)}]^2 \\ &= f(a'^{-1}a). \end{aligned}$$

$$\text{Finally, } (Q^0 - Q') = [L(a')Q^{(0)0} - L(a)Q^{(0)0}].$$

$$\text{So } \tilde{A}(a, t) = \int dt' \Delta(t') \int d\phi' \tilde{A}(\phi', t') \cdot f(a'^{-1}a).$$

⑪ Now, apply the $SL(2, \mathbb{C})$ addition theorem:

← but this is all first-kind!

$$D_{jm, j'm'}^{M\lambda}(a) = \sum_{j''=|M|}^{\infty} \sum_{m''=-j''}^{+j''} D_{jm, j''m''}^{M\lambda}(a') D_{j''m'', j'm'}^{M\lambda}(a' a)$$

and you should get

$$\tilde{A}_{jm, j'm'}^{M\lambda}(t) = \int dt' \Delta(t') \sum_{j''m''} \tilde{A}_{jm, j''m''}^{M\lambda}(t') \cdot \tilde{f}_{j''m'', j'm'}^{M\lambda}$$

⑫ But, look at these projections. Probably many things are forced to zero.

first: $\tilde{A}_{jm, j'm'}^{M\lambda}(t) \equiv \int da D_{jm, j'm'}^{M\lambda}(a) \tilde{A}(a, t)$

But $a = (\mu_1, \theta_1, \xi)$ only so $(\mu_2, \theta_2, \xi, \nu_2)$ not present.

Then write out as

$$\int d\mu_1 d(\cos \theta_1) (sh\xi)^2 d\xi \cdot \int d\mu_2 dz_2 dv_2 \cdot \sum_{n=-j}^j D_{m,n}^j(\mu_1, \theta_1, 0) d_{j'j''}^{M\lambda} D_{n,m'}^{j'}(\mu_2, \theta_2, \nu_2) \times \tilde{A}(\mu_1, \theta_1, \xi, -, -, -)$$

So $\int dg D_{nm'}^{j'}(g) = \delta_{j',0} \delta_{n,0} \delta_{m',0} \cdot c.$

$\Rightarrow \tilde{A}_{jm, j'm'}^{M\lambda}(t) = \delta_{j',0} \delta_{m',0} \int d\mu dz (sh\xi)^2 d\xi D_{m,0}^j(\mu, \theta, 0) d_{j0}^{M\lambda}(\xi) \tilde{A}(\mu, \theta, \xi).$

So $\tilde{A}_{jm, 00}^{M\lambda} = \int d\mu d(\cos \theta) (sh\xi)^2 d\xi Y_m^j(\theta, \mu) d_{j00}^{M\lambda}(\xi)$

So this is the only projection which can appear on LHS.

⑬ If $\tilde{A}(\mu, \theta, \xi) = \tilde{A}(-, -, \xi)$, then $\int d\mu dz Y_m^j(\theta, \mu) = \delta_{j,0} \delta_{m,0}$

But, since $j, j' \geq |M|$, we must have $M=0$.

$$\text{Then} \\ \text{vol} \\ A_{00,00}(t) = \int_0^{\infty} (\sin \xi)^2 d\xi \cdot d_{000}(\xi)$$

What is the connection between cosets, covariant fctns, and spherical harmonics?

① Note that $Y_m^l(\theta, \phi) = D_{m0}^l(\phi, \theta, -)$. As a fctn of $g = (\phi, \theta, \phi')$

$$f(u) = D_{m0}^l(\phi, \theta, \phi') = f(u \cdot R_z(\phi'))$$

On $f(u) = f(u \cdot \phi)$. The spherical harmonics are "right covariant" wrt R_z .

② Cosets? Let $G = SU_2$ and $H = SO(2)$; consider $S = G/H = SO(3)/SO(2)$. How do you "realize" this concept? Elements of S are "cosets", so

$$g_1 H = S_1 = \text{a coset labelled by } g_1$$

Suppose $f(g) = f(gh) \forall h \in H$, any g . f is constant on the cosets, defines a function on S . Also, this is the covariance condition. You can induce a rep of $SO(3)$ via $U(g_1) f(g) = f(g g_1)$ etc. But recall the duality idea.

③ Two ways to do induction: forget the bundle. Does one method tell you the $SO(3)$ matrix elements? Should we replace the covariance condition with

$$L^n(h) f(g) = f(gh) = e^{-inh} f(g)$$

Then $L^n(h) = e^{-inh}$ is the inducing representation. So what is the induced rep?

$$U(g_1) f(g) = f(\cancel{g g_1}) f(g_1^{-1} g)$$

But any g may be written $g = xh$ where $x = e^{-i\phi\sigma_3} e^{-i\theta\sigma_2}$

So:

$$U(g_1) f(xh) = f(\cancel{x h g_1}) = f(h^{-1} x^{-1} x h)$$

$$\text{or } U(g_1) f(x) = f(h^{-1} x^{-1} x)$$

Note: $x = (\phi, \theta)$ can serve as coset labels.

$$gH \xrightarrow{\sim} xH$$

Now: $f(h^{-1} x^{-1} x) = f(R_z(-h_1) R_y(-\theta) R_z(-\phi) R_z(\phi) R_y(\theta))$

$$\text{On: } g_1^{-1} g = g_2 = x_2 h_2.$$

What am I trying to do here? Want to see what rep of $O(3)$ is induced:

$$U(g_1) f(g) = f(g_1^{-1} g)$$

where $L(h)f(g) = f(gh) = e^{-i\mathbf{h}\cdot\mathbf{L}} f(g) = \text{inducing rep.}$

Norm? $\|f\|_1^2 = \langle f(g), f(g) \rangle = \text{constant on cosets.}$

① We need a measure-on-coset-space. Let's mimic Schwarz/Toller.

$k \in K = H = \text{inducing subgroup.}$ $\begin{pmatrix} e^{-i\phi} & 0 \\ 0 & e^{i\phi} \end{pmatrix}$

$a = kz \rightarrow u = hx$ $x = (\theta, \phi)$
 $h = \phi$

$u' \dots$

② How about $\|f\|_1 = \int d\mu(x) \langle f(x), f(x) \rangle$

x labels cosets. $g = xh$, so

$$= \int d\mu(x) \langle f(x), f(x) \rangle$$

$$= \int d\Omega \overline{f(x)} f(x).$$

now identify sphere with coset space!

We may work with functions on the surface!

③ In ST, coset space is labelled by z or u . Here, it is labelled by $x = (\theta, \phi)$.

$$g_1^{-1} g = g_2 = x_2 h_2$$

$$\Rightarrow U(g_1) f(x) = f(x_2) e^{-i\mathbf{h}_2 \cdot \mathbf{L}}$$

where x, x_2 lie on sphere. $x_2 h_2 = h_1^{-1} x_1^{-1} x h_1^{-1}$

$$x_2 = h_1^{-1} x_1^{-1} x h_1^{-1}$$

From this rep you could compute J^2 and J_3 as operators. Then could get $Y_{lm}(x)$ as basis fctns. Then:

$$\langle \psi_m^l, U(g, \cdot) \psi_{m'}^l \rangle = \int d\Omega \cdot \overline{\psi_m^l}(\Omega) \dots$$

yields usual rotation matrices.

- ⑦ Conclusion: The surface [sphere, $\mathbb{O}P$] has points which serve to label the cosets of $SO(3)$ wrt $SO(2)$. I.e., $S = G/H$ and $s_i = H(\theta, \phi) = Hx$. Here, s_i is the coset labelled by (θ, ϕ) .

The measure on the surface $d\Omega$ is identified with the coset space measure, + serves to define the norm, or scalar product, of a rep. of G induced from H .

Thus, one may regard the sphere as the homo. space which carries the $f(x)$. Similarly, ~~the~~ a future mass shell's points may be identified with $a \in L.G.$

The "harmonics" of the sphere serve as basis vectors

Eigenfunction Expansions Associated with the Second-Order Invariant Operator on Hyperboloids and Cones. III

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(Received 14 July 1966)

The eigenfunction expansions associated with the second-order invariant operator on hyperboloids and cones are derived. The global unitary irreducible representations of the $SO_0(p, q)$ groups related to hyperboloids and cones are obtained. The decomposition of the quasi-regular representations into the irreducible ones is given and the connection with the Mautner theorem and nuclear spectral theory is discussed.

1. INTRODUCTION

IN our two previous works^{1,2} the most degenerate irreducible infinitesimal representations of an arbitrary noncompact rotation group $SO_0(p, q)$ have been derived.^{3,4} These representations have been related to the homogeneous spaces $SO_0(p, q)/SO_0(p-1, q)$, $SO_0(p, q)/SO_0(p, q-1)$, and $SO_0(p, q)/T^{p+q-2}$ [S] $SO(p-1, q-1)$ which can be represented by the hyperboloids H_p^q , H_q^p and by the cone C_p^q , respectively. These homogeneous spaces are of rank one under the action of the group. The infinitesimal representations have been constructed by means of the sets of harmonic functions associated with the second-order invariant operators related to the above-mentioned manifolds.

The completeness of these sets of harmonic functions has been proved in the present work by using the classical Titchmarsh⁵-Kodaira⁶ eigenfunction

expansion theory associated with an ordinary second-order differential equation.

In Sec. 2 the review of the main results in the form convenient for applications is given. The other sections are devoted to the proofs. Thus Sec. 3 contains the proof of the essential self-adjointness of the Laplace-Beltrami operator on the linear manifold $\mathcal{D}(X)$, which was introduced earlier.² The proof of the completeness of the harmonic functions which were constructed in our previous papers^{1,2} is given in Sec. 4. In Sec. 5 we prove the unitarity and irreducibility of the representations of the group $SO_0(p, q)$ and consider the decomposition of the quasi-regular representations into irreducible ones. In Sec. 6 we show an interesting connection of our approach to the eigenfunction expansion and the general theory of the eigenfunction expansion developed by Gel'fand and Kostiuhenko,⁷ Maurin,⁸ and Gårding.⁹ Appendix I contains some auxiliary computations. In Appendix II we review the main results on the representations of the compact group $SO(p)$.

2. REVIEW OF MOST DEGENERATE REPRESENTATIONS OF $SO_0(p, q)$ GROUPS

We have considered in our previous papers^{1,2} three homogeneous spaces X of rank one under the action of the noncompact rotation group $SO_0(p, q)$ [see (2.2)¹ and (4.1)²]. They can be represented by the hyperboloids H_p^q and H_q^p and the cone C_p^q :

$$(x^1)^2 + \dots + (x^p)^2 - (x^{p+1})^2 - \dots - (x^{p+q})^2 = \begin{cases} 1 & \text{for the hyperboloid } H_p^q, \quad p \geq q, \\ 0 & \text{for the cone } C_p^q, \\ -1 & \text{for the hyperboloid } H_q^p, \quad p \geq q. \end{cases} \quad (2.1)$$

⁷ I. M. Gel'fand and A. G. Kostiuhenko, Dokl. Akad. Nauk SSSR 103, 349 (1955).

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⁹ L. Gårding, Seminar on Applied Mathematics, Boulder, Colorado (1957), pp. 1-30.

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¹ R. Rączka, N. Limić, and J. Niederle, J. Math. Phys. 7, 1861 (1966).

² N. Limić, J. Niederle, and R. Rączka, J. Math. Phys. 7, 2026 (1966).

³ The representations of the compact $SO(p)$ group related to the sphere and the Lorentz-type group $SO_0(p, 1)$ related to the real Lobachevski space (H_p^1 in our notation) are given in the book by N. J. Vilenkin: "Special Functions and Theory of Group Representation," "Nauka," Moscow (1965) (in Russian).

⁴ For example, the representations of some lower-dimensional $SO(p, q)$ groups are given in: J. A. C. Alcaraz and P. L. Ferreira, J. Math. Phys. 6, 578 (1965); V. Bargmann, Ann. Math. 48, 568 (1947); C. G. Bollini and J. J. Giambiagi, "Unified Representations of the Rotation and Lorentz (2 + 1) Groups," preprint, University of Buenos Aires (1965); J. Dixmier, Bull. Soc. Math. France 89, 9 (1961); A. Z. Dolginov, Soviet Phys.-JETP 3, 589 (1956); A. Z. Dolginov and J. N. Toptygin, *ibid.* 10, 1022 (1960); A. Z. Dolginov and A. N. Moskalev, *ibid.* 10, 1202 (1960); J. B. Ehrman, Proc. Cambridge Phil. Soc. 53, 290 (1956); N. T. Evans, J. Math. Phys. 8, 170 (1967); A. Kihlberg, Arkiv Fysik 27, 373 (1964), 28, 121 (1964), and 30, 121 (1965); A. Kihlberg and S. Ström, Arkiv Fysik 31, 491 (1966); A. Kihlberg, *ibid.* 30, 121 (1965); L. H. Thomas, Ann. Math. 42, 113 (1941).

⁵ E. C. Titchmarsh, *Eigenfunction Expansions* (Clarendon Press, Oxford, England, 1962), Part I.

⁶ K. Kodaira, Am. J. Math. 71, 921 (1949).

Comments on the LNR papers

- ① They represent the rep. space of $SO(p, q)$ as a hyperboloid which provides coordinates etc to work with. The hyp. is embedded in a nice E^n space whose metric you know. Now, "on" this hyperboloid H_q^p there exists a certain Casimir-like diff op called the "Laplace-Beltrami" oper. It is:

$$\Delta(H_q^p) = \frac{1}{(\cosh \theta)^{p-1} (\sinh \theta)^{q-1}} \frac{\partial}{\partial \theta} \left\{ (\cosh \theta)^{p-1} (\sinh \theta)^{q-1} \frac{\partial}{\partial \theta} \right\} + \frac{\Delta(S^{p-1})}{\cosh^2 \theta} - \frac{\Delta(S^{q-1})}{\sinh^2 \theta} \quad \theta \in [0, \infty).$$

where $\Delta(S^{r-1})$ = the LPO of $SO(r)$. These are given as (A3)(A4) of previous paper p 1874. Eg, $r=1$

$$\Delta(S^1) (\sim SO(2)) = (\cos \theta) \frac{\partial^2}{\partial \phi^2} + (\sin \theta) \frac{\partial^2}{\partial \theta^2}$$

No. $\Delta(S^1) = \frac{\partial^2}{\partial \phi^2}$ period, of $SO(2)$.

$$\Delta(S^2) = \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left[\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \right] + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \quad \text{of } SO(3).$$

ie, if you take as your homogeneous space for $SO(3)$ a sphere S^2 , [two dim surface], then if you realize your Lie algebra as diff ops on this space, then $\Delta(S^2)$ is the Casimir of the sphere.

So, to do $SO(p, q)$ he does roughly $\equiv SO(p) \cdot \Theta \cdot SO(q)$

For example:

$$SO(3, 2) \sim O(3) \cdot \Theta \cdot O(2).$$

- ② For $SO(p, 1)$ groups, there seems to be a special case. Here

$$SO(p, 1) \sim \Theta \cdot SO(p) \quad \text{so} \quad SO(2, 1) \sim \Theta \cdot SO(2)$$

The solution of the LPO gives sort of "spherical harmonics" on your surface.

$$\left(\alpha - \bigvee \right) \text{ etc.}$$

$$\uparrow \quad \uparrow \\ (\theta, \phi)$$

These harmonics then become basis functions for reps.

So consider again $SO(2,1) \sim O \cdot SO(2) = (\theta, \phi)$.
 Your reps are within a special class in O so as to
 be V.I.R's. Then only ϕ is left.

$$|l, m\rangle \rightarrow e^{im\phi}$$

$$\langle m | O | l, m \rangle = \int d\phi e^{im\phi} O e^{-im\phi}$$

Or, you could say = $\int_{\text{hyperbola}} d\Omega Y_l^m(\theta, \phi) Y_l^m(\theta, \phi)$

③ He is using $SO(n)$ as group which labels basis states, so "compact basis".

④ By the way: how do you do $SO(n)$? Refs given.

a) gives the LPO as function of two angles $\theta, \phi \quad \forall n$.

b) Casimir is: $\lambda = -l(l+p-2)$ for $SO(p)$.

The DE for $SO(p)$ eigenfunctions is in well, it always involves l as well as l of the next lower group.

$$\lambda_r = -l_r(l_r + p - 2) \quad r = \pm p \quad \text{or } r = \pm p - \frac{1}{2}$$

So: $SO(2)$: ~~$\lambda_1 = -l_1^2$~~ $\lambda_1 = -l_1^2$ usually called m^2 !

$$SO(3): \lambda_1 = -l(l+1)$$

$$SO(4): \lambda_2 = -l_2(l_2+2) \dots$$

So, the generalized "spherical harmonic" is:

$$\psi_{m_r; l_r}^{l_r}(\theta)$$

I assume goes $-l_r$ to l_r .

Example: $O(4)$ has its own $l=1$, plus the l of $O(3)$. Turns out that $l_3 \leq l_4$. So

$$\psi_{M; l}^{\lambda}(\theta). \quad [\text{Gegenbauer?}]$$

⑤ OK, so the general idea is to "build up" from lower groups

$$SO(p+1) \sim \Theta \cdot SO(p)$$

$$SO(4) \sim \Theta \cdot SO(3)$$

$$SO(3) \sim \Theta \cdot SO(2)$$

$\uparrow$ not real l except m
 l_3, m_3 happens that $m_3 = m$.

$$\begin{array}{c} \uparrow l_3 \\ l_4, m_4 \end{array}$$

$$SO(5) \sim \Theta \cdot SO(4) \Rightarrow \psi_{m_5, l_4}^{l_5}(\Theta) \dots \text{etc.}$$

$\underbrace{\quad}_{l_4, m_5}$

Then your reps will be:

$$\int \psi_{m, l}^{l_r}(\Theta) \psi_{m', l'}^{l_r}(\Theta) d\mu(\Theta)$$

$$= \delta_{m, l; m', l'}(\Theta)$$

So as for matrix els; $\mathbb{D}^{p+1} \sim SO(p) \cdot \Theta \cdot (SO(p)) \checkmark$

⑥ For $SO(p, q)$ $bf \sim \Theta \cdot SO(p) \cdot SO(q)$

Thus $SO(2, 1) \sim \Theta \cdot SO(2)$

$$\mathbb{D}^{SO(2,1)} \sim SO(2) \cdot \Theta \cdot SO(2) \checkmark$$

$$\mathbb{D}^{SO(3,1)} \sim SO(3) \cdot \Theta \cdot SO(3)$$

OK, they are always in the compact subgroups mode.
 For $SO(2, 1)$ only discrete basis is discussed, ergo.

⑦ To summarize, LNR have found all continuous + discrete IR's for $SO(p, q)$ in compact bases. For their basis functions are realized on hyperboloids as "SH's" and from these you can get mc 's.

example: $SO(3)$. $\psi_l^m(\theta, \phi) \sim P_l^m(\theta)$

$$\Delta \psi_l^m(\theta, \phi) = l(l+1) \psi_l^m$$

$\leftarrow$ Laplace-Beltrami invariant operator on sphere.

A.S. (72)

line going through the point $z = 0$ at the angle $2\pi/3$ in the direction of the greatest increase ($\arg z = 2\pi/3$) and decrease ($\arg z = -\pi/3$) of these functions. The property (A.9) for $f_1(z)$ presupposes that the cut is made along the $\arg z = 2\pi/3$ line.

ACKNOWLEDGMENTS

It is a pleasure to thank Dr. D. A. Kirzhnits and Dr. A. I. Nikishov for the discussion of this paper.

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Laplace Transform Diagonalization of Forward Scattering Absorptive Part Equations*

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Received May 10, 1971

We present a Laplace transform procedure for the partial diagonalization of Bethe-Salpeter-like absorptive part equations for the forward scattering of spinning particles. By using Laplace transforms the analysis allows directly for power growth in energy of the absorptive parts of the amplitude without any need for the analytic continuations required in a Fourier analysis. The integral equation with an arbitrary kernel is explicitly diagonalized, resulting in a reduction from a set of four-dimensional equations to coupled one-dimensional equations. As a tool for locating the proper definitions of partial wave amplitudes and for deriving a crucial addition theorem, the necessary representation theory of the little group for forward scattering, $SO(1, 3)$, is described.

1. INTRODUCTION

In a series of delightful papers published some time ago, Amati, Bertocchi, Fubini, Stanghellini, and Tonin [1] wrote down and extracted much of the physical consequence for multiperipheral integral equations. Their methods for attacking the equations were, however, restricted and somewhat beclouded by the approximations to phase space employed. Subsequent workers [2] have made the physics more transparent and much easier to come by through an exact analysis of the kinematical aspects of the equations. In particular, the role played by the little group of the momentum transfer has been clarified and the precise treatment of phase space utilizing this group theoretical wisdom has been formulated.

One real use of the group theoretical methods is to take a four-dimensional integral equation and perform a crossed-channel partial wave analysis [3] to reduce it to a two-dimensional equation for nonforward processes and to a one-dimensional equation for forward scattering. Defining the partial wave amplitudes, as is usually done, by projection onto the unitary irreducible representations of the little group,

* Research supported by the U. S. Atomic Energy Commission under Contract AT(30-1)-4159 and by the Air Force Office of Scientific Research under Contract AF 49(638)-1545.

† Alfred P. Sloan Foundation Research Fellow.

read on 2/2/77

AS '72 paper.

1. Introduction. Same overall astructure as their last paper. Usual remarks about the NSI problem and how they resolve it; spacelike momenta; spinors to come.

2. Prepare Integral Equ. The group variables are set up. Special $L(h)$ diagonal spin wavefunctions are selected. Matrix elements are defined. The double-propagator is defined and expanded onto the basis wavefunctions. The Clebsh's are handled. When the dust clears, we have equation (590.5) where the central variable dependence has been isolated, etc, etc. I have made detailed notes on most of this separately.

3. Laplace Transform. Projections onto the e functions are defined, and the integral equation is diagonalized using the second-kind addition theorem in terms of these e -functions, the result is given in (592.5). Section ends with comment about the quantum number $M = j_0$.

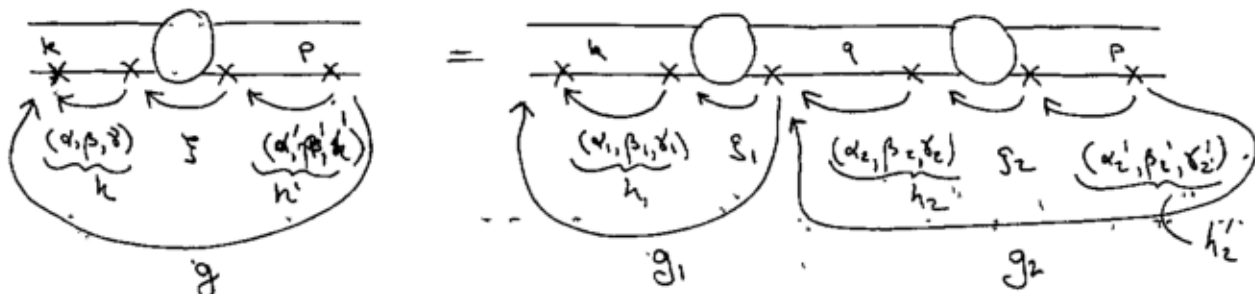
4. Discussion. Mention two possible applications: deep inelastic and regge inclusive stuff.

AppA: spinore wavefunctions in strange basis

AppB: $O(3,1)$ group theory (see notes).

Setup of integral equation.

1. I will just plug along and try to do this myself to see what happens.



This is my choice for convolution kinematics. (α, β, γ) is really (μ, ξ, η) of older.

2. The particles each carry a spin function. These wavefunctions are chosen to diagonalize J_3 and also to transform according to $O(2,1)$ finite dimensional representations. Thus we get (588.5) of AS.

Note the analogy to the simpler case. There, the basis spin functions were chosen to transform simply with the side group which was simply K_X . Here, the side group is a full $O(2,1)$. I wonder if they might consider using the continuous basis param for this side group... for my problem it might be better,

$$U^{(l,m)}(p) = U^{(s)}(p) = \text{single particle spin}$$

$$U^{(s)}(p) = \text{product of two such spin wavefunctions.}$$

$$A_{\alpha\beta}^{(S,S_p)}(k,p) = U^{(S)}(k) A(k,p) U^{(S_p)}(p)$$

matrix element matrix in 2x2 space.

3. As usual, it will be important to know how everything transforms under LT.

$$U_{\alpha}^{(l,m)}(kp) = \sum_{m'} D_{m'm}^{(l)}(h) U_{\alpha}^{(l,m')}(p)$$

$$h \in O(2,1)$$

spin index α unaffected.

Shorthand: double to U and S .

$$[U^{(S)}(kp) = \sum_{S'} D^{(S,S')}(h) U^{(S')}(p)]$$

meaning

$$U_{\alpha}^{(l,m)}(kp) U_{\beta\alpha'}^{(\tilde{l},\tilde{m})}(kp) = \sum_{m,\tilde{m}} D_{m'm}^{(l)}(h) D_{\tilde{m}\tilde{m}'}^{(\tilde{l})}(h) U_{\alpha}(p) U_{\alpha'}(p)$$

whereas not generally $U^{(S)}(gp) = L(g)U(p) \neq U(p)$

$$\begin{cases} U^{(s)}(gp) = L(g) U^{(s)}(p) \\ U^{(s)}(hp) = L(h) U^{(s)}(p) = \sum_{s'} D^{ss'}(h) U^{(s')}(p) \end{cases}$$

The point to be made here is that for general g , $L(g)$ is a matrix in spinor index space; but for special case of $h = O(2,1)$, $L(h)$ is diagonal in this space although it shuffled the spin label, ie, nondiagonal in spin J_3 label space m .

4. At any rate, we now know how everything transforms, and we have used these transformation properties to isolated the central variable dependence of the amplitudes which appear in the equation.

of course to really make this work, we have to be able to expand the double propagator in a consistent way. This means:

$$\begin{aligned} P(g) &= L(g) P(e) L(g^{-1}) \\ P(g) &= \sum_{s_1} C_{s_1} U^{(s_1)}(p) \bar{U}^{(s_1)}(p) \end{aligned} \quad \leftarrow \text{dot } g_1, e$$

single propagator

$$P(p)_{\alpha\beta} = \sum_{l,m} U_{\alpha}^{(l,m)}(p) \bar{U}^{(l,m)}(p) C_{lm}.$$

Do not confuse this with the sum in a completeness relation. Ie, in simpler case sum was sum on λ but was not the usual completeness imaginary integral. Here this is definitely a finite sum. In fact, for spinor or vector, this double sum is still only a sum over 4 components. Its just a fancy way to rearrange things. Actual components are shown in appendix A.

5. Integral equation now looks like this:

$$\begin{aligned} A^{(s,s_1)}(e,g) &= \sum_{s_2} \int d\tau \cdot C_{s_2} \int dg_1 A^{(s,s_2)}(e,g_1) \cdot \left[\bar{U}^{(s_1)}(g_1) I^{(s_1,s_2)}(g_1,g_1,g_2) U^{(s_1)}(g_1) \right] \\ I &= \bar{U}^{(s_1)}(g_1) L(g_1) I^{(s_1,s_2)}(e,g_2) L(g_1^{-1}) U^{(s_1)}(g_1) \\ &= \bar{U}^{(s_1)}(e) I^{(s_1,s_2)}(e,g_2) \bar{U}(e). \end{aligned}$$

← letter from $P(g_1)$

So:

$$A^{(s,s_1)}(e,g) = \sum_{s_2} \int d\tau \cdot C_{s_2} \int dg_1 A^{(s,s_2)}(e,g_1) \cdot I^{(s_1,s_2)}(e,g_2)$$

Now:

$$A^{(S_k, S_p)}(e, g) = U^{(S_k)}(e) A(e, g) U^{(S_p)}(g)$$

$$\Rightarrow A^{(S_k, S_p)}(e, hgh') = U^{(S_k)}(e) \underbrace{A(e, hgh')} U^{(S_p)}(hgh')$$

But

$$A(g^{(s)}, g^{(s)}) = L(g) A(k^{(s)}, p^{(s)}) L(g')$$

↑ operates in spin space.

$$\boxed{A(gg_1, gg_2) = L(g) A(g_1, g_2) L(g')}$$

Also

$$h p^{(s)} = 0 p^{(s)} \text{ because } p^{(s)} = (0, 0, 0, \sqrt{-t}).$$

So:

$$A^{(S_k, S_p)}(e, hgh') = U^{(S_k)}(e) \left[L(h) \underbrace{A(e, gh')} L(h') \right] U^{(S_p)}(hgh') \\ = A(e, gh') = A(e, g).$$

$$\text{Now } U^{(S_p)}(g) = L(g) U^{(S_p)}(e), \text{ so}$$

$$A^{(S_k, S_p)}(e, g=hgh') = U^{(S_k)}(e) L(h) A(e, g) L(h') U^{(S_p)}(e)$$

$$= \sum_{S_k' S_p'} D^{(S_k', S_k)}(h) \underbrace{D^{(S_p', S_p)}(h')}_{\text{not in spin space!!}}$$

$$= \sum_{S_k' S_p'} U^{(S_k)}(e) D^{(S_k', S_k)}(h)$$

$$= \sum_{S_k' S_p'} D^{(S_k', S_k)}(h)^\dagger D^{(S_p', S_p)}(h') \cdot \underbrace{[U^{(S_k)}(e) A(e, g) L(h') U^{(S_p)}(e)]}_{A(g) \text{ isolated.}} \\ = A^{(S_k, S_p)}(e, g)$$

This is of course exactly the same thing we got before. Everything falls into its proper place. If you want to expose all the D-functions, you could do so to get:

$$A^{(s_k, s_p)}(e, g = hgh') = \sum_{s_k' s_p'} D^{s_k' s_k}(h) D^{s_p s_p'}(h') a^{(s_k', s_p')}(g)$$

Then:

$$\sum_{s_k' s_p'} D^{s_k' s_k}(h) D^{s_p s_p'}(h') a^{(s_k, s_p)}(g)$$

$$= \sum_{s_q} \int dT \cdot \bar{C}_q \cdot \int dg_1 \cdot \sum_{s_k' s_q''} D^{s_k' s_k}(h_1) D^{s_q'' s_q}(h_1') a^{(s_k, s_q'')}(g_1) \\ \times \sum_{s_q' s_p'} D^{s_q' s_q}(h_2) D^{s_p s_p'}(h_2') a^{(s_q', s_p')}(g_2)$$

But, recall that $h_1' = e$, so

$$= \sum_{s_q s_q'} \int dT \cdot \bar{C}_q \cdot \int dg_1 \cdot \sum_{s_k' s_p'} D^{s_k' s_k}(h_1) D^{s_q' s_q}(h_2) D^{s_p s_p'}(h_2') \\ \times a^{(s_k, s_q)}(g_1) a^{(s_q', s_p')}(g_2)$$

6. I think it might be clearer to use the form of the integral equation given in 5 above and come up with some super addition theorem for $O(3,1)$. But the catch is that this addition theorem has to be in the continuous basis somehow. This is done in the appendix.

Basically, this is (590.5). Recall that they always set $h = h' = e$ on LHS. Also, Goldstein's shuffled, not very significant here.

Addition theorem in $O(3,1) > O(3,1) > O(2)$.

Appendix B

also should show (n, j_0) .

① Basics

$$|l, m, a\rangle \quad a = \pm$$

$$J_3 |l, m, a\rangle = m |l, m, a\rangle$$

$$J^2 |l, m, a\rangle = l(l+1) |l, m, a\rangle \quad \text{I presume.}$$

$$J^2 = J_3^2 + K_1^2 + K_2^2$$

= one of two Casimirs

$$L(h) |l, m, a\rangle = \sum_{m'=-\infty}^{\infty} D_{mm'}^{(l)}(h) |l, m', a\rangle \quad h \in SU(1,1)$$

So these basis vectors transform according to reps of $SU(1,1)$. Note that $SU(1,1)$ does not mix a index.

② Isotable completeness: $dn(l)$

$$1 = \sum_a \sum_{m=-\infty}^{\infty} \int dl(l+1) f(l) |l, m, a\rangle \langle l, m, a|$$

$$\langle l', m', a' | l, m, a \rangle = \delta(l-l') \delta_{m,m'} \delta_{a,a'}$$

③ Full matrix elements:

$$D_{l'm'a', l'm,a}^{(n,j_0)}(\xi) = \langle l'm'a' | B_Z(\xi) | l'm,a \rangle = \sum_{m,m'} \delta_{m,m'} d_{l'm'l'm'}^{(n,j_0)}(\xi)$$

Comment: note that J^2 defined above is the Casimir of $O(2,1)$, but is not one of the Casimirs of $O(3,1)$ because those Casimirs are $J \cdot K$ and $J^2 - K^2$. In particular, J^2 does not commute with K_3 . That is why l and l' are not set ~~matrix~~ equal in the above matrix element.

On the other hand, K_3 does commute with J_3 , so we get $m = m'$. So:

In terms of the bivalued indices a and a' , there are four such matrix elements, but as usual matrix is triangular for positive boost, and AS just look at the -- element. They identify:

$$\begin{array}{ccccccc} \overset{(n,j_0)}{d_{l'm'l'm'}}^{(n,j_0)}(\xi) & = & K_{--}(l'm;l'm;j_0 n; B_Z(\xi)) & \doteq & C_{l'm'l'm'}^{j_0 n}(\xi) & \doteq & a_{m'l'm'}^{j_0, n+1}(\xi) \\ \uparrow & & \uparrow & & \uparrow & & \uparrow \\ me. & & AS & & AS & & Toller \end{array}$$

So here we see that result I have always suspected, despite Ruhl's comment: the Toller second-kind function is actually the matrix element of the central boost of $O(3,1)$ when you work in a basis with $O(2,1)$ diagonalized. Thus:

$$g = hgh'$$

$$D_{l'm'a', l'm'a'}^{(j_0, n)}(g) = ?$$

$$D_{l'm'a', l'm'a'}^{(j_0, n)}(g_1 g_2) = \sum_{a''=\pm} \sum_{m''=\pm} \int du(l'') \cdot D_{l'm'a'', l'm'a''}^{(j_0, n)}(g_1) D_{l'm'a'', l'm'a'}^{(j_0, n)}(g_2)$$

Pause to remark: this is the group property of these new functions. The complete Plancherel measure of $O(2,1)$ appears inside. We may read $\int du(l'')$ as a sum over l'' . I imagine that the principle series of $O(3,1)$ representations, when done in this $O(2,1)$ basis rather than the usual $O(3)$ basis, has a bivalued multiplicity index. This same index arises when you treat $O(2,1)$ in the $O(1,1)$ basis, rather than the usual $O(2)$ basis. Exactly the same thing, dammit. Hyperboloid have two sides.

This is the analog to my second-kind addition theorem. This reduction should be symbolized thusly:

$$O(3,1) \supset O(2,1) \supset O(2)$$

The general group element is this:

$$D_{l'm'a', l'm'a'}^{(j_0, n)}(g = hgh') = \sum_{m''=-\infty}^{\infty} \int du(l'') \int du(l'') D_{l'm'a'', l'm'a''}^{(j_0, n)}(h) e_{l'm'a'', l'm'a'}^{(j_0, n)}(s) D_{l'm'a'', l'm'a'}^{(j_0, n)}(h')$$

Repeat:

$$D_{l'm'a', l'm'a'}^{(j_0, n)}(g) = \left(\sum_{m_1=-\infty}^{\infty} \int du(l_1) \right) \left(\sum_{m_2=-\infty}^{\infty} \int du(l_2) \right) D_{l'm_1 a_1, l'm_1 a_1}^{(j_0, n)}(h) e_{l'm_1 a_1, l'm_1 a_1}^{(j_0, n)}(s) D_{l'm_1 a_1, l'm_1 a_1}^{(j_0, n)}(h) \\ \delta(l_1 - l_1') \delta(a_1 - a_1') D_{l'm_1 a_1, l'm_1 a_1}^{(j_0, n)}(h) e_{l'm_1 a_1, l'm_1 a_1}^{(j_0, n)}(s) D_{l'm_1 a_1, l'm_1 a_1}^{(j_0, n)}(h) \\ \delta(l_2 - l_2') \delta(a_2 - a_2') D_{l'm_2 a_2, l'm_2 a_2}^{(j_0, n)}(h) e_{l'm_2 a_2, l'm_2 a_2}^{(j_0, n)}(s) D_{l'm_2 a_2, l'm_2 a_2}^{(j_0, n)}(h) \\ e_{l'm_1 a_1, l'm_2 a_2}^{(j_0, n)}(g) \delta_{m_1, m_2}$$

$$\Rightarrow D_{l'm'a', l'm'a'}^{(j_0, n)}(g) = \sum_{m=-\infty}^{\infty} D_{l'm a, l'm a}^{(j_0, n)}(h) e_{l'm a, l'm a}^{(j_0, n)}(s) D_{l'm a, l'm a}^{(j_0, n)}(h')$$

Rescale:

$$D_{lm, l'm'}^{(j, n)}(g = hgh') = \sum_{n=-\infty}^{\infty} D_{mn}^l(h) e_{ll'naa'}^{(j, n)}(\xi) D_{nm'}^{l'}(h')$$

This tells you the matrix elements of $O(3,1)$ for the principle series in the $O(2,1)$ to $O(2)$ reduction. When $a = a' = -$, you get the AS e function which is the same as the Toller second kind function.

In the more standard $O(3)$ to $O(2)$ reduction, the above formula in Ruhl would be something like:

$$D_{lm, l'm'}^{(j, n)}(g = u\xi u') = \sum_{n=-\infty}^{\infty} D_{mn}^l(u) e_{ll'n}^{(j, n)}(\xi) D_{nm'}^{l'}(u')$$

The big difference of course is that in this case the central function is a first kind function instead of the desired second kind function.

4. If z-boosts are kept positive, I get that hgh' defines a semigroup in $O(3,1)$. Since the 2×2 matrix is triangular, we will get the following addition theorem in the — sector, say:

$$Q_{lm, l'm'}^{(j, n)}(g_1 g_2) = \sum_{m''=-\infty}^{\infty} \int d\mu(l'') Q_{lm, l'm''}^{(j, n)}(g_1) Q_{l'm'', l'm'}^{(j, n)}(g_2)$$

where

$$Q_{lm, l'm'}^{(j, n)}(g) = \sum_n D_{mn}^l(h) e_{ll'n}^{(j, n)}(\xi) D_{nm'}^{l'}(h')$$

This is the true $O(3,1)$ analog of my second-kind addition theorem on the semigroup.

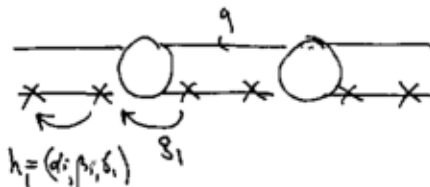
Note: no doubt the $t=0$ equation multiperipheral is supported on this semigroup which is the same as a coset space. Then this addition theorem will diagonalize the $t=0$ equation. Diagonalized equation looks just like the addition theorem.

AS'72 diag. in my language.

- ① I have reduced their integral equation to this form:

$$A(g) = \int dT \int dg_1 A(g_1) I(g_2)$$

What is dg_1 ?



central boost.

central $o(2,1)$

azimuthal.

Probably $d^4q = \text{loop phase space} = \text{tot. } d\beta_1 \cdot d(\alpha\beta_1) d\alpha_1 (sh\beta_1)^2$.
I am just guessing as to the Jacobian, but "educated guess".

- ② Now examine each of the three projections:

$$g = k g h' = (\alpha, \beta, \gamma) g (\alpha', \beta', \gamma') = 7 \text{ variables but } \gamma + \gamma' = \text{Total angle.}$$

$$dg = [d\alpha d(\alpha\beta)] dg (sh\beta) [d\alpha' d(\alpha\beta') d\gamma']. \quad \text{I guess.}$$

$$\text{So } dg = dg \cdot dh', \text{ usual "coset" sense.}$$

③ Now:

$$\int dg Q_{lm, l'm'}^{(j, n)}(g) \overline{A(g) \Theta(g)}$$

$$= \int dg Q_{lm, l'm'}^{(j, n)}(g) A(g)$$

$$= \int dg \cdot \int dh' \cdot \sum_n D_{mn}^l(h) e_{ll'm'}^{(j, n)}(g) D_{nm'}^{l'}(h') \\ \times \left[\sum_{s_k' s_l'} D^{(s_k' s_l')}(h) \cdot a(s) D^{(s_l' s_r')}(h') \right]$$

Stop. How are we going to combine these D-functions? Remember that D^l has l taking on values appropriate for the UIR's of $SU(1,1)$, whereas these other D's are products of finite dimensional ~~WIR~~ IR's of $SU(1,1)$.

It should be clear by now that l is the total angular momentum flowing across or into the left blob, so somehow it is correct to Clebsh the upper and lower spin states. AS couple things with $SO(2,1)$ FDR ~~BL~~ bashes on page 590 and footnote 8.

go to Clebschization. (next page)

Clebschization.

① Go back to:

$$A^{(s_k, s_p)}(e, g) = \sum_{s_k' s_p'} D^{s_k s_k'}(h) D^{s_p' s_p}(h') A^{(s_k', s_p')}(g)$$

Write out in detail:

$$\begin{aligned} \binom{l, m}{l', m'} A^{(j, n)}_{(j', n')} (e, g) &= \sum_{\tilde{m} \tilde{m}'} D^{l, m}_{\tilde{m} \tilde{m}'}(h) D^{l', m'}_{\tilde{m} \tilde{m}'}(h') \\ &\quad \sum_{\tilde{n} \tilde{n}'} D^{j, n}_{\tilde{n} \tilde{n}'}(h) D^{j', n'}_{\tilde{n} \tilde{n}'}(h') \times \binom{l, \tilde{m}}{l', \tilde{m}'} A^{(j, \tilde{n})}_{(j', \tilde{n}')} (g) \end{aligned}$$

② Now combine in pairs, as if were $O(3)$:

$$D^{l, m}_{\tilde{m} \tilde{m}'}(h) D^{l', m'}_{\tilde{m} \tilde{m}'}(h') = \sum_{LM} \langle ll' m m' | LM \rangle \langle L \tilde{M} | l l' \tilde{m} \tilde{m}' \rangle D^{L, M}_{\tilde{M} \tilde{M}'}(h)$$

$\underbrace{\quad}_{LM}$ ← usually only show $\frac{\tilde{M}'}{L}$

This looks about right.

Then

$$\begin{aligned} A(g) &= \sum_{\tilde{m} \tilde{m}'} \sum_{LM} \langle ll' m m' | LM \rangle \langle L \tilde{M} | l l' \tilde{m} \tilde{m}' \rangle D^{L, M}_{\tilde{M} \tilde{M}'}(h) \\ &\quad \sum_{\tilde{n} \tilde{n}'} \langle jj' m' m' | J N \rangle \langle J \tilde{N} | j j' \tilde{n} \tilde{n}' \rangle D^{J, N}_{\tilde{N} \tilde{N}'}(h') \cdot A(g) \end{aligned}$$

kill $\nwarrow$ $M = m + m'$ $\swarrow$ $\tilde{M} = \tilde{m} + \tilde{m}'$
kill $\nwarrow$

So:

$$\binom{l, m}{l', m'} A^{(j, n)}_{(j', n')} (e, g) = \sum_{LM} \sum_{JN} \left(\sum_{\tilde{m} \tilde{m}' \tilde{n} \tilde{n}'} \right)$$

$$\left[\frac{\langle ll' m m' | LM \rangle}{\langle ll' \tilde{m} \tilde{m}' | LM \rangle} \right] D^{L, M}_{\tilde{M} \tilde{M}'}(h) \cdot \left[\frac{\binom{l, \tilde{m}}{l', \tilde{m}'} A^{(j, \tilde{n})}_{(j', \tilde{n}')} (g)}{\binom{l, \tilde{m}}{l', \tilde{m}'} A^{(j, \tilde{n})}_{(j', \tilde{n}')} (g)} \right] D^{J, N}_{\tilde{N} \tilde{N}'}(h') \cdot \left[\frac{\langle jj' m' m' | J N \rangle}{\langle jj' \tilde{n} \tilde{n}' | J N \rangle} \right]$$

Group factors shown:

$$\sum_{\tilde{m} \tilde{m}' \tilde{n} \tilde{n}'} \langle ll' \tilde{m} \tilde{m}' | L \tilde{M} \rangle^* A(g) \langle jj' \tilde{n} \tilde{n}' | J \tilde{N} \rangle \equiv \binom{l, \tilde{m}}{l', \tilde{m}'} A^{(j, \tilde{n})}_{(j', \tilde{n}')} (g)$$

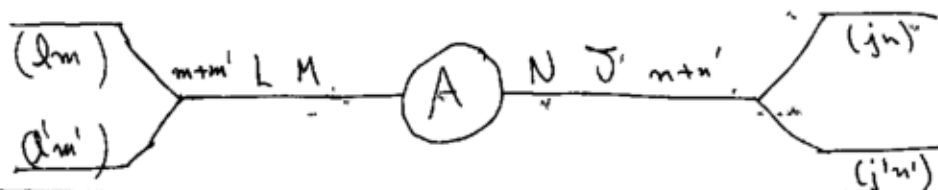
Then:

$$\binom{l, m}{l', m'} A^{(j, n)}_{(j', n')} (g) = \sum_{L \tilde{M} J \tilde{N}} \langle ll' m m' | L \tilde{M} \rangle^* D^{L, M}_{\tilde{M} \tilde{M}'}(h) A(g) D^{J, N}_{\tilde{N} \tilde{N}'}(h') \langle jj' m' m' | J \tilde{N} \rangle$$

Repeat:

$$\langle l'm' | A^{(j'm)}(g) | l'm' \rangle = \sum_{LJMN} \langle l'm' | L m m' \rangle D_{L m m', M}^L(h) \cdot A^{(L M)}(g) \cdot D_{N, m m'}^J(h') \cdot \langle J m m' | j j' m m' \rangle$$

$$\langle l'm' | L m m' \rangle D_{L m m', M}^L(h) \cdot A^{(L M)}(g) \cdot D_{N, m m'}^J(h') \cdot \langle J m m' | j j' m m' \rangle$$



Basically all we are doing is pre-combining the two particles on each side into a single $O(2,1)$ representation using Clebsches.

Now, what happens when we insert this same expansion three times into the integral equation?

$$\sum_{LJMN} \langle s_k | L \rangle D_{L M}^L(h) A^{(L M)}(g) D_{N, s_k}^J(h') \langle J | s_k \rangle = A^{(s_k, s_k)}(g) \text{ shortened}$$

Answer:

$$\left[\sum_{LJMN} \langle s_k | L \rangle D_{L M}^L(h) A^{(L M)}(g) D_{N, s_k}^J(h') \langle J | s_k \rangle \right] = \int dt \cdot C_{s_k} \int dg_1 \left[\sum_{L'J'M'} \langle s_k | L' \rangle D_{L' M'}^{L'}(h_1) A^{(L' M')}(g_1) D_{N', s_k}^{J'}(h'_1) \langle J' | s_k \rangle \right] \left[\sum_{L''J''M''} \langle s_k | L'' \rangle D_{L'' M''}^{L''}(h_2) A^{(L'' M'')}(g_2) D_{N'', s_k}^{J''}(h'_2) \langle J'' | s_k \rangle \right]$$

$\delta_{L', L''} \delta_{J', J''} \delta_{M', M''}$ kills $\sum_{L''}$

$$\rightarrow D_{L M}^L(h) A^{(L M)}(g) D_{N, s_k}^J(h') = \int dt \cdot C_{s_k} \int dg_1 \left[\sum_{L'J'M'} \langle s_k | L' \rangle D_{L' M'}^{L'}(h_1) A^{(L' M')}(g_1) D_{N', s_k}^{J'}(h'_1) \langle J' | s_k \rangle \right] \left[\sum_{L''J''M''} \langle s_k | L'' \rangle D_{L'' M''}^{L''}(h_2) A^{(L'' M'')}(g_2) D_{N'', s_k}^{J''}(h'_2) \langle J'' | s_k \rangle \right]$$

S₀:

$$\left[D_{S_k M_k}^{J_k} (h) A(s) D_{M_k S_p}^{J_p} (h') \right] = \sum_{J_q M_q M_q'} \int dT \int d\mathbf{g}_1$$

$$\left[D_{S_k M_k}^{J_k} (h_1) A(s_1) \right] \left[D_{M_q M_q'}^{J_q} (h_2) A(s_2) D_{M_q S_p}^{J_p} (h_2') \right]$$

This then is my view of AS equation (590.5). The clebschization has been performed. Of course they choose $h = h' = e$ so you don't see the d functions on the left hand side. Note that there are no CGC's in this equation, and that there are 3 D functions on the right side. As usual, $h_1' = e$ due to frames set up.

5. Having completed the Clebsching, we now proceed with the diagonalization. Again, we have to examine the projections:

$$\int d\mathbf{g} Q_{lm, l'm'}^{(j_0, n)}(g) A(g) O(g)$$

$$= \int d\mathbf{g} \int d\mathbf{h}' \left[\sum_n D_{mn}^l(h) C_{ll'm}^{(j_0, n)}(s) D_{nm'}^{l'}(h') \right] \left[D_{S_k M_k}^{J_k}(h) A(s) D_{M_q S_p}^{J_p}(h') \right]$$

$$= \left[\int d\mathbf{h} D_{mn}^l(h) D_{S_k M_k}^{J_k}(h) \right] \left[\int d\mathbf{h}' D_{nm'}^{l'}(h') D_{M_q S_p}^{J_p}(h') \right]$$

$$\times \sum_n \int d\mathbf{g} (shg)^2 C_{ll'm}^{(j_0, n)}(s) A(s) D_{M_q S_p}^{J_p}(s)$$

Now we have a clearer view of the problem we are up against. The left $O(2,1)$ group integral has been doctored up to full group integral. The problem is that, strictly speaking, l takes a value that is complex, principle series of $O(2,1)$, say, whereas J_k tends to be real, as for a FDR of $O(2,1)$, so not very clear what you get in such an integration.

Conjecture: these integrations give zero unless $l = J_k =$ a value in the discrete series of $O(2,1)$. Recall that the addition theorem does have full $O(2,1)$ Plancheral contributions. Then:

$$= \frac{1}{n} (\delta_{l, J_k} \delta_{m, S_k} \delta_{n, M_k}) (\delta_{l', J_p} \delta_{n, M_p} \delta_{m', S_p}) \int d\mathbf{g} (shg)^2 C_{J_k J_p m}^{(j_0, n)}(s) A(s)$$

$$= \frac{\delta_{S_k M_k} \delta_{S_p M_p}}{A} = \text{projection of LHS of Clebsched integral equation.}$$

Get nothing unless $l = J_k, l' = J_p, m = S_k, m' = S_p, n = M_k = M_p.$

Projection of I is similar. Projection of $A(q_1)$ is:

$$\int_{m_k} dg_1 [D(h_1) A(s_1)] Q_{lm, l'm'}(h_1) \\ = \delta_{l, l'} \delta_{m, m'} \int_0^\infty ds_1 (sh_1)^2 \underbrace{e^{(j_0, n)}_{J_K M_K J_P M_P} A(s_1)}_{\substack{= A^{(j_0, n)}_{J_K M_K J_P M_P}}} \\ = A^{(j_0, n)}_{J_K M_K J_P M_P}$$

So diagonalized equation looks like:

$$\left[(\delta_{l, l'} \dots \delta_{m, m'}) A^{(j_0, n)}_{J_K M_K J_P M_P} \right] = \sum_n \int dg_1 \dots \int dt. \\ \left[\delta_{l, l'} \dots \delta_{m, m'} A^{(j_0, n)}_{J_K M_K J_P M_P} \right] \cdot \left[(\delta_{l', l} \dots \delta_{m', m}) A^{(j_0, n)}_{J_K M_K J_P M_P} \right]$$

Here we are about to see the action happen. The addition theorem measure contains a helicity sum plus the full Plancherel for $O(2,1)$. BUT, the delta function inside the projection of $I(g_2)$ is going to freeze l at a value equal to J_3 . (which happens to be in the discrete series). Of course this variable is still summed over as you see on top of last page. I guess the helicity sum gets handled somehow and you are left with:

$$A^{(j_0, n)}_{J_K M_K J_P M_P} = \int dt. \sum_{J_q M_q} A^{(j_0, n)}_{J_K M_K J_q M_q} \cdot I^{(j_0, n)}_{J_q M_q J_P M_P}$$

Now, looking at projections on page 591, and considering that J_3 commutes with K_3 , we get helicity locking as I earlier suspected in that picture so that:

$$\delta_{M_K, M_P}$$

$$\delta_{M_K, M_q}$$

$$\delta_{M_q, M_P}$$

$$\Rightarrow A^{(j_0, n)}_{J_K M_K J_P} = \int dt. \sum_{J_q} A^{(j_0, n)}_{J_K M_K J_q} I^{(j_0, n)}_{J_q M_K J_P} \quad \leftarrow \text{result, minus } J_3$$

I have of course not really done things right, but I get the idea: my general semigroup addition theorem diagonalizes the equation, and then because of its special form, the projections are very simple and the j -integration goes away and is replaced with the J_q sum that appeared in the clebschization.

As usual, AS succeed by brute force, but they do not show elegance. There is something else going on here, the way that j -integral was replaced with a j -sum must be more than coincidence. Must be a better way to explain this!