

WAVE PACKETS, CROSS SECTIONS, BEAMS , ETC.

Contents:

1. Discussion of Incident State Wave Packet Business (typed, 3p)
2. Actual MS-type wave packet cross section derivation. (6p)
 - A. Detail on Momentum Space Wavefunctions.
 - B. Supportive material (see page 6 for cross section idea)
3. The Box Method and Fermi's Golden Rule of NRQM.
4. Comments on using final state wave packets. (typed)
 - A. The actual I-calculation.
5. Experimenter and his Beams.
 - a. Construction of packets going in arbitrary direction.
 - b. how he computes the luminosity integral
 - c. the very simple case of the "big uniform target plate"

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Discussion of the Wave Packet Cross Section Derivation

(Working Paper Only)

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1. In the beginning, you consider the scattering of two particles, where each particle is conceived of as a "little ball" travelling through space. Each "ball" is a wavepacket following a classical trajectory through space.

Each "ball" is characterized by a momentum space wavefunction normalized to unity. These wavefunctions should be thought of as strongly peaked at the classical momentum of the ball, but not infinitely peaked thereat. Having an infinite peak destroys the configuration space wavefunction by blurring it out so that it "fills all space with equal probability density". Clearly, such a spatial wavefunction has no meaning, even if you "put it in a box."

Now, once you have (theoretically) constructed your momentum wavefunctions, say with strong gaussian peaks, you may compute the spatial wavefunction. This function $f(x,t)$ should have the spacetime structure of one of these "balls" we have been talking about, travelling along a classical trajectory. Of course the normalization of the spatial wavefunction depends on how you define it, and in general will not be unity.

Once you have your spatial wavefunctions, you can derive densities. Each density should be such that at any time the spatial integral is unity. Notice that a density is not a Lorentz scalar. As you "fly over" some matter at high speed, the atoms appear more closely spaced in the longitudinal direction, so you measure a higher density than you would if your "plane" were at rest. But, the density divided by the energy of the observed atoms is a scalar because both increase by the same amount, gamma.

The upshot of all this is that the spacetime wavefunction squared, as defined by MS, is a Lorentz scalar. So is the momentum space wavefunction.

2. Now that these various wavefunctions are established, we can discuss the matrix elements of the T-matrix from an initial state consisting of the direct product of two single particle "distributed" state vectors, as discussed above, into a final state consisting of plane waves. We never have any need to construct wave-packets for the final states! Now, when this matrix element is squared and multiplied by an appropriate phase space, we have a direct interpretation:

$$/T^2 \times \text{Phasespace} = \text{probability that the reaction took place}$$

To elaborate a little, this is the probability that, when we sent in the single beam particle and the single target particle in their spatial wave packets (as described by the information in the momentum space wavefunctions and ultimately the momentum state vectors which appear in the T-matrix element), when we sent the incident particles in in this way, $/T^2 \times \text{Phasespace}$ = the probability that a certain set of final particles "appear" in the respective momentum cones.

So, suppose for each reaction this probability is .01 where we have set up very small "counters" which span each d^3p "cone". Then if we do this experiment 10,000 times, we expect to get 100 "events". An "event" is when all your counters go "ding".

3. Using our explicit expressions for the momentum space incident state vectors, we may rewrite $|T|^2$ in terms of the standard plane wave states everywhere. Then we are on familiar terrain and all remaining manipulations except for the cross section are easy and old hat.

For starters, you replace the T matrix element with the "alpha" type matrix element by going into the "little hilbert space". This of course produces various δ^4 delta functions. When all manipulations are finished, you end up with this equation:

$$\begin{aligned} \text{Prob.} &= |T|^2 \times \text{phase space} \\ &= \frac{(2\pi)^2}{2E_1 \cdot 2E_2} \cdot dQ_f \cdot |\langle \psi_f | T | \psi_i \rangle|^2 \cdot \int d^4x g_1(x) g_2(x) \end{aligned}$$

From comments made earlier, it is clear that this probability is lorentz invariant.

4. A little discussion of this equation is in order. We can have a genuine picture of what happens in the scattering process. The two incident particles start in a region where they have no spatial overlap of their density functions. This is what "no interaction in the asymptotic state" means.

Then, as time progresses, the flying "balls" begin to pass through each other. As soon as they overlap, the generation of the "final state" begins. In the time interval t to $t+dt$, there is "activity" in the overlap volume V . In that "spacetime volume" probability is being generated at a "level" which is proportional to the overlap. If we focus down on a very small spatial region, then in t to $t+dt$, probability is generated at a "level" which is proportional to the product of the densities. To get the total probability generated by this scattering, you integrate over all spacetime! Pretty clear!

So, the general picture is two balls of "gunk" passing through each other. Wherever and whenever the gunk of one ball touches the gunk of the other ball, new gunk is generated.

In this picture, the particles are definitely not "point" particles. Quantum mechanics tells us that exact spatial localization is simply not meaningful. Thus, even if you do a scattering experiment with an incident electron, that electron is spread out into a ball. Still, I am picturing the "interaction" as "local" in the sense that product is generated only where the incidents have regions which occupy the same spacetime volume.

More will be said on this subject later re "field theory".

5. Since experiments are not done with a single beam particle and a single target, we have to make some modifications to get the probability you would obtain in a real experiment.

The modification is easy, however. Consider the scattering of a "beam burst" with a "target". We need not worry about such things as whether the diameter of the "beam" is larger or smaller than the target, and all that. We need only conceive of the entire target as a large "particle", and the beam-burst as another "particle". We replace the "single ball density function" with a "target density function" and a "beam burst density function". These new density functions are simply sums of little balls, so they certainly are

not the smooth density functions of macrophysics.

So consider an instant in time when the beam burst is "in" the target. Where is "product" being generated? Microscopically, product is being generated wherever a "beam ball" overlaps a "target ball". At one time, such overlaps could be occurring at several places, and each overlap makes a contribution. Of course we have in mind a reasonably small physical target so that wherever the overlap occurs, the far away detectors subtend the same chunk of phase space. (This is called the "fiducial volume" I think).

Instead of explicitly showing where and when the various "balls" overlap, we shall allow our "macroscopic density functions" keep track of that for us. At any instant in time, the overlap of these macro density functions is really the sum of lots of little ball overlaps.

All this discussion has been designed to motivate this modified equation:

$$Prob. = \frac{(2\pi)^2}{2E_1 2E_2} dQ_f |T_{fi}|^2 \cdot \int d^4x \rho_B(x) \rho_T(x)$$

$$\int d^3x \rho_B(x) = N_B$$

$$\int d^3x \rho_T(x) = N_T$$

Now "Prob." is the number of events obtained per experiment. Eg, if prob = 27.23 by correct calculation (with knowledge of detector sizes etc). then you would expect that, if you were to perform the experiment one million times, the average number of events per experiment would be 27.23.

The question of how many experiments you have to do to measure Prob. to 1% is I think a purely statistical one having nothing to do with quantum mechanics or physics. Return to this later.

6. Finally we come to this matter of the cross section. Pick two regions of physical space, each shaped like a penny. Let these regions be very small and let them be flat touching one another. At some instant in time, there will be so many target particles in the "target penny", which we imagine to be at rest; "lab". In the "beam" penny there are a known number of beam particles, all travelling at the classical mean velocity. By fiddling around and finally integrating over all space and time, we find:

$$Prob. = N \cdot d\sigma \cdot \int d^4x \rho_T(x) \rho_B(x)$$

By comparison with the above expressions for prob, we quickly arrive at the "standard" cross section formula.

7. I am interested now to take the "plane wave limit" of all these equations to learn how to correctly interpret the plane wave derivation. This should be fun!

ie, incident packets only, as in MS. 2

We do calculation, wave packets, all-norm,

① Preliminaries: keep in mind that, in all-norm,

$$S = 1 + i C_1 T \quad \text{/_operators.}$$

$$\langle P | P' \rangle = C_3 \delta^4(P - P')$$

$$\langle P | P' \rangle = 2 E C_2 \delta^3(P - P')$$

$$T_{fi} = \prod_{\text{all}} C_4^{-1} \langle q_f | T | q_i \rangle$$

$$1 = \int \frac{d^3 p}{2 E C_2} |P\rangle \langle P|$$

$$= \int \frac{d^4 p \delta^+(p^2 - m^2)}{C_2} |P\rangle \langle P|$$

Notice that in any norm having E in C_2 , $|P\rangle \langle P|$ is not a Lorentz scalar operator, although this does not really hurt anything, just the covariance of presentation.

② Set up the matrix element:

$$\langle q_1 \dots q_n | S - 1 | \phi_1 \phi_2 \rangle = i C_1 \langle q_1 \dots q_n | T | \phi_1 \phi_2 \rangle$$

where

$$\rightarrow |\phi_i\rangle = \int \frac{d^4 p \delta^+(p^2 - m^2)}{C_2} \phi_i^{(P_i)}(p) |P\rangle$$

Here, $\phi_i^{(P_i)}(p)$ is a scalar distribution function strongly peaked at $p^\mu = p_i^\mu$. Notice:

$$\langle P' | \phi_i \rangle = \int \frac{d^3 p}{2 E C_2} \cdot \phi_i^{(P_i)}(p) \cdot 2 E C_2 \delta^3(p - P') = \phi_i^{(P_i)}(P')$$

Also,

$$\langle \phi_i | \phi_i \rangle = \int \frac{d^3 p}{2 E C_2} |\phi_i^{(P_i)}(p)|^2$$

Arbitrarily normalize $\phi_i^{(P_i)}(p)$ so that $\langle \phi_i | \phi_i \rangle = 1$.

Therefore: $\rightarrow \langle q_1 \dots | S - 1 | \phi_1 \phi_2 \rangle = i C_1$

$$\times \int \frac{d^4 p \delta^+(p^2 - m^2)}{C_2} \phi_1^{(P_1)}(p) \int \frac{d^4 p' \delta^+(p'^2 - m^2)}{C_2} \phi_2^{(P_2)}(p') \\ \times \langle q_1 \dots q_n | T | P, P' \rangle$$

Review the "all-norm" constants

distributed state $|\phi\rangle$

momentum space wavefunction $\phi(p)$

norm on ϕ

③ Now some manipulations: extract δ states here:

$$\langle q_1, \dots | T | p, p' \rangle = C_3 \delta^4(p + p' - \Sigma q_i) \cdot \langle \psi | T | \psi(p, p') \rangle$$

Next, square the whole works. [note that $|iC_1|^2 = 1$, goes away.]

$$\begin{aligned} |\langle q_1, \dots | S^{-1} | \Phi \Phi \rangle|^2 &= \int \frac{d^4 p}{C_2} \delta^4(p) \cdot \frac{d^4 p'}{C_2} \delta^4(p') \cdot \Phi_1(p) \Phi_2(p') \cdot C_3 \delta^4(p + p' - \Sigma q_i) \\ &\times \int \frac{d^4 k}{C_2} \delta^4(k) \cdot \frac{d^4 k'}{C_2} \delta^4(k') \cdot \Phi_1^*(k) \Phi_2^*(k') \cdot C_3 \delta^4(k + k' - \Sigma q_i) \\ &\times \langle \psi | T | \psi(p, p') \rangle \cdot \langle \psi | T | \psi(k, k') \rangle^* \end{aligned}$$

Now use strong peaked property to remove these from integral and set initial momenta to their mean values. Then:

$$\begin{aligned} \rightarrow |\langle q_1, \dots | S^{-1} | \Phi \Phi \rangle|^2 &= |\langle q_1, \dots | T | \Phi \Phi \rangle|^2 \\ &= |\langle \psi | T | \psi \rangle|^2 \times \int \frac{d^4 k}{C_2} \delta^4(k) \cdot \frac{d^4 k'}{C_2} \delta^4(k') \cdot \Phi_1^*(k) \Phi_2^*(k') \cdot C_3 \delta^4(k + k' - \Sigma q_i) \\ &\times \int \frac{d^4 p}{C_2} \delta^4(p) \cdot \frac{d^4 p'}{C_2} \delta^4(p') \cdot \Phi_1(p) \Phi_2(p') \cdot C_3 \delta^4(p + p' - \Sigma q_i) \end{aligned}$$

Next, stick on phase space:

$$\prod_{i=1}^n \left(\frac{d^3 q_i}{2\pi^3 C_2} \right) = \int d^4 p' \cdot \delta^4(p' - \Sigma q_i) \cdot \prod \left(\frac{d^3 q_i}{2\pi^3 C_2} \right) \equiv \int d^4 p' \cdot d\tilde{Q}$$

④ Before moving on, we must go to x -space. Define:

$$\phi_i(x) \equiv \int \frac{d^4 p}{C_2} \delta^4(p) \cdot \phi_i(p) \cdot e^{-ip \cdot x}$$

Compute the inversion:

$$\begin{aligned} \int d^4 x' \cdot \phi_i(x') \cdot e^{+ip' \cdot x'} &= \int d^4 x' \cdot \int \frac{d^4 p}{C_2} \delta^4(p) \cdot \phi_i(p) \cdot e^{i(p' \cdot x' - p \cdot x')} \\ &= (2\pi)^4 \int d^4 p \frac{\delta^4(p)}{C_2} \cdot \phi_i(p) \cdot \delta^4(p - p') = (2\pi)^4 \cdot \frac{\delta^4(p)}{C_2} \cdot \phi_i(p) \end{aligned}$$

Thus we have:

$$\rightarrow \delta^4(p^2 - m^2) \phi_i(p) = \frac{C_2}{(2\pi)^4} \cdot \int d^4 x \cdot \phi_i(x) \cdot e^{+ip \cdot x}$$

go to
little
Hilbert

$\phi(p)$
inversion

⑤ So far, then, we have arrived at:

$$|\langle q_1 \dots | T | \phi_1 \phi_2 \rangle|^2 = |\langle \alpha+1 | T | \alpha \rangle|^2 \times |I|^2$$

where:

$$I = \int \frac{d^4 k}{C_2} \cdot \frac{d^4 k'}{C_2} \cdot \delta^4 \phi_1(k) \cdot \delta^4 \phi_2(k') \cdot C_3 \cdot \delta^4(k+k' - \sum q_i)$$

Let's insert:

$$\begin{aligned} \delta^4 \phi_1(k) &= C_2 (2\pi)^4 \int d^4 x \phi_1(x) e^{+i k \cdot x} \\ \delta^4 \phi_2(k') &= C_2 (2\pi)^4 \int d^4 x' \phi_2(x') e^{+i k' \cdot x'} \end{aligned}$$

The C_2 's all cancel and we're left with:

$$I = (2\pi)^{-8} \cdot C_3 \cdot \int d^4 k d^4 k' d^4 x d^4 x' \phi_1(x) \phi_2(x') e^{i(k \cdot x + k' \cdot x')} \delta^4(k+k' - \sum q_i)$$

Do the $d^4 k'$ onto the δ^4 , so $(k \cdot x + k' \cdot x') \rightarrow (k \cdot x + [\sum q_i - k] \cdot x')$.

Thus,

$$I = (2\pi)^{-8} \cdot C_3 \cdot \int d^4 k d^4 x d^4 x' \phi_1(x) \phi_2(x') e^{i k \cdot (x-x') + i (\sum q_i) \cdot x'}$$

Next, do the $d^4 k$ to pull a $(2\pi)^4 \delta^4$, so:

$$I = (2\pi)^{-4} \cdot C_3 \cdot \int d^4 x d^4 x' \phi_1(x) \phi_2(x') e^{i \sum q_i \cdot x'} \delta^4(x-x')$$

Finally,

$$\rightarrow I = (2\pi)^{-4} \cdot C_3 \cdot \int d^4 x \phi_1(x) \phi_2(x) e^{i P \cdot x}$$

where $P \equiv \sum q_i$ (~~as used before in phase space~~) Now square this:

$$\rightarrow |I|^2 = C_3^2 \cdot (2\pi)^{-8} \cdot \int d^4 x d^4 y e^{i P \cdot (x-y)} \phi_1(x) \phi_1^*(y) \phi_2(x) \phi_2^*(y)$$

Notice now that

$$\rightarrow \int |I|^2 d^4 P = C_3^2 (2\pi)^{-8} \cdot (2\pi)^4 \cdot \int d^4 x |\phi_1(x)|^2 |\phi_2(x)|^2$$

⑥ Putting these pieces together, we have:

$$\begin{aligned} \int |\langle q_1 \dots | T | \phi_1 \phi_2 \rangle|^2 d^4 P' d^4 Q \\ = |\langle \alpha+1 | T | \alpha \rangle|^2 d^4 Q \cdot \int d^4 P \cdot |I|^2 \end{aligned}$$

Stop. Have to rethink this phase space step:

$$\prod_{i=1}^n \left(\frac{d^3 q_i}{2E_i c_2} \right) = d^4 P \cdot d\tilde{Q} \quad \bar{P} \equiv \sum q_i \leftarrow$$

where $d\tilde{Q}$ is "the rest" when you do this jacobian calculation.
Now apply $\delta^4(P_1 + P_2 - P)$ to both sides + integrate both sides over all phase space:

$$\int_{\text{all } q_i} \delta^4(P_1 + P_2 - P) \prod_{i=1}^n \left(\frac{d^3 q_i}{2E_i c_2} \right) \times f(q_1, \dots) = \int_{P \text{ "rest"}} \delta^4(P_1 + P_2 - P) d^4 P \cdot d\tilde{Q} \cdot \tilde{f}(\text{rest})$$

$$= \int_{\text{"rest"}} d\tilde{Q} \cdot f$$

So we write: $d\tilde{Q} \equiv \delta^4(P_1 + P_2 - P) \prod_{i=1}^n \left(\frac{d^3 q_i}{2E_i c_2} \right)$

Of course we mean that this is an equality when "appropriate" integration is performed. It is in the more precise sense that

$$d\tilde{Q} = d(\text{rest}) = \text{independent of } P.$$

For example, in the two body final state,

$$d\tilde{Q} = \frac{P}{4W} d\Omega$$

This was all to justify:

$$\left. \begin{aligned} \int_P |K(q_1, \dots)|^2 |\pi| |\Phi(q)|^2 d^4 P \cdot d\tilde{Q} &= |K(q_1, \dots)|^2 d\tilde{Q} \cdot \int_P d^4 P |I|^2 \\ &= |K(q_1, \dots)|^2 d\tilde{Q} \cdot C_3^2 (2\pi)^4 \cdot \int d^3 x |\phi_1(x)|^2 |\phi_2(x)|^2 \\ &= \text{Prob.} \end{aligned} \right\}$$

⑦ How are $\phi_i(x)$ normalized?

$$\int d^3 x |\phi_i(x)|^2 = \int d^3 x \cdot \int \frac{d^4 p \cdot \delta^4}{-c_2} \cdot \int \frac{d^4 p' \cdot \delta^4}{c_2} \phi_i(p) \phi_i^*(p') e^{-ip \cdot x + ip' \cdot x}$$

Go on shell $= \int d^3 x \cdot \int \frac{d^3 p}{2E c_2} \cdot \int \frac{d^3 p'}{2E' c_2} \phi_i(p) \phi_i^*(p') e^{-ix \cdot (p - p')}$

Do the d^3x :

$$= \int \frac{d^3p}{2E C_2} \int \frac{d^3p'}{2E C_2} \phi_i(p) \bar{\phi}_i(p') e^{-i \cdot x(E-E')} (2\pi)^3 \delta^3(k-k')$$

But $\vec{p}=\vec{p}' \Rightarrow E=E'$ since both on same mass shell.

$$= (2\pi)^3 \int \frac{d^3p}{(2E C_2)^2} |\phi_i(p)|^2 = \int |\phi_i(x)|^2 d^3x$$

Compare this to:

$$\int \frac{d^3p}{2E C_2} |\phi_i(p)|^2 = \langle \phi_i | \phi_i \rangle = 1$$

Suggests that: STOP. Again, we need "string peak" of ϕ_i to "pull out" one power of $(2E C_2)$:

$$\left[\int |\phi_i(x)|^2 d^3x = \frac{(2\pi)^3}{2E C_2} \cdot 1 \leftarrow \right.$$

So define $\rho_i(x)$ such that $\int \rho_i(x) d^3x = 1$.

Thus:

$$\frac{2E C_2}{(2\pi)^3} |\phi_i(x)|^2 = \rho_i(x) \quad |\phi_i(x)|^2 = \frac{(2\pi)^3}{2E C_2} \rho_i(x) = \text{scalar.}$$

⑧ So going back to our previous result:

$$\rightarrow \text{Prob.} = \int_{\mathcal{P}} | \langle q_1 \dots q_n | S-1 | \phi_i \phi_j \rangle |^2 \times \prod_f \left(\frac{d^3q_f}{2E_f C_2} \right)$$

$$= | \langle \alpha_4 | T | \alpha_i \rangle |^2 dQ \cdot C_3^2 \cdot (2\pi)^4 \cdot \frac{(2\pi)^6}{2E_1 C_2 \cdot 2E_2 C_2} \cdot \int d^4x \cdot \rho_1(x) \rho_2(x)$$

Finally, we can go into "p"-norm states via.

$$| \langle \alpha_4 | T | \alpha_i \rangle |^2 = \prod_{\text{all}} C_4^2 \cdot |T_{fi}|^2$$

Then we have:

$$\text{Prob.} = (2\pi)^2 C_3 \cdot \left(\frac{C_4^2}{C_2^2} \right) \left(\frac{C_4^2}{C_2^2} \right) \cdot \frac{1}{4E_1 E_2} \cdot dQ_f \cdot |T_{fi}|^2 \cdot \int d^3x \rho_1(x) \rho_2(x)$$

where:

$$dQ_f = C_3 \cdot \delta^4(\text{ext}) \cdot \prod_f \left(\frac{C_4^2}{C_2^2} \right) \frac{d^3p_f}{2E_f}$$

The last step is to replace: $\delta_i \equiv C_i^2 / C_z$, so:

$$\Rightarrow \text{Prob.} = \frac{(2\pi)^2 C_3 \gamma_a \gamma_b |T_{fi}|^2 dQ_f}{4 E_1 E_2} \times \int d^3x \rho_1(x) \rho_2(x)$$

⑨ As discussed in working paper, we raise to the macroscopic level by replacing

$$\rho_1 \rightarrow \rho_B$$

$$\rho_2 \rightarrow \rho_T$$

⑩ Cross section:



$$d(\text{Prob.}) = \left(\frac{d^3 N_T \cdot d\sigma}{dA} \right) \cdot d^3 N_B$$

$\nwarrow$ number in beam "penny"
 $\nwarrow$ area coverage fraction

(Use $d^3 N_i = \rho_i(x) \cdot dz_i \cdot dA$ for each, then replace one dz_i with $v_B dt$.
Then:

$$d(\text{Prob.}) = \rho_T(x) \rho_B(x) d\sigma \cdot d^3 x \cdot v_B dt.$$

$$\Rightarrow \text{Prob.} = d\sigma \cdot v_B \cdot \int d^3 x \rho_T(x) \rho_B(x). \leftarrow$$

Therefore, using $E_1 E_2 v_B = \text{flux}$, we get

$$d\sigma = \frac{(2\pi)^2 C_3 \gamma_a \gamma_b |T_{fi}|^2 dQ_f}{4 \cdot \text{flux}}$$

⑪ See the typed sheets for more dialogue + explanation.

Momentum Space Wavefunction

Try: $\phi^{(p)}(p) = e^{-(p-p)^2/\Delta} = \text{Lorentz scalar.}$

This is maximum when $p_1^\mu = p^\mu$ in all components. Insert into...
check normalization:

$$\langle p' | \phi \rangle = \int d^3p \delta(\cdot) \phi(p) \langle p' | p \rangle = \int \frac{d^3p}{2E} \phi(p) \cdot 2E \delta^3(p' - p) \quad \text{shell}$$

$$= \phi(p), \text{ on shell.}$$

So, notice that although $\phi(p, E)$ is defined everywhere by any gaussian, $|\phi\rangle$ is only defined on shell, and:

$$\langle p | \phi \rangle = \phi(p, E) \quad \text{where } p^2 = E^2 + m^2$$

Now: completeness reads:

$$1 = \int \frac{d^3p}{2E} |p\rangle \langle p| = \int d^3p \delta^+(p^2 - m^2) |p\rangle \langle p|$$

Therefore, $\langle \phi | \phi \rangle = \int d^3p \delta^+(p^2 - m^2) \langle \phi | p \rangle \langle p | \phi \rangle = \int d^3p \delta(\cdot) |\langle p | \phi \rangle|^2$

Let $\phi(p) = \langle p | \phi \rangle = K \cdot e^{-(p-p)^2/\Delta}$; then:

$$\langle \phi | \phi \rangle = |K|^2 \int d^3p \delta^+(p^2 - m^2) \cdot e^{-2(p-p)^2/\Delta}$$

Switch LS, go to rest frame of $p_1^\mu = (m, 0)$. Then:

$$= K^2 \int \frac{d^3p}{2E} \cdot e^{-\frac{2}{\Delta} \{ (m-E)^2 - (p)^2 \}}$$

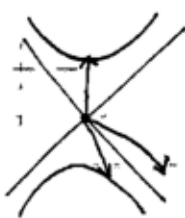
$$\{ \} = m^2 + E^2 - 2mE - p^2$$

$$= K^2 \int \frac{p^2 dp d\Omega}{2E} \cdot e^{-\frac{4m}{\Delta} (m-E)}$$

$$= 2m^2 - 2mE$$

$$= 2m(m-E)$$

This integral is going to diverge. Both p_1 and p are on shell future timelike, but $(p_1 - p)$ can be spacelike!



In fact, $e^{-(p_1-p)^2/\Delta} = e^{-(m-E)^2/\Delta + (p_1-p)^2/\Delta}$

$(p_1-p)^2 = 2m(m-E)$, it is always spacelike!

So, we renovate our gaussian: $\phi^{(p)}(p) = K \cdot e^{+(p-p)^2/\Delta}$

Then:

$$\begin{aligned} \langle \phi | \phi \rangle &= \int d^3p \delta^3 |\phi(p)|^2 = K^2 \cdot 4\pi \cdot \int_m^\infty \frac{p^2 dE}{2E} \cdot e^{-\frac{4m}{\Delta}(E-m)} \\ &= K^2 \cdot 2\pi \cdot \underbrace{\int_m^\infty \sqrt{E^2 - m^2} \cdot e^{-\frac{4m}{\Delta}(E-m)} dE}_I \end{aligned}$$

This convergent integral, I , determines K .

$$\langle \phi | \phi \rangle = 1 = 2\pi I \cdot K^2.$$

(This is now only "support material", do not read)

Wave packet derivation of standard cross section — all preliminary

① The secret is to avoid using plane wave states. Instead, use:

$$|\phi\rangle = \int d^3p \delta^3(p^2 - m^2) \phi(p) |p\rangle$$

Here, $\phi(p)$ is the "momentum distribution function", or, the "wavefunction in momentum space." In order to have $|\phi\rangle$ be a plane wave state, you would need:

$$\phi(p) = \delta^3(p - p_1)$$

note that with mass shell condition, this is a Lorentz invariant

We shall avoid such states!

② Now define a SPS ϕ for ^{incident} each particle involved, but ~~let's start off with a 2 → 2 process??~~ Maybe we only need such packets for the initial particles? Let's not make a distinction right now between S , S^{-1} , and T , as operators. Then what is:

$$|\langle p_1 \dots p_n | S | \phi_1 \phi_2 \rangle|^2 = \langle \phi_1 \phi_2 | S^\dagger | p_1 \dots p_n \rangle \langle p_1 \dots p_n | S | \phi_1 \phi_2 \rangle$$

Insert packets such that:

$$|\phi_1 \phi_2\rangle = \int d^3p_1 d^3p_2 \delta^3(1) \delta^3(2) \phi_1(p_1) \phi_2(p_2) |p_1 p_2\rangle$$

$$\langle \phi_1 \phi_2 | = \int d^3p'_1 d^3p'_2 \delta^3(1) \delta^3(2) \phi_1^*(p'_1) \phi_2^*(p'_2) \langle p_1 p_2 |$$

Then:

$$|\langle f, \text{plane} | S | i, \text{wave packets} \rangle|^2 = \int d^3p_1 d^3p_2 d^3p'_1 d^3p'_2 \delta^3(1) \delta^3(2) \delta^3(1') \delta^3(2')$$

$$\times \phi_1(p_1) \phi_2(p_2) \phi_1^*(p'_1) \phi_2^*(p'_2)$$

$$\times \langle p'_1 p'_2 | S^\dagger | q_1 \dots q_n \rangle \langle q_1 \dots q_n | S | p_1 p_2 \rangle$$

I think that this $|\langle S | \rangle|^2$ is the probability into a final plane wave state set $|q_1 \dots q_n\rangle$. Later we shall tack on some kind of phase space.

But first, we want to get T replaced by T_0

So, lets replace: $|p_1 p_2\rangle = |p_m\rangle \otimes |\alpha_{12}\rangle$
 $|q_1 \dots q_n\rangle = |p_n\rangle \otimes |\alpha_{1 \dots n}\rangle$

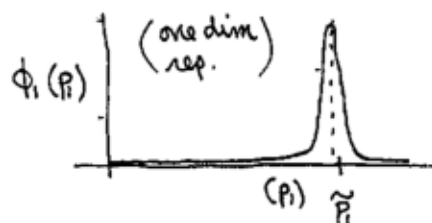
Then we have:

$$\begin{aligned} & \langle p'_1 p'_2 | T | q_1 \dots q_n \rangle \langle q_1 \dots q_n | T | p_1 p_2 \rangle \\ &= \langle p'_1 p'_2 | p_m \rangle \langle \alpha_{12} | T | \alpha_{1 \dots n} \rangle \langle p_n | p_{12} \rangle \langle \alpha_{1 \dots n} | T | \alpha_{12} \rangle \\ &= C_3 \cdot C_3 \cdot \delta^4(p'_1 + p'_2 - q_1 \dots q_n) \delta^4(q_1 \dots q_n - p_1 - p_2) \cdot \langle \alpha_{12} | T | \alpha_{1 \dots n} \rangle \langle \alpha_{1 \dots n} | T | \alpha_{12} \rangle \end{aligned}$$

Now lets "hold" with these α -states. We rewrite:

$$\begin{aligned} |S|^2 &= \int d^4 p_1 d^4 p_2 \delta^4(1) \delta^4(2) \delta^4(p_1 + p_2 - q_1 \dots q_n) \cdot C_3 \times \phi_1(p_1) \phi_2(p_2) \\ &\times \int d^4 p'_1 d^4 p'_2 \delta^4(1') \delta^4(2') \delta^4(p'_1 + p'_2 - q_1 \dots q_n) \cdot C_3 \times \phi_1^*(p'_1) \phi_2^*(p'_2) \\ &\times \langle \alpha_{12} | T | \alpha_{1 \dots n} \rangle \langle \alpha_{1 \dots n} | T | \alpha_{12} \rangle \end{aligned}$$

Now comes an important step. Assume that $\phi_1(p_1)$ has this form:



I.e., $\phi_1(\vec{p}_1)$ maps $\mathbb{R}^3 \rightarrow \mathbb{C}$, and we will allow ϕ_1 to be strongly peaked at $\tilde{p}_1$, the ideal momentum of particle 1.

Similarly for ϕ_2 . Then $|\alpha_{12}\rangle = |\alpha_{12}\rangle = |\tilde{\alpha}_{12}\rangle$, i.e., the state evaluated at $\tilde{p}_1$ and $\tilde{p}_2$, the central values.

This allows us to pull out of the integral, so

$$|S|^2 = |\langle \alpha_{1 \dots n} | T | \alpha_{12} \rangle|^2 \times \int d^4 p_1 d^4 p_2 \dots \times \int d^4 p'_1 d^4 p'_2 \dots$$

Now I leave this as a detail for later: define $\phi_i(x)$ by (4.29) and insert here to get:

$$|S|^2 = |\langle \alpha_{1 \dots n} | T | \alpha_{12} \rangle|^2 \cdot \frac{C_3^2}{(2\pi)^8} \int d^4 x d^4 y e^{-i P(x-y)} \phi_1(x) \phi_2(x) \phi_1^*(y) \phi_2^*(y)$$

At this point, I will shift to $|p\rangle$ states via C_4 , so that

$$|\langle \alpha_{1 \dots n} | T | \alpha_{12} \rangle|^2 = \prod_{\text{all}} \left(\frac{1}{C_4^2} \right) |T_{fi}|^2$$

Then we have:

$$|S|^2 = \prod_{\text{all}} (\bar{C}_4)^2 \cdot \left[\frac{C_3}{(2\pi)^4} \right]^2 \cdot |T_{fi}|^2 \cdot \int d^4x d^4y e^{-iP \cdot (x-y)} \phi_1(x) \phi_2(x) \phi_1^*(y) \phi_2^*(y)$$

Notice that we have succeeded in squaring $|S|$ without having to worry about delta fns!

Now add some phase space:

$$\prod_f \left(\frac{V_f d^3 p_f}{2E_f} \right) \times \underbrace{d^4 p \cdot \delta^4(p - \dots)}_{=1}$$

So we now sum our probability over phase space:

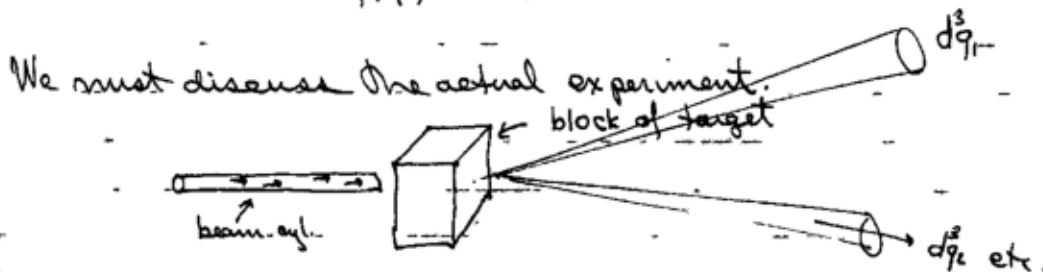
$$\begin{aligned} \int |S|^2 \prod_f \left(\frac{V_f d^3 p_f}{2E_f} \right) &= \prod_{\text{all}} (\bar{C}_4)^2 \cdot \left[\frac{C_3}{(2\pi)^4} \right]^2 \cdot |T_{fi}|^2 \cdot \prod_f \left(\frac{V_f d^3 p_f}{2E_f} \right) \delta^4(p - \dots) \\ &\quad \times \int d^4 p \cdot \int d^4x d^4y e^{-iP \cdot (x-y)} \phi \phi \phi^* \phi^* \\ &= (2\pi)^4 \int d^4x d^4y \delta^4(x-y) \phi \phi \phi^* \phi^* \\ &= (2\pi)^4 \int d^4x |\phi_1(x)|^2 |\phi_2(x)|^2 \end{aligned}$$

So:

$$|S|^2 \prod_f () = \prod_{\text{all}} C_4^2 \cdot C_3^2 \cdot \frac{1}{(2\pi)^4} \cdot |T_{fi}|^2 \cdot \prod_f () \delta^4(\text{ext}) \cdot \int d^4x |\phi_1|^2 |\phi_2|^2$$

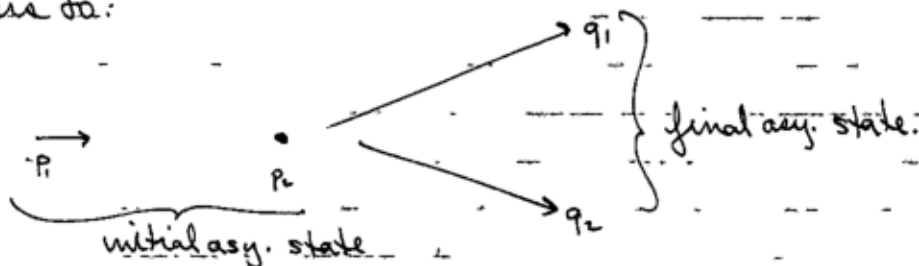
The norms look wrong, but come back later.

③ We still have to interpret $|S|^2 \prod_f ()$. But first, what is $\langle \phi | \phi \rangle$? Well, $|\phi\rangle$ was ...



How does $|S|^2 \prod_f ()$ connect to this picture?!

It refers to:



As I understand it, $|\langle f|S|i\rangle|^2$ phase space is the probability that, if you start out with two particles in an initial asymp. state as shown, you will end up with a final asymp state in the phase space momentum cone.

So, $|\langle f|S|i\rangle|^2 \frac{\Delta^3 q_1}{2E_1 c_1} \frac{\Delta^3 q_2}{2E_2 c_2} \dots =$ a true probability
 $0 \leq \text{prob} \leq 1$

Of course in an exper, you never deal with one beam particle + one target particle, so adjustments must be made.

Q: is $\vec{p}_1$ supposed to be heading right at target? When we write:

$$\langle f|T|i, p_1, p_2\rangle$$

There is no mention of the position of 1 and 2 in initial state.

Aha! But when you use $|p_i\rangle$ for a state, the position is "everywhere with equal probability".

$$\phi_i(x) = \int d^3p \cdot \delta(\vec{p} - \vec{p}_i) \cdot \phi_i(p) \cdot e^{ip \cdot x}$$

$$\text{If } \phi_i(p) = \delta^3(\vec{p} - \vec{p}_i), \text{ then } \phi_i(x) = e^{iEt} \cdot e^{-ip \cdot x}$$

And: $|\phi_i(x)|^2 = 1$, independent of x . This is specifically the situation which we are trying to avoid!

WAVE PACKETS

① MS. suggest:

$$|\phi(p)\rangle = \int d^3p \delta(p^2 - m^2) \phi(p; p) |p\rangle$$

I would like to read this as: $|\phi(\vec{p})\rangle = \int d^3p \phi(\vec{p}; \vec{p}) |\vec{p}\rangle$

$$\text{Let's take } \phi(\vec{p}; \vec{p}) = K e^{-(\vec{p}-\vec{p})^2/\Delta}$$



So Δ characterizes the width of this peak,
K is a normalizer.

What is happening meanwhile in configuration space?
Again, MS suggest:

$$\phi(x, t) = \int d^3p \delta^+(p^2 - m^2) e^{ip \cdot x} \phi(p)$$

Let's insert our ϕ & see what happens.

$$\phi_p(x, t) = \int d^3p \int d^3p' \delta^+(p'^2 - m^2) e^{ip' \cdot x} e^{-ip \cdot x} K e^{-(\vec{p}-\vec{p}')^2/\Delta}$$

$$\int d^3p \delta^+(p^2 - m^2) \dots = \int_0^\infty dp^0 \delta(p_0^2 - \vec{p}^2 - m^2) \text{ etc.}$$

$$\text{As is well known. } \int d^3p \delta^+(p^2 - m^2) = \int \frac{d^3p}{2E}$$

So:

$$\phi(x, t) = \int \frac{d^3p}{2E} e^{iEt} e^{-ip \cdot x} K e^{-(\vec{p}-\vec{p}')^2/\Delta}$$

However, $E = \sqrt{p^2 + m^2}$, so we cannot extract, so:

$$\phi^{(p)}(x, t) = K \int \frac{d^3p}{2\sqrt{p^2 + m^2}} e^{(i\sqrt{p^2 + m^2}t)} e^{-ip \cdot x} e^{-(\vec{p}-\vec{p}')^2/\Delta}$$

OK, let $d^3p = p^2 dp d\phi dz$, let the three vectors be arranged in this way:



$$\text{Then } (\vec{p}-\vec{p}')^2 = p^2 + p'^2 - 2pp' \cos \theta$$

$$\vec{p} \cdot \vec{x} = px \cos \theta$$

$$= -px \cos \theta$$

$$= px (\cos \theta \cos \phi - \sin \theta \sin \phi \cos \phi)$$

So, this wave packet integral clouds over very fast:

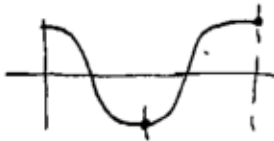
$$\phi = K \cdot \int \frac{p^3 dp d\phi dz}{2\sqrt{p^2+m^2}} \cdot \exp \left[i\sqrt{p^2+m^2}t - i p x \left(\beta z - \beta_3 \sqrt{1-z^2} C(\phi) \right) - \frac{1}{2} (p_1^2 + p_2^2 - 2 p_1 p_2 z) \right]$$

We could use $p^2 dp = p \cdot E dE$, but not much help. Can I do any of these integrals?

a) try $d\phi$: $\int d\phi \in [A + B \cos \phi]$

$$= \int_0^{2\pi} d\phi \in \cdot e^{A + B \cos \phi} = e^A \cdot \int_0^{2\pi} d\phi e^{B \cos \phi}$$

Let $y = \cos \phi$ $dy = -\sin \phi d\phi \Rightarrow d\phi = \frac{dy}{\sqrt{1-y^2}}$



Well, $\int_0^{2\pi} d\phi f(\cos \phi) = 2 \int_0^{\pi} d\phi f(\cos \phi)$, general

So $\phi=0 \Rightarrow y=1$ $\phi=\pi \Rightarrow y=-1$, so:

$$= 2 \cdot \int_1^{-1} \left[\frac{dy}{\sqrt{1-y^2}} \right] \cdot e^A \cdot e^{By} = 2e^A \cdot \int_{-1}^1 \frac{e^{By} dy}{\sqrt{1-y^2}}$$

Perhaps do this by parts:

$$\int_{-1}^1 dy \cdot \frac{1}{\sqrt{1-y^2}} \cdot e^{By} =$$

$$\int \frac{dy}{\sqrt{1-y^2}} = \sin^{-1} y, \text{ no help.}$$

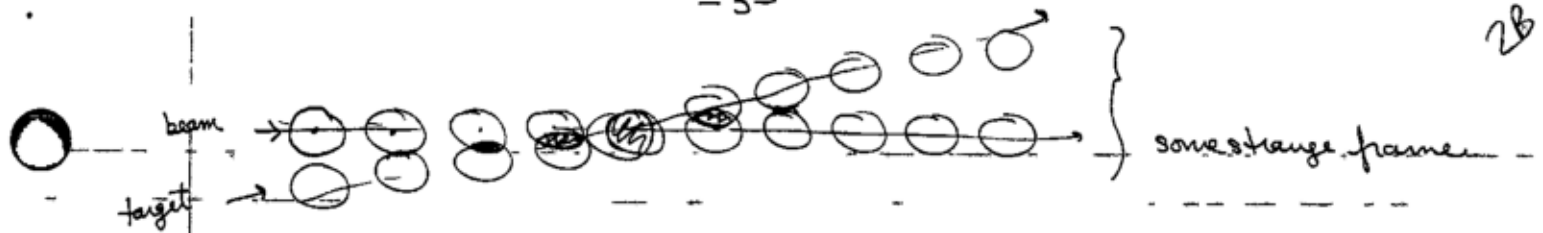
Another idea: X.

② I am confident that $\phi(x,t)$ would be sensible. At $t=0$, this should somehow be a "ball" centered at the origin. As time increases, this "ball" moves in the $\vec{p}_1$ direction and extends its size. This should model the motion of a "free particle".

I should be able to continue the cross section analysis without doing the above integral. Some day, do it.

③ Now, let's do an interpretation of:

$$\int dx |\phi_1(x)|^2 |\phi_2(x)|^2$$



$$\int d^3x |\phi_1(x)|^2 |\phi_2(x)|^2 = \text{in practice}$$

$$\int_{-\pi}^{+\pi} dt \cdot \underbrace{\int d^3x |\phi_1(x,t)|^2 \cdot |\phi_2(x,t)|^2}_{\text{overlap at time } t}$$

We imagine that

$$\int d^3x |\phi_1(x, -T)|^2 = 1$$

$$\int d^3x |\phi_2(x, -T)|^2 = 1$$

This says

"Before interaction, there is
one beam particle, and
one target particle"

So, the probability of producing some final state is proportional to the time integrated overlap. This makes sense to me.

⊕ Now let's normalize all our various states. What is $|\phi\rangle$? It is the linear combination of plane wave kets that makes up the state vector for that particle:

$$\langle p' | \phi \rangle = \int \frac{d^3p}{2E p_2} \phi(p) \underbrace{\langle p' | p \rangle}_{2E p_2 \cdot \delta^3(p' - p)} = \phi(p')$$

$$\text{So: } \langle p | \phi \rangle = \phi(p)$$

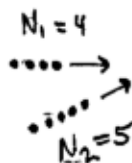
$$\text{Completeness is: } 1 = \int \frac{d^3p}{2E p_2} \cdot |p\rangle \langle p|$$

$$\text{Then: } \langle \phi | \phi \rangle = \int \frac{d^3p}{2E p_2} \cdot |\langle \phi | p \rangle|^2 = \int \frac{d^3p}{2E p_2} \cdot |\phi(p)|^2$$

Let us arbitrarily select the normalization of $|\phi\rangle$ so that

$$\langle \phi | \phi \rangle = N, \quad \text{the number of particles in the beam beam just a the number in the target burst.}$$

Clarify:



Why not use $N=1$.

Let's stick to MS norms for now.

So, scale $|\phi\rangle$ such that $\langle\phi|\phi\rangle = 1$. Then it turns out that, since $\phi(x,t)$ scales with $|\phi\rangle$,

$$\int |\phi(x,t)|^2 d^3x = \frac{(2\pi)^3}{2E} \quad \text{at any time!}$$

So, right here we must be assuming that $|S|^2$ is small enough so that the "beams" are not depleted.

$$\text{So: } \int \left[\frac{2E |\phi(x,t)|^2}{(2\pi)^3} \right] d^3x = 1 = \int g(x,t) d^3x.$$

So, as t changes, the spatial structure of $g(x,t)$ may change, but its integral stays the same. ["packet spreads out"].

Now return to:

$$|S|^2 \prod_f \left(\frac{d^3p_f}{2E_f} \right) = (2\pi)^4 |T_{fi}|^2 \underbrace{\prod_f (-) \delta^4(x_f)}_{dQ} \cdot \int d^3x \underbrace{|\phi_1|^2 |\phi_2|^2}_{\frac{(2\pi)^6}{2E_1 2E_2} g_1 g_2}$$

We have:

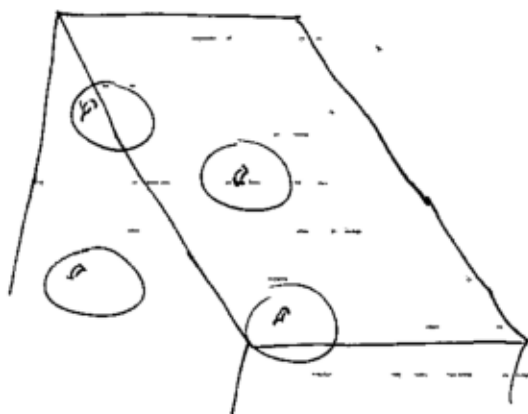
$$d(\text{Prob}) = \frac{(2\pi)^2}{2E_1 2E_2} dQ_f |T_{fi}|^2 \int d^3x g_1(x) g_2(x)$$

⑤ Now, what is "cross section"? Can I interpret g_1, g_2 for stationary target?



OK.

So far, g_2 has been the density function describing the spatial structure of one target particle. An actual target looks like



balls hanging within, far separated.

Why not define a fancier density function which describes the spatial structure of the entire target!



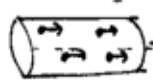
$$\rho_T(\vec{x}, t) \equiv \sum_{i=1}^N \rho_2(\vec{x}, t; \vec{x}_i)$$

↑ origin of i^{th} Ball is $\vec{x}_i$

Of course now $\int d^3x \rho_T(\vec{x}, t) = N_T$ instead of 1.

Usually, we think of $\rho_T(\vec{x}, t)$ as being independent of t .

Similarly, we can describe a beam:



$$\rho_B(\vec{x}, t) \equiv \sum_{i=1}^{N_B} \rho_1(\vec{x}, t; \vec{x}_i)$$

In the lab, $\rho_1(\vec{x}, t; \vec{x}_i)$ does have time dependence because each ball is moving at some velocity near $\vec{p}_i$.

Claim: suppose on a one-shot basis we blast this beam pulse through the target. Then the probability of obtaining a particular reaction configuration should be

$$d(\text{Prob}) = \frac{(2\pi)^2}{2E_1 2E_2} \cdot dQ_f \cdot |T_f|^2 \int d^4x \rho_B(\vec{x}, t) \rho_T(\vec{x}, t)$$

So now it is clear that if your beam misses your target, you should get zero here. This fact is not clear in the plane wave analysis.

Notice that we have done a simple sum here, incoherent!

Experimental meaning of $d(\text{Prob})$: let's set up Δ^3p counters all over. Assume we know $|T_f|^2$ etc etc. Then:

$d(\text{Prob})$ = probability that we get a "ding" in all counters when one pulse is sent through.

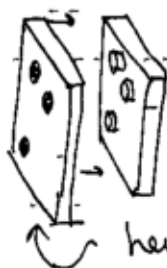
Presumably, if pulses & target have huge # of particles, $\Delta(\text{Prob})$ could be, say 27.36. This means that each counter gets 27 "dings", $\neq 2$, say.

So, let's replace $d(\text{Prob}) \rightarrow dN$ a $\Delta N = \text{number}$ of "events". Strictly, with specific counters,

$$N = \frac{(2\pi)^2}{2E \cdot 2E_L} \cdot \iiint_{\text{counter acceptance}} dQ_f \cdot |T_{fi}|^2 \cdot \int d^3x \cdot \rho_B(x) \rho_T(x)$$

This gives the number of events in an "experiment", i.e., when a beam-burst slams into the target.

⑥ Again, what is "cross section"? Do a target slice:



← how many targets in here?

$$\rho_T(x) dz dA = d^3 N_T$$

↪ here is a piece of the beam-burst cylinder. There are how many beam particles in there?

$$\rho_B(x) dz dA = d^3 N_B$$

Now, what fraction of dA is "covered" by targets?

$$\text{fraction} = \left[\frac{d^3 N_T \cdot d\sigma}{dA} \right]$$

So "fraction" = probability of an "event", per beam particle, during the time that these slabs pass through each other, from only ~~this~~ this target volume, at this time ~~at~~ of duration dt . So:

prob of "event" at (x, t) in target is

$$= \frac{d^3 N_T \cdot d\sigma}{dA} \cdot d^3 N_B = \rho_T(x) \rho_B(x) dz \cdot dz \cdot dA \cdot d\sigma$$

Now replace one dz with $dz = \sqrt{s} dt$ to get

$$= \rho_T(x) \rho_B(x) d^3x dt \cdot \sigma \cdot d\sigma$$

To get the total # events from this one "experiment", integrate over all space at one instant in time,

Then integrate over time. Then:

$$dN = N \cdot d\sigma \cdot \int d^4x g_T(x) g_B(x)$$

In passing, what is "flux"? = "Particle current density" at some place and time.

$$\text{Flux} = \frac{d^3N}{dA \cdot dt} = g_B(x, t) \cdot v_B^{\text{LAB}} = \text{flux}(x, t).$$

In our experiment, at a given point, say, in the target, flux looks like:



But we also have:

$$dN = \frac{(2\pi)^2}{2E_1 \cdot 2E_2} \cdot dQ_f \cdot |T_{fi}|^2 \cdot \int d^4x g_B(x) g_T(x).$$

Comparison then yields:

$$d\sigma = \frac{(2\pi)^2 dQ_f |T_{fi}|^2}{2E_1 \cdot 2E_2 \cdot v_{\text{LAB}}}$$

$$d\sigma = \frac{(2\pi)^2 dQ_f \cdot |T_{fi}|^2}{4 \cdot \text{flux}}$$

But $E_1^{\text{LAB}} m_2 v_{\text{LAB}} = p_1^{\text{LAB}} m_2 = \text{"flux"}$.

So we have derived: (in MS norms).

Now I have several tasks ahead:

- (1) rewrite this analysis
- (2) relate it to "box norm method"

The Box Method AND Fermi's Golden Rule

→ ① If we let $\phi(p)$ approach delta peaking, then

$$\phi(x) = \int \frac{d^3 p}{(2\pi)^3} \phi(p) e^{-i\vec{p}\cdot\vec{x}} = \int \frac{d^3 p}{2\pi^2 c^2}$$

Well, let $\phi(p) = \delta^3(p - \vec{p})$, then $\phi(x) = \delta^3(\vec{p} - m\vec{c}) \cdot \frac{1}{c^2} \cdot e^{-i\vec{p}\cdot\vec{x}}$. Thus, each packet is spread out over all space, so you cannot normalize. So, as usual, we imagine a large box with the "gunk" of each "beam" spread out smoothly.



picture

In this picture, interaction occurs at all points in the box equally, and at all times from, say, $-T/2$ to $+T/2$. We shall assume that V and T are so large that we may retain the continuum normalization without essential error. This means that:

$$2\pi\delta(0) = \int_{-T/2}^{+T/2} dt \, 1 = T \quad (2\pi)^3 \delta^3(0) = V$$

② This situation is so different from the physical wave approach that we may have to "go it alone" for a while. Let's try that:

$$\begin{aligned} \langle q_1 \dots | S^{-1} | p_1 p_2 \rangle &= \langle q_1 \dots | S^{-1} | p_1 p_2 \rangle = i C_1 \langle q_1 \dots | T | p_1 p_2 \rangle \\ &= i C_1 C_3 \delta^4(\text{ext}) \cdot \langle q_1 | T | i_1 \rangle \\ &= i C_1 C_3 \delta^4(\text{ext}) \cdot \prod_{\text{all}} C_4 \cdot T_{fi} \end{aligned}$$

$$|\langle q_1 \dots | S^{-1} | p_1 p_2 \rangle|^2 = |\langle q_1 \dots | T | p_1 p_2 \rangle|^2 = C_3^2 (2\pi)^4 \cdot (2\pi)^4 \delta^4(0) \cdot \delta^4(\text{ext}) \cdot \prod_{\text{all}} C_4^2 \cdot |T_{fi}|^2$$

This is the probability into a single zero measure final state. As we would expect, it is proportional to V and to T ; is proportional to the spacetime volume of the interaction region!

So:

$$\begin{aligned} \frac{|\langle q_1 \dots | T | p_1 p_2 \rangle|^2}{(2\pi)^4 \delta^4(0)} &= \frac{C_3^2 (2\pi)^4 \delta^4(\text{ext}) \prod C_4^2 |T_{fi}|^2}{(2\pi)^4 \delta^4(0)} \\ &= \text{rate per cm}^3 \end{aligned}$$

So, $|\langle q_1 \dots | T | p_1 p_2 \rangle|^2 =$ the total probability in this exper, except for phase space. Thus:

$$\text{Prob.} = C_3^2 (2\pi)^4 \delta^4(\text{ext}) \cdot [(2\pi)^4 \delta^4(0)] \cdot \prod C_4^2 \cdot |T_{fi}|^2 \cdot \prod \left(\frac{d^3 p_f}{2E_f C_2} \right)$$

→ ③ How do we arrive at "cross section"? I think the previous analysis still holds, namely:

$$\text{Prob.} = d\sigma \cdot V_B \cdot \int d^4x g_T(x) g_B(x).$$

cross
section

However, now $g_T(x)$ and $g_B(x)$ are constants over the box. Also, we have already assumed one beam p and one target because, e.g., $\langle p | p \rangle = 2E_C \delta^3(0) \times 1$. So $\leftarrow$ (error!)

$$g_T(x) = g_B(x) = \frac{1}{V} = \frac{1}{(2\pi)^3 \delta^3(0)}$$

This means:

$$\text{Prob.} = d\sigma \cdot V_B \cdot \frac{1}{(2\pi)^3 \delta^3(0)} \cdot \frac{1}{(2\pi)^3 \delta^3(0)} \cdot (2\pi)^4 \delta^4(0) \quad \checkmark$$

④ However, we have made an error back in ③. Recall that for 1 and 2 we used $|p_1\rangle$ and $|p_2\rangle$ states. And,

$$\langle p | p \rangle = 2E_C \delta^3(0)$$

But we want $\langle p | p \rangle = 1$ for one initial particle, so our previous result is "too large" in the sense that $2E_C \delta^3(0) \gg 1$. So we correct for that here:

$$\rightarrow \text{Prob.} = C_3^2 (2\pi)^4 \delta^4(0) \pi C_4^2 \prod_{\neq} \left(\frac{d^3 p_i}{2E_{C_i}} \right) \frac{[(2\pi)^4 \delta^4(0)] |T_f|^2}{2E_1 C_2 \delta^3(0) \cdot 2E_2 C_2 \delta^3(0)}$$

$$\rightarrow \text{Prob.} = d\sigma \cdot V_B \cdot \frac{(2\pi)^2 \cdot \delta^4(0)}{\delta^3(0) \cdot \delta^3(0)}$$

Upon comparison, all $\delta(0)$ factors cancel + we are left with

result $\rightarrow d\sigma = \frac{(2\pi)^2 \cdot C_3 \cdot \delta_a \delta_b \cdot |T_f|^2 \cdot dQ_f}{4 V_B E_1 E_2}$

This is the desired result.

⑤ Conventional derivation:

$$n_1 = \frac{2E_1 C_1 \delta^3(0)}{(2\pi)^3 \delta^3(0)} = \frac{2E_1 C_1}{(2\pi)^3} = \text{"density of beam" in box} = g_1$$

$$n_2 = 2E_2 C_2 / (2\pi)^3 = \text{"target density"} = g_2$$

$$\frac{|S-1|^2}{4\pi} = C_3^2 (2\pi)^4 \delta^4(\text{ext}) \pi C_4^2 |T_{fi}|^2$$

$$\frac{|S-1|^2}{T} = \text{rate} = C_3^2 (2\pi)^4 \delta^4(\text{ext}) \pi C_4^2 |T_{fi}|^2 \cdot V$$

This rate is, once again, "too large". We divide by $V g_2$ to get "rate per target particle". Then we say:

$$d\sigma \cdot \text{flux} = \text{Prob./sec.} \quad \text{flux} = g_1 \cdot v_B$$

$$\text{So:} \quad d\sigma = \frac{\text{Rate}}{g_1 g_2 v_B} = \frac{|S-1|^2}{T} \cdot \frac{1}{g_1 g_2 v_B V} = \frac{|S-1|^2}{4\pi} \cdot \frac{1}{g_1 g_2 v_B}$$

This then is the "conventional" box formula:

$$d\sigma = \frac{|S-1|^2}{4\pi} \cdot \frac{1}{g_1 g_2 v_B} \cdot \pi \left(\frac{d^3 p}{2E_1} \right)$$

If we insert these quantities, we find:

$$\begin{aligned} d\sigma &= \frac{C_3^2 (2\pi)^4 \delta^4(\text{ext}) \pi C_4^2 |T_{fi}|^2 \cdot V}{2E_1 C_1 \cdot 2E_2 C_2 \cdot (2\pi)^{-6} v_B} \\ &= \frac{C_3 (2\pi)^2 dQ_f |T_{fi}|^2}{4 v_B E_1 E_2} \end{aligned}$$

⑥ Note: in Gasnowski, $C_3 = (2\pi)^4$ and $C_4 = 1$, so he has:

$$\frac{|S-1|^2}{4\pi} = (2\pi)^4 \delta^4(\text{ext}) |T_{fi}|^2$$

① Note on "phase space" : I am always using:

$$\left(\frac{d^3p}{2EC_2} \right)$$

Classically, phase space is
 ↳ i.e. in usual QM a WM.

$$\frac{d^3p \cdot d^3r}{(2\pi)^3}$$

you can derive this by going to a box + counting plane wave modes of QM, then letting $V \rightarrow \infty$.

So in principle, we should be using final particle in our box method.

$$\frac{d^3p \cdot V}{(2\pi)^3}$$

for each

But, our states are named so that $\langle p|p \rangle = 2EC_2 \cdot V / (2\pi)^3$

We have to correct for this. We correct for the initial particles separately, + for the final particles via:

$$\frac{d^3p \cdot V}{(2\pi)^3} \frac{1}{2EC_2 V} \rightarrow \left(\frac{d^3p}{2EC_2} \right)$$

This is why we use this for phase space. This same idea holds in the wave packet method where final particles are treated as plane waves.

This is why you see $\prod \frac{1}{g_i}$ in Goss, p.142.

⑧ Notice that Gene Commins uses $C_2 = (2\pi)^3 / 2VE$ so that $\langle p|p \rangle = 1$!

⑨ [With $V \rightarrow \infty$, classical phase space is $d^3p \cdot \delta^3(0)$.] We correct this by $\div 2EC_2 \delta^3(0)$.

⑩ How about redoing analysis with packets in final state also!

⑪ With one particle in the box, flux = $(V\omega_0)$, period

② Still more on "conventional methods". [Use $\neq 1$ norm all the way through, pretending $= 1$, then correct at the end.] ← a nice "box method"

$$\text{Rate} = \frac{|S-1|^2}{T} \cdot \prod_f \left(\frac{d^3 p \cdot V}{(2\pi)^3} \right)$$

$$\text{Rate} = d\sigma \cdot \text{flux}$$

$$\text{flux} = g_i v_B = \frac{v_B}{V}$$

Thus:

$$d\sigma = \frac{|S-1|^2}{T} \cdot \frac{V}{v_B} \cdot \prod_f \left(\frac{d^3 p \cdot V}{(2\pi)^3} \right)$$

Now correct by multiplying RHS by:

$$\prod_{\text{all}} \left(\frac{1}{g_i} \right) = \prod_{\text{all}} \left(\frac{(2\pi)^3}{2E_i C_i V} \right)$$

Result is then:

$$d\sigma = \frac{|S-1|^2}{VT} \cdot \frac{1}{v_B} \cdot \frac{(2\pi)^6}{(2E_1 C_1)(2E_2 C_2)} \cdot \prod_f \left(\frac{d^3 p}{2E C_i} \right)$$

$$\text{Meanwhile, } |S-1| = (2\pi)^4 \delta^4(\text{ext}) |\tilde{T}|$$

$$\text{So: } \frac{|S-1|^2}{VT} = (2\pi)^4 \delta^4(\text{ext}) |\tilde{T}|^2$$

$$\text{Then: } d\sigma = (2\pi)^4 \delta^4(\text{ext}) |\tilde{T}|^2 \cdot \frac{(2\pi)^6}{v_B (2E_1 C_1)(2E_2 C_2)} \cdot \prod_f \left(\frac{d^3 p}{2E C_i} \right)$$

Here I have used $C_4 = 1$ and $C_3 = (2\pi)^4$. For fermions, if we take $C_2 = (2E)^{-1}$, we'd get BD result.

So, general formula might be:

$$g_i = 2E_i C_i \times \frac{V}{(2\pi)^3}$$

box result:

$$\left\{ \begin{aligned} d\sigma &= \underbrace{\frac{|S-1|^2}{T}}_{\text{rate}} \cdot \underbrace{\frac{1}{(v_B/V)}}_{\text{flux}} \cdot \underbrace{\prod_f \left(\frac{d^3 p \cdot V}{(2\pi)^3} \right)}_{\text{phase space}} \cdot \underbrace{\prod_{\text{all}} \left(\frac{1}{g_i} \right)}_{\text{norm correction}} \end{aligned} \right.$$

⑬ How does this relate to "Fermi's Golden Rules"?

This is a subject which usually occurs in NRQM. Eq,

$$i\dot{\psi}_{(s)} = (H_0^{(s)} + H_1^{(s)}) \psi_{(s)}$$

/SE.

normal
q.m.
perturbation
theory:

Go to the "interaction picture" via
 $H_0^{(s)} = H_0^{(z)}$, so: Ansatz:

$$\begin{cases} \psi_{(z)}^{(t)} = e^{iH_0 t} \psi_{(s)} \\ H_{(z)} = e^{iH_0 t} H_{(s)} e^{-iH_0 t} \end{cases}$$

$$\begin{aligned} e^{iH_0 t} \cdot i \frac{\partial}{\partial t} [e^{-iH_0 t} \psi_{(z)}^{(t)}] &= (H_0^{(z)} + H_1^{(z)}) \psi_{(z)}^{(t)} \\ &= e^{-iH_0 t} i \dot{\psi}_{(z)}^{(t)} + H_0^{(z)} \psi_{(z)}^{(t)} \end{aligned}$$

$$\Rightarrow i \dot{\psi}_{(z)}^{(t)} = H_1^{(z)} \psi_{(z)}^{(t)} \quad \leftarrow \text{usual interaction picture result.}$$

Next, one usually goes to the representation of the free Hamiltonian:

$$i\dot{\psi}_n = H_0 \psi_n \Rightarrow \psi = u_n e^{-i\epsilon_n t}$$

$$\text{So in general, } \psi_{(s)} = \sum_n a_n u_n e^{-i\epsilon_n t}$$

where $\{u_n\}$ = set of energy eigenstates of energy $\{\epsilon_n\}$.

$$|\psi_{(s)}\rangle = \sum_n a_n e^{-i\epsilon_n t} |u_n\rangle$$

Now go to interaction picture:

$$|\psi_{(z)}^{(t)}\rangle = \sum_n a_n(t) |u_n\rangle$$

where as we turn on H_1 , $a_n \rightarrow \tilde{a}_n(t)$. So, we can say:

$$i \sum_n \dot{\tilde{a}}_n(t) |u_n\rangle = H_1^{(z)} \sum_n \tilde{a}_n |u_n\rangle$$

$$\Rightarrow i \dot{\tilde{a}}_n |u_n\rangle = a_n H_1^{(z)} |u_n\rangle$$

Now close onto m' + assume $\langle u_{m'} | u_n \rangle = \delta_{m'n} = \langle n' | n \rangle$;

$$i \dot{\tilde{a}}_{m'} = i \dot{\tilde{a}}_n \langle u_{m'} | u_n \rangle =$$

Go back to:

$$i \sum_n \dot{a}_n(t) |u_n\rangle = \sum_n a_n H'(I) |u_n\rangle$$

$$\sum_{n'} |u_{n'}\rangle \langle u_{n'}|$$

So:

$$i \sum_n \dot{a}_n(t) |n\rangle = \sum_{n'} a_n |n'\rangle \langle n' | H' | n \rangle$$

Now close with $\langle n'' |$ to get:

$$i \dot{a}_{n''}(t) = \sum_n a_n \langle n'' | H' | n \rangle$$

So:

$$i \dot{a}_k(t) = \sum_n a_n \langle k | H' | n \rangle$$

/ see 35.5, Schiff

However, usually one uses $H'(s)$ in base $\uparrow$, so:

$$\langle k | H'(I) | n \rangle = e^{iE_k t} e^{-iE_n t} \langle k | H'(s) | n \rangle$$

Then:

$$\left[i \dot{a}_k(t) = \sum_n a_n(t) \cdot \langle k | H'(s) | n \rangle \cdot e^{i(E_k - E_n)t} \right]$$

Next comes the usual iterative solution:

usual results

$$a_n(t) = a_n^{(0)} + \lambda a_n^{(1)} + \lambda^2 a_n^{(2)} + \dots$$

If we use $a_n^{(0)} = \langle \text{stop} \rangle$.Without our perturbation, $\lambda=0$, assume system is in eigenstate $|m\rangle$. Then

$$\psi(s) = \sum_m e^{iE_m t} a_m |m\rangle$$

So $a_m^{(0)} = 1$, all other $a_n^{(0)} = 0$. Thus, $a_i^{(0)} = \langle m | i \rangle$ ✓Then $\dot{a}_k(t) \approx \dot{a}_k^{(1)}(t)$, so:

$$i \dot{a}_k^{(1)}(t) = a_m^{(0)} \langle k | H' | m \rangle e^{i\omega_{kt} t}$$

$$\text{Thus: } \left[a_k^{(1)}(t) = -i \int dt \langle k | H' | m \rangle e^{i(E_k - E_m)t} \right]$$

where $|m\rangle$ was the initial state.

Safar, then,

$$a_k(t) \approx -i \int dt \langle k | H'(t) | m \rangle e^{i(E_k - E_m)t}$$

Now suppose you assume $H'(t) \sim H'(0) e^{\pm i\omega t}$, and impose an "adiabatic" type cutoff whatever. Then.

$$a_k(t) = -i (2\pi) \delta(E_k - E_m \pm \omega) \cdot \langle k | H'(0) | m \rangle$$

$$|a_k(t)|^2 = T \cdot 2\pi \delta(E_k - E_m \pm \omega) \cdot |\langle k | H'(0) | m \rangle|^2$$

This leads directly to

$$\text{rate} = d\omega = \frac{|a_k(t)|^2}{T} = 2\pi \delta(\text{energy}) |\langle k | H'(0) | m \rangle|^2$$

into a final state. We have assumed unit norm for initial state,

$$W = 2\pi \delta(\text{energy}) |\langle f | H'(0) | i \rangle|^2 \cdot \mathcal{G}(E_f)$$

↑ phase space.

This is how one usually treats radiation in NRQM. The EM field is a classical perturbation, not quantized at all. The "atom" is first quantized, NR, - schrodinger.

→ Why the $2\pi \delta(\text{en})$?

(a) it comes out of this S.E. analysis

(b) there is only one particle in final state. Recall

$$\sum_x \text{ has } (2\pi) \delta(\text{en})$$

$$\int \text{ has } (2\pi)^4 \delta^4(\text{ext}) \cdot \left[\frac{d^3p}{(2\pi)^3} \right]$$

} → same

↑ hidden phase space.

4

Comments on the use of Wave Packets in the Final State too.

1. When we put wave packets in the final state as well as the initial state, we find that, assuming proper unit normalization of all distributed states, the probability for the reaction to take place is:

$$P = |\langle \psi_f | T | \psi_i \rangle|^2 / |I|^2$$

It turns out that I is simply an integral of the products of all the wavefunctions over space and time. (Note that C_3 has to do with the scale of the alpha states only).

2. Here are some things to note:

- a) unlike the case where packets are used only for the in state, the I we get here contains no exponentials.
- b) in this full-wave case, you do not add any "phase space". The final state is completely preset by your selection of wavefunctions.

3. How do we interpret an interaction? Again, two "balls" come in and start to overlap. During the time they overlap, final state is generated. I.e., there is generated a certain probability for n "balls" to be gradually created in the interaction volume. These balls then exit along their well-defined classical trajectories and end up in various detectors. As before, "final ball probability" is generated in all regions of overlap at any time instant. The total probability is obtained by integrating over all spacetime. In practice, this integration is restricted to the fiducial volume and the experiment duration.

We deal with strongly peaked wavefunctions so that the T-matrix element can be computed using the classical plane wave momenta. Thus, all the information about such things as whether the beam hits the target, or whether some final particle hits a detector, all this information is contained in the master overlap integral I .

As usual, a single experiment cannot tell you anything. You have to repeat the experiment lots of times so that the concept of "final ball probability" gets replaced with "average number of dings in the detector set".

4. In practice, if you wanted to compute this "number of dings", you could do one of two equivalent things. (1) you could work with a "standard" final state wavepacket, say much smaller than the detector. Then in computation you would sum over all possible final wavepackets which hit the area of your detector. (2) You could "custom design" a final wavepacket that exactly covers your detector, and just use it. Of course we are assuming the detector is small so the above average T approx is OK.

5. Actually, the use of the "final state wave packets picture" is not very useful experimentally. It is easiest to use the plane wave "idealized" packets and integrate them over your detector areas. I.e., it is best to show the "phase space" and then integrate as needed over whatever observation areas or detectors you are using. In effect, this is the "custom wavefunction" method described above, but we never have to construct the complete wavefunction. In effect we only construct it at one instant in time (i.e., at the detector) and we say where it came from.

6. On the other hand, for an understanding of a "one shot experiment", the "final state wave packets picture" is the only way to go. It says "there is a certain specific probability for the final particles to exit in those specified wave packets as opposed to some other set of wave packets (ie, some other star). In every experiment, some thing always happens, so some probability was realized into an "event". This leads us to wonder a little about Many Worlds vs Copenhagen, but now is not the time or place.

7. Although we can experimentally dispense with these final state wave packets, the initial state packets are indispensable. The theorist is able to generate a cross section for comparison with experiment, but the experimenter does not "measure" cross section directly. He "measures" the number of dings in his various detectors for a 3 hour run of some specified beam.

When I say that the initial wave packets are indispensable, that needs some qualification. No experimenter has actually had full knowledge of "the spatial wavefunction" in all its complexity, ie, its spreading properties, etc. He only needs to know the spatial shape (density function) of the packet near the time of interaction. During the interaction, he would assume that no change of shape occurs, the bursts just propagate like rigid bodies.

So, in a crossed beams experiment, one needs very precise knowledge of the burst density functions, including knowledge of "where the bursts are and when". It is not easy to know exactly where your bursts are all the time, much less to know their shape. That is why there is so much huffah about "knowing your luminosity".

In fact, at SLAC one really gives up on calculating the luminosity directly in this way. You end up computing it indirectly by simultaneously measuring Bhabha rates. Since you know from the theory the Bhabha cross section, you can deduce the luminosity from the Bhabha rate.

Calc. with all-packets.

① Define:

$$I = \prod_{i=1}^{n+2} \int \frac{d^4 p_i \delta^+(p_i^2 - m_i^2)}{C_2} \cdot \phi_i(p_i) \cdot C_3 \cdot \delta^4(p_1 + p_2 - p_3 - \dots - p_{n+2})$$

Later we shall see the significance of I ; now we just calculate it.

② Inject F.T. : $\delta^+(p_i^2 - m_i^2) \phi_i(p_i) = \int \frac{C_2}{(2\pi)^4} d^4 x_i \phi_i(p_i) e^{-i p_i x_i}$

Thus:

$$I = \prod_{i=1}^{n+2} \left(\int \frac{d^4 p_i d^4 x_i}{(2\pi)^4} \cdot \phi_i(x_i) \cdot e^{-i p_i x_i} \right) \cdot C_3 \cdot \delta^4(p_1 + p_2 - p_3 - \dots)$$

Extract the $d^4 p_i$ and $d^4 x_i$ integrals:

$$I = \prod_{i=2}^{n+2} \left(\int \frac{d^4 p_i d^4 x_i}{(2\pi)^4} \cdot \phi_i(x_i) e^{-i p_i x_i} \right) \cdot \underbrace{\int \frac{d^4 p_1 d^4 x_1}{(2\pi)^4} \phi_1(x_1) e^{-i p_1 x_1} C_3 \delta^4(\quad)}_{= \int \frac{d^4 x_1}{(2\pi)^4} \phi_1(x_1) e^{i(-p_2 + p_3 + \dots)x_1} C_3}$$

So:

$$\begin{aligned} I &= \frac{C_3}{(2\pi)^4} \int d^4 x_1 \phi_1(x_1) \cdot \left[\int \frac{d^4 p_2 d^4 x_2}{(2\pi)^4} \phi_2(x_2) e^{i p_2 (x_2 - x_1)} \right] \\ &\quad \times \prod_{i=3}^{n+2} \left(\int \frac{d^4 p_i d^4 x_i}{(2\pi)^4} \cdot \phi_i(x_i) e^{i p_i (x_i + x_1)} \right) \\ &= \frac{C_3}{(2\pi)^4} \int d^4 x_1 \phi_1(x_1) \cdot [\phi_2(x_1)] \cdot \prod_{i=3}^{n+2} \left(\phi_i(-x_1) \right) \end{aligned}$$

③ Later we show that $\phi_i(p) = \psi_i^*(p)$ for $i=3, 4, \dots, n+2$.

$$\begin{aligned} \text{Thus: } \phi_i(-x) &= \int \frac{d^4 p \delta^+}{C_2} \cdot \phi_i(p) e^{-i p \cdot x} \\ &= - \left[\int \frac{d^4 p \delta^+}{C_2} \cdot \psi_i(p) e^{+i p \cdot x} \right]^* \\ &= - [\psi_i(x)]^* \\ &= \psi_i^*(x) \end{aligned}$$

Thus, reverting to the $q_1 \dots q_n$ labelling scheme:

$$[I = \frac{C_3}{(2\pi)^4} \int d^4x \phi_1(x) \phi_2(x) \bar{\psi}_1^*(x) \bar{\psi}_2^*(x) \dots \psi_n^*(x)]$$

④ What is I ?

$$\langle \psi_1 \psi_2 \dots \psi_n | S-1 | \phi_1 \phi_2 \rangle =$$

$$\prod_{i=1,2} \int \frac{d^4p_i \delta^4(\cdot)}{C_2} \phi_i(p_i) \prod_{j=1+n} \int \frac{d^4q_j \delta^4(\cdot)}{C_2} \psi_j^*(q_j)$$

$$\times \langle q_1 \dots q_n | T | p_1 p_2 \rangle i \Omega_i$$

$$= C_3 \delta^4(\dots) \langle \alpha_s | T | \alpha_i \rangle$$

Now, if we relabel and define $\phi_i(p) = \psi_j^*(p)$, then also if we make the "strong peaking assumption", we get

$$\langle \psi_1 \psi_2 \dots \psi_n | T | \phi_1 \phi_2 \rangle = \langle \alpha_s | T | \alpha_i \rangle \cdot I$$

where I is defined in ①. That is why we wanted to know I .

⑤ Then, the probability for the reaction is:

$$[|T|^2 = |P|^2 \cdot |I|^2]$$

⑥ However, we have not yet mentioned the norm of the ϕ 's. If we follow MS+ say

$$\langle \phi | \phi \rangle = 1 = \int \frac{d^4p \delta^4(\cdot)}{C_2} |\phi(p)|^2$$

$$\langle \psi | \psi \rangle = 1$$

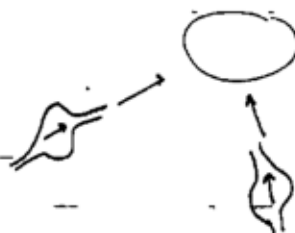
Then I think we are OK, and $|T|^2$ is the true probability.

The Experimenter vs. The Beam

① His problem, for a single-burst experiment, is to calculate:

$$\int d^3x g_B(\vec{x}, t) g_T(\vec{x}, t)$$

In a "lab" experiment, we could imagine a messy target ~ even in a non lab:



Here are two "bursts" heading for double.

I think we can assume that the physical shape of a burst, as "measured" just prior to its entry into the "interaction region" stays constant during the collision. What this means is

$$g(\vec{x}, t) = f(\vec{x} - \vec{v}t)$$

Thus, we only need the "shape" at one time:

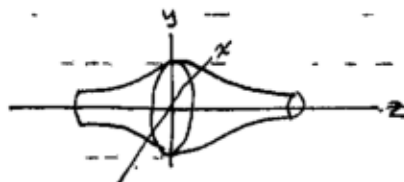
$$f(\vec{x}) = \text{[sketch of a Gaussian-like shape]} = K \cdot e^{-\vec{x} \cdot \vec{\Delta}}, \text{ eg.}$$

Here, the shape is completely specified by vector $\vec{\Delta}$, some of whose components vanish.

② I am not satisfied with this rep. of the packet. Define:

$$f(x, y, z) = e^{-(x^2+y^2)/\Delta r} \cdot e^{-z^2/\Delta z}$$

This defines a density fcn
one of whose "contours" is:



How do we make it move? $z \rightarrow (z - v_z t)$ makes it move in the z direction at velocity v_z .

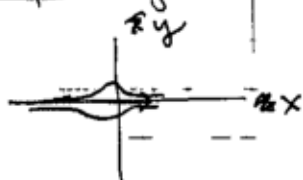
How do we make it move in an arbitrary direction?



answer:

$$g(\vec{r}, x, y, z) = f(\vec{R}^{-1}(\vec{r})\vec{r}) = f(x, y, z)$$

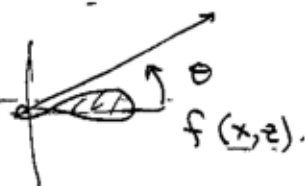
Example: go to two dimensions



$$f(x, y) = e^{-y^2/\Delta y} e^{-x^2/\Delta x}$$

$$f(x, y, t) = e^{-y^2/\Delta y} e^{-(x-vt)^2/\Delta x}$$

Think of that familiar atomic physics contour:
How do we make it point up at θ ?



$$f'(\vec{r}') = f(\vec{r}) \quad \vec{r}' = R_{act}(\theta)\vec{r}, \text{ so}$$

$$f'(\vec{r}') = f(R_{act}^{-1}(\theta)\vec{r}')$$

Also: $f'(\vec{r}') = f(R_{act}^{-1}(\theta)\vec{r}')$

So, in our example: $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ +\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = R_{act}(\theta) \begin{pmatrix} x \\ y \end{pmatrix}$

$$x' = \cos\theta x - \sin\theta y$$

$$y' = +\sin\theta x + \cos\theta y$$

Eg, $\vec{r} = (1, 0) \quad \vec{r}' = (\cos\theta, \sin\theta)$

So, $f'(x, y) = e^{-[\sin\theta x + \cos\theta y]^2/\Delta y} e^{-[\cos\theta x - \sin\theta y - vt]^2/\Delta x}$

Here is a packet travelling in $\hat{\theta}$ direction ✓

$$f'(x, y, t) = f(x', y', t) \quad \text{where} \quad \begin{pmatrix} x' \\ y' \end{pmatrix} = R_{act}^{-1} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$= e^{-y'^2/\Delta y} e^{-(x'-vt)^2/\Delta x}$$

So, back to 3dim, let

$$f(x, y, z; t) = e^{-(x^2+y^2)/\Delta r^2} \cdot e^{-z^2/\Delta z^2} \cdot (z - vt)^2$$

example of a "burst"

Then

$$g(\theta, \phi; x, y, z; t) = f(\bar{R}_{\text{ad}}(\theta, \phi) \vec{r}; t)$$

Now your packet is going "out" at velocity v .

Then:

$$h_{\vec{n}}^{\text{in}}(\vec{r}; t) = f(\bar{R}_{\text{ad}}(\theta, \phi) \vec{r}; -t)$$

incoming initial state 'packet'

This is a packet heading into the origin at v .

$$h_{\vec{n}}^{\text{in}, v}(\vec{r}; t) = h(x^{\mu}; \text{in}, \vec{v})$$



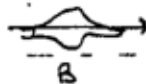
At $t=0$ it is at the origin. So, I am suggesting for a crossed-beam situation:

$$\int d^4x g_B(x^{\mu}) g_T(x^{\mu}) = \int d^4x h_1(x; \text{in}, \vec{v}_1) h_2(x; \text{in}, \vec{v}_2)$$

Of course most of the integration comes near $t=0$; but here is the explicit way that an experimenter would compute this important integral.

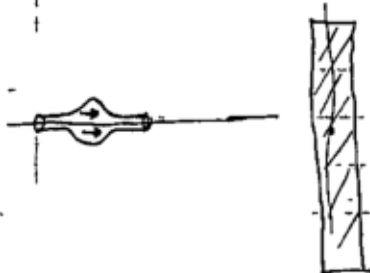
③ Now, do "target at rest". Then:

$$h_2 \rightarrow h_T(\vec{r}; \vec{0})$$



Still, you have to do an integration, so "shape" of both is important.

④ Suppose target is uniform and has a cross section large compared to the beams: (and a plate geometry.)

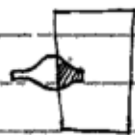


Then, for all practical purposes may write:

$$h_T(\vec{r}) = \text{const: } \begin{aligned} &\in (x; -\frac{b}{2}, +\frac{b}{2}) \\ &\in (y; -\frac{w_1}{2}, +\frac{w_1}{2}) \\ &\in (z; -\frac{w_2}{2}, +\frac{w_2}{2}) \end{aligned}$$

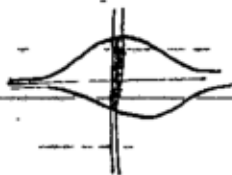
Then we have:

$$\int d^3x g_B g_T = g_T \int dt \cdot \int_{\text{plate}} d^3x \cdot h_1(x, t; \vec{v})$$



$$= g_T \cdot \int dt \cdot \int_0^L dz \cdot \int dx dy h_1(x, t; \vec{v})$$

$$= g_T \cdot \int_0^L dz \cdot \left[\int dt \cdot \int dx dy h_1(x, t; \vec{v}) \right]$$



And what!

$$\int dt \int dz f(z - vt)$$

We know that, eq, $dz = -v dt$, so

$$\int dt \cdot \int dx dy h_1 = \frac{1}{v} \int_{\text{all}} dx dy dz h_1 = \frac{1}{v} \cdot N_B$$

Then,

$$\left[\int d^3x g_B g_T = g_T \cdot L \cdot N_B \cdot \frac{1}{v_B} \right] \leftarrow \text{big target; result is independent of burst shape!!!}$$

What is the flux here?

$$\text{flux} = v_B \cdot g_B(x, t) \text{ , not a constant in space or time}$$

Conclusion: This integral, $\int d^3x g_B g_T$, is larger when you send the packet through slower. Recall that,

$$\text{Prob.} = d\sigma \cdot v_B \cdot \int d^3x f_T(x) g_B(x)$$

$$= d\sigma \cdot g_T \cdot L \cdot N_B \leftarrow$$

The # counts, or Prob, is independent of how fast you go thru!

So: in this case there is no calculation to be done except you need to ascertain N_B . (use a current probe).

If $f = \# \text{ bursts per sec}$, then:

$$\text{count rate} = \int d\sigma \cdot g_T \cdot L \cdot N_B \cdot f \leftarrow$$

Of course $N\sigma f = \text{total beam "current"}$, i.e.

$$i = q \cdot N\sigma \cdot f \leftarrow$$

So we find, for a target that is

(a) of large transverse size rel. to beam

(b) of uniform density ρ_T and length L :

absolute count rate. $\rightarrow \text{Rate} = (\rho_T \cdot L) \times (i/q) \times \int d\Omega$

(ideally, use only $\int d\Omega$ over counters).

Example: in a OCS exper, $d\sigma \sim dR$, so Rate $\propto \int dR = \text{your solid angle}$.

Note: $\rho_T \cdot L$ is the "gm/cm²" quantity that "they" always talk about. Eg, if you have 2 meters of hydrogen, then

$$\rho_T \cdot L = .07 \times 200 = 14 \text{ gm/cm}^2$$

The mass in a 1cm² area "tube" of length L .

Martin's approach to cross section

1. Use wave packets for the two initial particles. Thus:

$$\text{amp} = \int d^4 p_\mu d^4 q_\mu \bar{\phi}_1(p_\mu) \bar{\phi}_2(q_\mu) \langle p_1 p_2 \dots; \lambda | T | p q; \bar{\lambda} \rangle$$

where the states are in the usual invariant norm. Next, extract the momentum piece to get

$$\text{amp} = \langle \alpha_f | T_0 | \alpha_i \rangle \int d^4 p_\mu d^4 q_\mu \bar{\phi}_1(p_\mu) \bar{\phi}_2(q_\mu) \delta^4(p_\mu - p - q_\mu)$$

where the "narrow packet approx" has been used to extract $\langle 1 |$ from the integral. Next, introduce $\phi_1(x^\mu)$, $\phi_2(y^\mu)$ as F.T.'s and square to get:

$$|\text{amp}|^2 = |\langle \alpha_f | T_0 | \alpha_i \rangle|^2 \times (2\pi)^{-8} \times$$

$$\int d^4 x^\mu d^4 y^\mu e^{-i p_\mu (x-y)} \phi_1(x) \phi_2(x) \phi_1^*(y) \phi_2^*(y)$$

Next, put in the phase space stuff:

$$\prod_{\text{final}} \left(\frac{d^3 p_f}{2E_f} \right) \equiv d^4 P dQ$$

$$dQ = \prod_{\text{final}} \left(\frac{d^3 p_f}{2E_f} \right) \delta^4(p_1 + p_2 + \dots p_n - P)$$

Now do this:

$$\int d^4 p_\mu |\text{amp}|^2 = |\langle \alpha_f | T_0 | \alpha_i \rangle|^2 (2\pi)^{-4} \int d^4 x^\mu |\phi_1(x^\mu)|^2 |\phi_2(x^\mu)|^2$$

Now by examining the definition of $\phi_1(p)$, we see that $|\phi_1(x)|^2$ is related to the particle density:

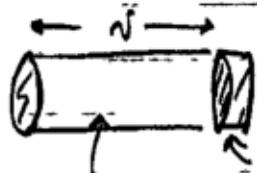
$$|\phi(x^\mu)|^2 = \frac{(2\pi)^2}{2E} \rho(x^\mu)$$

Inserting these densities we get

$$dP = |\langle \alpha | T | \alpha_i \rangle|^2 dQ \cdot \frac{(2\pi)^2}{2\bar{p}_0 \cdot 2\bar{q}_0} \int d^4x \rho_1(x) \rho_2(x) \quad (4.33)$$

Think about this! ρ_1 is definitely time dependent, ρ_2 is more or less fixed (the target). There is no fudging here! Keep in mind the meaning of $|\langle \alpha |$ states!!

Now, onto cross section: consider a piece of the target and a piece of the beam at time t :



$$n_{\text{targets}} = \rho_2(x) d^3x$$

$$\text{flux} = \frac{\text{pulse}}{\text{area}} = \frac{\rho_1(x) v}{1 \text{ sec}} = \rho_1 v$$

Thus:

$$\frac{dn_f}{dt} = [\rho_2(x) d^3x] \times \sigma_f \times [\rho_1 v]$$

$$\frac{dN_f}{dt} = \sigma_f v \int d^3x \rho_1(x) \rho_2(x)$$

Now integrate over time to get

$$N_f = v \sigma_f \int d^4x \rho_1(x) \rho_2(x)$$

Here N_f is the total count. Now we defined our initial states so that n = total particles in beam pulse, or in target. Thus, dP is the count of events (into a special sector of phase space). Hence;

$$dP = N_f = v \sigma_f \int \dots = |\langle \alpha | \dots \rangle|^2 \dots dQ$$

$$\frac{d\sigma}{dQ} = (2\pi)^2 v^{-1} (2\bar{p}_0 \cdot 2\bar{q}_0)^{-1} |\langle \alpha | T | \alpha_i \rangle|^2$$

This is the big result!

The invariant form is easy to get if $\vec{p}_1 \times \vec{p}_2 = 0$. It is:

$$\frac{d\sigma}{dQ} = \frac{\pi^2}{\sqrt{(p_1 p_2)^2 - m_1^2 m_2^2}} |\langle \alpha_f | T | \alpha_i \rangle|^2$$

In the CM frame where $W =$ total CM energy ("mass")

$$\left. \frac{d\sigma}{dQ} \right|_{\text{cms}} = \frac{\pi^2}{pW} |\langle \alpha_f | T | \alpha_i \rangle|^2$$

Now, try to describe the connection between T and M .
Compare:

STATE DEF.

$$\langle \vec{p}_1 \vec{p}_2 | \lambda | p_1 p_2 \rangle = (2\pi)^6 \delta(\sim) \delta(\sim) \delta$$

STATE DEF.

$$\langle \vec{p}_1 \vec{p}_2 | \lambda | p_1 p_2 \rangle = 2p_1^0 2p_2^0 \delta(\sim) \delta(\sim) \delta$$

DEF OF M_{fi}

$$\langle \vec{p}_1 \vec{p}_2 | S | p_1 p_2 \rangle = \langle \sim | \sim \rangle = 1 - i (2\pi)^4 \delta^4(\sim) \times \frac{1}{\sqrt{2E_1 \sqrt{2E_2}}} M_{fi}$$

DEF OF $\langle \alpha_f | T | \alpha_i \rangle$

$$\langle \vec{p}_1 \vec{p}_2 | S | p_1 p_2 \rangle = \langle \vec{p}_1 \vec{p}_2 | \lambda | p_1 p_2 \rangle + i \delta^4(\sim) \langle \alpha_f | T | \alpha_i \rangle$$

These are the definitions. How do we get the exact connection? If we said assume $2p_i^0 = (2\pi)^3$ for all particles, then the states would be the same "size" and we could compare. In general,

$$\frac{-i (2\pi)^4 M_{fi}}{(\sqrt{(2\pi)^3})^N \text{ total \# particles}} = +i \langle \alpha_f | T | \alpha_i \rangle$$

So in the case of $2 \rightarrow 2$, we get

$$-(2\pi)^{-2} M_{fi} = \langle \alpha_f | T | \alpha_i \rangle.$$