

Abers & Lee
Gauge Theories
1973

Jeremy Bernstein
1974

Part 1 of 2

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GAUGE THEORIES

Ernest S. ABERS and Benjamin W. LEE

*Institute for Theoretical Physics, State University of New York,
Stony Brook, N.Y. 11790, USA*



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Part I Abers and Lee notes:

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- Addon: notes from my own talk on this subject given in 1975. → next section

These notes are all original from Arjai-Lucht session in 1975.
Reviewed 3.11.78 after rereading Commins' book.

Introduction

① A quick history of the subject: The old but proven V-A theory of fermi is non renormalizable. Now, ~~we~~ we at last have renormalizable theories. They are all based on spontaneously broken gauge symmetry (S.S.B.)

The original work was done separately in 1968 by Steven Weinberg + Salam (the famous "Weinberg's Theory") The set (W^+, γ, W^-) of vector bosons is like the set b^i in Yang Mills; the dissimilarity between the weak + em is due to S.S.B.

Since 1964, many people have worked on this S.S.B. idea (Higgs). The usual phrases are here: degenerate vacuum, Goldstone bosons, gauge bosons.

But there was still a cloud over the renormalizability of a S.S.B.G.T., though it looked like it might go. So (we can summarize Weinberg/Salam by saying)

- a) they unified weak and em
- b) they suggested overall Renormalizability.

So things rested until 1971 when 't Hooft and others showed that yes, Yang Mills with S.S.B. was renormalizable. Then came the gauge theory explosion.

These late models predict some weird things:

- ✓ a) new heavy vector mesons
- ✓ b) heavy leptons
- ✓ c) weak neutral currents.

Also, we can now calculate higher order corrections, and even ~~em~~ mass differences!

The intro closes with a long outline. Several phrases are:

- a) Landau gauge
- b) Fadeev + Popov
- c) the R_ξ gauges.

1. Gauge invariance of classical field theories

① The first part of this section is almost exactly parallel to my own 230C paper. The $U(1)$ example of charge phase transformation is a symmetry of $\mathcal{L}$, so $\delta\mathcal{L}=0$. This leads to a conserved "current" and a conserved "charge" Q . Next, they do $SU(2)$ isospin example where things are now matrices. Again $\delta\mathcal{L}=0$. Finally, they generalize to any group. So far, they have only dealt with global, a "first kind" gauge transformation.

② Now comes the "local" a second kind transformation (Hey, I thought that was my own terminology...) If you start with a $\mathcal{L}$ where $\delta\mathcal{L}=0$ globally, then $\delta\mathcal{L}\neq 0$ locally because you pick up $\partial_\mu\theta(x)$ terms. But, by adding a gauge boson A_μ (i.e., photon), you can reinstate that $\delta\mathcal{L}=0$ locally. The gauge boson will transform as $\delta A^\mu = -\partial^\mu\theta(x)$. This is just as in Class. Elec-Dyn where you have gauge freedom to choose $A^\mu(x)$ up to a derivative. So in this $U(1)$ example, we could say that local gauge symmetry implies existence of a gauge boson, the photon. And as a bonus, we have $m_\gamma=0$! Very nice.

③ Next, they take this local idea into the non-abelian groups. They do $SU(2)$ Yang Mills, but at once generalize to any "gauge group" of the Lie type (as in Glashow + Gellmann). The prescription is: group G

- let $U(\theta) = \exp(-i\vec{L}\cdot\vec{\theta})$ act on a "plet" ϕ . Here there are N generators $\{\vec{L}_i\}$ and N "gauge functions" $\theta_i(x)$.
- start with an $\mathcal{L}(\phi) \Rightarrow \delta\mathcal{L}=0$ under $U(\theta)$ globally.
- make $\delta\mathcal{L}=0$ locally by adding N gauge bosons via the covariant derivative $D_\mu \Rightarrow (\partial_\mu - ig\vec{L}\cdot\vec{A}_\mu(x))$.
- insisting that $\delta\mathcal{L}=0$ locally puts a gauge condition on these gauge bosons: $\delta\vec{A}_\mu \equiv \partial_\mu\vec{\theta} + \vec{\theta} \times \vec{A}_\mu$ which involves only the structure constants.
- The K.E. term $\mathcal{L}_B$ these gauge bosons is formed then from $\vec{F}_{\mu\nu} = (\partial_\mu\vec{A}_\nu - \partial_\nu\vec{A}_\mu + g\vec{A}_\mu \times \vec{A}_\nu)$. A little bit messier than the $U(1)$ photon case where no change was needed in $\vec{F}_{\mu\nu}$. That's it!

If all the ϵ_i are small, then

$$e^{(i\vec{\epsilon} \cdot \vec{E}/\hbar)} \approx 1 + (i\vec{\epsilon} \cdot \vec{E})/\hbar$$

This is a matrix, so $\approx \vec{S} \cdot \vec{u} + (i\vec{\epsilon} \cdot \vec{E})/\hbar$

Apply to column vector $\vec{\phi} = (v+n)\vec{S} \cdot \vec{u}$:

First term gives: $(v+n)\vec{S} \cdot \vec{u} = (v+n)\vec{S} \cdot \vec{u}$

So whatever second term gives, add $(v+n)$ to bottom.

Second term gives:

$$(v+n)f(i\vec{\epsilon} \cdot \vec{E})/\hbar = \frac{v}{v+n}(i\vec{\epsilon} \cdot \vec{E})/\hbar = \frac{v}{v+n}i\vec{\epsilon} \cdot \vec{E}/\hbar$$

$$= \frac{v}{v+n}i\vec{\epsilon} \cdot (-i)[\vec{S} \cdot \vec{u} - \vec{S} \cdot \vec{u}]$$

$$= \frac{v}{v+n}i\vec{\epsilon} \cdot (-i)[\vec{S} \cdot \vec{u} - \vec{S} \cdot \vec{u}]$$

But: only average to $n-1$, so $\vec{S} \cdot \vec{u} = 0$. Thus $\sum \vec{S} \cdot \vec{u} = \vec{S} \cdot \vec{u}$

$$\approx \frac{v}{v+n} \cdot \vec{S} \cdot \vec{u} \approx \vec{S} \cdot \vec{u}$$

So, if n and $\vec{S} \cdot \vec{u}$ are small, get:

same old
full

$$\begin{bmatrix} \vec{S} \cdot \vec{u} \\ \vdots \\ \vec{S} \cdot \vec{u} \end{bmatrix} = \vec{\phi}$$

Why?

$$\langle \phi_m | = \langle n+v | = \langle v | \Rightarrow \langle n | = 0$$

[OK, all those new labels $\epsilon_1, \epsilon_2, \dots, \epsilon_{n-1}, n$! have gone!]

Fast Review of Gauge Theory:

with N generators.

- ① Choose a gauge group. Fit all particles into multiplets of that group. Write an invariant Lagrangian. (globally invar.)

$$\phi_i \rightarrow e^{-i \vec{L} \cdot \vec{\Theta}} \phi_i \quad \rightarrow \quad \mathcal{L} \rightarrow \mathcal{L}$$

- ② What happens if you make $\vec{\Theta} = \vec{\Theta}(x)$. Then

$$\phi_i \rightarrow e^{-i \vec{L} \cdot \vec{\Theta}} \phi_i \quad \text{but}$$

$$\partial_\mu \phi_i \rightarrow \partial_\mu \phi_i + [\text{piece proportional to } \partial_\mu \vec{\Theta}]$$

Now $\delta \mathcal{L} \neq 0$

- ③ Cure this by replacing $\partial_\mu \phi_i$ with $(\partial_\mu - i g \vec{L} \cdot \vec{A}_\mu) \phi_i$.
This adds N vector "gauge" bosons to the theory.
In order that $\delta \mathcal{L} = 0$, these must transform under G as:

$$\vec{A}_\mu \cdot \vec{L} \rightarrow x$$

$$\vec{A}_\mu \rightarrow \vec{A}_\mu + c_{ijk} \Theta^j \vec{A}_\mu^k - \frac{1}{g} \partial_\mu \vec{\Theta}^i$$

- ④ Now add A_μ terms to $\mathcal{L}$. No mass term because:

$$m A_\mu A^\mu \rightarrow m A_\mu A^\mu + \text{extra} \Rightarrow \delta \mathcal{L} \neq 0.$$

So we have N massless gauge vector mesons.
Add $\vec{F}_{\mu\nu} \cdot \vec{F}^{\mu\nu}$

Gauge Invariance

- ① Start with a Lie Group with N generators and an algebra:

$$[L_i, L_j] = C_{ijk} L_k$$

- ② Fit particles into multiplets. In general, $\{\phi_i\}$

- ③ Write a $\mathcal{L} \Rightarrow \delta\mathcal{L} = 0$ under:

$$\phi_i \rightarrow e^{i(L \otimes k)_{ij}} \phi_j$$

where Θ_k are N parameters (constants).

$$\Rightarrow \delta\phi_i = i \Theta_k \vec{L}_{ij} \phi_j$$

$$\delta(\partial_\mu \phi_i) = i \vec{\Theta} \cdot \vec{L}_{ij} (\partial_\mu \phi_j)$$

- ④ Now use local parameters $\Theta_k(x)$. Then

$$\delta\phi_i = i \vec{\Theta} \cdot \vec{L}_{ij} \phi_j \quad \text{but}$$

$$\delta(\partial_\mu \phi_i) = i \vec{\Theta} \cdot \vec{L}_{ij} (\partial_\mu \phi_j) + \text{extra piece}$$

$$\delta\mathcal{L} \neq 0.$$

- ⑤ Now couple in N $A_\mu^i = \vec{A}_\mu$

$$\partial_\mu \phi_i \rightarrow \partial_\mu \phi_i - ig \vec{L} \cdot \vec{A}_\mu \phi_i$$

Now if $\delta\mathcal{L} = 0$, can we say as before:

$$\delta A_\mu = C_{ijk} \Theta_j A_\mu^k - \frac{1}{g} \partial_\mu \Theta^i$$

QED

- ① Group is $U(1)$, $N=1$. Algebra is:

$$[Q, Q] = 0$$

(abelian group!)

- ② All reps of abelian group are 1dim, so all particles are charge singlets.

- ③ Try $\mathcal{L} = \bar{\psi}(i\gamma - m)\psi$.

$$\psi \rightarrow e^{iQ\Lambda} \psi$$

↑
just a
phase.

$$e^{iQ\Lambda} \psi = e^{iQ\Lambda} \psi$$

$$\text{then } \delta\mathcal{L} = 0.$$

- ④ $\delta\psi = i\Lambda \otimes \psi$

$$\delta(\partial_\mu \psi) = i\Lambda \otimes (\partial_\mu \psi)$$

$$+ i \psi \otimes (\partial_\mu \Lambda)$$

$$\delta\mathcal{L} \neq 0.$$

- ⑤ $N=1$, so couple in A_μ :

$$\partial_\mu \rightarrow \partial_\mu - ig \otimes A_\mu$$

$$\bar{\psi}(i\gamma - m)\psi$$

$$\rightarrow \text{same} + g \bar{\psi} \gamma_\mu \psi A_\mu$$

$$g = e!$$

History

1964 Higgs

1968 Weinberg + Salam

1954 Yang Mills

1971 't Hooft

Wade

Noether

global g.i.

local g.i.

$M_A = 0$

2. Spontaneously broken symmetries.

- ① The main idea here is that nature may choose the physical vacuum from a set of potential candidate vacua in the same way that a ferromagnetic domain chooses a direction for magnetization. These directions are all "equal energy", and just so, the various vacuums all minimize the "potential".
- ② My favorite example is $G = O(2)$, a sym of $\mathcal{L}$. If the potential is right, you find a ring of minima. Once you choose a place for the vacuum, say $(\phi_1, \phi_2) = (0, R)$, then the sym of the vacuum vector is $O(1)$, the zero parameter nothing trivial group. Here, $\sigma(G) = N = 1$, $\sigma(S) = \sigma(O(1)) = 0 = M$, so $N - M = 1$, there will be one Goldstone boson.
- ③ In general, if you start with a group G $\sigma(G) = N$, and if the spontaneously chosen vacuum has $\sigma(S) = M$, then the theory (when all the fields are mass-diagonalized) will have $N - M$ massless bosons. This is the Goldstone Theorem.
- ④ Usually when you "break" a symmetry, this means $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1$ and $\delta \mathcal{L}_1 \neq 0$. Here, $\delta \mathcal{L} = 0$ for the full symmetry group all the time. It is the vacuum field vector that fails to be invariant under group G , so the vacuum breaks the symmetry.
- ⑤ But, recall later comment that the "new" Lagrangian of new fields has 0-vac, but only sym under S , not G . The sym S was moved from the vac. to the $\mathcal{L}$.

The Goldstone Theorem (notes on proof)

① Start with $\mathcal{L} = \frac{1}{2} \partial_\mu \vec{\phi} \cdot \partial^\mu \vec{\phi} - V(\vec{\phi})$ where $\delta V = 0$ under some group G .

② $\vec{\phi}' = U(\vec{\theta}) \vec{\phi} = e^{-i \vec{\theta} \cdot \vec{L}} \vec{\phi} \Rightarrow \delta \vec{\phi} = -i (\vec{\theta} \cdot \vec{L}) \vec{\phi}$. This is how the field moves under an element $U(\vec{\theta})$ of G .

③ Invariance of $V \Rightarrow 0 = \delta V = \frac{\partial V}{\partial \phi_i} \delta \phi_i = \frac{\partial V}{\partial \phi_i} [-i \theta^\alpha L_{ij}^\alpha \phi_j]$.
This means

$$\frac{\partial V}{\partial \phi_i} L_{ij}^\alpha \phi_j = 0 \quad \forall \alpha \quad \text{so } N \text{ equations.}$$

④ Differentiate this once:

$$\frac{\partial^2 V}{\partial \phi_i \partial \phi_j} L_{ij}^\alpha \phi_j + \frac{\partial V}{\partial \phi_i} L_{ij}^\alpha \frac{\partial \phi_j}{\partial \phi_k} = 0 \quad \text{where } \frac{\partial \phi_j}{\partial \phi_k} = \delta_{jk}$$

$$\Rightarrow \frac{\partial^2 V}{\partial \phi_i \partial \phi_j} L_{ij}^\alpha \phi_j + \frac{\partial V}{\partial \phi_i} L_{ik}^\alpha = 0$$

⑤ There is presumably some vector $\vec{v}$ such that $V(\vec{v}) = \text{minimum of } V(\vec{\phi})$. This is your vacuum. This means

$$\frac{\partial V}{\partial \phi_i} \Big|_{\phi_i = v_i} = 0$$

⑥ So, from 4 we conclude that $\frac{\partial^2 V}{\partial \phi_i \partial \phi_k} \Big|_{\phi=v} L_{ij}^\alpha \phi_j = 0 \quad \forall k, \alpha$

This is a "somewhat" result.

⑦ Now expand V around v : Let $\vec{\epsilon} = \vec{\phi} - \vec{v}$.

$$V(\vec{\phi}) = V(\vec{v}) + \vec{\epsilon}_i \cdot \frac{\partial V}{\partial \phi_i} \Big|_{\phi=v} + \frac{1}{2} \epsilon_i \epsilon_j \frac{\partial^2 V}{\partial \phi_i \partial \phi_j} \Big|_{\phi=v} + \dots$$

Observe that these ϵ_i are new fields, so identify this term as the mass term:

$$\epsilon_i \epsilon_j \frac{\partial^2 V}{\partial \phi_i \partial \phi_j} \Big|_{\phi=v} = -\frac{1}{2} (M^2)_{ij} \epsilon_i \epsilon_j$$

⑧ Naively it looks like $(M^2)_{ij} = 0 \quad \forall i, j$, but be careful:

$$(M^2)_{ij} = \frac{\partial^2 V}{\partial \phi_i \partial \phi_j} \Big|_{\phi=v} \quad \text{or} \quad (M^2)_{ij} L_{ij}^\alpha \vec{v}_k = 0.$$

(more Goldstone theorem)

- ⑨ So far we have $V = -\frac{1}{2}(M^2)_{ij} \phi_i \phi_j + \text{const.} + \text{higher}$
and we have $(M^2)_{ij} L_{jk}^\alpha \phi_k = 0 \quad \forall \alpha$.

- ⑩ Let S be the subgroup of G that is still a sym of the vac. [recall that $G = O(n)$, $S = O(n-1)$ in last example.] Now $\vec{v}$ is that value of $\vec{\phi}$ that minimizes V .
 $\vec{v} = \vec{\phi}(\text{vacuum})$.

To say that " S leaves vac. invariant" is to say:

$$[e^{i\alpha L^\alpha}] \vec{v} = \vec{v} \quad \text{when } L^\alpha \in S$$

Thus, $L^\alpha \in S \Rightarrow L^\alpha \vec{v} = 0$. In this case (for these α) you cannot conclude that $m^2 = 0$, so if M is the order of S , N of G (before $N = n$, $M = n-1$), then $m^2 = 0$ only for the $N-M$ fields.

$$L^\alpha \notin S \text{ but } \in G \Rightarrow \underbrace{(M^2)}_{=0} \cdot (\vec{L}^\alpha \vec{v}) = 0 \cdot (\vec{L}^\alpha \vec{v}) = 0.$$

result. $\rightarrow$ { The mass matrix has $N-M$ zero eigenvalues. Thus, }
{ There are $N-M$ massless bosons in the theory. }

- ⑪ These massless dudes are the Goldstone bosons.

- ⑫ Now define $A^{\alpha\beta} = (L^\alpha \vec{v}, L^\beta \vec{v})$ for $\alpha, \beta \in G-S$ so that $L^\alpha \vec{v} \neq 0$. Then easy to show that $A^{\alpha\beta} = A^{\beta\alpha}$.
Diagonalize using O :

$$\tilde{A} = O \tilde{A} O^T$$

$$\begin{aligned} \tilde{A}_{\alpha\beta} &= O_{\alpha\gamma} \tilde{A}_{\gamma\delta} O_{\delta\beta}^T = O_{\alpha\gamma} (L^\gamma \vec{v}, L^\delta \vec{v}) O_{\delta\beta}^T \\ &= (O_{\alpha\gamma} L^\gamma \vec{v}, O_{\delta\beta} L^\delta \vec{v}) \Rightarrow \tilde{A}_{\alpha\alpha} = (Q_\alpha, Q_\alpha) \geq 0. \end{aligned}$$

This means that $\tilde{A}$ has $N-M$ non-zero diagonal elements, so this means you have a full $N-M$ dim space spanned by the $L^\alpha \vec{v}$ in $G-S$.

3. The Higgs mechanism

- ① We start off with a little example with $\phi = \text{Re}\phi + i\text{Im}\phi$ which is just like $\sigma + i\pi$. The gauge group is abelian $U(1)$, with $N=1$ parameter. We choose $\mu^2 < 0$ so there is a degenerate vacuum + we choose $\langle \phi \rangle_{\text{vac}} = \text{real} \neq 0$. That is, $\langle \text{Re}\phi \rangle = v$ $\langle \text{Im}\phi \rangle = 0$. These fields are recast into ξ and η . The new vacuum has $M=0$, so we would expect 1 Goldstone boson.
- ② But, because we got the gauge bosons A_μ into $\mathcal{L}$ at the start, to achieve L.G.I. $\delta\mathcal{L} = 0$, it is no longer clear that we do have simple Goldstone boson. In other words, before "gauging in" our A_μ 's, we did have a massless Goldstone ξ . But now A_μ complicates things.
- ③ The advantage of having a L.G.I. Lagrangian means we are free to transform our fields ϕ_i with any local gauge function(s) we wish. Now in this example, if we choose $\phi(x) = \xi(x)/v$ and look at the same Lagrangian in terms of the transformed fields, the Goldstone boson has blown down. In fact, $\phi' =$ only a real part, so there is no ξ . Meanwhile, the gauge boson A_μ acquires a mass.
- ④ The next example is the (non-abelian) $SU(2)$. Start with (ϕ_1, ϕ_2, ϕ_3) , but assume "ring" or "sphere" for vacuum degeneracy. Let $\langle \phi \rangle = (0, 0, v)$ along $\hat{\phi}_3$ "direction". So, $G = SU(2)$ with $N=3$, and now $S = U(1)$ with $M=1$. If there were no gauge bosons, we would expect 2 Goldstone bosons.
- ⑤ Now couple in your A_μ 's to achieve local G invariance. Now $\phi = (\phi_i) + (\xi_i, \xi_i, \eta)$, but if you write $\mathcal{L}$, you'll see masses & fields mixed up. So now choose your $2 = N-M$ gauge functions = ξ_1 and ξ_2 , and gauge these away. Then A_1 and A_2 pick up mass.
- ⑥ When you transform to the "gauge" where the Goldstones are gone, you have gone to the so-called V -gauge (one without mixed bilinear terms — all bilinears are mass terms).

The General Higgs Technique

- ① Start off with n scalar fields which are a representation of group G of order N . (need $n \geq [N-M]+1$)
- ② Assume the "vacuum vector" has the lower symmetry S of order M , where of course S is subgroup of G . Thus, if L^α is in S , then $L^\alpha v = 0$.
- ③ Divide your parametrization of the ϕ vector into two parts.

$$\phi = \begin{pmatrix} [i \frac{x_i + i y_i}{\sqrt{2}}] \\ \vdots \\ \begin{matrix} \text{sum only} \\ \text{over those} \\ L^\alpha \notin S \end{matrix} \left\{ \begin{matrix} 0 \\ \vdots \\ v_1 + \eta_1 \\ v_2 + \eta_2 \end{matrix} \right\} \end{pmatrix} \left. \begin{matrix} \left\{ \begin{matrix} 0 \\ \vdots \\ v_1 + \eta_1 \\ v_2 + \eta_2 \end{matrix} \right\} \right\} \begin{matrix} N-M \text{ of these} \\ n - [N-M] \text{ of these} \end{matrix} \end{matrix} \right\} n \geq [N-M]+1$$

There are $N-M$ ξ 's; $n - [N-M]$ v 's.

Remember that the L are $n \times n$ matrices.

- ④ If no gauge stuff, the Goldstone Theorem would say you now have $N-M$ massless goldstones, the ξ :

- ⑤ But, now do a special gauge xform on ϕ .

$$\phi' = e^{-i \xi L^\alpha / v} \phi = \begin{pmatrix} 0 \\ \vdots \\ v_1 + \eta_1 \\ v_2 + \eta_2 \end{pmatrix} \leftarrow \text{now the } \xi \text{ do not even appear here.}$$

- ⑥ This also means that we must change A to:

$$\vec{L} \cdot \vec{A}' = (0 \dots) [\dots] = (3.17).$$

Let's hunt for bilinear ξ terms in $\mathcal{L}$. Such terms must occur w — no, they clearly do not occur!!

$$\mathcal{L}' = \mathcal{L} = \frac{1}{2} [(\partial_\mu - ig' \vec{L}^\alpha A_\mu^\alpha) \phi' \cdot (\partial_\mu - ig' \vec{L}^\alpha A_\mu^\alpha) \phi']$$

because now the ξ appear nowhere.

- ⑦ What are the A masses now? Originally in $\mathcal{L}$ you had a "mass term"

$$-\frac{1}{2} [g^\alpha \vec{L}^\alpha \phi \cdot g^\beta \vec{L}^\beta \phi] A_\mu^\alpha A_\mu^\beta = -\frac{1}{2} g^\alpha g^\beta [L^\alpha \vec{\phi} \cdot L^\beta \vec{\phi}] A_\mu^\alpha A_\mu^\beta \quad \text{over!}$$

Now, in the original of this is not even a mass term. It is a quantum coupling.

After the gauge activity however you have

$$g' g^p [L^a \phi' \cdot L^p \phi'] A_\mu A_\mu$$

But now, $\phi' = \nu + \eta$ where $\eta = \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix}$ and $\eta = \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix}$ and $\eta = \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix}$ the η piece is still a quantum coupling, but now look at the piece

$$g' g^p [L^a \nu \cdot L^p \nu] A_\mu A_\mu$$

8 For $L^a \nu \cdot L^p \nu$ in the set S , $L^a \nu = 0$ and you still have massless neutrinos.

9 For α and β both in the set $G-S$ (the ghostive space), we have a mass matrix.

$$M^{\alpha\beta} = g' g^p [L^a \nu \cdot L^p \nu] \quad (\text{no sum})$$

Clearly $M^{\alpha\beta}$ is symmetric.

theorem: a symmetric matrix can always be completely diagonalized by an orthogonal transformation.

10 So do this diagonalization:

$$M^{\alpha\beta} = O^{\alpha\alpha'} M^{\alpha'p} O^{\beta\beta'} = g' g^p O^{\alpha\alpha'} [L^{a'} \nu \cdot L^{p'} \nu] O^{\beta\beta'}$$

$$= g' g^p [O^{\alpha\alpha'} L^{a'} \nu \cdot O^{\beta\beta'} L^{p'} \nu] = [O^{\alpha\alpha'} L^{a'} \nu \cdot O^{\beta\beta'} L^{p'} \nu]$$

The diagonal elements look like $[y \cdot y] = \sum y_i^2$. If everything is real, then diag elements are pos def.

11 Thus, the diagonalized gauge boson system will have

- a) M: massive gauge
- b) N-M: massive gauge
- c) no ghostlines.

4. Weak interaction phenomenology

① Some basic facts

- a) lepton number is conserved
- b) muon number is separately conserved
- c) $m_e \approx \frac{2}{3} \times m_\mu$
- d) data consistent with massless neutrinos.

② A massless particle of spin $\frac{1}{2}$ need not, by ordinary Lorentz invariance, occur in both helicities. But a massive one must occur in both. Since no one has ever seen a right handed neutrino, this is good evidence for $m_\nu = 0$.

③ If mass = 0, then $\psi' = -\gamma_5 \psi$ is a sym of the free neutrino Lagrangian. In any interaction term, if ψ always appears as $(1 \pm \gamma_5) \psi$, then that symmetry is preserved. Note that this is "full" parity violation in the Dirac form.

④ Of course $(1 \pm \gamma_5) \psi$ are the helicity states and we just said in (2) that only one of these need appear. The one that occurs on the earth at least is $(1 - \gamma_5) \psi = \psi_{\text{left}}$.

⑤ Since this $\psi \rightarrow -\gamma_5 \psi$ sym is good even for the full fields, one can show that to all orders, no mass appears in the neutrino propagator.

5. Mass weak phenomenology

- ① The usual photon Lagrangian is written down $(G/\sqrt{2}) J_\mu J_\mu^\dagger$, then current J_μ is broken into lepton piece J_μ^l plus a hadron piece J_μ^h . We borrow the lepton piece explicitly and it is entirely left handed.

- ② One can think of (γe) as "self-hand-isospin" 2-spin, same for the muon. Their compact way to write J_μ means J_μ^l for nucleon Lagrangian we wrote $\bar{\psi} \gamma_\mu \psi$, maybe as J_μ^l there, we come up with a "left-hand triplet" of currents
- $$J_\mu^l(x) = \frac{i}{2} [\chi_\mu^\dagger(e) \gamma_\mu \gamma_5 \chi_\mu(e) + \chi_\mu^\dagger(\mu) \gamma_\mu \gamma_5 \chi_\mu(\mu)]$$

Then one says that the weak leptonic current is J_μ^l . And of course J_μ^l will be J_μ^l . Question is: where is J_μ^l ? Does there exist a "neutral leptonic current"?

- ③ Then on the SU(2) symmetry. If there were a "neutral current" it would look like:

$$(J_\mu^3)^l = \chi_\mu^\dagger(e) \gamma_\mu \gamma_5 \chi_\mu(e) + \dots = \bar{\chi}_\mu^e \gamma_\mu \gamma_5 \chi_\mu^e - \bar{\chi}_\mu^{\nu} \gamma_\mu \gamma_5 \chi_\mu^{\nu} + \dots$$

Now, you see why it is "also neutral".

This is an open question.

- ④ Now look at the weak hadronic current. Break into vector and axial w. h. current. Now as before write each of these pieces using γ_5 isospin and γ_5 . Then:

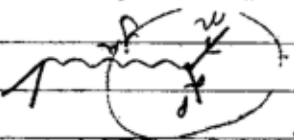
$$h_\mu^3 = v_\mu^3 + a_\mu^3$$

$$v_\mu^3 = \bar{\chi}_\mu^1 - i \bar{\chi}_\mu^2 \gamma_5 \chi_\mu^1 \approx \bar{\chi}_\mu^1$$

$$a_\mu^3 = \bar{\chi}_\mu^1 - i \bar{\chi}_\mu^2 \gamma_5 \chi_\mu^1 = \bar{\chi}_\mu^1$$

So we see clearly that v_μ^3 is the usual isospin current a_μ^3 is some kind of γ_5 -isospin current.

- ⑤ Now, since the $SU(2)_L$ is the weak current, we know that $J_\mu^3 = 0$ for strong interactions (conserved at the order γ_5).



(more weak phenomenology)

(2)

- ⑥ We must notice that j_1^3 is the usual nucleon electric current, with $g_e = 1$ one could say. If there is $SU(2)$ invariance here, then $j_1^\pm$ must also couple with the same g (up to possible $\sqrt{2}$). Then $g_1 = 1$ one would say.
- ⑦ So $g_1 = 1$ was based on isospin symmetry. For g_8 , there is not a similar sym. Turns out $g_8 = 1.24$.
- ⑧ Even with rad. corrections to pin G very closely from muon decay, still it seems that $g_8 = .97$ rather than true 1. Seems that CVC is a bit broken.
- ⑨ For strange decays, we find the old rules
- $\Delta S \leq 1$
 - $\Delta S \neq +\Delta Q$
 - $\lambda^{(1)}$ transforms as $T = 1/2$ (since $\Delta T_3 = \pm 1/2$ always).

⑩ It appears reasonable to say: $h_\mu^{(1)} = (V_\mu^4 - iV_\mu^5) - (A_\mu^1 - iA_\mu^2)$
So the $\Delta S = 1$ current is in the same $SU(3)$ octet.

⑪ Cabibbo showed that if $h_\mu = A h_\mu^0 + B h_\mu^1$, then data say $A^2 + B^2 = 1$. Thus:

$$h_\mu = \cos\theta_c h_\mu^0 + \sin\theta_c h_\mu^1 = e^{2i\theta_c F_8} h_\mu^0 e^{-2i\theta_c F_8}$$

It is this slight $SU(3)$ rotation that allows strangeness changing processes to occur. "A tilt."

⑫ Again, we have $V_1 \pm iV_2 = V^\pm$ currents, $V_4 \pm iV_5 = V^0$ currents. Are there T^0 and V^0 currents? (u-spin?)

6. A grab bag section including PCAC

① The unitarity contradiction: clearly, any σ will be $\sim G^2$. Since "point interaction", this is all s-wave, there is no t-dependence. Thus, dim. analysis says $\sigma \sim G^2$ since $d(G^2) = G d^4 = G d^4$. But, s-wave has bound $\sigma \sim 1/s$ as we well know. Thus, something is for sure not right.

② A specific example of 1. Consider $\bar{e} \bar{\nu}_e \rightarrow \bar{e} \bar{\nu}_e$ elastic. Direct compute gives $\bar{\sigma} = G^2 s / 3\pi$, but unitarity says $\bar{\sigma} \leq 24\pi/s$ (including helicity). The computed $\bar{\sigma}$ crashes thru this bound at $s = 2.5 \times 10^6 \text{ GeV}^2$. By the way, here $j=1$ because $S=1$ because $\mu_a = \mu_b = 1$.

③ Another example: $\nu_\mu \bar{e} \rightarrow \nu_\mu \bar{\nu}_e$. Here, $l=j=s=0$. Compute $\bar{\sigma} = G^2 s / \pi$, but bound is $\bar{\sigma} \leq 2\pi/s$, so here violation is at $s = 4 \times 10^5$.

④ The usual trick is to assume a massive vector boson is in there + comes out at large s . [Even so, you are still nonrenormalizable because higher graphs mess up due to $k_\mu k_\nu$]. I do not claim to understand why this is. Not clear that unitarity is rescued by W , says Gene.

⑤ Chirality invariance. We saw earlier that the absence of "neutral currents" blocked $SU(2)_L$ invariance of the theory. No such currents could still be there. Recall that in $SU(2)_L$, there are 3 generators corresponding to the "charges" associated with the 3 left-handed currents $\bar{\psi} \gamma^\mu (1-\gamma_5) \psi$. They call these charges T_L^i . Similarly, construct T_R^i .

⑥ Notice that when you add, you get the usual isospin with no γ_5 , so $T_L^i + T_R^i = T^i$. When you subtract, you get the "gamma-5 isospin" like $\bar{\psi} \gamma^\mu \gamma_5 \psi$. (γ^μ). Thus, you can construct an algebra. The sum + diff algebra is:

chiral

$$[T_L^i, T_L^j] = i\epsilon^{ijk} T_L^k$$

$$\alpha \quad V \otimes V \rightarrow V$$

$$[T_L^i, T_R^j] = i\epsilon^{ijk} T_S^k$$

$$\alpha \quad V \otimes A \rightarrow A$$

$$[T_S^i, T_S^j] = i\epsilon^{ijk} T^k$$

$$\alpha \quad A \otimes A \rightarrow V$$

This is the current algebra in its 'weak' form

→ ⑦ Somehow, the idea is that if the V and A stuff were both conserved (CVC + CAC), you would have exact chiral + a spont. s. b. G stones would → some goldstones. Now with PCAC, you

(6, con'd)

get almost-goldstones - the pions. This point is not dwelt on here.

⑧ Pion decay constant. Because the "strongest" in pion decay conserves parity $\pi \rightarrow \gamma \gamma$ or $\pi \rightarrow \ell \bar{\nu}$. Then only the A_μ part of the hadronic enters and crudely:

$$\langle 0 | A_\mu^i(x) | \pi^j \rangle \sim F_\pi \cdot P_\mu$$

⑨ From $\pi \rightarrow \ell \bar{\nu}$ decay, find that $F_\pi = 93 \text{ MeV}$.

⑩ Now comes PCAC and Goldberger-Treiman. See separate sheet.

⑪ Next PCAC example is the "Soft Pion Theorem". Using field theory + PCAC in the form $\langle A_\mu \rangle = F_\pi q_\mu \langle j_\pi \rangle + \text{CAC}$ + current algebra + CVC, they show that near threshold one expects

$$\xrightarrow{\text{square}} T_{\pi\pi} \approx \frac{1}{8\pi^2 F_\pi^2} p \cdot q \vec{T}_\pi \cdot \vec{q}$$

Corrections aside, it seems to work!

⑫ Next, "Adler's g_π -sum rule" relating g_π to $p \pi p$ cross sections.

⑬ Next a photon example using PCAC in which results fail. Anomalous Ward Identities?

PCAC etc

measured at 93 MeV.

① Consider pion decay:

$$\langle 0 | A^\mu | \pi \rangle = F_\pi q^\mu$$

$$\Rightarrow \langle 0 | \partial_\mu A^\mu | \pi \rangle = F_\pi m_\pi^2 = F_\pi q^2$$

and no pion decay on shell!

If m_π were 0, we would have CAC. From now on, assume $m_\pi = 0$ but write q^2 for general off shell case. The above equation is consistent with:

$$\langle 0 | A^\mu | \pi \rangle = F_\pi \langle 0 | \phi_\pi | \pi \rangle q^\mu = \frac{F_\pi q^\mu}{q^2} \langle 0 | j_\pi | \pi \rangle$$

$$\langle 0 | \partial_\mu A^\mu | \pi \rangle = F_\pi \langle 0 | \phi_\pi | \pi \rangle q^2 = F_\pi \langle 0 | j_\pi | \pi \rangle$$

② Now conjecture that these results are true for any states, even when no pions in sight. i.e. conjecture that the pion pole shows up in all $\langle b | A^\mu | a \rangle$ elements:

$$\langle b | A^\mu | a \rangle = \frac{F_\pi q^\mu}{q^2} \langle b | j_\pi | a \rangle + \text{analytic.}$$

q^2 : pion pole, $m_\pi = 0$.

I would expect this to be true only near the pole, $q^2 = 0$.

③ Now in particular look at $\langle p | A^\mu | n \rangle$. Remember that a and b are hadron states and that A^μ is the axial part of the hadronic weak current. Thus

$$\langle p | A^\mu | n \rangle = F_\pi \frac{q^\mu}{q^2} \langle p | j_\pi | n \rangle = \frac{F_\pi q^\mu}{q^2} g_{\pi nn}(q^2) \times \bar{u} \gamma_5 \not{q} u$$

④ But we also know that in general:

$$\langle p | A^\mu | n \rangle = \frac{1}{2} g_A(q^2) \bar{u} \gamma^\mu \gamma_5 u + \frac{1}{2} g_P(q^2) \bar{u} q^\mu \gamma_5 u$$

So take ∂_μ on both and "go to the pole" to get:

$$F_\pi \cdot g_{\pi nn}(0) \bar{u} \gamma_5 u = \frac{1}{2} g_A(0) \cdot 2M \bar{u} \gamma_5 u$$

$$\Rightarrow \boxed{F_\pi = \frac{M g_A(0)}{g_{\pi nn}(0)}} \approx \frac{938 \times 1.24}{13.5} \approx 86 \text{ MeV.}$$

← gottberger trieman -

7 The Weinberg-Salam Model

① First action: determine the guts of the model, the next section will interpret

② New way to look at spontaneous sym-breaking: You start with a ϕ

that does have perfect sym of some group G , but a heavy that

neutralizes that leaves some scalars whose vacuum is degenerate.

The group sym G prevents certain types of terms from appearing in ϕ .

Maybe the sym of the vacuum is S . Now, when you redefine

your scalars to achieve an invariant vacuum, the new

ϕ breaks G and respects only S . Any term in the original

that contained one of these scalars, now is changed.

③ Now in this particular example, the group G will be a gauge group. May

construct a globally gauge invariant ϕ using these objects.

$L =$ "lepton containing" ℓ and e_L ; $R =$ "neutrino containing" ν_R and e_R

$\phi = R \otimes L + L \otimes L$ = lepton kinetics

The gauge group is $SU(2) \otimes U(1)$. whose generators are the T_i and Y

Call these "isospin" & "hypercharge". Clearly L is an iso-

scalar and is hyper-invariant, eg. A lepton mass term

would be $L \otimes L$ in violation of the group. Note that

this is not true hypercharge + true isospin = parallel only

and R both have definite $Q = T_3 + \frac{1}{2}Y$

④ Next, add some gauge bosons to achieve local gauge symmetry

Consider each subgroup separately

$R \otimes L \rightarrow R \otimes L \rightarrow R \otimes L$ / no way to add isoscalar A

$L \otimes L \rightarrow L \otimes L \rightarrow L \otimes L$ / can add A

You could show explicitly that you do have local sym provided

the $A+B$ go accordingly. Also, add gauge, massless

kinetic process

⑤ Next, include some "Higgs scalars". Start with

$\phi =$ isospinor with ϕ^+ and ϕ^0 , complex fields

$$\text{Then first: } \mathcal{L}_{\text{Higgs}} = \partial_\mu \phi^\dagger \partial^\mu \phi - \mu^2 \phi^\dagger \phi - \lambda (\phi^\dagger \phi)^2 \\ - G_e [\bar{R} \phi^\dagger L + \bar{L} \phi R]$$

This is the most general Hermitian interaction term we can write

$$S = \int d^4x \mathcal{L} \quad \text{hermitean} \\ \uparrow \\ \text{unitary}$$

Note: unitarity S matrix $\Rightarrow$ hermitean action $\Rightarrow$ hermitean Lagrangian

⑥ But to achieve local gauge, you must also couple your gauge bosons to the $\partial_\mu \phi$:

$$\mathcal{L}_{\text{sc}} = \left(\partial_\mu + \frac{ig}{2} B_\mu + \frac{ig}{2} \vec{\sigma} \cdot \vec{A}_\mu \right) \phi^\dagger \left(\partial_\mu - \frac{ig}{2} B_\mu - \frac{ig}{2} \vec{\sigma} \cdot \vec{A}_\mu \right) \phi - V$$

⑦ Now assume $\mu^2 < 0$ on these naive Higgs scalars to get

$V = \dots$ Only $\text{Re } \phi^0 = c$ picks up a v.e.v. $\langle c \rangle = v/\sqrt{2}$
(see other sheet) Write now:

$$\phi = \begin{pmatrix} b \\ a \\ d \\ c \end{pmatrix} \quad \langle \phi \rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ v/\sqrt{2} \end{pmatrix} \quad \phi = \begin{pmatrix} b \\ a \\ d \\ c \end{pmatrix} = e^{i\vec{\xi} \cdot \vec{\sigma} / 2v} \begin{pmatrix} 0 \\ 0 \\ 0 \\ v + \eta \end{pmatrix}$$

⑧ Goldstone stage: redefine a, b, d into ξ_1, ξ_2, ξ_3 and η where ξ_i are the would be goldstones

⑨ But, Higgs stage. Now take ϕ to the V -gauge so

$$\phi' = U(\xi) \phi = e^{i\vec{\xi} \cdot \vec{\sigma} / 2v} \begin{pmatrix} 0 \\ 0 \\ 0 \\ v + \eta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ v + \eta \end{pmatrix}$$

The goldstones are now gone

⑩ Now in the V -gauge, carefully examine all terms in the $\mathcal{L}$:

$$\mathcal{L}_{\text{V-gauge}} = -\frac{1}{4} \vec{F}_{\mu\nu} \cdot \vec{F}^{\mu\nu} - \frac{1}{4} B_{\mu\nu} B^{\mu\nu}$$

(7. cont)

⑪ OK, let's list some of the results.

- a) electron acquires mass $m_e = Gv/\sqrt{2}$
- b) The leftover Higgs field η has $m_\eta = 2(-\mu^2)$ $\mu^2 < 0$.
- c) The W 's have mass $M_W = \frac{1}{2}gv$
- d) The photon A_μ is massless
- e) The 3rd massive gauge boson Z has $M_Z = \frac{1}{2}v\sqrt{g^2 + g'^2}$
- f) identify $e = gg'/\sqrt{g^2 + g'^2}$
- g) identify $G/\sqrt{2} = g^2/8M_W^2 = \frac{1}{2v^2}$!
- h) charge is still conserved.

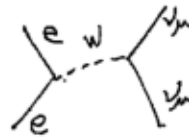
⑫ There are $\frac{5}{5}$ parameters: g, g', v, G, μ^2 and there are 3 conditions that we know: e, G, m_e . We will work on these later.

⑬ Look at new Lagrangian:

$$\begin{aligned} \mathcal{L} = & e A_\mu [\bar{e}_R \gamma^\mu e_R + \bar{e}_L \gamma^\mu e_L] \\ & + \frac{g}{\sqrt{2}} [\bar{\nu}_L \gamma^\mu e_L W_\mu^+ + \bar{e}_L \gamma^\mu \nu_L W_\mu^-] \\ & + \text{c.m.t.} \quad \sum_\mu (\underbrace{2\bar{e}_R \gamma^\mu e_R}_{\uparrow} + \underbrace{\bar{e}_L \gamma^\mu e_L}_{\uparrow} + \bar{\nu}_L \gamma^\mu \nu_L) + \dots \\ & + [\partial_\mu \eta \partial^\mu \eta - \frac{1}{2} m_\eta^2 \eta^2] + m_e \bar{e} e \end{aligned}$$

⑭ Here then is a summary. Start with 4 generators, break 3 of them leaving $U(1)$ charge group for S . The 3 would be goldstones are absorbed onto the 3 massive gauge bosons $W^\pm$ and Z . The fourth gauge boson stays massless + is the photon. There is one Higgs scalar left η . The weak and QED Lagrangians are reproduced.

8- scratch:



① We had: $e = gg' / \sqrt{g^2 + g'^2}$ and $g'/g = \tan \Theta_w$.

Define: $\sin \Theta = g/e$
 $\cos \Theta = g'/e$ } $\Rightarrow 1 = (g^2 + g'^2)/e^2 = (g^2 g'^2 / e^2) / e^2 = \frac{e^2 S^2 \cdot e'^2 C^2}{e^4}$
 $= S^2 C^2$??

$e = \frac{gg'}{\sqrt{g^2 + g'^2}} \Rightarrow e' = \sqrt{g^2 + g'^2}$ WALE $\mu_h = 400 \text{ GeV}$
 $S = Z_{mp} E_b \approx 2 \cdot 400$

Consider: $\left(\frac{g}{\sqrt{1}}\right)^2 + \left(\frac{g'}{\sqrt{1}}\right)^2 = 1$

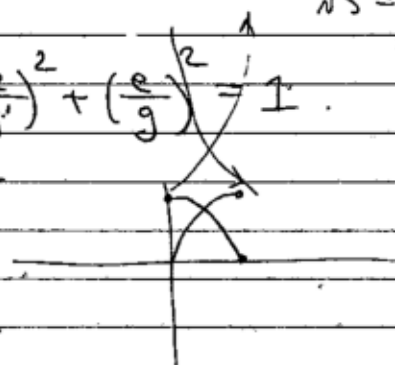
and:

$\frac{g}{\sqrt{1}} = \frac{e}{g'}$ so

$\left(\frac{e}{g'}\right)^2 + \left(\frac{e}{g}\right)^2 = 1$

$\sqrt{5} = \sqrt{800} = 30$

Thus: $\frac{g}{\sqrt{1}} = \frac{e}{g'} = \cos \Theta$
 $\frac{g'}{\sqrt{1}} = \frac{e}{g} = \sin \Theta$



Check: $e^2 = \frac{g^2 g'^2}{(\sqrt{1})^2} = ?$ $(\sqrt{1})^2 = g^2 + g'^2$

$g = e \cos \Theta$
 $g' = e \sin \Theta$

$e^2 = \frac{(e \cos \Theta)^2 (e \sin \Theta)^2}{(e \cos \Theta)^2 + (e \sin \Theta)^2} = e^2 \left[\frac{\cos^2 \Theta \sin^2 \Theta}{\cos^2 \Theta + \sin^2 \Theta} \right]$

$\left[\frac{1}{\sin^2 + \cos^2} \right] = 1 \checkmark$ so $g \geq e$ $g' \geq e \checkmark$

$M_W^2 = \frac{1}{4} g^2 v^2 = \frac{1}{4} g^2 \left(\frac{1}{2} \frac{\sqrt{2}}{G} \right)^2 = \frac{\sqrt{2}}{8} \frac{g^2}{G} = \frac{g \sin \Theta = e}{g' \cos \Theta = e}$

$g^2 = \frac{e^2}{5}$

$M_W = \frac{\sqrt{2}}{4 \cdot 8} \frac{e^2}{\sin^2 \Theta \cdot G}$

$\frac{g}{g'} = \cot \Theta$

$e^2 = 4 \pi \alpha$

$M_W = \frac{e}{\sin \Theta \cdot 2 \sqrt{2} \cdot G}$

Σ is a neutral current!

Weinberg-Salam (details)

① Show how electron acquires a mass.

We had: $\mathcal{L}_{int} = -G_e [\bar{R} \phi^\dagger L + \bar{L} \phi R]$

Now we have: $= -G_e [\bar{R}' \phi^\dagger L' + \bar{L}' \phi R']$

$$= -G_e [\bar{R} \left[\begin{pmatrix} 0 \\ v+n \end{pmatrix} \frac{1}{\sqrt{2}} \right]^\dagger U(1) L + \overline{[U(1) L]} \left[\begin{pmatrix} 0 \\ v+n \end{pmatrix} \frac{1}{\sqrt{2}} \right] R]$$

But keep primes on R+L.

$$\mathcal{L}_{int} \rightarrow \mathcal{L}'_{int} = -G_e [\bar{R}' \left\{ \begin{pmatrix} 0 \\ v+n \end{pmatrix} \frac{1}{\sqrt{2}} \right\}^\dagger L' + \bar{L}' \left\{ \begin{pmatrix} 0 \\ v+n \end{pmatrix} \frac{1}{\sqrt{2}} \right\} R']$$

$$= -G_e \bar{R}' (v+n) \frac{1}{\sqrt{2}} e_L + \bar{e} (v+n) \frac{1}{\sqrt{2}} R$$

$$= -G_e \bar{e}_R (v+n) e_L + \dots$$

$$= -G_e \bar{e} (1-\gamma_5) (1-\gamma_5) e$$

$$\sim -\left(G_e \frac{v}{\sqrt{2}}\right) \bar{e} e + \dots$$

② Write explicitly the Higgs potential for the ϕ .

$$V = \mu^2 (\phi^\dagger \phi + \phi^\dagger \phi^0) + \lambda (\phi^\dagger \phi + \phi^\dagger \phi^0)^2$$

$$\left. \begin{array}{l} \phi^\dagger = a+ib \\ \phi^0 = c+id \end{array} \right\} \Rightarrow V = \mu^2 ([a-ib][a+ib] + [c-id][c+id]) + \lambda (\text{same})^2$$

$$V = \mu^2 (a^2+b^2+c^2+d^2) + \lambda (a^2+b^2+c^2+d^2)^2$$

So its like 4 coords. Then choose $\left\langle \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \right\rangle = \begin{pmatrix} 0 \\ 0 \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}$ so vac. is in the "c direction"

$$\frac{\partial V}{\partial a} = 2\mu^2 a + 2(a^2+b^2+c^2+d^2)\lambda \cdot 2a = 2a [\mu^2 + 2\lambda R^2]$$

$$\left. \frac{\partial V}{\partial a} \right|_{vac} = 2a \left[\mu^2 + 2\lambda \frac{v^2}{2} \right] = 0$$

$$\left. \frac{\partial V}{\partial c} \right|_{vac} = 2 \frac{v}{\sqrt{2}} \left[\mu^2 + 2\lambda \frac{v^2}{2} \right] \Rightarrow v^2 = -\mu^2/\lambda$$

③ Write out V-gauge Higgs kinetic term with gauge bosons?

$$\mathcal{L}' = (\partial_\mu + \frac{ig'}{2} B_\mu + \frac{ig}{2} \vec{\tau} \cdot \vec{A}_\mu) \phi^\dagger (\partial^\mu - \frac{ig'}{2} B^\mu - \frac{ig}{2} \vec{\tau} \cdot \vec{A}^\mu) \phi$$

$$\mathcal{L}'_{\text{mas.}} = (\partial_\mu + \dots) \begin{bmatrix} a-ib \\ c+id \end{bmatrix} (\partial^\mu \dots) \begin{bmatrix} a+ib \\ c+id \end{bmatrix}$$

$$\mathcal{L}'_{\text{H}} = (\partial_\mu + \dots) \begin{bmatrix} 0 \\ \eta+v \end{bmatrix} (\partial^\mu \dots) \begin{bmatrix} 0 \\ \eta+v \end{bmatrix} \quad / \eta = \text{real scalar field.}$$

$$= (\partial_\mu + \dots)_{\text{on } (\eta+v)} \times (\partial^\mu \dots)_{\text{on } (\eta+v)}$$

$$= \partial_\mu \eta \partial^\mu \eta + \left[\frac{ig'}{2} B'_\mu \eta \partial^\mu \eta - \text{same} \right] + \left(\frac{ig'}{2} B'_\mu \eta \right) \left(-\frac{ig'}{2} B^{\mu'} \eta \right) + \frac{ig}{2} \vec{\tau} \cdot \vec{A}_\mu$$

OK, there are lots of terms. Drop primes on A, B:

$$\mathcal{L}' = \left[(\partial_\mu + \frac{ig'}{2} B_\mu + \frac{ig}{2} \vec{\tau} \cdot \vec{A}_\mu) (\eta+v) \right] \times \left[\right]^\dagger$$

$$= \left[\partial_\mu \eta + \frac{i}{2} \left(\frac{g'}{2} B_\mu + g \vec{\tau} \cdot \vec{A}_\mu \right) (\eta+v) \right] \left[\right]^\dagger = [\alpha + i\beta][\alpha - i\beta] = \alpha^2 + \beta^2$$

$$= \partial_\mu \eta \partial^\mu \eta + \frac{1}{4} \left\{ (g' B_\mu + g \vec{\tau} \cdot \vec{A}_\mu) (g' B^\mu + g \vec{\tau} \cdot \vec{A}^\mu) (\eta+v)^2 \right\}$$

$$= \partial_\mu \eta \partial^\mu \eta + \frac{1}{4} g'^2 B_\mu B^\mu (\eta^2 + 2v\eta + v^2) + \frac{1}{2} g g' (\vec{\tau} \cdot \vec{A}_\mu) B^\mu (\eta+v) + \frac{1}{4} g^2 (\vec{\tau} \cdot \vec{A}^\mu)^2 (\eta+v)^2$$

④ Now the potential term

$$(1) = 4 \quad (2) = \frac{4!}{2!2!} = \frac{4 \cdot 3}{2} = 6$$

$$\mathcal{L}'_{\text{pot}} = -V = - \left[\mu^2 \phi^\dagger \phi + \lambda (\phi^\dagger \phi)^2 \right] = - \left[\mu^2 (\eta+v)^2 + \lambda (\eta+v)^4 \right]$$

$$= - \left[\mu^2 \eta^2 + 2\mu^2 v \eta + \mu^2 v^2 + \lambda \eta^4 + \dots + 6\lambda v^2 \eta^2 \right] \quad \text{mass term.}$$

$$\mathcal{L} \propto - \left[\mu^2 \eta^2 + 6\lambda v^2 \eta^2 \right] = - \left[\mu^2 + 6\lambda v^2 \right] \eta^2 = - \left[\mu^2 - 6\mu^2 \right] \eta^2$$

$$v^2 = -\mu^2/\lambda \quad \lambda v^2 = -\mu^2 \text{ etc.}$$

How does the Z couple?

$$(\bar{e}_R \gamma^\mu e_R) \cdot Z_\mu \cdot \left\{ \frac{2g'^2}{2\sqrt{g^2 + g'^2}} \right\}$$

$$e_L = \frac{1}{2}(1 - \gamma_5) e$$

$$e_R = \frac{1}{2}(1 + \gamma_5) e$$

answer:

$$Z_\mu \bar{e} \gamma^\mu [d_V + d_A \gamma_5] e$$

$$Z_\mu \bar{\nu} \gamma^\mu [f - f \gamma_5] \nu$$

where:

$$d_V = \frac{1}{2} g' S_\theta + \frac{1}{4} g' S_\theta - \frac{1}{4} g C_\theta = \frac{3}{4} g' S_\theta - \frac{1}{4} g C_\theta$$

$$d_A = \frac{1}{4} g' S_\theta + \frac{1}{4} g C_\theta$$

$$f = \frac{1}{4} g' S_\theta + \frac{1}{4} g C_\theta = d_A$$

The axial couplings are the same:

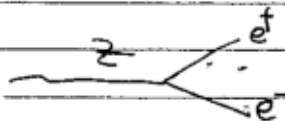
$$\frac{d_V}{d_A} = \frac{3g' S_\theta - g C_\theta}{g' S_\theta + g C_\theta} = \frac{3g'/g \tan \theta - 1}{g'/g \tan \theta + 1} = \frac{3 \tan^2 \theta - 1}{\tan^2 \theta + 1}$$

$$= \cos^2 \theta [3 \tan^2 \theta - 1] = [3 \sin^2 \theta - (1 - \sin^2 \theta)]$$

Thus, $d_A = f$ could be small; d_V large. $= [4 \sin^2 \theta - 1]$

$$d_V = [4 \sin^2 \theta - 1] d_A. \quad \text{Well, at best, } d_V = 3 d_A$$

$$\text{so } f = \frac{1}{3} d_V \Rightarrow \text{down by } \frac{1}{3} \text{ if}$$



This is my answer

Look at Bernstein who shows it more clearly.

Bernstein p35:

$$i 2 \gamma^\mu \left[\frac{\sqrt{g^2 + g'^2}}{4} + \frac{\sqrt{g^2 + g'^2}}{4} \gamma^5 \right] \gamma^\mu + i 2 \gamma^\mu \gamma^5 \left[\frac{3g'^2 - g^2}{4\sqrt{g^2 + g'^2}} - \frac{\sqrt{g^2 + g'^2}}{4} \gamma^5 \right] e$$

Well, $\frac{\sqrt{g^2 + g'^2}}{4} = \frac{1}{4} \frac{gg'}{e} = \frac{1}{4} \frac{g^2}{e} \tan \theta$ ✓

Cannot get absolutely.

⑤ → $g^2 = 8 M_W^2 \frac{G}{\sqrt{2}} = \frac{1}{4} \cdot \frac{1}{e} \cdot \tan \theta \cdot 8 M_W^2 \frac{G}{\sqrt{2}}$ ✓

But, $M_W^2 = e^2 \csc^2 \theta \frac{1}{4\sqrt{2}G}$ ✓ so

$= \frac{1}{4} \cdot \frac{1}{e} \cdot \frac{\sin \theta}{\cos \theta} \cdot \frac{8G}{\sqrt{2}} \cdot \frac{e^2}{\sin^2 \theta} \cdot \frac{1}{4\sqrt{2}G} = \frac{e}{4 \cos \theta \sin \theta}$ ✓

$f = \frac{1}{2} e \csc(2\theta)$ ✓

$= \frac{e}{2 \sin(2\theta)}$

Thus, f could be $\gg e$. - Use $M_W \Rightarrow 38 \csc \theta$ GeV.

$\Rightarrow \csc \theta = \left(\frac{M_W}{38} \right)$ $\sin \theta = \frac{38}{M_W}$

$M_Z = 38 \times 2 \csc(2\theta)$

$\left(\frac{M_Z}{76} \right) = \csc(2\theta) \geq 1 \Rightarrow M_Z \geq 76 \text{ GeV}$

Meanwhile, $\frac{3g'^2 - g^2}{4\sqrt{g^2 + g'^2}} = \frac{\sqrt{g^2 + g'^2}}{4} \cdot \frac{3g'^2 - g^2}{g'^2 + g^2}$

$\Rightarrow |dv| = |d\theta| \cdot [-1 + 4 \sin^2 \theta]$ ✓

$= \frac{3 \tan^2 - 1}{\tan^2 + 1} = (3 \tan^2 - 1) \cos^2$ ✓

$3(1 - \cos^4) - \cos^4 = 3 - 4 \cos^2 \theta$

$\leftarrow = 3 \sin^2 - \cos^2 =$
③ $= -1 + 4 \sin^2 \theta$

Branching ratios

$$f \cdot \Gamma_{\mu} \cdot \overline{\psi}(1+\gamma_5) \psi$$

$$f^2 \text{ tr} (\overline{u} \gamma_{\mu} (1+\gamma_5) \psi (1+\gamma_5) \psi) \Gamma_{\mu} \Gamma_{\nu}$$

$$f (\overline{u} \gamma_{\mu} (d_{\nu} - f \gamma_5) \psi (d_{\nu} - f \gamma_5) \psi)$$

$$= f^2 h (\overline{u} \gamma_{\mu} (c - \gamma_5) \psi (c - \gamma_5) \psi) \quad e = \frac{f}{d_{\nu}}$$

So the two diagrams:

$$\textcircled{1} = h (\overline{p} \gamma_{\mu} (1+\gamma_5) \psi (1+\gamma_5) \psi)$$

$$\textcircled{2} = h (\overline{p} \gamma_{\mu} (c - \gamma_5) \psi (c - \gamma_5) \psi)$$

$$\textcircled{1} = h (\overline{p} \gamma_{\mu} \psi (1+\gamma_5)) \cdot 2$$

$$\textcircled{2} = h (\overline{p} \gamma_{\mu} \psi (\frac{c+1}{2} - c \gamma_5)) \cdot 2$$

Go to cms? Quarks no ϕ polarization (helicity), charged lepton

$$p_{\mu} = (\frac{\sqrt{s}}{2}, \vec{0}) \quad \not{p} = \frac{\sqrt{s}}{2} \gamma_0$$

$$\textcircled{1} = \cancel{e} \cdot \cancel{e} \cdot h (\overline{p} \gamma_{\mu} \psi (1+\gamma_5)) \cdot 2 \cdot (\sqrt{s}/2)^2$$

$$\textcircled{2} = -\cancel{e} \cdot \cancel{e} \cdot h (\overline{p} \gamma_{\mu} \psi (\frac{c+1}{2} - c \gamma_5)) \cdot 2 \cdot (\sqrt{s}/2)^2$$

$$\textcircled{1} = h (\overline{p} \gamma_{\mu} (1+\gamma_5)) \cdot c = c \cdot 4 \cancel{e} \cdot \cancel{e}$$

$$\textcircled{2} = h (\overline{p} \gamma_{\mu} (c - c \gamma_5)) \cdot c = c \cdot 4 \cancel{e} \cdot \cancel{e} \cdot e$$

So $\textcircled{1}/\textcircled{2} = 1/e = e' = \frac{c^2+1}{2} = \frac{(d_{\nu}/f)^2+1}{2}$

$$\Rightarrow \frac{1}{1-e^2} = \frac{(d_{\nu}/f)^2+1}{2} = \frac{(4 \sin^2 \theta - 1)^2+1}{2}$$

$$- \text{But } \frac{d_{\nu}}{f} = \frac{dv}{d\theta} = (4 \sin^2 \theta - 1) \cdot \frac{1}{2\theta}$$

$$= \frac{16 \sin^4 \theta - 8 \sin^2 \theta + 2}{2} = \frac{8 \sin^4 \theta - 4 \sin^2 \theta + 1}{1}$$

$$\rho_{z \rightarrow e^+e^-} = 8\sin^4\theta - 4\sin^2\theta + 1, \quad \text{max} = 5$$

$$\rho_{z \rightarrow \nu\nu}$$



2a: Phenomenology of the W-S model

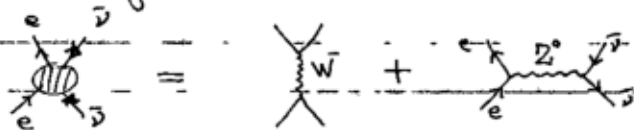
Weinberg angle.

① By playing with the constants, one finds easily that $M_W = \frac{38 \text{ GeV}}{\sin \Theta_W}$ where Θ_W is that mixing angle. $\Theta_W \in [0, \pi/2]$ at NAC, $(\sqrt{s})_{\text{max}} \approx 30 \text{ GeV}$, so you would never see a produced $W^\pm$. But at ISR, $(\sqrt{s})_{\text{max}} \approx 63$, so you might see it depending on the constant Θ_W .

② The Z has mass $M_Z = 76 \text{ GeV} / \sin(2\Theta_W) \geq 76 \text{ GeV}$.

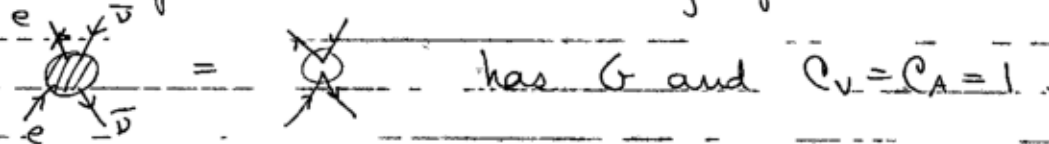
③ Since $G \rightarrow \nu \rightarrow G\nu$, we get $G_F = 2 \times 10^{-6}$ for ν -e-e coupling, so can forget these couplings compared to A-e-e, say.

④ What does the theory predict for $e\bar{\nu}$ elastic scattering? Here, the Z boson gives a neutral current contribution!



are sep. obs.

When you add these terms, there is an overall G and an "effective" C_V and C_A for the electron. Fermi theory says

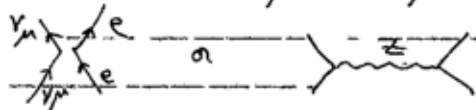


has G and $C_V = C_A = 1$.

But Weinberg gets: $C_V = \frac{1}{2} + 2 \sin^2 \Theta_W$ $C_A = \frac{1}{2}$.

The data of Reines search is consistent with both if $\Theta_W \leq 36^\circ$.

⑤ What about $\nu_\mu e \rightarrow \nu_\mu e$? Must be a neutral current process.



The VA here says $C_A = C_V = 0$, Weinberg says $C_A = \frac{1}{2}$ $C_V = \frac{1}{2} - 2 \sin^2 \Theta_W$. Same consistency so far from exper. upper bounds.

The next step is adding hadrons.

Problem: compute the Z contribution to $e\bar{\nu}_\ell$ elastic scattering.

- ① First, we need the eeZ and $\nu\nu Z$ terms from the $\mathcal{L}$.
Here, e is a full e

$$\begin{aligned} e &= e_L + e_R \\ -\gamma_5 e &= e_L - e_R \end{aligned} \Rightarrow \begin{aligned} e_L &= \frac{1}{2}(1-\gamma_5)e \\ e_R &= \frac{1}{2}(1+\gamma_5)e \end{aligned}$$

From 7.30,

$$\mathcal{L} = \frac{1}{2\sqrt{g^2 + g'^2}} \cdot \sum_{\mu} \left[g^2 (2\bar{e}_R \gamma^{\mu} e_R + 1\bar{e}_L \gamma^{\mu} e_L + \bar{\nu}_L \gamma^{\mu} \nu_L) - g^2 (\bar{e}_L \gamma^{\mu} e_L - \bar{\nu}_L \gamma^{\mu} \nu_L) \right]$$

- ② The electron term is:

$$\begin{aligned} &g' \sin \Theta_W \sum_{\mu} [\bar{e}_R \gamma^{\mu} e_R + \frac{1}{2} \bar{e}_L \gamma^{\mu} e_L] - g \cos \Theta_W \sum_{\mu} [\frac{1}{2} \bar{e}_L \gamma^{\mu} e_L] \\ &= \sum_{\mu} \cdot \left\{ g' \sin \Theta_W (\bar{e}_R \gamma^{\mu} e_R) + \frac{1}{2} (g' \sin \Theta_W - g \cos \Theta_W) (\bar{e}_L \gamma^{\mu} e_L) \right\} \end{aligned}$$

But $(\bar{e}_R \gamma^{\mu} e_R) = [\frac{1}{2}(1+\gamma_5)e]^{\dagger} \gamma_0 \gamma^{\mu} \frac{1}{2}(1+\gamma_5)e$

$$= \frac{1}{4} \cdot \bar{e} \gamma^{\mu} (1+\gamma_5) e = \frac{1}{2} \bar{e} \gamma^{\mu} (1+\gamma_5) e$$

And $\bar{e}_L \gamma^{\mu} e_L = \frac{1}{2} \bar{e} \gamma^{\mu} (1-\gamma_5) e$

So all in all

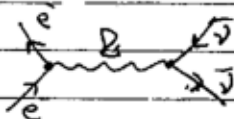
$$\begin{aligned} &= \sum_{\mu} \cdot \left\{ \overbrace{\left[\frac{1}{2} g' S_0 + \frac{1}{2} \cdot \frac{1}{2} (g' S_0 - g C_0) \right]}^{d_V} \bar{e} \gamma^{\mu} e + \overbrace{\left[\frac{1}{2} g' S_0 - \frac{1}{2} \cdot \frac{1}{2} (g' S_0 - g C_0) \right]}^{d_A} \bar{e} \gamma^{\mu} \gamma_5 e \right\} \\ &= \sum_{\mu} (d_V \bar{e} \gamma^{\mu} e + d_A \bar{e} \gamma^{\mu} \gamma_5 e) = \sum_{\mu} \bar{e} \gamma^{\mu} [d_V + d_A \gamma_5] e \end{aligned}$$

- ③ The neutrino term is:

$$g' S_0 \sum_{\mu} \frac{1}{2} \bar{\nu}_L \gamma^{\mu} \nu_L + g C_0 \sum_{\mu} \frac{1}{2} \bar{\nu}_L \gamma^{\mu} \nu_L = \sum_{\mu} \bar{\nu}_L \gamma^{\mu} [2f] \nu_L$$

where $2f = (\frac{1}{2} g' S_0 + \frac{1}{2} g C_0)$ $C = \sum_{\mu} \bar{\nu}_L \gamma^{\mu} (1-\gamma_5) \nu$

- ④ Matrix element is:



$$[\bar{e} \gamma^{\mu} [d_V + d_A \gamma_5] e] \cdot [f \bar{\nu} \gamma^{\mu} (1-\gamma_5) \nu] \cdot \left[\frac{-g_{\mu\nu}}{M_Z^2} \right]$$

8b: How do you include the hadrons?

① When you deal with hadrons, you have to think about strangeness. It is a fact of exper. that there are no "strangeness changing" neutral currents. This is what blocks the decay $K_L^0 \rightarrow \mu^+ \mu^-$. So, although theory can allow $\Delta S = 0$ neutral currents, it cannot allow $\Delta S = \pm 1$ neutral currents.

② Now deal with quarks. Put p_L and a Cabibbo lincomb of n_L, s_L into a left spinor of $SU(2)_L$. All others are $SU(2)_L$ singlets. Again, the $SU(2) \times U(1)$ rules out a quark mass term. [Why not a term like $m \bar{p} p$? Maybe violates parity for strongly int. quarks]. The only way to get $m \bar{p} p$ is $\bar{p}_R p_L + \bar{p}_L p_R$, but this violates $SU(2) \times U(1)$.

③ We ignore the strong quark-gluon term + go right to a quark-Higgs scalar interaction.

$$\tilde{\phi} = \begin{pmatrix} \phi^0 \\ \phi^- \end{pmatrix} = i\sigma_2 [\phi]^* \quad \phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} \text{ as before.}$$

$\mathcal{L}_{int} = G_1 [\bar{N}_L \tilde{\phi} p_R + \text{h.c.}] \dots$ why? Must satisfy $SU(2)$ and also $U(1)$. This is m.g.f. with p_R .

④ As before, let $\langle \phi^0 \rangle = v/\sqrt{2}$ + watch when $\phi^0 \rightarrow (v + \nu)/\sqrt{2}$. This causes mass terms to appear. Example:

$$\bar{N}_L \phi \lambda_R = (\bar{p}_L \bar{n}_L) \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} \lambda_R = \bar{p}_L \phi^+ \lambda_R + \bar{n}_L \phi^0 \lambda_R$$

$$\rightarrow \bar{n}_L v/\sqrt{2} \lambda_R = [\bar{n}_L \cos \theta_c + \bar{s}_L \sin \theta_c] \lambda_R \cdot v/\sqrt{2}$$

$$\rightarrow \sin \theta_c \frac{v}{\sqrt{2}} \bar{s}_L \lambda_R \sim \frac{v}{\sqrt{2}} G_3 \sin \theta \bar{\lambda} \lambda \quad \text{mass of } \lambda$$

By adjusting the G_i , you get just mass terms like $C \bar{n} n$. All masses are of course $\propto v$ since that is where they come from, and v is related to G_i .

⑤ The quark kinetic is $\mathcal{L}_{kin} = \bar{N}_L \not{\partial} N_L + \bar{p}_R \not{\partial} p_R + \text{etc.}$ You are supposed to couple in the $\tilde{A}_\mu$ and B_μ as before. When all is undone, you get $W^\pm$ terms and neutral terms.

$$e A_\mu j^{\mu(em)} + \sqrt{g^2 + g'^2} \sum_\mu [j^{\mu(3)} - \sin^2 \theta_W j^{\mu(em)}]$$

where $\rightarrow$

1) where $j_\mu^{(em)} = \bar{p} \gamma_\mu p$, the neutral em current, as expected.

2) where $j_\mu^{(3)} = \bar{p}_1 \gamma_\mu p_1 + \text{several other terms, some of which change strangeness!}$

⑥ Problem: The thing that connects to Z_μ is a $\Delta S \neq 0$ neutral current, just what we do not want.

Ellis, Ross, + Maiani

⑦ Glashow suggests a way out. Add a 4th quark, so

$$A_L = \begin{pmatrix} P_L \\ n_{cL} \end{pmatrix}$$

$$B_L = \begin{pmatrix} q_L^+ \\ \chi_{cL} \end{pmatrix}$$

new quark

Go through the whole deal, write $\mathcal{L}$, break, then identify the thing that couples to Z_μ .

Now $j_\mu^{(3)} [\text{Glashow}] = \text{all } \Delta S = 0!$ New Z_μ couplings are:

$$\Sigma^\mu \cdot \left\{ j_\mu^{(3)} - \sin^2 \theta_W j_\mu^{(em)} + \frac{1}{2} \bar{\nu} \gamma_\mu \nu_L \right\}$$

⑧ Now let's take this Glashow theory [1970] and try to compute neutrino-neutrino scattering.

$$\mathcal{M}(a\nu \rightarrow b\nu) = \sqrt{2} G_F [\bar{\nu} \gamma^\mu (1 - \gamma_5) \nu] \cdot [\langle b | j_\mu^{(3)} - \sin^2 \theta_W j_\mu^{(em)} | a \rangle]$$
$$= \text{diagram with } Z \text{ exchange}$$

From this formula in the GIM theory, we can compute that

$$\frac{\sigma(\nu p \rightarrow \nu p)}{\sigma(\nu n \rightarrow \nu p)} \approx .2 \quad (+ \text{this agrees? with some experi?})$$

⑨ There are other tests. The reason you can predict things is that a "model" is always a specific Lagrangian, and G is small so first order is plenty. Still, there are free parameters, namely θ_W .

9. Models with Heavy Leptons

- ① The motivation here is this: in both the Weinberg-Salam model, and in modified versions, you always end up with a neutral neutrino current coupling to Z_μ .

This coupling is very important in the theory because it renders the WS model renormalizable! This coupling cancels with:

$$\left(\text{diagram 1} + \text{diagram 2} \right) = 0$$

- ② But, exper. seems to be suggesting that this $\bar{\nu}\nu Z$ coupling does not exist. How do you then rescue the theory? Need some graphs to cancel $\bar{\nu}\nu Z$. Try more leptons.

- ③ These leptons must be heavier than K . ($M_K \approx 500 \text{ MeV}$) because they have not been observed in $\pi \rightarrow K$ decay. Hence, "heavy leptons".

- ④ There are already several models.

- The LPZ-model, uses E_L^+ and E_R^+ in a doublet with ν, e ...
- The PZ-II model, uses E^0 also.

- ⑤ One of the biggun heavy lepton gauge theories is that of Georgi + Glashow. Their gauge group is much simpler, namely $O(3)$.

The particles of the theory occur in vectors: (note: $T_3 = Q$) no need for Y , hence $O(3)$.

$$\vec{L} = (E_L^+, \nu_L \sin\beta + E_L^0 \cos\beta, e_L^-), \quad \vec{R} = (E_R^+, E_R^0, e_R^-)$$

So there are 3 heavy leptons E_L^+, E_R^+, E_R^0
 2 light leptons e_L^-, e_R^-
 1 neutrino ν_L } + antiparticles of course.

As in Yang-Mills, you gauge in a vector $\vec{A}$ of gauge fields [not 4 as in $SU_2 \times U_1$]. The Lagrangian reads.

$$\mathcal{L} = \bar{L}(\not{\partial} - \vec{A} \cdot \vec{T})L + \bar{R}(\not{\partial} - \vec{A} \cdot \vec{T})R$$

$$= \bar{L}\not{\partial}L + \bar{R}\not{\partial}R - \vec{A}_\mu [\bar{L}\gamma^\mu \vec{T}L + \bar{R}\gamma^\mu \vec{T}R]$$

$$\mathcal{L}_{int} = -A_\mu^{(0)} [\bar{L}\gamma^\mu T^{(0)}L + \dots] - A_\mu^{(+)} [\dots] - A_\mu^{(-)} [\dots]$$

⑥ So clearly, the $A^\pm$ are the $W^\pm$, the A^0 is the photon.

⑦ Many say $M_W \leq 53 \text{ GeV}$, an upper bound!

⑧ Now add a Higgs scalar, note ϕ which will break in the 3 direction. Redefine $(\phi_1, \phi_2, \phi_3) \rightarrow (S^+, \phi, S^-)$ where $S^\pm$ are the Goldstones. We are going to break $O(3)$ down to charge group $U(1)$, so expect two Goldstones.

⑨ These Higgs scalars were pre-coupled to L and R .

$$\mathcal{L} = \bar{G}_1 [L (\bar{L} \cdot \vec{\phi}) R] + \bar{G}_2 [L_s (\bar{L} \cdot \vec{R})]$$

because recall there a left-handed triplet sitting around in addition to L and R . When you "break", fermion mass terms will appear, in addition to the mass terms which are allowed at the start in the theory, m_0 . Now, "output" masses are functions of m_0 , angle β , G_1 , G_2 .

$$\begin{aligned} m_L^\pm &= m_0 - G_1 v \\ m_R^\pm &= \cos \beta \cdot m_0 - \sin \beta \cdot G_2 v \\ m_0 &= v \cdot m_0 + G_1 v \end{aligned}$$

$$\Rightarrow \begin{aligned} m_0 &= \frac{1}{2} (m_L^\pm + m_R^\pm) \approx \frac{1}{2} m_L^\pm \Rightarrow m_L^\pm \approx 2 m_0 \\ &\Rightarrow \text{also } m_L^\pm + m_R^\pm \approx m_L^\pm = 2 \cos \beta m_R^\pm \end{aligned} \quad \left. \begin{array}{l} \text{Some} \\ \text{"} \end{array} \right\}$$

⑩ So in closing, various integrations of these constants are needed.

⑪ Where should exper. look for heavy leptons?
a) of like SLAC
b) $W^+ \rightarrow \text{at NAL}$

10. Mass on Model Building

① First we have a concise review of the gauge strategy which I paraphrase as:

1. choose a gauge group G $\xrightarrow{N \text{ parameters}}$ $V(1) = \text{charge}$ [eg, $o(3)$ or $su(2)_c \otimes u(1)$]
2. write $\mathcal{L}$ invar under G (global) for leptons (+ quarks).
3. write $\mathcal{L}$ for Higgs scalars; must include one neutral. also, must have a degenerate vacuum type potential; also, renormalizable potential
4. Note: all particles must somehow be fitted into representations of the group G . (multiplets) Then invariant $\mathcal{L}$ is possible.
5. write Higgs-fermion invar couplings.
6. now insert gauge bosons in all kinetics (N of them). (Local gauge sym.)
7. Let $\langle \phi \rangle = v$ to break vacuum to group S , order M .
8. Redefine fields to face S onto $\mathcal{L}$.
9. The Higgs-fermion couplings will then create mass terms, even if these were disallowed in the original $\mathcal{L}$.
10. gauge away the $N-M$ goldstones, thereby create $N-M$ massive gauge bosons; still M massless (usually $M=1, 2$).
11. avoid $SS \neq 0$ neutral currents if possible.

② Second, a review of very recent research in gauge theory.

Part II

Physics Reports 9C

PART II

QUANTIZATION AND RENORMALIZATION OF GAUGE THEORIES

11. Path integral quantization?

One feels as Cavalieri must have felt calculating the volume of a pyramid before the invention of the calculus.

R.P. Feynman

(a)
Basic
Path
Integral
Idea.

In this section we develop the quantization procedure based on the notion of path integration. The first hint of this procedure appeared in a paper by Dirac in 1933; the method was perfected by Feynman in 1948. We shall first consider a quantum mechanical system with one degree of freedom, and generalize to quantum field theory in the next section.

Let $|q, t\rangle_H$ be the Heisenberg picture state vector describing a state which at time t is an eigenstate of the coordinate Q_H with eigenvalue q :

(H will have explicit t -dependence)

$$Q_H(t)|q, t\rangle_H = q|q, t\rangle_H = q(t)|q(t), t\rangle_H$$

$$Q_H(t) = e^{iHt} Q_S e^{-iHt} \quad (11.1)$$

where Q_S is the time-independent position operator in the Schroedinger picture; and H in the exponent is the Hamiltonian. The state

$$|q\rangle = e^{-iHt}|q, t\rangle_H$$

is an eigenstate of Q_S with eigenvalue q :

$$Q_S|q\rangle = q|q\rangle$$

and

$$|q, t\rangle_H = e^{+iHt}|q\rangle \quad (11.2)$$

The transformation matrix element

$$F(q', t'; q, t) = {}_H\langle q', t' | q, t \rangle_H = \langle q' | \exp\{-iH(t' - t)\} | q \rangle \quad (11.3)$$

plays a fundamental role in quantum mechanics. We are going to express $F(q', t'; q, t)$ as a path integral. We shall subdivide the time interval into $n + 1$ equal segments, and define

$$t_i = i\epsilon + t, \quad t' = (n + 1)\epsilon + t. \quad (11.4)$$

We make use of the completeness of the state vectors $|q_i, t_i\rangle$ to write

$$F(q', t'; q, t) = \int dq_1(t_1) \int dq_2(t_2) \dots \int dq_n(t_n) \langle q', t' | q_n, t_n \rangle \langle q_n, t_n | q_{n-1}, t_{n-1} \rangle \dots \langle q_1, t_1 | q, t \rangle \quad (11.5)$$

Here and in the following, we shall drop the subscript-H and understand the state $|q, t\rangle$ to mean that in the Heisenberg picture. For sufficiently large n , the time interval $t_i - t_{i-1}$ can be made as

$$1 = \int \mathcal{D}q_i(t_i) |q_i(t_i), t_i\rangle \langle q_i(t_i), t_i|$$

Index for Notes filed under Abers and Lee

- 1) original, old notes taken on Sections 11 thru 15, not very useful.
- 2) Section 11 summary: "path integral quantization" (regular QM)
 - a) about the state $/x,t$
 - b) relation of $/x,t$ to Greens Function = Propagator
 - c) comments about path integral idea
 - d) actual construction of path integral rep of a propagator = Greens
 - e) details of the harmonic oscillator example of using path integrals.
 - f) derivations of many equations in section 11.
- 3) Section 12 summary: "path integrals and field theory"
- 4) Section 13 summary: "Yang-Mills in the Coulomb gauge"
- 5) Section 14 summary: "Intuitive Approach to Quantization of Gauge theory"
 - a) the ghost loop expansion business
 - b) proof of relation (16.6), ie, connected Greens fctns and the $Z(J)$ object.
- 6) Section 15 summary: "equivalence of Landau and Coulomb gauge"
 - a) functional derivative facts
- 7) Section 16 summary: "proper vertices and effective potential, etc."
 - a) show (16.26), the power counting rule
 - b) algebra for section 16
- 8) Section 17 summary: renormalization in the ϕ^4 model
- 9) Section 18 summary: the BPHZ renormalization program
 - a) calculating δ_c and basic ideas.
- 10) Section 19 summary: dimensional regularizing Feynman integrals
- 11) Section 20 summary: Feynman Rules and Renorm, in SSB Gauge Theories (Landau gauge)
- 12) Section 21 summary: the R_ξ gauges and various kinds of ghosts
- 13) Section 22 summary: proof that S-matrix does not depend on gauge parameter ξ (and therefore no ghost poles in S-matrix)
- 14) Section 23 summary: computation of weak corrections to muon anom-moment in the Coleman-Glashow model.

.end!

11. Path Integral Quantization : Versenberg States

① Sadly, we start off on a note of confusion: Versenberg picture states that have time dependence I guess they do have time dependence if $H(p, q, t)$ has explicit time dependence.

② First, lets be convinced that S. type states do have time dependence. Let Q be some (S) operator $\Rightarrow$

$$Q|q\rangle_s = q|q\rangle_s$$

$$\begin{aligned} (= H|q\rangle_s &:= H|n, t, X, t, |q\rangle_s \\ &= E_n |n\rangle \langle n| q\rangle_s \end{aligned}$$

$\checkmark$ wave function
of q = position

Then one can show that: $-i \frac{d}{dt} |q\rangle_s = \sum_n E_n \langle n| q\rangle \langle n| \neq 0!$

where $|n, t\rangle = e^{-iEt} |n\rangle$ are the energy eigenstates

③ No go to the H-picture. Let $O_H \equiv e^{-iHt} O_S e^{iHt}$. Clearly $H_S = H_H \equiv H$. The S.E. says:

$$H_S |\alpha, t\rangle_S = -i \partial_t |\alpha, t\rangle_S \quad \text{for general state } |\alpha, t\rangle_S$$

$$\text{Insert } \bar{e}e \text{ to get: } \underbrace{\bar{e} H_S e}_{H_H} \underbrace{\bar{e} |\alpha\rangle_S}_{\equiv |\alpha\rangle_H} = -\bar{e} i \partial_t \underbrace{[e^\dagger \bar{e} |\alpha\rangle_S]}_{\equiv |\alpha\rangle_H}$$

$$\text{Now we have: } H |\alpha\rangle_H = -e^{-iHt} i \partial_t [e^{iHt} |\alpha\rangle_H] \checkmark$$

④ I want to see what the S.E. is in the Versenberg picture. If $H = H(t)$ explicit, then:

$$\begin{aligned} \partial_t [e^{iHt} |\alpha\rangle_H] &= \partial_t (e^{iHt}) |\alpha\rangle_H + e^{iHt} (\partial_t |\alpha\rangle_H) \\ &= iH e^{iHt} |\alpha\rangle_H + i t (\partial_t H) e^{iHt} |\alpha\rangle_H + e^{iHt} (\partial_t |\alpha\rangle_H) \end{aligned}$$

$$\text{Thus: } H |\alpha\rangle_H = H |\alpha\rangle_H + t \left(\frac{\partial H}{\partial t} \right) |\alpha\rangle_H - i \frac{d}{dt} |\alpha\rangle_H$$

Thus we have shown that

$$i \frac{d}{dt} |\alpha\rangle_H = t \left[\frac{\partial H}{\partial t} \right] |\alpha\rangle_H$$

Conclusion: if H has explicit time dependence, the $|\alpha\rangle_H$ also has time dependence. Since Akers will add $H' = J$ in (t, t') , $= 0$ elsewhere, this is explicit time dep, so notation $|q, t\rangle_H$ is justified. Q_{ver}

$|q\rangle = |q\rangle_s$ is just convenient for distinction.

⑤ While we're at it, should show:

$$\frac{d}{dt} Q_S = 0 \Rightarrow \frac{d}{dt} [e^{-iHt} Q_H e^{iHt}] = 0 \quad / \text{correct sign}$$

$$\Rightarrow \underbrace{\frac{d}{dt} Q_H}_{\text{usual rule}} = i [H, Q_H] + \underbrace{i t \left[\frac{\partial H}{\partial t}, Q_H \right]}_{\text{explicit piece}}$$

⑥ We know that $H|n,t\rangle_S = E_n|n,t\rangle_S$ and $H|n\rangle = E_n|n\rangle$.
What is $H|n\rangle_H$?

$$\begin{array}{ccc} H|n,t\rangle_S = E_n|n,t\rangle_S & \Rightarrow & \boxed{H|n\rangle_H = E_n e^{-iHt} |n\rangle_H} \\ \uparrow \quad \uparrow \quad \uparrow & & \uparrow \\ e^+ \quad e^- \quad e^+ & & e^- e^+ \end{array}$$

not same as S. case.

$$[e^{iHt} H] |n\rangle_H = E_n |n\rangle_H \quad \text{new eigenvalue equation}$$

The object - $F(q', t'; q, t)$

- ① Consider the state $|q, t\rangle_H$. If H had no explicit t dep, we could say

$$|q, t\rangle_H = |q, t'\rangle_H = |q\rangle_H$$

The state is the same at all times. It always remains an eigenstate of $Q_H(t)$ with eigenvalue q ? No! Although $|q\rangle$ is a constant in time, $Q_H(t)$ changes in time. So, eg, if $|q_0\rangle$ is es of $Q_H(0)$ at $t=0$, $|q_0\rangle$ is not an es of $Q_H(t)$ at time $t=t$. Why?

$$Q_H(0)|q_0\rangle_H = q_0|q_0\rangle_H$$

$$Q_H(t)|q_0\rangle_H \cong \{Q_H(0) + it[H, Q_H(0)]\}|q_0\rangle_H = q_0|q_0\rangle_H + \text{more!}$$

- ② Therefore, we might use this notation: The state $|q, t\rangle_H$ is an eigenstate of Q_H at time t . At some other time t' , there is an eigenstate $|q', t'\rangle_H$, so that in effect, " q " is a function of t . State $|q, t'\rangle_H$ is not an eigenstate.

- ③ When you speak of a completeness relation, you must use $\{|q, t\rangle_H\}$ that are eigenstates.

- ④ So, by repeatedly inserting complete sets of states, we show that the amplitude overlap of $|q, t\rangle$ with $|q', t'\rangle$ is:

$$\begin{aligned} \langle q', t' | q, t \rangle &= F(q', t'; q, t) \\ &= \int \left[\frac{dq dp}{2\pi} \right] e^{i \int_t^{t'} (pq - H) d\tau} \\ &= \int \left[\frac{dq}{\sqrt{2\pi i \epsilon}} \right] e^{i \int_t^{t'} L d\tau} \\ &= \int \left[\frac{dq}{\sqrt{2\pi i \epsilon}} \right] e^{i S(t, t')} \quad \text{action} \end{aligned}$$

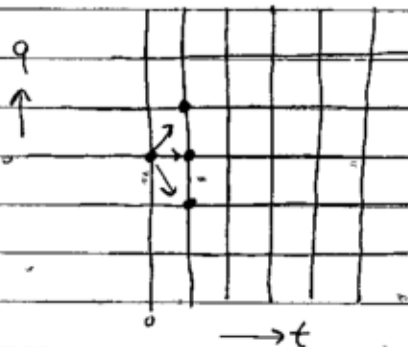
In these expressions, $q' = q_{\text{fin}} = q$ at $t = t'$. The phase space must — as far as I can tell — be computed microscopically.

- ⑤ F is the amplitude to go from q to q' in time $t'-t$, and for system to remain in a Q eigenstate.

More on $\langle q't' | qt \rangle$

- ① Think of a spacetime that is discrete.

Suppose $|00\rangle$ is an eigenstate of system at $t=0$. There are other eigenstates at $t=0$, but we choose the one with $q=0$ (hopefully there is one there).



- ② At time $t=1\epsilon$, the system may be in any of a number of eigenstates:

$$|q_i^1, \epsilon\rangle \quad i=1,2,3,\dots$$

These are shown as dots. So, I expect that

$$\sum_i |\langle q_i^1, \epsilon | 00 \rangle|^2 = 1$$

But presumably, the nearer dots have larger amplitudes.

- ③ I conjecture that at any time, the spectrum of operator $\hat{Q}_H$ is the same (all dots). It is the occupation of that spectrum that changes in time. Like the spreading out of a wave packet.

- ④ Example: The free particle propagator. Use results 11.40 and 11.41 with $\omega=0$ and $\tilde{\omega}=0$. One finds:

$$\langle q', t' | q, t \rangle = \sqrt{\frac{1}{2\pi i T}} \exp \left\{ i (q' - q)^2 / 2 T \right\} \quad T = t' - t.$$

In Saxon, p61, we saw this same result obtained from the S.E. There, $\langle q', t' | q, t \rangle$ was called "the kernel". Other people would call it "the free particle Green's Function."

$$\psi(x, t) = \int_{-\infty}^{\infty} \psi(x', 0) K(x', 0; x, t) dx \quad / \text{Saxon}$$

In either approach, you need a Hamiltonian in order to evaluate $F(q', t'; q, t)$. How about the "free particle propagator in coordinate space."

Screwing around with $W[J]$.

- ① First of all, $J(t)$ is an "external force" which is "on" between t and t' . i.e., while this force is present, we have:

$$\langle q', t' | q, t \rangle^J = \int \left[\frac{dp dq}{2\pi} \right] \exp \left\{ i \int_t^{t'} [p \dot{q} - H + Jq] dt \right\}$$

Thus, J perturbs the propagator a little.

- ② The ground state of our system is called $\phi_0(q, t)$. In general, the energy eigenstates are

$$|n, t\rangle_H = \cancel{\phi^n(q, t)} / \cancel{E_n}$$

$$\Rightarrow |0, t\rangle = \cancel{\phi^0(q, t)} / \cancel{E_0} = \cancel{\phi_0(q, t)} = \text{ground} \quad [\text{wrong on } H\text{-state}]$$

So $\langle q | 0, t \rangle = e^{-iE_0 t} \langle q | 0 \rangle$

$$\hookrightarrow \phi_0(q, t) = e^{-iE_0 t} \phi_0(q) = \text{ground}$$

- ③ Notice that the ground state $|0, t\rangle$ has the characteristic time dependence [as if it were a ~~schrod~~ state].

- ④ Define $W[J]$ by:

$$W[J] = \langle 0, t' | 0, t \rangle^J = \text{amplitude to stay in the ground state, once there, while } J \text{ is acting.}$$

- ⑤ By inserting $\int dq' |q', t'\rangle \langle q', t'|$ twice we find:

$$W[J] = \int dq \int dq' \phi_0^*(q', t') \langle q', t' | q, t \rangle^J \phi_0(q, t)$$

If we knew H , we could compute ϕ_0 . Then with J we could get $W[J]$, a functional!

- ⑥ There is still a third way to view $W[J]$. it is:

$$\begin{aligned} W[J] &= \langle q', -i\infty | q, +i\infty \rangle^J \\ &= \int [dq] \exp \left\{ i \int_{-i\infty}^{+i\infty} dt [L\phi + Jq] \right\} \quad 0, \text{ or } \infty \end{aligned}$$

All 3 ways are equivalent.

- ⑦ In 6, we could choose to Wick rotate that imag. time integral. Then we'd have a 4th way, but not quite same.

$$\cancel{W[J]} \neq \int [dq] \exp \left\{ \int_{-\infty}^{\infty} dt [L(q, \dot{q}) + Jq] \right\} \\ = W_E[J] \quad \text{"Euclidean"} \quad (\text{see comment 5-4})$$

- ⑧ Functional derivatives of $W[J]$ generate the time ordered coordinate operator products, taken between ground states:

$$\lim_{J \rightarrow 0} \frac{\delta^2 W[J]}{\delta J(t_1) \delta J(t_2) \dots} = (i)^n \langle 0, t | T(Q(t_1) Q(t_2) \dots) | 0, t \rangle$$

Later of course this will be used to get the free particle propagators in field theory.

- ⑨ A note on doing functional derivatives:

$$I \equiv \int_a^b dt f(q(t)) \quad \text{So if } f(q) = q^2, \text{ then}$$

$$\frac{\delta I}{\delta q(t_i)} \equiv \frac{\partial f}{\partial q(t_i)} \quad \frac{\delta I}{\delta q(t_i)} = 2q(t_i).$$

The harmonic-oscillator-with-external-force example:

- ① Given L for this problem, Abner is going to compute $W[J]$ and its second functional derivative via two methods

- 1) regular
- 2) euclidean

- ② The regular method. Using a lagrangian version of 4-1, he explicitly computes the kernel $\langle q(t) | q(t) \rangle$. This was the problem posed in Feynman's book. Then this result is inserted into version 4-5 to get $W[J]$. The result is:

$$W[J] = \exp \left\{ \frac{i}{2} \int dt_1 dt_2 J(t_1) D_+(t_1 - t_2) J(t_2) \right\}$$

$$\frac{1}{W} \frac{\delta^2 W}{\delta J(t_1) \delta J(t_2)} = D_+(t_1 - t_2) = \langle 0, t_2 | T(Q(t_2) Q(t_1)) | 0, t_1 \rangle$$

I do not have a significance to attach to this object $\uparrow$ in pre-field theory.

- ③ The euclidean route. Our starting point is 4-7 which says

$$W_E[J] = \int [dq] \exp \{-S_E\}$$

where S_E is the euclidean action:

$$S_E = \int_{-\infty}^{\infty} d\tau L_E(\tau) \quad \text{node.} \quad L_E = \frac{1}{2}(\dot{q})^2 - \frac{1}{2}\omega^2 q^2 + Jq$$

- 3a) Imagine that q_0 is the classical trajectory, and $q = q_0 + y$ is a nearby one. Abner shows us that:

$$W_E[J] \sim \exp \{-S_E(q_0)\}$$

i.e., the complicated sum goes away.

- 3b) So, what is the classical action? It is:

$$S_E(q_0) = -\frac{1}{2} \int_{-\infty}^{\infty} J(\beta) q_0(\beta) d\beta$$

Now all we need is the classical trajectory q_0 .

3c) So how do we find $q(t)$? Obers does this by solving the equation of motion (Lagrange) by a simple Reuss method. He gets:

$$q(t_1) = - \int_{-\infty}^{\infty} D_E(t_1 - t_2) \ddot{J}(t_2) dt_2$$

3d) when all these pieces are inserted to get $W_E[J]$, we get:

$$W_E[J] = N \exp \left\{ -\frac{1}{2} \int dt_1 dt_2 \ddot{J}(t_1) D_E(t_1 - t_2) \ddot{J}(t_2) \right\}$$

④ Comments: we do not expect $W[J] = W_E[J]$ because $\ddot{J}(t) \neq \ddot{J}(it)$. The usefulness of W_E is that it generates the same "propagators" via functional derivatives, except for $(i)^n$.

13. The Yang-Mills field in the Coulomb gauge.

① First, they write a Lagrangian for the Yang-Mills field. No nucleons, just $\vec{E}$ and $\vec{A}$. Of course not all these fields are independent. Use gauge invariance of $\mathcal{L}$ to restrict $\vec{A}$ to $\vec{\nabla} \cdot \vec{A} = 0$, the usual Coulomb gauge.

② Let the $\vec{A}_k$ be the three independent coords, where $\vec{A}_k$ has $\alpha=1,2,3$; $k=1,2$ since transverse. I.e. $A_3^\alpha = 0 \forall \alpha$. Then can show that $\vec{F}_{\alpha k}^T$ are the corresponding canonical momenta. Rename $\vec{F}_{\alpha k}^T = \vec{E}_k$. So $\vec{E}_k$ is an electric field with isospin indices!

$$\mathcal{H} = \sum \dot{p} \dot{q} - \mathcal{L}$$

$$\mathcal{H} = \sum_{\alpha,k} E_i^\alpha \dot{A}_i^\alpha - \mathcal{L}$$

③ The objects $\vec{F}_{\alpha k}^T$ [3 of them, $\alpha=1,2,3$] are now dependent. Define these dependent fields into $\vec{f} \Rightarrow \vec{F}_{\alpha k}^T = -\nabla_k \vec{f}$. Also, $\vec{A}_0$ is dependent. Explicitly can show:

$$\begin{cases} \vec{f} = g \mathcal{D}_c \cdot \vec{A}_k \times \vec{E}_k \\ \vec{A}_0 = \mathcal{D}_c \nabla_k^2 \vec{f} \end{cases}$$

Here, $\mathcal{D}_c$ is a fancy object. It is a 3×3 matrix in isospin space. It is a certain Green's function in coordinate space depending on the functions $\vec{A}_k(x)$. Here, it is considered as its own integral operator to avoid new symbols. The point is that given $\vec{A}_k$, we can compute the c.m. $\vec{E}_k$, then we can compute $\vec{f}$ and $\vec{A}_0$, truly dependent.

④ Next, construct $H = \frac{1}{2} \int d^3x [\vec{E}_i \cdot \vec{E}_i + \vec{B}_i \cdot \vec{B}_i + \mathcal{D}_i \vec{f} \cdot \mathcal{D}_i \vec{f}]$. Last term always shows up as Coulomb interaction term. I think it can be said to vanish by parts.

⑤ Now at last we call upon the path integral machinery as developed in sections 11. and 12. We have an $\mathcal{H}$, and we know exactly how to write the generating function $W[J]$ as a path integral. Recall (12.3), the major result of Sec. 12. $W[J]$ will ultimately tell us the Green's functions and the Feynman rules of the theory.

⑥ So in (13.21) we see $W_C[J]$ as $\int [dE^i][dA^i]$. You see only hamiverse integration variables because those are the only independent ones around. But, using some trickery, we can get the hamiverse aspect into some delta fctns.

$$W_C[J] = \int [dE^i][dA^i] \delta(2k E_k) \delta(2k A_k) e^{i \int d^4x [\dots]}$$

Notice of course that an $\vec{A} \cdot \vec{J}$ term has been inserted here, where $\vec{J}$ is the external force. Note also that this integral is extremely non-covariant in appearance.

⑦ Now comes a series of manipulations designed to make $W_C[J]$ look more covariant.

(a) introduce $\int [df] \delta(f - \dots) = \int [df] \det M_C \cdot \delta(\dots)$

(b) change int. var from E_i to F_i : $F_i \equiv E_i + P_i f \equiv F_{0i}$

(c) rewrite $\delta(\dots) = \int [dA_0] e^{\dots}$

(d) rewrite part of exp as $\int [dF_{ij}] \dots$

The end result is:

$$W_C[J] = \int [dA_\mu][dF_{\mu\nu}] \det M_C \prod_x \delta(2A_x) e^{i \int d^4x \{ \mathcal{L} + \vec{J}^\mu \cdot \vec{A}_\mu \}}$$

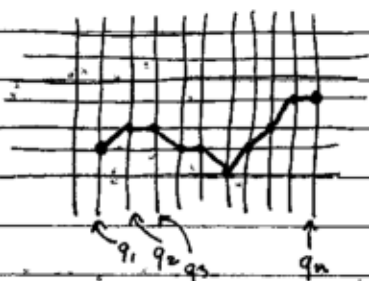
⑧ The tricky feature here is $\det M_C = \det [I + L]$ i.e. series in g . There are many strange functions flying around here.

14. Intuitive Approach to Quantization of Gauge Fields (Faddeev + Popov)

- ① In the usual path integral method, you are supposed to sum over all paths to get the generating function. Eg,

$$W[J] = \int [dq] \exp(i \int_{-\infty}^{\infty} dt [L_{eff} + Jq])$$

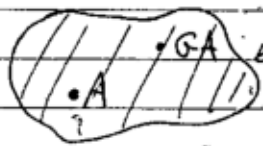
The notation $\int [dq]$ means $\int dq_1 dq_2 dq_3 \dots dq_n$ in the limit as $n \rightarrow \infty$. This is what a path means.



Here is shown one path. As all variables $q_i = q(0 + i\epsilon t)$ range from $-\infty$ to $+\infty$, you in effect sum over all paths.

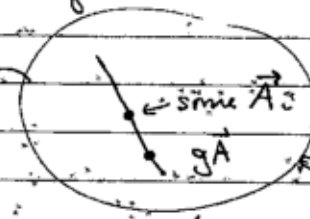
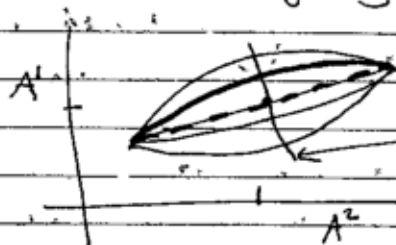
- ② In a standard field theory like scalar, the L_{eff} is a quadratic form in ϕ , $K_{\alpha\beta} \phi_\alpha \phi_\beta$ where α, β label cells in x^μ . Normally, $K_{\alpha\beta}$ has a known inverse, like ΔF , in which case $W[J] = \text{simple form}$.

- ③ Usually, the action contribution damps out very fast from nearby paths, + the classical trajectory only counts. In the gauge case, however, the action is constant over huge segments of the path space because of the gauge invariance. The segments are called "orbits of the gauge group".



← This surface represents the set $\{A_\mu^g\}$. It is a piece of the phase space $[dA_\mu]$.

- ④ So, the usual method is going to diverge because so many paths are strong:



← line is set $\{A_\mu^g\}$ an "orbit"

Red dash line is $f(\vec{A}) = 0$

Red solid is $f(g\vec{A}) = 0$ for some unique g .

- ⑤ So, when there is some group symmetry present, you can decompose the path integral as:

$$\int [d\vec{A}_\mu] = \int \underbrace{\Pi dg}_{\text{This restricts } dA \text{ integration to a particular equivalent path [like solid red]}} \int [d\vec{A}_\mu] \underbrace{\Delta f(A) \Pi \delta[f_a(A)]}_{\text{Then this sums over all equivalent paths.}}$$

It is only this (Πdg) that causes a divergence, so no harm in dropping it. There will be no change in $W[J]$.

- ⑥ So, after dropping this (Πdg) thing, you still have two new things in the integration:

- That $\Delta f(A)$ object [really a jacobian]
- That $\delta(f_a(A))$ hypersurface restrictor.

- ⑦ Now special case $G = SU(2)$ and they show that

$$\Delta f(A) = \det M_f \propto \det M_c$$

So that old $\det M_c$ is just this Δf factor. And of course

$$f_a(A_\mu) = 0 = 2i A_i^a \quad \text{in Coulomb Case}$$

↑ spin label; # of generators of the group

- ⑧ Next they compute $\Delta f(A)$ for the general Landau gauge $\partial_\mu A^\mu = 0$. This is a series in g , involves A propagators.

- ⑨ The extra $\det M_c$ can be put into the action and may be interpreted as the action arising from a set of complex scalar fields c [ghosts]. This is just a book-keeping device. 'massless'

- ⑩ The Feynman rules for the full theory are summarized in the figure. There are 3 kinds of particles and 3 interactions:

$$X = A^4 \quad Y = A^3 \quad Z = A c^2$$

15. Equivalences of the Landau and Coulomb gauges

- ① In (15.6), they claim to show the relation between $W_C[\psi]$ and $W_L[\psi]$. The Feynman diagrams do differ, because of an extra term. However, these differences only occur in self energy diagrams when external legs are taken to the mass shell. So,

$$\frac{(\text{external leg})_C}{(\text{external leg})_L} = 1$$

This is before things are renormalized. For an n -leg S -matrix element:

$$S_C = Z^n S_L$$

- ② But, if you renormalize external wavefunctions by $(Z_C)^{1/2}$ for Coul, and $(Z_L)^{1/2}$ for Lan, you get $S_{ren} = \text{same thing}$ in either gauge. This square root was demonstrated by Halpern in class.

- ③ Comment: so far, we have have not even thought about spontaneous sym. breaking. Just treated the $U(1)$ gauge theory in 2 gauges.

Summary of Section 11: Path Integral Quantization.

This is just the first 13 long sections which comprise the second part of Abers and Lee's review of gauge theory. I found the section very difficult, Many things I cannot derive without much work. This is the second time I have read this, last time was long time ago with Hassam Arfai.

I have decomposed the section into subsections because so much information is present here. The general domain here is first-quantized physics, like a first quantized harmonic oscillator. There are bras and kets and operators. There are operators like H the hamiltonian. Things are done mainly in the Heisenberg-States picture. The "system" of interest is described in this section by one coordinate q , operator Q . Thus, there are no lagrangian densities, no field theory at all.

If we divide physics into these areas: classical physics, quantum physics, classical field theory, quantum field theory, then this section falls into the second area. Path integrals are always about amplitudes, and that always means quantum physics. I presume that in the next section we will go over to quantum field theory by making these changes: coordinate operator Q replaced by field operator $\phi(x)$. Lagrangian replaced by ~~space~~ space integral of Lagrangian density. Otherwise I bet things are much the same, but I have not read that section yet.

(a) Path integral idea. Amplitude to go from here to there is divided into little times. At each time, you integrate over all possible positions. Hence (11.5)!. Since time evolution of a Heisenberg state may be written in the form (11.2), you can write the amplitude to move a little bit in time as in (11.9). Notice that things were converted to Schrodinger states and the Hamiltonian operator was put in in terms of Schrodinger operators P and Q , although this was never stated.

The main point of (11.9) is that you have expressed a quantum amplitude in terms of the c-number classical hamiltonian. Since each mini-amplitude is a p integration, when you stick all these into (11.5), you get the key result (11.11). Again, this gives the amplitude to go from here to there, a finite distance, in terms of the classical c-number Hamiltonian $H(p,q)$. The price paid to get this result however is a rather complicated calculational structure, namely the path integral.

In certain simple cases where $H(p,q)$ has simple form, you can execute all the dp integrations and get the same amplitude as a "path integral" $[dq]$ only of the a phasor whose phase is the classical action, ie, time integral of the lagrangian. Remember, this action is called classical only when you put in the classical trajectory $q(t)$.

(b) In cases where the Hamiltonian does not have the simple form of $\frac{1}{2}p^2 + V(q)$, you can still do the $[dp]$ integration and come up with an effective lagrangian and therefore an effective action under your ~~time~~ $[dq]$ integration. A famous calculation along this line was that of Yang and Lee in 1962. It seems obvious that you can always find an effective action, because you just do the dp integrals and see what results.

(c) Here we find a stunning result: it is very easy to write the matrix element between two states (here just Heisenberg q -states, but in general any states) of a time ordered product of coordinate operators. Usually it has been my experience that the time ordering complicated things, but here it makes the result very simple. You start with the path-integral representation of the $\langle q' | q \rangle$ amplitude. You spread it apart and stick in as many Q operators as you want, all in a TOP. In the path integral representation, all you need do is insert regular c-number coordinate functions like $q(t_1)$. Again, you see the role of the path-integral as converting from quantum mechanics to classical mechanics.

(d) Here we discuss the idea of adding a source term $J(t) q(t)$ to that Hamiltonian in the exponent. In the harmonic oscillator, $J(t)$ is precisely the driving function or external force. Of course $J(t)$ is a c-number function just like everything else up there in the exponent.

You can go on to compute the amplitude in the presence of this source term. As a path integral, it is (11.29), big deal. The precise definition of $W[J]$ is the amplitude for system to stay in ground state under presence of J . Ground state means lowest energy state, eigenstate of hamiltonian H , so JQ is a perturbation if you like. However, the end-twist integrations that convert from $\langle 0 | 0 \rangle^J$ to $\langle q' | q \rangle^J$ are in effect multiplicative constants, once you take certain large time limits. Hence you get result (11.35) which shows $W[J]$ as the $[dq]$ integral of the effective lagrangian deal with the source added in there.

The usefulness of the generating function $W(J)$ is that if you take its functional derivatives with respect to the source, you generate, after setting $J=0$, the matrix elements of time ordered products. In field theory, these will of course be the Greens Functions. So $W(J)$ is the Greens Functions generating functions.

(e) One slightly unpleasant aspect of the $W(J)$ thing is that you end up having to put large imaginary time limits on your action integral rather than real limits. This is avoided when you deal instead with the Euclidean version. Just compare (11.35) and (11.36) to see the difference between the regular and euclidean generating functions. The i 's are very important in the difference. In the case of W_E you have regular real time infinite endpoints.

(f) The final subsection works out the harmonic oscillator example. Any example is of course defined by its lagrangian or hamiltonian. So take this big machine and inject the lagrangian (11.38) and turn the crank. First off, you can calculate the thing $(q'/q)^J$ explicitly. The answer is given in (11.40) and (11.41) and I'll bet this is one hum-buster exercise.

Next, you fold in the ground state for the H.O. to get the exact $\phi x W[J]$ generating function. Answer takes the stupendously simple form shown in (11.45). No doubt I could do this entire calculation in detail. You can see that if you take the second derivative of this $W[J]$ you are going to get the propagator D_+ . I don't really know how to interpret this "time propagator" because we are not in quantum field theory, we are just in quantum physics. The form is very suggestive, I'm sure Feynman has some words on this thing.

Notice that the true propagator has phasing exponentials.

Now the whole thing is redone in the Euclidean case. The trick is used here for the first time observing that $W[J]$ is essentially the exponentiated classical action. I.e., if you put in the classical trajectory, the correction vanishes to first order. As far as J -dependence is concerned, the rest is just a constant. So we are down to (11.52).

Next, how to compute the classical action? The classical trajectory is found by solving the driven harmonic oscillator equation, i.e., (11.53). The solution is given in (11.56), and now you see that D_E is a Green's function in the differential equation sense, i.e., it solves (11.54). But this euclidean thing has no phase in it, totally damped.

So, having found the classical trajectory $q_0(t)$, you dump that back into S_E and compute the Euclidean classical action $S_E(q_0)$. This gives (11.57). Since $W(J)$ is the same as this expo'd classical action, you get (11.58).

Either way, you get the same basic form for the generating function, and you see what happens if you take a second functional derivative and set $0=0$.

Now I see the meaning of the propagator D : it is just the solution to the harmonic oscillator in the absence of source J . I.e., the propagator is the "free propagator" which describes how the system moves with no perturbation. Same concept is used in quantum field theory.

1. Understanding the state $|x, t\rangle$

- ① Consider the non-rel SE: $\vec{\nabla} \psi(x, t) = \nabla(i\hbar \frac{\partial}{\partial x}, p) \psi = i\hbar \frac{\partial}{\partial t} \psi(x, t)$.
Go to operator notation

$$\langle x | H | \psi(t) \rangle = \langle x | i\hbar \frac{\partial}{\partial t} | \psi(t) \rangle$$

OR $H | \alpha(t) \rangle = i\hbar \frac{\partial}{\partial t} | \alpha(t) \rangle$ so $\psi(x, t) = \langle x | \alpha(t) \rangle$

- ② ψ is a particular solution of the SE, so $|\alpha(t)\rangle$ is not just an arbitrary state in the QM HS. Identity: $|\alpha(t)\rangle = |\psi(t)\rangle =$ a vector which moves in time. This is all Sch. Picture, i.e.,

$$H | \alpha(t) \rangle_S = i\hbar \frac{\partial}{\partial t} | \alpha(t) \rangle_S$$

i.e., $|\alpha(t)\rangle_S = e^{-iHt/\hbar} |\alpha(0)\rangle_S$
↑ tells you how the state moves.

- ③ For states such as $|\alpha(t)\rangle_S$, you can define a state which does not move in time:

$$|\alpha(t)\rangle_H = |\alpha\rangle_H \equiv |\alpha(0)\rangle_S = e^{+iHt/\hbar} |\alpha(t)\rangle_S$$

But, if you in parallel define

$$|x(t)\rangle_H \equiv |x, t\rangle \equiv e^{+iHt/\hbar} |x\rangle_S$$

Then the state $|x, t\rangle$ does move in time. Only states which are solutions of the SE get "stopped" in the Heis. picture, i.e., states such as ~~$|\alpha(t)\rangle_S$~~ $|\alpha(t)\rangle_S$. [State $|x\rangle_S$ is not a SE solution.]

- ④ So the real meaning of the state $|x, t\rangle$ is this

$$|x, t\rangle = |\alpha(t)\rangle_H \equiv e^{+iHt/\hbar} |x\rangle_S$$

Derivation of the Green's Equation

① Define $|x, t\rangle_H \equiv e^{+i\hat{H}t/\hbar} |x\rangle_S$ where $\hat{H}$ denotes $H(\hat{P}, \hat{Q})$.

② Observe that $i\hbar \frac{d}{dt} |x, t\rangle_H = -\hat{H} |x, t\rangle_H$. Recall that the

usual S.E. reads $i\hbar \frac{d}{dt} |\alpha\rangle_S = +\hat{H} |\alpha\rangle_S$. Although the state

$|\alpha\rangle_S$ does not solve the SE, the state $|x, t\rangle_H$ does, but it does so with extra minus sign.

③ Define $G^+(x', t'; x, t) \equiv (-i) \Theta(t'-t) \langle x', t' | x, t \rangle$. Note that this yields the usual integral equation:

$$\begin{array}{ccc} \langle x, t | & = & \int d^3x' \langle x, t | x', t' \rangle \langle x', t' | \\ \uparrow & & \uparrow \\ \Theta(t-t') & & \Theta(t-t') \\ & & (+i)(-i) \end{array}$$

$$\Rightarrow \Theta(t-t') \langle x, t | = i \int d^3x' G^+(x, t; x', t') \langle x', t' |$$

$$\Rightarrow \Theta(t-t') \psi(x, t) = +i \int d^3x' G^+(x, t; x', t') \psi(x', t')$$

This integral equation is valid for any wavefunction; $|\psi\rangle$ need not be a SE solution.

④ Now, compute:

$$\vec{\chi}_x G^+(x', t'; x, t) = -i \Theta(t'-t) \langle x', t' | \hat{H} | x, t \rangle \quad \text{not very hard!}$$

Next, $i\hbar \frac{\partial}{\partial t} G^+(x', t'; x, t) = (-i) i\hbar \frac{\partial}{\partial t} (\underbrace{\Theta(t'-t)}_{-\delta(t'-t)}) \underbrace{\langle x', t' | x, t \rangle}_{\delta^3(x'-x)}$

$+ (-i) \Theta(t'-t) \langle x', t' | i\hbar \frac{\partial}{\partial t} | x, t \rangle$

$= -\hat{H} | x, t \rangle$

$= -\delta(t'-t) \delta^3(x'-x) + i \Theta(t'-t) \langle x', t' | \hat{H} | x, t \rangle = i\hbar \frac{\partial}{\partial t} G^+(x', t'; x, t)$

$= -\vec{\mathcal{H}}_x G^+(x', t'; x, t)$

Thus: $[-i\hbar \frac{\partial}{\partial t} - \vec{\mathcal{H}}_x] G^+(x', t'; x, t) = +\delta^4(x'-x)$

Here, operators act on second pair of arguments.

⑤ Could also define: $\bar{G}(x', t'; x, t) \equiv (+i) \Theta(t-t') \langle x', t' | x, t \rangle$.

↑ note.

Then above arguments would yield:

$$\Theta(t'-t) \Psi(x, t) = -i \int d^3x' \bar{G}(x, t; x', t') \Psi(x', t')$$

For the DE, you have made two changes

- sign of G
- sign of arg of Θ , hence sign of δ^4 .

Thus, should get

$$[-i\hbar \frac{\partial}{\partial t} - \vec{\mathcal{H}}_x] \bar{G}(x', t'; x, t) = +\delta^4(x'-x)$$

Thus, same equation! [Reason for putting in $(+i)$].

So:

$$[G^+(x', t'; x, t)]^* = +i \theta(t' - t) \langle x, t | x', t' \rangle = + G^-(x, t; x', t')$$

$$\text{Then } [-i\hbar \frac{\partial}{\partial t} - \vec{\nabla}_x] G^-(x', t'; x, t)^* = + \delta^4(x' - x)^*$$

$$\Rightarrow [+i\hbar \frac{\partial}{\partial t} - \vec{\nabla}_x] G^+(x, t; x', t') = \delta^4(x' - x).$$

where now you operate only on first arguments.

Summary:

$$G^{(\pm)}(x', t'; x, t) \equiv (\mp i) \theta[\pm(t' - t)] \langle x', t' | x, t \rangle$$

$$\theta[\pm(t' - t)] \psi(x, t) = (\pm i) \int d^3x' G^{(\pm)}(x, t; x', t') \psi(x', t')$$

$$\{G^{(\pm)}(x', t'; x, t)\}^* = + G^{(\mp)}(x, t; x', t')$$

$$[-i\hbar \frac{\partial}{\partial t} - \vec{\nabla}_x] G^{(\pm)}(x', t'; x, t) = + \delta^4(x' - x)$$

$$[+i\hbar \frac{\partial}{\partial t} - \vec{\nabla}_x] G^{(\pm)}(x, t; x', t') = + \delta^4(x' - x)$$

Footnote: From $|x, t\rangle_H \equiv e^{+i\hat{H}t/\hbar} |x\rangle_S$, you can at once show that

$$\underline{\langle x', t | x, t \rangle = \delta^3(x' - x)}$$

$$\text{so } \underline{1 = \int d^3x' |x', t\rangle \langle x', t|} \quad \text{at any time } t'.$$

3. Path Integral Quantization.

comments.

1. The Greens function or propagator contains all the information of the Hamiltonian. To find G , you can write out the differential Green's equation, putting in $H(p,x)$ where $p = i\hbar/dx$. I.e., the fact that $(p,x) \neq 0$ is already built into ordinary Schrodinger equation quantum mechanics.

With the Hamiltonian, we have a differential equation for wavefunctions, with G we have an integral equation for the wavefunctions, big deal. The physics is the same. The integral equation is more useful, however, for scattering problems and therefore for particle physics in general.

2. All the above has no connection with "second-quantization", which is to say, with theories which allow the creation and destruction of particles. In regular Schrodinger theory, even in integral form with G functions, you have only wavefunctions, no field operators in a Fock space. Of course you can interpret the Sch theory as the 1 particle sector of the second-quantized theory.

So, the above stuff has only to do with "first-quantization", the fact that $(p,x) \neq 0$. Second quantization usually proceeds by interpreting the field operator as a "coordinate" and saying $(\pi, \phi) \neq 0$. But in the present context, coordinate means x , has nothing to do with field operators.

3. The Green's Function $G(x,t; x',t')$ is therefore an object in the realm of first-quantized physics, i.e., regular quantum mechanics. We can compute G by dealing with a "quantized" Hamiltonian $H(p=i\hbar/dx, x)$. However, there is a trick for directly computing the propagator G in terms of the classical Hamiltonian $H(p,q)$. Of course you get the same result. The trick is just playing with the (x,t) states and has the name "path integral technique". Since you calculate quantum-mechanical stuff directly from classical H , you are in effect performing "first-quantization", hence the name path integral quantization.

Path Integral Construction of $\langle x', t' | x, t \rangle$.

① Your first reaction is to do this: $\langle x', t' | x, t \rangle = \langle x' | e^{i\hat{H}(t-t')} | x \rangle_s$.
 where $\hat{H} = H(\hat{p}, \hat{x})$. You cannot compute this thing directly because you may have terms in $\hat{H}$ like $\hat{p}\hat{x}\hat{p}$. Or, in the term $(\hat{H})^2$ you will have $\hat{p}\hat{x}\hat{p}\hat{x}$ say.

② So here is a trick to work this out. Divide time interval $t' - t$ into n equal slices of width ϵ . Insert completeness n times to get

$$\langle x', t' | x, t \rangle = \int dx_n \dots dx_2 dx_1 \langle x', t' | x_n, t_n \rangle \langle x_n, t_n | x_{n-1}, t_{n-1} \rangle \dots \langle x_2, t_2 | x_1, t_1 \rangle \langle x_1, t_1 | x, t \rangle \quad / (n+1) \text{ factors.}$$

As $n \rightarrow \infty$, $\epsilon \rightarrow 0$. Look at one factor, keep only terms of order ϵ .

$$\langle x_2, t_2 | x_1, t_1 \rangle = \langle x_2 | e^{i\hat{H}(t_2-t_1)} | x_1 \rangle_s = \langle x_2 | e^{-i\epsilon \hat{H}} | x_1 \rangle$$

$$= \underbrace{\langle x_2 | x_1 \rangle}_{\downarrow} - i\epsilon \underbrace{\langle x_2 | \hat{H} | x_1 \rangle}_{\curvearrowright} + \mathcal{O}(\epsilon^2)$$

$$\int \frac{d^3 p}{(2\pi)^3} \langle x_2 | p \rangle \langle p | x_1 \rangle$$

$$\int \frac{d^3 p}{(2\pi)^3} \langle x_2 | H(p, x_1) | p \rangle \langle p | x_1 \rangle = \int \frac{d^3 p}{(2\pi)^3} \langle x_2 | p \rangle \langle p | x_1 \rangle H(p, x_1)$$

$$\approx \int \frac{d^3 p}{(2\pi)^3} e^{i\vec{p} \cdot (\vec{x}_2 - \vec{x}_1)} [1 - i\epsilon H(p, x_1) + \mathcal{O}(\epsilon^2)]$$

$$\Rightarrow \langle x_2, t_1 + \epsilon | x_1, t_1 \rangle = \int \frac{d^3 p}{(2\pi)^3} \exp i \left[\vec{p} \cdot (\vec{x}_2 - \vec{x}_1) - \epsilon H(\vec{p}, \vec{x}_1) \right] + \mathcal{O}(\epsilon^2)$$

So add ss "2" to get

$$\langle x_2, t_2 | x_1, t_1 \rangle = \int \frac{d^3 p_2}{(2\pi)^3} \exp i \left[\vec{p}_2 \cdot (\vec{x}_2 - \vec{x}_1) - \epsilon H(\vec{p}_2, \vec{x}_1) \right] + \mathcal{O}(\epsilon^2).$$

on x_2 OK.

③ Insert this for each of the $(n+1)$ factors:

$$\langle x', t' | x, t \rangle = \left(\prod_{i=1}^n d^3 x_i \right) \left(\prod_{j=1}^{n+1} \frac{d^3 p_j}{(2\pi)^3} \right) \cdot \underbrace{\prod_{j=1}^{n+1} \left(\exp i \left[\vec{p}_j \cdot (\vec{x}_j - \vec{x}_{j-1}) - \epsilon H(\vec{p}_j, \vec{x}_{j-1}) \right] \right)}_A$$

$$A = \exp i \left[\sum_{j=1}^{n+1} \epsilon \left\{ \vec{p}_j \cdot \frac{(\vec{x}_j - \vec{x}_{j-1})}{\epsilon} - H(\vec{p}_j, \vec{x}_{j-1}) \right\} \right]$$

④ For each factor, take limit as $n \rightarrow \infty$. Then

$$A = \exp i \left[\int_t^{t'} dt \underbrace{\left\{ \vec{p}(t) \cdot \frac{d\vec{x}(t)}{dt} - H(\vec{p}(t), \vec{x}(t)) \right\}}_{\mathcal{L}(\vec{p}, \vec{x})} \right]$$

So: $\langle x', t' | x, t \rangle = \int \frac{[d\vec{x}][d\vec{p}]}{[(2\pi)^3]} e^{i \int_t^{t'} dt \mathcal{L}(\vec{p}, \vec{x})}$

Extremely symbolic in the limit.

⑤ Suppose $H = \frac{p^2}{2m} + V$. Then go way back to:

$$\begin{aligned} \langle x_2, t_2 | x_1, t_1 \rangle &= \int \frac{d^3 p_2}{(2\pi)^3} \exp i \left[\vec{p}_2 \cdot (\vec{x}_2 - \vec{x}_1) - \epsilon \frac{\vec{p}_2 \cdot \vec{p}_2}{2m} - \epsilon V(\vec{x}_1) \right] \\ &= \exp i [-\epsilon V(\vec{x}_1)] \cdot \int \frac{p^2 dp d\Omega}{(2\pi)^3} e^{-i \left[\frac{\epsilon}{2m} p^2 - p(\vec{x}_2 - \vec{x}_1) \cos \theta \right]} \end{aligned}$$

Stop: Recall:

$$\int d^3 p e^{-[a p^2 + b \cdot \vec{p} + c]} = \left(\int_{-\infty}^{\infty} dp_x e^{-[a p_x^2 + b_x p_x + \frac{c}{3}]} \right) () ()$$

$$\sqrt{\frac{\pi}{a}} \exp \left[\frac{b_x^2 - 4ac/3}{4a} \right]$$

$$\text{So } \int d^3p e^{-[ap^2 + \vec{b} \cdot \vec{p} + c]} = \left(\frac{\pi}{a}\right)^{3/2} \exp\left[\frac{b^2 - 4ac}{4a}\right] \quad b^2 = \vec{b} \cdot \vec{b}.$$

$$\text{So let } a = i \frac{\epsilon}{2m} \quad \vec{b} = (-i)(\vec{x}_2 - \vec{x}_1) \quad c = +i\epsilon V(\vec{x})$$

$$\text{So } \langle x_2, t_2 | x_1, t_1 \rangle = e^{-i\epsilon V(\vec{x}_1)} \cdot \frac{1}{(2\pi)^3} \cdot \left(\frac{\pi 2m}{i\epsilon}\right)^{3/2} \cdot e^{\left[\frac{-|\vec{x}_2 - \vec{x}_1|^2 + 4\epsilon^2 V(\vec{x})}{4i \frac{\epsilon}{2m}}\right]}$$

Get factor $e^{-i\epsilon V}$ either by pulling out at start, or by leaving in. So:

$$= e^{i\left(\frac{4\epsilon}{2m}\right)^{-1} |\vec{x}_2 - \vec{x}_1|^2} \cdot e^{-i\epsilon V(\vec{x})}.$$

$$\langle x_2, t_2 | x_1, t_1 \rangle = \left(\frac{2m\pi}{i\epsilon}\right)^{3/2} \frac{1}{(2\pi)^3} \cdot e^{-i\epsilon V(x_1)} e^{2mi \tau_{21}^2 / 4\epsilon}$$

$$= \left(\frac{m}{2\pi i\epsilon}\right)^{3/2} \exp i \left[-\epsilon V(\vec{x}_1) + 2m \frac{(\vec{x}_2 - \vec{x}_1)^2}{4\epsilon^2} \right]$$

$$= \left(\frac{m}{2\pi i\epsilon}\right)^{3/2} \exp \left[i\epsilon \left\{ \frac{1}{2} m \left(\frac{\vec{x}_2 - \vec{x}_1}{\epsilon} \right)^2 - V(\vec{x}_1) \right\} \right].$$

Then put this into thing to get:

$$\begin{aligned} \langle x', t' | x, t \rangle &= \left(\prod_{i=1}^n dx_i \right) \left(\prod_{i=1}^{n+1} \left(\frac{m}{2\pi i\epsilon} \right)^{3/2} \right) e^{i \sum_{i=1}^{n+1} \epsilon \left\{ \frac{1}{2} m \left(\frac{\vec{x}_i - \vec{x}_{i-1}}{\epsilon} \right)^2 - V(\vec{x}_i) \right\}} \\ &\equiv \int \left[\left(\frac{m dx}{2\pi i\epsilon} \right)^{3/2} \right] \rightarrow e^{i \int_t^{t'} dt \left[\frac{1}{2} m \dot{\vec{x}}^2 - V(\vec{x}) \right]} \\ &\quad \mathcal{L}(\vec{x}, \dot{\vec{x}}). \end{aligned}$$

note ∇ one extra factor here!

Schematically:

$$\langle \vec{x}', t' | \vec{x}, t \rangle = \int \left[\left(\frac{m}{2\pi i\epsilon} \right)^{3/2} dx \right] e^{i \int_t^{t'} d\tau \mathcal{L}(\vec{x}, \dot{\vec{x}})}$$

Harmonic Oscillator as Application of Path Integral Technique.

1. One problem you can do is to compute the propagator $G(x', t'; x, t)$ in regular quantum mechanics taking as your Hamiltonian a particle in a harmonic potential well with an added driving force term. This is a path integral calculation, if you like, and you can do the whole thing in detail; the answer is quoted in Abers and Lee. This example shows how you can compute a quantum amplitude (the propagator) in terms of the classical action.
2. But here we are more interested in computing the generating functional called $W(J)$. Roughly speaking, this is the limit $T \rightarrow +i\infty$ of the propagator. Rigorously it is the ground state to ground state amplitude. If we do not care about multiplicative factors, then we can use Coleman's vulgar gaussian path integral idea to see what we get.
3. The exact result is derived on page 68 of AL. I.e., take the exact Feynman-Hellmann quantum propagator and fold in ground states, then take time limits. Result is (11.45) which is also the result I will get below.
4. My first method is to identify the operator $\hat{A}$ by immediately rotating the contour with no variable change. Then I apply Sidney's formula and get an answer. The problem with this method is that
 - a) the integral is in fact phasing, not a dying gaussian, so the whole thing is questionable.
 - b) that fact causes you to need an extra rule to define your propagator. You have to let ω_0 be a little below real axis.
5. The second method is different. Here, I actually change variables and get something that looks more like a gaussian. The operator $\hat{A}$ is different, and there is no longer an ambiguity. This is the Euclidean method.
6. But either way works fine; the Euclidean thing just serves to define more clearly what you are really doing.
7. So this example is a combination of Abers-Lee with Coleman's integral trick. The transition to field theory is made by Coleman, and he redoes everything with a free Klein-Gordon field plus source (field theory analog of harmonic oscillator + source). Of course now you see d^4x stuff, and things are covariant.

Harmonic Oscillator Calculations

① Let $W(J) = \int [dx] e^{i \int_{-\infty}^{\infty} dt \left[\frac{1}{2} m \dot{x}^2 - \frac{1}{2} k x^2 + Jx \right]}$

where $L = T - V =$ h.o. lagrangian with external source. I.e.,

$$L = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} k x^2 + Jx \quad \Rightarrow \quad \frac{\partial L}{\partial \dot{x}} = m \dot{x} \quad \frac{\partial L}{\partial x} = -kx + J$$

Thus, Lagrange motional is: $m\ddot{x} = -kx + J$ external driving force.

② As first approach, lets say we can harmlessly Wick rotate the contour to $(-\infty, +\infty)$, no change in variable. Then define:

$$\langle x, y \rangle = \int_{-\infty}^{\infty} dt x(t) y(t)$$

so that $W[J] = \int [dx] \exp \left\{ i \langle \frac{1}{2} m \dot{x}, \dot{x} \rangle - i \langle \frac{1}{2} k x, x \rangle + i \langle J, x \rangle \right\}$

But write $\langle \dot{x}, \dot{x} \rangle = -\langle x, \ddot{x} \rangle$ by parts, + then

$$\begin{aligned} W[J] &= \int [dx] \exp \left\{ -\langle x, (i \frac{1}{2} m \frac{d^2}{dt^2}) x \rangle - \langle x, i \frac{1}{2} k x \rangle + \langle iJ, x \rangle \right\} \\ &= \int [dx] \exp \left\{ -\frac{1}{2} \langle x, \underbrace{\left[+i \frac{1}{2} m \frac{d^2}{dt^2} + \frac{1}{2} i k \right]}_A x \rangle - \underbrace{\langle -iJ, x \rangle}_b \right\} \end{aligned}$$

This is now in Coleman's standard form where $A = (+im \partial_t^2 + ik)$ and $b = -iJ$, $c = 0$. Result will thus be:

$$W[J] \approx \exp \left[-\frac{1}{2} \langle b, A^{-1} b \rangle \right]$$

Now, if $A = +im \frac{d^2}{dt^2} + ik = +i[m\omega_c^2 + k] = +im[\omega_c^2 + \omega_0^2]$,

Then $A^{-1} = \frac{1}{(-im)} \left[-\frac{1}{2\omega_0} \right] e^{-\omega_0|t|}$

omit. $I.e., A^{-1} = G(t, t') = \frac{1}{2im\omega_0} e^{-\omega_0|t-t'|}$

So result is $\exp \left[- \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dt' [-iJ(t)] \cdot \left[\frac{e^{-\omega_0|t-t'|}}{2im\omega_0} \right] [-iJ(t')] \right]$

Then $A^{-1} = g(t, t') = \frac{1}{im} \cdot \frac{i}{2\omega_0} \left[\theta(t-t') e^{-i\omega_0(t-t')} + \theta(t'-t) e^{+i\omega_0(t-t')} \right]$
 $\equiv D(t-t')$

So result is $w[J] = \exp \left[-\frac{1}{2} \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dt' [-iJ(t)] \cdot \left[\frac{1}{i} D(t-t') \right] [-iJ(t')] \right]$

But $-\frac{1}{2} (-i)^2 \frac{1}{i} = +\frac{1}{2i} = -\frac{1}{2}i \Rightarrow$

$w[J] = \exp \left[-\frac{1}{2} i \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dt' J(t) D(t-t') J(t') \right] \quad (\text{agrees (11.45)})$

Conclusion: in case that $L(x, \dot{x}) = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} k x^2 + Jx$, we have shown that

$w[J] \equiv \int [dx] e^{i \int_{-\infty}^{\infty} dt L(x, \dot{x})} = e^{-\frac{1}{2} i \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dt' J(t) D_+(t-t') J(t')}$

where $D_+(t) \equiv \frac{-1}{2im\omega_0} \left\{ \theta(t) e^{-i\omega_0 t} + \theta(-t) e^{+i\omega_0 t} \right\}$

③ Now as second approach, do not Wick rotate, just rename the variable:

$$i \int_{-\infty}^{\infty} dt \left[\frac{1}{2} m \dot{x}^2 - \frac{1}{2} k x^2 + Jx \right] = - \int_{-\infty}^{\infty} dt \left[-\frac{1}{2} m \dot{x}^2 - \frac{1}{2} k x^2 + Jx \right]$$

Perhaps $J(t) \rightarrow J(\pm it)$ but I will call result $J(t)$ anyway. Now do parts again so that

$$= - \int_{-\infty}^{\infty} dt x(t) \cdot \left[\frac{1}{2} m \partial_t^2 - \frac{1}{2} k \right] x - \int_{-\infty}^{\infty} dt Jx.$$

Let same scalar product, so $= -\frac{1}{2} \langle x, [m \partial_t^2 - k] x \rangle - \langle J, x \rangle.$

$$\text{Now } A = m [\partial_t^2 - \omega_0^2] \Rightarrow A^{-1} = \underbrace{-\frac{1}{2\omega_0 m} e^{-\omega_0 |t+t'|}}_{\equiv D_E(t-t')}.$$

And $b = J$ and $c = 0$, so

$$W_E[J] \approx e^{-\frac{1}{2} \int_{-\infty}^{\infty} dt dt' J(t) D_E(t-t') J(t')} \quad / \text{ agrees (11.58)}$$

Conclusion

$$W_E[J] \equiv \int [dx] e^{-\int_{-\infty}^{\infty} dt L(x, \frac{dx}{dt})} = e^{-\frac{1}{2} \int dt dt' J(t) D_E(t-t') J(t')}.$$

$$\text{where } D_E(t) = -\frac{1}{2m\omega_0} e^{-\omega_0 |t|}.$$

the various pictures

Notes on Section (11.a).

$$i\hbar \frac{d}{dt} |\alpha\rangle_S = H_S |\alpha\rangle_S$$

$$H_H = e^{iHt/\hbar} H_S e^{-iHt/\hbar} = H_S = H.$$

$$|\alpha\rangle_H = e^{iHt/\hbar} |\alpha\rangle_S.$$

Set $\hbar=1$.

$$|\alpha\rangle_S = e^{-iHt} |\alpha\rangle_H$$

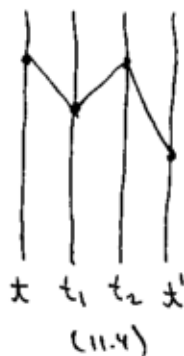
Let $Q = \text{operator}$. Then:

$$\begin{cases} Q_H |q\rangle_H = q |q\rangle_H & (11.1a) \\ Q_S |q\rangle_S = q |q\rangle_S & (11.1d) \end{cases}$$

$$\begin{cases} Q_S |q\rangle_S = q |q\rangle_S & (11.1d) \end{cases}$$

No loss by assuming $|q\rangle_H$ depends explicitly on time.

$$\langle q', t' | q, t \rangle_H \equiv F(q', t'; q, t) = \text{amplitude} = \langle q' | e^{-iHt'} e^{+iHt} | q \rangle_S. \quad (11.3)$$



$$\text{But } \langle q', t' | q, t \rangle = \int dq_1(t_1) \langle q', t' | q_1, t_1 \rangle \langle q_1, t_1 | q, t \rangle.$$

etc, hence: (11.5)

Now what is adjacent amplitude?

$$\langle q', t' | q, t \rangle = \langle q' | e^{-iH(t'-t)} | q \rangle = \langle q' | e^{-iH\epsilon} | q \rangle \approx \langle q' | q \rangle - i\epsilon \langle q' | H | q \rangle. \quad (11.6)$$

But

$$\begin{aligned} \langle q' | H | q \rangle &= \langle q' | \frac{1}{2} P^2 + V(Q) | q \rangle = \langle q' | \frac{1}{2} P^2 + V(q) | q \rangle \text{ or } V(q'), \text{ very close} \\ &= \int \frac{dp}{2\pi} \int \frac{dp'}{2\pi} \langle q' | p' \rangle \langle p' | \frac{1}{2} P^2 + V(q) | p \rangle \langle p | q \rangle = \int \frac{dp}{2\pi} \int \frac{dp'}{2\pi} e^{-ip'q'} e^{ipq} \delta(p-p') \left[\frac{1}{2} p^2 + V(q) \right] \\ &= \int \frac{dp}{2\pi} e^{ip(q'-q)} \left[\frac{1}{2} p^2 + V\left(\frac{q+q'}{2}\right) \right] = \int \frac{dp}{2\pi} e^{ip(q'-q)} H(p, \frac{q+q'}{2}) \end{aligned} \quad (11.8)$$

Also, $\langle q' | q \rangle = \int \frac{dp}{2\pi} e^{ip(q'-q)}$

so $\langle q', t' | q, t \rangle = \int \frac{dp}{2\pi} e^{ip(q'-q)} \left[1 - i\epsilon H(p, \frac{q+q'}{2}) \right] \approx \int \frac{dp}{2\pi} e^{ip(q'-q)} e^{-i\epsilon H(p, \frac{q+q'}{2})} \quad (11.9)$

where here, $t' = t + \epsilon$. $\approx \int \frac{dp}{2\pi} \exp[i(p(q'-q) - \epsilon H(p, \frac{q+q'}{2}))]$

Put one of these in everywhere:

$$F(q', t'; q, t) = \int dq_1 \dots dq_n \langle q' | q_n \rangle \langle q_n | q_{n-1} \rangle \dots \langle q_1 | q \rangle$$

$$= \prod_{i=1}^n \int dq_i \cdot \prod_{i=1}^n \int \frac{dp_i}{2\pi i\epsilon} e^{i p_i (q_i - q_{i-1}) - \epsilon H(p_i, \frac{1}{2}(q_i + q_{i-1})) (t_i - t_{i-1})} \quad (11.10)$$

Now take $\epsilon \rightarrow 0$, $n \rightarrow \infty$, $\epsilon n = t' - t$.

$$= \lim_{n \rightarrow \infty} \left(\prod_{i=1}^n \int \frac{dq_i dp_i}{2\pi} \right) \cdot e^{\sum [i p \dot{q} \Delta t - H(p, \frac{1}{2}(q + q')) \Delta t]}$$

$$= \lim_{n \rightarrow \infty} \prod_{i=1}^n \int \frac{dq_i(t_i) dp_i(t_i)}{2\pi \hbar} \cdot e^{\frac{i}{\hbar} \int_t^{t'} [p \dot{q} - H(p, q)] dt} \quad (11.11), (11.12)$$

Lagrangian

Now back off + try to do some p -integrations.

$$\prod_{i=1}^n \int dq_i e^{i p_i [q_i - q_{i-1}] - \epsilon \left[\frac{p_i^2}{2} + V(q_i) \right]}$$

$$= \prod_{i=1}^n \left[\int dq_i \cdot e^{i p_i q_i - \epsilon \frac{p_i^2}{2}} \right] \cdot e^{-\epsilon V(q)} = \prod_{i=1}^n \left[e^{-\epsilon V(q_i)} \cdot \frac{e^{i \frac{1}{2} \frac{q_i^2}{\epsilon}}}{\int 2\pi i \epsilon} \right]$$

$$= \frac{e^{i \frac{1}{2} q_i^2 n}}{[2\pi i \epsilon]^{n/2}} e^{-\epsilon V(q)}.$$

Well, confusion of subscripts. Go back to (11.10):

$$F(q', t'; q, t) = \lim_{n \rightarrow \infty} \prod_{i=1}^n \int dq_i \prod_{j=1}^{n+1} \left[\frac{dp_j}{2\pi} \cdot \exp \left\{ i \sum_{j=1}^n p_j (q_j - q_{j-1}) - \left(\frac{p_j^2}{2} + V(q_j) \right) \epsilon \right\} \right]$$

For each q_i , you have a factor .. More detail needed, see next page.

$$F(q'_j, t'_j; q, t) = \prod_{i=1}^n \left[\int dq_i(t_i) \right] \cdot \prod_{j=1}^{n+1} \left[\langle q_j, t_j | q_{j-1}, t_{j-1} \rangle \right]$$

where $q_{j-1} = q_{1-1} = q_0 = q$ and $q_{n+1} = q'$. Now replace:

$$\langle q_j, t_j | q_{j-1}, t_{j-1} \rangle = \int \frac{dp_j}{2\pi} e^{i \left[p_j (q_j - q_{j-1}) - \epsilon H(p_j, q_j) \right]}$$

Then:

$$\begin{aligned} & \prod_{j=1}^{n+1} \left[\int \frac{dp_j}{2\pi} e^{i \left[p_j (q_j - q_{j-1}) - \epsilon H(p_j, q_j) \right]} \right] * \\ &= \prod_{j=1}^{n+1} \left[\int \frac{dp_j}{2\pi} \right] \cdot \prod_{j=1}^{n+1} \left[e^{i \left[p_j (q_j - q_{j-1}) - \epsilon H(p_j, q_j) \right]} \right] \\ &= \prod_{j=1}^{n+1} \left[\int \frac{dp_j}{2\pi} \right] e^{i \sum_{j=1}^{n+1} \left[p_j (q_j - q_{j-1}) - \epsilon H(p_j, q_j) \right]} \end{aligned}$$

Therefore, precise results is

$$F(q'_j, t'_j; q, t) = \lim_{n \rightarrow \infty} \left\{ \prod_{i=1}^n \left[\int dq_i(t_i) \right] \cdot \prod_{i=1}^{n+1} \left[\int \frac{dp_i}{2\pi} \right] \cdot e^{i \sum_{j=1}^{n+1} \left[p_j (q_j - q_{j-1}) - \epsilon H(p_j, q_j) \right]} \right\} \quad (11.10) \text{ improved}$$

Now, assume $H(p_j, q_j) = \frac{p_j^2}{2} + V(q_j)$. Then you have:

$$\prod_{i=1}^{n+1} \left[\int \frac{dp_i}{2\pi} \right] \times e^{i \sum_{j=1}^{n+1} \dots} \quad \text{stop.} \quad \text{Easier to do this at stage *}$$

$$\int \frac{dp_j}{2\pi} e^{i \left[p_j (q_j - q_{j-1}) - \epsilon \left(\frac{p_j^2}{2} + V(q_j) \right) \right]} = e^{-i\epsilon V(q_j)} \int_{-\infty}^{\infty} \frac{dp_j}{2\pi} e^{i \left[p_j \dot{q}_j - \frac{p_j^2}{2} \right] \epsilon}$$

Look up integral: $\int_{-\infty}^{\infty} dx e^{-ax^2 - bx} = \sqrt{\frac{\pi}{a}} e^{b^2/4a} \quad a = +\frac{i\epsilon}{2} \quad b = -i\epsilon \dot{q}_j$

$$\begin{aligned} \text{So get } & e^{i\epsilon V(q_j)} \cdot \frac{1}{2\pi} \cdot \sqrt{\frac{\pi \cdot 2}{+i\epsilon}} \cdot e^{\frac{4(\dot{q}_j)^2 \epsilon / 4 + 4i\epsilon}{2}} = e^{-i\epsilon V(q_j)} \left[\frac{1}{i2\pi\epsilon} \right]^{\frac{1}{2}} e^{-\epsilon^2 (\dot{q}_j)^2 / 2i\epsilon} \\ &= \left[e^{-i\epsilon V(q_j)} \frac{1}{\sqrt{2\pi i\epsilon}} \exp \left[\frac{1}{2} i\epsilon \dot{q}_j^2 \right] \right] \end{aligned}$$

So I have now showed that:

$$\int \frac{dp_j}{2\pi} e^{i[p_j \dot{q}_j - \epsilon H(p_j, q_j)]} = \frac{1}{\sqrt{2\pi i\epsilon}} e^{\frac{1}{2} i \epsilon \dot{q}_j^2 - i \epsilon V(q_j)}$$

So how do you multiply n of these then take limit?

$$\begin{aligned} \lim_{n \rightarrow \infty} \prod_{j=1}^{n+1} \left[\frac{1}{\sqrt{2\pi i\epsilon}} e^{\frac{1}{2} i \epsilon \dot{q}_j^2 - i \epsilon V(q_j)} \right] \\ = \frac{1}{\sqrt{2\pi i\epsilon}} e^{i \sum_{j=1}^{n+1} \left[\frac{1}{2} \epsilon \dot{q}_j^2 - \epsilon V(q_j) \right]} \end{aligned}$$

But if $H = \frac{p^2}{2} + V(q)$, then $L = p\dot{q} - \frac{p^2}{2} - V(q)$

Well $\dot{q} = \frac{\partial H}{\partial p} = \frac{1}{2} p$ so $L = \frac{1}{2} \dot{q}^2 - V(q)$ (11.15)

So we get:

$$= \left[\frac{1}{\sqrt{2\pi i\epsilon}} \right]^{n+1} e^{i \int_t^{t'} dt L(q, \dot{q})}$$

in this special case!

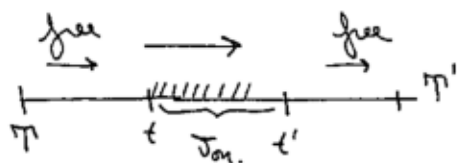
Put back the dq_j + get (11.14).

(11.14).

Amplitude is "the sum over paths" of a phasor whose phase is the action. (11.14)

Just as in Feynman Hells.

(d)



$$\langle Q', T' | Q, T \rangle^J = \int dq' \int dq \langle Q', T' | q, t' \rangle \cdot \langle q, t' | q, t \rangle^J \langle q, t | Q, T \rangle \quad (11.29)$$

(11.30)

Now: $\langle q, t | Q, T \rangle = \langle q | e^{-iH(t-T)} | Q \rangle$
↑ *Schröd.*

But $1 = \sum_n |n\rangle \langle n|$ and $H|n\rangle = E_n|n\rangle$ in *Schröd.*, so

$$\langle q, t | Q, T \rangle = \sum_{n, n'} \langle q | n \rangle \langle n | e^{-iH(t-T)} | n' \rangle \langle n' | Q \rangle = \sum_n \underbrace{\phi_n(q) \phi_n^*(Q)}_{\text{regular wave function, time indep.}} e^{-iE_n(t-T)}$$

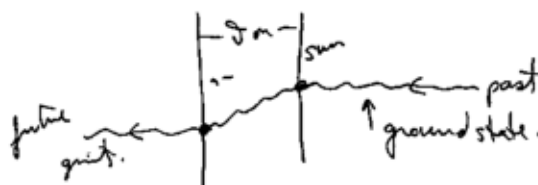
Now $\lim_{T \rightarrow i\infty} \sum_n \phi_n(q) \phi_n^*(Q) e^{-iE_n t} e^{iE_n T} = \phi_0(q) \bar{\phi}_0(Q) e^{-iE_0 t} e^{iE_0 T}.$

So $\lim_{T \rightarrow i\infty} [e^{-iE_0 T} \langle q, t | Q, T \rangle] = \phi_0(q, t) \bar{\phi}_0(Q) \quad (11.32)$

This limit is a trick to pull out lowest term in the sum.

So
$$\lim_{\substack{T \rightarrow i\infty \\ T' \rightarrow -i\infty}} \left[\frac{\langle Q', T' | Q, T \rangle^J}{e^{iE_0 T} \cdot e^{-iE_0 T'}} \right] = \int dq' \int dq \underbrace{\bar{\phi}_0(q', t')}_{\text{future}} \underbrace{\phi_0(Q')}_{\text{past}} \langle q', t' | q, t \rangle^J \underbrace{\phi_0(q, t)}_{\text{past}} \underbrace{\bar{\phi}_0(Q)}_{\text{past}}$$
(11.33)

So $w[J] = \int dq dq' \langle q, t' | \langle q', t' | q, t \rangle^J \langle q, t | 0 \rangle.$



So $w[J] = \langle 0 | 0 \rangle^J =$ amplitude to stay in ground state under temporary effect of J .

By the way, $\mathcal{T}$ acts (by definition) in this way:

$$\langle q', t' | q, t \rangle^{\mathcal{T}} = \int \left[\frac{dp dq}{2\pi} \right] e^{i \int_t^{t'} dt [p \dot{q} - H(p, q) + \mathcal{T} q]}$$

Therefore, you could do a variation. Ex,

$$1. \begin{cases} f = \int dx g(x) h(x). \\ \frac{\delta f}{\delta g} \equiv h(x) \end{cases}$$

$$2. \quad f = \int dx \int dy$$

Well, how do you define I ?

$$I(p, q) = \int_t^{t'} dt [p \dot{q} - H(p, q) + \mathcal{T} q]$$

$$\Rightarrow \frac{\delta I}{\delta \mathcal{T}} = q$$

$$\text{Then if } K = e^{iI}, \quad \delta K = e^{i(I+\delta I)} - e^{iI} = e^{iI} (i\delta I)$$

$$\text{so } \delta K = K(i\delta I) = iq K.$$

So perhaps

$$\frac{\delta \langle q', t' | q, t \rangle^{\mathcal{T}}}{\delta \mathcal{T}} = \int \left[\frac{dp dq}{2\pi} \right] q \cdot e^{i \int \dots}$$

↑ pulled down.

But $\mathcal{T} = \mathcal{T}(t)$, so something in both places.

Now do n derivatives as in (11.34) then set $\mathcal{T} = 0$. Pull down q 's.

$$\text{so } (11.34) = i^n \langle 0 | T [Q(t_1) Q(t_2) \dots Q(t_n)] | 0 \rangle.$$

Later "coordinates Q " will be fields, so Q 's will be of great interest.

3 Main results:

$$\langle q', t' | q, t \rangle = F(q', t'; q, t) = \int \left[\frac{dq dp}{2\pi\hbar} \right] \exp \left\{ \frac{i}{\hbar} \int_t^{t'} (p\dot{q} - H(p, q)) d\tau \right\} \quad (11.11)$$

$$= \int \left[\frac{dq}{2\pi i \epsilon} \right] \exp \left\{ i \int_t^{t'} L_{\text{eff}}(q, \dot{q}) d\tau \right\} \quad (11.22), (11.23)$$

$L_{\text{eff}} = L = p\dot{q} - H$ only when H has the form $H(p, q) = \frac{1}{2}p^2 + V(q)$.

Otherwise you have to deduce L_{eff} by doing $[dp]$ integration.

$W[J]$: definition is this:

$$\begin{aligned} W[J] &= \int dq dq' \langle 0 | q' \rangle \langle q' | q \rangle^J \langle q | 0 \rangle = \langle 0 | 0 \rangle^J \\ &= \int dq dq' \langle 0 | q' \rangle \cdot \left[\int \left[\frac{dp dq}{2\pi} \right] \exp i \left\{ \int_t^{t'} [p\dot{q} - H + Jq] \right\} \right] \langle q | 0 \rangle. \end{aligned}$$

So W involves the energy ground state $|0\rangle$. But from (11.33),

$$\begin{aligned} W[J] &\sim \langle Q', -i\infty | Q, +i\infty \rangle \\ &\sim \lim: \int [dq] \exp \left\{ i \int_T^{T'} [L_{\text{eff}}(q, \dot{q}) + J(t)q(t)] \right\} \\ &\sim \lim: \int \left[\frac{dp dq}{2\pi} \right] \exp \left\{ i \int_T^{T'} [p\dot{q} - H(p, q) + J(t)q(t)] \right\} \end{aligned}$$

So

$$\frac{\delta W[J]}{\delta J(t_1)} = \lim \int \left[\frac{dp dq}{2\pi} \right] q(t_1) \exp \{ \dots \}$$

And

$$\frac{\delta^2 W[J]}{\delta J(t_1) \delta J(t_2)} \Big|_{J=0} = \lim \int \left[\frac{dp dq}{2\pi} \right] q(t_1) q(t_2) \exp \{ \dots \} \Big|_{J=0} \quad (11.28)$$

$$= \lim \langle Q', T' | T[Q(t_1)Q(t_2)] | Q, T \rangle$$

So functional derivatives of W at $J=0$ will be TOP's!

Since we took T, T' to imag. values, maybe take all t 's there:

$$\langle q, t | q', t' \rangle = \lim_{n \rightarrow \infty} \int \prod_{i=1}^n \frac{dq_i}{\sqrt{2\pi i \epsilon}} \exp \left\{ i \sum_{i=1}^n \epsilon L_{\text{eff}} \left(\frac{q_i + q_{i+1}}{\epsilon}, \frac{q_i - q_{i-1}}{\epsilon} \right) \right\}$$

This was (11.14) where $\epsilon = \frac{t-t'}{n}$ if t, t' sep. by n space.

No $\epsilon = \frac{t-t'}{n} = \frac{\text{full width}}{\# \text{ slices}} \checkmark$.

Try to continue $x_i = \text{real}$ to $x_i = \text{imag.}$, so $x_i = -i\tau_i$ ↙ real.

Define $\epsilon' = \frac{\tau - \tau'}{n+1} = i \frac{(t-t')}{n+1} = i\epsilon$ so $\epsilon' = i\epsilon$

$$\Rightarrow \langle q, t | q', t' \rangle \Big|_{\text{cont'd}} = \lim_{n \rightarrow \infty} \int \prod \frac{dq_i}{2\pi i \epsilon'} \exp \left\{ \underbrace{\epsilon' \sum L_{\text{eff}} \left(q_i, \frac{q_i - q_{i-1}}{-i\epsilon'} \right)}_{(11.35.5)} \right\}$$

Now take $\epsilon' \rightarrow 0$ to get.

$$= \exp \left\{ \int_{\tau}^{\tau'} d\tau L_{\text{eff}} \left(q, i \frac{dq}{d\tau} \right) \right\}$$

no longer an i here. i here

In this case, you have:

$$\langle Q, T | Q', T' \rangle^J = \int [dq] \exp \left\{ \frac{i}{\hbar} \int_{iT}^{iT'} d\tau L_{\text{eff}} \left(q, i \frac{dq}{d\tau} \right) \right\}$$

So that

$$W[J] \sim \lim_{\substack{T \rightarrow i\infty \\ T' \rightarrow -i\infty}} \langle Q, T | Q', T' \rangle^J = \int [dq] \exp \left\{ \int_{-\infty}^{\infty} d\tau L_{\text{eff}} (q, i\dot{q}) + Jq \right\}$$

This is the Euclidean version $W_E[J]$.

Basically, you can work with either one. (11.37)

Check:

$$\int_{-\infty}^{\infty} d\tau \int_{-\infty}^{\infty} d\sigma \, \mathcal{J}(\tau) \mathcal{J}(\sigma) \left[\frac{1}{2i\omega} \theta(\tau-\sigma) e^{-i\omega(\tau-\sigma)} + \frac{1}{2i\omega} \theta(\sigma-\tau) e^{i\omega(\sigma-\tau)} \right]$$

$$= \underbrace{\int_{-\infty}^{\infty} d\sigma \int_{\sigma}^{\infty} d\tau \, \mathcal{J} \cdot \mathcal{J} \cdot \left[\frac{e^{-i\omega(\tau-\sigma)}}{2i\omega} \right]}_{\text{same form}} + \underbrace{\int_{-\infty}^{\infty} d\sigma \int_{-\infty}^{\sigma} d\tau \, \mathcal{J} \cdot \mathcal{J} \cdot \left[\frac{e^{+i\omega(\tau-\sigma)}}{2i\omega} \right]}_{\text{desired form.}}$$

$$= \int_{-\infty}^{\infty} d\tau \int_{-\infty}^{\sigma} d\sigma \dots$$

$$= 2 \cdot [\text{desired result}]. \quad (11.45), (11.46).$$

So interesting result:

$$W[\mathcal{J}] = \langle 0, \text{future} | 0, \text{past} \rangle^{\mathcal{J}} = \exp \left\{ \frac{-i}{2} \iint d\tau d\sigma \, \mathcal{J}(\tau) D_+(\tau-\sigma) \mathcal{J}(\sigma) \right\}.$$

More detail in a Feynman paper.

Now what is

$$\frac{\delta^2 W[\mathcal{J}]}{\delta \mathcal{J}(t_1) \delta \mathcal{J}(t_2)} = \langle 0 | T [Q(t_1) Q(t_2)] | 0 \rangle$$

$$= ?$$

$$W = \exp \left\{ \iint \mathcal{J} \cdot D \cdot \mathcal{J} \right\}$$

$$\frac{\delta W}{\delta \mathcal{J}} = \exp \left\{ \iint \mathcal{J} \cdot D \cdot \mathcal{J} \right\} \cdot \mathcal{J} D \cdot 2$$

$$\frac{\delta^2 W}{\delta \mathcal{J}^2} = \exp \left\{ \iint \mathcal{J} \cdot D \cdot \mathcal{J} \right\} \cdot 2 \mathcal{J} D + \exp \left(\iint \mathcal{J} \cdot D \cdot \mathcal{J} \right) \cdot 2 D.$$

Set $\mathcal{J} = 0$:

$$\frac{\delta^2 W}{\delta \mathcal{J}^2} = 0$$

So this shows $\langle 0 | T (Q(t_1) Q(t_2)) | 0 \rangle \approx D_+(t_1 - t_2)$

Compute (11.39) using (11.38)

$$\langle q', t' | q, t \rangle^J = \lim_{n \rightarrow \infty} \int \prod_{i=1}^n \frac{dq(t_i)}{\sqrt{2\pi i \epsilon}} \exp \left\{ i \int_t^{t'} d\tau \left[\frac{1}{2} (\dot{q}^2 - \omega^2 q^2) + Jq \right] \right\}$$

One way might be to undo the time integral.

$$\int_t^{t'} d\tau f(q, \dot{q}) = \sum_{i=1}^{n+1} \epsilon f\left(\frac{q_i + q_{i-1}}{2}, \frac{q_i - q_{i-1}}{\epsilon}\right) \quad \text{as in (11.14).}$$

Maybe go back one more step + put this sum into product:

$$\prod_{i=1}^n \left[\int \frac{dq(t_i)}{\sqrt{2\pi i \epsilon}} \exp \left\{ i \epsilon \cdot \left[\frac{1}{2} \left(\frac{q_i - q_{i-1}}{\epsilon} \right)^2 - \frac{1}{2} \omega^2 \left(\frac{q_i + q_{i-1}}{2} \right)^2 + J \left(\frac{q_i + q_{i-1}}{2} \right) \right] \right\} \right]$$

You can do this integral. Then do a product. I will accept the answer.

$$\langle q', t' | q, t \rangle^J = \left[\frac{\omega}{2\pi i \sin(\omega T)} \right]^{\frac{1}{2}} \exp \{ i Q \}$$

where Q = long mess involving the function J .

What is J ?

$$L = \frac{1}{2} m \dot{q}^2 - \frac{1}{2} k q^2 + Jq$$

$$\Rightarrow \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} = \frac{\partial L}{\partial q} \quad \text{or} \quad \frac{d}{dt} [m \dot{q}] = -kq + J$$

$$\Rightarrow m \ddot{q} + kq = J.$$

So J is like the driven harmonic oscillator function.

Next, they compute $W[J]$ itself.

$$\begin{aligned} W[J] &= \langle 0, t' | 0, t \rangle^J = \exp \left\{ i \int_{t'}^{t'} d\sigma \int_t^\sigma d\tau J(\sigma) \left[\frac{i}{2\omega} \exp \{ -i\omega(\sigma - \tau) \} \right] \right\} \\ &= \exp \left\{ i \int_{-\infty}^\infty d\sigma \int_{-\infty}^\sigma d\tau J(\sigma) \left[\frac{i \exp \{ -i\omega(\sigma - \tau) \}}{2\omega} \right] J(\tau) \right\}. \end{aligned}$$

Nov. 25, 1977

Section 12: Path Integral Formulation for Field Theory

The basic path integral formula for the generating function is given in (12.2). It shows integration over both the fields and the canonical field momenta. The argument of the action here is really the Lagrangian written in terms of the p and q . In certain cases, you can do the p integrations to get the more standard form (12.9) which Coleman uses.

The Euclidean stuff is mentioned here. You have to regularize these integrals by Wick rotation and so on.

The two kinds of generators $W(J)$ and $Z(J)$ are defined, and their relation to "connected" feynman diagarams is stated. Greens functions are written as j derivatives in the usual way.

On page 74 the field integrations are attempted. The characteristic determinant appears out front, and you get the standard form $J \times \text{Prop} \times J$ in the exponent. In fact, the propagator is even written as K^{-1} which I like. This was a free field case.

For the interacting case, they do the standard trick of an operator on the free action as in (12.16). I was once very confused as to why this is the Feynman rules, but studying Colamanna notes straightened this out. I think the rules are more obvious in the $d/d\phi$ notation. The fact that there is no need for a Wick theorem is mentioned, and now I understnad the remark.

We will not go into anti-commuting c-numbers. Apparantly just a minor technical detail.

Section 13: Yang-Mills in the Coulomb gauge.

(a) Standard Quantization.

1. First, the YM Lagrangian is written in the usual way, only YM fields present. However, the F and A are treated as independent objects. Of course this means that the Euler equation on the F will set them equal to expected object as in (13.3). Under local gauge transformations, F transforms as a simple isovector as in (13.2), whereas A goes as an isovector with the little extra piece. The $u(x)$ are the gauge functions.
2. Consider all the Euler equations. Separate out those which have time derivative, see (13.5) and (13.6). Computing the momenta, you find that F_{0i} are momenta of A_i , but the momentum of A_0 vanishes, thus A_0 is a dependent variable; just as in QED.
3. Now look at the constraint equations. (13.8) clearly tells you that F_{ij} are dependent variables determined completely by the A_i . Then (13.9) puts a condition on the F_{0i} variables so that not all three of them are independent. If the F_{0i} are not all independent, then their corresponding fields A_i cannot be independent. So you impose some kind of gauge condition on A , here just the usual Coulomb gauge that $\text{div} A = 0$, (13.10).
4. So only the transverse piece of A_i is true variables. I think you can just set $A^L = 0$ as was done in BD. So, break F_{0i} into T and L pieces as in (13.11). The quantity F_{0i}^T (1) = F_{Ti} has no curl, and may therefore be written as a gradient of some f as in (13.12). So the thing f is like the electrostatic potential. And then F_T is called E .
5. What goes on here? The true canonical variables are E_i and $A_i = A_i^T$. So you want to solve equation (13.13) for the potential f . You go ahead and solve this by the Green's method to get f as in (13.15). Then once we know f , we get F_T from (13.12). Finally, you write down (13.17) and solve for A_0 again by a greens method. ~~So I was wrong in saying that $A_0 = 0$, but maybe you could still gauge it away. But A_L .~~
6. Summary to this point: for the Yang Mills Lagrangian, we have identified the fields which are independent and which we want to quantize. All the other fields are dependent:

$$\text{EQ} \left\{ \begin{array}{l} \vec{A}_i = \vec{A}_i^T \text{ and } \vec{A}_i^L = 0 \text{ so that } \vec{\nabla}_i \cdot \vec{A}_i = 0, \text{ Coulomb gauge.} \\ \uparrow \\ \text{true field.} \end{array} \right.$$

$$\text{EQ} \left\{ \begin{array}{l} \vec{E}_i = \vec{F}_{0i}^T = \text{conjugate momentum to } \vec{A}_i. \end{array} \right.$$

Other fields are:

$$\left\{ \begin{array}{ll} \vec{F}_{0i}^L & \text{via (13.12) and (13.15)} \\ \vec{F}_{ij} & \text{via (13.8)} \\ \vec{A}_0 & \text{via (13.17a)} \end{array} \right.$$

7. Next, they compute the Hamiltonian and get (13.20) which looks right.

8. The procedure up to this point was the standard quantization procedure. I.e, you would go ahead and impose equal time commutators on your independent objects and go from there.

(b) Path integral Method.

1. Now, write down the full path integral generating function: only the dynamical fields enter, hence you get ~~(13.21)~~ (13.21). Makes sense to me.
2. With very little work, rewrite this generating path integral as in (13.24). Now nominally you integrate over all E_i and all A_i spatial components, but really the two delta functions restrict you to transverse fields only. The determinant turns out to be field-independent so is simply ignored. So I am happy with (13.24). Of course it is still not covariant because you are only integrating over spatial components of P and Q , so to speak.
3. Next, write 1 in fancy way (13.25). Here f is in isolation. Rewrite the delta as the equation involving f , and compensate by jacobian determinant $\det M_c$ where c stands to remind you we are in coulomb gauge. The result of this step is (13.28) which looks just fine. There are now three deltas.
4. The rest of the manipulations are very artificial, no reason to check them out. We have three deltas. Do the $[df]$ integration to remove one delta, hence get (13.31). Now you see all spatial field components present. The goal now is to add the time components to get a covariant formulation. The time components are dummied in in (13.32) and (13.33) and the final result is very simple, see (13.34).
5. The final result says: integrate over all the fields, but don't forget the Popov determinant $\det M_c$, and also don't forget the gauge delta. Then the exponent is simply your lagrangian.
6. The last step is to exponentiate the Popov determinant. A good method for dealing with such determinants is given using the old trace theorem. But the effective lagrangian is not yet discussed. You can see that the exponentiated determinant is just going to correct the lagrangian from (13.38).
7. Apparently, the material of this section was the way Fadeev and Popov originally presented some of their work. Started from just the dynamical fields, and then dummied in all the extra stuff to arrive at the simple form (13.34). Obviously there must be some simpler and more direct way to get this result. Must be some easy way to explain the determinant.

Section 14: Intuitive Approach to Quantization

1. First we copy down $W(J)$ from last section. Can integrate trivially over the F_{uv} so that only $[dA_u]$ integration remains. Plus you have the gauge delta function and the FP determinant. Question: why does this $W(J)$ not look like it does in a "normal" field theory? This is the question that Fadeev and Popov answered.

2. Intuitive comments: we know that in a gauge theory, the operator K (Coleman called A) which appears in the action is a projection operator and is therefore non-invertible. The exponent is "immune" to variations of the fields in the "longitudinal" direction. Usually, as you let some field go to infinity, the exponent is damped. This was the requirement that the "matrix" A be positive definite in Coleman notes. Here it seems that in effect some of our diagonal matrix elements vanish! I.e., the exponent is constant as you go off in A_L direction. Obviously, the longitudinal integration then gives a divergence.

To repair the problem, you should in effect divide out the orbit volume. Another way to say this is that you should restrict the integration to a hypersurface or cross section $f(A) = 0$ which intersects each orbit only once.

3. To illustrate the situation, Abers and Lee tell us to define a determinant which is essentially the jacobian from the f of $f(A)$ to the gauge parameter functions called u . By playing with the "normal form" (14.8), you can expose the orbit volume as in (14.10). You just divide this out to get your answer!

4. What is the determinant? You first choose a gauge, this determines $f(A) = 0$. You then vary the gauge functions causing a variation in f . Then $\det$ is $\det(df/du)$. For example, in Coulomb gauge you can read off M_c from (14.14). Notice that M has the for Δ^2 times (1+ something), where something is field dependent. Thus, you cannot just ignore the determinant! M for Landau gauge is also A -dependent, (14.15).

5. Finally, on page 86 it is shown how the determinant can be exponentiated by introducing fermion scalar ghost fields called c . The effect is to add S_c to the action you already have. Finally, you trivially exponentiate the gauge delta function to get another lagrangian correction of the form $(d_u A_u)^2$ in case of Landau. Once you get everything into the exponent, you can then read off the feynman rules! These rules are shown in the table on page 88, and I think I understand now, at last. To go perturbation theory with a gauge theory, you will in general need ghosts! I.e., you have to include the ghosts as a symbolic part of your feynman diagrams. In QED which is abelian, it turns out that the determinant is A -indep, so no ghosts needed.

How to understand the ghost loop expansion business.

1. The ghosts are created to simulate a determinant. If the determinant is $\det M$, then the ghost action will be something like $\int d^4x \bar{c}^+ M c$ where M is some operator. For the ghosts, this is a sort of ϕ^2 field theory. Ghosts start out being charged scalars, so it is ϕ^2 theory with arrows on the lines. It turns out that the non A -dependent part of the operator M is just $\Box$, so the ghost propagator is same as that of a massless Klein Gordan field, ie, $1/k^2$.

2. So consider for the moment what a ϕ^2 field theory looks like. First obvious fact is that number of external particles on a Greens function must be even, just as with usual electron case. Here, the total incoming charge must be zero and all particles have charge $+$ or $-$, hence even. But the object of my present interest is the vacuum bubbles expansion. I know that $W[0]$ is the sum of all vacuum bubbles. Lets show a few terms:

$$W[0] = \int [dc][d\bar{c}] e^{iS_c} = \langle 0 | e^{iS_c} | 0 \rangle \quad \text{where } S_c = \int d^4x \cdot g : \phi^2 :$$

Order g^2 : $\langle 0 | : \underbrace{\phi^2 : \phi^2} : | 0 \rangle = \left[\text{bubble} \right] \times 2!2!$

Order g^3 : $\langle 0 | : \underbrace{\phi^2 : \phi^2 : \phi^2} : | 0 \rangle = \left[\text{figure-eight} \right] \times 2!2!2!$

Order g^4 : $\langle 0 | : \underbrace{\phi^2 : \phi^2 : \phi^2 : \phi^2} : | 0 \rangle = \left[\text{figure-eight with two loops} \right] \times (2!)^4$
 $+ \left[\text{two bubbles} \right] \times (2!)^4 + \text{permutational.}$

Order g^5 : $\left[\text{figure-eight with three loops} \right] + \left[\text{bubble and figure-eight} \right] + \dots$

Order g^6 : $\left[\text{figure-eight with four loops} \right] + \left[\text{two bubbles and figure-eight} \right] + \left[\text{bubble and figure-eight with two loops} \right] + \left[\text{three bubbles} \right].$

The interesting fact about ϕ^2 theory is that the most complicated vacuum graph is just a product of loops. It is not hard to imagine that you could write all such graphs in this way:

$$e^{[\text{bubble} + \text{figure-eight} + \dots]} = 1 + [\text{single loops}] + [\text{double loops}] + \dots$$

This looks reasonable, but I have never read about such a thing. Maybe, though, I am now on the way to making sense out of the series in the exponent.

3. Lets now do the functional integral to get the Fp determinant, and then expand the determinant using the trace theorem:

$$W(0) = \int [dc] [d\bar{c}] e^{i\tilde{S}_c} = (\det M)^{-1} = (e^{\hbar [\ln M]})^{-1} = e^{-\hbar [\ln M]}$$

Let $M = 1 + L$, then expand

$$e^{-\hbar \text{trace} [\ln(1+L)]} = \exp \left[-\hbar \left\{ \cancel{L} + L^2 - \frac{L^2}{2!} + \frac{L^3}{3!} \dots \right\} \right]$$

$$= \exp \left[-\hbar(L) + \frac{1}{2} \hbar(L^2) - \frac{1}{3!} \hbar(L^3) + \dots \right] = W(0).$$

I have the desire to identify $\text{trace}(L)$ with a loop with 2 dots, $\text{trace}(L^2)$ with the three-dotted loop and so on. Then you might sort of see why you should put in a minus sign for the fermions.

4. Also, you get the feeling that $Z()$ should then be the sum of closed loops, just as Coleman said. Still, there are pieces missing. Ie, I dont really know how to interpret the object called L , and so on. Also, I made a basic mistake. The ghost action is not an interaction hamiltonian like the ϕ^2 I was talking about. The ghost thing is like a non-interacting theory kinetic energy term, and there is no perturbation series because there are no interactions.

So many loose screws need to be located, defined, and tightened to make this stuff go.

Induction Proof of (16.6)

① Recall that $\frac{\delta^n W}{\delta J_1 \dots \delta J_n} = \frac{\langle 0 | T(\phi_1 \dots \phi_n) | 0 \rangle^J}{\langle 0 | 0 \rangle^J}$

'If J is "n" + $\delta\phi$ is part of the Hamiltonian. Now, assume (16.6) is true for n , show true for $(n+1)$.

$$\begin{aligned} \frac{\delta^{n+1} Z(J)}{\delta J_1 \dots \delta J_{n+1}} &= (i) \frac{\langle 0 | T(\bar{\phi}_1 \dots \bar{\phi}_n) | 0 \rangle^J}{\langle 0 | 0 \rangle^J} = (i) \frac{\langle 0 | T(\bar{\phi}_1 \dots \bar{\phi}_n) | 0 \rangle^J}{W[J]} \\ &= [W(J)]^{-1} \cdot \int [d\phi] \bar{\phi}_1 \bar{\phi}_2 \dots \bar{\phi}_n e^{i \int dx [\mathcal{L} + J \cdot \phi]} \quad (i)^{n+1} \end{aligned}$$

Take one more derivative:

$$\begin{aligned} \frac{\delta^{n+1}}{\delta J_1 \dots \delta J_{n+1}} &= \left\{ [W(J)]^{-1} \cdot \int [d\phi] \bar{\phi}_1 \dots \bar{\phi}_n i \phi_{n+1} e^{iS} \right. \\ &\quad \left. - [W(J)]^{-2} \frac{\partial W}{\partial J_{n+1}} \cdot \int [d\phi] \bar{\phi}_1 \dots \bar{\phi}_n e^{iS} \right\} (i)^{n+1} \\ &= [W(J)]^{-1} \left\{ \int [d\phi] \bar{\phi}_1 \dots \bar{\phi}_n i \phi_{n+1} e^{iS} - i \cdot \int [d\phi] \bar{\phi}_1 \dots \bar{\phi}_n e^{iS} \right\} (i)^{n+1} \\ &= [W(J)]^{-1} \left\{ \int [d\phi] \bar{\phi}_1 \bar{\phi}_2 \dots \bar{\phi}_n \phi_{n+1} e^{iS} \right\} (i)^n \\ &= \frac{\langle 0 | T(\bar{\phi}_1 \dots \bar{\phi}_{n+1}) | 0 \rangle^J}{\langle 0 | 0 \rangle^J} (i)^n \quad \text{QED.} \end{aligned}$$

② Show for case $n=1$. $\frac{\partial Z}{\partial J_1} = \frac{\langle 0 | \phi_1 | 0 \rangle^J}{\langle 0 | 0 \rangle^J} ? = \frac{\langle 0 | \phi_1 | 0 \rangle^J - \langle 0 | 0 \rangle^J v_1}{\langle 0 | 0 \rangle^J}$

$$= \frac{\langle 0 | \phi_1 | 0 \rangle^J}{\langle 0 | 0 \rangle^J} - v_1 = \frac{1}{W} \frac{\partial W}{\partial J_1} - v_1 \quad \text{not true.}$$

On attached sheet I show true for $n=2$. Thus,
proved for $n=2, 3, \dots$

Show for $n=2$

~~Induction proof of statement about $Z(x)$~~

$$① \left. \frac{\partial W}{\partial \tau} \right|_{\tau=0} = i \langle 0 | \phi | 0 \rangle ; \quad W = e^{iZ} ; \quad \frac{\partial W}{\partial \tau} = W \cdot i \frac{\partial Z}{\partial \tau}$$

$$\Rightarrow \left. \frac{\partial Z}{\partial \tau} \right|_{\tau=0} = \left[\frac{1}{iW} \frac{\partial W}{\partial \tau} \right]_{\tau=0} = \frac{i \langle 0 | \phi | 0 \rangle}{i \langle 0 | 0 \rangle} = \phi_{\text{classical}, \tau=0} \equiv v$$

$$\text{So } \boxed{\left. \frac{\partial Z}{\partial \tau} \right|_{\tau=0} \equiv v} \quad \checkmark$$

$$\begin{aligned} ② \frac{\partial}{\partial \tau_1} \left(\frac{\partial W}{\partial \tau_2} \right) &= \frac{\partial}{\partial \tau_1} \left[W \cdot i \frac{\partial Z}{\partial \tau_2} \right] = (iW) \frac{\partial^2 Z}{\partial \tau_1 \partial \tau_2} + \left(\frac{\partial Z}{\partial \tau_2} \right) i \left[W \frac{\partial Z}{\partial \tau_1} \right] \\ &= W \cdot \left[i \frac{\partial^2 Z}{\partial \tau_1 \partial \tau_2} - \frac{\partial Z}{\partial \tau_1} \frac{\partial Z}{\partial \tau_2} \right] \quad \checkmark \end{aligned}$$

Better allow fields to have components!

$$\frac{1}{W} \frac{\partial^2 W}{\partial \tau_i \partial \tau_j} = \frac{i}{\partial \tau_i \partial \tau_j} \frac{\partial^2 Z}{\partial \tau_i \partial \tau_j} - \left(\frac{\partial Z}{\partial \tau_i} \right) \left(\frac{\partial Z}{\partial \tau_j} \right) \checkmark = i Z_{ij} - v_i v_j \quad \checkmark$$

$$\text{So } \left. \frac{W_{,ij}}{W} \right|_{\tau=0} = i Z_{,ij}|_{\tau=0} - v_i v_j$$

③ Check claim for $n=2$:

$$\text{Therefore: } \left. \frac{i \partial^2 Z}{\partial \tau_i \partial \tau_j} \right|_{\tau=0} = \frac{i^2 \langle 0 | T(\bar{\phi}_i \bar{\phi}_j) | 0 \rangle}{\langle 0 | 0 \rangle}$$

$$\begin{aligned} \frac{\langle 0 | T(\bar{\phi}_i \bar{\phi}_j) | 0 \rangle}{\langle 0 | 0 \rangle} &= \frac{\langle 0 | T(\phi_i - v_i)(\phi_j - v_j) | 0 \rangle}{\langle 0 | 0 \rangle} = \\ &= \frac{\langle 0 | T(\phi_i \phi_j) | 0 \rangle}{\langle 0 | 0 \rangle} - v_i v_j - v_i v_j + v_i v_j \end{aligned}$$

$$\text{Or, } \frac{\langle 0 | T(\phi_i \phi_j) | 0 \rangle}{\langle 0 | 0 \rangle} = \frac{\langle 0 | T(\bar{\phi}_i \bar{\phi}_j) | 0 \rangle}{\langle 0 | 0 \rangle} + v_i v_j$$

$$\Rightarrow \frac{i^2 \langle 0 | T(\phi_i \phi_j) | 0 \rangle}{\langle 0 | 0 \rangle} = \frac{i^2 \langle 0 | T(\bar{\phi}_i \bar{\phi}_j) | 0 \rangle}{\langle 0 | 0 \rangle} - v_i v_j$$

④ So here is what I just showed:

$n=2$ ✓

$$\left. \frac{\partial^2 Z}{\partial \bar{\psi}_i \partial \psi_j} \right|_{\bar{\psi}=0} = (i)^{2-1} \frac{\langle 0 | T(\bar{\psi}_i \psi_j) | 0 \rangle}{\langle 0 | 0 \rangle}$$

stop here.

⑤ Hopefully, this is the case $n=2$ of a general result,

$$\left. \frac{\partial^n Z}{\partial \bar{\psi}_i \partial \bar{\psi}_{i_2} \dots \partial \bar{\psi}_{i_n}} \right|_{\bar{\psi}=0} = (i)^{n-1} \frac{\langle 0 | T(\bar{\psi}_i \bar{\psi}_{i_2} \dots \bar{\psi}_{i_n}) | 0 \rangle}{\langle 0 | 0 \rangle}$$

It remains, therefore, to show $k \Rightarrow k+1$. But cannot carry through because cannot "undo" $\bar{\psi}=0$ to take next derivative.

⑥ Go back to:

Section 15: Equivalence of the Landau and Coulomb gauges.

1. I thought that Coleman had more or less convinced me that the path integral was independent of which surface you took, and therefore it was obvious that all gauge choices give the same generating function. But here, they explicitly compute the relation between W_L and W_C and the result is not that they are equal, but almost equal according to (15.6). In fact they are not equal, and the unrenormalized S-matrices are also not equal. After renormalization things become equal again. This is a lot more complicated than I thought. Right now I know nothing about renormalization theory.

More Functional Facts

① Let $\mathcal{F}[f] = \int dx F(f(x))$ where

$F =$ a function of f

$\mathcal{F} =$ a functional of f .

$$\text{Then } \frac{\delta \mathcal{F}[f]}{\delta f} \equiv \frac{dF(f)}{df}$$

The functional derivative of a functional is a function, not a distribution.

2. However, the functional derivative of a function is a distribution:

Consider: $\mathcal{F}[f] = \int dx \overbrace{[F_1(f) \delta(x-x_0)]}^{F(f)} = F_1(f(x_0))$

$$\text{Then } \frac{\delta \mathcal{F}[f]}{\delta f} = \frac{dF}{df} = \underbrace{\frac{dF_1(f)}{df} \cdot \delta(x-x_0)}_{\text{distribution}} = \frac{\delta F_1(f(x_0))}{\delta f(x)}$$

$\swarrow$ formal deriv
of F_1 , a
function of f .

Special case: $\frac{\delta f(x_0)}{\delta f(x)} = \delta(x-x_0).$

Section 16: Generating Functionals for Green's functions and proper vertices.

1. This is a very meat and potatoes section, cram packed with useful ideas.
2. Theorem: The derivatives of the $Z[J]$ functional give the connected greens functions of the shifted fields ϕ . This fact resolves earlier conflict I had.

The first derivative of Z defines the classical field ϕ in presence of source J . Set $J=0$ to get classical field = v_1 , the vacuum expectation value of the quantum field.

3. The first Z derivative gives the classical field ϕ in presence of J . In effect, this defines $\phi(J)$. To invert this equation, you use the Legendre transform trick. The dual function is called GAMMA, dual to Z . The first derivative of Gamma is $-J$. The higher ϕ derivatives of gamma yield the proper vertices $G^{(n)}$ with external props amputated. This is still a slightly amazing fact! Since the various derivatives starting with $n=2$ of $G[\phi]$ give the proper vertices, you can represent G as in (16.20), Gamma is called "the proper vertex generating function". Notice that in the derivatives you set $\phi = v$ just like you set $J=0$ in getting the regular greens functions.

4. The superpotential is also a generating function of sorts, defined in (16.23). The coefficient of each term in the sum is the momentum-space proper vertex greens function (legs amputated) evaluated at zero momentum. Why zero momentum? Because then, as in (16.25), the curvature of this function gets identified with mass! Ie, the inverse renormalized propagator evaluated at zero momentum. Notice that the superpotential is really almost the same as the generating proper vertex functional. So curvature equals mass, and slope = 0 when $\phi = v$. Therefore, this is a good candidate to apply the classical field theory analysis on SSB! In effect, we have proved Goldstones to all orders of perturbation theory, for a "regular" field theory (ie, no gauge group).

5. Obviously, one would like to know, given some lagrangian, what the superpotential $V(\phi)$ looks like, so you can find its minima and then see if there is a superconducting vacuum or not. There is a perturbative method to calculate this V , called the "loop expansion". The first term is just the regular potential of the lagrangian. To this same lowest order, the Gamma generator is just the action, so proper vertices are just the coupling constants in the lagrangian! Also, to this order the classical field is the same as the solution field of the classical field equations. When you go to the one loop level, the Z is corrected by a fancy trace. This causes Gamma to be corrected by the same trace, as shown in (16.38). This trace correction then finds its way into the superpotential.

6. The point is made that , although it appears that the first few terms in the potential are divergent, in fact they can be absorbed into renormalized constants.

7. From (16.44), it appears that to the extent that λ is small, the regular potential is a good estimate for the superpotential.

Show (16.26)

① Write:

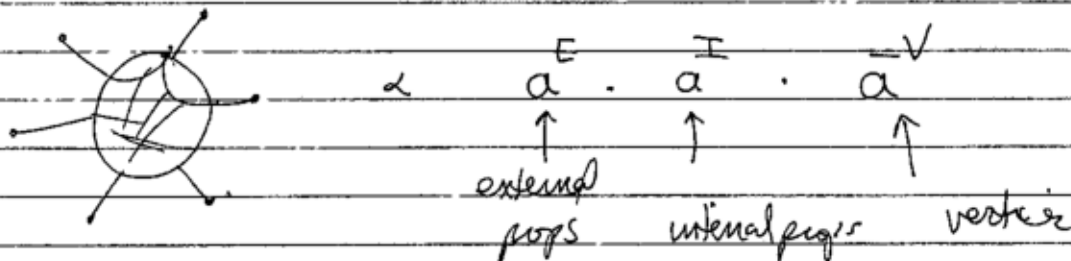
$$\begin{aligned}
 W &= \int [d\phi] e^{i \int d^4x \left[\frac{1}{2} \partial_\mu \phi \partial^\mu \phi + \frac{1}{2} \partial_\mu \phi \partial^\mu \phi + \mathcal{L}(\phi) \right]} \\
 &= e^{i \int d^4x \left[\frac{1}{2} \partial_\mu \phi \partial^\mu \phi + \mathcal{L}(\phi) \right]} \cdot \underbrace{\int [d\phi] e^{i \int d^4x \left[\frac{1}{2} \partial_\mu \phi \partial^\mu \phi + \mathcal{L}(\phi) \right]}}_{=1} \\
 &= \int [d\phi] e^{-\left(\phi, \frac{A}{2} \phi\right) - \left(\phi, J\right)} \\
 &= e^{\left(J, \left[A/2\right]^{-1} J\right)} = e^{\int d^4x d^4y J(x) \underbrace{a A^{-1}}_{\cdot \Delta_F} J(y)}
 \end{aligned}$$

Thus, the Δ_F appears correctly in (16.26).

② Interpret Diagrams:

- a) each prop Δ_F gets an a
- b) each vertex ϕ^n gets a^n .

So general diagram (with external props,



So power of a graph is a^{E+I-V}

③ Notice $I - V + 1 = 3 - 3 + 1 = 1 = \# \text{ loops}.$

Page 92 algebra.

$$\sum_{j,i} \frac{\delta \mathcal{L}_j}{\delta \phi_k} = \delta_{ik} \quad \sum_{j,i} \frac{\partial \mathcal{L}_j}{\partial \phi_k} \frac{\partial \phi_k}{\partial \mathcal{L}_i} = \underbrace{\delta_{ik} \frac{\partial \phi_k}{\partial \mathcal{L}_i}}_{\delta_{il} = 1} = \frac{\partial \phi_i}{\partial \mathcal{L}_i}$$

$$\text{So } \sum_{j,i} \frac{\partial \mathcal{L}_j}{\partial \mathcal{L}_i} = \frac{\partial \phi_i}{\partial \mathcal{L}_i} \Rightarrow \text{so}$$

$$\text{So } \boxed{\frac{\delta \phi_i}{\delta \mathcal{L}_j} = \frac{\delta \mathcal{L}}{\delta \mathcal{L}_j \delta \mathcal{L}_i} = [-X^{-1}]_{ij}} \quad \text{Just diff (16.3)!}$$

$$\boxed{\frac{\delta \mathcal{L}_i}{\delta \phi_m} = -\frac{\delta^2 \mathcal{L}}{\delta \phi_i \delta \phi_m} = [-X]_{im}} \quad \text{Just diff (16.8)}$$

Left term shows obvious inverse relation!

Now take $\partial/\partial \mathcal{L}_i$ on top box:

$$\frac{\delta^2 \mathcal{L}}{\delta \mathcal{L}_i \delta \mathcal{L}_j \delta \mathcal{L}_l} = \frac{\partial}{\partial \mathcal{L}_i} \left[\frac{\partial \phi_i}{\partial \mathcal{L}_j} \right] = -\frac{\partial}{\partial \mathcal{L}_i} (X^{-1})_{ij}$$

Easier to rewrite first box as:

$$\frac{\delta^2 \mathcal{L}}{\delta \mathcal{L}_i \delta \mathcal{L}_j} \frac{\delta \phi_j}{\delta \phi_k} = \delta_{ik}$$

Then diff:

$$\sum_{j,i,l} \frac{\partial}{\partial \mathcal{L}_i} [-X]_{jk} + \sum_{j,i} \frac{\partial}{\partial \mathcal{L}_i} \left[\frac{\partial \mathcal{L}_j(\phi)}{\partial \phi_k} \right] = 0$$

$$\downarrow$$

$$[-X^{-1}]_{ij} \frac{\partial \phi_m}{\partial \mathcal{L}_i} \frac{\partial}{\partial \phi_m} \left[\frac{\partial \mathcal{L}_j}{\partial \phi_k} \right]$$

$$[-X^{-1}]_{ml} [-\Gamma_{jkn}]$$

So by -1.

$$\sum_{j,i,l} \underbrace{(X)_{jk}}_{\text{by -1}} + \underbrace{(X^{-1})_{ij}}_{\text{by -1}} \underbrace{\Gamma_{jkn}}_{\text{by -1}} \underbrace{(X)_{ml}}_{\text{by -1}} = 0$$

$$- \sum_{ijl} (X)_{jk} = (X^{-1})_{ij} \Gamma_{jkn} (X^{-1})_{nl}$$

Mult by $(X^{-1})_{kr}$ and get

$$- \sum_{ijl} (X^{-1})_{kr} (X^{-1})_{ij} (X^{-1})_{nl} \Gamma_{jkn}$$

On:

$$- \sum_{j3} (X^{-1}) (X^{-1}) (X^{-1}) P_{j3}$$

Now add i's

$$\sum_{j3} = (iX^{-1})(iX^{-1})(iX^{-1}) (iP_{j3}) \quad \lambda^4 = +1$$

Thus, 16.15

So set $\gamma = 0$:

by (16.6)

$$\text{Green}(3) = \frac{1}{i\epsilon} \sum_{j3} \Delta_F' \Delta_F' \Delta_F' [P_{j3}]$$

So $\text{green}(3) = P_{j3}$ (16.16)

Section 17: Renormalization in the σ -model.

1. First, the punch line. If you know how to renormalize the "symmetric σ -model" (ie, the $U(1)$ or $O(4)$ invariant lagrangian without that linear term which gives PCAC), in the regular mode, then you also know how to do renormalization in the Goldstone mode, or in the model with the linear term added and either mode activated. The fact that I don't know how to renormalize anything should not block me from getting the gist of this section.
2. Notice the trick of having one instead of three pions for didactic purposes. You thereby avoid all the pain of chirality and $O(4)$ etc etc.
3. So here is how it goes. You write down your potential with or without the linear term. If no linear term, then the theory has the usual regular and goldstone mode, depending on the sign of μ^2 . When you turn on the linear term so that $c \neq 0$, it appears from (17.5) that you are forced to a goldstone mode ie to a theory with superconducting vacuum, ie, sigma field has nonzero VEV.
4. Consider the generating function for the symmetric theory. If you take derivatives and then set $J=0$, ie, $J = (0,0)$, you get the connected Greens functions of the symmetric theory. On the other hand, by evaluating the same derivatives at $J = (u,0)$, you get the greens functions of the theory including the linear term!
5. Now several other rules get slightly modified. Look at (17.15). Usually we identify this derivative as the slope of the potential and we expect it to be zero when $\phi = u$, ie, at the minimal point. The reason it is not zero in (17.15) is that Γ is the vertex generating function of the symmetric theory, not of the theory-with-linear-term. The idea is to solve (17.15) for u . Ie, given that constant c in the linear term, what number u causes (17.15) to be true? Then evaluate higher derivatives of Γ at this point $\phi=u$ and you will Green vertices of the theory-with-linear-term.
6. Restate this last point. If you were doing the symmetric σ -model without the linear term, you would set $c=0$ and solve (17.15) for some u . Ie, in general there is some function $u(c)$. Ie, $u = u(c)$. $u = u(0)$ is the correct u for the symmetric theory and evaluating at this u would give the symmetric greens functions in (17.16). But evaluating (17.16) at $u = u(c)$ with $c \neq 0$ gives vertices for the σ -model.
7. Thus, if you like, (17.16) gives the general vertex as a function of $u = u(c)$. By putting in different u 's, you get the vertices of different theories.
8. Look at the equation above (17.19). This relates $\Gamma(u)$ to $\Gamma(0)$ via simple Taylor series. If your theory is regular in absence of c term, then regular theory is described by this $\Gamma(0)$. Adding the c term gives $\Gamma(u)$. Thus, if you know the regular theory, you also know the sigma model.
9. The point being made here is also the point made by Coleman: if your symmetric lagrangian is renormalizable, the fact that it may have a mode with SSB does not affect renormalizability. Here, we have shown this fact by showing that the regular theory vertices (to all orders) determine the theory at $u \neq 0$. Also, the $u \neq 0$ may be caused either by SSB (internal breaking) or by external linear term. In either case, renormalization follows from renormalization of the $u=0$ theory. This in turn is well known.

$\frac{d}{dJ} = c(u)$
 $\frac{d}{dJ} = c(u)$
 $\frac{d}{dJ} = c(u)$

10. Say it once more: you can drive u away from 0 by either internal or external symmetry breaking. Regardless of the cause, the theory is still renormalizable as long as the $u=0$ theory is renormalizable.

11. Ward identities: If you rotate your source functions J at the same time as your fields, the term $J\phi$ stays invariant. This simple fact at once makes a relation between two first order derivatives of Γ . Higher derivatives then give you the more familiar Ward identities.

12. Recall in QED how the Ward identity was helpful in the renormalization process. Same thing will happen here. Again, all these Ward identities are functions of u . At $u=0$ they apply to the regular theory, at some other u they apply to the broken theory, spontaneously or not.

The next task is to learn something about how to do renormalization!

Section 18. The BPHZ renormalization program.

1. References here are: the 1959 book of Bogoliubov and Shirkov; a 1965 paper by Hepp; some 1970 lectures by Zimmermann. Also Symanzik has a foot in this somewhere.

2. Many things are defined: most are OK:

- proper diagram
- superficial degree of divergence D
- the processed integrand R and its corresponding finite integral J
- primitively divergent diagram
- operation $(1-t)$ to effect subtractions
- renormalization part
- the contraction notation
- Bogoliubov's R operation: (pre-process integrand, then subtract with $(1-t)$).
- Zimmermann's forest theorem
- propagator regularization

3. When is a theory renormalizable? Ie, what is the punchline of all this high technology? Answer: look back at the definition of the index δ . Recall Coleman's remark that this thing is defined in complete disregard of the dimensions of the coupling constant. Thus, if $L_I = a \phi^5$, ie, PHI-FIFTH theory, $\dim(L) = 5$ and $\delta = +1$. Of course $\dim(L) = 4$ if you include dimensions of the couplings constant that is not the point.

The point is this: if all δ_i of the various interaction terms are ≤ 0 , the theory is renormalizable. A simple result to remember!

Technically this is so because then only a finitenumber of counterterms are necessary to implement the R -operation and achieve renormalization. Obviously ϕ^5 theory is not renormalizable, but ϕ^4 and ϕ^3 theories are. QED has $\bar{\psi}\gamma_\mu\psi A^\mu$ which is $3/2+3/2 + 1 = 4$, hence yes, QED is renormalizable. Same goes for the sigma model since all terms are like ϕ^4 .

Dont forget this very simple and useful result!

4. The rest of the section is to technical for me to follow at this time.

Notes on BPHZ Renormalization Theory

$\mathcal{L}_i = i^{\text{th}}$ interaction Lag. $\mathcal{L}_I = \sum_i \mathcal{L}_i$.

$b_i = \#$ boson fields in $\mathcal{L}_i$

$f_i = \#$ fermion fields

$d_i = \#$ derivatives ∂_μ .

Note: $\dim \mathcal{L}_i = b_i \times 1 + f_i \times \frac{3}{2} + d_i \times 1$.

How do I know then $\mathcal{L} = \bar{\psi} \gamma_\mu \psi$; $\mathcal{L} = m^2 \phi^2$ is obvious.
 $\uparrow$ $\uparrow$ $\uparrow$
 E $E^{\frac{3}{2}}$ E'
 each each

Usually, $\mathcal{L}_i$ has dimension 4. So define

$\delta_i = \dim(\mathcal{L}_i) - 4 =$ deviation from usual dimension

$E_B = \#$ External bosons

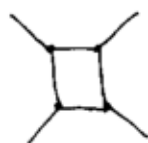
$E_F = \#$ External fermion

$\Gamma =$ OPI graph of some sort

$I_B = \#$ Internal boson lines in graph

$I_F = \dots$

$n_i = \#$ vertices in graph.



$$E_B = 4$$

$$I_B = 4$$

$$n = 4$$

$$b = 3$$

$n_b = \#$ boson lines ending in vertices.

Thus, $n_b = E_B + 2I_B$

Each internal line accounts for two such lines.

Thus, (18.2,3) now clear.

Power Counting:



$$I_B \rightarrow E^{-2}$$

$$I_F \rightarrow E^{-1}$$

$$\text{deriv} \rightarrow k^\mu \rightarrow E^{+1}$$

$$\text{loop} \rightarrow d^4k \sim k^3 dk d\Omega \rightarrow E^4$$

Thus, $4I - 2I_B - I_F = \text{measure of divergence} = D_{\text{Bjorken hell.}}$

But $I = I_B + I_F - V + 1$, so

$$D(P) = 4[I_B + I_F - V + 1] - 2I_B - I_F = 2I_B + 3I_F - 4V + 4 \quad (18.4).$$

If you also have gradients, add in $\sum n_i d_i$.

$$\text{Now: } D = \sum n_i d_i + 2[-E_B + \sum n_i b_i] + \frac{3}{2}[-E_F + \sum n_i f_i] - 4V + 4$$

$$= \sum n_i [d_i + b_i + \frac{3}{2} f_i] - E_B - \frac{3}{2} E_F - 4V + 4 + 4 \sum n_i$$

$$= \sum n_i \delta_i - E_B - \frac{3}{2} E_F + \{4 \sum n_i + 4 - 4V\} = (18.5)$$

I_P = integral of feynman graph P .

F_P = the integral

ΔF = prop.

P = vertex.

R_P = processed integral (eg, subtracted)

J_P = resultant integral (finite).

- Γ is "primitively divergent" $\Leftrightarrow$
- a) graph is proper, i.e., OPI.
 - b) superficially divergent $D(\Gamma) \geq 0$.
 - c) becomes convergent if you break a line.

For such graphs [which seem to be almost convergent], the rule to get the finite piece is:

$$R_\Gamma = (1 - t_\Gamma) I_\Gamma$$

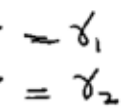
↑
i.e., subtract off something.

Some formula for subtractions,
not clear to me how to connect to what I remember.

"normalization part" $\equiv$ proper diagram with $D \geq 0$.

"disjoint diagrams" γ_1, γ_2 are d.d. if $\gamma_1 \cap \gamma_2 = \emptyset$.

$F = \Gamma / \{\gamma_1, \gamma_2, \dots, \gamma_c\}$ means: start with a graph Γ + shrink each of the listed subdiagrams to a point to get a new diagram.

Eg,  = Γ .  = γ_1 ,  = γ_2 $\rightarrow F =$ .

Bogolubov says: if Γ has $D \geq 0$, then do $R_\Gamma = (1 - t^\Gamma) \bar{R}_\Gamma$

where $\bar{R}_\Gamma = I_\Gamma + \sum_{\{\gamma_1, \dots, \gamma_c\}} I_{\Gamma/\{\gamma_1, \dots, \gamma_c\}} \prod_{i=1}^c O_{\gamma_i}$

I.e., just you add some other terms to your integrand, then you do subtractions.

"overlap" — same as for channels.

Section 19: Dimensional Regularization.

1. The idea is to continue in the number of space-time dimensions from $n=4$ to complex n ; do your integrals which can then converge, then take limit as $n \rightarrow 4$. The divergences will then appear as poles. These poles are precisely removed by the R-prescription.
2. Technically, here is what you do. ~~Param~~ Take a particular Feynman graph. Put all propagators into usual exponential parametrization. Do the usual completion of the square trick or whatever (recall ELOP) so that you can do the d^4k integrals. Only remaining integrals are the d^4 's. The UV divergence now appears as end-point singularity caused by denominator shown. This is the usual procedure.
3. Here is the dimensional procedure. Write $d^n k = d^4 k \, d^{n-4} k$ where you extend to larger space. Set up the alpha parameters as before. Then do the $d^{n-4} k$ integral directly. Your result then has those potential endpoint singularities. Then choose n small enough so that these endpoint sings go away. Then you have desired integral as a function of n .
4. A reason this method is nice is that it preserves at every step your various symmetries, since these don't depend on dimension of spacetime. Recall in BD vol 1 how they kept doing things in such a way as to preserve gauge invariance. I guess that is desirable.
5. Big footnote on the Adler anomalies. Apparently even the fancy dimensional regularization cannot handle these guys. You really want to avoid anomalies in a gauge theory. Some little theorems:
 - a) if triangle anomaly absent, then all anomalies are absent.
 - b) if the triple-gluon fermion loop is anomaly-free, your NAGT is OK.In fact you can write down an explicit condition for the ~~triple~~ loop of interest. You can gauge anomaly cancellation by choosing particles carefully. I think charmed particle does this in Georgi Glashow model.
6. An example of this t'hooft Veltman regularization is done: the two graphs which add to give lowest order vacuum polarization in scalar ED theory. You have two terms. By going to complex n , you find that the second term = 0? I thought you were supposed to get a pole? Well, I really don't have any details here. The idea is clear though.

Section 20: Feynman Rules and renormalization of SSB gauge theories. Landau gauge.

1. We consider a specific model of a gauge theory. Model has $O(3)$ as gauge group, so non-abelian. The only carrier particles are some scalar pions with isospin 1, ie, $SU(2)$ group. Add in the gauge fields and the ghosts and figure out the Feynman rules. For no SSB this is rather straightforward.

2. A trick to locate and identify all the necessary counterterms is to scale all the bare fields and couplings in a certain way. Then (20.6) and (20.7) gives you all the counterterms. Then you renormalize the theory in the "usual way", ie, you show how the counterterms fix things up to any order. For more details see Lee and Zinn-Justin.

3. Now, what happens if you are in the Goldstone mode, ie, there is ssb. In the usual way go ahead and define your proper vertex generating functional; we now have a classical field for the isovector of scalars and for the isovector of gauge fields.

Suppose there were a linear $\xi \cdot \phi$ term in the lagrangian. Here ξ is a constant source and plays the role of constant c in the sigma model. Choose ξ in z -direction in isospace. Assume that this causes the field ϕ to have a nonzer VEV called u . So you solve (20.15) so that, given a ξ , you can find u . u points also in the z direction.

Since you have therefore broken two gauge generators I_1 and I_2 by assuming that ϕ has a VEV pointing in the I_3 direction, you expect to get two massless goldstones. These are referred to as χ fields. Look at (20.18). It says: suppose that constant $\xi = 0$ so there really was no explicit breaking in the lagrangian. But suppose $u \neq 0$ still. You are forced to conclude that $m_\chi = 0$; just statement of the goldstone theorem.

So, what do you do to get feynman rules and so on? As usual you shift the z piece of the ϕ field to something called χ , so now you replace the three fields ϕ with the three fields $\chi(2)$ and $\chi(1)$. Rewrite the lagrangian in terms of these shifted fields. Only problem now is that, even though these are all physical fields after the shift, you seem to still have a χ tadpole term in there. Somehow you make this thing go away.

Then once this tadpole term is killed off... you renormalize the theory in exactly the same way as before for the theory without SSB. Notice that there are no massive vectors here because we are not in the unitary gauge. We are in the Landau gauge where the χ field has some mass and is still in there. Ie, I know that this fields .. no wait. Look carefully at (20.25). You do see a mass term for some of the vector gauge fields. But you also see the χ - fields as massless goldstones. In the U-gauge these goldstones would have been gauged away. Here they are still present. Somehow they must decouple.

From this lagrangian you read off the feynman rules for this theory in the goldstone mode. You see that two of the gauge fields are now massive, as just noted; one is still massless (photon); you also have a massive χ propagator. In addition you have massless ghost c field and that goldstone fields χ .

4. But, when you rewrite the massive vector boson propagator, you see that it too has some kind of piece with poles at $k^2=0$. These are some kind of awful negative probability scalars, ie, ghosts in another sense.

As you would expect, when you go to the ~~xxx~~ S-matrix, you get a glorious cancellation of the three kinds of massless bosons sitting in this theory: the ghosts c , the ghosts just mentioned, and the goldstones χ . The goldstones "decouple" as expected.

Summary of this section: Given a gauge theory, you have to consider both possibilities of no-SSB and SSB. In the first case you read off your Feynman rules in the usual way and you show theory is renormalizeable in usual way. But, in SSB case you have to shift field with nonzero VEV. This creates goldstones. You then have to show that theory is still renormalizeable, and that the goldstones decouple.

Section 21: The R_ξ-Gauges.

1. The model is set up in rather general terms. Let ϕ be some gauge group vector of mesons, say of dimension m . These are the Higgs scalars because one will break the symmetry. Couple in your gauge fields to these scalars in the usual way; add the F_{uv} gauge term. Assume that the Higgs potential is such that it has a minimum so you have SSB. The vacuum vector v is still invariant under M rotations, but breaks $N-M$ of them. Recall that this means that the vector v is annihilated by the $N-M$ generators in the goldstone grouping. So shift the field vector ϕ to ϕ' and observe that there is now a gauge-field mass-matrix called μ^2 . For the $m-(N-M)$ Higgs fields that are not converted to goldstones, there will be a mass matrix M^2 . The fancy projection operator is in there merely to separate the goldstones from the other Higgs scalars.

Now, notice in (21.14) that after the shift for SSB, there is a direct coupling of the gauge fields A_μ to the Higgs scalars $d_\mu \phi$. They are going to choose a fancy gauge that kills off this coupling. The gauge surface is stated in (21.23) and is almost the Landau gauge, but it has a piece added onto it involving the ϕ' field. I guess there is nothing wrong with choosing F in this way; recall that it just had to be a gaussian to be reasonable. So, exponentiation-of-the-delta creates a term which cancels that just-mentioned $A-\phi'$ coupling; it also adds a little constant term to the gauge propagator so that is just like the $\frac{1}{2}$ thing earlier, now called ξ . I.e., this parameter ξ is a free parameter in the gauge condition. The last term in $F^T F$ from the delta causes an additional ϕ masslike term which you see in (21.24).

Finally, you have to compute the FP determinant and then use ghosts to exponentiate it. This gives (21.32). So, now you are in a position to read off all the Feynman rules of the theory.

- a) gauge propagator is (21.27); recall that μ^2 is a matrix, so this shows that some of the gauge fields have mass. There is some spurious pole at $\xi^{-1} \mu^2$.
- b) the Higgs particle propagators are given in (21.28). The still massive ones propagate at masses det by mass matrix M^2 . The ones in the Goldstone space seem to have mass $\xi^{-1} \mu^2$ instead of being really massless goldstones.
- c) the ghosts c now have mass $\xi^{-1} \mu^2$ also, and there is a new $c-c-\phi$ coupling.

2. What have we done? We have taken an arbitrary gauge theory with an arbitrary multiplet of Higgs scalars. We have done SSB, and have chosen a peculiar gauge surface. The gauge surface has a parameter ξ . The reason for doing this will become clear in the next section, I presume. As $\xi \rightarrow \infty$, obvious that you go to the original Landau gauge. They claim that if you take $\xi \rightarrow 0$ in the end, you have the unitary gauge.

If they can show that the S-matrix is independent of ξ , then in effect they have shown that all those poles of the gauge, ghost, and goldstones have somehow managed to cancel.

Notes on Section 21, the R_ξ -gauge.

① Let $\mathcal{L} = \frac{1}{2}(\partial_\mu \phi^\dagger)(\partial^\mu \phi) = \text{kinetic part of } \phi \text{ lag, } \phi = \text{group vector.}$

Then by our usual rules,

$$\delta \phi = -i \vec{\partial} \cdot \vec{G} \phi$$

$$\partial_\mu \rightarrow (\partial_\mu - i g \vec{G} \cdot \vec{A}_\mu).$$

where $[G_i, G_j] = f_{ijk} G_k = i C_{ijk} G_k$. Let $G'_i = g G_i$, then $C'_{ijk} = g C_{ijk}$. You can just rescale G 's and C_{ijk} to absorb g .

So:

$$\mathcal{L} = \frac{1}{2}(\partial^\mu \phi^\dagger + i \phi^\dagger \vec{G} \cdot \vec{A}^\mu)(\partial_\mu \phi - i \vec{G} \cdot \vec{A}_\mu \phi) - V(\phi)$$

Note: $G_i^\dagger = G_i = \text{hermitian}$ for all generators.

② Now let's say $\left. \frac{\partial V}{\partial \phi_i} \right|_{\phi=\vec{v}} = 0$, so let $\phi' = \phi - v$ and rewrite things.

$$\mathcal{L} = \frac{1}{2}(\partial^\mu \phi'^\dagger + i(\phi'^\dagger + v^\dagger)(\vec{G} \cdot \vec{A}^\mu))(\partial_\mu \phi' - i(\vec{G} \cdot \vec{A}_\mu)(\phi' + v)) - V(\phi' + v).$$

Extra terms are: $\frac{1}{2} i v^\dagger \underbrace{\vec{G} \cdot \vec{A}^\mu}_{(1)} (\partial_\mu \phi') - \frac{1}{2} i (\partial_\mu \phi')^\dagger \underbrace{(\vec{G} \cdot \vec{A}^\mu)}_{(2)} v$

+ $\frac{1}{2} v^\dagger \underbrace{(\vec{G} \cdot \vec{A}^\mu)^2}_{(3)} v$. Define $(a, Ab) = a^\dagger Ab$ as scalar product.

Then

$$(1) = \frac{1}{2} i (v, G_a \partial_\mu \phi') A_a^\mu$$

$$(2) = -\frac{1}{2} i (\partial_\mu \phi', G_a v) A_a^\mu$$

$$(3) = \frac{1}{2} (v, G_a G_b v) A_a^\mu A_{b\mu}$$

$$+ \frac{1}{2} i \phi'^\dagger (\vec{G} \cdot \vec{A}^\mu) (-i) (\vec{G} \cdot \vec{A}^\mu) v =$$

$$= \frac{1}{2} (\phi', G_a G_b v) A_a^\mu A_{b\mu} \quad (4)$$

$$+ \frac{1}{2} (v, G_a G_b \phi') A_a^\mu A_{b\mu} \quad (5)$$

real so add. $\Rightarrow + (v, G_a G_b \phi') A A$

③ If $\phi = \text{real group vector}$, easy to show that the
 hermitian generators $\hat{G}_{ij}$ are imaginary + antisymmetric.
 Thus:

$$\begin{aligned} (\partial_\mu \phi', G_a v) &= (G_a \partial_\mu \phi', v) = (v, G_a \partial_\mu \phi')^* \\ &= -(v, G_a \partial_\mu \phi') \end{aligned}$$

So we may combine terms ① + ② to get.

$$+i (v, G_a [\partial_\mu \phi']) A_a^\mu$$

So we now have derived (21.6).

Section 22. Proof that the renormalized S-matrix is independent of ξ

This section has several distinct parts which can be separated:

1. First there is an elaborate set of manipulations which all lead to the result (22.18). That is one business. Then comes another business of interpreting the resultant (22.18).

2. Lets examine this first. Equation (22.18) tells us that the net effect of changing the gauge surface from F to $F + \Delta F$ is to add couplings of the source J to the fields. I can see that indeed (22.18) does say this.

What I do not understand is the next conclusion, namely, that when you add terms which are an interaction of fields with the source J , that merely has the effect of shifting your various Z 's. These shifts are then compensated when you go to the renormalized S-matrix, since you divide by the same Z 's.

Thus, the conclusion they claim is this: if you change your generating function by adding couplings between J and the fields ϕ , you simply do not change the renormalized S-matrix one iota. I do not really understand why this is so, but I do see the flow pattern of the logic here. It is this:

1. Show that changing gauge adds only $J \cdot \text{fields}$ term to action
2. Claim that this does not change renormalized S-matrix.
3. Since changing parameter ξ is like changing gauge, the renormalized S-matrix cannot depend on ξ .
4. Therefore the spurious poles must cancel because they were located at a ξ -dependent position, i.e., at $\xi^{-1} p^2$.

3. Now, how is this equation (22.18) derived? One starts with the generating functional $W[J]$ in an arbitrary gauge F . One then shifts the fields ϕ to new fields ϕ' in a rather peculiar way such that the group parameters which I call Θ (and they call u) depend on the fields in a certain way. This little trick enables you to derive the Ard-Takahashi identity for this theory, ie appears as either (22.10) or (22.11). Basically, this is a statement of gauge invariance in terms of the generating function. I am sure there is an easier way to comprehend this thing.

One then computes the change in W if you change F by a small amount. This is shown in (22.16). The WTI is then recast in the form (22.17), and you finally arrive at (22.18) which is the desired result.

4. It seems to me that Coleman convinced me that the generating functional was somehow independent of the gauge surface you chose. But now it seems that that statement is true only after you renormalize.

Section 23: Weak corrections to muon magnetic moment.

1. In the Weinberg-Salam type model, the anomalous magnetic moment due to the weak graphs which convert muon into W boson before hitting vertex, these contributions are all of order Gm^2 where m = mass of muon. But $Gm_W^2 = g^2 = e^2$, so roughly the weak corrections are of the order $\alpha(m/m_W)^2$. The regular electromagnetic anomaly is of order α , so for the muon, the weak corrections are down by $(.1/30)^2 = 10^{-5}$. But the experimental muon anomaly is only known to 4 places, so this will not be too relevant. Notice that for the electron, you will be down an additional 10^{-4} because electron mass so small, thus weak anomaly for electron in Weinberg Salam is order 10^{-9} well beyond any experiment.

2. In models with heavy leptons like the Coleman Glashow, the E-exchange diagrams like those shown on page 133 are stronger than the Weinberg graphs by power (m_E/m_μ) which ratio could be $(2/.1) = 20$, so maybe these graphs in this theory will contribute something of order 10^{-4} to the muon correction, and you could measure it??

3. The actual calculation is done in an R_ξ gauge. You read off the necessary vertices from your set of Feynman rules, and your final result comes out as shown in (23.47), or (23.51) as a number. Answer is about $.5(M/m)$ in last place, so if ratio were 50, you would have a correction of about 25 units. The experimental error is 32 units.

4. One thing you see is this: ,if M_+ is larger than 10 GeV, the created correction will be large and take you out of range of error. Thus, you have ~~xxxxxx~~ an upper bound on the M mass in this theory.

Jeremy Bernstein

1974

Spontaneous symmetry breaking, gauge theories, the Higgs mechanism and all that

Jeremy Bernstein*

Stevens Institute of Technology, Hoboken, New Jersey

A more or less self-contained introductory review is presented of the so-called Higgs phenomenon. This is the mechanism by which, in a certain class of gauge theories, the "photon" and would-be Goldstone scalar mesons conspire together to produce massive vector mesons via a "spontaneous" breaking of gauge invariance. It is conceivable that this is the way in which nature has chosen to unify weak and electromagnetic interactions. It is hoped that a reader of this review will come to understand the meaning of the first three sentences in this abstract and will then be able to proceed to confront a rapidly growing literature in the subject of gauge theories.

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I. INTRODUCTION

Elementary particle theory seems to proceed from fashion to fashion in intervals of two to three years. A few familiar names from the recent past will give the general idea: renormalizable field theories, dispersion relations, conserved and partially conserved currents, current algebras, Regge poles, etc., etc. At the moment when these specialities are at the height of their activity, most practitioners have neither the time nor inclination to go back and read the very early literature in the discipline, so that a group of standard references is arrived at and these become oft quoted and rarely read. When one actually does go back to read these early papers, one is often amazed by how much their authors knew or conjectured, and one comes to the conclusion that these papers rearranged and unified are probably the best introduction to the subject. That will be the spirit and methodology of this review. Anything novel on the part of the present author is unintentional.

The plan of attack in this review is as follows. We begin (in Sec. II) in 1960 when Nambu (Nambu, 1960; Nambu and Jona-Lasinio 1961) observed that the natural interpretation of a conserved $\Delta S = 0$ axial vector current in weak interactions is in the limiting case of a world in which the mass of the meson is set equal to zero. At essentially the same time, Gell-Mann and Lévy (1960) produced several field theoretic models in which this phenomenon was shown to occur as a consequence of the field equations appropriate to the model. Of these models, the so-called "σ" model is, in the present context, the most interesting. In its simplest version, there are three basic fields in the Lagrangian: an isotopic vector pion, an isotopic doublet nucleon, and an isotopic singlet, Lorentz scalar, σ meson. The latter has the quantum numbers of "a vacuum state" of the strong interactions. We use the phrase a vacuum state advisedly since in this class of theories there are in general an infinite number

of orthogonal states with the property that

$$H|0\rangle = 0, \quad (1.1)$$

where H is the Hamiltonian of the theory, and $|0\rangle$ is one of the vacua. Since σ has the quantum numbers of a vacuum, its vacuum expectation value is not forced by any symmetry principle to vanish, i.e., we may have

$$\langle 0|\sigma(x)|0\rangle = \langle 0|\sigma(0)|0\rangle = \lambda \neq 0, \quad (1.2)$$

where λ is a real number and σ is an Hermitian operator with dimensions of a mass. We assume that

$$\sigma(x) = \exp[-i(Px)]\sigma(0)\exp[i(Px)], \quad (1.3)$$

with—note the metric convention +, +, +, - to be used throughout—

$$(Px) = P \cdot x - P_0 t, \quad (1.4)$$

where the P_μ are the generators of space-time displacements. We always assume that

$$P_\mu|0\rangle = 0. \quad (1.5)$$

If we want to give the σ field a particle interpretation, we are forced to redefine it in such a way that

$$\langle 0|\sigma'(0)|0\rangle = 0 \quad (1.6)$$

with

$$\sigma'(x) = \sigma(x) - \lambda$$

otherwise the "vacuum" and the one-particle state will not be orthogonal. As we shall see, it is possible to arrange the Lagrangian of the σ model in such a way that, at the outset the pion and σ have a "bare" mass. The nucleons appear to be massless but, in fact, acquire a mass that is proportional to λ , the vacuum expectation value of the σ. To achieve exact conservation of the axial vector current, A_μ , in this theory, it is necessary to give the pion zero bare mass. This, it turns out, corresponds to a limit in which the theory is invariant with respect to the group $SU(2) \times SU(2)$; exact chiral invariance. Hence, in this way of realizing the symmetry, zero mass bosons—in this case the pions—make their appearance. In 1961, Goldstone conjectured that such zero mass bosons would be an inevitable consequence of a symmetry realization of theories like the σ-model in which the Lagrangian would be fully invariant with respect to a continuous group, but in which the vacuum would not be invariant

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Bernstein, Introduction.

11/77

3.11/78 revised ✓

3.16 FIN.

1. The only kind of field that can have a non-zero VEV is a field with vacuum quantum no's.
2. The " σ -model" was invented by Gellmann & Levy in 1960. In this model, you have the nucleons as iso-doublet and pion as iso-triplet, and single field called σ . At the start, both the pion and sigma have mass, but the nucleons do not. After shifting the sigma field due to humped potential, nucleons acquire mass. In process, I think the sigma loses its mass, or maybe the pion does. Yes. pion loses mass. This is just a goldstone example. This theory also has C.A.C. since it is chiral invariant. The idea that PCAC results from non-zero pion mass is introduced here maybe for first time. This Gellmann-Levy thing came before Goldstone paper.
3. In 1962, it was thought that the Goldstone Theorem was proved and that you were stuck therefore with massless bosons if you have a superconducting vacuum. Higgs found the loophole. This loophole has something to do with a subtlety of gauge invariance and gauge choice, the idea that A_μ is not a 4-vector if you sit in radiation gauge which is non-covariant.
4. Higgs makes a simple model: charged scalar field ϕ_1 and ϕ_2 and photon. Both scalars start out massless, as does the photon. But after doing supercond vacuum shift, ϕ_2 gets a mass, ϕ_1 disappears, and the photon becomes a massive vector meson. You always need a scalar field like ϕ to get going because you need a field that can have a non-zero VEV, as noted above. Such fields are Higgs fields. Some of them are "eaten" by the "photons" but others stay around in the theory.
5. Later Higgs in 1966 shows how to write lagrangians directly in terms of the massive vector fields.
6. Important point: in the shifted Lagrangian, the original symmetry is no longer present and is extremely broken. In Nature, the symmetry may be undetectable directly.
7. In 1964 Englert and Brout do a covariant trip and find that photon propagator gets shifted. This is (Higgs mechanism in disguise.) No doubt lots of messy field theory here.
8. Idea that conserved current may not in certain cases imply a conserved charge! This is new to me. GHK showed this in 1964. Kibble worked this up in 1967.
9. In Weinberg unified theory, one might be surprised that the photon and W have such tremendously different mass. But "broken symmetry" in the spontaneous sense does not imply slight mass differences like conventional H' symmetry breaking term in a Hamiltonian.
10. Combined unified theory of weak and EM is for some reason renormalizable, free bonus.

(2.) The Goldstone Theorem (10 ha-pages)

1. In a gauge theory you have currents and charges. Consider $Q|0\rangle$. Usually this is zero, vacuum has no charge of any kind. But, in SSB analysis, if some scalar field ϕ has a non-zero VEV, then "any charge which drives that field" through the usual commutator has this property: $Q|0\rangle \neq 0$. I never thought of this before.

2. Proof of Goldstone theorem. In (2.27) is defined the FT of commutator of $J(x)$ with $\phi(0)$. Insert complete set of states, assume positive norm etc. I gather that these intermediate states $|n\rangle$ can be at most single particle, because ϕ can create at most one particle. Also, any such states $|n\rangle$ must have $J=0$, based on rotation properties.

The proof here (Gilbert's of 1964) assumes (2.32) which seems obvious to me, and (2.29) seems wrong to me at the moment. If Gilbert can show that there is some contribution to (2.28), then he knows there is a spinless particle floating around. The fact that some field has a nonzero VEV allows him to prove that there must be some term in that sum. Moreover, if this current J is conserved, then (2.32) tells you at once that that state must be massless. So, to summarize: if your theory has a conserved current, but a vacuum which is not killed off by the charge of that same current (ie, non zero VEV of some scalar field), then you must have massless spin-zero particle in your theory.

Maybe the catch here is this: (2.32) is correct only if J_μ is an honest 4-vector. This is the point precisely!

3. The " ρ - model" of GellMann-Levy, 1960. Write a lagrangian of $\vec{\phi}$ and ψ meson fields which is chiral-invariant, ie, $SU(2) \times SU(2)$, sort of isospin and axial isospin together. This theory has a conserved current V_μ and A_μ . Hence charges Q and Q_5 type.

If there is no SSB, then all mesons have same mass; if there is SSB, then you have massless pions come out. This is application of normal Goldstone Theorem: you here have originally 6 conserved currents. From (2.59) you see that, if sigma has a nonzero VEV, then the three axial charges have property that $Q_5|0\rangle \neq 0$. Thus, you expect to get three goldstones, the three pions. Notice that, in this theory, shifted lagrangian still has isospin invariance; you only get a Goldstone boson for each conserved current you "break". Here you started with 6, and you break down to three, so you get three Goldstones. Each "broken" charge will no longer kill the vacuum.

4. Finally nucleons are added. If no SSB, then you have massless nucleons and equal mass pions and sigmas. Experimentally terrible because implies massless nucleons and scalar mesons degenerate with pion. In the SSB view, nucleons acquire mass and pion is massless, looks much better.

5. The Gellman-Levy model is an example of ~~xxx~~ a model with SSB in which the Goldstone Theorem works in the way you'd expect. You break some symmetries and thereby create massless particles. The theory is nice because you want to think of these mass less guys as the true pions. Ie, add a small perturbation to the lagrangian and get very light pion.

6. What we shall really be interested in is the following: under what conditions can the naive Goldstone theorem be violated? That is the Higgs business of the next section.

Comments on Gilbert's 1964 Goldstone Theorem proof.

1. I have worked out the details on the attached sheets.
2. What are the main ingredients?
 - a) if J_μ is a true 4-vector (manifest covariance) then (2.32) is correct. In gauge theories this step will fail if you choose a noncovariant gauge. Condition (2.32) leads at once to (2.33) with no other assumptions.
 - b) one assumption skipped over is that vacuum does not contribute in sum over intermediate states. I am not very clear on this; they say something about positive definite states only are allowed. Certainly you cannot put the vacuum in a unitarity sum.
 - c) Next, this chosen J_μ (a current corresponding to some gauge generator called Q) is assumed to be conserved. This fact leads to (2.39). At this point here is what we know: if $M_\mu \neq 0$, then there must be some scalar and massless particle contributing. The massless part came from the conservation of the current. The spinless came from the fact that ϕ is a scalar field. Maybe I should add that.
 - d) the field ϕ is a true scalar field.
 - e) finally, assume that $\langle 0|\phi(0)|0\rangle \neq 0$, ie, SSB. Then from (2.46) you get $c_1 \neq 0$ and you are done. Ingredient here is basically (2.41) which says: the fields transform in some definite way under the charges. Then (2.43) is a restatement of this fact.
3. Overview: in a theory there are many gauge charges and conserved currents. By examining the way in which the charges "drive" the fields, and by deciding which fields you want to have non-zero VEV, you decide which of the charges Q fail to kill vacuum. Ie, you decide for which currents (2.42) is true.
4. What was not clear from the proof here is that each "broken charge" gets its own goldstone. We only proved that if you have at least one broken charge, then you have at least one goldstone.

Goldstone theorem proof.

① Define $M_{\mu}^{ij}(k) = \int d^4x e^{ikx} \langle 0 | [J_{\mu}^i(x), \phi^j(0)] | 0 \rangle$. (2.27)
Labels i, j denote gauge group labels like iso spin. $\phi = \text{scalar field}$

② Insert $1 = \sum_n |n\rangle\langle n|$. Should you include vacuum here?

$\langle 0 | J_{\mu}^i(x) | 0 \rangle$
I guess we must presume that J_{μ} is made of normal fields? But usually J_{μ} made of the ϕ . So I guess you have to assume that $|0\rangle\langle 0|$ is not allowed.

Then only single meson states allowed, $|n\rangle$ such that $\langle n | \phi(0) | 0 \rangle \neq 0$. So put in $1 = \int \frac{d^3p}{2E} |p\rangle\langle p| = \int d^4p \delta^4(p^2 - m^2) |p\rangle\langle p|$.
Then you get

$$M_{\mu}(k) = \int d^4x e^{ikx} \cdot \int d^4p \delta^4(p^2 - m^2) \{ \langle 0 | J_{\mu}(x) | p \rangle \langle p | \phi(0) | 0 \rangle - \langle 0 | \phi(0) | p \rangle \langle p | J_{\mu}(x) | 0 \rangle \}$$

But $J_{\mu}(x) = e^{-i\hat{p} \cdot x} J_{\mu}(0) e^{+i\hat{p} \cdot x}$ so $\langle 0 | J_{\mu}(x) | p \rangle = \langle 0 | J_{\mu}(0) | p \rangle e^{ipx}$
etc, so

$$M_{\mu}(k) = \int (2\pi)^4 d^4p \delta^4(p^2 - m^2) \delta^4(p+k) \langle 0 | J_{\mu}(0) | p \rangle \langle p | \phi(0) | 0 \rangle - \int (2\pi)^4 d^4p \delta^4(p^2 - m^2) \delta^4(p-k) \langle 0 | \phi(0) | p \rangle \langle p | J_{\mu}(0) | 0 \rangle$$

$$= (2\pi)^4 \delta^4(k^2 - m^2) \langle 0 | J_{\mu}(0) | -k \rangle \langle -k | \phi(0) | 0 \rangle - (2\pi)^4 \delta^4(k^2 - m^2) \langle 0 | J_{\mu}(0) | k \rangle \langle k | \phi(0) | 0 \rangle$$

$p = -k$
 $p = +k$

$$= \left[(2\pi)^4 \delta^4(k^2 - m^2) \left[\langle 0 | J_{\mu}(0) | -k \rangle \langle -k | \phi(0) | 0 \rangle - \langle 0 | \phi(0) | +k \rangle \langle +k | J_{\mu}(0) | 0 \rangle \right] = M_{\mu}(k) \right]$$

Now use manifest covariance. do say:

$$\langle 0 | J_\mu(0) | p \rangle = a(p^2) p_\mu$$

$$\text{Then } \mathcal{M}_\mu(k) = (2\pi)^4 \delta^4(k^2 - m^2) \left\{ a(k^2) [-k_\mu] \langle -k | \phi(0) | 0 \rangle \underbrace{\Theta(-k^0)}_{\frac{1}{2}(1 - \epsilon(k^0))} \right. \\ \left. - \bar{a}(k^2) k_\mu \langle 0 | \phi(0) | +k \rangle \underbrace{\Theta(+k^0)}_{\frac{1}{2}(1 + \epsilon(k^0))} \right\}$$

Now write: $\epsilon(x) = \Theta(x) - \Theta(-x)$

$$\begin{aligned} \text{So } \epsilon(x) &= \Theta(x) - \Theta(-x) \\ 1 &= \Theta(x) + \Theta(-x) \end{aligned} \Rightarrow \begin{aligned} \Theta(x) &= \frac{1}{2}(1 + \epsilon) \\ \Theta(-x) &= \frac{1}{2}(1 - \epsilon) \end{aligned}$$

Regroup

$$\mathcal{M}_\mu(k) = (2\pi)^4 \delta^4(k^2 - m^2)$$

$$\left\{ \epsilon(k_0) \left[+\frac{1}{2} k_\mu a(k^2) \langle -k | \phi(0) | 0 \rangle - \frac{1}{2} \bar{a}(k^2) k_\mu \langle 0 | \phi(0) | +k \rangle \right] \right. \\ \left. + \left[-\frac{1}{2} k_\mu a(k^2) \langle -k | \phi(0) | 0 \rangle - \frac{1}{2} \bar{a}(k^2) k_\mu \langle 0 | \phi(0) | +k \rangle \right] \right\}$$

$$\text{So } \mathcal{M}_\mu(k) = \epsilon(k_0) k_\mu \left\{ \overbrace{\left(\frac{(2\pi)^4}{2} \delta^4(k^2 - m^2) \left[\frac{1}{2} a(k^2) \langle -k | \phi(0) | 0 \rangle - \bar{a}(k^2) \langle 0 | \phi(0) | +k \rangle \right] \right)}^{S_1} \right\} \\ + k_\mu \left\{ \underbrace{-\frac{(2\pi)^4}{2} \delta^4(k^2 - m^2) \left[a(k^2) \langle -k | \phi(0) | 0 \rangle + \bar{a}(k^2) \langle 0 | \phi(0) | +k \rangle \right]}_{S_2} \right\} \quad (2.33) \checkmark$$

② Now, if $\partial_\mu \partial^\mu = 0$, then from (2.27), $k^\mu M_\mu(k) = 0 \Rightarrow$ (2.36)
 $\Rightarrow$ (2.37). Thus you know that

$$k^2 g_1(k^2) = 0$$

$$k^2 g_2(k^2) = 0$$

At same time, you know that $g_i \propto \delta(k^2 - m^2)$, no δ' stuff. Thus, must conclude that

$$g_i = c_i \delta(k^2) \Rightarrow m^2 = 0.$$

Thus, your scalar particles causing $M_\mu(k)$ are massless!
 This follows from $\partial_\mu \partial^\mu = 0$.

So:
$$g_1 = \delta(k^2) \cdot \underbrace{\frac{(2\pi)^4}{2} \left\{ a(k^2) \langle -k | \phi(0) | 0 \rangle - \bar{a}(k^2) \langle 0 | \phi(0) | +k \rangle \right\}}_{\equiv c_1}.$$

③ Now want to compute c_1 . Consider:

$$\begin{aligned} M_\mu(k) &= \epsilon(k_0) k_\mu c_1 \delta(k^2) + k_\mu c_2 \delta(k^2) \\ &= k_\mu \delta(k^2) [c_1 \epsilon(k_0) + c_2]. \end{aligned}$$

Calculate:

$$\left. \int_{-\infty}^{\infty} dk^0 M_0(k) \right|_{\vec{k}=0} = \int_{-\infty}^{\infty} dk^0 \cdot k^0 \underbrace{\delta(k_0^2) \delta(\vec{k})}_{\frac{\delta(k_0 - |\vec{k}|)}{2|\vec{k}|} + \frac{\delta(k_0 + |\vec{k}|)}{2|\vec{k}|}} [c_1 \epsilon(k_0) + c_2] \Big|_{\vec{k}=0}$$

$$= \frac{|\vec{k}|}{2|\vec{k}|} [c_1 \epsilon(|\vec{k}|) + c_2] + \frac{(-|\vec{k}|)}{2|\vec{k}|} [c_1 \epsilon(-|\vec{k}|) + c_2]$$

$$= \frac{1}{2} c_1 \{ \epsilon(+|\vec{k}|) - \epsilon(-|\vec{k}|) \} + \frac{1}{2} (c_2 - c_2)$$

But $\epsilon(x) = \theta(x) - \theta(-x)$
 $\epsilon(-x) = \theta(-x) - \theta(x)$

so $\epsilon(x) - \epsilon(-x) = 2$, so

$$\left[\int_{-\infty}^{\infty} dk^0 M_0(k) = c_1 \quad \text{for any value of } t_2 \right]$$

On the other hand,

$$\begin{aligned} \int_{-\infty}^{\infty} dk^0 M_0(k) &= \int_{-\infty}^{\infty} dk^0 \cdot \int d^4x e^{ikx} \langle 0 | [\mathcal{J}_0(x), \phi(0)] | 0 \rangle \\ &= 2\pi\delta(t) \cdot \int d^4x \cdot \langle 0 | [\mathcal{J}_0(\vec{x}, 0), \phi(0)] | 0 \rangle e^{i\vec{k} \cdot \vec{x}} \\ &\quad \text{equal time!} \\ &= \delta^3(\vec{x}) \phi(0) \quad / \text{assume} \end{aligned}$$

$$= \int d^4x \cdot 2\pi\delta(t) \cdot \delta^3(\vec{x}) e^{i\vec{k} \cdot \vec{x}} \langle 0 | \phi(0) | 0 \rangle.$$

$$= 2\pi \langle 0 | \phi(0) | 0 \rangle. \quad / \text{no need to set } t_2 = 0.$$

So, I get: $\left[c_1 = 2\pi \langle 0 | \phi(0) | 0 \rangle. \right] \quad (2.46)$

④ But why did I say:

$$[\mathcal{J}_0(\vec{x}, 0), \phi(0)] = \delta^3(\vec{x}) \phi(0).$$

Well, really, nothing in terms of charges:

$$[Q, \phi(0)] = \tau \cdot \phi(0) \quad \leftarrow \text{ie, fields transform in some way under gauge generators.}$$

Bernstein: (3.) The Higgs Loophole.

1. When you do the usual Coulomb gauge quantization with $\vec{\nabla} \cdot \vec{A} = 0$, you in effect force the object A_μ to transform not as a regular 4-vector, but as some weird object with the extra piece as shown in (3.16). This extra piece is found by B. in two ways: first, by requiring that the gauge condition be true in any two frames; second, by just looking at the way you quantize with δ^{tr} object.
2. Another viewpoint is this: since you are choosing this non-covariant gauge condition, you in effect pick out the time direction as special. The gauge condition is still rotationally covariant, but not boost covariant. Thus, in effect, you can imagine that anything you have can also be a function of the 4-vector $\eta_\mu = (1, 0, 0, 0)$. I.e., you go to any frame but this vector stays as it is, i.e., you are violating boost covariance. This same η_μ shows up in the Feynman propagator but has no effect on S-matrix, argument repeated from BD vol 2.
3. The presence of this extra vector, i.e. the loss of boost covariance, ruins the proof of the Goldstone theorem. From (3.42) in momentum space you find:
 $J_\mu(0) = (k \cdot A) k_\mu - k^2 A_\mu(0)$. You see that J is conserved, but since A is not a true 4-vector, neither is J. Clearly this eliminates a key step in the Goldstone proof which now gets replaced by: $\langle 0 | J_\mu(0) | 0 \rangle = a(k^2 k_\mu) + b(k^2 k_\mu) \eta_\mu$. Another result is that the charge Q corresponding to this current is now Q(t) and moves in time, something very new. However, the current is still conserved.

(4.) The Higgs Mechanism.

1. The preceding section of this paper showed that the usual Coulomb gauge allowed presence of η_μ and this then allowed an "escape" from the Goldstone Theorem; this is the Higgs Loophole, so called. In this section we will examine the Higgs idea in more detail. Section is a long 15 half-pages.
2. First interesting trick is to write a Lagrangian without showing second-order derivatives. This is just a Schwinger trick to eliminate extra work. Of course second-order derivs reappear when you vary everything. This is the same as the idea in Abers Lenn of treating A_μ and $F_{\mu\nu}$ as independent fields for a while.
3. The model of this section is a charged Klein-Gordon scalar field with photons, i.e., scalar ED. Charged means that scalars come in a doublet. Mass term is put in with wrong sign, intentionally. A finite charge rotation (gauge transformation) is identified in its effect on the scalar doublet; just a simple rotation. Some gauge function of course appears in gauge trans. for A_μ . The covariant $D_\mu \phi$ is identified as usual, standard gauge theory.
4. Now we allow breaking, i.e., let ϕ_2 have nonzero VEV. Several paths are then followed from this point.

5. In the first path, you examine the equations after linearizing them. This linearization is justified after shifting field because fields are "small" in some sense. You then assume the covariant Lorentz gauge and you find that you do have a Goldstone (massless scalar ϕ_1') but this thing solves a free KG equation and is thus a free Goldstone, ie, uncoupled to rest of theory.

Meanwhile, if you define B_u to be A_u plus d_u of ϕ_1' , this Goldstone, you find that B_u is a massive vector meson.

So you get the idea that in this gauge, although Goldstone theorem is valid and you do get your Goldstone, it is decoupled. Moreover, your gauge field has acquired a mass. Your other meson also has mass, the ϕ_2' field.

The ratio of the masses of vector meson and scalar meson is set by the two theory couplings, the ϕ^4 self coupling f and the gauge coupling e_0 .

6. Now lets redo this in the Coulomb gauge. Again play with the equations to see what you can see. First, ϕ_1' is now identified with d/dt of A_0 , so ϕ_1' is not even a true Lorentz scalar!

However, B_u constructed as before is a true 4-vector (covariant). The field ϕ_1' has the same mass as B_u , still decoupled. Thus, Goldstone Theorem is expected to fail since loss of covariance, and Goldstone particle now has a mass. All fits together.

In this gauge, the physical results are the same: you have a massive scalar and a massive vector with masses ratioed as before.

Finally, explicit failure of Goldstone Theorem is shown in this gauge. I wont bother to follow it. Hinges on fact that ϕ_1' is not a true scalar. Hence it can couple the vector B_u to vacuum.

The explicit commutator of the charge and ϕ_1' is computed and you see explicit time dependence of charge, even though current is conserved.

7. Idea to remember: the reason you get $\dot{Q} \neq 0$ is surface terms. These surface terms are likely to be present when you have ~~massive~~ massless particles in your theory. Gauge fields are massless particles.

8. The third path is the U-gauge. You start by choosing a gauge function that rotates your ~~XXXX~~ ϕ_1' field to zero so it is just plain gone. This same transformation takes your A_u into $A_u' = B_u$. Only ϕ_2' and B_u are left. You then do SSB and shift ϕ_2' field to CHI and CHI becomes your massive scalar. This is the U-gauge where S-matrix in terms of CHI and B fields is manifestly unitary. This path is easy to generalize proof and analysis to all orders.

9. Last piece here adds the baryons. Global gauge invariance gives you baryon conservation and some "S-number" conservation for meson. You do SSB and break the S-number, but maintain baryon conservation. Now the B_u field is driven by the baryon current. More physical theory. Big deal.

10. This section follows Higgs own work closely.

(5.) To show how particles acquire mass in perturbation theory.

1. In the last section we studied in detail a little model with a charged scalar Klein-Gordon field and some photons. In the U-gauge we made one of the mesons "go away" by a carefully chosen gauge rotation. Then we shifted the other meson from ϕ_2 to CHI, and the field B_u picked up the remnants of the Goldstone.

The question here is: how does all this look in perturbation theory? You start as shown in (5.17) with a lagrangian in which SSB has a ready taken place and a shifted field ϕ' appears. Neither the ϕ' field nor the A_u field have explicit mass terms in this lagrangian's L_0 part. Of course the mass terms do appear in the L_I part because we have already shifted. The game here is to treat these mass terms as interactions. Thus, your bare ϕ and A_u are massless and so have $1/q^2$ propagators. Then the game is to watch how the masses of both these particles appear as you do perturbation theory.

2. The lowest order corrections to the bare massless photon propagator are shown in figure 4. You see the "point" coupling which is really the direct mass term, and then you see a contribution of the massless meson in there. Notice that you can couple a scalar to a vector if you have gradient coupling. The meson term causes a pole in $\Pi_{\mu\nu}$ and this pole then causes the renormalized photon prop. (renorm to first order) to have a shifted pole! This is how the photon acquires its mass.

3. Notice that the "mass term" $A_u A^u$ itself does not shift the pole. But of course it must be there to yield gauge invariance of $\Pi_{\mu\nu}$.

4. Next, he shows how the meson itself acquires mass. In the meson case, the trivial point interaction does cause the mass shift. There is a difference between the two: notice the q^2 factors in (5.26) versus (5.6) for the photon. Thus, you don't need massless photon pole to generate mass on your meson. The whole difference is related to the u and v indices, gauge invariance, conserved tensor, etc.

5. This section is based on the work of Englert and Brout, 1964. Recall that these guys independently discovered the Higgs Mechanism in perturbation theory. I.e., they discovered that one of the mesons attaches to the vector and makes it massive.

6. Further work along the lines of this section was done by Coleman and Weinberg 1973.

7. A detail I omitted: notice that the "massive photon" has a pole at some mass, but it does not have the usual propagator you would associate with a massive vector meson. In particular, it has $q_u q_v / q^2$ rather than $q^u q^v / M^2$. Thus, it will be more open to renormalization than a true massive vector meson! I.e., it will not have the obvious UV problem.

8. It is also noted that, in this theory, if you do not allow SSB, you have massless photons and scalars and theory is an IR nightmare.

(6.) Non-abelian cases.

1. Section opens with review of Yang Mills. Start with some fermion fields in isospin doublet as usual (nucleons). Global isospin rotations yield the usual conserved isotopic spin current. Local isospin invariance requires addition of three gauge fields, the b^i_u $i=1,2,3$ since three SU(2) isospin generators. As usual, the extra non-abelian term is added to F_{uv} to get a true isovector F_{uv} and usual gauge fields term. The b fields must be massless, although nucleons could have mass. This is where Yang Mills stopped.

2. But Bernstein now goes on to do some SSB. Add an iso-doublet of Higgs scalar fields. These fields add some terms to the full lagrangian, called L_H .

3. At this point, what are our currents? The usual electric charge type transformation (phases on charged fields, nothing on b_u) yields a conserved electric or hypercharge current called Y_u (global transformation). Also there is the isospin current T_u . Each current has a charge, and we set up the electric charge in usual G_M-N manner.

4. Next step is to allow field ϕ to break vacuum. We go to a U-gauge and kill off ϕ^+ field. Then we let the ϕ^0 field break, then shift by η to field χ . What is the result?

a) formerly massless Higgs scalars now do this:

a1) one Higgs field is gone away

a2) the other Higgs field is χ and has acquired mass.

b) all three gauge fields b have the same mass, nonzero.

5. Comments: as shown in (6.47), we have broken three isospin generators, hence three goldstones, but these were all eaten by gauge fields. There are no massless gauge fields left in this model after SSB takes place.

(7.) Weinberg's 1967 Model.

1. Multiplets: lefthanded electron and neutrino put into a weak isospin doublet, both are massless. Right part of electron put into singlet. The requirement of local weak-isospin invariance brings in the compensating fields b_u as in Yang-Mills. This is weak-isospin, not isospin however.
2. So far lagrangian has only lepton fields L and R, and gauge fields b. There is at once a weak isostatic spin current which is conserved. This current has two pieces: the lepton obvious piece, and the gauge -field purely b piece. This latter is called J with charges τ . The lepton piece is called J' with charges τ' . Total weak-isospin current is called T. The charges of T_u are called T, and these are the conserved charges which generate the SU(2) part of Weinberg's gauge group.
3. Now the "hypercharge-photon" a_u is added; this is the U(1) part of the gauge group. Hypercharge of course means "weak hypercharge" and is related to the "weak T_3 " component of weak-isospin to give the true charge Q.
4. Argument is given for why this hypercharge photon couples with g' to R, but with $g'/2$ to L. I do not follow argument, but could if I had to.
5. Now lagrangian has these fields: L, R, b, a. If you like, you can regroup things, taking linear combinations, so the fields L, R, b and a are replaced with e, ν , Z, W & A. Z is found by looking at what couples to $\bar{\nu} \dots \nu$. The photon is then the orthogonal field, couples only to $\bar{e} \dots e$.
6. Now comes the Higgs fields. They come in a weak iso-doublet. The scalar Higgs lagrangian is then made local invariant under entire U(1) x SU(2) by coupling in the b and a fields to the Higgs fields. Higgs fields are massless.
7. Now, the field ϕ^+ is gauged away and somehow the field ϕ^0 is related with a Higgs field called R(x). The final Higgs lagrangian in terms now of this field R and the A, Z and W is shown in (7.82). There is no term like AAR. Thus, when we later do SSB with R, photon will not acquire mass.
8. However, you do want your electrons to acquire mass. Thus, coupling $\bar{e} R e$ is added by hand, coupling G, a new free parameter.
9. So finally you do SSB and shift field R into usual CHI. The results of this SSB are enumerated: electron acquires mass, so does Higgs CHI field. So do the fields W and Z; Weinberg angle defined; photon stays massless as noted.
10. A problem noted with this theory is that the A and Z fields can mix, since both couple to the $\bar{e}e$ system. Thus some kind of mass-matrix diagonalization is needed.
11. Rest of this section not very interesting to me. Situation reanalyzed in terms of linearized equations, look at decoupled Goldstones in the Lo entz gauge.
12. Last section is about possibility of magnetic moment and minimal coupling. In certain cases it seems that minimal prescription does allow an anomalous moment term.??? Skip for now.

(8.) conclusions.

1. Bjorkens comments on "believable" gauge theories.
2. Division of theorems into Class 1 with neutral vector carrier like the Z_0 , and Class 2 without neutral vectors, but with heavy leptons.
3. The Z_0 implies neutral currents, Bernstein in 1974 was not sure there were any such animals. At the time this was argument in favor of Class 2 models like the $SU(2)$ GeorgiGlashow with its single heavy lepton.
4. Bernstein shows how the Z_0 of the Weinberg model cures the unitarity problem which the W bosons themselves do not cure.
5. He gives a few other examples of how gauge theories mysteriously "cure" problems or diseases of weak interactions.