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(1973)

TRANSFORMATION PROPERTIES OF PROPER VERTICES IN GAUGE THEORIES

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Received 2 July 1973

The Ward-Takahashi identities for (single-particle irreducible) proper vertices are derived for non-Abelian gauge theories; the renormalization of such theories, previously discussed in terms of the Ward-Takahashi identities for Green functions, can now be considerably simplified.

In discussing the renormalizability and unitarity of the S -matrix in (spontaneously broken) gauge theories, the Ward-Takahashi (WT) identities satisfied by Green functions play a central rôle. The renormalization procedure is usually stated in terms of proper (i.e., single-particle irreducible) vertices, so that the WT identities written in terms of proper vertices would enormously simplify the discussion of the renormalization procedure. It is the purpose of this note to provide a derivation of such identities. In previous discussions [1, 2] of the renormalizability of these theories, one has to contend with^{†1} the WT identities for Green functions, and a considerable amount of work was necessary to extract information on proper vertices therefrom.

We shall adopt the convention previously used [3] that all fields [gauge fields $b_\mu^\alpha(x)$ and scalar fields $\psi_a(x)$] are represented by ϕ_i where i stands for all attributes of the fields. Thus for the gauge field $b_\mu^\alpha(x)$ i stands for the group index α , the Lorentz index μ and the spacetime variable x , collectively. Summation and integration over repeated indices shall be understood. Thus, for example,

$$\phi_i^2 = \phi_i \phi_i \\ = \int d^4x \left\{ - \sum_\alpha b_\mu^\alpha(x) b_\nu^\alpha(x) g^{\mu\nu} + \sum_a \psi_a(x) \psi_a(x) \right\}$$

The infinitesimal local gauge transformation of ϕ_i may be written as

$$\phi_i \rightarrow \phi_i' = \phi_i + (\Lambda_i^\alpha + t_{ij}^\alpha \phi_j) \omega_\alpha$$

where $\omega_\alpha = \omega_\alpha(x)$ is the space-time dependent parameter of a compact Lie group G . Λ_i^α is of the form

$$\Lambda_i^\alpha = \begin{cases} \frac{1}{g} \partial_\mu \delta^4(x_\alpha - x), & \text{if } \phi_i = b_\mu^\beta(x), \\ 0 & \text{otherwise,} \end{cases}$$

where $g_{\alpha\beta} = g_\alpha \delta_{\alpha\beta}$ is the gauge coupling matrix^{†2}. We have

$$t_{ik}^\alpha (t_{kj}^\beta \phi_j + \Lambda_k^\beta) - t_{ik}^\beta (t_{kj}^\alpha \phi_j + \Lambda_k^\alpha) \\ = f_{\alpha\beta\gamma} (t_{ik}^\gamma \phi_k + \Lambda_i^\gamma)$$

where $f_{\alpha\beta\gamma}$ is the structure constant of the local gauge group. The invariance of the Lagrangian under this gauge transformation may be formulated as

$$(\Lambda_i^\alpha + t_{ij}^\alpha \phi_j) \frac{\delta L[\phi]}{\delta \phi_i} = 0. \quad (1)$$

We shall always use a real basis for ϕ_i , so that the representation matrix of the Lie algebra $[t^\alpha]_{ij} = t_{ij}^\alpha$ is real antisymmetric.

In quantizing such a Lagrangian, we must choose a gauge. We consider a gauge condition linear in ϕ :

$$F_\alpha[\phi] = F_{\alpha i} \phi_i = 0.$$

^{†1} This was especially true in the first-cited reference in [1].

^{†2} If the group G is a direct product of n simple groups, $G_1 \otimes G_2 \otimes \dots \otimes G_n$, there are in general n gauge couplings $g_1, g_2, \dots, g_n$. Within the same factor group G_i , for course $g_\alpha = g_\beta$.

* Supported in part by the NSF grant GP-32998X.

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Usually, one chooses $F_{\alpha i}$ to be

$$F_{\alpha i} = \frac{1}{\sqrt{\alpha}} g^{\alpha\beta} \Lambda_i^\beta$$

if ϕ_i is a gauge field,

$$= \frac{\sqrt{\alpha}}{\xi} c_i^\alpha \quad (2)$$

if ϕ_i is a scalar field, where α, ξ are gauge fixing parameters, and c_i^α are some constants. In this gauge, the Feynman rules for constructing Green functions are deduced from the effective action S_{eff} [4]

$$S_{\text{eff}}[\phi] = L[\phi] - \frac{1}{2} \{F_\alpha[\phi]\}^2 + \bar{c}_\alpha M_{\alpha\beta}[\phi] c_\beta \quad (3)$$

where $\bar{c}$ and c are fictitious anticommuting fields which generate the so-called de Witt-Faddeev-Popov ghost loops [5, 6] and $M_{\alpha\beta}$ is given by

$$M_{\alpha\beta}[\phi] = F_{\alpha i}(\Lambda_i^\beta + t_{ij}^\beta \phi_j) \quad (3')$$

The operator M is in general not Hermitian so that the ghost line is orientable.

In the following we shall be dealing with unrenormalized fields and coupling constants, and dimensionally regularized [7] quantities. As has been shown elsewhere [3], the generating functional $W[J]$ of Green functions satisfies the WT identity^{†3}

$$\left\{ -F_\alpha \left[\frac{1}{i} \frac{\delta}{\delta J} \right] + J_i \left(\Lambda_i^\beta + t_{ij}^\beta \frac{1}{i} \frac{\delta}{\delta J_j} \right) M_{\beta\alpha}^{-1} \left[\frac{1}{i} \frac{\delta}{\delta J} \right] \right\} W[J] = 0. \quad (4)$$

The quantity $M^{-1} [\delta/i\delta J] W[J]$ is the Green function for the fictitious field c in the presence of the external source J .

The functional $Z[J]$ defined by

$$W[J] = \exp(iZ[J])$$

is the generating functional of the connected Green functions. The Legendre transform $\Gamma[A]$ of $Z[J]$:

$$\Gamma[A] = Z[J] - J_i A_i, \quad A_i = \delta Z[J] / \delta J_i, \quad -J_i = \delta \Gamma[A] / \delta A_i, \quad (5)$$

is the generating functional of proper vertices [8]. For later use we define $X_{ij}[A]$ by

$$X_{ij}[A] = -\delta A_i / \delta J_j \quad (6)$$

$$X_{ik}[A] \delta^2 \Gamma[A] / \delta A_k \delta A_j = \delta_{ij}$$

^{†3} An equivalent diagrammatic derivation of eq. (4) is given in ref. [2].

i.e., $X_{ij}[A]$ are the propagators when the fields ϕ are constrained to have the vacuum expectation values A .

Now we are in a position to write down the WT identity for $\Gamma[A]$. First observe that

$$M_{\beta\alpha}^{-1} \left[\frac{1}{i} \frac{\delta}{\delta J} \right] \exp(iZ[J]) M_{\beta\alpha}^{-1} \left[A + \frac{1}{i} \frac{\delta}{\delta J} \right] \quad (7)$$

where

$$A_i + \frac{1}{i} \frac{\delta}{\delta J_i} = A_i + iX_{ij}[A] \frac{\delta}{\delta A_j} \\ = \exp(iZ[J]) \frac{1}{i} \frac{\delta}{\delta J_i} \exp(iZ[J]).$$

Recalling that F_α is linear in ϕ , we write eq. (4) as

$$-F_\alpha[A] - \frac{\delta \Gamma[A]}{\delta A_i} \left[\Lambda_i^\beta + t_{ij}^\beta \left(A_j + iX_{jk}[A] \frac{\delta}{\delta A_k} \right) \right] \\ \times M_{\beta\alpha}^{-1} \left[A + iX \frac{\delta}{\delta A} \right] = 0. \quad (8)$$

Now define $\Gamma_o[A]$ by

$$\Gamma[A] = \Gamma_o[A] - \frac{1}{2} \{F_\alpha[A]\}^2. \quad (9)$$

Then from eq. (8) follows

$$0 = \frac{\delta \Gamma_o[A]}{\delta A_i} \left[\Lambda_i^\beta + t_{ij}^\beta \left(A_j + \frac{1}{i} \frac{\delta}{\delta J_j} \right) \right] M_{\beta\alpha}^{-1} \left[A + \frac{1}{i} \frac{\delta}{\delta J} \right]. \quad (10)$$

We must now examine

$$t_{ij}^\beta \frac{1}{i} \frac{\delta}{\delta J_j} M_{\beta\alpha}^{-1} \left[A + \frac{1}{i} \frac{\delta}{\delta J} \right] \\ = i t_{ij}^\beta X_{jk}[A] \frac{\delta}{\delta A_k} M_{\beta\alpha}^{-1} \left[A + iX \frac{\delta}{\delta A} \right] \quad (11) \\ \equiv \gamma_i^\beta[A] M_{\beta\alpha}^{-1} \left[A + iX \frac{\delta}{\delta A} \right].$$

Upon defining^{†4}

$$G_{\beta\alpha}[A] = M_{\beta\alpha}^{-1} \left[A + iX \frac{\delta}{\delta A} \right]$$

so that $G_{\beta\alpha}[A]$ is the ghost propagator when the fields ϕ are constrained to have the vacuum expectation values A , we can solve eq. (11) for $\gamma_i^\alpha[A]$:

^{†4} Here one must distinguish between an operator equation such as $(1+x\partial)^2 = 1 + \partial + x^2\partial^2$ and the equation resulting from its action on a certain function of x , such as $(1+x\partial)^2 1 = 1$.

$$\gamma_i^\alpha[A] = -it_{ij}^\beta X_{jk}[A] G_{\beta\gamma}[A] \frac{\delta}{\delta A_k} G_{\gamma\alpha}^{-1}[A]. \quad (12)$$

Now eq. (10) may be written as

$$L_{\alpha i}[A] \frac{\delta}{\delta A_i} \Gamma_0[A] = 0 \quad (13)$$

where

$$L_{\alpha i}[A] = \Lambda_i^\alpha + t_{ij}^\alpha A_j + \gamma_i^\alpha[A] \quad (14)$$

and

$$G_{\beta\alpha}^{-1}[A] = F_{\beta i} L_{\alpha i}[A]. \quad (15)$$

Note that $G^{-1}[A]$ is the generating functional of proper vertices with two ghost lines, so that

$$G_{\beta\alpha}^{-1} \equiv G_{\beta\alpha}^{-1}[0]$$

is the inverse ghost propagator:

$$\gamma_{\beta\alpha, i} \equiv \frac{\delta^2 G_{\beta\alpha}^{-1}}{\delta A_i} [0]$$

is the proper vertex of two ghosts at α and β and the field at i , etc.

The system, eqs. (12)–(15), is our main result. Eq. (13) defines a non-linear gauge transformation which leaves $\Gamma_0[A]$ invariant. It is generalization of eq. (1) and reduces to it in the tree approximation.

On the basis of eqs. (12)–(15), the considerations of ref. [1] can be generalized readily to any compact Lie groups and arbitrary representations of scalar

fields^{†5}. Furthermore it can be shown that the divergent parts of the renormalization constants are insensitive to the parameter ξ in eq. (2) and therefore the renormalized S -matrix is independent of the gauges^{†6,7}. The details have been worked out and will be reported elsewhere.

I thank S. Coleman, G. 't Hooft, M. Veltman and J. Zinn-Justin for much help they have given me.

^{†5} The discussions in refs. [1] were tailored to the group $SU(2)$ and other completely reducible compact Lie groups (i.e., groups in which the product of two irreducible representations R and R' contains a third R'' at most once).

^{†6} The renormalizability in the most general case, and the gauge independence of the renormalized S -matrix have also been worked out, prior to me, by G. 't Hooft and M. Veltman by means of their diagrammatic analysis (private communication) and by J. Zinn-Justin (unpublished manuscript) on the basis of eq. (4).

^{†7} I have not full investigated the case in which F_α is of dimension two and not linear in fields.

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Ben Lee's letter

The purpose of this letter is very clear and simple: convert the WT identities from the old ones in terms of $W[J]$ (which thus involve the Greens functions) to new ones in terms of $\text{GAMMA}[A]$ involving the proper vertices. Obviously the proofs of renormalizeability are going to involve the proper vertices so nice to have Wards in terms of these vertices. This stuff is of course contained in Lee's review, but I like the idea of having the original letter just to pin dates.

GAUGE DEPENDENCE OF RENORMALIZATION GROUP PARAMETERS IN GHOST-FREE NON-ABELIAN GAUGE THEORIES

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Received 30 September 1975

We discuss the gauge dependence of the renormalization group parameters in a class of ghost-free non-abelian gauge theories. We show, using the n -dimensional regularization with the "minimal" renormalization procedure, that these parameters are gauge independent.

The renormalization group equations [1] play an important role in the study of the asymptotic behaviour of renormalized field theories, especially in view of the fact that non-abelian gauge theories are asymptotically free [2]. An interesting problem is the gauge independence of the renormalization group parameters, or lack thereof. This was already discussed by a number of authors [3] for the class of covariant Lorentz gauges.

Here, using the n -dimensional regularization scheme [4], we would like to discuss this question in the context of non-covariant "axial" gauges, which do not lead to the appearance of Faddeev-Popov ghosts [5]. For simplicity, we will restrict ourselves to a pure Yang-Mills theory [6], since all essential features are already present here. This theory is described by the (unrenormalized) Lagrangian:

$$L^0 = -\frac{1}{4}(\partial_\mu A_\nu^{0a} - \partial_\nu A_\mu^{0a} + g^0 f^{abc} A_\mu^{0b} A_\nu^{0c})^2 - \frac{1}{2\beta} (n_\mu^0 A_\mu^{0a})^2 \quad (1)$$

where f^{abc} are the totally antisymmetric structure constants of the gauge group G and n_μ^0 is any fixed four-vector ($n_\mu^0 n_\mu^0 \neq 0$). In order to obtain finite Green functions we rescale the fields A_μ^{0a}

$$A_\mu^{0a} = Z_3^{1/2} A_\mu^a \quad (2)$$

In the dimensional regularization scheme it is convenient to rescale g^0 such that the renormalized coupling constant is dimensionless. As a consequence of the Slavnov-Taylor identities [7], which hold in the absence of ghosts, one finds $Z_1 = Z_3$, so that we can write:

$$g^0 = \mu^{\epsilon/2} Z_3^{-1/2} g \quad (3)$$

where $\epsilon = 4 - n$ and μ is an arbitrary mass, which sets the scale of the theory. Furthermore, it is necessary to rescale the gauge parameter n_μ^0 such that the gauge fixing term in (1) remains invariant. This requires:

$$n_\mu^0 = Z_3^{-1/2} n_\mu \quad (4)$$

We will determine the (dimensionless) counterterm Z_3 using the "minimal" regularization scheme [8], which in this case contains only inverse powers of ϵ in an expansion about $\epsilon = 0$.

The propagator for the vector-boson fields, for $\beta = 0$:

$$D_{\mu\nu}^{ab}(p, n) = \delta^{ab} \frac{1}{p^2} \left[\delta_{\mu\nu} - \frac{n_\nu p_\mu + n_\mu p_\nu}{n \cdot p} + n^2 \frac{p_\mu p_\nu}{(n \cdot p)^2} \right] \quad (5)$$

leads, by power counting arguments, to a renormalized theory. Remark that, since $D_{\mu\nu}^{ab}$ is a function of zeroth degree in n , it has the important property of being invariant under the scaling $n \rightarrow sn$, where s is an arbitrary ($s \neq 0$) parameter:

$$D_{\mu\nu}^{ab}(p, n) = D_{\mu\nu}^{ab}(p, sn) \quad (6)$$

Using this relation, we will show that Z_3 is independent of the gauge parameter n , which implies, as we will see, the gauge independence of the renormalization group parameters. The proof is inductive.

Consider a Feynman diagram, which is made finite, order by order in a loop expansion, up to l loops, by counterterms which are independent of n . This is trivially true in the tree-approximation ($l = 0$), where $Z_3 = 1$. To order $(l + 1)$ the diagram will exhibit poles (up to degree $l + 1$), with the property that the residues

are polynomial in the external momenta [4, 9]. Furthermore, using (6) and by the induction hypothesis, all terms, including the residues, must be invariant under the scaling $n \rightarrow sn$. Note that this property must be satisfied independently by rational and logarithmic functions. We now observe (from (3)), that μ only appears as a power of $\mu^{\epsilon/2}$, multiplying coupling constants. When the Laurent expansion of each Feynman integral is made, only logarithms of μ will appear. In view of the first property, the residues could depend on μ , via these logarithmic functions, only through dimensionless ratios of n^2/μ^2 . However, these terms cannot satisfy the second requirement, mentioned above, so that we conclude that the residues are independent of μ (for a discussion of this point see also ref. [10]). We now remark that Z_3 is a dimensionless quantity, which can depend on n only via dimensionless ratios of n^2/μ^2 . As its poles are defined to be precisely those needed to subtract the poles associated with the above residues, which are independent of μ , Z_3 must also be μ -independent. It follows then, that also to order $(l+1)$, we can remove the divergences with counterterms independent of n , which concludes the proof.

We can reach the same conclusions by observing that, since the theory is renormalizable, the residues must have the same form as the counterterms generated in the Lagrangian (1) by the rescalings (2), (3) and (4). Separating out explicitly the dependence on the external momenta, we see that the residues could depend on n only through functions of n^2/μ^2 . For the reasons discussed above, such functions cannot be present, whence we conclude the gauge independence of Z_3 .

As a consequence of this property, we can easily show with the help of (5) that any N -particle (one-particle irreducible) Green function $\Gamma^N(p_i, g, n, \mu)$ satisfies the relation:

$$n_\alpha \frac{\partial}{\partial n_\alpha} \Gamma^N(p_i, g, n, \mu) = 0. \quad (7)$$

On the other hand, Γ^N satisfies the renormalization group equation [1]:

$$\left(\mu \frac{\partial}{\partial \mu} + \beta \frac{\partial}{\partial g} + \delta_\nu \frac{\partial}{\partial n_\nu} - N\gamma \right) \Gamma^N(p_i, g, n, \mu) = 0 \quad (8)$$

where β , δ_ν and γ are given by:

$$\beta = \mu \frac{\partial g}{\partial \mu} \Big|_{g_0, n_0}; \quad \delta_\nu = \mu \frac{\partial n_\nu}{\partial \mu} \Big|_{g_0, n_0}; \quad (9)$$

$$\gamma = \mu \frac{\partial}{\partial \mu} \log Z_3^{1/2} \Big|_{g_0, n_0}.$$

Although the dimensionless parameters β , $\delta = \delta_\nu n_\nu/n^2$, γ could a priori depend on n via functions of n^2/μ^2 , we will now show that they are, in fact, independent of the gauge parameter n . To see this, let us apply the operator $n_\alpha \partial/\partial n_\alpha$ on eq. (8). Using relation (7) we arrive at:

$$\left[\left(n_\alpha \frac{\partial \beta}{\partial n_\alpha} \right) \frac{\partial}{\partial g} + \left(n_\alpha \frac{\partial \delta_\nu}{\partial n_\alpha} - \delta_\nu \right) \frac{\partial}{\partial n_\nu} - N \left(n_\alpha \frac{\partial \gamma}{\partial n_\alpha} \right) \right] \Gamma^N = 0. \quad (10)$$

We have imposed no new conditions on Γ^N , so that this equation must be satisfied identically. Thus, the coefficients vanish and we obtain the relations:

$$n_\alpha \partial \beta / \partial n_\alpha = 0, \quad (11a)$$

$$n_\alpha \partial \delta / \partial n_\alpha = 0, \quad (11b)$$

$$n_\alpha \partial \gamma / \partial n_\alpha = 0, \quad (11c)$$

which imply the gauge independence of the parameters β , δ and γ since they are momentum independent.

From (9) we see that (11c) is trivially satisfied as the counterterm Z_3 is gauge independent. Eqs. (11a) and (11b) can also be derived by noting that, from eqs. (3), (4) and (9), we have (in the limit $\epsilon \rightarrow 0$):

$$\beta = \gamma g, \quad \delta = \gamma. \quad (12)$$

Using these relations we see that (11a) and (11b) follow from (11c).

As an illustration, we obtain, without making any assumption about n_μ , in the lowest order of perturbation theory:

$$Z_3 = 1 - \frac{11C}{24\pi^2} g^2 \frac{1}{n-4}, \quad (13) \text{ do I agree?}$$

where C is the value of the quadratic Casimir operator in the regular representation of G and is given by $\delta_{ab} C = f_{acd} f_{bcd}$. Relation (13) shows that Z_3 is independent of the gauge parameter n and yields, using (9) and (12), the same gauge invariant result for the β -

function, as the one obtained in covariant Lorentz gauges [1].

Finally, we would like to remark that, despite the gauge invariance of the result (13), it is not possible to choose the gauge $n^2 = 0$, which would greatly simplify the calculations (see eq. (5)). In fact, an explicit calculation performed in this gauge yields for Z_3 a different (and, hence, incorrect) result. The reason for this difference is due to the singularities n^{-2} which appear when computing the Feynman integrals [11]. These singularities, when multiplied by the factor n^2 , which appears in the propagators, yield well defined results which, in general, do not vanish. Due to this fact, in order to obtain consistent results, it is (unfortunately) necessary to use the full expression (5) for the vector boson propagator, with $n^2 \neq 0$.

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Frenkel's letter of Dec 1975

Interesting and perhaps relevant to our simple ansatz. He claims that, when you work in the dim reg framework with that mass scale parameter MU , you can show that in the axial gauge the Z_3 and the little renorm group functions like BETA are independent completely of the vector ETA or n_u . Thus you cannot have Z (our gamma).

The Z 's are of course the residues of poles at $n=4$; as an example he computes what is I think our Born loop and shows that the residue (ie Z) is in fact independent of n_u . I should check this calculation. You only need the leading term.

I should also go read about AF now that I am with the dim reg stuff.

Another comment make in passing is that you cannot take $n^2=0$ because something goes singular and you get the wrong answer. This sounds similar to something we did wrong once.

As a bonus, I am reinforced about renorm when he claims that our form for the propagator makes the theory, by power counting argument, renormalizeable. I agree.

This is a good paper to study in more detail. See if you can duplicate its basic results.

Lee + Zinn - Justin

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³As an example, we consider the case $l=3$, $m_{k+1}=2$, $m_{k+2}=1$, $m_j=0$ otherwise. Then we have

$$R^{\alpha_1\alpha_2\alpha_3} = P_{k+1}^{\alpha_1} P_{k+1}^{\alpha_2} P_{k+2}^{\alpha_3} + q P_{k+1}^{\alpha_1} P_{k+2}^{\alpha_2} P_{k+1}^{\alpha_3} + q^2 P_{k+2}^{\alpha_1} P_{k+1}^{\alpha_2} P_{k+1}^{\alpha_3}.$$

⁴In Eq. (24), C , the polynomial in the crossed variables, was written in a factorized form. If we invert the ordering of vectors in both $L_{\alpha_1\alpha_2\cdots\alpha_l}$ and $R^{\alpha_1\alpha_2\cdots\alpha_l}$, the result will remain valid, i.e., we have

$$C = \bar{L}_{\alpha_1\alpha_2\cdots\alpha_l} \hat{R}^{\alpha_1\alpha_2\cdots\alpha_l}, \quad (24')$$

where

$$\begin{aligned} \bar{L}_{\alpha_1\alpha_2\cdots\alpha_l} &= \frac{(P_{1\alpha_1} P_{1\alpha_2} \cdots P_{1\alpha_{(n_2)}})}{f(n_2)} \\ &\times \frac{(P_{2\alpha_{(n_2)+1}} \cdots P_{2\alpha_{(n_2+n_3)}})}{f(n_3)} \cdots \\ &\times \frac{(P_{k-1\alpha_{(l-n_k+1)}} \cdots P_{k-1\alpha_{(l)}})}{f(n_k)} \end{aligned} \quad (20')$$

and

$$\begin{aligned} \hat{R}^{\alpha_1\alpha_2\cdots\alpha_l} &= \sum_{\alpha P} q^{T_P} (P_{P(N-l)}^{\alpha_1} \cdots P_{P(N-l)}^{\alpha_{(N-l)}}) \cdots \\ &\times (P_{k+1}^{\alpha_{(l-m_{k+1}+1)+0}} \cdots P_{k+1}^{\alpha_{(l)}}). \end{aligned} \quad (25')$$

The amplitude of Eq. (30) follows from the factorized form of Eq. (24'). For notational convenience, we have also made the change $n_i \rightarrow n_{i-1}$.

PHYSICAL REVIEW D

VOLUME 5, NUMBER 12

15 JUNE 1972

Spontaneously Broken Gauge Symmetries. I. Preliminaries

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(Received 10 March 1972)

This is the first of a series of papers addressed to the renormalizability question of spontaneously broken gauge theories. We give a brief outline of the motivation for such an investigation and describe the manner in which the renormalizability of such theories will be proved in the sequel. Put briefly, we will show that in an appropriate gauge, ultraviolet divergences of a spontaneously broken gauge theory are removed completely by the gauge-invariant counterterms in the Lagrangian which would make the Green's functions of the corresponding unbroken gauge theory finite, that the S matrix computed in this gauge is unitary, and that the S matrix is independent of the gauge chosen. In this paper, the renormalizability question of the unbroken gauge theory is considered. We derive the Ward-Takahashi identities of the theory. We discuss several ways of regulating divergent Feynman integrals of the theory without destroying gauge invariance. Infrared divergences are avoided by the device of intermediate renormalization, wherein we choose as subtraction points some points where external momenta are Euclidean. This suffices to establish that the Bogoliubov-Parasiuk-Hepp renormalization will give renormalized Green's functions which satisfy the Ward-Takahashi identities. The existence of finite, renormalized Green's functions satisfying the Ward-Takahashi identities provides us with the means of proving the renormalizability of the spontaneously broken symmetry case. The Ward-Takahashi identities were previously derived for the gauge bosons by Slavnov. We present here a new derivation. The discussions on regularization methods and intermediate renormalization procedure and the renormalization conditions for matter fields, we believe, are new contributions of the present paper.

I. INTRODUCTION

This is the first of a series of papers which will deal with the renormalizability of spontaneously broken gauge symmetries. The intriguing possibilities of unifying electromagnetic and weak interactions in terms of Yang-Mills gauge bosons,¹⁻⁹

whose masses are generated by spontaneous breakdown of gauge invariance of the second kind,^{4,5} and of constructing a finite theory of weak interactions^{4,7-9} prompt a closer examination of the quantization and renormalization questions of theories of this genre.

In the sequel of this series, we wish to examine

1. Introduction. Motivation is unification of weak and EM interactions. No mention of QCD. There will be (1) Discussion of Higgs and stability (2) Theorem: renormalizability of unbroken gauge theory implies renorm of broken theory. Actually, this first paper will not get into the broken theories. (3) In the Renorm gauge, unitarity will be un-obvious; nevertheless, unitarity will be verified (4) the Renorm and Unitary gauges will be shown equivalent. This is the program for the entire series of papers.

The aim of this particular paper is to derive the Ward Identities and then construct Greens functions that satisfy them. These should be renormed and finite greens functions, so they will have to discuss how they are made finite, where they are subtracted, etc. The BPHZ will be used.

Intermediate renorm means that they will choose as spacelike renorm point some point other than where the IR problem lies. No big deal. Ward Identity derivation credited to Slavnov.

2. Quantization. The usual stuff. The FP determinant is written as $\exp[\text{Tr} \phi \log \dots]$ and then $\text{Tr} \ln(\dots)$ appears as part of the Lagrangian. It is noted that this extra term can be interpreted as ghosts in the usual way, and I could if I wanted figure out why they are fermion-like. It is shown how greens functions are obtained by functional derivatives on $Z[J]$, which by the way is the thing that gives connected graphs.

3. Ward-Takahashi Identities--I. I have made detailed notes on derivations of equations in this section. I derived everything except, unhappily, I was unable to execute the final step to get the key result, that will have to wait because I simply don't know how to do it right now. Here is how things went: W_0 is the genfunc without the ghosts: the FP ghost det is pulled out as in (3.1), a new idea to me. Now, obviously making a group variation on integration variable of an integration cannot change the number that is the integral, so RHS of (3.4) is zero. On the other hand, your variator reaches in and "varies" parts of the action which are not gauge-invariant. Recall that, although F_{uv} does rotate as a color vector, A_u does not, so the gauge-fixing term (arising from the delta) and the source term are gauge-violating as shown in (3.3). Thus, in effect, the gauge-fixing variation must cancel against the source-term variation, and thus you have Eq. (3.4) which is WTI for W_0 . But of course you want WTI for the full W . Putting the ghosts back in is accomplished by the horrendous identity (3.5) which was derived in appendix A. Thus, you get (3.8), where H is the greens function of operator Dd . This result is manipulated finally into (3.13) which is the result he will use. This is an amazingly complicated, functional-derivative condition on the full $W[J]$ for a pure Yang-Mills theory quantized correctly. I get a result close to (3.13), but I have a sign difference, plus I can't dump the outside Dd operator. $\sim$ OK.

This (3.13) is the family of WTI's for the Greens Function genfunc. The claim is that the simpler result for the 1PI functions is much more complicated! Slavnov is credited with first derivation of this. they are called Taylor-Slavnov identities.

4. Ward-Takahashi Identities-II. Basically, the idea here is to define the renorm parts of the theory (they are: ghost-prop, gauge-prop, or respective self-energies, the triple-gluon vertex, the gluon/ghost/ghost vertex, and the 4-gluon vertex.) In their discussion, only the 1PI versions of the 4-gluon and 3-gluon enter. Called Γ . The ghost/gluon triple vertex is called γ and is also 1PI.

In the middle of this section we have the ghost self-energy Dyson, (4.14). I don't see a gluon-Dyson or other Dyson equations.

The major Ward identities are (4.14) [which relates gluon-prop, ghost-prop, and 3-gluon and ghost-gluon vertex] and also (4.17) which relates the 4-gluon to 3-gluon and the gluon-prop. Strangely, the ghosts don't seem to enter this one at all.

Why are we doing all this? REcall how in QED you use the Ward identity to simply avoid worrying about the electron self-energy. I assume the Fancy Wards will be utilized in a similar way in NAGT.

Notice of course that all functions so far defined are "unrenormalized". Later I imagine he will rewrite these equations for the renormed objects, and will state renorm conditions, defining various Z 's and so on. This is the part I am waiting for!!!

5. Regularization. I gather that the dimreg method was not yet popular, perhaps tHV had not come out yet. So here is the gist of this section: First they convince themselves that they can render finite any graph involving only one loop. This was done in a paper of Hooft earlier. The method is this: just add some "regulator fields" with masses m_1^2 (spinor and scalar fields may both be necessary). Then adjust the couplings of these regular fields, and their masses, to satisfy certain linear conditions such as (5.3) through (5.5). Then you discover that, when you include Feynman graphs which include these regulator fields, you have cancelled all divergences at the one-loop level.

I presume that you can interpret these "regulator fields" as terms added to the original lagrangian, a finite number of terms, so this is OK to do. But, it seems that this "regulator fields method" will fail for higher-loop diagrams. So more stuff is needed. That "more stuff" is to add fancy derivative YM coupling terms to the Lagrangian as shown in (5.7); cutoff appears in these terms. The effect of the addition of such terms is: 1) regulates the gluon propagator down to power k^{-6} instead of k^{-2} so this makes higher-loop graphs tend to converge fast since lots of propagators.; 2) but you also have added all kinds of couplings, like 8-gluon couplings! Nevertheless, it is claimed that the result of this propagator modification is: only one-loop graphs are divergent!!

Thus, by so changing the glue propagator (by adding those derivative couplings), you have put all divergence problems into the one-loop sector. But these one-loop graphs can be rendered finite by the regulator field method! Thus, you have solved the whole thing.

Moreover, a major point, the idea is that the counterterms you have added to the Lagrangian to do this regulation were gauge invariant! Eg, you know that F is a true color vector, dF is not, but DF is, and only DF type stuff appears in (5.7). Thus, these counterterms have no effect on the Ward Takahashi Identities !!!! You have regulated without hurting gauge invariance, and this is the main point. Of course the dimreg does this too in a much fancier way.

You can go on to add higher propagator derivative couplings to improve the UV convergence of your propagator, but this never makes the one-loopers finite.

Appendix D shows a little how you power count graphs with this propagator modification and you draw your own conclusions, fine.

6. Renorm Conditions and IR Divergences. Suddenly, without warning, all the Greens functions they are using are to be understood as "renormalized". Thus, first working with $q^2=0$ as their renorm point, they choose $J(k^2)=1$ at $k^2=0$. Obviously if we had an unrenormed theory you could not arbitrarily set this, it would be some divergent thing depending on the cutoff. But in the functional method, you can just let everything be renormalized. Recall that the multiplicative renormalization is accomplished by adding counterterms to the Lagrangian.

So, $J=1$ at $k^2=0$ is the renorm condition on the glue-prop (or self-energy). The ghost residue is also set to unity. The triple-glue vertex is proportional to the bare vertex since other tensors involve more p 's and are thus smaller! The renorm condition on the triple-glue is that at the zero point it is G times bare tensor.

What about the ghost-glue vertex? It starts out with its own constant G' , but then the Ward sets this to same G as in triple glue.

And what about the 4-glue? It has two tensor forms with constants F and F' . But then the other Ward identity puts $F=G^2$ and $F' = 0$. So for all the vertices, there is only one renormalized charge G .

Now they want to shift the whole thing to a new renorm point so things are off-shell at the renorm point. It is noted that holding things off shell tends to remove IR divergences. This is easy to see with a little ϕ^3 loop graph $d^4k/(k^2k'^2)$ which only goes IR divergent if $q=0$. So the motivation to "renorm" off shell is clear. How is this implemented? For the glue and ghost prop, its easy. You just choose $J=1$ and $Z_g=1$ at $k^2=-a^2$. Trivial. The vertices are now not so trivial. The triple-glue now has three tensor forms, the bare vertex plus two more. Constants are called G, H, J . And the

ghost-gluon vertex now has G' plus six more constants to worry about. Then the first Ward identity relates all these constants.

But, as you can see, only G and G' are divergent constants because those other tensor forms have higher powers of momenta in them. So you can read equation (6.9) to also say: $[G]_{\text{div}} = G'_{\text{div}} \dots$ But wait, these constants are not divergent. OK, I think they have left something out: ie, they never defined divergent Z 's, but I think I now see how to do it.

By the way, notice that they say nothing about the 4-gluon vertex in this off-shell renorm case. Probably it has a million tensor forms. One of my little problems is to use group theory to try and find these tensor forms.

Conclusion: good to see some real live renorm points being selected.

7. Renorm of the matter fields. Up to now we have had pure Yang Mills fields only. In this section we throw in a vector of scalars (massive) belonging to the adjoint rep. Then the theory has more vertices which are given names. Then a renorm condition is made for the new vertices. Ie, you have a bunch of constants multiplying tensor forms, as usual.

Now, the Ward identity including the new fields connects the new vertices only with the triple-gluon, which you recall had a G constant. When you stuff your new renorm conditions into the Ward, you get certain results like $C=G$ etc. Again, everything can be related back to G .

Of course the new fields couple with the same bare coupling g_0 , so this result does not surprise me.

Finally, L&ZJ go back and redo all this stuff using $-a^2$ as renorm point because the first time through they used $q^2=u^2$ the scalar mass, and this has the usual IR problem.

Then suddenly the paper ends.

[One of the main points is this: to write down your Renorm Conditions, you simply pick a renorm point and you write down all relevant tensor forms and you give each of them a constant multiplier. Then you related the various constants as much as you can using the various Ward identities. When you do do, everything boils down to just one constant, the G on the triple gluon, the renormed charge.]

Appendices: I have already done A and D in the midst of the above. Appendices B and C deal with the idea of proper vertices and Wards in terms of them. I will keep in mind that these appendices are available, but now is not the time for me to read them.

(3.3) (L2I)

$$\left[\frac{1}{2\alpha} \partial_\mu \vec{A}^\mu \cdot \partial_\nu + \vec{J}_\nu \cdot \right] \underbrace{\vec{\delta A}^\nu}_{\substack{\text{drop} \\ (-\frac{1}{g}) \vec{D}^\nu \vec{\Theta}}}$$

$$= \left\{ \frac{1}{2} \partial_\mu \vec{A}^\mu \cdot \partial_\nu + \vec{J}_\nu \right\} \cdot \vec{D}^\nu \vec{\Theta}$$

$$= \left\{ -\frac{1}{2} (\partial_\nu \partial_\mu \vec{A}^\mu) + \vec{J}_\nu \right\} \cdot \vec{D}^\nu \vec{\Theta} \quad \text{parts die, get - and index reversal.}$$

$$= -\vec{\Theta} \cdot \vec{D}^\nu \cdot \left\{ -\frac{1}{2} (\partial_\nu \partial_\mu \vec{A}^\mu) + \vec{J}_\nu \right\}$$

$$= \vec{\Theta} \cdot \left\{ \frac{1}{2} \vec{D}^\nu \partial_\nu \partial_\mu \vec{A}^\mu - \vec{D}^\nu \vec{J}_\nu \right\} = (3.3) \quad \checkmark$$

(2.8)

Start with

$$\underline{\underline{M}} = \left(-\frac{1}{g}\right) \underline{\underline{D}}^{\mu} \underline{\underline{\partial}}_{\mu}$$

(=) = space matrix
(~) = color matrix

ie, $M_{ab}(x,y) = -\frac{1}{g} \times D_{ab}^{\mu} \partial_{\mu}^{(x)} \delta(x-y)$

maybe backwards!

Now: $\underline{\underline{D}}^{\mu} = \left[\underline{\underline{1}} \underline{\underline{\partial}}^{\mu} - ig \underline{\underline{G}} \cdot \underline{\underline{A}}^{\mu} \right]$

where $(\underline{\underline{\partial}}^{\mu})_{xy} = \partial_{\mu}^{(x)} \delta(x-y)$ and

$(\underline{\underline{A}}^{\mu})_{xy} = A^{\mu}(x) \delta(x-y)$

Thus:

$$\begin{aligned} \underline{\underline{M}} &= \left(-\frac{1}{g}\right) \left[\underline{\underline{1}} \underline{\underline{\partial}}^{\mu} - ig \underline{\underline{G}} \cdot \underline{\underline{A}}^{\mu} \right] \underline{\underline{\partial}}_{\mu} \\ &= -\left(\frac{1}{g}\right) \left[\underline{\underline{1}} \underline{\underline{\partial}}^2 - ig \underline{\underline{G}} \cdot \underline{\underline{A}}^{\mu} \underline{\underline{\partial}}_{\mu} \right] \\ &= \left[-\frac{1}{g} \underline{\underline{\partial}}^2\right] \cdot \underbrace{\left[\underline{\underline{1}} \underline{\underline{1}} - ig (\underline{\underline{\partial}}^2)^{-1} \underline{\underline{G}} \cdot \underline{\underline{A}}^{\mu} \underline{\underline{\partial}}_{\mu} \right]}_{\equiv \underline{\underline{M'}}} \end{aligned}$$

Thus, dropping the ∞ deal,

$$\det \underline{\underline{M}} = \det \left[-\frac{1}{g} \underline{\underline{\partial}}^2 \underline{\underline{1}}\right] \det [\underline{\underline{M'}}]$$

and then

$$\det(\underline{\underline{M'}}) = e^{\text{Tr. ln.} \left[\underline{\underline{1}} - ig (\underline{\underline{\partial}}^2)^{-1} \underline{\underline{G}} \cdot \underline{\underline{A}}^{\mu} \underline{\underline{\partial}}_{\mu} \right]}$$

But, $\text{Tr}(ABC) = \text{Tr}(\text{cyclic}) \Rightarrow \text{Tr ln} [1 - (ABC)] = \text{Tr} \cdot \left[-\sum \frac{(ABC)^n}{n} \right] = \text{cyclic.}$

so you can put your three space matrices in any cyclic order!

$$\begin{aligned} \det(\underline{\underline{M'}}) &= e^{\text{Tr ln} \left[\underline{\underline{1}} - ig (\underline{\underline{\partial}}^2)^{-1} \underline{\underline{G}} \cdot \underline{\underline{A}}^{\mu} \underline{\underline{\partial}}_{\mu} \right]} = e^{\text{Tr ln} \left[\underline{\underline{1}} - ig \underline{\underline{G}} \cdot \underline{\underline{A}}^{\mu} \underline{\underline{\partial}}_{\mu} (\underline{\underline{\partial}}^2)^{-1} \right]} \\ &= e^{\text{Tr ln} \left[\underline{\underline{1}} - ig \underline{\underline{\partial}}_{\mu} (\underline{\underline{\partial}}^2)^{-1} \underline{\underline{G}} \cdot \underline{\underline{A}}^{\mu} \right]} \end{aligned}$$

see notes due.

loop expansion in (2.8)

$$\det \underline{\underline{M}}' = e^{-\sum_{n=1}^{\infty} \frac{(ig)^n}{n} \cdot \text{Tr} \left[\underline{\underline{G}} \cdot \underline{\underline{A}}^n \underline{\underline{\partial}}_{\mu} (\underline{\underline{\partial}}^{\mu})^{-1} \right]^n}$$

Look at $n=3$. Exponent is

$$-\frac{(ig)^3}{3} \text{tr} \cdot \int dx_1 dx_2 dx_3 dx_4 dx_5 dx_6$$

$$\left[\underline{\underline{G}} \cdot \underline{\underline{A}}^{(x_1)} \underline{\underline{\partial}}_{\mu} \delta(x_1 - x_2) g(x_2, x_3) \right.$$

$$\underline{\underline{G}} \cdot \underline{\underline{A}}^{(x_3)} \underline{\underline{\partial}}_{\mu} \delta(x_3 - x_4) g(x_4, x_5)$$

$$\left. \underline{\underline{G}} \cdot \underline{\underline{A}}^{(x_5)} \underline{\underline{\partial}}_{\mu} \delta(x_5 - x_6) g(x_6, x_1) \right]$$

$$= -\frac{(ig)^3}{3} \text{tr} \int dx_1 dx_3 dx_5 \left[\underline{\underline{G}} \cdot \underline{\underline{A}}^{\mu_1}(x_1) \underline{\underline{\partial}}_{\mu_1} g(x_1, x_3) \right.$$

$$\underline{\underline{G}} \cdot \underline{\underline{A}}^{\mu_3}(x_3) \underline{\underline{\partial}}_{\mu_3} g(x_3, x_5)$$

$$\left. \underline{\underline{G}} \cdot \underline{\underline{A}}^{\mu_5}(x_5) \underline{\underline{\partial}}_{\mu_5} g(x_5, x_1) \right]$$

$$= -\frac{(ig)^3}{3} \text{tr} \cdot \int dx_1 dx_2 dx_3 \left[\underline{\underline{G}} \cdot \underline{\underline{A}}^{\mu_1}(x_1) \underline{\underline{\partial}}_{\mu_1} g(x_1, x_2) \underline{\underline{G}} \cdot \underline{\underline{A}}^{\mu_2}(x_2) \underline{\underline{\partial}}_{\mu_2} g(x_2, x_3) \right.$$

$$\left. \underline{\underline{G}} \cdot \underline{\underline{A}}^{\mu_3}(x_3) \underline{\underline{\partial}}_{\mu_3} g(x_3, x_1) \right]$$

$$= -\frac{(ig)^3}{3} \text{tr} \left[\underline{\underline{G}}^{a_1} \underline{\underline{G}}^{a_2} \underline{\underline{G}}^{a_3} \right]$$

$$\times \int dx_1 dx_2 dx_3 A_{a_1}^{\mu_1}(x_1) [\underline{\underline{\partial}}_{\mu_1} g(x_1, x_2)] A_{a_2}^{\mu_2}(x_2) [\underline{\underline{\partial}}_{\mu_2} g(x_2, x_3)] A_{a_3}^{\mu_3}(x_3) [\underline{\underline{\partial}}_{\mu_3} g(x_3, x_1)]$$

$$f_{\text{act}n} = -\frac{(ig)^3}{3} \sum_{b_1 b_2 b_3} (G^{a_1})_{b_1 b_2} (G^{a_2})_{b_2 b_3} (G^{a_3})_{b_3 b_1}$$

$$= -\frac{(ig)^3}{3} \cdot (-i)^3 \cdot C_{a_1 b_1 b_2} C_{a_2 b_2 b_3} C_{a_3 b_3 b_1} = -\frac{g^3}{3} c c c$$

⇒

$n=3$ loop is: e^x where

$$x = -\frac{g^3}{3} C_{b_1 b_2 a_1} C_{b_2 b_3 a_2} C_{b_3 b_1 a_3} \times \int dx_1 dx_2 dx_3 A_{a_1}^{\mu_1}(x_1) \left[\partial_{\mu_1}^{(x_1)} g(x_1, x_2) \right] A_{a_2}^{\mu_2}(x_2) \left[\partial_{\mu_2}^{(x_2)} g(x_2, x_3) \right] A_{a_3}^{\mu_3}(x_3) \left[\partial_{\mu_3}^{(x_3)} g(x_3, x_1) \right]$$

Note:

a) overall minus

b) color label order is: (dotted ghost, ghost, gluon) as in L.R.

$$(A.3), (A.3)', (2.6), (3.6), (3.7), (3.5)$$

(A.3) of appendix A.

$$\left\{ e^{+Tr \cdot \ln(\dots)} \frac{\delta}{\delta \xi_{\lambda}^a(x)} e^{-Tr \cdot \ln\left(1 + i g \frac{t^a}{\approx} \xi_{\lambda}^a \frac{\partial^{\lambda}}{\approx} (\partial^2)^{-1}\right)} \right\} f(\xi, \partial/\partial \xi)$$

$$= \frac{\delta}{\delta \xi_{\lambda}^a(x)} f(\xi, \frac{\partial}{\partial \xi}) + [-Tr \cdot \left\{ \left(\frac{\partial^2}{\approx} \right)^{-1} i g \frac{t^a}{\approx} \frac{\partial^{\lambda}}{\approx} (\partial^2)^{-1} \right\}] f$$

from $[\ln(x)]' = x^{-1}$

Thus:

$$e^{+Tr \cdot \ln(\dots)} \frac{\delta}{\delta \xi_{\lambda}^a(x)} e^{-Tr \cdot \ln(\dots)}$$

$$= \frac{\delta}{\delta \xi_{\lambda}^a(x)} - i g \text{Tr} \left\{ \left(\frac{\partial^2}{\approx} \right)^{-1} \frac{\delta}{\delta \xi_{\lambda}^a(x)} \frac{t^a}{\approx} \xi_{\lambda}^a \frac{\partial^{\lambda}}{\approx} (\partial^2)^{-1} \right\}$$

$\begin{matrix} y & z \\ b & c \end{matrix} \qquad \begin{matrix} z & s \\ c & b \end{matrix}$

$$\text{Term} = -i g \sum_{b,c} \int dy dz ds \langle b, y | (\dots)^{-1} | c, z \rangle \frac{\delta}{\delta \xi_{\lambda}^a(x)} (t^a)_{cb} \xi_{\lambda}^a(z) \frac{\partial^{\lambda}}{\approx} (\partial^2)^{-1}_{sy}$$

$\uparrow \qquad \uparrow$
 $= \delta(z-x)$

$$= -i g \sum_{b,c} \int dy ds \langle b, y | (\dots)^{-1} | c, x \rangle (t^a)_{cb} \frac{\partial^{\lambda}}{\approx} \delta(x-s) g(s, y)$$

change to s, then parts to g.

$$= -i g \sum_{b,c} \int dy ds \langle y, b | (\dots)^{-1} | c, x \rangle (t^a)_{cb} \delta(x-s) \frac{\partial^{\lambda}}{\approx} g(s, y) \quad / \text{now } s \rightarrow z$$

$$= -i g \sum_{b,c} \int dy dz \langle y, b | (\dots)^{-1} | c, z \rangle (t^a)_{cb} \delta(x-z) \frac{\partial^{\lambda}}{\approx} g(z, y) = \boxed{(A.3)}$$

$$= -i g \text{tr} \cdot \int dy \left[\frac{\partial^2}{\approx}^{-1}_{yx} (t^a) \frac{\partial^{\lambda}}{\approx} (\partial^2)^{-1}_{xy} \right]$$

$$= -i g \text{tr} \left[\frac{t^a}{\approx} \cdot \int dy \left[\frac{\partial^2}{\approx}^{-1}_{yx} \frac{\partial^{\lambda}}{\approx} (\partial^2)^{-1}_{xy} \right] \right]$$

~ ~

● Rewrite (A.3) as matrix.

$$\text{Term} = -ig \text{Tr} \left\{ \left[\underline{\underline{1}} + ig \underline{\underline{\vec{x}}} \cdot \underline{\underline{\vec{\nabla}}} \underline{\underline{\partial}} (\underline{\underline{\partial}}^2)^{-1} \right]^{-1} \underline{\underline{\vec{x}}} \underline{\underline{\partial}}^x \underline{\underline{\partial}}^1 (\underline{\underline{\partial}}^2)^{-1} \right\}$$

where $\underline{\underline{(\partial^x)}}_{y_1 y_2} = \delta(x-y_1) \delta(y_1-y_2).$

so that, eg,
$$\text{Tr} [\underline{\underline{A}} \underline{\underline{\partial}}^x \underline{\underline{B}}] = \int dy_1 dy_2 dy_3 A(y_1, y_2) \delta(x-y_2) \delta(y_2, y_3) B(y_3, y_1)$$

$$= \int dy_1 dy_2 A(y_1, y_2) \underbrace{\delta(x-y_2)}_{\substack{\uparrow \\ \text{just jams in a delta to} \\ \text{some fixed external value.}}} B(y_2, y_1)$$

Now cyclic to get

$$\text{Term} = -ig \text{Tr} \left\{ \underline{\underline{\partial}}^x \underline{\underline{\partial}}^1 \underbrace{(\underline{\underline{\partial}}^2)^{-1} [\dots]^{-1}}_{\substack{[\dots \partial^2]^{-1} \\ = [\underline{\underline{1}} \underline{\underline{\partial}}^2 + ig \underline{\underline{\vec{x}}} \cdot \underline{\underline{\vec{\nabla}}} \underline{\underline{\partial}}]^{-1}}} \underline{\underline{\vec{x}}} \right\}$$

$$= -ig \text{Tr} \left\{ \underline{\underline{\vec{x}}} \underline{\underline{\partial}}^x \underline{\underline{\partial}}^1 \underbrace{[\underline{\underline{1}} \underline{\underline{\partial}}^2 + ig \underline{\underline{\vec{x}}} \cdot \underline{\underline{\vec{\nabla}}} \underline{\underline{\partial}}]^{-1}}_{\underline{\underline{H}}[-i\vec{\nabla}]} \right\} = -ig \text{Tr} \left\{ \underline{\underline{\vec{x}}} \underline{\underline{\partial}}^x \underline{\underline{\partial}}^1 \underline{\underline{H}}(-i\vec{\nabla}) \right\}$$

$$= -ig \text{tr} \left\{ \underline{\underline{\vec{x}}} \int dy_1 d\vec{p}_2 dy_3 \left[\delta(x-y_1) \delta(y_1-y_2) \right] \left[\partial_{y_2}^1 \delta(y_2-y_3) \right] \underline{\underline{H}}(y_3, y_1; -i\vec{\nabla}(y_3)) \right\}$$

$$= -ig \text{tr} \left\{ \underline{\underline{\vec{x}}} \int dy_3 \left[\partial_x^1 \delta(x-y_3) \right] \underline{\underline{H}}(y_3, x; -i\vec{\nabla}(y_3)) \right\}$$

$$= -ig \text{tr} \left\{ \underline{\underline{\vec{x}}} \int dy_3 \delta(x-y_3) \partial_{y_3}^1 \underline{\underline{H}}(y_3, x; -i\vec{\nabla}(y_3)) \right\}$$

$$\underline{\underline{= -ig \text{tr} \left\{ \underline{\underline{\vec{x}}} \cdot \left[\partial_y^1 \underline{\underline{H}}(y, x; -i\vec{\nabla}(y)) \right]^{y=x} \right\}}}$$

my (A.3)'

But, $\left[\frac{\partial}{\partial y} f(y, x) \right]^{y=x} = \left[\frac{\partial}{\partial x} f(x, y) \right]^{y=x}$ so agree exactly.

Footnote: L2J's g is minus mine, also $\vec{t} = -i \vec{G}$.

<u>L2J</u>	<u>me+AL</u>
g	$-g$
$\vec{t}$	$-i \vec{G}$

Thus:

$$\underline{D}^{\mu}[\vec{A}]|^{me} = [\frac{1}{\approx} \partial^{\mu} - ig \vec{G} \cdot \vec{A}^{\mu}] = [\frac{1}{\approx} \partial^{\mu} + g \vec{t} \cdot \vec{A}^{\mu}]$$

but

$$\underline{D}^{\mu}[\vec{A}]|^{L2J} = [\frac{1}{\approx} \partial^{\mu} - g \vec{t} \cdot \vec{A}^{\mu}] \quad (\text{as in their } [2.6]).$$

So consider:
(L2J)

$$\begin{aligned} & \underbrace{[\underline{D}^{\mu}[\vec{A}] \underline{\partial}_{\mu}]}_{\text{operator}} \underbrace{H[\vec{A}]}_{\text{greens}} = \frac{1}{\approx} \quad \text{as in } [3.6] \\ & = [\frac{1}{\approx} \partial^{\mu} - g \vec{t} \cdot \vec{A}^{\mu} \underline{\partial}_{\mu}] \end{aligned}$$

Thus,

$$\underline{H}[\vec{A}] = [\frac{1}{\approx} \partial^2 - g \vec{t} \cdot \vec{A}^{\mu} \underline{\partial}_{\mu}]^{-1} \quad (\text{as in their } [3.7])$$

Now restate (A.3) Theorem with J :

$$\begin{aligned} & \Delta_L[-i\vec{F}] \left(\frac{\delta}{\delta \mathcal{F}_{\lambda}^a(x)} \right) \Delta_L^{-1}[-i\vec{F}] \\ & = \left(\frac{\delta}{\delta \mathcal{F}_{\lambda}^a(x)} \right) - ig \text{tr} \left\{ \vec{t}^a \left[\underline{\partial}_{\mu}^{\lambda} \underline{H}(y, x; -i\vec{F}(y)) \right]^{y=x} \right\} \end{aligned}$$

Then let $\left(\frac{\delta}{\delta \mathcal{F}_{\lambda}^a(x)} \right) = J_{\lambda}^a(x)$
 $-i\vec{F} = \left[\frac{i\delta}{\delta \vec{F}} \right]$ } to get [3.5]

(3.8)

Eq (3.4) tells us that

$$\left[\frac{1}{\alpha} D_{ab}^{\lambda} \left[i \frac{\delta}{\delta \mathcal{J}} \right] \partial_{\lambda} \partial_{\mu} \frac{i \delta}{\delta \mathcal{J}_{\mu}^b(x)} - D_{ab}^{\mu} \left[i \frac{\delta}{\delta \mathcal{J}} \right] \mathcal{J}_{\mu}^b(x) \right] \underbrace{\left(\nabla_L^{\lambda} \left[i \frac{\delta}{\delta \mathcal{J}} \right] W[\mathcal{J}] \right)}_{\text{from (3.1)}} = 0$$

apply $\Delta_L(i \delta / \delta \mathcal{J})$ since RHS = 0.

Note: $[f(\frac{d}{dx}), g(\frac{d}{dx})] = 0$ as long as not $f(x, \frac{d}{dx})$.

Thus, Δ_L commutes with D etc. At 1st term just says, $\Delta \Delta^{-1} = 1$.
On second term, use (3.5). So

$$\Rightarrow \left[\frac{1}{\alpha} D_{ab}^{\lambda} \left[i \frac{\delta}{\delta \mathcal{J}} \right] \partial_{\lambda} \partial_{\mu} \frac{i \delta}{\delta \mathcal{J}_{\mu}^b(x)} - \overset{\lambda}{D_{ab}^{\mu}} \left[i \frac{\delta}{\delta \mathcal{J}} \right] \cdot \left(\underset{\lambda}{\mathcal{J}_{\mu}^b(x)} - i g \text{tr} \left\{ \overset{\lambda}{t^b} \left[\underset{\lambda}{\partial_{\lambda}} H(x, y, i \frac{\delta}{\delta \mathcal{J}}) \right] \right\} \right) \right] W_{||} = 0$$

$$\Rightarrow \overset{\lambda}{D_{ab}^{\lambda}} \left(i \frac{\delta}{\delta \mathcal{J}} \right) \left[\frac{1}{\alpha} \overset{(\lambda)}{\partial_{\lambda}} \overset{(\mu)}{\partial_{\mu}} \frac{i \delta}{\delta \mathcal{J}_{\mu}^b(x)} - \mathcal{J}_{\lambda}^b(x) + i g \text{tr} \left\{ \overset{(\lambda)}{t^b} \left[\overset{(\mu)}{\partial_{\lambda}} H(x, y, i \frac{\delta}{\delta \mathcal{J}}) \right] \right\} \right] W[\mathcal{J}] = 0$$

/(3.8).

Derivation of (A.4).

① I find it useful to expand:

$$[\partial_\lambda^{(x)} H(x, y)]|^{y=x} = \int dy \delta(y-x) \partial_\lambda^{(x)} H(x, y)$$

Then the ∂_x^λ hits this and makes two terms like so:

$$\begin{aligned} \partial_x^\lambda [\partial_\lambda^{(x)} H(x, y)]|^{y=x} &= \partial_x^\lambda \int dy \delta(x-y) [\partial_\lambda^{(x)} H(x, y)] \\ &= \int dy [\partial_x^\lambda \delta(x-y)] [\partial_\lambda^{(x)} H(x, y)] + \int dy \delta(x-y) [\partial_x^2 H(x, y)] \\ &= \int dy \delta(x-y) \{ \partial_y^\lambda \partial_\lambda^{(x)} + \partial_x^2 \} H(x, y) \\ &= [\partial_y^\lambda \partial_\lambda^{(x)} H(x, y)]|^{y=x} + [\partial_x^2 H(x, y)]|^{y=x} \equiv "Q" \end{aligned}$$

The delta fn really gives a clear instruction.

② Now expand all:

$$ig \left[\underbrace{\delta_{ab} \partial_x^\lambda}_{\downarrow} - g(t^e)_{ab} A_e^\lambda(x) \right] t^{cbd} [\partial_\lambda^{(x)} H^{dc}(x, y)]|^{y=x}$$

$$ig t^{cad} \cdot ["Q"] = \text{the first two terms in RHS of (A.4).}$$

$$\begin{aligned} \text{Last term is } & -ig^2 (t^e)_{ab} t^{cbd} A_e^\lambda(x) [\partial_\lambda^{(x)} H^{dc}(x, y)]|^{y=x} \\ & = t^{aeb} \text{ says (2.1)} \quad t^{aeb} = \epsilon^{aeb} = \text{my } C^{aeb}. \end{aligned}$$

Thus we have (A.4).

Derive (A6) + (3.9)

① Insert (A5) into (A4) + use $t^{\text{cad}} \delta^{\text{dc}} = 0$ to get the left-hand-side of (A6) as stated in paper. It is correct, I did it on scratch paper. Now insert Jacobi as shown to get

$$= ig \text{tr} \left\{ \underline{\underline{t}}^a \left[\left(\partial_x^\mu \partial_\mu^\nu \underline{\underline{H}}(x,y) \right) \Big|_{y=x} + \left(\partial_x^\mu \underline{\underline{H}}(x,y) \right) \Big|_{y=x} g \underline{\underline{t}}^a \cdot \vec{A}^\mu \right] \right\}$$

Now let $y=x$ be done by a delta, so

$$= ig \text{tr} \left\{ \underline{\underline{t}}^a \int dy \delta(x-y) \left[\partial_x^\mu \partial_\mu^\nu \underline{\underline{H}}(x,y) + \partial_x^\mu \underline{\underline{H}}(x,y) g \underline{\underline{t}}^a \cdot \vec{A}^\mu(x) \right] \right\}$$

② This is not really a Tr , so leave as is. Hence, $\approx \boxed{\text{A.6}}$ $\uparrow$

② Now, parts ∂_μ^ν over to the delta to get

oh-do a γ ✓.

$$= ig \text{tr} \left\{ \underline{\underline{t}}^a \int dy \left[\partial_x^\mu \underline{\underline{H}}(x,y) \right] \underbrace{\left[-\partial_\mu^\nu + g \underline{\underline{t}}^a \cdot \vec{A}^\mu(y) \right]}_{-\hat{D}_\mu^\nu[A]} \delta(x-y) \right\}$$

$$= -ig \text{tr} \left\{ \underline{\underline{t}}^a \cdot \int dy \left[\partial_x^\mu \underline{\underline{H}}(x,y) \right] \hat{D}_\mu^\nu[A] \delta(x-y) \right\}$$

Now let $\vec{A} \rightarrow i\delta/\delta\vec{J}$ and this is $\boxed{\text{RHS of (3.9)}}$ $\uparrow$

LHS (3.9) is then LHS (A4) ✓

derivation of (3.12)

Recall that
$$\underline{\underline{H}} = \left[\frac{1}{2} \underline{\underline{\partial}}^2 - g \underline{\underline{\vec{x}}} \cdot \underline{\underline{\vec{A}}} \underline{\underline{\partial}}_\mu \right]^{-1} = \left[\underline{\underline{D}}^\mu \cdot \underline{\underline{\partial}}_\mu \right]^{-1}$$

Thus, we get
$$\underline{\underline{G}} = \left[\frac{1}{2} \underline{\underline{\partial}}^2 - g \underline{\underline{\partial}}_\mu \underline{\underline{\vec{x}}} \cdot \underline{\underline{\vec{A}}}^\mu \right]^{-1} = \left[\underline{\underline{\partial}}_\mu \cdot \underline{\underline{D}}^\mu \right]^{-1}$$

Now
$$\begin{aligned} \underline{\underline{G}}^\top &= \left(\left[\frac{1}{2} \underline{\underline{\partial}}^2 - g \underline{\underline{\partial}}_\mu \underline{\underline{\vec{x}}} \cdot \underline{\underline{\vec{A}}}^\mu \right]^\top \right)^{-1} \\ &= \left(\frac{1}{2} (\underline{\underline{\partial}}^2)^\top - g (\underline{\underline{\vec{A}}}^\mu)^\top \cdot (\underline{\underline{\vec{x}}})^\top (\underline{\underline{\partial}}_\mu)^\top \right)^{-1} \end{aligned}$$

But $(\underline{\underline{\partial}}^2)_{xy} = \partial_x^2 \delta(x-y) = (\underline{\underline{\partial}}^2)_{yx}^\top \quad \underline{\underline{A}}^\mu = A^\mu(x) \delta(x-y) = \underline{\underline{A}}^{\mu\top}$

$(\underline{\underline{\partial}}_\mu^\dagger)_{xy} = \partial_\mu^x \delta(x-y) = -(\underline{\underline{\partial}}_\mu)_{xy}^\top$

$\underline{\underline{\vec{x}}}^\top = -\underline{\underline{\vec{x}}} \quad \text{since T.A.}$

Thus,

$$\underline{\underline{G}}^\top = \left(\frac{1}{2} \underline{\underline{\partial}}^2 - g \underline{\underline{\vec{A}}}^\mu \cdot \underline{\underline{\vec{x}}} \underline{\underline{\partial}}_\mu \right)^{-1} = \underline{\underline{H}}$$

So

$$\boxed{\underline{\underline{G}}^\top = \underline{\underline{H}}}$$

Derivation of (3.10)

Comment: I had a lot of trouble with this because there is a lot of stuff going on here. Finally, however, I have made it work.

We want to show that (3.10) duplicates (3.8). The first term of 3.10 obviously is the same as the first term of (3.8) so cancel those. It remains to show that terms 2+3 of (3.8) equal term 2 of (3.10). But, notice that either side of (3.9) is exactly term 3 of (3.8). So here is what we want to show:

$$[(3.8) \text{ term } 2] + [\text{RHS of (3.9)}] = [(3.10) \text{ term } 2] \quad (*)$$

Normal order: why have they done this? The meaning is that all those d/dJ 's sitting inside the $::$ are supposed to not act on that J^d guy, but are supposed to only go to the right and hit onto $W[J]$. You wonder: why didn't they just write the J^d on the left and dispense with the $::$. Reason: don't forget that D_y contains a regular derivative d_y which acts on $J^d(y)$ so you have a problem: whichever way you order these guys, one of the derivatives won't be right. You would have to back-arrow the d_y inside D_y .

But, I could still fix up (3.10) by first partsing the d_y in D_y over onto the H (getting a minus), then switching the indices from cd to dc (getting another minus on other term for overall minus). Then 3.10 could be written like this:

$$D_{ab}^{\lambda} \partial_{\lambda} \left\{ \frac{i}{2} \partial_{\mu} \frac{\delta}{\delta J_{\mu}(x)} + \int dy J_{\lambda}^d(y) \overline{D_y^{\mu} [i \frac{\delta}{\delta J}]}^{dc} H^{bc}(x, y; i \frac{\delta}{\delta J}(x)) \right\} W = 0$$

The price paid for getting rid of the $::$ is that now the color indices don't match correctly. But now you can replace H with G^T and get the indices to match, so you get:

$$D_{ab}^{\lambda} [i \frac{\delta}{\delta J(x)}] \partial_{\lambda} \left\{ \frac{i}{2} \partial_{\mu} \frac{\delta}{\delta J_{\mu}(x)} + \int dy J_{\lambda}^d(y) \overline{D_y^{\mu} [i \frac{\delta}{\delta J}]}^{dc} G^{cb}(y, x; i \frac{\delta}{\delta J}(x)) \right\} W[J] = 0.$$

How you now go from this to (3.13) is a story I don't know yet, so let's get back to the present problem. I will keep the normal ordering form of (3.10) since I know what it means.

see next section →

Proof of above equation (*): Look at (3.10). Forget for the moment the action of the outer operator Dd on the $::...J::$. Then the Dd goes right into the integral and goes inside the normal ordering and hits onto H . By (3.6), this makes a delta and the net result is this: $- : D(id/dJ) J :$ However, I claim this is exactly the same as (3.8) term 2. Actually there you see $-D J$ without the normal order. But, the term in which the d/dJ hits the J vanishes by color symmetry (just did it).

Therefore, all that remain is to show that , when you let the outside d/dJ in (3.10) hit the explicit J inside the $...$ factor, this must yield RHS of (3.9). It does, as is shown on page attached to this, not very hard to do.

Aside: notice that $[d_x, D_x(id/dJ(x))]\neq 0$. It matters which order these appear in. That is why G/H , for example.

$$- \dot{D}_x^{(x)} \left[i \frac{\delta}{\delta \bar{\psi}(x)} \right] \partial_x^{(x)} \int dy: H^{bc}(x, y; \frac{\delta}{\delta \bar{\psi}(x)}) D_{(y)}^{(x)} \left[i \frac{\delta}{\delta \bar{\psi}(y)} \right] \bar{\psi}^d(y):$$

only

$$= - \dot{D}_x^{(x)} \left[i \frac{\delta}{\delta \bar{\psi}(x)} \right] \partial_x^{ab} \int dy: \left[\partial_x^{(x)} H^{bc}(x, y; \frac{\delta}{\delta \bar{\psi}(x)}) \right] D_{(y)}^{(x)} \left[i \frac{\delta}{\delta \bar{\psi}(y)} \right] \bar{\psi}^{cd}(y):$$

Now, expand

$$- \left[\delta^{ab} \partial_x^{(x)} - g(t^e) i \frac{\delta}{\delta \bar{\psi}_x^e(x)} \right]$$

Let me hit $\bar{\psi}_\mu^d(y)$ to get

$$- i g t^{eab} \int dy: \left[\partial_x^{(x)} H^{bc}(x, y; \frac{\delta}{\delta \bar{\psi}(x)}) \right] D_{(y)}^{(x)} \left[i \frac{\delta}{\delta \bar{\psi}(y)} \right] \left(\delta_x^\mu \delta_e^d \delta(x-y) \right)$$

= (3.9) RHS ✓

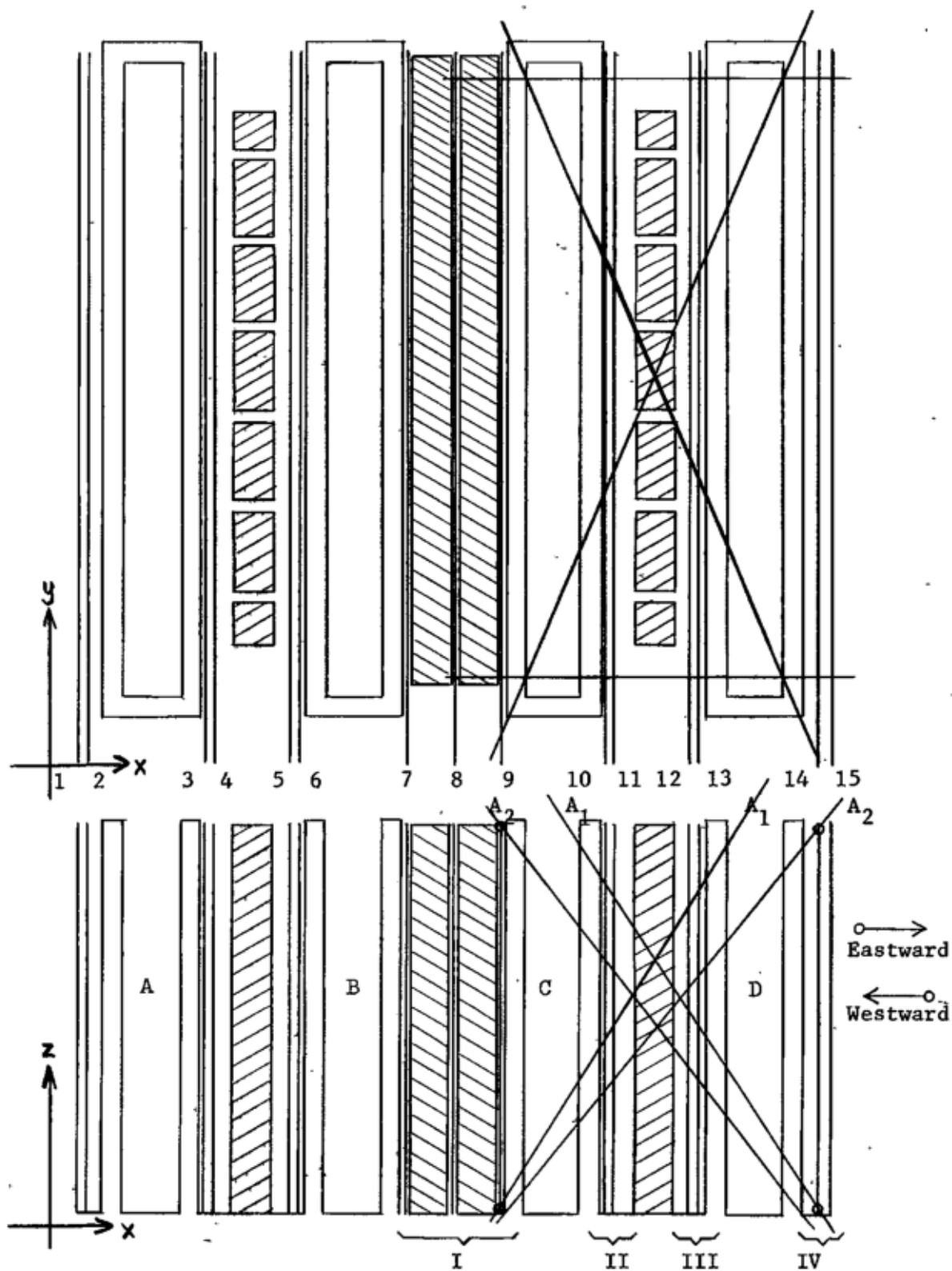


Figure 15. Schematic of Restricted Aperture.

Sign Error in LeeZinnJustin-I

1. First, notice that you always have a convention on how you will define your sources and how you put in your gauge fixing term. There is also a question of sign of metric. I always use the metric which implies $p^2 - m^2$ in propagators. So does Coleman. Also, Coleman shows that if you Wick rotate correctly with this metric, you get:

$$d^4x = -i d^4x_E$$

Thus, I am able to justify the sign of Coleman's gauge fixing term:

$$e^{iS} = e^{+i \int d^4x \left[S_0(x) - \frac{1}{2\alpha} (\partial_\mu A^\mu)^2 \right]}$$

$$\text{But} \quad +i \int d^4x \left[-\frac{1}{2\alpha} (\partial_\mu A^\mu)^2 \right] = - \int d^4x_E \left[\frac{1}{2\alpha} (\partial_\mu^E A^\mu)^2 \right]$$

and then the idea is that, for $\alpha > 0$ at least, this exponent is negative and vaguely speaking you have a convergent integral if you go over to Euclidean space.

Anyway, this is the sign of the gauge fixing term used by Coleman, and be LZJ-1, and also in Abers and Lee. When this is combined with the usual $L_0 = -(1/4) F^{uv} F_{uv}$ you get the following bare propagator:

$$\left[(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}) + \frac{\alpha k_\mu k_\nu}{k^2} \right]$$

This is what I get, Coleman gets, Abers and Lee get, and LZJ-1 get (see 2.11). I.e, we all have that $g_{\mu\nu}$ has the same sign as alpha.

We do not all agree on the sign of the source, but no matter for now.

2. Fact: when I work through LZJ computation in detail, keeping exact track of all signs, here is what I get for my final result :

$$\text{holog.} \quad D_{\phi a}^\lambda \left[i \frac{\delta}{\delta J} \right] \partial_\lambda \left[\frac{i}{\alpha} \partial_\mu^{(x)} \frac{\delta}{\delta J_\mu^a(x)} + \int dy \, J_\lambda^c(y) D_{(y)}^\mu \left[i \frac{\delta}{\delta J} \right]^{cb} G^{ba}(y, x; i \frac{\delta}{\delta J}) \right] W[J] = 0 \quad (3.13)$$

This is my equation (3.13). The relative plus sign between the terms is consistent with LZJ's equation (3.10) because you get a minus when you switch the action of D from J over onto H, etc etc. So, I am willing, without justification, to claim that the square bracket above equals 0. I wonder: does my result agree with Slavnovs result?

On a page attached I have examined Slavnov's paper and have converted his result into a functional result so I can compare it with mine. First, I have to replace his source with $J = -n$ because his "source-sign convention" is opposite LZJ's. When I do this, I find that Slavnov's result has a relative minus sign between the terms, whereas I had a relative plus. BUT that means we agree because Slavnov's α is minus the α used by LZJ and me and everyone. Ie,

$$\alpha_{\text{Slavnov}} = -\alpha_{\text{LZJ-1}}$$

$$n_{\text{Slavnov}} = -J_{\text{LZJ-1}}$$

Thus, my result agrees with Slavnov's result.

Conclusion: Lee and Zinn Justin have the sign wrong in their (3:13). Everything else they have done is correct, except typos I have already marked.

Compare Slavnov's Result to Lee ET.

① Slavnov uses:

$$Z[\eta] = \int dB \Delta(B) e^{i \int dx \left[\mathcal{L}(x) + \underbrace{\vec{\eta} \cdot \vec{B}}_{\equiv S} + \frac{1}{2\alpha} (\partial_\mu \vec{B})^2 \right]}$$

This result is (20) which I write as

$$0 = \int dB \Delta(B) \cdot \left[\frac{1}{\alpha} (\partial_\mu B_d^{(y)}) + \int dz \cdot \eta_\mu^a(z) \left\{ \partial_\mu^{(z)} \delta^{ac} + g \epsilon^{abc} B_\mu^b(z) \right\} (\tilde{M}^{-1})^{cd} \right] e^{iS}$$

Now he must use $D_\mu^{(z)ac}[B] = \delta^{ac} \partial_\mu^{(z)} + g \epsilon^{abc} B_\mu^b(z)$

$$= \delta^{ac} \partial_\mu^{(z)} - ig [G_b^{adj}]^{ac} B_\mu^b(z)$$

$$= i[G_b]_{ca}$$

= same as my convention.

② So Slavnov really says

$$0 = \int [dB] \Delta(B) \cdot \left\{ \frac{1}{\alpha} (\partial_\mu B_d^{(y)}) + \int dz \eta_\mu^a(z) D^{(z)\mu}_{[B]ac} (\tilde{M}^{-1})^{cd}_{zy} \right\} e^{iS}$$

Now, replace $B \rightarrow \left(\frac{1}{i} \frac{\partial}{\partial \eta} \right)$. So he would say

$$0 = \left\{ \frac{1}{\alpha} \left[\partial_\mu^{(y)} \left[\frac{1}{i} \frac{\partial}{\partial \eta_d^{(y)}} \right] + \int dz \left[\eta_\mu^a(z) D^{(z)\mu}_{\left[\frac{1}{i} \frac{\partial}{\partial \eta} \right] ac} (\tilde{M}^{-1})^{cd}_{zy} \left(\frac{1}{i} \frac{\partial}{\partial \eta} \right) \right] \right\} W[\eta]$$

Now to make this look like L2J, let $\eta = -J$. Then:

$$S = S_0 - \vec{J} \cdot \vec{B} + \frac{1}{2\alpha} (\partial_\mu \vec{B})^2$$

③ $\Rightarrow$
$$0 = \left\{ \frac{1}{\alpha} \left[\partial_\mu^{(y)} \left[\frac{1}{i} \frac{\partial}{\partial J_d^{(y)}} \right] - \int dz J_\mu^a(z) D^{(z)\mu}_{\left[\frac{1}{i} \frac{\partial}{\partial J} \right] ac} \tilde{M}^{-1} \left(\frac{1}{i} \frac{\partial}{\partial J} \right)^{cd}_{zy} \right] \right\} W[J]$$

This is exactly (3.13). But their α 's are opposite sign !!

III. More WTIs.

notes.

A. $n=2$. Very strange stuff.

- ① (4.5) says that inverse propagator has no scalar fctn to worry about. I didn't need WTI to learn this, or did I? Maybe yes because ghosts enter the vector Dyson!

$$J(k^2) = \text{sole scalar fctn in } [\Delta^{-1}(k)]_{\mu\nu}.$$

$$\mathcal{D}(k^2) = \text{the ghost propagator.} \quad \Delta_{\mu\nu} = \text{gauge prop}$$

B. $n=3$

$$\gamma_{\mu}^{abc}(p, q, r) = \text{diagram} = g g - v \text{ vertex, 1PI.}$$

$$= p^{\lambda} \gamma_{\lambda\mu}(\dots)$$

① fuddle $\Rightarrow$ x-space
no twid $\Rightarrow$ k-space.

$$(4.14) \text{ reads: } \text{diagram} = \text{diagram}$$

This is the x-space ghost self-energy Dyson!

$$\Sigma_g = -k_{\mu} \Sigma_g^{\mu}(k) \quad / \text{perhaps by dimension}$$

[4.15] = some kind of Ward which says

[ugh!!]

$$f[J(k^2), \Gamma_{abc}^{\lambda\mu\nu}(p, q, r), \mathcal{D}(k^2), \gamma_{\lambda\mu}^{cab}] = 0$$

$\uparrow$ gauge prop $\uparrow$ 3-gluon vertex $\uparrow$ ghost prop $\uparrow$ 1-gluon/2-ghost vertex

Comments after redoing FP.

Looking at (3.11) I see

$$\begin{aligned} \partial^\mu \overset{ab}{D}_\mu [A(x)] G^{bc}(x, y; \vec{A}) &= \int dz \partial^\mu_{(x)} \overset{ab}{D}_\mu [A(x)] \delta(x-z) G^{bc}(z, y; \vec{A}) \\ &= \int dz \cdot \{-g M_{ab}(x, z)\} G^{bc}(z, y) = \delta_{ac} \delta(x-y) \end{aligned}$$

Or: $[-g \underline{\underline{M}}][\underline{\underline{G}}] = \underline{\underline{1}}$

Therefore, beyond any doubt,

$$G_{ab}(x, y; \vec{A}) = -\frac{1}{g} \overset{-1}{M}_{ab}(x, y; \vec{A})$$

You can see that

$$H_{ab}(x, y) = -\frac{1}{g} [M^T]^{-1}_{ab}(x, y) = G^T_{ab}(x, y) \checkmark$$

If $\partial_\mu A_\mu = 0$, then $G = G^T$.

By the way, we know from our propagator-identification page that this function $M^{-1}_{ab}(x, y)$ is the dressed ghost propagator! Recall from BD vol I that any propagator is the Greens function of a differential operator. Here that operator is dD . As written above, G is the amplitude for propagation to x from a source at point y . Lee's remark about "outgoing boundary conditions" merely refers to the fact that you want the positive energy component of your "wave" $G(x, y)$ to radiate outward from the source, to respect causality. In non-rel theory this just means $G=0$ for time in the wrong sequence, and in positron theory its more complicated but same is true for each energy piece.

Lee & Zinn-Justin

II

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PHYSICAL REVIEW D

VOLUME 5, NUMBER 12

15 JUNE 1972

Spontaneously Broken Gauge Symmetries.

II. Perturbation Theory and Renormalization

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 (Received 10 March 1972)

The second paper in this series is devoted to the formulation of a renormalizable perturbation theory of Higgs phenomena (spontaneously broken gauge theories). In Sec. II, we reformulate the renormalization prescription for massless Yang-Mills theories in terms of gauge-invariant renormalization counterterms in the action. Section III gives a group-theoretic discussion of Higgs phenomena. We discuss the possibility that an asymmetric vacuum is stable, and show how the symmetry of the physical vacuum determines the mass spectrum of the gauge bosons. We show further that in a special gauge (*U* gauge), all unphysical fields can be eliminated. Section IV discusses the quantization of a spontaneously broken gauge theory in the *R* gauge, where, as we show in Sec. V, Green's functions are made finite by the renormalization counterterms of the symmetric theory (in which the gauge invariance is not spontaneously broken). The *R*-gauge formulation makes use of redundant fields for the sake of renormalizability. Section VI is a discussion of the low-energy limits of propagators in the *R*-gauge formulation. In Sec. VII we show that the particles associated with redundant fields peculiar to the *R*-gauge formulation are unphysical, i.e., they do not contribute to the sum over intermediate states.

I. INTRODUCTION

In this paper we give a renormalization method and a proof of finiteness of renormalized Green's functions of spontaneously broken gauge theories. For definiteness we consider a very simple model in which $SU(2)$ gauge bosons are coupled to a triplet of scalar mesons. There is an extra complica-

tion when chiral fermions are included in the model, as pointed out by Veltman,¹ and more recently by Gross and Jackiw.² This difficulty can be circumvented in a realistic model of electromagnetic and weak interactions. We shall not discuss this problem further in this paper, but postpone the discussion until we deal with the renormalizability of a realistic theory in a sequel to this paper.

1. Introduction. These guys have figured all this out in the SIGMA model, and are now generalizing to arbitrary NAGT. Actually, in this paper they will use a simple SU(2) color group with obvious Higgs fields ϕ coupled in as you would imagine. These are the only matter fields they will use, I think.

The gauge wherein the Goldstones are absorbed into gauge masses via the Higgs-Kibble mechanism is called the U-gauge. The claim is that in this gauge, unitarity of the theory is obvious, but renormalizability is not (after all, you have massive vector propagators screwing you up.) But then there will be something called the R-gauge where the renorm is easily proved, but in which unitarity is not easy to see. I imagine the authors will prove renormalizability in the R-gauge only.

Then the game is to show that the two gauges are equivalent, and I think that is the topic of the next paper, No. III.

As an added complication, authors allow for the possibility of SSB. After all, it is SSB which creates the would-be Goldstones. Discuss this below.

II. Gauge-Invariant Counterterms. In this section they show how you add counterterms to the lagrangian to get a finite theory. Finally this is clear to me. It is totally un-obvious why you get a finite theory unless you have studied say ϕ^3 . It is just the usual multiplicative renorm deal, except of course you have to consider both the glues and the ghosts. Good discussion of choosing renorm points. The both Wards are used to show that only one charge g is needed in the end. I think the Wards will be used again later.

It is noted that if you define renormed sources, you can get renormed Greens by differentiating the renormed genfun W , which now includes those counterterms. It is noted that the counterterms, including the regulator counterterms discussed in last paper, are all gauge invariant.

A new version of I 3.13 is written (general Ward) but this time for renormed objects. I have not yet checked this result, because I am hurrying to see where this series of papers is headed.

III. Group Theory of Higgs Phenomena. A quick review of the usual stuff. Start with a group sym with N generators. Assume the Higgs field has nonzero VEV, so the group of the vacuum has only M generators. The difference is $m = N - M$ is the number of "would-be Goldstones" (Goldstone Theorem: there must be m massless particles. I.e., m of the Higgs scalars must be massless.) Then you choose a fancy gauge so that the Goldstones are eaten by some of the glues, so you get m massive vectors. This is Higgs-Kibble Mechanism. Here all discussed in terms of classical fields.

4. Quantization of Higgs Phenomena. Take it slow, it all makes sense. First, write down Lagrangian including Higgs fields. Higgs self coupling now called λ and there is a Higgs mass counterterm. Whole thing is a gauge scalar. When this Lagrangian is combined with gauge-fixer and ghosts, you get action called S_λ since parameter λ is still in there.

Major point: write down the genfunc for this action, putting in source J for the glues and source K for the Higgs. Then the Ward Identity (4.5) is true whether there is SSB or not. The Functional Ward doesn't care where the potential has its minimum.

Now we want to avoid imaginary masses in our Feynman rules. Obviously, then you want to expand your potential around its minimum, as in classical theory. First, write each field as a sum of "long" and "trans" parts, where these refer to the direction chosen by the vacuum in isospin space (these have nothing to do with long/trans with respect to some k^μ as in a gauge propagator; the gauge prop always has long and trans pieces in this sense).

Then, imagine that "long" refers to the z or 3 direction in isospace. So you expect your Higgs field to have a non-zero VEV for ϕ_3 . The difference of $\phi_3 - v$ which is the VEV is called ψ . The long part of color vector $\vec{A}$ is called A .

So what are we doing? To get sensible Feynman rules, we are replacing the field ϕ_3 with the field ψ ; then all fields in L will have zero VEV. Then rewrite action in terms of all these new fields. (only one new field here). New action is (4.15) plus (4.16). I haven't checked all the terms but IK could.

Next, if you want to do perturbation theory you pull out your interaction action and claw in functional derivatives, as in (4.17). The non-interaction action can then be recast in the usual way into propagators, as in (4.19). You see that there are 5 kinds of propagators in the new theory.

So, we now know the "sensible" Feynman rules for the SSB broken theory. Great. This is a hairy combination of many ingredients: 1) functional formalism to get Feynman rules for a perturbation expansion; 2) Yang-Mills theory coupled to Higgs scalars, all gauge scalar; 3) gauge-fixing and FP det ghosts stuff; 4) SSB breaking possibility so fields must be shifted.

Now on top of all this stuff we have to ask: how do you renormalize the damn thing? It's really no problem. First, you are supposed to fiddle with the Higgs mass (counterterm) to make sure that, in any order, your field ψ really has zero VEV. I.e., the tadpoles with one ψ field must be cancelled. See * term in (4.16) to see that this mass counterterm is in a position to accomplish this task.

So choose that mass counterterm as just mentioned, then do a standard multiplicative renormalization on the theory. Here I'll just list things off:

Higgs: $(\psi, \psi, \vec{\phi}_k) = Z_2^{-\frac{1}{2}} (\psi, \psi, \vec{\phi}_k)$ so Z_2 rela to Higgs prop.

Gluon: $^0 A_\mu = Z_3^{-\frac{1}{2}} A_\mu$

Ghost: $^0 c_\mu = Z_3^{-\frac{1}{2}} c_\mu$ / they don't do this

Σ_1 rela 3-gluon vertex. ($= \Sigma_5$)
 Σ_1 rela 2-gluon / 1-ghost vertex. ($= \Sigma_5$)
 Σ_4 rela. 4-Higgs vertex.

Σ
 $\underbrace{\Sigma_3, \Sigma_1}_{\text{gluon}}$; $\underbrace{\tilde{\Sigma}_3, \tilde{\Sigma}_1}_{\text{ghost}}$; $\underbrace{\Sigma_2, \Sigma_4}_{\text{Higgs}}$

detail: $\Gamma_{\text{Higgs}}^{(4)} = \Gamma$.

$$\Gamma = \Sigma_2 \Gamma \quad \Gamma_S = \Sigma_4 \lambda_0$$

$$\Gamma_3 = \lambda \Rightarrow \lambda = \Sigma_2 \Sigma_4 \lambda_0$$

$$\Rightarrow \lambda_0 = \lambda \Sigma_4 / \Sigma_2^2 \quad (4.26)$$

So we define six Z's in this standard way. Then we can take the Lagrangian and replace all fields and charges with their expressions in terms of renormed fields and renormed charges, then we regroup into terms plus counter-terms. The result will be (4.27) which shows only the counter-terms. Notice that the FP ghosts don't get mixed around with other things. You can imagine adding this counterterm lagrangian to the thing shown in (4.15) plus (4.16).

Comment: Now things are all set up. If I were at this stage with a simpler theory, the remaining work would be to study integral equations etc and find that if you iterate order by order the theory really is finite. I.e., I would want to show that the multiplicative renorm scheme really does yield a finite theory. I carried this through for ϕ^3 ; BD carry it through for QED. Now it looks like L and ZJ are going to carry it through for this complicated gauge theory which has everything: non-abelian so ghosts, Higgs and SSB, etc. This I have got to see!!!

(an applied external field!)

5. Proof of Finiteness. I am very confused but will try to make sense out of this anyway. Usually when you have sources you say: compute a derivative and then set sources = 0. Now he says: set source $J=0$, but set source $\vec{K} = \vec{\gamma}$. Why I don't yet know. So, at the point $(0, \gamma)$ for the sources, $dZ/dK_1 =$ some Higgs field value v_1 which is not zero. Call this $v_c \hat{n}$, the Higgs VEV deal point in the $\hat{n}$ direction, fine. Then define field χ as in 5.5 and rewrite Lagrangian in terms of these new fields as in (5.7'). Notice that this gives just the kinetic and mass terms of the lagrangian; the interaction terms are the same as in 4.16 with v replaced by our new v_c .

First, notice the significance of symbol m : it is the mass of the trans Higgs fields. Thus, these are the guys that would be zero if Goldstones (start with 3, break 2 down to 1 so 2 goldstones; golds are the transversals here). So we are curious to see whether $m=0$ or no.

Meanwhile, we are told to assume as will be proven later that $\vec{\gamma}$ will be proportional to the vector $\vec{\gamma}$ so we can say that $\vec{\gamma} = c \hat{n}$, so symbol c is this thing. As c goes to zero, you return to the usual sources=0 place to define greenses.

Small detail: the linear χ term in our new lagrangian contains extra terms and turns out to be as shown in (5.9). Again the idea will be to adjust the mass counterterm to kill off all tadpole graphs, as stated in 5.10.

Now, off we go into proper vertices. The classical fields are called $\vec{A}, \vec{\Phi}_1, \vec{\Phi}_2$ where these last two refer to the Higgs guys. You can form the proper vertex genfunc and then its func derivs will give you proper vertices, as says (5.18).

Then comes the step I don't get yet: if write the proper vertex for the theory with $v = v$, and its a power series in v , and the coefficients are certain proper vertex functions of the theory with $v=0$. Lets accept this fact for the time being and see where it is trying to lead us. If 5.20 is true, so what??? Well, 5.21 states how you renormalize the symmetric theory. This paper is assuming that the finiteness has been shown for the symmetric theory (symmetric means no SSB so $v=0$). Ie, you can choose the counterterm Z 's to get a finite theory.

Now define the renormed objects shown on LHS of (5.22). Then you get (5.23) which is nice: if the v_r are finite, then the object on the RHS is finite because the proper vertices PI of the symmetric theory are assumed to be finite.

Now there is a discontinuity in the discussion, a note added in proof is added. Try to pick it up again. They want to select a value for Z_2 so that (5.24) is true. Thus they have slightly changed the definition of Z_2 (by a finite amount). So, at $k^2=0$ they want the Kiggs prop to be $-m^{-2}$ (and m better be some finite number!).

We now start on top of second column, page 3147:

This column is just about the Goldstone theorem. I have derived (5.26) from (5.25) and thus I agree, the things are parallel. With the new definition of Z_2 , I also agree with (5.26) and thus (5.27). "Spontaneous" breaking means all sources are zero, so when you now take the K source $\vec{\zeta} = \hat{c}\vec{n}$ to zero, you see that either $m=0$ or $v_r=0$. I.e., either you have Goldstones (massless Higgses) or you have no breaking of the vacuum.

But that was just Goldstone stuff. The main work of the section is already done and is contained in (5.23) which says: if the symmetric theory is renormalizable, then so is the theory with SSB present! It is assumed that one already knows completely how to renormalize the "symmetric theory" i.e. the one with no SSB.

In fact, I think that even with c_r held away from zero (so there is a K source present) you can renorm the theory.

Recap of the first 5 sections of this paper.

Lets look back and see just what has been done. In Section 2 they list off the counterterms you would add to the Lagrangian to renormalize a normal theory with glues and a triplet of Higgs with normal mass. This section does not properly allow for the possibility of SSB because if taken directly, it would imply imaginary mass in Feynman rules etc. As we know, you really should define new, shifted fields if there is SSB.

So, skipping over the Higgs-Kibble review section, we come to section 4. Here, they write down the same action as above for the glue + Higgs system. Then they define potentially shifted fields (the PSI field) and set up the perturbation theory in general, allowing for the possibility that there is SSB. Equ. (4.20) shows all the propagator in the perturbation theory, for example. Then on the next page they show all the counterterms (now written in terms of the shifted, physical fields) you should add to achieve renormalization.

If there were no SSB, then $v=0$ and shifted field reduces to regular field. The Z's which render the theory finite can be computed in terms of the cutoff. These are "the Z's which renormalize the symmetric theory". These are the Z's you would use if $u^2 = \text{positive}$.

Now suppose there is SSB, how do you know you can still renorm the theory? You have a bunch of counterterms written down, how do you know that they really make the theory finite? Answer: you derive equation (5.23) which states: the proper renormed vertices of the SSB theory can be written in terms of the proper renormed vertices of the regular (symmetric) theory. So choose the Z's that renorm the regular theory, and then your SSB proper vertices will also be finite!! Thus, the same Z's will renormalize the theory regardless of the sign of u^2 !!!

One interesting point which I think will be important later is this: the Ward Identities have the same form no matter what u^2 is. Whether or not you have SSB, the Wards look the same. Moreover, since the counterterms are all gauge-scalars, they have no effect on the Ward identity, thus, in the theory with these counterterms included, your greens will satisfy Wards of the same form as the wards of the unrenormed theory. In other words, the finite, renormed Greens functions satisfy the same old Wards we have been using all the time.

So, Wards survive renormalization and SSB!

So, these sections of the paper constitute a "proof" that ~~BYEM~~ SSB does not affect the renormalizability of a theory. If the symmetric theory is Renormalizable, then so is the broken theory! Note that this proof is carried out in a normal ϕ -Landau gauge. However, as (4.20) shows, you have "redundant fields" because you have both goldstone Higgs and massive glues present. One wonders if these massless Higgs fields don't mess up unitarity somehow. In the U-gauge these goldstones are completely gone so unitarity is easier to show.

Section 6: Low energy behavior of propagators.

In the next section authors want to show how the negmetric + goldstones + FGghosts (all massless "particles" present in the theory in the R-gauge) all cancel in unitarity sums, so you never get an amplitude having a pole at zero. Thus, in this section they want to find out how to express the behavior of the negmet and goldstone propagators as they go near mass shell (ie, $k^2=0$), because mass shell is relevant for unitarity quaestions. The results are shown in (6.10).

How are these results arrived at? They write general scalar-function forms for three "inverse propagators" as shown in (6.7). These are related to the propagators through a matrix equation (6.6) which reminds me of the similar thing in Kummer. Thus, you can use the same symbols A,B,C,D in your propagators (these are scalar functions). It seems that C=0 identically due to color symmetry somehow; from this you learn that for small k^2 , the function $\hat{A}(k^2) = k^2/\alpha$ where α is the gauge parameter. The upshot is that the negativemetric propagator contains function A, whereas the goldstone prop contains function D. So far there seems to be no correlation between functions A and D, but there is as shown later.

Section 7: Unitarity

First example is look at 1-particle intermediate state. They show that the goldstone cancels exactly agains the negmetric (ie, the long part of the glue propagator). In this example of course you cant have FP ghosts because they can only occur in pairs.

To show this little cancellation, use is made of a very simple case of the Ward Identity. Thus, very important that this identity be true for the renormalized theory (and possibly SSB broken !). Here is where the Ward really helps: showing these ghost cancellations. They explicitly consider the case of two massless and then three massless. In each case the Ward allows them to show cancellation. They dont have notation that allows a general case to be done, but seems clear that it works.

Thus, the only poles at zero will come from the "phtons" of the theory, gauge particles that did not acquire mass. The other massless particles always cancel out.

Also in this section they claim to show that S-matrix elements cannot depend on paramter alpha.

Comment: In this paper, authors show both renorm and unitarity entirely within the R-gauge. They mention that there is a U-gauge where goldstones are completely eaten, but they never use that gauge in this paper.

The Famous Z/Z ratio equation for NAGT in Covariant Gauge.

1. This equation appears as (2.2) in L-ZJ-II paper. I finally understand where it comes from and I will derive it here. The only ingredient I have not yet derived is the Ward Identity which appears as Eq. (4.15) in LZJ-I paper. You must understand that this identity is derived using the functional formalism from a certain Lagrangian, namely the YMT Lagrangian with F.F and gauge fixing and FP det. This lagrangian does not have the renorm counterterms in it yet, so the Ward Identities you get always relate the unrenormalized Greens functions! Basically, this Ward Identity (scalewise) reads:

$$\overset{\circ}{\Delta}^{\mu\nu} = \overset{\circ}{G}^{\mu\nu} \quad (I 4.15)$$

2. Glue Propagator. Even though there are ghosts to worry about, the longitudinal part of the glue propagator is not "renormed", which means it is the same as in the bare prop. I have not yet showed this, but JZJ showed it functionally (and so does Kummer later in the axial gauge). Thus, your unrenormed propagator looks like this:

$$\overset{\circ}{\Delta}_{\mu\nu}(k) = \frac{1}{k^2} \left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right) J(k^2) + \alpha_0 \frac{k_\mu k_\nu}{k^2} \quad / \text{unren.}$$

$$\Delta_{\mu\nu}(k) = \frac{1}{k^2} \left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right) J(k^2) + \alpha \frac{k_\mu k_\nu}{k^2} \quad / \text{ren.}$$

These guys are related by:

$$\Delta = Z_3 \overset{\circ}{\Delta} \Rightarrow \begin{cases} J(k^2) = Z_3 \overset{\circ}{J}(k^2) \\ \alpha_0 = Z_3 \alpha \end{cases}$$

$$A^0 = Z_3^{\frac{1}{2}} A \quad \checkmark$$

Renorm condition:

$$J(-a^2) = 1 \Rightarrow \Delta_{\mu\nu}(-a^2) = -\frac{1}{a^2} \left[g_{\mu\nu} + \frac{k_\mu k_\nu}{a^2} \right] (1) + \dots$$

3. Ghost Propagator.

$$\overset{\circ}{\mathcal{Q}}(k^2) = \frac{\overset{\circ}{L}(k^2)}{k^2} \quad \mathcal{Q} = \sum_3 \overset{\circ}{\mathcal{Q}}$$

$$\mathcal{Q}(k^2) = \frac{L(k^2)}{k^2} \quad L(-a^2) = 1 \quad \Rightarrow \quad \mathcal{Q}(-a^2) = \frac{1}{(-a^2)} \quad \checkmark$$

Ren. Cond.

4. Triple Glue Vertex.

$$^{\circ}\Gamma_S = g_0 \sum_5^{-1} \tilde{\Gamma}_S^{\text{Bare}} + \underbrace{(\text{other tensor forms})}. \quad S = \text{sym point} \Rightarrow q_i^2 = -q^2 \text{ and } \dots$$

these can be projected to zero by a choice of ϵ^μ pol. vectors (like $\tilde{u}(p)$ in QED).

$$^{\circ}\Gamma = \sum_3^{-3/2} \Gamma$$

$$\Rightarrow \Gamma_S = \sum_3^{3/2} ^{\circ}\Gamma_S = \left(\sum_3^{3/2} \cdot \sum_5^{-1} g_0 \right) \tilde{\Gamma}_S^{\text{Bare}} + \sum_3^{3/2} (\text{other stuff})$$

Renorm cond.: $\tilde{\Gamma}_S = \tilde{\Gamma}_S^{\text{Bare}}$, when projected correctly.

$$\Rightarrow \boxed{g = \sum_3^{3/2} \sum_5^{-1} g_0} \quad \text{since } \tilde{\Gamma}_S = \frac{1}{g} \Gamma_S \text{ as usual.}$$

So this is the renormed "charge", the triple glue vertex coupling constant/. I few details are not yet clear: like, what happens to those extra terms? Sure, you can exclude them from your renorm condition by projecting them out, but they are still present in the vertex itself. Moreover, due to the powers of momenta (power counting), I think you can show that the object above called (other tensor forms) is a function only of g_0 and does not depend on the cutoff. Thus, it seems likely to me that the term gets multiplied by a power of Z_3 and therefore will probably be cutoff dependent in the renormed vertex. Somehow, it must turn out that these other terms never contribute to an S-matrix element, just like the "gauge terms" in a glue propagator. Later I will look at this problem in more detail.

For now, we have chosen a charge g and have insisted that the triple-glue vertex be this number g (times bare tensor, ~~modulo~~ modulo projection) at the symmetric renorm point. Earlier we insisted that the renormed propagator have "unit residue", ie, that $J=1$ at the renorm point.

But is this possible? Actually, if you hold g_0 fixed (and $m_0=0$) and you have a given cutoff, the quantities Z_3 and Z_5 are already determined to any order in g_0 and are divergent. Nevertheless, since prop is rescaled exactly by Z_3 , you can still insist that $J=1$. However, g comes out as it will. But now reverse it and insist on a g ; then g_0 gets forced to some value. Fine.

5. Ghost-glue vertex.

$$^{\circ}\gamma_S = g_0 \sum_5^{-1} \gamma_S^{\text{Bare}} + (\text{possible other forms}).$$

↑ same g_0 since g_0 appears in bare γ coupling.

$$^0\gamma = (\tilde{Z}_3^{-\frac{1}{2}})^2 (\tilde{Z}_3^{\frac{1}{2}})^1 \gamma$$

since one particle at this vertex is a glue.

$$\Rightarrow \gamma_S = \tilde{Z}_3 \tilde{Z}_3^{\frac{1}{2}} \gamma_S = (\tilde{Z}_3 \tilde{Z}_3^{\frac{1}{2}} \tilde{Z}_5^{-1} g_0) \tilde{\gamma}_S^{\text{Base}} + \text{other.}$$

define $\tilde{Z}_5$
↓

$$^0\gamma_S = \tilde{Z}_5^{-1} g_0 \gamma_S^{\text{Base}} + \text{other.}$$

Renorm Cond:

$$\gamma_S = \tilde{g} \tilde{\gamma}_S^{\text{Base}} \iff \tilde{\gamma}_S = \tilde{\gamma}_S^{\text{Base}} \Rightarrow$$

$$\boxed{\tilde{g} = (\tilde{Z}_3 \tilde{Z}_3^{\frac{1}{2}} \tilde{Z}_5^{-1} g_0)}$$

Again, suppose g_0 = fixed. Then all four Z's are determined, and g and $\tilde{g}$ come out where they may. Or, you might choose a value of g , adjust g_0 , then you are stuck with whatever $\tilde{g}$ comes out in the wash. You cannot set $\tilde{g}$ also. Luckily, as we shall see, $\tilde{g} = g$.

6. Apply Ward Identity:

$$\boxed{\Delta \cdot \Gamma = G \cdot \gamma} \Rightarrow [Z_3 \Delta] [Z_3^{-\frac{3}{2}} \Gamma] = [\tilde{Z}_3 G] [\tilde{Z}_3^{-1} \tilde{Z}_3^{\frac{1}{2}} \gamma]$$

Go to renorm pt:

$$\Rightarrow [Z_3 \cdot 1] [Z_3^{-\frac{3}{2}} g \tilde{\gamma}_S^{\text{Base}}] = [\tilde{Z}_3 \cdot 1] [\tilde{Z}_3^{-1} \tilde{Z}_3^{\frac{1}{2}} \tilde{g} \tilde{\gamma}_S^{\text{Base}}]$$

$$\Rightarrow Z_3 \tilde{Z}_3^{-\frac{3}{2}} g = \tilde{Z}_3 \tilde{Z}_3^{-1} \tilde{Z}_3^{\frac{1}{2}} \tilde{g} \Rightarrow \boxed{g = \tilde{g}}$$

Now look back so

$$Z_3^{-\frac{3}{2}} \tilde{Z}_5^{-1} g_0 = \tilde{Z}_3 \tilde{Z}_3^{\frac{1}{2}} \tilde{Z}_5^{-1} g_0$$

$$\Rightarrow \boxed{\frac{Z_3}{\tilde{Z}_5} = \frac{\tilde{Z}_3}{\tilde{Z}_5}}$$

Thus, well even before this we knew that:

$$\boxed{\Delta \cdot \Gamma = G \cdot \gamma}$$

The Ward Identity in terms of renormed objects must be satisfied simply due to the fields scaling. I.e, Unrenorm Ward implies Renorm Ward. Then you go to the renorm point and you find that g 's are the same, and you get the famous Z ratio fact. Perhaps all the other tensor forms project to zero in the explicit Ward identity. Obviously you want to renorm your various Greens functions at the same point so you can use the Ward to related constants.

7. The Four-Glue vertex. We can observe that $Z_3^{-2} T = {}^0T$ for this guy. We could say that at its sym point the renormed T is equal to some new g' times the bare tensor plus other tensors. Then we would apply the ward (4.17) and we would then conclude that $g' = g^2$. So I think the relation between the 3-point and the 4-point is the same as in bare perturbation theory. Of course there are tons of other tensors around that you have to do something about.

8. The Lagrangian counterterms. L and ZJ show all the counter-terms that you must add to implement the above renormalizations. This is their (2.1) of paper II. Lets see if we can't verify these things:

$$-\frac{1}{4} (Z_3 - 1) (\partial^\mu \vec{A}^\nu - \partial^\nu \vec{A}^\mu)^2$$

$\uparrow$ replace with this so that
 $\uparrow$ cancell off term already present.

$$\Delta^{\text{bare}} = \sum_3 {}^0 \Delta^{\text{bare}}$$

Then $\Delta = \sum_3 \Delta$ because factor scales through.

Note that they show renormed fields and renormed charge in their lagrangian. I treated ϕ^3 theory in this way once. All you have to do is this: start with the bare lagrangian in terms of bare fields and bare charges, then do the obvious rescaling and all the terms will drop out.

$$\mathcal{L}_{\text{bare}} = -\frac{1}{4} [\partial^\mu \vec{A}^\nu - \partial^\nu \vec{A}^\mu - g_0 \vec{A}^\mu \times \vec{A}^\nu]^2 - \frac{1}{2\alpha_0} (\partial_\mu \vec{A}^\mu)^2 + \text{ghost stuff}.$$

$$\text{Let } {}^0A = Z_3^{\frac{1}{2}} A \quad g_0 = Z_1 Z_3^{-3/2} g \quad (\alpha_0 = Z_3 \alpha)$$

$$\Rightarrow \mathcal{L}_{\text{bare}} = -\frac{1}{4} Z_3 [\partial^\mu \vec{A}^\nu - \partial^\nu \vec{A}^\mu - Z_1 Z_3^{-3/2} g \cdot Z_3^{\frac{1}{2}} \vec{A}^\mu \times \vec{A}^\nu]^2 - \frac{1}{2\alpha} (\partial_\mu \vec{A}^\mu)^2 + g^2$$

$$= -\frac{1}{4} Z_3 (\partial^\mu \vec{A}^\nu - \partial^\nu \vec{A}^\mu)^2$$

$$- \frac{1}{4} \frac{Z_1^2}{Z_3} g^2 (\vec{A}^\mu \times \vec{A}^\nu)^2$$

$$+ \frac{1}{2} Z_1 g (\partial^\mu \vec{A}^\nu - \partial^\nu \vec{A}^\mu) \cdot (\vec{A}_\mu \times \vec{A}_\nu)$$

All we are doing is rewriting the bare lagrangian in terms of renormed charges and fields. Then the claim is that as you do perturbation theory, in each order you will have no divergences because they will be cancelled by the counterterms. As usual, it was good for me to have done this first in ϕ^3 . All is well so far.

Exercise: derive (5.26) from (5.25)

(5.25) says: $\epsilon^{abc} \int dx \left[J_\mu^b(x) \frac{\delta}{\delta J_\mu^c(x)} + K^b(x) \frac{\delta}{\delta K^c(x)} \right] W[J, K] = 0.$

Apply $\frac{\delta}{\delta K^d(y)}$ then set $J=0$. There seem to be 4 terms.

But $\delta J / \delta K = 0$ so: 1 = 0

Then $J \delta^2 / \delta K \delta J = 0$ when $J \rightarrow 0$ so 2 = 0

Thus you get only: $0 = \epsilon^{abc} \int dx \left\{ \delta^{bd} \delta(x-y) \frac{\delta}{\delta K^c(x)} + K^b(x) \frac{\delta^2}{\delta K^d(y) \delta K^c(x)} \right\}$

Rewrite:

$$0 = \epsilon^{adc} \frac{\delta W}{\delta K^c(y)} + \int dx \epsilon^{abc} K^b(x) \frac{\delta^2 W}{\delta K^d(y) \delta K^c(x)}$$

Now set $K^b(x) = \gamma^b$, $\frac{\delta W}{\delta K} = iW \left(\frac{\delta Z}{\delta K} \right) = iW \cdot \Phi = iW \vec{v}$

Then, $\frac{1}{W} \frac{\delta^2 W}{\delta K \delta K} \Big|_{K=\gamma} = i \Delta$ so we now have

$$0 = i \epsilon^{adc} \vec{v}_c + \int dx \epsilon^{abc} \gamma^b i \Delta^{dc}(y-x)$$

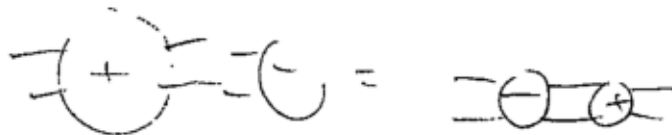
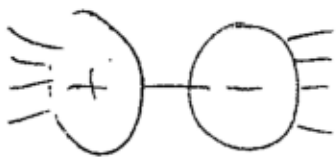
$$\Rightarrow 0 = \epsilon^{adc} v^c + \epsilon^{abc} \gamma^b \int dx \Delta^{cd}(x-y)$$

But $\Delta^{cd} = \delta^{cd} \Delta^d$ so $\Rightarrow 0 = \epsilon^{adc} \sqrt{v^c} + \epsilon^{abd} \gamma^b \int dx \Delta^d(x-y)$

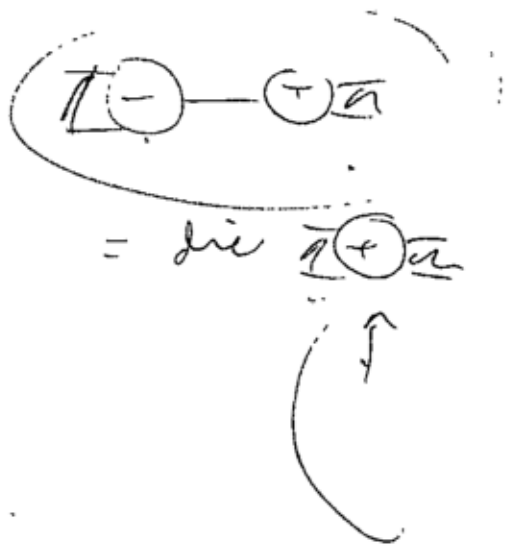
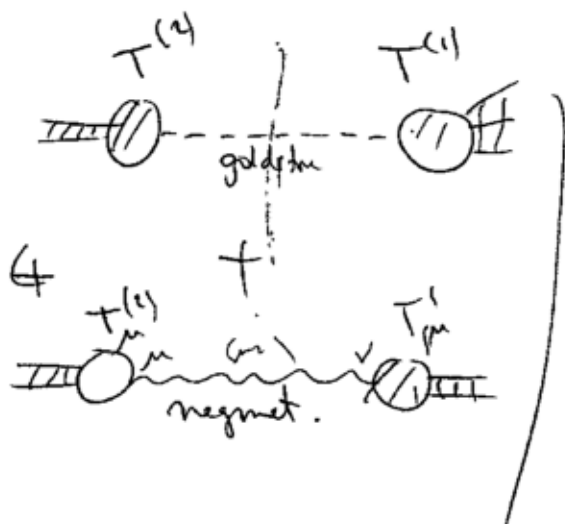
$$\Rightarrow 0 = \epsilon^{adc} \left[v^c - \gamma^c \int dx \Delta^d(x-y) \right] \text{ no sum on } d.$$

Now apply ϵ with. Let's guess that $\int dx \Delta^d$ indep. of d ?

Then $\vec{v} = \gamma \int dx \Delta(x-y) \Rightarrow \vec{v} = -\gamma \sum_x \vec{v}_x(0)$
 $= -\vec{\Delta}(0) \text{ somehow} \quad (\text{fudged!})$



$$\text{disc } \boxed{+} = \uparrow$$



Now go to shell,

$$\frac{1}{r^2}$$

input AND gate, whose output generates another forty-nanosecond gate, which is now used as the "wall" output. Obviously, a wall output occurs only when the required odd, even, and total coincidence requirements are met.

Fig. 22 indicates the master coincidence system where the individual wall pulses are routed and meshed. Each wall has an input which goes to various four-input gates as shown and also to the wall unit. Again, switches select the coincidence conditions to be satisfied in a given run. The outputs of the four-input AND gates are combined in an OR circuit, whose output drives two univibrators: a 120-millisecond UV used to hold off the coincidence system long enough to write the event on tape, and a 100-nanosecond UV used as the master coincidence output. Emitter followers couple this output to the various units requiring it. On the input a holdoff from the recorder is also provided to prevent an event from coming in during the recording time or while the recorder is writing an inter-record gap, which occurs in our data every 2^4 events. This is an easily-varied parameter.

The wall coincidence information system, shown in Fig. 23, amplifies the wall coincidence outputs, feeds them into 400-nanosecond gates, and then into two-input AND gates, where they are required to be in coincidence with the master coincidence output. This eliminates spurious wall noise pulses from being recorded. The outputs of these AND gates are amplified and routed to the readout logic system where they are used to set RS flip flops which are later read onto tape ahead of the pulse-height analyser information.

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and since any vector to be contracted with μ or ν of the above propagator may be expressed as a linear combination of $p_\mu(p_\nu)$ and $q_\mu(q_\nu)$ we see that the contributions of two massless particles to the self-energy are necessarily of order $p_\mu p_\nu$, disregarding logarithmic factors.

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PHYSICAL REVIEW D

VOLUME 5, NUMBER 12

15 JUNE 1972

Spontaneously Broken Gauge Symmetries. III. Equivalence

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We discuss the equivalence of the S matrix in the R - and U -gauge formulations of spontaneously broken gauge theories. We give definitions of the U -gauge Green's functions in terms of the R -gauge ones, for both Abelian and non-Abelian cases. Based on the equivalence theorem, we give a renormalization prescription of the U -gauge formulation.

I. INTRODUCTION

In this paper, we wish to demonstrate the equivalence of the S matrix in the R - and U -gauge formulations of spontaneously broken gauge theories. We have discussed the advantages and disadvantages of the two formulations in a previous paper (paper II).

We shall carry out this demonstration by expressing Green's functions in the U gauge in terms of those in the R gauge. What we shall show in this paper is a concrete realization of the remarks made previously by Weinberg¹ and by Salam and

Strathdee² about the equivalence of the two formulations. But more importantly, the present work gives definitions of the U -gauge Green's functions in terms of the well-defined R -gauge ones.

This paper is organized as follows. In Sec. II we consider the equivalence of the two formulations for the Abelian model considered previously. In Sec. III, we give some illustrations of the equivalence and formulate the renormalization prescription in the U gauge. In Sec. IV, we deal with the generalization to non-Abelian cases.

It is empirically known that the T matrix for the Abelian case, computed in the U gauge is finite.^{1,3}

related by finite constant?
1. Introduction. They will consider a simple theory in two gauges and show that the S-matrix elements are "the same". In the R-gauge there are redundant fields wandering around but renorm is straightforward. In the U-gauge the massless Goldstone fields have been absorbed onto the vectors and unitarity is easier to show. Something tells me this paper was not done with quite the effort of the previous papers.

2. Abelian Case. The sample theory here is scalar QED. There is the usual electric charge, and there is the ϕ scalar self-coupling λ ; we have a field theory with two fields and two couplings. If the ϕ field has imaginary mass, we get SSB so let's assume this is the case. ~~In the~~

In the R-gauge you just shift the field as was done in the last paper. Choose usual Lorentz gauge, do anything you like. Write down the generating functional W_R and do perturbation theory.

In the U-gauge you do the little expon thing and choose gauge as shown, then the goldstone vanishes completely from the theory. The final result is that somehow you can relate the amputated Greens functions computed in these two gauges as shown in (2.15).

The main idea is to show how you related the perturbation expansions in the two differing gauges. It is not done very well in my opinion.

3. Renormalization. Does this only for the U-gauge, since I guess the last paper did it for the R-gauge. Rewrite Lagrangian in terms of the U-gauge fields called U and ρ . Define a whole pile of Z-like quantities in an ambiguous way. State some renorm conditions for fixing the Z's. Write down a perturbation expansion for the Z's. Not very interesting to me and too concise, just the opposite of their usual fault (ie, too much detail).

4. Non-abelian Case. A Rehash of the above in general ^{NA} case where you start with a larger group and break down to a smaller one. You get a result which says that $T(u)=T(r)$, modulo some Z's. I gather that the ratio of these Z's is proven to be finite in this paper so here is the conclusion: finiteness of T-matrix elements in one gauge implies they are finite in the other gauge. This is the "equivalence" that they have been waiting to show. The Wards are not used that I can see. The main use of the Wards was to show that the R-gauge was unitary, to get all those ghosties to cancel out in unitarity sums.

Comment: let's just cool this paper until its contents are needed. I now know what it is trying to do, so that's enough for the time being.

Comparison of the two gauges.

① Suppose we wanted $\Delta_{\mu\nu}$ in the V-gauge? We would do this:

$$\Delta_{\mu\nu}^V = \langle B_\mu B_\nu \rangle = \frac{\delta^2 W}{\delta J_\mu \delta J_\nu} \Big|_{J=K=0} = \int [dB][d\epsilon][d\zeta] \pi \delta(\zeta) \delta(\epsilon) B_\mu B_\nu e^{iS_V}$$

as you deduce from (2.7). On the other hand,

$$\Delta_{\mu\nu}^R = \langle A_\mu A_\nu \rangle = \int [dA][dx][d\eta] \pi \delta(\eta) A_\mu A_\nu e^{iS_R}$$

Why should these be the same? I think that $B_\mu = A_\mu + (\text{longitudinal part})$, so if you are only talking transverse, you use (2.11) to get.

$$\Delta_{\mu\nu}^{V-T} = \langle A_\mu A_\nu \rangle = \quad , \text{ hence the same.}$$

② On the other hand,

$$\Delta^V = \langle \phi\phi \rangle = \frac{\delta^2 W^V}{\delta K \delta K} = \langle XX \rangle + [\text{other stuff}] \leftarrow \text{because } \frac{\delta S}{\delta K}$$

$$\Delta^R = \langle XX \rangle = \left(\frac{\delta^2 W^R}{\delta K \delta K} \right) \quad \text{brings down more than just } X.$$

So here is I think the point: the W^U and W^R gefuncs can be related to one another as done in (2.11). As far as glue propagators are concerned (transverse), they are the same in either gauge. But the scalar propagators are different. Thus, if you define Z_2 and Z_3 in each gauge separately, you will find that the Z_3 's are the same, but the Z_2 's are different. This is what (2.12) and (2.14) are saying.

Therefore, consider what happens when you renormalize each theory to get finite Greens functions. Ignoring the implementation of renorm counterterms etc, you know that the end result is simply that you rescale the fields with the correct Z 's, and you take all charges to renormalized charges. I guess in these two theories we are comparing, you conspire to end up with the same renormalized charge λ but the Z 's are different for the scalars. (Actually, easier to keep λ_0 fixed and then say that there are two different charges λ_R and λ_V and also $Z_2^R \neq Z_2^U$).

So if you want to compare renormed finite greens functions, all you have to do is rescale your fields properly, which is just "wavefunction" renorm. The glue rescalings will cancel between the two gauges, but the scalars will not cancel, hence you get the nice result (2.15) which now seems quite reasonable, but each theory still has its own charge as far as I can figure.

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IV

formulated the text in terms of the less customary point transformation, in order to facilitate the comparison with paper I (Ref. 3).

³G. Carmi, preceding paper, Phys. Rev. D **7**, 1038 (1973), which is henceforth referred to as "paper I." Its equations will be quoted, as e.g., (I2.3).

⁴Professor Donald Newman, private communication.

⁵This is hindsight, of course. At the time paper I was written, the solution had to be found by trial and error.

⁶Thus the equilibrium-thermodynamics (i.e., $\sum_n \langle n | e^{-\beta H} | n \rangle$) cannot be evaluated exactly. However, if the system is first transformed by the two Bohm-Pines transformations [D. Bohm and D. Pines, Phys. Rev. **92**, 609 (1953)] and the random-phase approximation (RPA) (and, possibly, also [G. Carmi and A. J. Lock, Phys. Rev. A **5**, 1447 (1972)] the Bogoliubov approximation) is used, the residual interaction is small and the transformation (3.1) will (if it is chosen in such a way as to fulfill the other criteria of Ref. 1) describe also the thermodynamics of the system quite well (Ref. 1).

⁷This definition is obviously not the widest possible generalization of the corresponding definition in Sec. II, but it serves our purposes here.

⁸Again, this is not the most general system for which Lemma 1_n could be proved. More generally, one could write $f_i(x_1, \dots, x_{i-1}, x_i+1, x_{i+1}, \dots, x_n)$ on the left-hand side of (3.12).

⁹The main restriction is that the corresponding Lie element $pf + fp + g$ has a nonvanishing domain in L_2 , as operator with range in L_2 , and that this remains so for the Lie products of such elements.

¹⁰Robert Hermann, *Lie Groups and Physics* (Benjamin, New York, 1966), p. 139.

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¹² $(pf_1 + f_1p + g, pf_2 + f_2p + g_2) = pf_3 + f_3p + g_3$, with $f_3 = 2(f_1f_2 - f_2f_1)$, etc.

¹³These results do not seem to be recorded in the literature but, judging from their simple nature, they must have occurred to almost everybody who has encountered this algebra.

¹⁴Within the underlying associative algebra (with respect to ordinary operator multiplication) the algebra can be spanned by a basis of two elements only, e.g., $A_{-2} = pe^{-2ix} + e^{-2ix}p$ and $B_1 = e^{ix}$.

PHYSICAL REVIEW D

VOLUME 7, NUMBER 4

15 FEBRUARY 1973

Spontaneously Broken Gauge Symmetries IV. General Gauge Formulation

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(Received 30 October 1972)

The advent of the dimensional-regularization procedure allows the study of renormalizability of spontaneously broken gauge theories formulated in a wide class of gauges. We derive and study the Ward-Takahashi identities appropriate to such gauges. A consequence of the Ward-Takahashi identities is that the physical S matrix is invariant under a variation of the gauge condition. As remarked before, since the variation of a parameter in the R_ξ -gauge formulation shifts the masses of unphysical excitations, the above result, the ξ independence of the physical S matrix, implies that the unphysical excitations cannot contribute to the sum over intermediate states, establishing the unitarity of the S matrix. We also give the renormalization procedure of a model formulated in the R_ξ gauge.

I. INTRODUCTION

The advent of a very powerful regularization procedure for Feynman integrals – the so-called dimensional regularization¹ – permits us to discuss intelligently the renormalizability question of spontaneously broken gauge theories formulated in a fairly general class of gauges.² The present paper is dedicated to the derivation of the Ward-Takahashi (WT) identities in such gauges, which can be used to prove the renormalizability and unitarity

of the theory in question. It has been observed³ that in the so-called R_ξ -gauge formulation invariance of the physical S matrix under the variation of a gauge-specifying parameter (i.e., ξ) implies the unitarity of the S matrix, that is to say, that unphysical excitations do not contribute to sums over intermediate states. Thus, the ability to formulate quantum theory of spontaneously broken gauge symmetry in a general class of gauge conditions, in a way that reflects the gauge invariance of the action as expressed through the WT identi-

1. Introduction. Note that this paper, although called IV, was written 7 months after the other three papers. In the intervening time, tH-V dim reg came out. You no longer had to make long-winded comments about gauge invariance of counterterms and so on to get the Wards to be the same for renormed greens functions.

The main idea of this paper is to simply state the Ward Identities for a general gauge rather than any particular gauge. This derivation is done in the first section. Other sections summarized below.

2. The WT Identities. For the first time authors go to the compact notation (1) where all fields including the gauge guys are put into same vector ϕ_i . The general gauge condition is (2) where F_q is, I presume, any function of the fields that you want, and a_q is a vector of functions at some point. Note that there are as many gauge conditions as there are group generators, the q index. However, as noted later, when you get the gauge delta up into the action where it appears squared, you observe that the dimension of F better not be more than 2, or you have non-renormalizable Lagrangian!

Next there ensues the standard discussion of the exponentiation of the gauge delta. There is no limit that any parameter goes to zero. So far, then, we have (11). This column is much like my own notes, and the function H is just taken to be a gaussian though other things might do. *[But I had to change my view on this!]*

Next, they observe (for the first time!) that the $\det M$ can be interpreted using ghost fields with $[dcd\bar{c}]$ functional integral. It is my guess that the authors did not have access to the FP paper (unpublished) at time of earlier LZJ papers: the FP letter does not mention the ghost fields c . Maybe Feynman's thing does.

Now comes a lightning fast derivation of the WT identity: recall how tedious and simply awful their other derivation was (back in paper I). The trick here is to utilize some kind of constraint (I have not yet studied how this works). When you vary subject to this constraint, out pops (17) which is the Ward in full generality in any gauge!! Quite a powerful and general result! Notice that M^{-1} appears in there and that this is the ghost propagator, so somehow the ghost propagator is going to appear all over in your specific ward identities!

This derivation is so short and neat and general that I should now probably invest some time to see what I can do with it, maybe make some general notes on the thing. Their previous derivation was so messy that just not worth it.

Next, they specialize to gauge conditions linear in the fields (as are all the gauge conditions whose use I know of). This leads to result (20) which I think is just a special case of the general ward or something, not yet studied it.

III. Consequences of Ward Identities. With the dim reg deal, you can renorm your theory in the usual multiplicative way, and your renormed Greens functions will satisfy the Ward's too, and these are obtained from the unrenormed Wards simply by rescaling fields and sources and charges. What could be easier!

The rest of this section shows the following result: if you vary your gauge conditions a little, the renormalized S-matrix (called S) does not change at all. This point was a little confused in paper III, I think. This is a very important result and it is rather easily proven: change the gauge, ^{it} don't change your S-matrix! Thus, once this is proven, it no longer seems necessary to do things like showing that the R-gauge and the U-gauge are equivalent, and so on. All gauges give the same S-matrix.

But perhaps the catch is that this is a formal proof. The dimreg is supposed to justify all manipulations, but maybe something does not work. I don't know what the Gribov disease is but it may be connected somehow.

4. Renormalization. Here is an application of the above results. The idea is to show that, if you choose a gauge which depends on a parameter ξ , then of course you know right away that the resultant S-matrix is independent of ξ . That's what was just proved in the last section, S-matrix is independent of gauge.

So, go into this ξ gauge, whatever it is. It turns out that the ghosts, the neg metrics, and the goldstones (called $\tilde{\chi}$) all have poles at $p^2 = m^2/\xi$. The argument now goes as follows: since S-matrix independent of ξ , these ghostie particles must always cancel by hook or crook, otherwise there will be singularities which depend on ξ in the S-matrix, and this is impossible.

This is called the R_ξ gauge because it is treated in the usual way where you have these redundant fields like the goldstones. Multiplicative Renormalization works and yields a finite, ξ -independent result by arguments similar to the R-gauge analysis of earlier paper II. Then as you take $\xi \rightarrow 0$ the poles of the ghosties recede to infinity and in effect you have arrived at the U-gauge. Similarly, $\xi \rightarrow \infty$ is probably the R-gauge. Thus, the two gauges have been interpolated in a continuous fashion and you can watch how you go smoothly from one to the other without the S-matrix every knowing about its gauge choice.

The particular model happens to have an $SU(2)$ group which is completely broken so that there are 3 goldstones called $\tilde{\chi}$. But there are originally 4 Higgs fields, the fourth is called χ and is a gauge-group scalar so it is the one that gets NZVEV.

About the constraint on the gauge functions, and Appendix Stuff.

1. Equation (1) shows the first order change in fields induced by a gauge transformation by an arbitrary gauge function $g_\alpha(x)$. Normally, one thinks of these gauge functions as being arbitrary functions. However, you can make use of this arbitrariness to your advantage. Suppose, as suggested by equation (7), you make your gauge functions $g_\alpha(x)$, what I usually call $\theta_a(x)$, depend on the fields!

In other words, pick a point x^u . At that point, the form of the gauge function or rather the value of the gauge function depends on the value of the fields at that point. Suppose you know the fields at all points, $\phi_i(x)$. Pick a gauge. Go off and try to compute the object M . In a general gauge, I think this can be a function of $g_\alpha(x)$, so in general M^{-1} is a function of the fields and $g_\alpha(x)$. To now insist that (7) be true is a constraint on g ; it does not determine g . It forms some sort of equation that g must satisfy. It looks like a highly non-linear condition. How do I know that a $g(x)$ exists that satisfies this constraint?

Due to this constraint, the gauge functions $g_\alpha(x)$ are actually functions of the fields ϕ_i . If you were to change the fields ϕ a little bit, you change the constraint, and thus you change any solution to the constraint. Thus, write $g_\alpha(x; \phi_i)$.

2. Now consider going from fields ϕ to fields ϕ^g via a transformation (gauge) which satisfies the constraint. How does the path integral change? We would like to compute the jacobian to go from $d\phi_a$ to $d\phi_a^g$ variables.

$$\begin{aligned} \text{Jacobian} &= \det \left(\frac{\delta \phi_i^g}{\delta \phi_j} \right) = \det \left[\delta_{ij} + \underbrace{\Gamma_{ij}^\alpha g_\alpha + (\Gamma_{ij}^\alpha \phi_j + \Lambda_i^\alpha) \frac{\partial g_\alpha}{\partial \phi_j}}_{\equiv X_{ij}} \right] \\ &= \det(1+X) = e^{\text{tr} \ln(1+X)} \approx e^{\text{tr} X} \end{aligned}$$

if X is first order in g .

$$\Rightarrow \text{Jac.} \equiv \exp \left[\Gamma_{ii}^\alpha g_\alpha + (\Gamma_{ik}^\alpha \phi_k + \Lambda_i^\alpha) \left(\frac{\partial g_\alpha}{\partial \phi_i} \right) \right] \quad (A.2)$$

But (7) says

$$M(\phi)_{\beta\alpha} g_\alpha(\phi) = \lambda_\beta$$

$$\Rightarrow \left(\frac{\delta M}{\delta \phi_i} \right)_{\beta\alpha} g_\alpha + (M)_{\beta\alpha} \left(\frac{\delta g_\alpha}{\delta \phi_i} \right) = 0 \Rightarrow \left(\frac{\delta g_\alpha}{\delta \phi_i} \right) = -(\bar{M}^{-1})_{\alpha\beta} \left(\frac{\delta M}{\delta \phi_i} \right)_{\beta\gamma} g_\gamma$$

3. If we were merely doing a linear xform (with no constraint), we would

know that $J=1$. In that case, $J \equiv \exp(\Pi_{ii}^{\alpha} g_{\alpha}) = 1$. So

that is true here also, so we have for our non-linear xform,

$$J \equiv \exp \left[-(\Pi_{ik}^{\alpha} \phi_k + \Lambda_i^{\alpha}) (\tilde{M}^{-1})_{\alpha\beta} \left(\frac{\delta M}{\delta \phi_i} \right)_{\beta\gamma} g_{\gamma} \right] \quad (A.3) \checkmark$$

get A.4

$$\Delta_F^g = \det(M^g) = \det(M + \delta M) = \det(M) (1 + \tilde{M}^{-1} \delta M)$$

$$= \Delta_F \cdot \exp \left[\text{Tr} \ln(1 + \tilde{M}^{-1} \delta M) \right] \approx \Delta_F \exp \left[\text{Tr} [\tilde{M}^{-1} \delta M] \right] \quad (A.4) \checkmark$$

get A.7

$$\delta(\text{mass}) = \Delta_F' [d\phi]' - \Delta_F [d\phi] = [\Delta_F' J - \Delta_F] [d\phi]$$

$$= \frac{[d\phi]}{\Delta_F} \left\{ \frac{\Delta_F'}{\Delta_F} J - 1 \right\} = \frac{[d\phi]}{\Delta_F} \left\{ \left(\frac{\Delta_F' - \Delta_F + \Delta_F}{\Delta_F} \right) J - 1 \right\}$$

$$= \frac{[d\phi]}{\Delta_F} \left\{ \left(\frac{\delta \Delta_F}{\Delta_F} + 1 \right) J - 1 \right\} = \frac{[d\phi]}{\Delta_F} \left\{ \underbrace{\frac{\delta \Delta_F}{\Delta_F}}_{= \delta[\ln \Delta_F]} + \underbrace{(J-1)}_{\approx \ln J} \right\}$$

since $J=1+\epsilon$. Hence, LHS of (A7).

Rest seems OK.

get A.6

$$\Delta_F^g \approx \Delta_F + \Delta_F \text{Tr}(\tilde{M}^{-1} \delta M) \Rightarrow$$

$$\boxed{\frac{\delta \Delta_F}{\Delta_F} = \text{Tr}(\tilde{M}^{-1} \delta M) = \delta[\ln \Delta_F]} \quad (A.6) \checkmark$$

(L25-4)

Proof that Equation (A11) is wrong.

1. Remember that for the gauge fields the symbol Λ represents a derivative on the group parameters, as in equation (1). So, the claim of (A11) is this:

$$\left[G_{ab}^{\gamma} \left(\frac{\nabla \hat{A}_b^{\alpha}(x)}{\nabla \Theta_{\gamma}(y)} \right) - G_{ab}^{\delta} \left(\frac{\nabla \hat{A}_b^{\alpha}(x)}{\nabla \Theta_{\delta}(y)} \right) \stackrel{?}{=} i C_{\gamma\delta\alpha} \left(\frac{\nabla \hat{A}_a^{\alpha}(x)}{\nabla \Theta_{\alpha}(y)} \right) \right] \quad (\#1)$$

OR

$$G_{ab}^{\gamma} D_{b\gamma}^{\mu} - G_{ab}^{\delta} D_{b\delta}^{\mu} = i C_{\gamma\delta\alpha} D_{a\alpha}^{\mu} \quad [\text{all acting on } (-\frac{1}{g}) \delta(x-y)]$$

Not obviously true, so expand.

$$\begin{aligned} G_{ab}^{\gamma} [\delta_{b\delta} \partial^{\mu} - i g G_{b\delta}^{\dagger} A_f] - G_{ab}^{\delta} [\delta_{b\gamma} \partial^{\mu} - i g G_{b\gamma}^{\dagger} A_f] \\ \stackrel{?}{=} i C_{\gamma\delta\alpha} [\delta_{a\alpha} \partial^{\mu} - i g G_{a\alpha}^{\dagger} A_f] \end{aligned}$$

Regroup:

$$G_{a\delta}^{\gamma} \partial^{\mu} - G_{a\gamma}^{\delta} \partial^{\mu} \stackrel{?}{=} i C_{\gamma\delta\alpha} \partial^{\mu} \quad (1)$$

$$-i g \{ G_{ab}^{\gamma} G_{b\delta}^{\dagger} - G_{ab}^{\delta} G_{b\gamma}^{\dagger} \} A_f \stackrel{?}{=} -i g \{ i C_{\gamma\delta\alpha} G_{a\alpha}^{\dagger} \} A_f \quad (2)$$

We want the sum of these equations to be true. Ill look at #2 + use cyclic property of G 's

$$\begin{aligned} \left(\frac{(2)}{-ig} \right) \Rightarrow \{ -G_{ab}^{\gamma} G_{b\delta}^{\dagger} + G_{ab}^{\delta} G_{b\gamma}^{\dagger} \} A_f \stackrel{?}{=} \{ i C_{\gamma\delta\alpha} G_{a\alpha}^{\dagger} \} A_f. \\ \text{OR} \quad \underline{\underline{G_{ab}^{\delta} G_{b\gamma}^{\dagger}}} - \underline{\underline{G_{ab}^{\gamma} G_{b\delta}^{\dagger}}} \stackrel{?}{=} +i C_{\gamma\delta\alpha} \underline{\underline{G_{a\alpha}^{\dagger}}} \quad \checkmark \end{aligned}$$

So (2) is true. Thus I need (1) true. But

$$\begin{aligned} (1) \Rightarrow (G_{a\delta}^{\gamma} - G_{a\gamma}^{\delta}) \partial^{\mu} &= -G_{\delta a}^{\gamma} \partial^{\mu} = +G_{a\delta}^{\gamma} \partial^{\mu} \\ \Rightarrow -G_{a\gamma}^{\delta} \partial^{\mu} &= 0 \quad \text{not true!} \end{aligned}$$

Therefore equation (A11) is simply not true. It might be true if both sides were to act on a function that was either 1) independent of x , so the d_u 's do nothing, or 2) is antisymmetric in any pair of hanging indices. But neither of these is the case the way they use it!

1. Refer to FP Jacobian notes. There I wrote a "gauge condition" as $f(\phi, g) = a$, where $a =$ some number (one color, etc). Suppose you have a field ϕ and a g such that $f(\phi, g) = a$. If you were to now make a change on the field and keep g the same, your gauge condition would no longer be satisfied. For example, $f(\phi^g, g) \neq a$. Or, $f(\phi + d\phi, g) \neq a$. By going to new fields, you have "left the gauge" that you started in. Eg, suppose you had an axial gauge $n^\mu A_\mu = 0$ and you rotate your fields by an arbitrary rotation, then no longer true that $n \cdot A = 0$, you have left this particular axial gauge, although you are now in a new axial gauge $n'^\mu A_\mu = 0$.

2. However, suppose while you change your field from ϕ to $\phi + d\phi$, you simultaneously change your g to $g + dg$. If you choose dg correctly, you can maintain the original gauge condition!!! Thus, you are correlating your change of dg to your change of ϕ . I think this is precisely the "constraint" they are talking about, let's see:::

3. Choose your gauge $f(\phi, g) = a$ so that, for a given ϕ , this thing is solved by $g =$ the identity (zero for parameters!). Always consider it as parameters from now on. Then we can say:

$$f(\phi, g) \approx f(\phi, \tilde{g}) + (g - \tilde{g}) \left(\frac{df(\phi, g)}{dg} \right) \Big|_{\tilde{g}} \quad \text{for } g \text{ near } \tilde{g}.$$

$$\Rightarrow f(\phi, g) \approx a + g \cdot \left(\frac{df}{dg} \right)^{\tilde{g}} = a + \left(\delta f \right)_{\delta a}$$

$$\Rightarrow g = \frac{f(\phi, g) - f(\phi, 0)}{\left(\frac{df}{dg} \right)^0} = \left[\left(\frac{df}{dg} \right)^0 \right]^{-1} \delta f$$

We can express δf as δa [change that would be caused in a] or " λ ".

But, when you generalize to many colors, this "solution for g " is exactly equation (7) of LZJ-4. This equation (7) tells you what g must be (g near zero in param space) in order that the gauge condition $f(\phi, g) = a$ continue to be satisfied. So we write

$$f(\phi, g(\phi)) = a \quad \text{true even as } \phi \text{ is varied by small amount.}$$

$$\text{no, give } \lambda + \left(\frac{\delta a}{\delta \phi} \right)_{\text{omit}} \quad \text{no matter how you vary}$$

Start at the top. They define the FP-object, as I call it, and I think this thing is really also a function of a , so I would label it $\Delta_{F,a}[A]$. Next, they correctly observe that, when evaluated for field on the gauge surface, the Fp-object is equal to a Jacobian object, hence Eq. (4). It is then easy to derive equation (6). In equation 6, since you have a gauge surface delta, it doesn't matter whether you put in the FP-object or the Jacobian object. They put in the latter, called $\det M$.

Now comes the big question: how do you "exponentiate the delta"???? I will accept for the moment that $dA \det M$ is in fact invariant under their constrained gauge transformation: note that the Jac det object is ~~being~~ being talked about here, not the FP object. I know that the FP object is invariant, but I don't know much about $\det M [A]$ as you move A off the gauge surface.

Similarly, I thought I had shown that dA was invariant, but then that proof sort of fell apart. Now I am beginning for the first time to believe that the pieces are separately not invariant.

So, assuming that $dA \det M$ are invariant, look what you can do:

$$\begin{aligned} W(a) &= \int [d\phi] \det M[\phi] e^{iS[\phi]} \delta(F(\phi) - a) \\ &= \int [d\phi'] \det M[\phi'] e^{iS[\phi']} \delta(F(\phi') - a) \quad \left. \begin{array}{l} \text{change of variable} \\ \text{still pinned from above.} \end{array} \right\} \\ &= \int \underbrace{[d\phi] \det M[\phi]}_{\text{claimed invariant}} \underbrace{e^{iS[\phi]}}_{\text{no source}} \delta(F(\phi') - a) \end{aligned}$$

But
$$F(\phi') = F(\phi) + \underbrace{\frac{\partial F}{\partial \phi}}_M \underbrace{\frac{\partial \phi}{\partial g}}_{M^{-1}\lambda} \frac{\partial g}{\partial \lambda} = F(\phi) + \lambda$$

So
$$W(a) = \int [d\phi] \det M[\phi] e^{iS[\phi]} \delta(F(\phi) - (a + \lambda))$$

$$= W(a + \lambda).$$

You can see how this gauge transformation was chosen: the gauge function F changes by a constant (independent of ϕ !!!). Thus, you have proven that $W(a)$ is invariant against shifts in parameter a to first order. But this is true at any point a . A function which has zero derivative at all values of the argument is a constant! Thus, $W(a)$ is completely independent of a !!!! Unlike the FP-object! Note by the way that $\det M$ does not even know about a !

Now the rest is easy. Start off with $\int da H(a) = 1$. To the right of $H(a)$, some integrable function, insert $W \doteq W$. Thus,

$$1 = \int da H(a) \Rightarrow W = \int da H(a) \cdot \underbrace{W(a)}_{\text{indep of } a} \\ = \int da H(a) \int d\phi e^{iS(\phi)} \det M(\phi) \delta(F(\phi) - a)$$

$$\boxed{W = \int d\phi e^{iS(\phi)} \det M(\phi) H(F(\phi))} \quad 5.22.79.$$

Finally we have a correctly done "exponentiation of the delta"!!! And notice that it really is the Jacobian object and not the FP object that goes in here!!!

I am happy with this if I can really show that $dA \det M$ is an invariant!

[See whole section on this in gauge bundle!]

Derivation of the Ward Identities.

1. I have spent a great deal of time on this business of showing the invariance of the product $d\phi \det M$. The whole point of this was to be able to derive the Ward Identities, so here goes that derivation.

2. Once the gauge surface delta has been exponentiated and the source turned on, we get:

$$W = \int [d\phi] \det M(\phi) e^{i S_{tot}(\phi)}$$

$$S_{tot}(\phi) = S(\phi) - \frac{1}{2} F_\alpha F^\alpha + \phi_i J^i$$

$$= S(\phi) + \int d^4x \left[-\frac{1}{2} F_\alpha(\phi(x))^2 + \phi_i(x) J^i(x) \right].$$

Remember that this holds for an arbitrary gauge and arbitrary number of fields. Thus there is a field source for each component of each type of fields including the gauge fields. As before, ϕ_i is a "supervector". Now:

$$W = \int [d\phi'] \det M(\phi') e^{i S_{tot}(\phi')}$$

where ϕ' related to ϕ by C.G.T. Change of variables does not change W .
Now use invariance to say

$$W = \int d\phi \det M e^{i S_{tot}(\phi)}$$

Now subtract to get

$$0 = \int [d\phi] \det M(\phi) \left[e^{i S_{tot}(\phi')} - e^{i S_{tot}(\phi)} \right]$$

But

$$h(\phi') \equiv e^{i S_{tot}(\phi')} \approx h(\phi) + \delta\phi_i \frac{\partial h}{\partial \phi_i} = e^{i S_{tot}(\phi)} + \delta\phi_i e^{i S_{tot}(\phi)} i \frac{\delta S}{\delta \phi_i}$$

$\Rightarrow$

$$0 = \int [d\phi] \det M(\phi) \left[\cancel{e^{i S_{tot}(\phi)}} + \delta\phi_i e^{i S_{tot}(\phi)} i \frac{\delta S}{\delta \phi_i} - \cancel{e^{i S_{tot}(\phi)}} \right]$$

$$\Rightarrow 0 = \int [d\phi] \det M(\phi) e^{i S_{tot}(\phi)} \underbrace{\int \left(\frac{\nabla S}{\nabla \phi_i(x)} \right) \delta\phi_i(x) d^4x}_{= \delta S(\phi)}.$$

But

$$\delta S_{\text{tot}}(\phi) = \underbrace{\delta S_0(\phi)}_{=0} - \frac{1}{2} \cdot 2 F_\alpha(\phi) \cdot \lambda^\alpha + J^i(x) \delta \phi_i(x)$$

$\delta F_\alpha(\phi) = \lambda^\alpha$ for this special gt.

$$\delta F_\alpha = \frac{\partial F_\alpha}{\partial \phi_i} \delta \phi_i = \frac{\partial F_\alpha}{\partial \phi_i} (\Gamma_{ij} \phi_j + \lambda) g_i = m^{-1} \lambda \dots = \lambda_\alpha.$$

see below.

$$S_0 \quad \delta S_{\text{tot}}(\phi) = \int d^4x \left[-F_\alpha(\phi) \lambda^\alpha(x) + J^i(x) [\Gamma_{ij}^\alpha \phi_j + \lambda_i] g_\alpha \right]$$

$[m^{-1}(\phi)]_{\alpha\beta} \lambda_\beta.$

3. I see how this goes, but I think I will do it in my own notation to make sure I don't get confused with where there are and are not integrals. So:

$$S_{\text{eff}}[\phi] = S_0[\phi] + \int dx \left[J_i(x) \phi_i(x) - \frac{1}{2} F_\alpha(\phi(x))^2 \right]$$

$$\delta S_{\text{eff}}[\phi] = \int dx \left[J_i(x) \delta \phi_i(x) - F_\alpha(\phi) \delta F_\alpha(\phi) \right]$$

But,

$$\delta \phi_i(x) = -\frac{1}{g} \overrightarrow{D}_{ib}(x) \Theta_b(x)$$

$$\delta F_\alpha(\phi) = \int dx \frac{\nabla F_\alpha(\phi)}{\nabla \phi_i(x)} \delta \phi_i(x)$$

$$\Rightarrow \delta F_\alpha(\phi(z)) = \int dx dy \frac{\nabla F_\alpha(\phi(z))}{\nabla \phi_i(x)} \frac{\nabla \phi_i(x)}{\nabla \Theta_b(y)} \Theta_b(y)$$

$$= \left[\frac{\nabla F_\alpha}{\nabla \phi_i} \right] \left[\frac{\nabla \phi_i}{\nabla \Theta_b} \right] [\Theta_b]$$

$$= [\underline{M}_{\alpha b}(\phi)] [\underline{\Theta}_b]$$

$$= [\underline{M}_{\alpha b}(\phi)] [\underline{M}_{bc}^{-1}(\phi)] [\underline{\lambda}_c]$$

$$= \lambda_\alpha(z)$$

Then also:

$$\begin{aligned}\delta\phi_i(x) &= -\frac{1}{g} [\underline{Dib}] [\underline{\phi_b}] \\ &= -\frac{1}{g} [\underline{Dib}] [\underline{M_{bc}^{-1}}] [\underline{\phi_c}] \\ &= -\frac{1}{g} \int dy dz D_{ib}(x,y) M_{bc}^{-1}(y,z) \lambda_c(z)\end{aligned}$$

So we learn:

$$\begin{aligned}\delta S_{\text{eff}}[\phi] &= \int dx \cdot [J_i(x) \left\{ -\frac{1}{g} \int dy dz D_{ib}(x,y) M_{bc}^{-1}(y,z) \lambda_c(z) \right\} \\ &\quad - F_a(\phi) \lambda_a(x)] \\ &= \int dx dy dz J_i(x) \left(-\frac{1}{g} \right) D_{ib}(x,y) M_{bc}^{-1}(y,z) \lambda_c(z) - \int dx F_a(\phi(x)) \lambda_a(x) \\ &= \int dx dy dz J_i(y) \left(-\frac{1}{g} \right) D_{ib}(y,z) M_{bc}^{-1}(z,x) \lambda_c(x) - \int dx F_a(\phi(x)) \lambda_a(x) \\ &= \int dx \left\{ \int_{dydz} J_i(y) \left(-\frac{1}{g} \right) D_{ib}(y,z) M_{bc}^{-1}(z,x) - F_c(\phi(x)) \right\} \lambda_c(x)\end{aligned}$$

This is the variation due to arbitrary $\lambda_c(x)$. In particular, let $\lambda_c(x) = \lambda_c$.
Then

$$\delta S_{\text{eff}}[\phi] = \lambda_c \cdot \left[\int dx \left\{ -F_c(\phi(x)) - \frac{1}{g} \int J_i(y) D_{ib}(y,z) M_{bc}^{-1}(z,x) dy dz \right\} \right]$$

Well, go back to

$$0 = \int [d\phi] \det M(\phi) e^{i S_{\text{tot}}(\phi)} \left[\int dx \lambda_c(x) \left\{ \int dy dz (J_i(y) \left(-\frac{1}{g} \right) D_{ib}(y,z) M_{bc}^{-1}(z,x) - F_c(x) \right\} \right]$$

$$\Rightarrow 0 = \int dx \lambda_c(x) \cdot \int [d\phi] \det M e^{i S} \left\{ \int dy dz J_i(y) \left(-\frac{1}{g} \right) D_{ib}(y,z) M_{bc}^{-1}(z,x) - F_c(x) \right\}$$

I think here all we need is that λ be independent of fields ϕ ; it can be spacetime function. The above must be true for any function λ ! Therefore we conclude:

$$\bigcirc \int d\phi \det M e^{iS_{\text{tot}}} \left\{ -F_c(x) - \frac{1}{g} \int dy dz J_i(y) D_{ib}(y,z) \tilde{m}_{bc}^{-1}(z,x) \right\} = 0$$

Be more complete:

$$\int [d\phi] \det M(\phi) e^{iS_{\text{eff}}(\phi)} \left\{ -F_c(\phi(x)) - \frac{1}{g} \int dy dz \mathcal{T}_i(y) \mathcal{D}ib(y, z; \phi) \mathcal{M}_{bc}^{-1}(z, x; \phi) \right\} = 0$$

Here we are taking care to show all functional dependence on the fields ϕ . Since these can be replaced with $1/i \ d/dJ$ we can say:

$$\left[-F_c \left(\frac{1}{i \nabla J} \right) - \frac{1}{g} \int dy dz J_i(y) D_{ib}(y, z; \frac{1}{i \nabla J}) \tilde{M}_{bc}^{-1}(z, x; \frac{1}{i \nabla J}) \right] W[J] = 0$$

where $w[\tau] = \int [d\phi] \det M(\phi) \exp i \left[S[\phi] + \int dx \cdot [\bar{\tau}_i \phi_i - \frac{1}{2} F_\alpha(\phi)^2] \right]$

This result is similar to that derived in LZJ-1. There recall that they derived the thing without using the trick gauge transformation. There also, only gauge fields appeared, and covariant gauge was assumed. This result is valid for all gauges and in multi-field case. I think probably this is the most general Ward Identity you can have!

This is an operator acting on $W[J]$ to give zero. Eventually this will lead to Ward identities on the Green's functions themselves.

Frادkin + Tyutin

S Matrix for Yang-Mills and Gravitational Fields

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(Received 19 January 1970)

A method is suggested (and applied to the Yang-Mills and gravitational fields) for the construction of the generating functional (S matrix) for fields possessing an invariance group. The unitarity and gauge independence of the S matrix on the mass shell are seen explicitly.

I. INTRODUCTION

THERE has lately been considerable intensification in the study of theories partially or completely invariant under non-Abelian groups of transformations. This is in connection with the discovery of vector mesons and their classification into multiplets, with the use of vector mesons to account for the form factors of particles, and with the current-algebra approach. Intermediate vector bosons are introduced in many schemes of weak interaction. An important example of a theory with a non-Abelian group of invariance is that of the gravitational field.

In the present paper a procedure for constructing the Feynman rules is proposed for theories possessing a gauge group, such as the theories of the massless Yang-Mills and gravitation fields. It is known that some additional (gauge) condition must be imposed on the dynamical variables in order that a consistent quantum field theory may be formulated on the basis of a Lagrangian density invariant under a local transformation group. In covariant gauges this can conveniently be done by the use of Lagrange multipliers. The basic idea of the method proposed is to choose the Lagrange multiplier in such a way that one is led to free equations of motion for the additional field. This fact guarantees the unitarity of the S matrix in physical space. The Feynman rules obtained coincide with those proposed in Refs. 1-5. The difference of the method under consideration from that of Refs. 2-5 is that we have succeeded in obtaining a set of consistent dynamical equations completely describing the theory. On the one hand, these equations make it possible to elucidate the reason for the additional diagrams to appear, and, on the other hand, guarantee the unitarity of the physical S matrix. Section II is devoted to the construction of the S matrix for the massless Yang-Mills field in arbitrary gauge and to the proof of the gauge invariance of the S matrix. In Sec. III constructing the Feynman rules in the Coulomb and axial gauges is considered on the basis of the canonical quantization procedure. The S matrix obtained coincides with that found in Sec. II,

which fact reaffirms its unitarity. It is shown, furthermore, that taking the additional conditions consistently into account makes it possible to obtain self-consistent equations for the massless Yang-Mills field in the presence of an external source.

In Sec. IV the Feynman rules for the gravitational field are constructed in covariant gauges. These rules coincide with those suggested in Refs. 2, 3, and 5. In the framework of our approach we also obtain the S matrix for a noncovariant (Dirac) gauge, for which the Feynman rules have been obtained by Popov and Faddeev⁶ using a method closely connected with the canonical formulation of the gravitational field. In addition, by our method, the equivalence of the S matrix in covariant and noncovariant gauges is proved.

We use the following notation. Greek $\mu, \nu, \lambda, \dots$ and the Latin i, j, k indices take the values 0, 1, 2, 3 and 1, 2, 3, respectively. In Secs. II and III, $g_{\mu\nu}$ means the Minkowski tensor (+, ---) and δ_{ik} means the unit tensor. By the summation over repeated indices is everywhere meant $a_\mu b_\mu = a_0 b_0 - a_k b_k$; $\partial_\mu \equiv \partial/\partial x^\mu$; $\square \equiv \partial_\mu \partial_\mu$; $\nabla \equiv \partial_k \partial_k$. In Sec. IV, $g_{\mu\nu}$ means the metric tensor, and the Minkowski tensor is designated as $\delta_{\mu\nu}$. The usual summation over repeated indices means $a_\mu b^\mu \equiv \sum_{\mu=0}^3 a_\mu b^\mu$. We use the system of units $\hbar=c=1$ in Secs. II and III and $c^2/16\pi k=1$ in Sec. IV (where k is the gravitational constant).

II. GENERAL THEORY OF CONSTRUCTION OF FEYNMAN RULES FOR MASSLESS YANG-MILLS FIELD. GAUGE INVARIANCE OF S MATRIX

In this section the general theory for construction of a unitary S matrix for massless Yang-Mills fields is considered.

The classical Lagrangian for a Yang-Mills⁷ field has the form

$$L_0(x) = -\frac{1}{4} G_{\mu\nu}^a(x) G_{\mu\nu}^a(x). \quad (2.1)$$

Here $G_{\mu\nu}^a$ is the field-strength tensor,

$$G_{\mu\nu}^a(x) = \partial_\mu A_\nu^a(x) - \partial_\nu A_\mu^a(x) + \lambda \hat{A}_\mu^a(x) A_\nu^b(x), \quad (2.2)$$

$$\hat{A}_\mu^a(x) = f^{abc} A_\mu^c(x). \quad (2.3)$$

The f^{abc} are the structure constants of the arbitrary finite-dimensional compact simple Lie group G . The f^{abc}

⁴ V. N. Popov and I. D. Faddeev (to be published).

⁷ C. N. Yang and R. L. Mills, Phys. Rev. 96, 191 (1954).

¹ R. P. Feynman, Acta Phys. Polon. 24, 697 (1963).

² S. DeWitt, Phys. Rev. 162, 1195 (1967); 162, 1239 (1967).

³ L. D. Faddeev and V. N. Popov, Phys. Letters 25B, 30 (1967);

⁴ V. N. Popov and I. D. Faddeev, ITI report, Kiev, 1967 (unpublished).

⁵ S. Mandelstam, Phys. Rev. 175, 1580 (1968).

⁶ S. Mandelstam, Phys. Rev. 175, 1604 (1968).

rec'd Jan 1970

Comment: This paper occurs before the Lee sequence. I am only interested in section II which discusses YMTh. Later sections discuss comparison with Canonical quantization procedure which I don't care about now, and also gravitational applications.

Section II:

1. First off they write out the lagrangian for Yang Mills theory: no other particles are put in here. They show the Euler equation that the fields tensor F_{uv}^a should solve, and they also show a general identity that any F_{uv} solves, regardless of whether it solves the Euler equations. This is the first time I have even thought about Euler equations for YMT, so was worth reading. See gauge notes for derivation.
2. Next, the usual problem with quantization is noted: you cannot treat all field components on equal footing and impose the canonical commutators: it just doesn't work. You are forced to assume a gauge condition of some sort.
3. Their general technique will be to inject a Lagrange multiplier field which decouples from all physical fields. This method works simply in QED; the Euler equation for the fictitious field effects the gauge condition. The same general idea, with certain variations, also works for YMTh. I have discussed this technique on a separate sheet, see also notation translator. The net effect is this: the covariant gauge generating functional is (2.37). The important facts are: (a) There is never a gauge-surface delta; the gauge fixing term appears in a natural way from the gaussian integral; (b) the FP determinant $\det M$ appears, not the FP object which is invariant.

So this method of derivation is a sort of alternative to the Fadeev Popov group method. It gives the same result! The fictitious fields are the ghosts, and the usual Feynman rules drop out.

4. Next, they show that the on-shell S-matrix elements are independent of your chosen gauge surface function Ψ which I call f . I have not tried to follow their proof in detail, but seems reasonable.
5. Next, they specialize results ^{first} to the axial gauge (n points in 3-direction only), Coulomb gauge, and then they consider the alpha-Landau gauge system. In passing they show that the S-matrix is independent of alpha. They do this last deal by using a fancy constrained gauge transformation: it was this section which helped me understand how $\det M d\phi$ is invariant in Lee's papers.

1. In QED you know you have to impose a gauge condition like $dA=0$ to remove some of the fields from consideration in the canonical quantization procedure. This can be implemented as suggested in (2.17) by adding a "Lagrange Multiplier" field term to the Lagrangian. Adding such a field just adds new Euler equations, and you see at once how the gauge condition appears as the Euler equation in B. The field B so used turns out to be a free field. Thus, if you don't include B-fields in your external states, they will never enter the S-matrix because they don't couple to any physical particles. In short, they are just a tool to get you correct quantization.

2. The idea is to somehow do the same trick in QCD. If you try it in the same way you run into trouble because you find that your B-field does couple to A. So a new method is needed. The method is discussed starting at (2.21).

You begin by postulating a lagrange multiplier field $\bar{B}$ and you add certain terms to the lagrangian as in (2.23). Note that a particular gauge function has been chosen in (2.21). So you have added a new field to the lagrangian and you can now get the Euler Lagrange equations (2.25) and (2.26). These equations are easy to derive.

The trick is going to be this: arrange for field $\bar{B}$ to be connected to another field B via an operator D as in (2.24), and choose D cleverly so that B comes out being a free field. Then you can ignore field B and you still maintain "unitarity of the S-matrix" physical.

Examine now the sequence of events as you go down the right hand column. If we apply a fat D onto the first term of (2.26) we get zero because of the general identity (2.13). Thus the second term also dies and we have $DR\bar{B} = 0$ as shown in (2.28). We can define $Q = DR$. Essentially, this is the transpose of the famous object I call M. Then Q^0 is just Q with $A=0$ everywhere.

Next, script-D is defined as the Greens function of Q in (2.31), this it is the transpose of my M^{-1} . If you then make bold-D the combination shown in (2.32) you get:

$$Q\bar{B} = 0 = Q[DB] = Q[Q^{-1}Q^0B] = Q^0B = 0.$$

This last equality is a field equation for B, and since Q^0 does not involve A, B is seen to be a free field, as desired.

Derivation of (2.37) So, stick this lagrangian with the $\bar{B}$ terms added into your functional integration to get a generating functional as in first line of (2.37) where we integrate over both the gauge field A and the free field B. The first thing you do is replace:

$$dB = \det(dB/d\bar{B}) d\bar{B} = /D^{-1}/ d\bar{B}.$$

where D here is the bold D, so $D^{-1} = (\vec{Q}_r^{(q)})^{-1} \cdot \underline{Q}_r$. Writing this determinant as expo you get what is shown in 2.37.

Note that the order of the two operators in the trace doesn't matter! You now are left with a trivial gaussian integral in variable $\bar{B}$ which you can do using Sidney Matrix Theorem which yields the term I have circled in 2.37. I recognize that term as the exponentiated gauge surface delta function: so now I see why they chose those added $\bar{B}$ terms the way they did. Also, I see something that at least resembles the FP determinant, which I will think about later.

The point is this: the authors have obtained Fadeev and Popov's result using the Lagrange multiplier field. They never factored out a group integration or anything like that. I.e., it's just a different approach.

And note that it's the FP det, not the FP object, that appears in (2.37).

Notation Translator.

F&T

PhL

$G_{\mu\nu}^a$

$+F_{\mu\nu}^a$

f^{abc}

C^{abc}

λ

g

u^a

Θ_a

Note: their equations (2.2), (2.3), (2.8), (2.5')

[but not consistent with (2.9) & (2.5)]

∇_{μ}^{ab}

D_{μ}^{ab}

$\hat{\Sigma}_{ab}$

$-C^{abc}X_c$

/as in (2.2).

ψ_a

f_a

/gauge surface fctn.

Q_{ab}

$g M_{ab}^T$

D_{bc}

$\frac{1}{g} M_{bc}^{T-1}$

Generalized Ghost Idea for any gauge.

- ① According to F&T, the real thing whose det you want is not M but their D , because D is the thing that relates B to $\bar{B}$.

And this thing turns out to be:

$$D = (M^T)^{-1} \overleftarrow{(\partial_\mu R_\mu[A=0])}$$

$$\begin{aligned} \bar{D}^{-1} &= (\partial_\mu R_\mu[A=0])^{-1} (M^T) \\ &= \underbrace{[\partial_\mu R_\mu[A=0]]^{-1}}_{(Q^0)^{-1}} \cdot \underbrace{\left[\frac{1}{g} (\partial_\mu R^\mu[A] - ig \underline{G} \cdot \hat{A} R_\mu[A]) \right]}_Q \end{aligned}$$

You see that this first factor is somehow removing the "inessential part" of M . I.e. when you take a determinant, you only care about the A -dependent piece in quantization, so you are allowed to extract any A -independent operator you want from M .

Note that in covariant gauge, $R^\mu = d^\mu$ so the removed factor is just $(\text{box})^{-1}$ which I am used to removing.

2. Now, when you write this out in trace log notation you get:

$$\begin{aligned} |\bar{D}| &= \exp(\text{Tr}[\ln(Q Q^{\bar{0}})]) \\ &= \exp(-\text{Tr} \left[\sum_{n=2}^{\infty} \frac{1}{n!} \{1 - Q Q^{\bar{0}}\}^n \right]) \\ &= \{ (Q^{\bar{0}})^{-1} (Q^0 - Q) \}^n \end{aligned} \quad \ln(1-x) = -\sum_{n=1}^{\infty} \frac{x^n}{n}$$

For $n=3$,

$$\exp \left[-\text{Tr} \left[\frac{1}{3!} (Q^{\bar{0}})^{-1} (Q^0 - Q) (Q^{\bar{0}})^{-1} (Q^0 - Q) (Q^{\bar{0}})^{-1} (Q^0 - Q) \right] \right]$$

Here you see the famous minus sign for the loop, and you see that whatever Q_0^{-1} happens to be, it is the ghost propagator. In fact that is just what FT equation (2.33) tells us. Like $(p^2 - m^2) \phi(x) = 0$ is motional, then propagator is inverse of that operator. So now we know what is the "ghost propagator" in any gauge!!!! Obviously for covariant gauge this reduces to the result I know.

Try to lock FT notation to Luchit notation.

① Inconsistency #1: They call f^{abc} "st. const" but they say they are real. But they never write their LA so no way to lock onto definition of their f 's!

② They say $L = -\frac{1}{4} G_{\mu\nu}^a G_{\mu\nu}^a$ so I am sure of this: $G_{\mu\nu}^a = \pm F_{\mu\nu}^a$. But then (2.2) sets this sign so

$$\boxed{G_{\mu\nu}^a = + F_{\mu\nu}^a} \quad \text{me.}$$

③ Now look again at extra term in (2.2) [with (2.3)]

$$\underbrace{+\lambda f^{acb} A_\mu^c A_\nu^b}_{\text{them.}} = \underbrace{+g C_{abc} A_\mu^b A_\nu^c}_{\text{me}}$$

reverse base.

$$= +\lambda f^{abc} A_\mu^b A_\nu^c$$

Compare to get

$$\boxed{\lambda f^{abc} = g C_{abc}} \quad (1)$$

(2.2) + (2.3).

④ Now look at (2.8).

$$D_\mu^{ab} = \delta^{ab} \partial_\mu + \lambda f^{acb} A_\mu^c \quad \leftarrow \text{them}$$

$$D_\mu^{ab} = \delta^{ab} \partial_\mu - \frac{1}{2} g C_{abc} A_\mu^c \quad \leftarrow \text{me.}$$

also. ✓

⑤ Next, look at (2.5')

$$\delta A = -\frac{1}{\lambda} D_\mu^{ab} u^b$$

← then

$$\delta A = -\frac{1}{g} D_\mu^{ab} \theta^b$$

← me.

$$\Rightarrow \boxed{\frac{u^a}{\lambda} = \frac{\theta^a}{g}} \quad (2)$$

⑥ Next, by (2.7)

$$S(u)_{ab} \simeq \delta_{ab} + i f^{acb} u_c$$

The

$$V(\theta)_{ab} = \delta_{ab} - i \vec{G}_a \cdot \vec{\theta}$$

$$= \delta_{ab} - i G_{ab}^c \theta_c$$

$$= \delta_{ab} \theta_c - f_{cab} \theta_c$$

$$= \delta_{ab} + C_{acb} \theta_c$$

→

$$\boxed{i f^{acb} u_c = C^{acb} \theta_c} \quad (3)$$

But (1,2,3) are not consistent!

(1) says: $f^{abc} = \left(\frac{g}{\lambda}\right) C^{abc}$

(2) says $u^a = \left(\frac{\lambda}{g}\right) \theta^a$

Then (1) x (2) says

$$u^a f^{abc} = \theta^a C^{abc}$$

which conflicts with (3)!

① Now about (2.5). They say

$$\underline{\hat{A}'} = \underline{S} \underline{\hat{A}} \underline{S}^{-1} + \frac{i}{\lambda} \partial_\mu \underline{S} \cdot \underline{S}^{-1}$$

2 says

$$\vec{G} \cdot \vec{A}' = v [\vec{G} \cdot \vec{A}] \vec{v}^{-1} + \frac{1}{ig} (\partial_\mu v) \vec{v}^{-1}$$

Well

$$\begin{aligned} (\vec{G} \cdot \vec{A})_{ab} &= G_{ab}^c A^c = -i C_{cab} A^c = +i C_{acb} A^c \\ &= +i \left(\frac{\lambda}{g}\right) f^{acb} A^c = +i \left(\frac{\lambda}{g}\right) \hat{A}_{ab} \end{aligned}$$

So ~~mult~~ mult by ~~by~~ So mult by

$$i \frac{\lambda}{g} \hat{A}' = i \frac{\lambda}{g} v \hat{A} \vec{v}^{-1} + \frac{1}{ig} (\partial_\mu v) \vec{v}^{-1}$$

Mult by $\frac{g}{i\lambda}$, then mine says "

$$\hat{A}' = \cancel{0} \hat{U} \hat{A} \tilde{V}' + \underbrace{\frac{1}{i\lambda} \left(\frac{g}{i\lambda} \right) (2\mu\tilde{V}) \tilde{V}'}_{= -\frac{1}{\lambda} (2\mu\tilde{V}) \tilde{V}'}$$

Again, this is inconsistent with (2.5) by (i).

Conclusion: either my system is inconsistent, or their system is inconsistent. For the time being, I will assume theirs is wrong. Thus, I will not read their paper "close" on notation.

Slavnov

A. A. Slavnov

Generalized Ward-Takahashi identities are obtained for gauge theories of the type of the massless Yang-Mills field. It is shown that all the divergences of the Yang-Mills theory can be removed by means of a renormalization of the charge and the wave function.

A relativistically invariant formalism for gauge theories of the type of the massless Yang-Mills field was first constructed in [1, 2]. In this formalism the S matrix is constructed from an effective Lagrangian, which is a nonlocal function of the fields. Despite this, the gauge invariance ensures unitarity of the S matrix. The perturbation series contains a finite number of types of divergent diagram and the theory can therefore be renormalized by means of the Bogoliubov-Parasiuk R operation [3]. However, to ensure unitarity of the renormalized S matrix it is necessary to take into account the Ward identities, which guarantee gauge invariance of the theory, when carrying out the renormalization.

In the present paper we obtain Ward identities that relate diagrams with a different number of external lines and accordingly renormalization constants of different vertices. We shall show that all divergences can be eliminated by a renormalization of the charge and the wave function of a vector particle (for simplicity we consider a selfinteracting Yang-Mills field. Allowance for an interaction with other fields does not lead to additional difficulties). Ward identities have been considered in [4]. However, in [4] only relations of the type of transversality conditions for the matrix elements are obtained and there are no explicit relations between the renormalization constants.

The method of obtaining the Ward identities developed in the present paper is applicable to any gauge theory, including the gravitational field. To illustrate the method we derive the well-known Ward identities in electrodynamics in the first section. In the second section we consider the Yang-Mills field.

In this investigation we have been concerned solely with the problem of ultraviolet divergences and have ignored the difficulties associated with infrared divergences. To give the expressions we have obtained a well-defined mathematical meaning it may be assumed, for example, that the subtractions are not made on the mass shell. Of course, this does not solve the problem of infrared divergences and this question remains open at the present time.

1. Electrodynamics

The generating functional for the Green's functions in quantum electrodynamics is determined by the functional Feynman integral

$$Z(\eta_\mu, \xi, \bar{\xi}) = \int \exp i \left\{ \int \left[\mathcal{L}(x) + \frac{1}{2\alpha} (\partial_\mu A_\mu)^2 + \eta_\mu A_\mu + \bar{\xi} \psi + \xi \bar{\psi} \right] dx \right\} dA d\bar{\psi} d\psi. \quad (1)$$

Here $\mathcal{L}(x)$ is the gauge-invariant Lagrangian of the electromagnetic and electron-positron fields. The term $(1/2\alpha)(\partial_\mu A_\mu)^2$ fixes the gauge. In particular, $\alpha = 0$ corresponds to the transverse gauge; $\alpha = 1$, to the Feynman gauge. The Green's functions can be expressed in terms of the generating functional (1):

$$(i)^n \langle T A_\mu(x_1) \dots \bar{\psi}(x_i) \dots \psi(x_n) \rangle = \frac{\delta^n Z}{\delta \eta_\mu(x_1) \dots \delta \bar{\xi}(x_i) \dots \delta \xi(x_n)} \Big|_{\eta=\xi=\bar{\xi}=0} \quad (2)$$

V. A. Steklov Mathematics Institute, Academy of Sciences of the USSR, Moscow. Translated from *Teoreticheskaya i Matematicheskaya Fizika*, Vol. 10, No. 2, pp. 153-161, February, 1972. Original article submitted June 23, 1971.

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In the first section, the QED Ward identity is derived using the functional formalism. It is all very simple, though I have not computed things.

In the second section this idea is generalized to pure Yang Mills Theory. Using the constraint method, the complete Ward statement is (20). He then differentiates this various ways to get desired relations.

(a) as usual, the longitudinal part is unrenormed

(b) the simplest non-trivial ward is (25) which relates the gauge field propagator and vertex to the ghost quantities.

It is this identity which produces the famous statement~~x~~ that $Z_1/Z_2 = Z_1/Z_2$. These Z's are clearly defined as tensor coefficients of certain quantities.

(c) the 4-point Ward is equation (34). Results are found for its Z's also.

The major conclusion is this: these Ward identities relate the Z's and reduce the number of divergent constants you have to worry about. The proof of renormalization is then simplified, I guess.

Better: when you renormalize a field theory, you have to pin down certain things: eg, you have to decide where your 3-vertex is equal to g . These are called the renormalization conditions. The Ward identities just tell you that you can't and should'nt independently set renorm conditions that are related~~x~~.

Comparison of Slavnov Notation to my Notation: Hunting for Errors?

1. The first thing to keep in mind is that Slavnov's symbol α is the negative of mine. This just follows from the sign chosen in the gauge-fixing lagrangian. Looking at Slavnov's Eq. (13) you can see how other things are related:

<u>Slavnov</u>	<u>me</u>	
B_μ^a	A_μ^a	/gauge field
ϵ^{abc}	C^{abc}	
g	g	
α_s	$-\alpha_{ME}$	
M_s^{ab}	$(-\sqrt{\alpha} g) M_{ME}^{ab}$	
$M_{yx}^{-1 ba}$	$(-\sqrt{\alpha} g)^{-1} \tilde{M}_{ME}^{-1 ba}(y, x)$	
$\eta_\mu^a(x)$	$J_\mu^a(x)$	/source

2. Now lets looks at Slavnov's equations one by one and tie them to mine.

Eq.(20): This is the basic Ward Identity. I have already checked elsewhere that this agrees with mine.

Eq. (21): I think Slavnov has omitted an i on the left which I have filled in. Now I will convert to my notation as follows: first change sign of alpha, then rescale M^{-1} 's to get:

$$\frac{+i}{\alpha_{ME}} \langle T \partial_\mu B^\mu \cdot B_\nu \rangle = \left[\frac{1}{-\sqrt{\alpha} g} \right] \left(\langle T \partial_\nu \tilde{M} \rangle + g C \cdots \langle T B \tilde{M} \rangle \right)$$

$$\Rightarrow -\frac{1}{\sqrt{\alpha}} \langle T \partial_\mu B^\mu \cdot b_\nu \rangle = -\frac{1}{g} i \langle T \partial_\nu \tilde{M} \rangle - \frac{1}{g} C i \langle T B \tilde{M} \rangle.$$

And this is my first-level Ward Identity, all factors OK except I am not checking bolor label ordering.

Eq. 22: With alpha sign change and i included as shown, this agrees with my example 1 on page 5e, exactly.

Eq 23: have not checked exactly. The object D_0 is FT of $1/k^2$ so looks right.

Eq 24: Here Slavnov has the i's correct. Move the i to the RHS and it then incorporates into M^{-1} to give ghost propagators. This is my level-two Ward identity, see page 5a.

Eq. 25: This is what I call the Slavnov Identity. I finally proved it on page 17.

Eq. 26: Go back to 25 first. Slavnov's D is my iD , so he has $iD^{tr} \dots = G \times iD^G \times$
According to (26), his $G = D^G$. So his identity should read (converted to my notation),
 $D^{tr} \dots = -i D^G \dots$ My result shows a plus sign on the right because I have $k_1 = -k_2 - k_3$.
We agree!

Eq 27: I accepted this to convention my Slavnov Object $\gamma u v$.

Eq. 34: This is my level-3 Ward identity, but he is missing an i. Page 5b.

Eq. 35: This looks about right for my 4 -derivatives on the 4 -glue. The terms on the right are from the disconnected parts on the left, if you like. See page 18-19.