

Binder 1

Kummer (1974)

&

Ball & Zachariasen (1978)

QCD problem

Binder 1: "Kummer + BZ"

This binder contains the source papers + notes on them. For Kummer, there is a lot of fancy functional derivative jazz leading to a fancy functional Ward Identity which contains all Ward identities of the theory. Also a lot of stuff on tensors, then more on renormalization. All is done of course in the axial gauge. This Kummer paper provides much of the backup for BZ paper.

Ball+Lack paper then writes the Dyson for self energy (propagator) and uses a Ward Identity "solution" to make this a closed equation. Reduces to two equations, the "en dot en" (2.18), and the "hase eq." (2.19). To facilitate solution, certain ansatz's are made, like $\alpha = \gamma$, $\beta = 1$ and $A = B^2$, $B = Z$ so that full prop = bare prop. mult. by Z , as shown in (3.1). Then the nTn is written as

$$\tilde{Z}^{-1}(1-1/r) = (1-1/r) + \tilde{Z}^{-1}L + K + (d,e).$$

The problem was to try $Z = (A^2/q^4)^\alpha$ for small q^2 and see if consistent for some α . The problem was to show that simple power was a consistent solution at least for small q^2 ("infrared"). This "consistency" then led to equation

$$\tilde{Z}^{-1} = \frac{1+K}{1+L} \approx \frac{K}{L} \Rightarrow \boxed{-1 = \frac{\partial K}{\partial L}} \text{ "ratio equation"}$$

We then tried to implement this "ratio equation" to show that power for Z was consistent (see next binder).

Misc. stuff in this binder.

- 1) Kummer - BZ translation table.
- 2) derivation of bare gluon propagator + Ward identity. [axial gauge QCD ...]
- 3) partial derivation of Dyson Equation.
- 4) show that Ball's thing does solve Ward. (Ball's vertex)
- 5) antisym of Γ^3 -vertex.
- 6) attempt to do O(3) diagonalization.
- 7) attempt to understand the Z -stuff, renormalization.

Binder 2 "The B.D.S.A.M."

The simple ansatz model. In section "Main" I set up the naive condition in terms of variables r, z, x and K, L . Then came the question of what is a constant: in $K(r) \sim$ because such are removed by subtraction with the (d,e) terms. I made some "rules" leading to the idea of $\Delta K = K(0) - K(\infty)$. Many pages here on the subtraction problem.

In Section "Math" are the integration variables and basic (a) integrals. These are all x-integrals. We always factored in $(n, K), (n, K')$ there did these integrals. Math contains various expansions + other integrals. There are z-integrals, mainly. Some done generally, some done for small r or large r .

We soon discovered that our r -integrals were infrared divergent as $\ln \epsilon$ for $s=1$, and this was seen as desirable because then infrared dominates infrared. Technically, this allowed us to avoid doing r -integrals. The x, z integrals were easy + we could do everything.

We are, at this time, still using the BD vertex Γ .

Our first real calculation was done with $\alpha = \hat{q} \cdot \hat{n} = 0$. In the n th equation we got the result $-1 = 2\pi / (-2\pi)$ in this limit, + we were ecstatic. I guess we then moved directly to the general- α case. The result of that calculation was

$$\text{ratio} = \frac{+2\pi\beta^2 - \pi\alpha\beta \sin'(\beta)}{-\pi\beta^2 + \pi\alpha\beta \sin'(\beta)}$$

We fought hard to make this work but it refused. Also, the denominator at $\alpha=0$ did not yield -2π , so we concluded that the $\alpha=0$ case was not to be trusted.

We tried to do integrals exactly rather than by expansion, could find no discrepancy. Result is summarized in " $\alpha=\alpha$ " section.

7/2 Sections L and K show how we built + computed $L(r, z; \alpha), K(r, z; \alpha)$ when we were novice users of Reduce.

Section "SAMamp" contains a remake of the $\alpha=\alpha$ calculation where the computer does all. Rows K, L are in arrays C, A, computer does (8/6) expansions + gives KIR, KIV, LIR . Same result obtained.

over

8/12 In Section "trace" I considered for the first time the trace equation.
I wrote expressions for tensors L, K^+, K^0 for general model,
then specialized to the SAM.

In a sequence of 8 programs I generated L and K appropriate
for the trace equation. (analogous to section L, K for nTn).
Then TRSAM computed $\Delta K, L(0)$ and here was the result.

$$\text{ratio} = \frac{-\pi\beta^2 + \alpha\beta J_{-1}}{+2\pi\beta^2 + \alpha\beta J_{-1}}$$

in agreement with J_{in} . Not only factor of 2, but wrong sign on J_{-1} !

And so ended the first stage of our work.

Binder 3 "More General Models"

Since the SAM failed (at least with BD vertex), we went on to try to change the model. First I tried to be very general, keeping α, β as general functions. (see first few pages in binder). But this was too hard because you got unknown functions like $\alpha(\hat{n} \cdot \hat{k})$ inside integrands.

Model C. So I restricted to $\alpha, \beta, \delta = \text{constants}$. Conclusions not clear here. L, K^F, K^G all exist. I think this led to undoable integrals, so dropped.

σ -class. This class of models had $\alpha = \sigma(x) \cdot x$ and $\beta = \sigma(x) \cdot 1$. The advantage here is that $K^F \equiv 0$ as in the SAM. You quickly arrive at a new ratio equation as shown on last page of this section. In essence, this model class uses $Z_{ij} = Z(k/n, x) = 2 \cdot \sigma(x)$, so you have that unpleasant x -dependent Z .

The following models are all in this σ -class.

model 1: $\sigma(x) = x$. This lead to $K(r \rightarrow) = \text{const}/r$ and was thus rejected.

model 2: $\sigma(x) = x^2$. Constructed L and K , got ratio $= -1 = -9\beta^2/\alpha^2$, so model failed. It was easy to build L, K from the forms obtained in the SAM.

model 4: $\sigma(x) = x^4$. Got a mess, ratio failed.

8/8 model $N \geq 2$: I decided that I could do general case $\sigma(x) = x^N$ all at once. Ran various cases (output retained). $N = \text{odd}$ always gave $K(0) = c/r$. Did $N = 2, 4, 6, 8$ + always got a mess in the ratio.

So all these attempts failed + Jim heads for Japan.

Computer: Here I have stored my subroutine listings, directory listing at the time, and my Z -integral insertions like $Z^2 \rightarrow \pi/8$. Later this list of insertions grew longer.

Binder 4: "msam, psam"

(Jim returns from Japan)

At this point we observed that the B2 vertex had kin-sings as $q, q_3 \rightarrow 0$, eg. We thought that maybe these were causing our Ratio calculation to go haywire. So we set about constructing a new "Modified" vertex to solve the Ward Identity. (msam series)

In Section "NewVert" I made up several Ward-Null tensors $P_1, P_2, \dots, P_6$.

Of these, only P_1 and P_2 have no n^2 . I then found some coefficient functions for P_1 and P_2 such that $P_0(B2) + P_1 + P_2 = \text{no poles if } q_3 \rightarrow 0$.

This was our new vertex!

Then with this new vertex we redid the ratio calculation ~ we were very hopeful. We created L and K for the ntm for general s and we found:

$$\text{nm: } \text{Ratio} = \frac{3}{1-2s} \quad s=2 \text{ worked!}$$

Then we did the same for the trace equation. Sadly, $s=2$ did not work for the trace. (msam programs) [Sept 19-20]

Then we noticed that both the ntm and the trace had $\Delta K = \frac{3}{2} \pi \beta^2$, so

that the difference equation had $\Delta K = 0$. Then we noticed that this was true for the entire σ -class.

Thus, you could never get a reasonable ratio (with this new vertex) unless you could find a $\sigma(x)$ such that $LIR = 0$ for nm-trace. In an effort to find such a σ , we tried $\sigma = (P + x)$ where $P = \text{constant}$ (the PSAM's). Sept 21.

But we could not find a combination of P, s such that $LIR = 0$. Thus, we were stuck. For this Ball vertex P , we could not get a reasonable ratio because $\Delta K = 0$ for nm-trace, and we could not make $L = 0$.

The final entry in this binder concerns the fact that $P^0 \Delta \sigma P^0$ does not seem to be symmetric, ie, that $T(q) \neq 0$. This led us to reconsider the constancy of the d,e equations.

Binder 5: "Bloop + Fred + Beta"

Having discovered this asymmetry problem in the Dyson RHS, we began to consider all 5 equations implied by the Dyson Π , i.e., the $\pi\pi$, $\pi\pi\pi$, and 3 involving d, e , and possibly h . We then said: let's calculate these 5 RHS's in lowest order, i.e., the "Born Loop" or Bloop. For this case no hay asymmetry, so $h=0$.

So this was done in Section "Bloop". We set out to calculate the Λ^2 and $\ln \Lambda^2$ terms in a, b, c, d, e to order g^2 . I did this and got some results.

I then noticed that the $d=2e$ condition was violated by the result. I.e., $d = \frac{\pi}{2} \Lambda^2$ but $e=0$. I showed that in any SAM with a Δ_1 , you must get $d=e=0$, if folding legal. It then turned out that folding the integrand can change the coefficient of Λ^2 and this was the problem. I tried other methods of regulating but could not get useful method.

At this point I lost interest in Λ^2 terms in a result. My attitude became: whenever you see them, ignore them (set $=0$, or subtract them.) From this point of view, the Born Loop results were:

$$a = \frac{4}{3} \pi \ln(\Lambda^2/g^2)$$

$$b = \frac{4}{3} \pi \ln(\Lambda^2/g^2)$$

$$c = 0$$

$$d = 0$$

$$e = 0$$

The fact that $c=0$ was nice: it meant that at least to lowest order pert. theory there was no $c \equiv f \equiv K^+$ term generated. Thus, $\Pi = \Pi^0 \times \ln \Lambda^2$, i.e., if you look at the $\ln \Lambda^2$, you get $\Pi =$ multiple of bare Π^0 . I now see this as part of what

must be true if theory is renormalizable. Later same thing happens with triangle.

At this point we received the new Fred Vertex. (9/25). I checked that it satisfied the Ward and constructed the h and K tensors implied by this vertex. I think we shifted our interest to this vertex because it had no $q^2 \rightarrow 0$ singularity whereas we thought the Ball vertex did have such singularity (later we found that it did not). Also, Ball failed ratio test!! (The $SK=0$ problem).

Question: what happens when you insert Fred vertex into RHS of the $\pi\pi\pi$ and trace equation? First we did VT calc. Here we discovered a non-uniformity problem + learned that to do VT, you must fold away the I2 and I3 terms first. Also, at this point we stopped doing separate L and K (i.e., ratio approach) and just considered $L+K$, so to speak.

To our surprise, we found that in the $\nu\bar{\nu}$ neither the nn or $face$ had any Λ stuff, i.e., no Λ^2 , no $\ln\Lambda^2$. Te, the Fred vertex was $\nu\bar{\nu}$ finite.

Next we did the IR and were excited to get $0 \cdot \ln\epsilon$. (Whereas the Ball vertex did not give $0 \cdot \ln\epsilon$ as we learned in the ratio method for the difference equation where $sK=0$ but $L \neq 0$.) So Fred's vertex was looking good. By the way, $0 \cdot \ln\epsilon$ for Fred Vertex in both nn and $face$ equation!

At this point I think we thought that these integrals (right sides of nn and $face$) were identically zero. So we were ready to improve our result by going to $\bar{Z} = (\Lambda^2/q^2 + \beta)$.

In Section Beta we used this $\bar{Z}$ and began to recalculate RHS's. There were three terms on right. X = original result with old Z (which we thought = 0), the X' term, and the X'' term, Born loop like.

For the X' term (PREP1) my results were:

$$\begin{array}{ll} nn: & \frac{\pi}{2} \Lambda^2 + \frac{5}{3} \pi (\alpha^2 - 1) q^2 \ln(\Lambda/q) \\ face: & 2\pi \Lambda^2 + \pi (6\alpha^2 - 5) q^2 \ln(\Lambda/q) \end{array} \quad / \nu\bar{\nu} \text{ only.}$$

For the X'' term I stole results from Bloch; + ignored Λ^2 stuff. Then on last page I combined these to get the full $\ln\Lambda$ results in this Beta model.

$$RHS \text{ of } n\bar{n}, \ln\Lambda \text{ only} = \frac{\beta}{(1+\Lambda q^2)} \frac{5}{3} \pi (\alpha^2 - 1) q^2 \ln \frac{\Lambda^2}{q^2} + \frac{q^2 \beta^2}{\Lambda^2 (1+\Lambda q^2)} \frac{11}{6} \pi (\alpha^2 - 1) q^2 \ln \left(\frac{\Lambda^2}{q^2} \right)$$

I was then confused because no $\ln\Lambda$ on the LHS. So result not consistent. But it was gauge invan. i.e.,

$$LHS = RHS, \text{ small } q^2 \rightarrow -q^2 (\alpha^2 - 1) = q^2 (\alpha^2 - 1) \beta \frac{5}{6} \pi \ln(\Lambda^2/q^2)$$

Binder 6:

At this point we knew that, with fixed vertex, RHS of both Γ and Trace were IR and UV finite. I made up a little proof that RHS's $\equiv 0$. But proof was incorrect due to uniformity.

Next I worked thru Baker's IR theorem which shows that any vertex which is "soft", i.e., no $1/R$ terms, satisfies $\int d^4k \Delta_{\mu\nu}(k) = 0$ and is therefore IR finite. Thus, our computer work in last binder was just confirming this now obvious fact.

Marshall also has a UV theorem which I tried to show but never could.

Next we did a counterexample. We studied:

$$\int \frac{d^4k'}{k'^2} \left[2 + \frac{1}{(n \cdot k')^2} \right] = \text{UV and IR finite} = \frac{\pi}{2} [1 + \ln(4\alpha^2)]$$

This was an example to violate my theorem.

So we then said: let's compute directly the RHS of the ren equation. First we did all the z -integrals which we gave names like HA1, HA2...

Most of these z -integrals were trivial, but the $\ln(R1)$'s were hard. By the way, I folded FNN so as to get only $1/a^1$ and $1/a^2$. I got rid of $1/a$ and $1/a^2$ [$1/a = I2$, $1/a^2 = I3$]. Earlier I did this because you must do so to get correct UV result.

So, the roughest integrals were $\ln(R1)$. We found:

$$HA1 = (-1/4r^2\beta) \int_{-1}^1 dz \frac{\ln(R1)}{(z-\alpha)^2} = \frac{-\pi}{4\beta r^2} \left[-2\sin^{-1}\beta + \sin^{-1}\left(\frac{1+\beta\alpha}{\alpha+\beta}\right) - \sin^{-1}\left(\frac{1-\beta\alpha}{\alpha-\beta}\right) \right]$$

$$HA2 = (1/8r^2\beta) \int_{-1}^1 dz \frac{\ln(R1)}{(z-\alpha)^2} = \dots \text{simple, etc.}$$

Next, we had to do the R -integrals. By folding this meant $\int dR \dots$. I looked only at the HA1 terms; Jim used Tony to do all other terms via reduce. I.e., all other terms were integrals of rational forms plus $\sqrt{a^2-1} = AR = (1-R^2)/2R$. Thus, just rational. No square roots. So "other terms" = elementary form of α .

over

For the nn equation, the HAI terms were:

$$\int_0^1 dr [-2\sin^2\beta \dots] (r + \frac{2}{r} + \frac{1}{r^3})$$

But the $\frac{1}{r}$ yields $4(\sin^2\beta)$, Labodhesky. This is the only non-elem term on RHS.

So I found:

$$\text{HAI terms} = (\text{elem}) + \frac{1}{2} \alpha \ln \epsilon^2 + \frac{1}{\beta} 4(\sin^2\beta)$$

$$\text{other terms} = (\text{elem}) - \frac{1}{2} \alpha \ln \epsilon^2$$

$$\text{total RHS} = (\text{elem}) + \frac{1}{\beta} 4(\sin^2\beta) \quad ?$$

Thus, RHS of nn-equation had a term $\left(\frac{q^2}{\beta}\right) 4(\sin^2\beta)$. Clearly you never get 4 on the left, so consistency to order q^2 fails unless you subtract something.

programs of this binder:

fed88, c, d fed30 lobe

By the way, did computer numerical check on lobe at $\beta = .4$.

Binder 7

Motivation: compute vertex to order g^3 in pert theory to see if there are $n^4 m^4 q^4$ terms. I.e., this was to be a sort of test of the form of the peak vertex tensor form.

First I tried putting the full Δ into computer, overflowed case. Next we decided to keep only UV and leading terms. This led to the use of R as scaling var, and the choice of $q_3 = 0$.

In TRIAN I was able to get the leading log "guts" of the vertex with g . I then had to do about 20 tensor integrals and edit them in. Got a simple result which agreed with Jim, but it seemed to violate Ward relative to Born loop.

I then suggested we add bubble graphs. In BIBI got bubble guts + edited in the same integrals. I guessed the factor 3 + added things together to get the $22/3$ P_0 result. Was excited + called Jim.

Next, showed that g -triangle $\neq$ Born loops as Jim thought.

Then in TRIFI I redid for general Q_1, Q_2, Q_3 . Made some new integrals etc + got very simple result. Then I (unnecessarily) redid the bubble and got a complex result.

Finally, I "did the color" + showed factor of 3, earlier guessed.

But then we decided that result was obvious due to renormalizability of the theory.

Kummer (1974)

Ghost-Free Nonabelian Gauge Theory*

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Summary

The advantage of the special axial gauge $n^\mu A_\mu = 0$ for a nonabelian gauge field A_μ^a is that no Faddeev-Popov ghosts must be considered. This leads to important simplifications in certain field-theoretic applications, which are especially important, if such theories are used as (e.g. asymptotically free) models of the hadronic interactions. We have in mind Wilson-expansions in terms of gauge-invariant operators etc. It is shown here that the field-theoretic machinery can be developed consistently in this gauge. Appropriate generalized Ward-identities can be derived and form the basis for a discussion of possible counter-terms in the Lagrangian. Although, at first, this seems to be an example of a (presumably) renormalizable field-theory with not *multiplicatively* renormalizable self-energy, in the end its properties turn out to be extremely simple. One obtains, e.g., the "naive" Ward-identity $Z_A = Z^{(3)} = Z^{(4)}$ between the divergent parts of the wave-function renormalization and the renormalizations of the three-vertex and of the four-vertex.

1. Introduction

Nonabelian gauge theories of the Yang-Mills type [1] are now in the focus of interest, because they could provide not only a unification of weak and electromagnetic interactions [2], but they also seem to be the only renormalizable field theories showing asymptotical freedom [3], which seems to be a good approximation to the observed scaling properties of the structure of hadrons.

In general in the treatment of *hadronic* gauge theories even more knowledge about general properties of field theories is necessary. E.g., the discussion of the asymptotical behaviour of the structure functions in deep electroproduction [4] requires the construction of a set of gauge-invariant operators, to be used in a Wilson expansion [5] of operators in connection with a Callan-Symanzik equation [6], in order to arrive at gauge-invariant anomalous dimensions. □

* Dedicated to Prof. Dr. P. URBAN on the occasion of his 70th birthday.

Kummer: Non-Abelian Gauge Theory Without Ghosts. (rec'd Oct 74, publ. 1975)

1. Introduction. In certain calculations (eg, deep inelastic structure functions) where you use renormalization group and Wilson expansions, you want to have a set of operators handy. You would like these operators to be gauge invariant. A problem with the usual ghost-field approach to NAGT is that the ghost field is not gauge invariant, and this renders difficult the task of constructing simple invariant Wilson operators. This seems to be one of Kummer's principle motivations for working up the ghost-free version, ie, for going into the axial gauge. It is noted that the ghosts do not damage the renormalizability of the theory (as was shown by t'Hooft, Lee, Zinn-Justin etc using Slavnov identities).

Kummer's remarks about $\beta \rightarrow 0$ still confuse me, because I don't see why you would try to do (1.7) with (1.8). But I do understand that there is a great practical motivation for doing NAGT in the axial gauge because ghosts are a mess. By the way, in my notes on axial gauge gauge theory (stored in BZ notes somewhere), I have derived the fact that (1.9) is the " β -axial gauge" propagator for the gluons. (Terminology in the sense of the α -Landau gauge).

2. Slavnov Identity. Here Kummer sets up his theory. The $n \cdot A = 0$ gauge surface delta function is replaced by an effective lagrangian term as in (1.10), and now there is a path integration over the dummy variable fields $[dC]$. Remember that the ghosts were also dummy variable fields in this sense. So now you can talk about things like "mixed C-A propagator" and so on. Each field gets a source, J for A as usual, and K for C.

The major equation of this section is (2.7). Basically (2.7) is merely a statement that $\delta W = 0$ under a gauge transformation, where W = generating functional. Ie, the theory is gauge invariant. I will for the time being refer to (2.7) as the Slavnov equation. Essentially, it contains all the Ward identities of the theory. It will soon be replaced with the simpler (3.11) but let's wait on that. If you take (2.7) by itself with no further differentiation and set sources = 0, you get $0 = 0$, as shown in (2.9), sort of an exercise in consistency. If you first take a d/dJ derivative of (2.7), then set sources = 0 you get (2.10), which says that the C-A propagator does not renormalize. By the way, remember that these functional derivatives are giving you fully renormalized Greens functions.

Next, it is noted that all Greens functions are "transverse" to the gauge vector n^μ , as shown in (2.12). In particular, this is true of the propagator, see (2.14). This seems very obvious to me when I think of the prop or greens functions as Tau functions, you imagine n^μ reaching into the VEVTOP and dying against any A.

I think the analog of (2.7) in the regular (Lorentz)gauge cum ghosts is the tool used by Zinn-Justin. Apparently that result is rather messy as you take higher derivatives to see your Ward identities.

3. Proper Vertex Identities. As I know by now, the generating functional W or Z can be replaced with a function Γ = proper-vertex generating function. Several

exciting results drop out at once just by reformulating the theory in this way. :

(1) You can define an "inverse propagator" as a second derivative of Γ , ie, the proper 2-point function with two external propagators removed is an inverse propo. This inversemess is expressed in equation (3.7). However, this is now a matrix equation in the space of field types, A and C , in addition to being a matrix equation with Lorentz $\mu\nu$ indices. Although the inverse of the field matrix Δ does exist and is Π , you cannot invert the gluon propagator by itself $(\Delta_{AA})_{\mu\nu}$, so that Π_{AA} is not really the inverse of Δ_{AA} in the Lorentz index space. This fact later appears as equation (4.8) where you have an extra term.

(2) The second exciting result is that the Slavnov Equation (2.7) now has the simple form (3.11) when expressed in terms of Γ . Obviously derivatives of (3.11) with respect to the A field d/dA . (I am not sure why Kummer now takes his fields A and C to lower case a and c). You can see looking at (3.11) that sometimes a derivative will pick off in $D(a)$ and not raise the number of external particles on proper vertex. This Slavnov is going to give all the Ward identities, ie, it will relate different vertices like $\Gamma^{(3)}$ to $\Gamma^{(2)} = \Pi$, this being the historical Ward identity.

The principle ward identities (3.15) and (3.16) result from 2 and 3 d/dA applied to (3.11). The C - A inverse prop is given by (3.12), again you see that it does not renormalize. A nother simple result of (3.11) is that the various inverse propagators are transverse to k_μ , just as in QED. However, we are warned that $\Pi_{\mu\nu}$ will not quite be the self energy in the QED simple way; wait on that.

4. Kinematic Structure of n-point functions ($n=2,3,4$). From the vectors n and q you can construct 4 symmetric tensors (including $g_{\mu\nu}$). It is convenient to rearrange these into $R, S, T, 1$ as needed because these tensors have both projection and transversality properties, see notes for details. The prop-full must look like (4.1) because that is the most general symmetric, transverse matrix. Similar comment applies to (4.6) which is transverse with respect to q rather than n . Eq. (4.8) is the "almost-inverse" equation referred to above. It forces a relation between (A, B) and (a, b) . There are only two unknown functions.

Since the self-energy need not be transverse, it can in general have extra terms. The coefficients are called $\bar{a}, \bar{b}, \bar{d}, e$ where I have added e as in BZ. Note $\bar{d}=d$ but the others differ a little. You put in this most-general-form for the self energy and turn the crank on the mini-dyson (4.12). Then you know that a and b must be in terms of these more basic input variables in the self-energy. The hatted variables are discussed in my notes.

For the 3-vertex, oh my. 4.16 looks like the correct Ward, but something looks wrong with later equations in this section.

(a) I have checked Jim's form for the vertex, and it does not seem to have a $(1+L)^{1/2}$ piece where $1+L = (b-c)$. Also (b) Jim has a 3- q type term which Kummer does not show. Also, (c) the second-line of (4.20), the nnn term, seems dimensionally wrong in powers of n , not that we ever use this term for anything.

The logic of this section escapes me. Is he claiming he can solve the Ward identity backwards for the vertex, or what. I skipped the next part on the 4-point vertex.

5. Counterterms. First, Kummer makes an IR comment: unless you do SSB, you will have infrared divergences, maybe you should choose $p^2 = -u^2$ as your renorm point, we will not dwell on this. Kummer says zilch about "containment" in this entire paper not even a speculative remark.

Now, how do you renormalized the theory? Quotes a Theorem which says: if it is renormalizeable (by power counting), then you can implement renormalization by adding a finite number of counterterms of dimension less than 4. Somehow Kummer build things down to just the ~~xxx~~ Z_A and Z_A' "kinetic" counterterms shown in (5.1). Recall that these are there to "undo" the Z which accumulates to the propagator-full when you sum over all graphs, ie, solve the mini-dyson. Ie, you thereby remove a divergent constant from the theory. The vertex charge renormas seem correct to me, eg (5.2), although (5.3) seems wrong.

Now comes a question: since this is not simple QED, you might wonder whether the counterterms (5.1) really do what you think. To find out, you first show

that as a direct result of these counterterms, the bare feynman-rules propagator has been changed to have the new form (5.5). You see the characteristic Z^{-1} factors. Do these inverse Z factors then propagate into the full propagator? The answer is yes, but Kummer really hazes up the issue. You have to put in some mass counterterms also, it seems to me, so that the full propagator still has a pole at 0. Then the Z_R^{-1} goes into the A numerator, and Z_S^{-1} goes into the B numerator. By the way, later Kummer will argue that consistency requires that $Z_A' = 1$, ie, there really cannot be a counterterm of that type. Then $Z_S = Z_R = Z_A$ and there is only one Z in the theory. This is also the 3-point and 4-point vertex Z as in (5.21), these following from the Ward identities.

So, to summarize, the effect of the Z_A kinetic counterterm is to cause a Z_A^{-1} to appear in the numerator of the full propagator. This then cancels against the Z_A factor that naturally accumulates there do to adding graphs. Ie, Z_A arises in the numerator when you solve the mini-dyson because of the effect of the self energy. This fact is stated here in equation (5.13) where you see the Z's being given in terms of the self-energy parameters. So the counterterm removes this Z, hence removes divergent factor from the propagator, ie, you have renormalized the propagator.

6. Conclusions: What are the advantages of the axial gauge approach?

- 1) naive Ward identities are true, ie, $Z^A = Z^3 = Z^4$. In ghost gauges the Wards are I gather much less transparent.
- 2) An argument is given as to why Z is expected to be gauge invariant (ie, not a function of $\checkmark$ I presume!)
- 3) He comments that it has not yet been really proved that S-matrix elements are independent of n_μ vector, when you go to SSB to avoid zero mass gauge particles.

Appendix A: First, Kummer gives the principle part prescription for the single and double poles. This is just what we need?! to see if calculation is correct. I will do that right now after finishing this. Claims other people got same prescription see ref 27. The rest of this appendix deals with the fact that when $q \cdot n = 0$ things go a little haywire, the power counting seems to change and you might lose Renorm. I think he shows here that things are OK and fit Weinbergs Theorem. However, he notes that $n^2=0$ light-like gauge still has some problems. This problem with n parallel n has long given the axial gauge a "bad reputation".

Summary of Riemann Tensors

$$\gamma = n^2 q^2 / (q \cdot n)^2$$

① Definitions

$$N_\mu = n_\mu - q_\mu n^2 / (q \cdot n)$$

$$Q_\mu = q_\mu - n_\mu q^2 / (q \cdot n)$$

$$N^2 = n^2 (\gamma - 1)$$

$$Q^2 = q^2 (\gamma - 1)$$

$$R_{\mu\nu} = N_\mu N_\nu / N^2$$

$$T_{\mu\nu} = Q_\mu Q_\nu / Q^2$$

$$S_{\mu\nu} = [g_{\mu\nu} - R_{\mu\nu} - n_\mu n_\nu / n^2] = g_{\mu\nu} - R_{\mu\nu} - M_{\mu\nu}$$

$$= [g_{\mu\nu} - T_{\mu\nu} - q_\mu q_\nu / q^2] = g_{\mu\nu} - T_{\mu\nu} - L_{\mu\nu}$$

$$n_\mu R_{\mu\nu} = 0$$

$$q_\mu T_{\mu\nu} = 0$$

$$n_\mu S_{\mu\nu} = q_\mu S_{\mu\nu} = 0$$

$$R^2 = R$$

$$S^2 = S$$

$$T^2 = T$$

$$R \perp S$$

$$T \perp S$$

$$RTR = R/\gamma$$

$$M_{\mu\nu} \equiv n_\mu n_\nu / n^2$$

$$L_{\mu\nu} \equiv q_\mu q_\nu / q^2$$

$$V_{\mu\nu} \equiv \frac{R_{\mu\nu} + n_\mu q_\nu}{(q \cdot n)}$$

$$(\gamma - 1) R_{\mu\nu} = M_{\mu\nu} - V_{\mu\nu} + \gamma L_{\mu\nu}$$

$$(\gamma - 1) T_{\mu\nu} = L_{\mu\nu} - V_{\mu\nu} + \gamma M_{\mu\nu}$$

$$R_{\mu\nu} - T_{\mu\nu} = L_{\mu\nu} - M_{\mu\nu}$$

BZ/Kummer Translation Table.

BZ

Kummer

True in either notation

d

$=$

$\bar{d}$

$$a = \frac{b-a}{1-\gamma}$$

b

$=$

$$-b/q^2$$

$$C = \frac{B-A}{1-\gamma}$$

c

$=$

$$-c/q^2$$

a

$=$

$$-a/q^2$$

$$A = \gamma/a$$

B

$=$

$$-q^2 B$$

$$B = 1/b$$

A

$=$

$$-q^2 A$$

C

$=$

$$-q^2 C$$

$(1-b)$

$=$

$$\frac{\hat{A}}{\hat{b}}$$

d/q^2

$=$

$$\frac{\hat{A}}{\hat{d}}$$

$-d/q^2 - c$

$=$

$$\frac{\hat{A}}{\hat{c}}$$

$-c$

$=$

$$(\frac{\hat{A}}{\hat{c}} + \frac{\hat{A}}{\hat{d}})$$

$b-c-1$

$=$

$$(\frac{\hat{A}}{\hat{c}} + \frac{\hat{A}}{\hat{d}} - \frac{\hat{A}}{\hat{b}}) = L$$

$(b-c)$

$=$

$$(L+1)$$

$$\frac{f_i}{(n;q)_i}$$

$=$

$$f_i$$

$$p_{\text{symmetric}}^{(0)}$$

$=$

$$K_{\text{symmetric}}^{(s)}$$

Some Kummer Notation

$$\tilde{\Delta}_{\mu\nu} = \tilde{\Delta}'_{\mu\nu} = \Delta'_{\mu\nu} = \text{"full" propagator, i.e., solves mini-Dyson}$$

$$= \text{the } n=2 \text{ Greens Function}$$

$$\Pi_{\mu\nu} = \tilde{\Pi}_{\mu\nu} = \Pi'_{\mu\nu} = \text{"inverse" of full prop in sense of (4.8)}$$

$$\Delta_{\mu\nu} = \text{bare propagator}$$

$$\tilde{\Pi}_{\mu\nu} = \text{self-energy} = \text{the } n=2 \text{ proper Greens Fctn.}$$

$$\Delta_r = \text{bare propagator with (5.1) counterterms included.}$$

$$\Delta'_r = \text{full prop with (5.1) and } \Pi_r \text{ ct's}$$

$$\tilde{\Pi}_r = \text{self energy w/ ct.}$$

I prefer BL notation!

Kummer, Section 2.

Defines this generating functional:

$$W[\gamma, K] = \int [dA][dc] e^{-i \int d^4x [\mathcal{L}(A) + (m_A)^2 C^a + K^a(x) C^a(x) - \mathcal{L}_m^a(x) A_\mu^a(x)]} \quad (*)$$

I gather that Kummer means to imply that $\int [dc] e^{i \int d^4x [n \cdot A^a(x)] C^a(x)} = \delta(n \cdot A)$ I have never seen this, but it looks possible. Let's see if we can prove it.

Yes, it's very easy:

$$\prod_x \left(\int dc(x) \right) e^{i \sum_x f(x) c(x)} = \prod_{x_i} \left[\int dc_i e^{i f_i c_i} \right] = \prod_{x_i} 2\pi \delta(f_i) = 0$$

$$= \prod_x 2\pi \delta(f(x)) \doteq \delta(f) \quad \text{, can omit the } 2\pi \text{ if you like.}$$

Conclusion: (*) is a viable generator for axial-gauge theory. You can always add source term for each field. Let's recall some basic facts:

$$\frac{1}{W(\gamma)} \frac{\delta^n W}{\delta \gamma_1 \dots \delta \gamma_n} = (i)^n G_n^c(x_1, \dots, x_n)$$

↑
connected

$$Z = -i \ln W$$

$$\frac{\delta^n Z}{\delta \gamma_1 \dots \delta \gamma_n} = (i)^{n-1} G_n^c(x_1, \dots, x_n)$$

Now, let's do a gauge transform on A

$$\delta A_\mu^a(z) = -i f_{bd}^a \theta^b(z) A_\mu^d(z) - \frac{1}{g} \partial_\mu \theta^a(z) \quad \text{by my own notes.}$$

$$= -\frac{1}{g} \left[\partial_\mu \theta^a(z) + i g f_{bd}^a \theta^b(z) A_\mu^d(z) \right]$$

$$\delta A_\mu^a(z) = \left[\partial_\mu \delta^{ab} + i g f_{bc}^a A_\mu^c(z) \right] \theta^b(z) \left[-\frac{1}{g} \right]$$

Now define Kummer's thing to get $\delta A_\mu^a(z) = D_\mu^{ab} \delta \omega^b(z)$,

$$+i f_{bc}^a = -f_{ab}^c$$

$$\delta \omega^b(z) \equiv \theta^b(z) \left[-\frac{1}{g} \right]$$

where

$$D_\mu^{ab} = \partial_\mu \delta^{ab} - g t^{ab}_c A_\mu^c \quad \text{as in (1.5)}$$

Let's check:

$$t^{ab}_c = -i t^{ba}_c = (-i) i C_{bca} = C_{bca} = C_{abc} \quad \checkmark$$

Thus I can make sense of $(1.5) + (1.6) + (2.6)$

So we now write an equation for W .

$$\begin{aligned} \frac{\delta W}{\delta A_\mu^a} = 0 &= \frac{\delta}{\delta A_\mu^a} \left[\int [dA][dC] e^{i \int d^4x [\mathcal{L}(A) + (m \cdot A) C + K \cdot C - \mathcal{J} A]} \right] \\ &= \int [dA][dC] \left\{ n^\mu C^a - \mathcal{J}^\mu \right\} e^{i \int d^4x [\mathcal{L}(A) + (m \cdot A) C + K \cdot C - \mathcal{J} A]} \end{aligned}$$

Try again:

$$\begin{aligned} 0 = \delta W &= \int [dA][dC] \left[\frac{\delta C^i}{\delta A^a} \frac{\delta \mathcal{L}}{\delta A^a} \right] \\ &= \int [dA][dC] D_\mu^{ab}(A) \delta \omega^b \left\{ n^\mu C^a - \mathcal{J}^\mu \right\} e^{iS} \end{aligned}$$

$$C^a = -i \frac{\delta}{\delta K^a} \quad A = +i \frac{\delta}{\delta \mathcal{J}}$$

$$0 = \int [dA][dC] D_\mu^{ab} \left(i \frac{\delta}{\delta \mathcal{J}} \right) \delta \omega^b \left\{ -n^\mu i \frac{\delta}{\delta K^a} - \mathcal{J}^\mu \right\} e^{iS}$$

$$\text{So } 0 = \left[D_\mu^{ab} \left(i \frac{\delta}{\delta \mathcal{J}} \right) \left\{ \mathcal{J}^\mu + i n^\mu \frac{\delta}{\delta K^a} \right\} \right] W[\mathcal{J}, K].$$

acts only on W ! not here

Again

$$\left[0 = \delta W = \left\{ \mathcal{J}^\mu + i n^\mu \left(\frac{\delta}{\delta K^a} \right) \right\} \cdot D_\mu^{ab} \left(i \frac{\delta}{\delta \mathcal{J}(x)} \right) \cdot \delta \omega^b(x) \cdot W[K, \mathcal{J}] \right]$$

Kummer's statement is ambiguous, this $\uparrow$ is not.

Let's try for more accuracy in (2.7).

$$\begin{aligned}\delta W &= \delta \left\{ \int [dA][dc] e^{iS[A,c]} \right\} \\ &= \int [dA][dc] \delta \left\{ e^{iS[A,c]} \right\} \\ &= \int [dA][dc] i \delta \{ S[A,c] \} e^{iS[A,c]}\end{aligned}$$

But $\delta \{ S[A,c] \} = \delta \left\{ \int d^4x \mathcal{L}(A,c) \right\} = \int d^4x \delta [\mathcal{L}(A,c)]$

Now $\mathcal{L}(A,c) = \mathcal{L}(A) + (n \cdot A)C + KC - \mathcal{I}A$

$$\Rightarrow \delta \mathcal{L}(A,c) = \delta \mathcal{L}(A) + n^\mu C^a \delta A_\mu^a - \mathcal{I}_\mu^a \delta A_\mu^a = [n^\mu C^a - \mathcal{I}_\mu^a] \delta A_\mu^a$$

But $\delta A_\mu^a = D_\mu^{ab}[A] \delta \omega^b$ so we now have

$$\delta W = \int [dA][dc] \left[i \int d^4x [n^\mu C^a(x) - \mathcal{I}_\mu^a(x)] D_\mu^{ab}[A(x)] \delta \omega^b(x) \right] e^{iS[A,c]}$$

Now do a parts integration so the ∂_μ in D_μ acts on the other term

$$\delta W = \int [dA][dc] \left[i \int d^4x \delta \omega^b(x) \tilde{D}_\mu^{ab}[A(x)] (n^\mu C^a(x) - \mathcal{I}_\mu^a(x)) \right] e^{iS[A,c]}$$

where

$$\tilde{D}_\mu^{ab}[A] = -\partial_\mu \delta^{ab} - g t_{ab}^c A_\mu^c = -D_\mu^{ba}[A]$$

clever! So we can now switch $a \leftrightarrow b$ to get

$$\delta W = \int [dA][dc] \left[i \int d^4x \delta \omega^a(x) D_\mu^{ab}[A(x)] (n^\mu C^b(x) - \mathcal{I}_\mu^b(x)) \right] e^{iS[A,c]}$$

Now note $A = i \frac{\nabla}{\nabla \cdot \mathcal{I}}$ and $C = -i \frac{\nabla}{\nabla \cdot K}$ so we then can back it:

$$\delta W = -i \int d^4x \delta \omega^a(x) \left[D_\mu^{ab} \left[i \frac{\nabla}{\nabla \cdot \mathcal{I}} \right] \left(\mathcal{I}_\mu^b \left\{ -i \frac{\nabla}{\nabla \cdot K} \right\} - \left\{ \frac{\nabla}{\nabla \cdot \mathcal{I}} \right\} \right) \right] W[\mathcal{I}, K]$$

Since this must be true for arbitrary $\delta \omega^a(x)$, we get

$$D_\mu^{ab} \left[i \frac{\nabla}{\nabla \cdot \mathcal{I}(x)} \right] \left(\mathcal{I}_\mu^b(x) + i n^\mu \frac{\nabla}{\nabla \cdot K(x)} \right) W[\mathcal{I}, K] = 0$$

This is the real (2.7)

Comments: why is $dW=0$ under this variation of A ? Just look I am confused, I thought adding source terms would break the gauge invariance of your action, so why should you get $dW=0$??

Consider group theory w) invar measure:

$$I = \int dg f(g) = \int dg f(g_1 g)$$

by measure invariance,
not because $f = \text{scalar}$.

Thus $\frac{\delta I}{\delta g_1} = 0$

Similarly, let

$$W = \int [dA] f[A] = \int [dA] f(A^g)$$

So $\frac{\delta W}{\delta g} = 0 = \frac{\delta W}{\delta A_g} \left(\frac{\delta A^g}{\delta g} \right) D$
 $\uparrow = 0$

Yes. The fact that $dW=0$ has nothing to do with the action $S(A) = S(A^g)$. In fact this is generally not true, only true for the $S_0(A)$ term. The fact that $dW=0$ follows from the invariance of the measure under transformations of A to A^g , ie, under gauge transformations.

Now, look at (2.7). at $T=0$, first term vanishes so we have:

$$n^{\mu} \left[\partial_{\mu} \delta^{ab} - g f^{cab} \left(i \frac{\delta}{\delta y^c} \right) \right] i \frac{\delta}{\delta K^b} W = 0$$

or,

$$(n \cdot \partial) \frac{\delta W}{\delta K^b} - ig (t^c)_{ab} n^{\mu} \frac{\delta^2 W}{\delta y^c \delta K^b} = 0$$

So $\left[(n \cdot \partial) \frac{\delta W}{\delta K^b(x)} \right] \Big|_0 = ig (t^c)_{ab} n^{\mu} \left(\frac{\delta^2 W}{\delta y^c(x) \delta K^b(x)} \right) \Big|_0 \quad (2.9)$

Claim: based on color-tensor availability, this must be $\sim \delta^{cb}$, so
 RHS = 0. On left, you have $\langle 0 | C^b | 0 \rangle = 0$, obvious unless you have SSB.

(2.10). Now go back to (2.7)

$$\left[\delta^{ab} \partial_\mu - g t_{ab}^c i \frac{\delta}{\delta J_c^\mu} \right] \left[J^{ab} + n^a i \frac{\delta}{\delta K^b} \right] W = 0$$

Apply $\delta/\delta J_v^d(y)$ to get

$$\frac{\delta}{\delta J_v^d(y)} \left[\delta^{ab} (\partial_\mu J^{ab} W) + \delta^{ab} n^a \partial_\mu \left(i \frac{\delta W}{\delta K^b} \right) - g t_{ab}^c i J^{ab} \frac{\delta W}{\delta J_c^\mu} + g t_{ab}^c n^a \frac{\delta^2 W}{\delta J_c^\mu \delta K^b} \right] = 0$$

$$\Rightarrow \delta^{ab} n^a \partial_\mu \left(i \frac{\delta^2 W}{\delta K^b(x) \delta J_v^d(y)} \right) + \delta^{ab} \partial_\mu \left(\frac{\delta J^{ab}(x)}{\delta J_v^d(y)} \delta W \right) + \delta^{ab} (\partial_\mu J^{ab}) \frac{\delta W}{\delta J_v^d(y)} = 0$$

$$+ -g (t_{ab}^c)^i \delta^{ab} \frac{\delta W}{\delta J_c^i} - g t_{ab}^c i J^{ab} \frac{\delta^2 W}{\delta J_c^\mu \delta J_v^d} + g t_{ab}^c n^a \frac{\delta^3 W}{\delta J_c^\mu \delta J_v^d \delta K^b} = 0$$

assume = 0 for the moment.

$$\Rightarrow (n \cdot \partial) \frac{i \delta^2 W}{2 K^a(x) \delta J_v^d(y)} + \delta^{ad} \partial_\mu \delta(x-y) W = 0$$

Let $d \rightarrow b$, $v \rightarrow \mu$, $\div W$

$$-i \frac{(n \cdot \partial) \delta^2 W}{W \delta K^a(x) \delta J_\mu^b(y)} = \delta^{ab} \partial_\mu \delta(x-y)$$

$$a \quad \frac{-i \delta^2 W}{W_0 \delta K^a(x) \delta J_\mu^b(y)} \Big|_0 = \delta^{ab} (n \cdot \partial)^{-1} \partial_\mu \delta(x-y) \quad (2.10)$$

↑ formal.

$$\equiv \tilde{\Delta}^{a(b)}_\mu(x-y) = \text{"mixed propagator"}$$

Now look at the "double"

$$(t_{ab}^c) n^a \int [d\bar{c}] [dA] A_d^\nu(y) A_c^\mu(x) C^b(x) e^{iS[A, \bar{c}]}$$

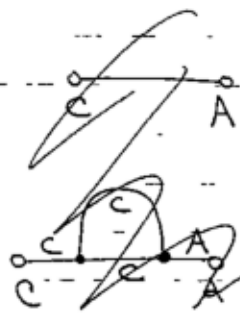
$$= t_{ab}^c n^a \int [dA] A_d^\nu(y) A_c^\mu(x) \cdot \delta'_{n \cdot A} [n \cdot A^b] e^{iS[A, c]} \delta[n \cdot A^{i \neq b}]$$

from $\int [d\bar{c}^b]$ from other $[d\bar{c}^i]$

$$= t_{ab}^c \int [dA] A_d^\nu(y) (n \cdot A^c(x)) \frac{\partial}{\partial (n \cdot A^b)} \delta[n \cdot A^b] \delta[n \cdot A^{i \neq b}] e^{iS[A, c]}$$

study later.

What does (2.10.) say?



← This propagator cannot have higher
contribution, E ,

$$A \text{ --- } \text{[shaded circle]} \text{ --- } C = A \text{ --- } C + \text{[?]}$$

$$\left(A \text{ --- } C \text{ --- } A \text{ --- } C \right) \leftarrow \text{not allowed because } \exists \text{ no } C \text{ propagator!}$$

$$\left(A \text{ --- } \text{[circle]} \text{ --- } C \right) \text{ does not count since waveform.}$$

Key, idea for the check

$$n^a \frac{\delta^3 W}{\delta J_\mu^a(x) \delta K^b(x) \delta J_\nu^d(y)} \Big|_0 = n^a \langle 0 | T A_\mu^c(x) K^b(x) J_\nu^d(y) | 0 \rangle$$

$\equiv 0$ because $n \cdot A^c = 0$, regardless of $(t^c)_{ab}$!

Comment: in Landau gauge, $\partial_\mu A^\mu = 0 \Rightarrow k_\mu \langle 0 | A_\mu A_\nu \dots | 0 \rangle = 0$

in axial gauge, $n_\mu A^\mu = 0 \Rightarrow n_\mu \langle 0 | A_\mu \dots | 0 \rangle = 0$

→ But, for some reason this argument is only valid when $\alpha = 0$
in Landau, $\alpha \beta = 0$ in axial. Why I don't know. □

Kummer, Section 3

Look at (3.7) which says: $\tilde{\Delta}_{ij} \tilde{\Pi}_{jk} = \delta_{ik}$. These indices label the C or A fields. Thus, eg,

$$\Delta_{cc} \Pi_{cc} + \Delta_{cA} \Pi_{Ac} = 1$$

Also, $\Delta_{AA} \Pi_{AA} + \Delta_{Ac} \Pi_{cA} = 1$, but $\Delta_{AA} \Pi_{AA} \neq 1$.

Taken alone, there is no inverse propagator. But I already know that because when you take $\beta \rightarrow 0$, the "usual" inverse propagator diverges. In the various Γ manipulations you must include all fields in your various derivatives. Thus:

$$J_i = - \frac{\delta \Gamma}{\delta \phi_i} \quad \text{is true for each field separately, but}$$

$$\delta_{ij} = \frac{\delta J_i}{\delta J_j} = - \frac{\delta}{\delta J_j} \left(\frac{\delta \Gamma}{\delta \phi_i} \right) = - \sum_{k=C,A} \frac{\delta^2 \Gamma}{\delta \phi_k \delta \phi_i} \frac{\delta \phi_k}{\delta J_j}$$

$$\Rightarrow -\delta_{ij} = \left(\frac{\delta^2 \Gamma}{\delta \phi_k \delta \phi_i} \right) \left(\frac{\delta^2 \mathcal{L}}{\delta J_k \delta J_j} \right)$$

$$\Rightarrow -\delta_{ij} = \Pi_{ki} \Delta_{kj}$$

You only have inverse in the 2×2 matrix sense

Now, let us convert (3.7) to these new objects

$$i \frac{\delta W}{\delta J} = i \frac{\delta e^{i\mathcal{L}}}{\delta J} = - \frac{\delta \mathcal{L}}{\delta J} \cdot W = + W \cdot a$$

$$J = - \frac{\delta \Gamma}{\delta a} \quad \frac{i \delta W}{\delta K^b} = - W \frac{\delta \mathcal{L}}{\delta K^b} = - W \cdot c$$

Wait. Can we just convert (3.7) from W to $\mathcal{L}$?

$$[A + B + \dots] W = [A + B + \dots] e^{i\mathcal{L}}$$

If all terms were derivatives you would be OK. Do it with W anyway

$$\cancel{D_\mu^{ab}(-W_a)}$$

$$D_\mu^{ab} \left(i \frac{\delta}{\delta \bar{\psi}} \right) \left[\bar{\psi}(x) + n^\mu i \frac{\delta}{\delta K^b(x)} \right] W = 0$$

$$\Rightarrow D_\mu^{ab} \left(i \frac{\delta}{\delta \bar{\psi}} \right) \left[+ \frac{\delta P(a, c)}{\delta a_\mu^b(x)} W + n^\mu c^b(x) W \right] = 0$$

$$\Rightarrow D_\mu^{ab} (+W_a) \left[\frac{\delta P(a, c)}{\delta a_\mu^b(x)} + n^\mu c^b(x) \right] = 0$$

Let's repair signs

$$\mathcal{L}(\phi_i) = P(\phi_i) + \int dx [Kc - a\bar{\psi}]$$

$$\frac{\delta \mathcal{L}}{\delta \bar{\psi}} = -a \quad \left. \vphantom{\frac{\delta \mathcal{L}}{\delta \bar{\psi}}} \right\} = \phi_i \Rightarrow \phi_i = (-a, c)$$

$$\frac{\delta \mathcal{L}}{\delta K} = +c$$

Then go the other way:

$$P(\phi_i) = \mathcal{L}(\phi_i) - \int dx [Kc - a\bar{\psi}]$$

$$\frac{\delta P}{\delta c} = -K$$

$$\frac{\delta P}{\delta \phi} = -\bar{\psi}$$

$$\frac{\delta P}{\delta a} = +\bar{\psi}$$

So

$$\boxed{\phi_i = (-a^a, c^a)}$$

Signs repaired above.

Finally: $D_\mu^{ab} \left(i \frac{\delta}{\delta \bar{\psi}} \right) W = \dots$ start again:

I have arrived at this statement:

$$\bigcirc \quad D_{\mu}^{ab} \left(i \frac{\delta}{\delta y} \right) \left[\frac{\delta P(a, c)}{\delta q_{\mu}^b(x)} - n^{\mu} c^b(x) \right] [W] = 0$$

Write this out:

$$\left[\partial_{\mu} \delta^{ab} - t_{ab}^c \left(i \frac{\delta}{\delta q_{\mu}^c(x)} \right) \right] [W] = 0$$

$$= \delta^{ab} \partial_{\mu} [W] - t_{ab}^c [W] \cdot \left(i \frac{\delta W}{\delta q_{\mu}^c(x)} \right) + W a^c$$

$$= \delta^{ab} \partial_{\mu} [W] - t_{ab}^c [W] a^c$$

$$= \underbrace{\left[\delta^{ab} \partial_{\mu} - t_{ab}^c a^c \right]}_{D_{\mu}^{ab}(a)} \left[\frac{\delta P(a, c)}{\delta q_{\mu}^b(x)} - n^{\mu} c^b(x) \right] W = 0.$$

↑
now this is just
multiplication

$$\bigcirc \quad \text{Then} \quad \left[D_{\mu}^{ab}(a) \left[\frac{\delta P}{\delta q_{\mu}^b(x)} - n^{\mu} c^b(x) \right] = 0 \right] \quad (3.11) \checkmark$$

Write it out:

$$\left\{ \begin{aligned} & \delta^{ab} \partial_{\mu} \left[\frac{\delta P(a, c)}{\delta q_{\mu}^b(x)} \right] - \delta^{ab} (n \cdot \partial) [c^b(x)] \\ & - t_{ab}^c q_{\mu}^c \frac{\delta P}{\delta q_{\mu}^b(x)} + t_{ab}^c q_{\mu}^c n^{\mu} c^b(x) = 0. \end{aligned} \right\}$$

If we just set $a=c=0$, this says $\delta P / \delta a = 0$, obviously. Now do a c -derivative:

$$\cancel{\delta^{ab}} \partial_{\mu} \left[\frac{\delta^2 P(a, c)}{\delta q_{\mu}^b(x) \delta c^d(y)} \right] \Big|_0 - \cancel{\delta^{ab}} (n \cdot \partial) [\delta^{bd} \delta(x-y)] = 0$$

$$\bigcirc \Rightarrow \partial_{\mu} \left[\frac{\partial^c P(a, c)}{\delta q_{\mu}^a(x) \delta c^b(y)} \right] \Big|_0 = \delta^{ab} (n \cdot \partial) \delta(x-y)$$

$- \prod_{\mu}^{a,b}$

/go to top next page.

Now try $\partial/\partial a$ first:

$$\delta^{ab} \partial_\mu \left[\frac{\delta^2 \Gamma(a, c)}{\delta a_\mu^b(x) \delta a_\nu^c(y)} \right] = 0$$

$$\Rightarrow \partial_\mu \tilde{\Pi}_{\mu\nu}^{ab} = 0$$

Next we do two derivatives:

$$\delta^{ab} \partial_\mu \left[\frac{\delta^3 \Gamma(a, c)}{\delta a_\mu^b(x) \delta a_\lambda^c(y) \delta a_\sigma^d(z)} \right]$$

$$= t_{ab}^c \frac{\delta}{\delta a_\lambda^c(y)} \left[\frac{a_\mu^c(x) \delta^2 \Gamma}{\delta a_\mu^b(x) \delta a_\sigma^d(z)} + \delta^{dc} \delta_\sigma^\mu \delta(x-z) \frac{\delta \Gamma}{\delta a_\mu^b(x)} \right] = 0$$

$$- t_{ab}^c \left[\delta^{ce} \delta_\lambda^\mu \delta(x-y) \frac{\delta^2 \Gamma}{\delta a_\mu^b(x) \delta a_\sigma^d(z)} + \right.$$

$$\left. + \delta^{dc} \delta_\sigma^\mu \delta(x-z) \frac{\delta^2 \Gamma}{\delta a_\lambda^c(y) \delta a_\mu^b(x)} \right]$$

Now take $\begin{smallmatrix} bcd \\ \mu \lambda \sigma \end{smallmatrix} \rightarrow \begin{smallmatrix} abc \\ \mu \nu \rho \end{smallmatrix}$ to get, no, rewrite first as is:

$$- \partial_\mu \tilde{\Pi}_{\mu\lambda\sigma}^{aed}(x, y, z) - t_{ab}^c \left[\delta^{ce} \delta_\lambda^\mu \delta(x-y) \tilde{\Pi}_{\mu\sigma}^{bd}(x, z) + \delta^{dc} \delta_\sigma^\mu \delta(x-z) \tilde{\Pi}_{\lambda\mu}^{eb}(y, x) \right] = 0$$

$$\Rightarrow \partial_\mu \tilde{\Pi}_{\mu\lambda\sigma}^{aed}(x, y, z) = - t_{ab}^e \delta(x-y) \tilde{\Pi}_{\lambda\sigma}^{bd}(x, z) - t_{ab}^d \delta(x-z) \tilde{\Pi}_{\lambda\sigma}^{db}(y, x)$$

$$\Rightarrow \left[\partial_\mu \tilde{\Pi}_{\mu\nu\rho}^{abc}(x, y, z) = - t_{ad}^e \left[\delta_{eb} \delta(x-y) \tilde{\Pi}_{\nu\rho}^{dc}(x, z) + \delta_{ce} \delta(x-z) \tilde{\Pi}_{\nu\rho}^{bd}(y, x) \right] \right] \quad (3.15)$$

Agrees (3.15) except he produces a g ?!

Shu (3.11):

From earlier section, $W = \int [dA] F dS \cdot \vec{C} = \int dx [\mathcal{L}(A) + (n^\mu A_\mu) C + K C - \vec{\mathcal{D}}^\mu A_\mu]$

ie, $\left[S_{eff} = \int dx [\mathcal{L}(A) + (n^\mu \vec{A}_\mu) \cdot \vec{C} + K \cdot \vec{C} - \vec{\mathcal{D}}^\mu \vec{A}_\mu] \right]$

When you take $\vec{\delta A}_\mu = \vec{D}_\mu(\delta\omega)$, what happens?

$$\delta S = n^\mu D_\mu \delta\omega \cdot C - \vec{\mathcal{D}}^\mu D_\mu \delta\omega = [n^\mu C - \vec{\mathcal{D}}^\mu] D_\mu(\delta\omega)$$

Because $\delta \mathcal{L}(A) = \delta(F_{\mu\nu} F^{\mu\nu}) = 0$. Thus, if you insist that the complete effective action be a gauge scalar, you get

$$\int dx [n^\mu C(x) - \vec{\mathcal{D}}^\mu(x)] (\partial_\mu - g \epsilon A_\mu) [\delta\omega] = 0$$

Put gauge indices back in:

$$0 = \int d^4x [n^\mu C^a(x) - \vec{\mathcal{D}}^{\mu,a}(x)] (\partial_\mu \delta^{ab} - g \epsilon^{abc} A_\mu^c) [\delta\omega^b]$$

Those parts as $\partial_\mu \rightarrow -\partial_\mu$ and

$$0 = \int d^4x [-\partial_\mu \delta^{ab} - g \epsilon^{abc} A_\mu^c] [n^\mu C^a - \vec{\mathcal{D}}^{\mu,a}] \times (\delta\omega)^a = 0$$

Thus, $(-\partial_\mu \delta^{ab} - g \epsilon^{abc} A_\mu^c) (n^\mu C^a - \vec{\mathcal{D}}^{\mu,a}) = 0$

$$\Rightarrow (-\partial_\mu \delta^{ba} + g \epsilon^{bac} A_\mu^c) (n^\mu C^a - \vec{\mathcal{D}}^{\mu,a}) = 0$$

$$\Rightarrow D_\mu^{ba} [n^\mu C^a - \vec{\mathcal{D}}^{\mu,a}] = 0 \Rightarrow D_\mu^{ab} [n^\mu C^b - \vec{\mathcal{D}}^{\mu,b}] = 0$$

So, once again: $D_\mu^{ab} [\eta_\mu^b - \tilde{J}_\mu^b] = 0$

Now, (3.5) says $-\tilde{J}_\mu^b = \frac{\delta \Gamma}{\delta \phi_\mu^b} = \frac{\delta \Gamma}{\delta q_\mu^b}$

$$S \quad \boxed{D_\mu^{ab} \left[\frac{\delta \Gamma}{\delta q_\mu^b} + \eta_\mu^b \right] = 0}$$

Off by a minus sign! Repair later if needed

Show (3.12). Accept (3.11) Note that

$$D_\mu^{ab} = (\partial_\mu \delta^{ab} - g \epsilon^{abc} A_c) = \text{unrep of } C \text{ field}$$

So do $\delta/\delta C^e$ on (3.11):

$$D_\mu^{ab} \left[\frac{\delta \Gamma}{\delta C^e \delta q_\mu^b} - \eta_\mu^b \delta^{eb} \delta(x-y) \right] = 0$$

Now set $a=c=0$, so $D_\mu = \partial_\mu$. Then, from (3.9)

$$\partial_\mu \delta^{ab} \left[\tilde{\Pi}_{b\mu}^a = \eta_\mu^a \delta^{ab} \delta(x-y) \right] = 0$$

$$\text{or } \boxed{\partial_\mu \tilde{\Pi}_{b\mu}^a = \cancel{\partial_\mu \eta_\mu^a} (\eta_\mu^a) \delta^{ab} \delta(x-y)}$$

But (3.15) shows extra sign

Note: $\eta_\mu = \text{spacetime constant}$

Kummer Section 4

$$\Delta'_{\mu\nu} = -\delta^{ab} \frac{1}{k^2} \left[g_{\mu\nu} + \frac{n^2 k_\mu k_\nu}{(k \cdot n)^2} - \frac{(n_\mu k_\nu + n_\nu k_\mu)}{(k \cdot n)} \right]$$

= bare propagator.

What is the most general form for the full propagator?

$$\eta_\mu \Delta_{\mu\nu}(P) = 0.$$

What tensors can you make using η_μ and p_μ ? $g_{\mu\nu}$.

Let $N_\mu \equiv \eta_\mu - p_\mu \frac{n^2}{p \cdot n}$ so that $n \cdot N = 0$.

Then what? Well, use N_μ to replace p_μ . Then

$$\Delta_{\mu\nu} = A g_{\mu\nu} + B \eta_\mu \eta_\nu + C N_\mu N_\nu + D (\eta_\mu N_\nu + n_\nu N_\mu)$$

Then $\eta_\mu \Delta_{\mu\nu} = 0 \Rightarrow$

$$A n_\nu + B n^2 n_\nu + D n^2 N_\nu = 0 \Rightarrow D = 0 \text{ and } A + B n^2 = 0$$

Then

$$\Delta_{\mu\nu} = A g_{\mu\nu} - A \frac{\eta_\mu \eta_\nu}{n^2} + C N_\mu N_\nu$$

Again:

$$\left[\Delta_{\mu\nu}(k) = A \left(g_{\mu\nu} - \frac{\eta_\mu \eta_\nu}{n^2} \right) + C N_\mu N_\nu \right]$$

This by the way is what Ball & Zach use, two are false. By similar argument, we know that

$$k_\mu \Pi_{\mu\nu} = 0$$

So replace set k_μ, η_μ with K_μ, n_μ where $K_\mu \equiv k_\mu - \frac{\eta_\mu k^2}{k \cdot n}$ so $n \cdot K = 0$, and $k \cdot K = 0$. Then

$$\left[\Pi_{\mu\nu}(k) = A \left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right) + B K_\mu K_\nu \right]$$

What are the invariants?

$$k_\mu, \eta_\mu \Rightarrow k^2, n^2, n \cdot k$$

Comment: If you take $(n \rightarrow 2n)$ nothing can change. This just changes scale of generator W . Therefore

$f(n^2, k^2, n \cdot k)$ is too general.

ie $f(n^2, k^2, n \cdot k) = f(4n^2, k^2, 2n \cdot k)$ we know.

So $f(n^2, k^2, n \cdot k) = f(\alpha^2 n^2, k^2, \alpha n \cdot k)$

$= F(k^2, \frac{(n \cdot k)^2}{n^2})$, only two variables really.

$= G(\frac{n^2 k^2}{(n \cdot k)^2}, k^2)$ if you like

$= \underline{G(\gamma_k, k^2)}$

What is the C-A prop?

$$\tilde{\Delta}_{\mu}^{ab}(x-y) = \delta^{ab} (n \cdot \partial)^{-1} \partial_{\mu} \delta(x-y) \quad (2.10)$$

or $\Delta_{\mu}(x-y) = (n \cdot \partial)^{-1} \partial_{\mu} \delta(x-y)$

Comment this to p-space:

$$= \frac{1}{(2\pi)^4} \int d^4 k e^{ikx} \Delta_{\mu}(k) = (n \cdot \partial)^{-1} \partial_{\mu} \cdot \int d^4 k e^{ikx} \frac{1}{(2\pi)^4}$$

$$\Rightarrow \Delta_{\mu}(k) = (n \cdot ik)^{-1} (ik)_{\mu} = \frac{k_{\mu}}{(n \cdot k)}$$

Now we are ready to look again at (3.1)

$$\tilde{\Delta}_{\mu\nu} \tilde{T}_{\nu\lambda} + \tilde{\Delta}_{\mu} \tilde{T}_{\lambda} = \delta_{\mu\lambda} \quad / \text{summe all cols.}$$

So $-\Delta_{\mu\nu} T_{\nu\lambda} - \frac{p_{\mu}}{p \cdot n} T^{\lambda} = -\delta_{\mu\lambda}$

Apply n_{μ} :

$$0 - T^{\lambda} = -n^{\lambda}$$

$$\Rightarrow \left[\Delta_{\mu\nu} T_{\nu\lambda} = \left(\delta_{\mu\lambda} - \frac{p_{\mu} n_{\lambda}}{p \cdot n} \right) \right]$$

↖ Bullloch 2.4

so

$$+\Delta_{\mu\nu} T_{\nu\lambda} + \frac{p_{\mu} n^{\lambda}}{p \cdot n} = +\delta_{\mu\lambda} \quad \text{Kummer (4.5) + (4.7)}$$

Sec 4.

A careful study of Kummer tensors.

$$R_{\mu\nu} \equiv -\frac{N_{\mu}N_{\nu}}{n^2(1-r)}$$

$$N^2 = n^2(1-r)$$

$$R_{\mu\lambda}R_{\lambda\nu} = \frac{N_{\mu}N_{\nu}N^2}{n^4(1-r)^2} = -\frac{N_{\mu}N_{\nu}}{n^2(1-r)} = R_{\mu\nu} \quad [\text{he says } R^2 = R \checkmark]$$

$$\begin{aligned} \text{By one way, } N_{\mu}N_{\nu} &= \left(n_{\mu} - q_{\mu} \frac{n^2}{q \cdot n}\right) \left(n_{\nu} - q_{\nu} \frac{n^2}{q \cdot n}\right) \\ &= n_{\mu}n_{\nu} - [q_{\mu}n_{\nu} + q_{\nu}n_{\mu}] \frac{n^2}{q \cdot n} + q_{\mu}q_{\nu} \frac{n^4}{(q \cdot n)^2} \end{aligned}$$

So it seems that

$$S_{\mu\nu} = g_{\mu\nu} + \frac{1}{(1-r)} \left[\frac{n_{\mu}n_{\nu}}{n^2}(1-r) + \frac{N_{\mu}N_{\nu}}{n^2} \right] = g_{\mu\nu} - R_{\mu\nu} - \frac{n_{\mu}n_{\nu}}{n^2}$$

$$\text{Then } S_{\mu\lambda}R_{\lambda\nu} = [g_{\mu\lambda} - R_{\mu\lambda} - \frac{n_{\mu}n_{\lambda}}{n^2}] [-\frac{N_{\lambda}N_{\nu}}{n^2(1-r)}]$$

$$= \left[-\frac{1}{n^2(1-r)} \right] (N_{\mu}N_{\nu}) \text{ stop}$$

$$\begin{aligned} S_{\mu\lambda}R_{\lambda\nu} &= [g_{\mu\lambda} - R_{\mu\lambda} - \frac{n_{\mu}n_{\lambda}}{n^2}] R_{\lambda\nu} \quad \text{Then } SR = 0 = RS. \\ &= R_{\mu\nu} - R_{\mu\nu} - 0 = 0 \checkmark. \end{aligned}$$

$$\begin{aligned} \text{Next: } S_{\mu\lambda}S_{\lambda\nu} &= [g_{\mu\lambda} - R_{\mu\lambda} - \frac{n_{\mu}n_{\lambda}}{n^2}] [g_{\lambda\nu} - R_{\lambda\nu} - \frac{n_{\lambda}n_{\nu}}{n^2}] \\ &= g_{\mu\nu} - R_{\mu\nu} - \frac{n_{\mu}n_{\nu}}{n^2} - \cancel{R_{\mu\nu}} + \cancel{R_{\mu\nu}} + \cancel{R_{\mu\nu}} \\ &\quad - \cancel{n_{\mu}n_{\nu}/n^2} + 0 + \cancel{n_{\mu}n_{\nu}/n^2} = S_{\mu\nu}. \end{aligned}$$

Summary:

$$\begin{aligned} n_{\mu}R_{\mu\nu} &= 0 \\ n_{\mu}S_{\mu\nu} &= 0 \end{aligned}$$

$$R_{\mu\nu} \equiv -\frac{N_{\mu}N_{\nu}}{n^2(1-r)}$$

$$S_{\mu\nu} \equiv [g_{\mu\nu} - R_{\mu\nu} - \frac{n_{\mu}n_{\nu}}{n^2}]$$

$$R^2 = R$$

$$S^2 = S$$

$$RS = SR = 0$$

$$n_{\mu}R_{\mu\nu} = 0$$

$$n_{\mu}S_{\mu\nu} = 0$$

Next: $q_{\mu} R_{\mu\nu} = \left(\frac{1}{n^2(r-1)} \right) q_{\mu} N_{\mu} N_{\nu} = \frac{(q \cdot N)}{(N \cdot N)} N_{\nu}$

Data:

$$n \cdot Q = N \cdot q = Q \cdot N = (n \cdot q)(1-r)$$

$$N \cdot N = n^2(r-1)$$

$$Q \cdot Q = q^2(r-1)$$

So $q_{\mu} R_{\mu\nu} = \frac{(n \cdot q)(1-r) N_{\nu}}{n^2(r-1)} = -\frac{(n \cdot q)}{n^2} N_{\nu}$

Then $q_{\mu} S_{\mu\nu} = q_{\mu} [q_{\mu\nu} - R_{\mu\nu} - n_{\mu} n_{\nu} / n^2] = q_{\nu} + \frac{(n \cdot q)}{n^2} N_{\nu} - \frac{(n \cdot q)}{n^2} n_{\nu}$
 $= \cancel{q_{\nu}} - \cancel{n_{\nu}} \frac{(n \cdot q)}{n^2} + \frac{(n \cdot q)}{n^2} [\cancel{n_{\nu}} - \cancel{q_{\nu}} \frac{n^2}{q \cdot n}] = 0 !$

Next: Let $T_{\mu\nu} = \frac{+1}{q^2(r-1)} Q_{\mu} Q_{\nu} = \frac{Q_{\mu} Q_{\nu}}{Q^2}$

Then obviously $q_{\mu} T_{\mu\nu} = 0$.

Next: $T_{\mu\nu} T_{\nu\lambda} = \frac{Q_{\mu} Q_{\nu}}{Q^2} \cdot \frac{Q_{\nu} Q_{\lambda}}{Q^2} = T_{\mu\lambda} \checkmark$

$$T_{\mu} S_{\mu\nu} = T_{\mu} [q_{\mu\nu} - R_{\mu\nu} - n_{\mu} n_{\nu} / n^2]$$

$$= T_{\mu\nu} - T_{\mu} R_{\mu\nu} - T_{\mu} n_{\mu} n_{\nu} / n^2$$

Now, $T_{\mu} R_{\mu\nu} = \frac{Q_{\mu} Q_{\lambda}}{Q^2} \cdot \frac{N_{\lambda} N_{\nu}}{N^2} = \frac{Q_{\mu} N_{\nu} (Q \cdot N)}{Q^2 N^2}$

$$T_{\mu} n_{\mu} = \frac{Q_{\mu} Q_{\lambda}}{Q^2} n_{\lambda} = \frac{Q_{\mu} (Q \cdot n)}{Q^2}$$

So $T_{\mu} S_{\mu\nu} = \frac{Q_{\mu} Q_{\nu}}{Q^2} - \frac{(Q \cdot N)}{Q^2 N^2} Q_{\mu} N_{\nu} - \frac{Q_{\mu} n_{\nu} (Q \cdot n)}{Q^2 n^2}$

$$= \frac{Q_\mu}{Q^2} \left[Q_\nu - \frac{(n \cdot q)(1-\epsilon) N_\nu}{N^2} - \frac{(n \cdot q)(1-\epsilon) n_\nu}{n^2} \right]$$

$$= \frac{Q_\mu}{Q^2} \left[\left(q_\nu - n_\nu \frac{q^2}{q \cdot n} \right) + \frac{(n \cdot q)(1-\epsilon)}{n^2 (1-\epsilon)} \left(n_\nu - q_\nu \frac{n^2}{q \cdot n} \right) - \frac{(n \cdot q)(1-\epsilon) n_\nu}{n^2} \right]$$

$$= \frac{Q_\mu}{Q^2} \left[q_\nu \left\{ 1 - \frac{1}{1} \right\} + n_\nu \left\{ -\frac{q^2}{q \cdot n} + \frac{n \cdot q}{n^2} - \frac{(n \cdot q)}{n^2} \left(1 - \frac{n^2}{(q \cdot n)^2} \right) \right\} \right] = 0$$

Thus $T_{\mu\nu} S_{\mu\nu} = 0 = S_{\mu\nu} T_{\mu\nu}$ and $TS = ST = 0$.

Summary

$$\begin{aligned} T_{\mu\nu} &\equiv \frac{Q_\mu Q_\nu}{Q^2} & q_\mu T_{\mu\nu} &= 0 & q_\mu S_{\mu\nu} &= 0 \\ T^2 &= T & TS &= ST = 0 \end{aligned}$$

So how to summarize this

$$R_{\mu\nu} \equiv \frac{N_\mu N_\nu}{N^2} \quad T_{\mu\nu} \equiv \frac{Q_\mu Q_\nu}{Q^2} \quad S_{\mu\nu} \equiv \left[g_{\mu\nu} - R_{\mu\nu} - \frac{n_\mu n_\nu}{n^2} \right]$$

$$\begin{aligned} n \cdot R &= 0 & q \cdot T &= 0 \\ n \cdot S &= 0 & q \cdot S &= 0 \end{aligned}$$

$$R^2 = R \quad T^2 = T \quad S^2 = S$$

$$R \perp S \quad T \perp S$$

Theorem: There are only three symmetric tensors you can make from q^μ and n^μ . Take them to be R, T, S .

Now let $\Delta_{\mu\nu} = AR_{\mu\nu} + BS_{\mu\nu}$ and $\Delta = AR + BS$
 $\Pi = aT + bS$

Then $\Delta\Pi = (AR + BS)(aT + bS) = AaRT + BbS$

see →
2 pgs
for more

$$(\Delta T)_{\mu\nu} = (g_{\mu\nu} - \frac{g_{\mu\nu} n^2}{n^2 q^2}) \text{ is required.}$$

$$\begin{aligned} (\Delta T)_{\mu\nu} &= Aa (RT)_{\mu\nu} + Bb S_{\mu\nu} \\ &= Aa (R_{\mu\lambda} T_{\lambda\nu}) + Bb S_{\mu\nu} \end{aligned}$$

$$\rightarrow \frac{N_\mu N_\lambda}{N^2} \cdot \frac{Q_\lambda Q_\nu}{Q^2} = N_\mu Q_\nu \frac{(N \cdot Q)}{N^2 Q^2}$$

So

$$(\Delta T)_{\mu\nu} = \frac{Aa (N \cdot Q)}{N^2 Q^2} N_\mu Q_\nu + Bb \left[-g_{\mu\nu} - \frac{N_\mu N_\nu}{N^2} - \frac{\eta_\mu \eta_\nu}{n^2} \right] = g_{\mu\nu} - \frac{g_{\mu\nu} n^2}{n^2}$$

First set $Bb=1$ so $g_{\mu\nu}$ matches. Then by $Aa=r$.

$$\frac{r (N \cdot Q)}{N^2 Q^2} N_\mu Q_\nu - \frac{N_\mu N_\nu}{N^2} - \frac{\eta_\mu \eta_\nu}{n^2} = - \frac{g_{\mu\nu} n^2}{n^2 q^2} ?$$

$$r \frac{(n \cdot q)}{n^2 (q-1)} \frac{N_\mu Q_\nu}{q^2 (r-1)} \frac{(r-1)}{(r-1)}$$

$$\text{so } \frac{-r(n \cdot q)}{n^2 q^2 (r-1)} N_\mu Q_\nu - \frac{N_\mu N_\nu}{n^2 (r-1)} = \frac{(\eta_\mu \eta_\nu - g_{\mu\nu} n^2)(r-1)}{n^2 q^2}$$

$$\frac{r(n \cdot q)}{q^2} N_\mu Q_\nu + N_\mu N_\nu = (r-1) \left(\frac{q^2 n^2}{q \cdot n} - \eta_\mu \eta_\nu \right) ?$$

$$N_\mu \left[\frac{r(n \cdot q)}{q^2} (q_\nu - n_\nu \frac{q^2}{q \cdot n}) + (n_\nu - q_\nu \frac{n^2}{q \cdot n}) \right]$$

$$= N_\mu \left[q_\nu \left\{ \frac{r(n \cdot q)}{q^2} - \frac{n^2}{(q \cdot n)} \right\} + n_\nu \{ 1 - r \} \right] = N_\mu n_\nu (1-r)$$

$$\frac{\left\{ \frac{r^2 n^2 (n \cdot q)}{(q \cdot n)^2} - \frac{n^2}{q \cdot n} \right\}}{0}$$

$$= \left[n_\mu - q_\mu \frac{n^2}{q \cdot n} \right] n_\nu (1-r)$$

$$= \left[n_\mu n_\nu - \frac{n^2}{q \cdot n} g_{\mu\nu} \right] (1-r)$$

Therefore

$$Bb=1 \quad Aa=r$$

Compare Ball Zech to Kummer. on A, B etc.

Kummer says:

$$\Delta_{uv} = A R_{uv} + B S_{uv} = B [g_{uv} - R_{uv} - \eta_{\mu\nu}/n^2] + A R_{uv}$$

$$= B (g_{uv} - \eta_{\mu\nu}/n^2) + (A-B) R_{uv}$$

$$= [-Bq^2] \left(\frac{g_{uv} - \eta_{\mu\nu}/n^2}{-q^2} \right) + (-q^2) \frac{(A-B)}{(r-1)} \frac{N_{\mu} N_{\nu}}{n^2 (-q^2)}$$

$$= -\left(\frac{1}{q^2}\right) \left[\underbrace{\left(g_{uv} - \frac{\eta_{\mu\nu}}{n^2} \right) [-Bq^2]}_B + \underbrace{\frac{N_{\mu} N_{\nu}}{n^2} \left[\frac{(-q^2)(A-B)}{r-1} \right]}_G \right]$$

So $\left[\overset{\nwarrow BE}{B} = -q^2 B \quad \overset{\nwarrow BE}{G} = \frac{(-q^2)(B-A)}{(1-r)} \right]$

Kummer says:

$$\Pi_{uv} = a T_{uv} + b S_{uv} = \frac{a Q_{\mu} Q_{\nu}}{Q^2} + b \left[g_{uv} - R_{uv} - \frac{\eta_{\mu\nu}}{n^2} \right]$$

Can we write S in terms of T .

$$R_{uv} + \frac{\eta_{\mu\nu}}{n^2} = \frac{N_{\mu} N_{\nu}}{N^2} + \frac{\eta_{\mu\nu}}{n^2}$$

$$= \frac{(\eta_{\mu} - q_{\mu} \frac{n^2}{q \cdot n}) (n_{\nu} - q_{\nu} \frac{n^2}{q \cdot n})}{n^2 (r-1)} + \eta_{\mu\nu}/n^2$$

$$= \frac{\eta_{\mu\nu}}{n^2} \left[1 + \frac{1}{r-1} \right] + [q_{\mu} n_{\nu} + q_{\nu} n_{\mu}] \frac{1}{(q \cdot n)(r-1)}$$

Simple idea. Two basis vectors are q^{μ}, n^{μ} . Define "switch" in observing.

quant		new q
q^{μ}	$q \leftrightarrow n$	n^{μ}
n^{μ}		q^{μ}
Q^{μ}		N^{μ}
δ		δ
R_{uv}		T_{uv}
S_{uv}		S_{uv}

What happens to $S_{\mu\nu}$

$$S_{\mu\nu} \rightarrow g_{\mu\nu} + (1-\alpha)^{-1} \left[\alpha \frac{p_\mu p_\nu}{p^2} + \frac{n_\mu n_\nu p^2 n^2}{(p \cdot n)^2 n^2} - (p_\mu n_\nu + p_\nu n_\mu)/(p \cdot n) \right]$$

$$= g_{\mu\nu} + (1-\alpha)^{-1} \left[\frac{n_\mu n_\nu \alpha}{n^2} + \frac{n^2}{(p \cdot n)^2} p_\mu p_\nu + \text{etc} \right]$$

So $S_{\mu\nu} \rightarrow S_{\mu\nu}$!

Thus

$$S_{\mu\nu} = g_{\mu\nu} - R_{\mu\nu} - n_\mu n_\nu / n^2$$

$$\rightarrow g_{\mu\nu} - T_{\mu\nu} - q_\mu q_\nu / q^2 = S_{\mu\nu} \text{ also!}$$

$$\begin{aligned} S_{\mu\nu} &= g_{\mu\nu} - R_{\mu\nu} - n_\mu n_\nu / n^2 \\ &= g_{\mu\nu} - T_{\mu\nu} - q_\mu q_\nu / q^2 \end{aligned}$$

So we now have

Kummer says : $T_{\mu\nu} = a T_{\mu\nu} + b S_{\mu\nu}$

$$= a T_{\mu\nu} + b [g_{\mu\nu} - T_{\mu\nu} - q_\mu q_\nu / q^2]$$

$$= b \left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \right) + (a-b) T_{\mu\nu}$$

$$= (-q^2) \left[\frac{b}{(-q^2)} \right] \left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \right) + \frac{(a-b)}{q^2(1-\alpha)} q_\mu q_\nu$$

$$= (-q^2) \left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \right) \underbrace{\left[\frac{b}{-q^2} \right]}_{\beta} + \underbrace{\left[\frac{(b-a)}{(-q^2)(1-\alpha)} \right]}_{\alpha} q_\mu q_\nu$$

$$\text{So } \left[\beta = -b/q^2 \quad \alpha = -\frac{(b-a)}{(1-\alpha)} \frac{1}{q^2} \right]$$

Summary: let primes denote BD quantities

$$B' = (-q^2)B$$

$$b' = b/(-q^2)$$

$$C' = (-q^2) \left[\frac{B-A}{1-\delta} \right]$$

$$c' = \frac{(b-a)/(-q^2)}{(1-\delta)}$$

Now use Kummer rule to get

$$B'b' = Bb = 1$$

$$C'c' =$$

Kummer, Sec 4. could [starting page 323].

$\Delta_{\mu\nu} = A R_{\mu\nu} + B S_{\mu\nu} =$ the full gluon propagator

$\Pi_{\mu\nu} =$ the full inverse prop in sense 1) $(\Delta')_{\mu\alpha}(\Pi)_{\alpha\nu} = (g_{\mu\nu} - \frac{q_\mu q_\nu}{q \cdot n})$

$\Delta_{\mu\nu} =$ the bare propagator with $\beta=0$.

Examine this again:

$$\Delta_{\mu\nu} = -\left(\frac{1}{k^2}\right) \left[g_{\mu\nu} + k_\mu k_\nu \frac{n^2}{(k \cdot n)^2} - \frac{[\eta_\mu k_\nu + \eta_\nu k_\mu]}{(k \cdot n)} \right]$$

Note that

$$\eta_\mu \Delta_{\mu\nu} = \left(-\frac{1}{k^2}\right) \left[\cancel{\eta_\nu} + k_\nu \frac{\cancel{\eta^2}}{k \cdot n} - \cancel{\eta^\mu} \frac{k_\mu}{k \cdot n} - \cancel{\eta_\nu} \right] = 0$$

Thus you must be able to write this $\Delta_{\mu\nu}$ in a simpler way:

$$\Delta_{\mu\nu} = A^\circ R_{\mu\nu} + B^\circ S_{\mu\nu}$$

Conjecture: $\Delta_{\mu\nu} = -\left(\frac{1}{k^2}\right) \left[\left(\delta_{\mu\nu} - \frac{\eta_\mu \eta_\nu}{n^2} \right) + \frac{\eta_\mu \eta_\nu}{n^2} \right]$ ☑

Check: $\eta_\mu \eta_\nu = (\eta_\mu - q_\mu \frac{n^2}{q \cdot n})(\eta_\nu - q_\nu \frac{n^2}{q \cdot n}) = \eta_\mu \eta_\nu + q_\mu q_\nu \frac{n^4}{(q \cdot n)^2} - \frac{n^2}{q \cdot n} [\eta_\mu q_\nu + q_\mu \eta_\nu]$

Then

$$[] = g_{\mu\nu} - \frac{\eta_\mu \eta_\nu}{n^2} + \frac{\eta_\mu \eta_\nu}{n^2} + \frac{q_\mu q_\nu n^2}{(q \cdot n)^2} - \frac{1}{q \cdot n} [k_\mu k_\nu + \dots]$$

Now solve for A° and B° :

$$\Delta_{\mu\nu} = A^\circ R_{\mu\nu} + B^\circ \left[g_{\mu\nu} - R_{\mu\nu} - \eta_\mu \eta_\nu / n^2 \right]$$

$$= (A^\circ - B^\circ) \frac{\eta_\mu \eta_\nu}{n^2 (\gamma - 1)} + B^\circ \left(g_{\mu\nu} - \frac{\eta_\mu \eta_\nu}{n^2} \right)$$

$\Rightarrow$

$$B^\circ = 1/(-k^2) \quad \frac{A^\circ - B^\circ}{\gamma - 1} = 1/(-k^2) = B^\circ \Rightarrow A^\circ - B^\circ = B^\circ (\gamma - 1)$$

$$\Rightarrow A^\circ = \gamma B^\circ$$

Thus $\Delta_{\mu\nu} = \left(-\frac{1}{k^2}\right) [\gamma R_{\mu\nu} + S_{\mu\nu}]$ (4.13) ✓

Summarize

$$\Delta_{\mu\nu} = \left[g_{\mu\nu} + k_\mu k_\nu \frac{n^2}{(k \cdot n)^2} - \frac{[k_\mu n_\nu + k_\nu n_\mu]}{(k \cdot n)} \right] / (-k^2)$$

$$\Delta_{\mu\nu} = \left[\left(g_{\mu\nu} - \frac{n_\mu n_\nu}{n^2} \right) + \frac{N_\mu N_\nu}{n^2} \right] / (-k^2)$$

$$\Delta_{\mu\nu} = [\delta R_{\mu\nu} + S_{\mu\nu}] / (-k^2)$$

Base propagator

$$n \cdot \Delta = 0$$

Now define a new object $\bar{\Pi}$ by

$$\Delta' (1 + \bar{\Pi} \Delta) = \Delta \quad \text{so } \bar{\Pi} \text{ is related to self energy!}$$

Solve for $\bar{\Pi}$:

$$\Pi_{\alpha\mu} (\Delta')_{\mu\lambda} [\delta_{\lambda\nu} + [\bar{\Pi} \Delta]_{\lambda\nu}] = \Delta_{\mu\nu} \cdot \Pi_{\alpha\mu}$$

$$\{ = \Delta'_{\lambda\mu} \Pi_{\mu\alpha} = \left(\delta_{\lambda\alpha} - \frac{k_\lambda n_\alpha}{k \cdot n} \right)$$

$$\text{So } \left[\delta_{\lambda\alpha} - \frac{k_\lambda n_\alpha}{k \cdot n} \right] [\delta_{\lambda\nu} + [\bar{\Pi} \Delta]_{\lambda\nu}] = \Delta_{\mu\nu} \Pi_{\alpha\mu}$$

$$\Rightarrow \delta_{\lambda\alpha} - \frac{k_\lambda n_\alpha}{k \cdot n} + [\bar{\Pi} \Delta]_{\alpha\nu} - \frac{k_\lambda n_\alpha}{k \cdot n} [\bar{\Pi}_{\lambda\mu} \Delta_{\mu\nu}] = \Delta_{\mu\nu} \Pi_{\alpha\mu}$$

Come back to this later.

b) Three-Vertex: Go back to (3.15) and transform: how do you Fourier this thing?

$$2i \tilde{\Pi}_{\mu\nu\lambda}^{abc}(x, y, z) = \frac{1}{(2\pi)^{12}} \int d^4p d^4k d^4q e^{ip \cdot x} e^{ik \cdot y} e^{iq \cdot z} (ip^\mu)_{\mu\nu\lambda}^{abc}(p, k, q)$$

$$\begin{aligned} (\delta(x-y) \tilde{\Pi}_{\nu\lambda}^{dc}(x, y, z)) &= \frac{1}{(2\pi)^{12}} \int d^4p d^4k d^4q e^{ip \cdot (x-y)} e^{ik \cdot x} e^{iq \cdot z} \tilde{\Pi}_{\nu\lambda}^{dc}(k, q) \\ &= \frac{1}{(2\pi)^8} \int d^4p e^{ip \cdot (x-y)} \int d^4k e^{ik \cdot (x-z)} \tilde{\Pi}_{\nu\lambda}^{dc}(k) \end{aligned}$$

Try some more:

$$\tilde{\Gamma}_{\mu\nu g}^{abc}(x, y, z) = \tilde{\Gamma}(x-z, y-z, 0) = f(x-z, y-z)$$

$$= \frac{1}{(2\pi)^8} \int dp dq e^{ip(x-z)} e^{iq(y-z)} \hat{f}(p, q) \quad / \text{define } \hat{f}$$

$$\text{So } \partial_\mu \Gamma_\mu(x, y, z) = \frac{1}{(2\pi)^8} \int dp dq (ip_\mu) e^{ip(x-z)} e^{iq(y-z)} \hat{f}(p, q) \quad (1)$$

Next:

$$\delta(x-y) \tilde{\Pi}(x-z) = \frac{1}{(2\pi)^8} \int dp dq e^{ip(x-y)} e^{iq(x-z)} \Pi(q) \quad (2)$$

$$\delta(x-z) \Pi(y-x) = \frac{1}{(2\pi)^8} \int dp dq e^{ip(x-z)} e^{iq(y-x)} \Pi(q) \quad (3)$$

Now, in (2) let $p \leftrightarrow q$ do get (and $q \rightarrow -q$)

$$(2) = \frac{1}{(2\pi)^8} \int dp dq e^{ip(x-z)} e^{iq(y-x)} \Pi(p) \quad (y-x)$$

Now let $p \rightarrow p+q$. Then $ip(p+q)(x-z) + iq(y-x) = ip(x-z) + iq(y-z) = \text{desired br}$

$$\text{So } (2) = \frac{1}{(2\pi)^8} \int dp dq e^{ip(x-z)} e^{iq(y-z)} \Pi(p+q)$$

Next, do (3). $p \rightarrow p+q$.

$$(3) = \frac{1}{(2\pi)^8} \int dp dq e^{ip(x-z)} e^{iq(y-z)} \Pi(q)$$

Now read off: note $p+q+k=0$.

$$ip_\mu \hat{f}(p, q) = \Pi(p+q) + \Pi(q)$$

But note from general form of Π that $\Pi(k) = \Pi(-k)$ [reality in x], so we have

$$ip_\mu \Gamma_{\mu\nu g}^{abc}(p, q, k) = -g t_{ad}^c \delta_{eb} \Pi_{\nu g}^{dc}(k)$$

$$-g t_{ad}^c \delta_{ec} \Pi_{\nu g}^{bd}(q)$$

$$= -g \left[\underbrace{t_{ad}^b}_{cabd} \Pi_{\nu g}^{dc}(k) + \underbrace{t_{ad}^c}_{acdb} \Pi_{\nu g}^{bd}(q) \right]$$

But Π diagonal in color, so

$$i p_\mu \Gamma_{\mu \nu \rho}^{abc}(p, q, k) = -g \left[C^{abc} \Pi_{\nu \rho}(k) + C^{acb} \Pi_{\nu \rho}(q) \right]$$

$$\Rightarrow p_\mu \Gamma_{\mu \nu \rho}^{abc}(p, q, k) = ig C^{abc} \left[\Pi_{\nu \rho}(k) - \Pi_{\nu \rho}(q) \right]$$

$$\Rightarrow \boxed{p_\mu \Gamma_{\mu_1 \mu_2 \mu_3}^{a_1 a_2 a_3}(p_1, p_2, p_3) = -ig C^{a_1 a_2 a_3} \left[\Pi_{\mu_2 \mu_3}(p_2) - \Pi_{\mu_2 \mu_3}(p_3) \right]}$$

Comments: These things are all "full" objects. Thus, this Ward-type identity tells us the color dependence of the full vertex !!
The only fact you need is the color diagonality of Π , which follows from that of Δ' , just don't do it to see !!

Dyson Equation

$$\mu \text{---} \text{---} \nu = \mu \text{---} \text{---} \nu + \mu \text{---} \text{---} \nu$$

wrong, not the Dyson equation!!

This thing reads, in Kummer notation,

$$(\Delta'_{\mu\nu} \delta^{ab}) = (\Delta_{\mu\nu} \delta^{ab}) + \Delta_{\mu\alpha}^{(k)} \delta^{ad} C[iP^0](q, -k, -k') C^d$$

Restat: color is continue, no need for so many labels:

$$\mu \text{---} \text{---} \nu = \mu \text{---} \text{---} \nu + \mu \text{---} \text{---} \nu$$

$$\Delta'_{\mu\nu}(q) = \Delta_{\mu\nu}^0(q) + \Delta_{\mu\mu'}^0(q) \int dk P_{\mu'\nu'}^0(q, -k, -k') \Delta'_{\nu\nu'}(k) \Delta'_{\nu'\nu'}(k')$$

$$\sum_{bb'} C^{abb'} C^{abb'} = \text{some number, call it } C^2 \cdot S_0$$

$$\Delta_{\mu\nu}(q) = \Delta_{\mu\nu}^0(q) + \Delta_{\mu\mu'}^0(q) \left[\int dk P_{\mu'\nu'}^0(q, -k, -k') \Delta_{\nu\nu'}(k) \Delta_{\nu'\nu'}(k') \right. \\ \left. \times P_{\nu'\nu}(k, k', -q) \right] \Delta_{\nu\nu}^0(q) \cdot C^2$$

Now how do we pull off these extra free propagators?? Recall that free prop at $\beta=0$ is not invertible. Rather:

$$\Delta_{\mu\nu}(q) \Pi_{\mu\nu}^0(q) = \left(\delta_{\mu\nu} - \frac{q_\mu q_\nu}{q \cdot n} \right) = \Delta_{\mu\nu}^0(q) \Pi^0(q) \\ \equiv A_{\mu\nu}$$

then we get

$$\Pi^0 \Delta \Pi^0 = \Pi^0 \Delta^0 \Pi^0 + \Pi^0 \Delta^0 \left[\int dk P^0 \Delta \Gamma \right] \Delta^0 \Pi^0$$

$$\text{or } \Pi^0 \Delta \Pi^0 = \Pi^0 A + A \left[\int dk P^0 \Delta \Gamma \right] A$$

Is A invertible?

I need to invert $A_{\mu\nu} = (\delta_{\mu\nu} - \frac{q_\mu q_\nu}{q \cdot n})$.

General: $B_{\lambda\lambda} = A \cdot \delta_{\lambda\lambda} + b q_\mu n_\lambda + c q_\lambda n_\nu + d q_\mu q_\lambda + e n_\mu n_\lambda$

$$\begin{aligned} \text{So } \delta_{\mu\lambda} &= a \delta_{\mu\lambda} + b q_\mu n_\lambda + c n_\mu q_\lambda + d q_\mu q_\lambda + e n_\mu n_\lambda \\ &= a \delta_{\mu\lambda} + q_\mu n_\lambda \left[\cancel{b} - \frac{a}{q \cdot n} \cancel{b} + e \frac{n^2}{q \cdot n} \right] \\ &\quad + n_\mu q_\lambda [c] + q_\mu q_\lambda \left[\cancel{d} - c \frac{n^2}{q \cdot n} \cancel{d} \right] + n_\mu n_\lambda [e] \end{aligned}$$

Solution is:

$$\begin{aligned} (a=1, a=0) &\text{ inconsistent...} \\ e=0 &\checkmark \\ c=0 &\checkmark \end{aligned}$$

Now I see: $\det A_{\mu\nu} = 0$, so cannot invert A!! correct.

Idea: let's check

$$\begin{aligned} \Delta_{\mu\lambda} \Pi_{\lambda\nu} &= \left(-\frac{1}{q^2} \right) \left(\delta_{\mu\lambda} - \frac{n_\mu n_\lambda}{n^2} + \frac{N_\mu N_\lambda}{n^2} \right) \left(-q^2 \right) \left(\delta_{\lambda\nu} - \frac{q_\lambda q_\nu}{q^2} \right) \\ &= \left(\delta_{\mu\lambda} - \frac{n_\mu n_\lambda}{n^2} + \frac{N_\mu N_\lambda}{n^2} \right) \left(\delta_{\lambda\nu} - \frac{q_\lambda q_\nu}{q^2} \right) \end{aligned}$$

$$= \delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} - \frac{n_\mu n_\nu}{n^2} + \frac{n_\mu q_\nu (q \cdot n)}{n^2 q^2} + \frac{N_\mu N_\nu}{n^2} - \frac{N_\mu q_\nu (q \cdot N)}{n^2 q^2}$$

Do again

$$\Delta_{\mu\lambda} \Pi_{\lambda\nu} = \left[\delta_{\mu\lambda} + k_\mu k_\lambda \frac{n^2}{(k \cdot n)^2} - \frac{k_\mu n_\lambda}{k \cdot n} - \frac{k_\lambda n_\mu}{k \cdot n} \right] \left[\delta_{\lambda\nu} - \frac{q_\lambda q_\nu}{q^2} \right]$$

$$= \delta_{\mu\nu} + k_\mu k_\nu \frac{n^2}{(k \cdot n)^2} - \frac{k_\mu n_\nu}{k \cdot n} - \frac{k_\nu n_\mu}{k \cdot n} - \frac{q_\mu q_\nu}{q^2} - k_\mu k_\nu \frac{n^2}{(k \cdot n)^2}$$

$$+ \frac{k_\mu k_\nu}{k^2} + \frac{n_\mu k_\nu}{k \cdot n}$$

$$= \delta_{\mu\nu} - \frac{k_\mu n_\nu}{k \cdot n} \checkmark$$

Summary:

$$\Delta_{\mu\lambda}^0 \Pi_{\lambda\nu}^0 = \left(\delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \right) \quad \Delta^0 \text{ not invertible}$$

where

$$\Delta_{\mu\lambda}^0 = \left[\delta_{\mu\lambda} - \frac{q_\mu q_\lambda}{q^2} + \frac{N_\mu N_\lambda}{n^2} \right] / (-q^2) = \text{see earlier}$$

$$\Pi_{\lambda\nu}^0 = \left[\delta_{\lambda\nu} - \frac{q_\lambda q_\nu}{q^2} \right] / (-q^2)$$

Return to:

$$[\Pi^0 \Delta \Pi^0]_{\mu\nu} = [\Pi^0 A]_{\mu\nu} + A_{\mu\lambda} \left\{ \sum_{\lambda\lambda'} A_{\lambda\lambda'} \right\} \quad A_{\mu\nu} = \delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}$$

Dot with N_ν on the right:

$$[\Pi^0 \Delta \Pi^0]_{\mu\nu} N_\nu = [\Pi^0]_{\mu\lambda} \underbrace{A_{\lambda\nu} N_\nu}_{N_\lambda} + A_{\mu\lambda} \left\{ \sum_{\lambda\lambda'} A_{\lambda\lambda'} \right\} N_\nu$$

stucked again! No break.

$$\Delta = \Delta^0 + \Delta^0 [] \Delta^0 = \Delta^0 [1 + \Delta^0 []] \Delta^0 = \Delta^0 [1 + [] \Delta^0]$$

Kummer says do this: $\Delta + \Delta \bar{\Pi} \Delta_0 = \Delta^0$

So that $\Delta - \Delta^0 = -\Delta \bar{\Pi} \Delta_0$ Then I would get:

$$-\Delta \bar{\Pi} \Delta_0 = \Delta^0 [] \Delta_0 \Rightarrow -\Delta \bar{\Pi} = \Delta^0 [], \text{ optimistically.}$$

Will the true dyon please stand up!

$$\text{on } \textcircled{\text{diagonal}} \text{ no} = \text{on } \text{---} + \text{on } \textcircled{\text{diagonal}} \text{ no} + \text{etc}$$

$$\Delta = \Delta^0 + \Delta^0 \bar{\Pi} \Delta^0 + \text{etc}$$

where we define: $\bar{\Pi} = \textcircled{\text{diagonal}} = \text{proper entity, } \textcircled{\text{diagonal}}$
 $\Rightarrow \text{proper self-energy part.}$

$$\text{Thus } \Delta = \Delta^0 [1 + (\bar{\Pi} \Delta^0) + (\bar{\Pi} \Delta^0)^2 + \dots]$$

$$= \Delta^0 [1 - \bar{\Pi} \Delta^0]^{-1} \text{ wait.}$$

Lets be even more precise

$$\text{blob} = \text{blob} + \text{blob} \otimes \text{blob} + \dots$$

$$i\Delta = i\Delta^0 + i\Delta^0 i\bar{\Pi} i\Delta^0 + \dots$$

$$\Rightarrow (i\Delta) = (i\Delta^0) [1 - (i\Delta^0)(i\bar{\Pi})]^{-1}$$

$$\Rightarrow \Delta = \Delta^0 [1 + \bar{\Pi} \Delta^0]^{-1}$$

Summary:

$$\begin{aligned} \text{blob} &= i\Delta \\ \text{wavy line} &= i\Delta^0 \\ \text{cross-hatched blob} &= i\bar{\Pi} \\ \Delta &= \Delta^0 [1 + \bar{\Pi} \Delta^0]^{-1} \end{aligned}$$

The Dyson Equation says:

$$\text{cross-hatched blob} = \text{wavy line} \text{ with three blobs attached}$$

is correct.

$$(i\bar{\Pi}) = \int dk (i\bar{\Pi}^0) (i\Delta) (i\Delta) (i\bar{\Pi}^0) \Rightarrow \left(\bar{\Pi} = i \int dk (\bar{\Pi}^0) \Delta \Delta (i\bar{\Pi}^0) \right)$$

pulled out

So

Dyson equation: $\bar{\Pi}_{\mu\nu} = C^2 \int d^4k \bar{\Pi}_{\mu\nu}^0$ etc.

The $\frac{1}{2}$ is for identicals. I.e., don't count twice!

I now accept BallZach (2.11),
but as yet $\bar{\Pi}$ is the unknown
solution to the above equation.

Kummer: want to solve (4.12) for $\bar{\Pi}$. Solve for Π^0 in standard form:

$$\begin{aligned}\Pi^0_{\mu\nu} &= a^0 T_{\mu\nu} + b^0 S_{\mu\nu} = \left(\delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \right) \times (-q^2) \\ &= a^0 T_{\mu\nu} + b^0 \left[g_{\mu\nu} - T_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \right] \\ &= (a^0 - b^0) T_{\mu\nu} + b^0 \left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \right) \\ \Rightarrow a^0 &= b^0 = (-q^2)\end{aligned}$$

S.

$$\boxed{\Pi^0_{\mu\nu} = (-q^2) T_{\mu\nu} + (-q^2) S_{\mu\nu}}$$

Now lets accept Kummer's solution (4.14) for the moment. Then

$$\begin{aligned}\bar{\Pi}_{\mu\nu} &= (a + q^2 + (1-\alpha)\bar{d}) T_{\mu\nu} + (b + p^2) S_{\mu\nu} + \bar{d} L_{\mu\nu} \\ &= \Pi_{\mu\nu} - \Pi^0_{\mu\nu} + (1-\alpha)\bar{d} T_{\mu\nu} + \bar{d} L_{\mu\nu}\end{aligned}$$

$$(1-\alpha)\bar{d} \frac{q_\mu q_\nu}{q^2} + \bar{d} \frac{p_\mu p_\nu}{q^2}$$

$$= \frac{(1-\alpha)\bar{d}}{(\alpha-1)q^2} \frac{q_\mu q_\nu}{q^2} + \bar{d} \frac{q_\mu q_\nu}{q^2} = -\bar{d} \frac{q_\mu q_\nu}{q^2} + \bar{d} \frac{q_\mu q_\nu}{q^2}$$

$$= \bar{d} \left(\frac{q_\mu q_\nu}{q^2} - \frac{q_\mu q_\nu}{q^2} \right) \quad \text{Fried shows another form!}$$

So lets now solve

$$\bar{\Pi}_{\mu\nu} = \text{symmetric}$$

Then: There are four symmetric tensors.

$$g_{\mu\nu} \quad q_\mu q_\nu \quad \eta_\mu \eta_\nu \quad q_\mu \eta_\nu + q_\nu \eta_\mu$$

$$g_{\mu\nu} \quad S_{\mu\nu} \quad R_{\mu\nu} \quad T_{\mu\nu}$$

Of these 4, only two are $\perp$ to n
only two are $\perp$ to η

So, let us assume that

$$T_{\mu\nu} = \bar{a} T_{\mu\nu} + \bar{b} S_{\mu\nu} + \bar{d} L_{\mu\nu} + e \left(\frac{\eta_{\mu\nu}}{n^2} \right) \equiv M_{\mu\nu}$$

Then this is a most general form. Slide it in to get:

$$[AR + BS]$$

$$[AR + BS] \left[1 + (\bar{a}T + \bar{b}S + \bar{d}L + eM) \right] \left(-\frac{\gamma}{p^2} R - \frac{1}{p^2} S \right)$$

$$= \left(-\frac{\gamma}{p^2} R - \frac{1}{p^2} S \right)$$

Use these facts, write out

$R \perp S$

$T \perp S$

$$M \cdot R = 0$$

$$M \cdot S = 0$$

$$n \cdot R = 0$$

$$n \cdot S = 0$$

$$q \cdot T = 0$$

$$q \cdot S = 0$$

$$L \cdot T = 0$$

$$L \cdot S = 0$$

$$AR + A\bar{a}RT + A\bar{b}RS + \text{stop}$$

$$(\bar{a}T + \bar{b}S + \bar{d}L + eM) \left(-\frac{\gamma}{p^2} R - \frac{1}{p^2} S \right)$$

$$= (\bar{a}T + \bar{b}S + \bar{d}L + eM) (\gamma R + S) / (-p^2)$$

$$= [\bar{a} \gamma TR + \bar{b} \gamma SR + \bar{d} \gamma LR + e \gamma MR + \bar{a} T \gamma S + \bar{b} S \gamma S + \bar{d} L \gamma S + e M \gamma S] / (-p^2)$$

$$= [\bar{a} \gamma TR + \bar{d} \gamma LR + \bar{b} S] / (-p^2)$$

So boils down to

$$[AR + BS] [(-p^2) + \bar{a} \gamma TR + \bar{d} \gamma LR + \bar{b} S] / (-p^2)$$

$$= \{ A(-p^2)R + B(-p^2)S + \bar{a} \gamma A R T R + \bar{d} \gamma A R L R + A \bar{b} R S$$

$$+ \bar{a} \gamma B S T R + \bar{d} \gamma B S L R + B \bar{b} S^2 \} / (-p^2)$$

$$= \{ A(-p^2)R + B(-p^2)S + \bar{a} \gamma A R T R + \bar{d} \gamma A R L R + B \bar{b} S \} / (-p^2)$$

$$= -\frac{\gamma}{p^2} R - \frac{1}{p^2} S$$

Now expand:

$$\begin{aligned}
 [RTR]_{\mu\nu} &= R_{\mu\lambda} T_{\lambda\sigma} R_{\sigma\nu} = \frac{N_\mu N_\lambda}{N^2} \cdot \frac{Q_\lambda Q_\sigma}{Q^2} \cdot \frac{N_\sigma N_\nu}{N^2} \\
 &= \frac{N_\mu N_\nu}{N^2} \frac{(N \cdot Q)^2}{Q^2 N^2} = R_{\mu\nu} \cdot \frac{(n \cdot q)^2 (1-r)^2}{q^2 (r+1) n^2 (r-1)} \\
 &= R_{\mu\nu} / \delta
 \end{aligned}$$

$$R.T.R = \bar{\delta}^{-1} R$$

Next:

$$\begin{aligned}
 RLR &= \frac{N_\mu N_\lambda}{N^2} \frac{q_\lambda q_\sigma}{q^2} \frac{N_\sigma N_\nu}{N^2} = \frac{N_\mu N_\nu}{N^2} \frac{(N \cdot q)^2}{N^2 q^2} \\
 &= \frac{N_\mu N_\nu}{N^2} \frac{(n \cdot q)^2 (1-r)^2}{n^2 (r-1) q^2} = R_{\mu\nu} \cdot \frac{1}{\delta} \cdot (r-1) \\
 &= R_{\mu\nu} \cdot (1 - \bar{\delta}^{-1})
 \end{aligned}$$

$$RLR = (1 - \bar{\delta}^{-1}) R$$

So, our result is

$$A(-p^2) \bar{R} + B(-p^2) \bar{S} + \bar{a} A \bar{R} + \bar{d} A (r-1) \bar{R} + \bar{B} \bar{S} = \delta R + S$$

$$\text{or } R [A(-p^2) + \bar{a} A + \bar{d} A (r-1) - \delta] + S [\bar{B} - 1] = 0$$

$$\Rightarrow -A p^2 + \bar{a} A + \bar{d} A (r-1) - \delta = 0$$

$$\bar{B} + B(-p^2) - 1 = 0$$

$$\text{One second says: } \frac{1}{b} [\bar{A} - p^2] = 1 \Rightarrow (b = \bar{A} - p^2)$$

First says:

$$-\frac{\bar{A}}{a} p^2 + \bar{a} \frac{\bar{A}}{a} + \bar{d} \frac{\bar{A}}{a} (r-1) = \delta$$

$$-p^2 + \bar{a} + \bar{d} (r-1) = a \quad (a = \bar{a} - p^2 + \bar{d} (r-1))$$

Comment: The e -term in $\Pi_{\mu\nu}$ is at once killed when it hits Δ since $n \cdot \Delta = 0$. Thus, you cannot evaluate e in this way

Conclusion: if you solve the mini-Dyson which relates Δ , Δ_0 and $\bar{\Pi}$, as in (4.12) Kummer, you find that $\bar{\Pi}$ must have the form:

$$\bar{\Pi}_{\mu\nu} = \Pi_{\mu\nu} - \Pi_{\mu\nu}^0 + \bar{d} \left(\frac{q_\mu q_\nu}{q^2} - \frac{Q_\mu Q_\nu}{Q^2} \right) + e \frac{\eta_\mu \eta_\nu}{\eta^2}$$

where $\bar{d}$ and e are not determined by this mini-Dyson. This is most-general form of $\bar{\Pi}_{\mu\nu}$.

Kummer Section 5 on Renormalization

① First I will do (5.1), (5.5), (5.6) and App. B.

a) for $(Z_A - 1)$ see (BS notes) $\in$ (Renorm) $\in$ (Genel F Th).

b) for axial-gauge QCD see (3.1) $\in$ (B2 notes).

Equation (5.14) is what you add to the original $\mathcal{L}$ to get "operational" $\mathcal{L}$.
It is like this:

$$\mathcal{L}' = (m^2 - m_0^2) \phi^2 / 2$$

Now, in axial gauge I know that the kinetic $\mathcal{L}$ reads

$$\mathcal{L}_{prop} = \frac{1}{2} A_\mu^a(x) \left[\delta_{ab} \left(g_{\mu\nu} \partial_\mu - \partial_\mu \partial_\nu - \frac{1}{2} \eta_{\mu\nu} \right) \right] A_\nu^b(x)$$

$\Rightarrow$

$$\bar{D}_{\mu\nu} = \left(g_{\mu\nu} \square - \partial_\mu \partial_\nu - \frac{1}{2} \eta_{\mu\nu} \right), \text{ prime col.}$$

⊙ /warning: $\bar{D}$ not same as Π of B2 /.

The $(g_{\mu\nu} \square - \partial_\mu \partial_\nu)$ arises from $(\partial_\mu A_\nu - \partial_\nu A_\mu)^2$ of course, so Z_A of (5.1) modifies this whole shebang.

Whereas, $(\partial_\mu A_\nu)^2$ alone is just $\square g_{\mu\nu}$ part; thus, given (5.1), we would get

$$\bar{D}_{\mu\nu} \rightarrow \left[Z_A' g_{\mu\nu} \square - \partial_\mu \partial_\nu \right] \cancel{Z_A} - \frac{1}{2} \eta_{\mu\nu}$$

Well, not quite:

$$[g_{\mu\nu} \square - \partial_\mu \partial_\nu] + (Z_A - 1) [g_{\mu\nu} \square - \partial_\mu \partial_\nu] + (Z_A' - 1) g_{\mu\nu} \square$$

$$= Z_A [g_{\mu\nu} \square - \partial_\mu \partial_\nu] + (Z_A' - 1) g_{\mu\nu} \square$$

$$\odot = \left([Z_A + (Z_A' - 1)] g_{\mu\nu} \square - Z_A \partial_\mu \partial_\nu - \frac{1}{2} \eta_{\mu\nu} \right) = \bar{D}_{\mu\nu}.$$

So we now have to reconsider the problem of solving for $\bar{D}_{\mu\nu}^{+1}$.

(over)

Te, we have:

$$[2s g_{\mu\nu} q^2 - 2A q_{\mu} q_{\nu} - \frac{1}{2} \eta_{\mu\nu}] = \bar{D}_{\mu\nu}^{-1}$$

Define $\alpha = \frac{1}{\beta} \frac{q^2}{n^2}$ to balance dimensions. Then

$$\bar{D}_{\mu\nu}^{-1} = q^2 [2s g_{\mu\nu} - 2A l_{\mu\nu} - \frac{1}{\beta} M_{\mu\nu}]$$

Replace: $l_{\mu\nu} = g_{\mu\nu} - T_{\mu\nu} - S_{\mu\nu}$
 $M_{\mu\nu} = g_{\mu\nu} - l_{\mu\nu} - S_{\mu\nu}$

$$\Rightarrow \bar{D}^{-1} = q^2 [2s g - 2A(g - T - S) - \frac{1}{\beta}(g - R - S)]$$

$$= q^2 [(2s - 2A - \frac{1}{\beta})g + 2AT + \frac{1}{\beta}R + (2A + \frac{1}{\beta})S]$$

We seek D , and we know due to n^+ condition that $D = AR + BS$. So

$$g = \bar{D}D = (AR + BS) \cdot q^2 [(\overset{x}{2A' - 1 - \frac{1}{\beta}})g + 2AT + \frac{1}{\beta}R + (\overset{y}{2A + \frac{1}{\beta}})S]$$

Of course $g \cdot 1 = 1$ since $g \cdot M = M \cdot g = M$. So

$$q^2 \cdot 1 = q^2 \bar{D}D = (AR + BS)(xg + 2AT + \beta'R + yS)$$

$$= AxR + A2AT + A\beta'R + BxS + ByS \quad \begin{matrix} x+y = 2A+2A'-1 \\ = 2s \end{matrix}$$

$$= RA(x + 2A\beta' + \beta'') + BS(x+y)$$

$$x + \beta' = 2A' - 1$$

$$\Rightarrow (1/q^2) \underline{1} = \underline{RA}(x + 2A\beta' + \beta'') + B(x+y) \underline{S}$$

$$\Rightarrow \bar{q}^2 \underline{1} = A(2A'-1)\underline{B} + A2A \underline{R} \underline{I} + B2s \underline{S}$$

Want to solve this for A and B . First, right multiply by R , use $RTR = R/\gamma$:

$$\bar{q}^2 \underline{R} = A(2A'-1) \underline{R} + A2A \underline{R}/\gamma \Rightarrow A[2A'-1 + 2A/\gamma] = \bar{q}^2$$

$$\Rightarrow A = \frac{\gamma}{q^2 [2A + \gamma(2A'-1)]} = \boxed{\frac{\gamma}{q^2 2R} = A}$$

Now left mult by S :

$$\bar{q}^2 \underline{S} = B2s \underline{S}$$

$$\boxed{\frac{1}{q^2 2s} = B}$$

Thus, putting overall - sign back into $\tilde{D}$ (I omitted it) we get

$$\underline{\Delta}_r = -\bar{q}^2 \left[\underline{S}/\underline{Z}_S + \chi \underline{R}/\underline{Z}_R \right] \quad \text{which is (5.5).}$$

aside: why does (5.1) show all possible quadratic counterterms? Look at original $\mathcal{L}$:

$$\mathcal{L}_{\text{quad}} = (\partial_\mu A_\nu - \partial_\nu A_\mu)^2 - \frac{1}{2} (\eta_\mu A^\mu)^2$$

The only possibilities are: $(\partial_\mu A_\nu)^2$, $(\partial_\mu A_\nu \partial_\nu A_\mu)$, $(\eta_\mu A^\mu)^2$

But $\eta \cdot A = 0$ in F-rules, thus these are the only two, regroup, then done!

aside: I can now omit app B since I derived (5.5) OK.

② Look at (5.2). See renorm notes. The renorm group invariance xform reads:

$$\begin{cases} \Delta \rightarrow \Delta_r = \underline{Z}_3^{-1} \Delta & \text{--- taken care of by (5.1)} \\ \Gamma \rightarrow \Gamma_r = \Gamma \cdot \underline{Z}^{(3)[R^3]} & \text{3-vertex where "3 } R^3 \text{ meet"} \\ g_0 \rightarrow g_{\text{ren}} = g_0 \cdot (\underline{Z}_3)^{3/2} / \underline{Z}^{(3)[R^3]} \end{cases}$$

For the R^3 , $\underline{Z}_3 = \underline{Z}_R$ as you see from (5.5) and base (4.13). Thus:

$$g_0 = g_{\text{ren}} \cdot \underline{Z}^{(3)[R^3]} / \underline{Z}_R^{3/2} \quad (5.2)$$

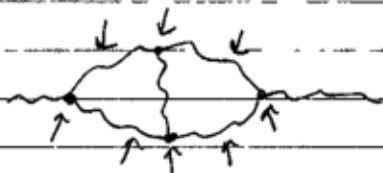
I would say that you could express this as a counterterm like so:

$$\text{counterterm} = \underset{\substack{\text{installs} \\ \text{the new}}}{(g_{\text{ren}} - g_0)} \underset{\substack{\uparrow \\ \text{cancels original}}}{\Gamma^0} \cdot \text{AAAC}$$

$$= g_{\text{ren}} \left[1 - \underline{Z}^{(3)[R^3]} / \underline{Z}_R^{3/2} \right] \text{AAAC}$$

But this does not quite agree with (5.3). Almost.

③ Next, derive (5.10). These are in scratch. The idea is this: The counterterms of (5.1) cause the bare propagator to become (5.5). Also, these same counterterms along with the various vertex counterterms will cause a change in the self energy $\bar{\Pi}$, eg,



causing $\bar{\Pi} \rightarrow (\bar{\Pi})_r$ described by $\bar{a}_r, \bar{b}_r, \bar{d}_r$ in place of $\bar{a}, \bar{b}, \bar{d}$.

Suppose you feed both these changes into the mini-Dyson (4.12), then solve for the "full" Δ . What do you get?

I.e., you want to solve this for A, B:

$$[AR + BS] \left[-p^2 + (\bar{a}T + \bar{b}S + \bar{d}L) (\underline{Z}_s^{-1}S + \gamma \underline{Z}_R^{-1}R) \right] = [\underline{Z}_s^{-1}S + \gamma \underline{Z}_R^{-1}R]$$

$$(\bar{d}1 + (\bar{a} - \bar{d})T + (\bar{b} - \bar{d})S)$$

$$\underline{Z}_s^{-1} \bar{b} S + \gamma \underline{Z}_R^{-1} \bar{d} R + \gamma \underline{Z}_R^{-1} (\bar{a} - \bar{d}) \underline{I} R$$

$$\underline{R} [-p^2 A + A \gamma \bar{d} \underline{Z}_R^{-1} + A \underline{Z}_R^{-1} (\bar{a} - \bar{d})] + \underline{S} [-p^2 B + B \bar{b} \underline{Z}_s^{-1}]$$

$$\Rightarrow \begin{array}{l} A_r = \frac{\gamma}{[-p^2 \underline{Z}_R + \bar{a}_r + (\gamma - 1) \bar{d}_r]} = \frac{\gamma \underline{Z}_R^{-1}}{-p^2 + [\bar{a}_r + (\gamma - 1) \bar{d}_r] \underline{Z}_R^{-1}} \\ B_r = \frac{1}{[-p^2 \underline{Z}_s + \bar{b}_r]} = \frac{\underline{Z}_s^{-1}}{-p^2 + (\bar{b}_r \underline{Z}_s^{-1})} \end{array}$$

↑ remove by mass counterterm.

Were $\underline{Z}_R = \underline{Z}_s = 1$ (ie, before doing counterterms), we would have gotten ... We could absorb everything to the self energy if we like, ie.

$$\begin{array}{ccc} (\bar{a}_r, \bar{b}_r, \bar{d}_r) + (\underline{Z}_s, \underline{Z}_R) & \approx & (a_{eff}, b_{eff}, d_{eff}) + (1, 1) \\ \uparrow & & \uparrow \\ \text{modify } \bar{\Pi} & & \text{no mod of } \Delta_{bare} \\ & & \text{different mod of } \bar{\Pi} \end{array}$$

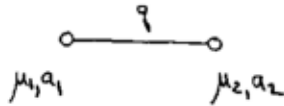
Hence (5.11)

Feynman Rules and Ward Identities for QCD.

(no ghosts included)
(ie, axial gauge)

(gluons only)

① Propagator:



$$= i \Delta_{\mu_1 \mu_2}^{a_1 a_2}(q) = i \delta^{a_1 a_2} \Delta_{\mu_1 \mu_2}^{\circ}(q)$$

$$N_{\mu} = n_{\mu} - q_{\mu} / (n \cdot q)$$

$$n^2 = 1 \quad n \cdot N = 0$$

where $\Delta_{\mu_1 \mu_2}^{\circ}(q) = -\frac{1}{q^2} [g_{\mu_1 \mu_2} - n_{\mu_1} n_{\mu_2} + N_{\mu_1} N_{\mu_2}]$

② "Inverse" Propagator:

$$= -\frac{1}{q^2} [g_{\mu_1 \mu_2} - \frac{(n_{\mu_1} q_{\mu_2} + n_{\mu_2} q_{\mu_1})}{n \cdot q} + \frac{q_{\mu_1} q_{\mu_2}}{(n \cdot q)^2}]$$

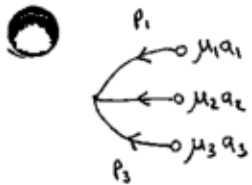
$$\Pi_{\mu_1 \mu_2}^{\circ}(q) = -(q^2 g_{\mu_1 \mu_2} - q_{\mu_1} q_{\mu_2})$$

$$\Delta_{\mu\lambda}^{\circ} \Pi_{\lambda\nu}^{\circ} = g_{\mu\nu} - \frac{q_{\mu} n_{\nu}}{(n \cdot q)}$$

$$n \cdot \Delta = 0$$

$$q \cdot \Pi = 0$$

③ Three-Vertex



$$= i \Gamma_{\mu_1 \mu_2 \mu_3}^{a_1 a_2 a_3}(p_1, p_2, p_3) = C^{a_1 a_2 a_3} i \Gamma_{\mu_1 \mu_2 \mu_3}^{\circ}(p_1, p_2, p_3)$$

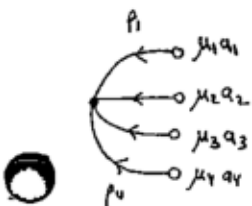
where $i \Gamma_{\mu_1 \mu_2 \mu_3}^{\circ}(p_1, p_2, p_3) = [g_{\mu_1 \mu_2} (p_1 - p_2)_{\mu_3} + \text{cyclic.}] g$

④ The 3/2 Ward Identity:

$$(p_3)^{\mu_3} i \Gamma_{\mu_1 \mu_2 \mu_3}^{\circ}(p_1, p_2, p_3) = [\Pi_{\mu_1 \mu_2}^{\circ}(p_1) - \Pi_{\mu_1 \mu_2}^{\circ}(p_2)] g$$

/verified ✓

⑤ The Four-Vertex



$$= i \Gamma_{\mu_1 \mu_2 \mu_3 \mu_4}^{a_1 a_2 a_3 a_4}(p_1, p_2, p_3, p_4) = \overset{\text{sym.}}{\underset{\text{sym.}}{(12, 34)}} + (13, 24) + (14, 23)$$

where $(12, 34) = -ig^2 C^{a_2 a_3 b} C^{a_1 a_4 b} [g_{\mu_1 \mu_3} g_{\mu_2 \mu_4} - g_{\mu_1 \mu_4} g_{\mu_2 \mu_3}]$

Note: both 3 and 4-vertex are sym. under $(1 \leftrightarrow 2)$, ie, $(p_1, \mu_1, a_1) \leftrightarrow (p_2, \mu_2, a_2)$ since identical bosons.

⑥ The 4/3 Ward Identity:

(verified g^2).

$$(p_4)^{\mu_4} i \Gamma_{\mu_1 \mu_2 \mu_3 \mu_4}^{a_1 a_2 a_3 a_4}(p_1, p_2, p_3, p_4) = ig \left[C^{a_4 a_1 b} i \Gamma_{\mu_1 \mu_2 \mu_3}^{b a_2 a_3}(p_1 + p_4, p_2, p_3) + (1 \leftrightarrow 2) + (1 \leftrightarrow 3) \right]$$

Can write to look like $\circ \Gamma$ as follows:

$$\hookrightarrow = -ig \left[C^{a_1 a_2 b} C^{a_3 a_4 b} i \Gamma_{\mu_1 \mu_2 \mu_3}^{b a_3 a_4}(p_1 + p_4, p_2, p_3) + (1 \leftrightarrow 3) + (2 \leftrightarrow 3) \right]$$

For convenience, we can write:

$$i \Gamma_{\mu_1 \mu_2 \mu_3 \mu_4}^{a_1 a_2 a_3 a_4}(p_1, p_2, p_3, p_4) = -ig^2 C^{a_1 a_2 b} C^{a_3 a_4 b} \left[g_{\mu_1 \mu_3} g_{\mu_2 \mu_4} - g_{\mu_1 \mu_4} g_{\mu_2 \mu_3} \right] + (1 \leftrightarrow 3) + (2 \leftrightarrow 3)$$

⑦ Rules for color tensors:

- 1) $C^{a_1 a_2 b} C^{a_3 a_4 b} - (2 \leftrightarrow 3) - (2 \leftrightarrow 4) = 0$ /and cyclic, Jacobi
- 2) $C^{a b m} C^{b m l} = k \delta^{ab} = \text{trace}[G^a G^b]$
- 3) $C^{m a l} C^{l b k} C^{k c m} = \frac{1}{2} k C^{abc} = -\text{trace}[G^a G^b G^c]$

Verification of the 4/3 Ward.

$$\textcircled{1} \text{ LHS} = -ig^2 \left\{ C^{a_1 a_2 b} C^{a_3 a_4 b} [P_1^{\mu_2} \delta_{\mu_1 \mu_3} - P_1^{\mu_1} \delta_{\mu_2 \mu_3}] \right.$$

$$+ C^{a_3 a_2 b} C^{a_1 a_4 b} [P_1^{\mu_2} \delta_{\mu_1 \mu_3} - P_1^{\mu_3} \delta_{\mu_2 \mu_1}]$$

$$\left. + C^{a_3 b} C^{a_2 a_4 b} [P_1^{\mu_3} \delta_{\mu_1 \mu_2} - P_1^{\mu_1} \delta_{\mu_2 \mu_3}] \right\}$$

$$= -ig^2 \left\{ \delta_{\mu_1 \mu_2} P_1^{\mu_3} [C^{a_1 a_3 b} C^{a_2 a_4 b} - C^{a_3 a_2 b} C^{a_1 a_4 b}] \right.$$

$$+ \delta_{\mu_2 \mu_3} P_1^{\mu_1} [-C^{a_1 a_2 b} C^{a_3 a_4 b} - C^{a_3 a_3 b} C^{a_2 a_4 b}]$$

$$\left. + \delta_{\mu_3 \mu_1} P_1^{\mu_2} [C^{a_1 a_2 b} C^{a_3 a_4 b} + C^{a_3 a_2 b} C^{a_1 a_4 b}] \right\}$$

$$\textcircled{2} \text{ RHS} = ig^2 \left\{ C^{a_1 a_4 b} C^{a_2 a_3 b} [\delta_{\mu_1 \mu_2} (P_1 - P_2)^{\mu_3} + \delta_{\mu_2 \mu_3} (P_2 - P_3)^{\mu_1} + \delta_{\mu_3 \mu_1} (P_3 - P_1)^{\mu_2}] \right.$$

$$+ C^{a_1 a_2 b} C^{a_3 a_4 b} [\delta_{\mu_1 \mu_2} (P_2 - P_1)^{\mu_3} + \delta_{\mu_1 \mu_3} (P_1 - P_3)^{\mu_2} + \delta_{\mu_3 \mu_2} (P_3 - P_2)^{\mu_1}]$$

$$\left. + C^{a_3 b} C^{a_2 a_4 b} [\delta_{\mu_3 \mu_2} (P_3 - P_2)^{\mu_1} + \delta_{\mu_2 \mu_1} (P_2 - P_1)^{\mu_3} + \delta_{\mu_3 \mu_1} (P_1 - P_3)^{\mu_2}] \right\}$$

$$= ig^2 \left\{ \delta_{\mu_1 \mu_2} (P_1 - P_2)^{\mu_3} [C^{a_1 a_4 b} C^{a_2 a_3 b} - C^{a_1 a_2 b} C^{a_3 a_4 b} - C^{a_3 a_3 b} C^{a_2 a_1 b}] \right.$$

$$+ \delta_{\mu_2 \mu_3} (P_2 - P_3)^{\mu_1} [C^{a_1 a_4 b} C^{a_2 a_3 b} - C^{a_1 a_2 b} C^{a_3 a_4 b} - C^{a_3 a_3 b} C^{a_2 a_1 b}]$$

$$+ \delta_{\mu_3 \mu_1} (P_3 - P_1)^{\mu_2} [\text{same}]$$

$$+ \delta_{\mu_1 \mu_2} (P_1)^{\mu_3} [C^{a_1 a_4 b} C^{a_2 a_3 b} + C^{a_1 a_2 b} C^{a_3 a_4 b}]$$

$$+ \delta_{\mu_2 \mu_3} (P_2)^{\mu_1} [-C^{a_1 a_4 b} C^{a_2 a_3 b} + C^{a_1 a_2 b} C^{a_3 a_4 b}] \checkmark$$

$$+ \delta_{\mu_3 \mu_1} (P_3)^{\mu_2} [-C^{a_1 a_4 b} C^{a_2 a_3 b} - C^{a_1 a_2 b} C^{a_3 a_4 b}] \checkmark$$

$$\text{Now, } [cc - cc - cc] = [C^{a_1 a_4 b} C^{a_2 a_3 b} - (1 \leftrightarrow 2) - (1 \leftrightarrow 3)] \text{ by origin} \\ = 0 \text{ by Jacobi!}$$

And other terms match exactly, so QED.

Ball & Zachariasen (1978)

Fried 2. TH2601 Dec 78
 "IR Beh. of g^{rod} in YMTH"
 talk given at Erice Workshop.

1. Introduction If $\text{prop} \sim g(q^2)/q^2 \sim 1/q^2$, then crudely a linear potential. Our method of solving Dyson is attrib. to C+T in QED, see Ph. Rev. D10. In fact, Fried outlines the QED analysis ~ uses Ward to eliminate Γ + concludes that:

$$\tilde{S}(p) = \frac{1}{(-p^2 - m^2)} \left(1 + \frac{g}{4} (1 + \frac{1}{2}\xi) \right)$$

"well-known result" see Fradkin '66.
 / scalar QED; as $p^2 \rightarrow m^2$

where ξ is const chosen in $D_{\mu\nu} = D(k) [\delta_{\mu\nu} - \xi k_\mu k_\nu / k^2]$. In the solution you use Laplace's Sm.

$$[-\text{blob}] = [-] + \text{diagram with blob and wavy line} \quad (1.2)$$

$$\Gamma_1(p, p-k, k) = (2p-k)_\mu P_1 + (k_\mu - \frac{(2p-k)_\mu k^2}{(2p-k)^2}) P_2$$

Solve the Ward:

$$P_1 = \frac{\tilde{S}(p) - \tilde{S}(p-k)}{p^2 - (p-k)^2}$$

Now look near IR. $D \rightarrow \frac{2}{k^2}$, absorb $e^2 P_2 = e^2$. Then since P_2 no anomaly, you get a linear integral equation for S which you solve by Laplace method.

Next, it is noted that AF applies to $g(q^2)$ but this fails to yield the correct IR behavior because of some fixed point...

Then comes the comment that a) the VT signs of objects are det by Ward etc + are in D ; b) the IR signs are probably also det by D , by little's R-group arg. Then, Ward ought to det. IR sign.

Comments

a) Fried inv. includes all 4 glue. (not $\frac{1}{2}$ in leaf!)

b) that Ward is enough for Γ in IR is clearly an assumption.

2. Prop+Vertex Fctn Usual setups, but

a) notes that Thm Dyson has overlap-problems which make it useless for a VT analysis, but still OK for IR.

b) Fred gives "most general" vertex, + notes that the "homog." terms look as though they vanish as any $q_i \rightarrow 0$, so that an arg. on why maybe no contribution in IR.

3. Int Equa + sol'n 3460

This is not up to date, Oct. 78 but OK.

Fred claims that $PI(0)=0$, but we computed it to be Lobe, even though it is UV and IR finite. Now I think he says that Lobe will cancel against d and e, now that the 4-gluon is in there giving us (for now) more free play.

Also, he does not really have a most-general form of the vertex, and also we no longer can use the L/K method.

Infrared Properties of the Glue Propagator

in Non-Abelian Gauge Theories

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ABSTRACT

In a pure Yang-Mills theory, the Dyson equation for the gluon propagator is studied in the infrared regime, under the assumption that, as in QED, only those parts of the proper gluon vertex functions determined by the Ward identities are relevant. The calculations are all carried out in axial gauge. With a number of simplifying assumptions the resulting integral equation for the gluon propagator can be solved in the IR regime. The solution displays a power singularity in the IR for the renormalized coupling constant $g(q^2)$.

Some notes regarding this paper are filed with Kummer paper!

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I. Introduction

P1 There is a widespread belief that confinement of color in non-Abelian gauge theories is a consequence of their highly singular infrared behavior, which is itself a consequence of the non-linear gluon couplings in the theory. From this point of view one expects confinement to be produced by the purely gluon sector of the theory. Quarks, while they feel the consequences of color confinement, play no role in generating it.

P2 There is a further belief that confinement is not a perturbative phenomenon; one must go beyond perturbation calculations to see it. This is not to say that the numerous perturbation studies of the infrared behavior of non-Abelian gauge theories have not been of value; indeed they have provided much guidance. It is only to say that one must not be overly constrained by perturbative results in looking for new phenomena in the theory.

P3 To begin with, let us briefly review one convenient way of obtaining the infrared (IR) behavior - that is, behavior near the mass shell - of the electron propagator in ordinary QED¹⁾. One first writes the Dyson equation, which expresses the propagator in terms of itself, the photon propagator, and the proper vertex function. One notes that the near mass shell behavior is dominated by small virtual photon momenta, and that in QED since the photon is coupled only to (massive) electrons the photon propagator has no IR singularities: for small k , $D(k) \rightarrow Z_3/k^2$. The only problem is the vertex function.

P4 The vertex function satisfies the Ward identity, and this allows one to express part of it (the charge form factor) in terms of the propagator. The other part of the vertex (the moment form factor) is not determined. But in the IR regime, the charge form factor part of the vertex dominates. Therefore to study the IR behavior, one may eliminate the vertex in terms of the propagator, and obtain an integral equation for the propagator alone (with the

photon propagator Z_3/k^2 as a given input) which may be solved to yield the usual near mass shell expression.

P5

What would be involved in carrying out the analogous procedure in the non-Abelian case? Suppose, in view of our comment at the start, we limit ourselves to pure glue - no quarks. Let us also work in axial gauge to make the analogy with QED as close as possible²⁾. Then there are no ghosts. The Dyson equation for the glue propagator expresses it in terms of itself and the triple and quadruple glue vertex functions. In axial gauge the Ward identity looks like that in QED; part of the triple glue vertex is expressed in terms of the propagator, and so is part of the quadruple glue vertex. If, as in QED, only these parts are relevant in the IR regime, the Dyson equation becomes a non-linear integral equation for the glue propagator. Note that here, in contrast to QED, no external input analogous to the photon propagator is required; the equation is entirely self contained.

P6

The integral equation thus obtained is non-linear; it in general may have non-perturbative as well as perturbative solutions, and these may shed light on confinement. Indeed, since in axial gauge the propagator includes the renormalization constant Z , and since $g^2 = Zg_0^2$, one should be able to extract the behavior of the renormalized coupling constant as a function of renormalization point in the IR regime. A singularity in this behavior is often taken as a suggestion of confinement (3) (4).

P7

The crucial ingredient in the program as outlined above is that the only part of the vertex function which we need to know in the IR regime is the part determined by the Ward identities. In conventional QED this assumption is justified by the fact that small virtual photon momenta govern the IR behavior, and therefore invariant functions in the vertex multiplying spinological coefficients having larger numbers of photon momenta in them, such as the moment form factor, play no IR role.

P8 Invariant functions multiplying the minimum number of spinological photon momenta are all that matter. But these are also the invariants fixed by the Ward identity, because they are precisely the UV divergent invariants.

P9 In massless theories, however, small internal momenta do not constitute the IR regime; indeed, perturbation examples suggest that internal momenta comparable to external ones are what is relevant. Therefore in a massless theory, such as a pure Yang-Mills theory, the justification (if there is one) of our basic assumption must be put on a somewhat different footing.

P10 Such a justification may eventually be seen to lie in the following remarks. In unrenormalized perturbation theory, the only mass scale against which to measure IR singularities is provided by the UV cutoff. (Upon renormalization, this scale changes to the renormalization point.) Thus again UV and IR singularities must be correlated. If it were not for the existence of UV divergent insertions in invariant functions which would otherwise be finite, we would conclude that invariants which are UV infinite are also IR infinite. But insertions do exist, so we cannot yet draw such a conclusion, and can only hope that it is nevertheless true. If it is, then, since in Yang-Mills theory, as in ordinary QED, the UV divergences in the vertex and propagator are connected by the Ward identities, the same will be true of the IR divergences.

P11 Needless to say, these issues are presently under active study; in the meantime, and pending their resolution, we shall simply assume that the Ward identity may be used to eliminate the vertex function in favor of the propagator in the IR regime in Yang-Mills theory, as in conventional QED, and investigate the consequences of this assumption.

If this assumption is correct, then the Dyson equation coupled with the Ward identity yields a non-linear integral equation determining the IR behavior of the glue propagator. No further assumptions are in principle necessary to

obtain this behavior. In practice, however, the equation is complicated, and we shall find it convenient to make some simplifying ansatzes regarding the solutions we are looking for. These are not further assumptions; they are guesses as to the nature of the solution which can be checked against the integral equation. The first of these is the following.

We use axial gauge throughout, since, as we mentioned above, in this gauge the parallel with QED is most obvious. The Ward identity is simple; the divergent part of Z is gauge invariant; there are no ghosts. The price we pay for these virtues is spinological complexity. The unrenormalized glue propagator contains two independent invariant functions, A and B , and these are functions of two independent scalar variables q^2/Λ^2 and $\gamma = q^2 n^2 / (q \cdot n)^2$ where Λ is the UV cutoff and n is the direction of the gauge choice²⁾. These functions renormalize according to

$$\begin{aligned} A(q^2/\Lambda^2, \gamma, g_0) &= Z(M^2/\Lambda^2) A_{\text{ren}}(q^2/M^2, \gamma, g(M^2)), \\ B(q^2/\Lambda^2, \gamma, g_0) &= Z(M^2/\Lambda^2) B_{\text{ren}}(q^2/M^2, \gamma, g(M^2)), \end{aligned} \quad (1.1)$$

with a common Z whose divergent part is independent of γ . We have

$$g^2(M^2) = Z(M^2/\Lambda^2) g_0^2. \quad (1.2)$$

At $q^2 = M^2$, the renormalized quantities A_{ren} and B_{ren} become specified functions of the gauge variable γ and $g(M^2)$.

Eq. (1.1) is valid for any M . In particular we can choose $M^2 = q^2$. Then (1.1) reads

$$A(q^2/\Lambda^2, \gamma, g_0) = Z(q^2/\Lambda^2) A_{\text{ren}}(1, \gamma, g(q^2)), \quad (1.3)$$

and similarly for B . If $g(q^2)$ had a finite limit as $q^2 \rightarrow 0$, then the IR singularities in both A and B would be common, gauge invariant, and confined

to the renormalization constant $Z(q^2/\Lambda^2)$. In massless QED this is what actually occurs. In Yang-Mills theory, in perturbation calculations it apparently does not occur; it seems that the leading log IR singularities are gauge invariant, but non-leading ones are not⁵⁾. Nevertheless, we are searching for non-perturbative solutions to Yang-Mills theory, and we must not be too heavily inhibited by perturbation theory. Therefore our first ansatz is that there are solutions in which all IR singularities are confined to the factor $Z(q^2/\Lambda^2)$ in eq. (1.3). This ansatz can be checked for consistency by seeing if, when the vertex is eliminated from the Dyson equation, the resulting equation for the propagator really has such solutions.

Our first ansatz, then, is that in the IR,

even though

$$A(q^2/\Lambda^2, \gamma, g_0) = Z(q^2/\Lambda^2) \alpha(\gamma)$$

$$\lim_{q^2 \rightarrow 0} g(q^2) = \infty$$

$$B(q^2/\Lambda^2, \gamma, g_0) = Z(q^2/\Lambda^2) \beta(\gamma) \quad , \quad (1.4)$$

where α and β are functions only of the gauge direction. With this ansatz, the Dyson equation becomes a non-linear integral equation for the small q^2 behavior of $g^2(q^2) = Z(q^2/\Lambda^2) g_0^2$. We shall argue that this equation permits solutions with the behavior

$$g^2(q^2) \rightarrow (q^2)^{-\alpha} \text{ as } \underbrace{g^2}_{q^2} \rightarrow 0 \quad , \quad (1.5)$$

with $\alpha > 0$. The power α is in principle determined by the equation, though we have not yet in practice ascertained what value it actually has. We shall, however, compute it in a model.

Our second, and stronger, ansatz is based on the following observation. We know that a certain combination of the functions A and B in the propagator has no UV singularities. This combination is $1/B(\gamma)/A$. Perhaps, therefore, it also has no IR singularities. In fact, if it is zero, then the spin structure of the exact

propagator becomes that of the free propagator. Thus our second ansatz is
that in the IR, the exact propagator consists of an IR singular factor, namely
 $Z(q^2/\Lambda^2)$, multiplied by the free propagator. We note that this property is true
of the IR behavior of scalar, spinor, and vector QED. The ansatz amounts to
guessing that $\alpha(\gamma) = \gamma$ and $\beta(\gamma) = 1$, in eq. (1.4). Again, of course, this
guess is to be checked against the integral equation to see if such solutions
really do exist.

Finally, in this paper we shall make the simplification of neglecting the
quadruple glue vertex term. This modification is again not in principle necessary;
it is made solely for convenience and the consequences of retaining it will be
reported elsewhere. (We note that in leading log perturbation calculations,
the quadruple glue term does not contribute to the IR behavior.) Our hope,
and feeling, is that the gross IR properties of the propagator are not altered
by this simplification.

One final comment. In a massless theory, where in the UV $\log q^2/M^2$
approaches $+\infty$ and in the IR it approaches $-\infty$, the leading log UV and IR be-
haviors are formally the same, though of course the leading log result is
correct only in the regime in which the theory is free. The Yang-Mills theory
is UV free, and we know the expression for g in the UV regime:

$$g^2(q^2) = \frac{g^2(M^2) \leftarrow \text{fixed point.}}{1 - b g^2(M^2) \log q^2/M^2 \leftarrow \text{correction}}$$

see Abomb note.
(1.6)

with $b < 0$. Formally, then, the leading log sum in the IR regime must be the
same, though since $b < 0$ the expression has a singularity for $\log q^2/M^2 < 0$.

II. Derivation of the Integral Equation

A. Notation

We shall generally use the notation employed by Kummer⁶⁾. We write the unrenormalized glue propagator as

$$\Delta_{\mu\nu}^{ab}(q) = -\frac{\delta_{ab}}{q^2} \left\{ \left(\delta_{\mu\nu} - \frac{n_\mu n_\nu}{n^2} \right) B(q^2/\Lambda^2, \gamma, g_0) + \frac{N_\mu N_\nu}{n^2} C(q^2/\Lambda^2, \gamma, g_0) \right\} \quad (2.1) \checkmark$$

where N_μ is a vector orthogonal to n_μ :

$$N_\mu = n_\mu - \frac{q_\mu n^2}{q \cdot n} \quad (2.2) \checkmark$$

The functions B and C are related to Kummer's $\tilde{A}$ and $\tilde{B}$ by

$$C = \frac{\tilde{B} - \tilde{A}}{1 - \gamma} \quad B = -q^2 \tilde{B} \quad (2.3) \checkmark$$

In eq. (2.1) a and b are color indices. We shall for simplicity generally suppress them; they play no significant role.

The inverse propagator, connected to $\Delta_{\mu\nu}$ by the relation

why not covariant notation?

$$\Delta_{\mu\lambda} \Pi_{\lambda\nu} = \delta_{\mu\nu} - \frac{q_\mu n_\nu}{q \cdot n} \quad (2.4) \checkmark$$

can be written

$$\Pi_{\mu\nu}(q) = -q^2 \left(\delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \right) b(q^2/\Lambda^2, \gamma, g_0) - Q_\mu Q_\nu c(q^2/\Lambda^2, \gamma, g_0) \quad (2.5) \checkmark$$

where

$$b = 1/B, \quad a = \gamma/A \quad \text{Kummer} \quad (2.6) \checkmark$$

and c is related to Kummer's a through

$$c = \frac{b-a}{1-\gamma} \quad (2.7)$$

The function c is actually UV finite²⁾. The vector Q_μ is orthogonal to

q_μ :

$$Q_\mu = q_\mu - \frac{n_\mu q^2}{q \cdot n} \quad (2.8) \checkmark$$

The Dyson equation is an equation for the inverse propagator except for some additional trivial terms. We define

$$\bar{\Pi}_{\mu\nu}(q) = \Pi_{\mu\nu}(q) - \Pi_{\mu\nu}^0(q) + d \left(\frac{q_\mu q_\nu}{q^2} - \frac{Q_\mu Q_\nu}{q^2} \right) + e \frac{n_\mu n_\nu}{n^2}, \quad (2.9) \checkmark$$

where $\Pi_{\mu\nu}^0$ is the free inverse propagator

$$\Pi_{\mu\nu}^0(q) = - (q^2 \delta_{\mu\nu} - q_\mu q_\nu) \quad q_\mu \Pi_{\mu\nu}^0(q) = 0 \quad (2.10) \checkmark$$

As we shall verify below, d and e are constants, which in perturbation theory are quadratically divergent. The Dyson equation reads

$$\bar{\Pi}_{\mu\nu}(q) = \frac{1}{2} \int dk \, i\Gamma_{\mu\lambda\lambda}^0(q, -k, -k') \Delta_{\lambda\sigma}(k) \Delta_{\lambda'\sigma'}(k') i\Gamma_{\sigma\sigma'\nu}(k, k', -q), \quad (2.11) \checkmark$$

and is represented graphically in Figure 1(a). We write $k+k' = q$. As mentioned in the introduction, we have neglected the quadruple glue term shown in Figure 1(b). There is also a cutoff⁷⁾ implicit on the right-hand side; we do not display it explicitly. We use the shorthand dk to stand for

$$i g_0^2 C_A \frac{(d^4 k)}{(2\pi)^4} \quad g_0 \text{ extended} \quad (2.12)$$

where C_A is the Casimir eigenvalue of the adjoint representation of the color group. All color effects are included in C_A , so explicit color indices can be suppressed in (2.11). The $1/2$ in (2.11) is from Bose statistics for the virtual gluons. Γ^0 and Γ are the bare and unrenormalized vertex functions: we have

$$i\Gamma_{\lambda_1\lambda_2\lambda_3}^0(q_1q_2q_3) = (q_1 - q_2)_{\lambda_3} \delta_{\lambda_1\lambda_2} + \text{cyclic permutations} \quad (2.13) \checkmark$$

$+ (q_2 - q_3)_{\lambda_1} \delta_{\lambda_2\lambda_3}$
 $+ (q_3 - q_1)_{\lambda_2} \delta_{\lambda_3\lambda_1}$

why not $g_{\lambda_1\lambda_2}$?
agree with my notes

B. The Ward Identity

The Ward identity relates Γ and Π : (color remove)

$$i\Gamma_{\sigma_1\sigma_2\sigma_3}(q_1q_2q_3)(q_3)_{\sigma_3} = \Pi_{\sigma_1\sigma_2}(q_1) - \Pi_{\sigma_1\sigma_2}(q_2) \quad (2.14) \checkmark$$

Using this and the Dyson equation (2.11), it is easy to show that d and e are independent of q . Evidently

$$n_\mu \bar{\Pi}_{\mu\nu} q_\nu = \checkmark n \cdot q (d+e) = \checkmark \int dk n \cdot (k-k') \Delta_{\lambda\lambda}(k') \quad (2.15) \checkmark$$

and

$$\begin{aligned} q_\mu \bar{\Pi}_{\mu\nu} q_\nu &= \checkmark q^2 d + \frac{(n \cdot q)^2}{n^2} e \\ &= \checkmark \int dk (k^2 - k'^2) \Delta_{\lambda\lambda}(k') - \int dk (k_\lambda k_\lambda - k'_\lambda k'_\lambda) \Delta_{\lambda\lambda}(k') \end{aligned} \quad (2.16) \checkmark$$

The spinology on the right-hand side of these is readily decomposed, and one obtains

$$= -\frac{2}{3} \int \frac{dk}{k^2} \Delta_{\mu\mu}(k) = \checkmark d = \checkmark 2e = -\frac{2}{3} \int \frac{dk}{k^2} [A(k^2/\Lambda^2, \gamma_k, g_0) + 2B(k^2/\Lambda^2, \gamma_k, g_0)] \quad (2.17) \checkmark$$

which is a constant. (Here we use the notation $\gamma_k = k^2 n^2 / (k \cdot n)^2$.)

Two other scalar equations, for the physically interesting functions in $\bar{\Pi}$, are obtained by multiplying (2.11) by $n_\mu n_\nu$ and $\delta_{\mu\nu}$ respectively. These are

$$\begin{aligned} -q^2(a-1)(1-1/\gamma) + (2-\gamma)d + e \\ = \frac{1}{2} \int dk \frac{n \cdot (k-k')}{n^2} \Delta_{\lambda\sigma}(k) \Delta_{\lambda\sigma}(k') i\Gamma_{\sigma\sigma',\nu}(k, k', -q) n_\nu \quad (2.18) \checkmark \end{aligned}$$

and

$$\begin{aligned} -q^2(a-1) - 2q^2(b-1) + \checkmark d + e \\ = \frac{1}{2} \int dk i\Gamma_{\mu\lambda\lambda'}^0(-k, -k', q) \Delta_{\lambda\sigma}(k) \Delta_{\lambda'\sigma'}(k') i\Gamma_{\sigma\sigma',\mu}(k, k', -q) \quad (2.19) \checkmark \end{aligned}$$

the trace equation, ie, $\bar{\Pi}_{\mu\mu} = \dots$

In the first of these, we have made use of the fact that n is orthogonal to Δ , and therefore terms with $\delta_{\mu\lambda}$ or $\delta_{\mu\lambda}'$ in the bare vertex do not contribute.

The starting point for all of our subsequent discussion is the "solution" of the Ward identity. It is of course possible to write a triple glue vertex which satisfies the Ward identity in many different forms. Any two such forms differ by something which vanishes when inserted in the Ward identity. But our basic hypothesis is that, as in QED, the only part of the vertex relevant in the IR is that determined by the Ward Identity; hence any part of the vertex which yields zero in the Ward identity is irrelevant in the IR regime. That is to say, any two forms for the vertex compatible with the Ward identity will give rise to the same IR behavior.

We have found a convenient form to choose for the vertex to be the following:

$$\begin{aligned}
 i\Gamma_{\sigma_1\sigma_2\sigma_3}(q_1q_2q_3) = & \\
 & + \delta_{\sigma_1\sigma_2} \left\{ \frac{(q_1)_{\sigma_3}}{q_1 \cdot q_3} g_1 - \frac{(q_2)_{\sigma_3}}{q_2 \cdot q_3} g_2 \right\} \\
 & + \frac{g_1}{q_1} (q_1)_{\sigma_2} (q_1)_{\sigma_3} \left\{ \frac{(q_2)_{\sigma_1}}{q_1 \cdot q_3} - \frac{(q_3)_{\sigma_1}}{q_1 \cdot q_2} \right\} \\
 & + \delta_{\sigma_1\sigma_2} n_{\sigma_3} \left\{ \frac{f_2 - f_1}{q_3 \cdot n} \right\} \\
 & + n_{\sigma_1} n_{\sigma_2} \left\{ \frac{(q_2)_{\sigma_3}}{q_1 \cdot n} \left[\frac{f_2}{q_2 \cdot n} + \frac{f_3}{q_3 \cdot n} \right] - \frac{(q_1)_{\sigma_3}}{q_2 \cdot n} \left[\frac{f_1}{q_1 \cdot n} + \frac{f_3}{q_3 \cdot n} \right] \right\} \\
 & + n_{\sigma_1} n_{\sigma_2} n_{\sigma_3} \left\{ \frac{f_1}{(q_1 \cdot n)^2} \left(\frac{q_1 \cdot q_2}{q_3 \cdot n} - \frac{q_1 \cdot q_3}{q_2 \cdot n} \right) \right\} \\
 & + \text{cyclic permutations.}
 \end{aligned}
 \tag{2.20}$$

Here we use the notation $f_1 = f(q_1)$, $g_1 = g(q_1)$ etc., and the functions f and g are given by

$$f = \frac{1}{2} q^2 c \quad , \quad (2.21) \checkmark$$

$$g = - q^2 (b-c) \quad , \quad (2.22) \checkmark$$

in terms of those previously defined associated with $\Pi_{\mu\nu}$.

Let us first focus our attention on eq. (2.18). From eq. (2.20), we see that we can write

$$\begin{aligned} i\Gamma_{\sigma\sigma'}(k, k', -q)n_\nu = & \\ & \delta_{\sigma\sigma'} \left\{ \frac{n^2}{q \cdot n} f(k) - \frac{n \cdot k}{q \cdot k} g(k) \right\} \\ & + \left\{ \frac{k_\sigma k_{\sigma'}}{k^2} \frac{n \cdot k}{q \cdot k} + \frac{q_\sigma k_{\sigma'}}{q \cdot k} \frac{n \cdot k}{k \cdot k'} \right\} g(k) \\ & + \left\{ \frac{q_\sigma q_{\sigma'}}{q^2} \frac{n \cdot k'}{q \cdot k} \right\} g(q) \\ & - (\text{same with } k \leftrightarrow k', \sigma \leftrightarrow \sigma') \quad . \end{aligned} \quad (2.23) \checkmark$$

where we have dropped terms proportional to n_σ or $n_{\sigma'}$, as they will vanish in eq. (2.18). Because of the symmetry in eq. (2.18), the antisymmetric term in eq. (2.23) simply gives rise to a factor of two. Then if we define

$$K_{\sigma\sigma'}^g = \frac{n \cdot (k-k') n \cdot k}{n^2 q \cdot k} \left\{ -\delta_{\sigma\sigma'} + \frac{k_\sigma k_{\sigma'}}{k^2} + \frac{q_\sigma k_{\sigma'}}{k \cdot k'} \right\} \quad , \quad \checkmark (2.24)$$

$$K_{\sigma\sigma'}^f = \frac{n \cdot (k-k')}{n \cdot q} \left\{ \delta_{\sigma\sigma'} \right\} \quad , \quad \checkmark (2.25)$$

$$L_{\sigma\sigma'} = \frac{n \cdot (k-k') n \cdot k'}{n^2 q \cdot k} \left\{ \frac{q_\sigma q_{\sigma'}}{q^2} \right\} \quad , \quad \checkmark (2.26)$$

then eq. (2.18) takes the form

$$\begin{aligned}
 & -q^2(a-1)(1 - 1/\gamma) + (2 - \gamma)d + e \\
 & = \int dk \Delta_{\lambda\sigma}(k) \Delta_{\lambda\sigma}(k') \\
 & \left\{ K_{\sigma\sigma}^f f(k) + K_{\sigma\sigma}^g g(k) + L_{\sigma\sigma} g(q) \right\} .
 \end{aligned}
 \tag{2.27}$$

One comment on this expression is in order. In perturbation theory there are quadratic divergences in the K term; these give rise to constant quadratically infinite pieces and contribute only to the constant d and e terms on the left-hand side. Further, this is the only effect of d and e; once the quadratically infinite constants are removed from the right-hand side, one can drop d and e on the left-hand side. Contributions to a are therefore only logarithmically infinite.

We recall eqs. (2.21), (2.22), and (2.6). Thus $a = \gamma/A$, Δ is a linear function of A and B, and f and g are linear functions of $1/A$ and $1/B$. Hence eq. (2.27) has the form (disregarding the trivial constant terms d and e).

$$\begin{aligned}
 & = ? \\
 & \frac{1}{A_i(q)} \bigoplus \sum_{jkl} \int dk K_{jk}^{il} \frac{A_j(k) A_k(k')}{k^2 k'^2} \frac{1}{A_l(k)} \\
 & + \sum_{jkl} \frac{1}{A_l(q)} \int dk L_{jk}^{il} \frac{A_j(k) A_k(k')}{k^2 k'^2} ,
 \end{aligned}
 \tag{2.28}$$

where we label $A_1 = A$ and $A_2 = B$. A little thought makes it evident that eq. (2.19) will also lead to an equation of this form. Thus we have a pair

sentence?
meaning?

of coupled equations for the two unknown functions. [The kernels in eq. (2.28) are all dimensionless, in both momentum and n , and functions of the scalars which can be constructed from k , q , and n .] One may take these to be k^2/q^2 , $\gamma_q = q^2 n^2 / (q \cdot n)^2$, $\gamma_k = k^2 n^2 / (k \cdot n)^2$, $(k \cdot q)^2 / k^2 q^2$. Of these, only the first is not dimensionless individually in k and q ; hence only the first could be relevant to the IR properties of the equation. But it is easy to see by explicitly looking at the kernels that they are well behaved both as $k^2/q^2 \rightarrow \infty$ and as $k^2/q^2 \rightarrow 0$. Therefore the infrared behavior is entirely controlled by the behavior of the functions A_i themselves.

C. The Integral Equation

The coupled integral equations (2.28) are complicated. Therefore let us make the ansatz, as discussed in the introduction, that the IR behavior is all in the renormalization constants Z . We have the renormalization prescription

$$A_i(q) = Z(M^2/\Lambda^2) \alpha_i(q^2/M^2, \gamma, g(M)) \quad , \quad (2.29) \text{ ansatz}$$

with a common factor Z . We choose $M^2 = q^2$, and then assume that the dimensionless renormalized functions $\alpha_i(1, \gamma, g(q))$ are IR finite, and simply functions of γ in the IR limit. Thus our ansatz is

$$\begin{aligned} A(q) &= Z(q^2/\Lambda^2) \alpha(\gamma) \checkmark, \text{ or } a(q) = \frac{1}{Z(q^2/\Lambda^2)} \frac{\gamma}{\alpha(\gamma)} \checkmark \\ \text{and} \quad B(q) &= Z(q^2/\Lambda^2) \beta(\gamma) \checkmark, \text{ or } b(q) = \frac{1}{Z(q^2/\Lambda^2)} \frac{1}{\beta(\gamma)} \checkmark \end{aligned} \quad (2.30) \checkmark$$

To check whether this ansatz works, it is necessary to insert (2.30) into the pair of coupled integral equations (2.28) and show that we can find Z , α , and β satisfying the equations. We have not been able to do this; what we can do is the more limited problem of showing that if we have found an α and a β which work, then we can find a Z that works.

To do this, we go back to eq. (2.27). Under the integral, we replace $A(k)$ by $Z(k^2/\Lambda^2) \alpha(\gamma_k)$ and similarly for $B(k)$.

We also replace f and g as follows:

$$f(k) = \checkmark k^2 \checkmark \tilde{c}(k) = \checkmark \frac{k^2}{Z(k^2/\Lambda^2)} \frac{1/\beta(\gamma_k) - \gamma_k/\alpha(\gamma_k)}{1 - \gamma_k} \checkmark$$

$$g(k) = \checkmark - k^2 (b(k) - c(k))$$

$$= \checkmark - \frac{k^2}{Z(k^2/\Lambda^2)} \left(\frac{\gamma_k}{1-\gamma_k} \right) \left(\frac{1}{\alpha(\gamma_k)} - \frac{1}{\beta(\gamma_k)} \right) + \cancel{\frac{k^2}{Z(q^2/\Lambda^2)}} \quad (2.31) \checkmark$$

Then eq. (2.27) becomes

$$\frac{1}{Z(q^2/\Lambda^2)} \frac{\gamma-1}{\alpha(\gamma)} = (1 - 1/\gamma)$$

$$+ \frac{1}{Z(q^2/\Lambda^2)} \frac{\gamma}{1-\gamma} \left(\frac{1}{\alpha(\gamma)} - \frac{1}{\beta(\gamma)} \right) \int dk \frac{Z(k^2/\Lambda^2) Z(k'^2/\Lambda^2)}{k^2 k'^2} L$$

$$+ \frac{1}{2} \int dk \frac{Z(k'^2/\Lambda^2)}{k'^2} \left(K^g \frac{\gamma_k}{1-\gamma_k} \left[\frac{1}{\alpha(\gamma_k)} - \frac{1}{\beta(\gamma_k)} \right] \right.$$

$$\left. + K^f \frac{1}{1-\gamma_k} \left[\frac{\gamma_k}{\alpha(\gamma_k)} - \frac{1}{\beta(\gamma_k)} \right] \right) \frac{1}{q^2} \left[(2-\gamma)d + e \right] \quad (2.18)''$$

(2.32) ✓

$$\alpha = \delta$$

$$\beta = 1$$

with the definitions

I assume β' means $\beta(\delta_{k'})$
 β means $\beta(\delta_k)$? or $\beta(\delta)$?

$$K_{\sigma\sigma'}^{\delta} = K_{\sigma\sigma'}^{\delta} \left[\beta(\delta_{\lambda\sigma} - \frac{n_{\lambda} n_{\sigma}}{n^2}) + \frac{(\beta-\alpha) N_{\lambda} N_{\sigma}}{(1-\gamma) n^2} \right] \left[\beta'(\delta_{\lambda\sigma'} - \frac{n_{\lambda} n_{\sigma'}}{n^2}) + \frac{(\beta'-\alpha') N'_{\lambda} N'_{\sigma'}}{(1-\gamma') n^2} \right] \quad (2.33) \checkmark$$

and similarly for K^f and L . This is still apparently a somewhat formidable equation; however much of the complexity is illusory because, as we remarked earlier, the only important variable on which the various kernels depend is k^2/q^2 , and their dependence on this is trivial. The unknown dependence on γ will not influence the IR behavior of Z , except in detail. It is also important to note that the apparent $1/q^2$ singularity is also illusory: as we have commented earlier, the constants d and e in eq. (2.32) are just such as to cancel the $1/q^2$ terms in the second integral.

III. Outline of the Solution of the Integral Equation

We shall approach the solution of (2.32) in steps. First we make an even stronger ansatz than (2.30): we look for a solution in which the propagator in the IR regime is simply an IR singular function times the free propagator. (This behavior, we note, is what actually occurs in scalar, spinor, and vector QED)¹⁾. Thus we assume

$$\Delta_{\mu\nu}(q) = \left(-\frac{Z(q^2/\Lambda^2)}{q^2} \right) \left(\delta_{\mu\nu} - \frac{n_{\mu} n_{\nu}}{n^2} + \frac{N_{\mu} N_{\nu}}{n^2} \right) = \Delta_{\mu\nu}^0(q) \quad \checkmark(3.1) \quad \text{ansatz}$$

and consequently

$$\Pi_{\mu\nu}(q) = \left(-q^2/Z(q^2/\Lambda^2) \right) \left(\delta_{\mu\nu} - q_{\mu} q_{\nu}/q^2 \right) \cdot \Pi_{\mu\nu}^0 \quad \checkmark(3.2)$$

This means we look for a solution like (2.30) but with $\alpha = \gamma$, $\beta = 1$. Then the integral equation (2.32) takes the form

green shows signs
after (2.18) even corrected.

$$\frac{1}{Z(q^2/\Lambda^2)} (1 - 1/\gamma) = (1 - 1/\gamma)$$

$$+ \frac{1}{Z(q^2/\Lambda^2)} \int dk \frac{Z(k^2/\Lambda^2) Z(k'^2/\Lambda^2)}{k^2 k'^2} L$$

(2.18)^{III}
(3.3)✓

$$+ 1/q^2 \int dk \frac{Z(k'^2/\Lambda^2)}{k'^2} K$$

$$+ \cancel{1/q^2} [(2-\gamma)d + e]$$

Note that the K^f term in (2.32) has disappeared. The remaining kernels are

$$L = \frac{n \cdot (k-k') n \cdot k}{n^2 q \cdot k} \left(-\frac{q_\sigma q_{\sigma'}}{q^2} \right) \left(\delta_{\sigma\sigma'} - \frac{n_\sigma n_{\sigma'}}{n^2} + \frac{N_\sigma N_{\sigma'} + N'_\sigma N'_{\sigma'}}{n^2} + \frac{N \cdot N' N_\sigma N'_{\sigma'}}{n^4} \right)$$

(3.4)✓

and (writing K for K^g)

$$K = \frac{n \cdot (k-k') n \cdot k}{n^2 q \cdot k} \left(-\delta_{\sigma\sigma'} + \frac{k_\sigma k_{\sigma'}}{k^2} + \frac{q_\sigma k_{\sigma'}}{k \cdot k'} \right)$$

$$\left(\delta_{\sigma\sigma'} - \frac{n_\sigma n_{\sigma'}}{n^2} + \frac{N_\sigma N_{\sigma'} + N'_\sigma N'_{\sigma'}}{n^2} + \frac{N \cdot N' N_\sigma N'_{\sigma'}}{n^4} \right)$$

(3.5)✓

The structure of (3.3) is quite transparent, though some spinological complexities are still present in view of (3.4) and (3.5). Therefore, to see how things work, let us first arbitrarily simplify this equation by simply pretending that $-L = k^2 K/q^2 = (1-1/\gamma)$.^{OK} We also ignore the constants d and e. Then (3.3) reads

$$\frac{1}{Z(q^2/\Lambda^2)} = 1 - \frac{1}{Z(q^2/\Lambda^2)} \int dk \frac{Z(k^2/\Lambda^2) Z(k'^2/\Lambda^2)}{k^2 k'^2} + \int dk \frac{Z(k'^2/\Lambda^2)}{k^2 k'^2}$$

(3.6)✓

We solve for $Z(q^2/\Lambda^2)$:

$$\frac{1}{Z(q^2/\Lambda^2)} = \frac{1 + \int dk \frac{Z(k'^2/\Lambda^2)}{k^2 k'^2}}{1 + \int dk \frac{Z(k^2/\Lambda^2) Z(k'^2/\Lambda^2)}{k^2 k'^2}} \quad (3.7) \checkmark$$

Let us try a power solution, in which ⁸⁾

$$Z(q^2/\Lambda^2) = (\Lambda^2/q^2)^\alpha \quad (3.8) \text{ ansatz}$$

for small q^2 . It is easy to evaluate the integrals on the right-hand side. We have

$$\int \frac{d^4 k}{(k^2)^{1+\alpha} (k'^2)^{1+\beta}} = i\pi^2 \frac{\Gamma(\alpha+\beta)}{\Gamma(1+\alpha)\Gamma(1+\beta)} \left(\frac{1}{2}\right)^{\alpha+\beta} \frac{1}{q^2} \int_0^1 \frac{dz}{z^\alpha (1-z)^\beta} \quad (3.9) \checkmark$$

Thus eg. (3.7) becomes

where $q^2 \equiv \left(\frac{q^2}{\Lambda^2}\right)$

$$(q^2)^\alpha = \frac{1 + i\pi^2 \frac{\Gamma(\alpha)}{\Gamma(1+\alpha)} \frac{1}{1-\alpha} \left(\frac{1}{2}\right)^\alpha}{1 + i\pi^2 \frac{\Gamma(2\alpha)}{(\Gamma(1+\alpha))^2} \frac{\Gamma^2(1-\alpha)}{\Gamma(2-2\alpha)} \left(\frac{1}{2}\right)^{2\alpha}} \quad (3.10) \checkmark$$

The two sides match as $q^2 \rightarrow 0$ provided that there exists a positive α such that

$$\frac{\Gamma(\alpha)}{\Gamma(2\alpha)} \frac{\Gamma(1+\alpha)}{1-\alpha} \frac{\Gamma(2-2\alpha)}{\Gamma^2(1-\alpha)} = 1 \quad (3.11) \checkmark$$

This indeed happens for $\alpha = 1/2$.[✓] Hence, as $q^2 \rightarrow 0$ (3.7) is satisfied by

$$Z(q^2/\Lambda^2) = (\Lambda^2/q^2)^{1/2} \quad (3.12)^{\checkmark}$$

This simple model provides a guide to how we may expect Z to behave in the real situation.^{o✓} Let us return to (3.3). We first expect the constants d and e to remove from the integrals the quadratically divergent terms which occur when $Z \rightarrow 1$ in the UV. We then expect the γ dependence to disappear from the equation, upon inserting the expressions (3.4) and (3.5) and carrying out the spinology. The resulting equation is then much like (3.7) except that there will be dimensionless kernels involving k^2 and $k \cdot q$ appearing in the numerator and denominator integrals. If we try a power solution in the IR like (3.8), then we will again have a consistent solution provided the equation replacing (3.11) has a solution for a positive α . Thus we will again find

$$Z(q^2/\Lambda^2) \rightarrow (\Lambda^2/q^2)^\alpha \quad (3.13)$$

with $\alpha > 0$ as $q^2 \rightarrow 0$.

To illustrate how this procedure works, let us describe a more realistic version of (3.3) than (3.6). Namely, let us keep only the following simple spinological expressions for L and K ⁹⁾:

$$K = L = \frac{n \cdot (k-k') n \cdot k}{n^2 q \cdot k} \quad (3.14)$$

We wish to solve eq. (3.3) with these choices.

At first glance it might be thought that such a truncation of the true kernels in (3.3) would prevent satisfaction of the γ dependence of the equation. But in fact straightforward manipulation of the integrals yields the relations

$$\int \frac{n \cdot (k-k')}{n^2 q \cdot k} \frac{Z(k^2) Z(k'^2)}{k^2 k'^2} dk$$

$$= \frac{1}{3} \left(1 - \frac{1}{\gamma}\right) \int \frac{1}{q \cdot k q \cdot k'} (k^2 q^2 - (q \cdot k)^2) \frac{Z(k^2) Z(k'^2)}{k^2 k'^2} dk \quad (3.15)$$

and ¹⁰⁾

$$\int \frac{n \cdot (k-k') n \cdot k}{n^2 q \cdot k} \frac{Z(k'^2)}{k'^2} dk$$

$$= -\frac{2}{3} \left(1 - \frac{1}{\gamma}\right) \int \frac{k \cdot k'}{q \cdot k q \cdot k'} \frac{Z(k'^2)}{k'^2} dk \quad (3.16)$$

Thus the γ dependence is consistent, and (3.3) reduces to the integral equation

$$\frac{1}{Z(q^2)} = \frac{1 + \frac{2}{3} \int dk \frac{k \cdot k'}{q \cdot k q \cdot k'} \frac{Z(k^2)}{k^2}}{1 + \frac{1}{3} \int dk \frac{1}{q \cdot k q \cdot k'} (k^2 q^2 - (q \cdot k)^2) \frac{Z(k^2) Z(k'^2)}{k^2 k'^2}} \quad (3.17)$$

This equation is very much like (3.7), with some minor spinological complications resulting from our more realistic (though still not exact) treatment of the kernels L and K .

As before we try a solution

$$Z(q^2) = (\Lambda^2/q^2)^\alpha$$

with α positive. The power α is determined by an equation analogous to eq. (3.11), but more complicated due to the spinology in eq. (3.17). A numerical integration yields $\alpha = .52$, not very different from the value $1/2$ obtained in the simpler model.

An analyses of eq. (3.3) including the full spinological glory of J and K is in progress and will be reported elsewhere.

If, when we turn to the full equation (3.3), the ansatz (3.1) fails to yield a consistent value of α , or if the γ dependence fails to cancel out of the equations, then we can go back to eq. (2.32) and try the same solution for Z there. Again it will work, provided that the functions $\alpha(\gamma)$ and $\beta(\gamma)$ have the appropriate properties. Of course the equation, analogous to eq. (3.11), which determines the power α is now much more complex, and we are unable to explicitly write it down. The value of the power resulting from eq. (2.32) therefore remains unknown¹¹⁾.

IV. Conclusions

Can we draw any firm conclusions from all of this? Not yet. To recapitulate, we have made two assumptions, one essential, the other an ansatz which is in principle and perhaps in practice verifiable. The essential assumption is that, as in QED, only that part of the vertex function determined through the Ward identity counts in the IR regime. The ansatz is that the Dyson equation then admits a factorized solution of the form $A(q) = Z(q^2/\Lambda^2)^\alpha(\gamma)$ and $B(q) = Z(q^2/\Lambda^2)^\beta(\gamma)$. With more strength, and a study of the trace equation, eq. (2.19), this ansatz can be checked.

Given the above assumptions, we are able to show that the equation admits a solution near $q^2 = 0$ which behaves like $\sim q^2$.

$$Z(q^2/\Lambda^2) = (\Lambda^2/q^2)^\alpha \quad (4.1)$$

The (unknown) functions $\alpha(\gamma)$ and $\beta(\gamma)$ determine the power; the solution exists provided they are such that the power is positive. The power, we note, is independent of the coupling constant; it is also independent of which color group is employed, as long as it is non-Abelian.

If eq. (4.1) is indeed valid, then we have

$$g^2(q^2) = (q^2)^{-\alpha} \quad (4.2)$$

in the IR regime. Such a result is sometimes taken to imply confinement⁽³⁾. This would suggest that confinement is indeed a consequence of the singular IR properties of Yang-Mills theories associated with the fact that glue can couple to itself. There is no obvious relation to some of the more esoteric features of Yang-Mills theories, such as instantons, merons and the like. But since the solution we obtain is nonperturbative, there may nevertheless be one.

In addition to clarifying the relationship, if any, of the Dyson equation/Ward identity approach to these more popular topics in gauge theories, other more mechanical things also remain to be done. It is necessary to incorporate the four-gluon vertex term into the Dyson equation, and to verify that its presence does not alter our basic conclusions. It is necessary to study the true equations, (2.18) and (2.19), and check that (3.1), or at least (2.30), is consistent with them, and that the power solution for $Z(q^2)$ still obtains. Finally, if our present conclusions continue to hold up, it is necessary to

understand more clearly just what, if any, implications a power singularity in $g(q^2)$ at small q^2 really has for confinement.

We are indebted to M. Baker, J. M. Cornwall and D. P. Crewther for advice and comments.

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References and Footnotes

1. J. M. Cornwall and G. Tiktopoulos, Phys. Rev. D10, 2937 (1977);
R. Delbourgo, J. Phys. A10, 1369 (1977).
2. { See for example: W. Kummer, Acta Physica Austriaca II, 313 (1975);
J. Schwinger, Phys. Rev. 130, 402 (1963);
E. Fradkin and I. Tyutin, Phys. Rev. D2, 2841 (1972).
3. For example, see Cornwall and Tiktopoulos, Ref. 1.
4. More precisely, Z is gauge invariant within the class of axial gauges; that is, it does not depend on the direction of the axial gauge. The coupling constant defined by $g^2 = Zg_0^2$ is renormalized at equal masses for all three gluon legs: $g^2(M,M,M) = Z(M)g_0^2$. This g is therefore also independent of the direction of the axial gauge, but it is not truly gauge invariant. Only the "on shell" g (which does not exist due to IR singularities) is gauge invariant. Therefore the suggestion that a singular behavior of $g(M,M,M)$ as $M \rightarrow 0$ implies confinement is one that must be treated with some suspicion. It may, of course, be that the infrared singular part of $g(M)$ as $M \rightarrow 0$ is actually gauge invariant.
5. For a particularly relevant example, see J. Frenkel and R. Meuldermans, Physics Letters 65B, 64 (1976).
6. We modify Kummer's notation primarily by extracting factors of q^{-2} from A and B, so that our A and B are dimensionless.
7. That is, we insert sufficient powers of $\Lambda^2/k^2 + \Lambda^2$ under the integral on the right-hand side of eq. (2.11) to make it converge. This introduces a mass scale into the equation, and permits the solutions to have a non-trivial dependence on q^2/Λ^2 . Note that we could also have used a dimensional cutoff, though in a slightly unusual form since we have an integral equation, not an integral, to deal with. It seems to us slightly simpler to use a mass cutoff, even though in perturbation theory that leads to non-gauge invariant quadratically divergent terms. These contribute to d and e in eq. (2.9).

on axial gauge

8. The mass scale determining "large" and "small" q^2 , is the renormalization group invariant mass μ , which for large M can be expressed through $\mu^2 = M^2 e^{-1/bg^2(M^2)}$. This is the only physically meaningful mass in the theory. In evaluating integrals such as those in eq. (3.7) there are three regimes: $0 < k^2 < q^2$; $q^2 < k^2 < \mu^2$; and $\mu^2 < k^2 < \Lambda^2$ where Λ^2 is the UV cutoff. For $k^2 < \mu^2$, we can use $Z(k^2) = (k^2)^{-\alpha}$; for very large k^2 we expect $Z(k^2)$ to be proportional to $(\log k^2)^{-1}$, because of asymptotic freedom. The dominant IR behavior of the integrals comes from the small k^2 region. *otherwise example is useless.*
9. These forms arise naturally from a slightly different, but IR equivalent, choice for the form of the vertex function than (2.20), together with simply using $\delta_{\sigma\sigma'}$ as the spinology factor associated with $\Delta_{\lambda\sigma} \Delta'_{\lambda\sigma'}$.
10. In deriving eq. (3.16) we have discarded some constant (q^2 independent) terms which, in perturbation theory, would be quadratically divergent. These are lumped with the constants d and e . The integral in eq. (3.16) still contains an additional such constant piece. In evaluating it we will discard this piece as well by cancelling it against d and e . Having performed this service, d and e may now be disregarded.
11. Again, of course, it is unknown that there exists a positive α satisfying this condition. Unless there does, the power law is not actually a solution.

Figure Caption

- Fig. 1: (a) Shows the Dyson equation with only the triple glue term included.
 (b) Shows the (neglected) quadruple glue term.

$$(\text{wavy line with a shaded circle})^{-1} = (\text{wavy line})^{-1} + \frac{1}{2} \text{ (loop diagram with four shaded circles) }$$

(a)

$$\frac{1}{6} \text{ (loop diagram with six shaded circles) }$$

(b)

Figure 1

Equation Derivations

Index of Equation Derivations:

- (2.20) the most general form of the vertex (almost())
- (2.23) the vertex dotted into n , for the nn equation
- (2.27) equation 2.28 (the nn) with n .vertex inserted
- show that vertex (2.20) solves the Ward (2.14)
- (2.16) the qq equation involving d and e
- (2.17) solution for d and e
- (2.15) the nq equation involving d and e
- (2.18) the nn equation, original form
- (2.19) the trace equation
- (2.32) the nn equation with general α/β ansatz
- (2.33) the tensor I call $W_{1s}W'_{1s}$, arising from pair of propagators
- (3.3) the nn equation in the SA model
- (3.4) the expression for L
- (3.5) the expression for K
- (3.6) the nn equation in the fake trivial model
- (3.7) solution of same for $Z(q)$, fake trivial
- (3.10) the power solution, fake trivial
- (3.11) equation which says $\alpha = 1/2$, fake trivial model
- (3.1) the SA form of propagator involves only Z
- (3.2) the SA form of "inverse propagator"
- (2.4) equation relating propagator to its "inverse"
- (2.10) form of bare "inverse" propagator

(2.23) + (2.27)

RHS (2.18) = $\frac{1}{2} \int dk \frac{n_\mu(k-k')}{n^2} \Delta_{\lambda\sigma}(k) \Delta_{\lambda'\sigma'}(k') [i\Gamma(k, k', -q)]_{\sigma\sigma'} n_\nu$
 $\uparrow$ 15 terms

$$i\Gamma(k, k', -q)_{\sigma\sigma'} = \delta_{\sigma\lambda'} \left\{ \frac{k_\nu}{-k \cdot q} g(k) - \frac{k'_\nu}{-k' \cdot q} g(k') \right\} + \delta_{\sigma\sigma'} n_\nu \left\{ \frac{f(k') - f(k)}{(-q) \cdot n} \right\} \\ + \delta_{\sigma'\nu} \left\{ \frac{k'_\sigma}{k' \cdot k} g(k') - \frac{(-q)_\sigma g(-q)}{(-q) \cdot k} \right\} + \delta_{\sigma'\nu} n_\sigma \left\{ \frac{f(-q) - f(k)}{k \cdot n} \right\} \\ + \delta_{\nu\sigma} \left\{ \frac{(-q)_{\sigma'}}{(-q) \cdot k'} g(-q) - \frac{(k)_{\sigma'} g(k)}{k \cdot k'} \right\} + \delta_{\nu\sigma} n_{\sigma'} \left\{ \frac{f(k) - f(-q)}{k' \cdot n} \right\}$$

$$+ \frac{g(k)}{k^2} (k)_\sigma (k')_\nu \left\{ \frac{k'_\sigma}{k \cdot (-q)} - \frac{(-q)_\sigma}{k \cdot k'} \right\}$$

$$+ \frac{g(k')}{(k')^2} (k')_\nu (k)_\sigma \left\{ \frac{(-q)_{\sigma'}}{k' \cdot k} - \frac{(k)_{\sigma'}}{k' \cdot (-q)} \right\}$$

Other 6 terms do not contribute because, eg, $\Delta_{\lambda\sigma}(k) n_\sigma = 0$, since $A_\sigma(x) n_\sigma = 0$, axial gauge.

So 9 terms survive in Γ .

$$+ \frac{g(-q)}{(-q)^2} (-q)_\sigma (-q)_{\sigma'} \left\{ \frac{(k)_\nu}{(-q) \cdot k'} - \frac{(k')_\nu}{(-q) \cdot k} \right\}$$

Next: $\Delta_{\lambda\sigma}(k) = -k^2 \left\{ \left(\delta_{\lambda\sigma} - \frac{n_\lambda n_\sigma}{n^2} \right) B(k) + \frac{N_\lambda N_\sigma}{n^2} C(k) \right\}$ note $n \cdot N = 0$

$$\Delta_{\lambda'\sigma'}(k') = -(k')^2 \left\{ \left(\delta_{\lambda'\sigma'} - \frac{n_{\lambda'} n_{\sigma'}}{n^2} \right) B(k') + \frac{N_{\lambda'} N_{\sigma'}}{n^2} C(k') \right\}$$

Thus there are $2 \times 2 \times 9 = 36$ terms in RHS of (2.18). Let's do the n_ν on the 9 terms to see if simple.

$$i\Gamma(k, k', -q)_{\sigma\sigma'} n_\nu = \delta_{\sigma\lambda'} \left[\frac{k \cdot n}{k \cdot (-q)} g(k) - \frac{k' \cdot n}{k' \cdot (-q)} g(k') + \frac{n^2}{(-q) \cdot n} (f(k') - f(k)) \right] \\ + n_{\sigma'} \left[\text{stuff, but then this goes since } n_{\sigma'} \Delta_{\lambda\sigma'} = 0 \right] \\ + n_\sigma \left[\text{this also goes} \right]$$

$$+ \frac{g(k)}{k^2} (k)_\sigma (k')_\nu \left\{ \frac{k'_\sigma}{k \cdot (-q)} - \frac{(-q)_\sigma}{k \cdot k'} \right\}$$

Down basically to 4 terms.

$$+ \frac{g(k')}{(k')^2} (k')_\nu (k)_\sigma \left\{ \frac{(-q)_{\sigma'}}{k' \cdot k} - \frac{(k)_{\sigma'}}{k' \cdot (-q)} \right\}$$

$$+ \frac{g(-q)}{(-q)^2} (-q)_\sigma (-q)_{\sigma'} \left\{ \frac{k \cdot n}{(-q) \cdot k'} - \frac{k' \cdot n}{(-q) \cdot k} \right\}$$

Pause to look at symmetry of P : --

$$i\Gamma_{\mu\nu\sigma}^{(0)}(p, q, k) = (p-q)_\sigma \delta_{\mu\nu} + (q-k)_\mu \delta_{\nu\sigma} + (k-p)_\nu \delta_{\sigma\mu}$$

= antisynsmebic under $(p \leftrightarrow q : \text{and. } \mu \leftrightarrow v)$.

$$= [p_\mu \delta_{\mu\nu} - k_\mu \delta_{\nu\alpha} - p_\nu \delta_{\alpha\mu}] - [p \rightarrow q, \mu \leftrightarrow \nu].$$

Why is this so? Vaguely, we intuitively say

$$\left[\begin{array}{c} m_1, j_1 \\ m_2, j_2 \end{array} \right] = \langle j_1 j_2 m_1 m_2 | J_3 m_3 \rangle \times [\text{rot. scalar}]$$
$$= (-1)^{j_3 - j_1 - j_2} \times \left[\begin{array}{c} m_1, j_1 \\ m_2, j_2 \end{array} \right]$$
 and $(-1)^{|j_1-j_2|} = -1$

Can I make this idea rigorous? // Yes, see some other sheet!!

/go to next page:-

Now knowing about antisymmetry, we can say:

$$i\Gamma(k, k', -q)_{\alpha\alpha'}(n) = \left\{ \delta_{\alpha\alpha'} \left[\frac{k \cdot n}{k \cdot (-q)} g(k) - \frac{n^2 f(k)}{(-q) \cdot n} \right] \right.$$

$$+ \frac{g(k)}{k^2} (k \cdot n) (k')_{\alpha'} \left\{ \frac{(k')_{\alpha}}{k \cdot (-q)} - \frac{(-q)_{\alpha}}{k \cdot k'} \right\} + \frac{g(-q)}{(-q)^2} (-q)_{\alpha'} \frac{(k \cdot n)}{(-q) \cdot k'} \left. \right\}$$

minus { terms with $k \leftrightarrow k'$ and $\alpha \leftrightarrow \alpha'$ }

$$= \left[\delta_{\alpha\alpha'} \left\{ \frac{n^2}{q \cdot n} f(k) - \frac{n \cdot k}{q \cdot k} g(k) \right\} \right. \checkmark$$

$$+ g(k) \left\{ \frac{(k \cdot n) (k)_{\alpha'} (k')_{\alpha}}{-k^2 (k \cdot q)} + \frac{(k \cdot n) (k)_{\alpha'} (q)_{\alpha}}{k^2 (k \cdot k')} \right\}$$

$$+ g(-q) \left\{ \frac{q_{\alpha} q_{\alpha'} (k \cdot n)}{-q^2 (q \cdot k')} \right\} \Big] - [\text{reverse term}]$$

Can replace this with its reverse. $\Rightarrow g(-q) \left\{ \frac{q_{\alpha} q_{\alpha'} (k' \cdot n)}{+q^2 (q \cdot k)} \right\} \checkmark$

The second term needs work:

$$\left\{ \right\}_2 = \frac{(k)_{\alpha'} (k')_{\alpha}}{-k^2 (k \cdot q)} + \frac{(k)_{\alpha'} (q)_{\alpha}}{k^2 (k \cdot k')}$$

Try: $(k)_{\alpha'} (k')_{\alpha} = (k)_{\alpha'} (q - k)_{\alpha} = (k)_{\alpha'} (q)_{\alpha} - (k)_{\alpha'} (k)_{\alpha}$

Then $\left\{ \right\}_2 = \frac{(k)_{\alpha'} (q)_{\alpha}}{-k^2 (k \cdot q)} - \frac{(k)_{\alpha'} (k)_{\alpha}}{-k^2 (k \cdot q)} + \frac{(k)_{\alpha'} (q)_{\alpha}}{k^2 (k \cdot k')} \quad / k \cdot n \text{ removed}$

$$= + \frac{(k)_{\alpha} (k)_{\alpha'}}{k^2 (k \cdot q)} + \frac{(q)_{\alpha} (k)_{\alpha'}}{k^2} \left[\frac{1}{(k \cdot k')} - \frac{1}{(k \cdot q)} \right]$$

$$= \frac{(q)_{\alpha} (k)_{\alpha'}}{k^2} \left[\frac{k \cdot (q - k')}{(k \cdot k') (k \cdot q)} \right] = \frac{(q)_{\alpha} (k)_{\alpha'}}{k^2} \left[\frac{k'}{k \cdot k' \cdot q} \right]$$

So $\left\{ \right\}_2 = \frac{(k \cdot n)}{(k \cdot k') (k \cdot q)} (q)_{\alpha} (k)_{\alpha'} + \frac{(k)_{\alpha} (k)_{\alpha'}}{k^2 (k \cdot q)} (n \cdot k) \quad \checkmark$

So (2.23) is verified.

Now put this into (2.18):

$$\int_{-\infty}^{\infty} dk \underbrace{\frac{n \cdot (k-k')}{n^2} \Delta_{\lambda\alpha}(k) \Delta_{\lambda\alpha'}(k')}_{g_{\alpha\alpha'}(k,k') = \text{anti-symmetric}} \underbrace{f_{\alpha\alpha'}(k,k')}_{\text{antisym.}} \quad q = k+k'$$

$$\int_{-\infty}^{\infty} dk f_{\alpha\alpha'}(k,k') = - \int_{-\infty}^{\infty} dk' f_{\alpha\alpha'}(k,k'), \text{ so } \int dk = \int dk' \text{ in } \checkmark$$

So

$$\text{RHS}(2.18) = \int dk \frac{n \cdot (k-k')}{n^2} \Delta_{\lambda\alpha}(k) \Delta_{\lambda\alpha'}(k')$$

$$\times \left[g(k) \left\{ -\delta_{\alpha\alpha'} \frac{n \cdot k}{q \cdot k} + \frac{k_\alpha k_{\alpha'}}{k^2} \frac{n \cdot k}{q \cdot k} + \frac{q_\alpha k_{\alpha'}}{q \cdot k} \frac{n \cdot k}{k' \cdot k} \right\} \right. \\ \left. + g(q) \left\{ \frac{q_\alpha q_{\alpha'} n \cdot k'}{q^2 q \cdot k} \right\} + f(k) \left\{ \delta_{\alpha\alpha'} \frac{n^2}{q \cdot n} \right\} \right]$$

$$= \int dk \Delta_{\lambda\alpha}(k) \Delta_{\lambda\alpha'}(k') \left\{ f(k) \times \underbrace{\left[\delta_{\alpha\alpha'} \frac{n \cdot (k-k')}{q \cdot n} \right]}_{\equiv K_{\alpha\alpha'}^f} \right. \\ \left. + g(q) \times \underbrace{\left[\frac{q_\alpha q_{\alpha'} n \cdot (k-k') n \cdot k'}{q^2 q \cdot k n^2} \right]}_{\equiv L_{\alpha\alpha'}} \right. \\ \left. + g(k) \underbrace{\left[\frac{n \cdot (k-k')}{n^2} \frac{n \cdot k}{q \cdot k} \left(-\delta_{\alpha\alpha'} + \frac{k_\alpha k_{\alpha'}}{k^2} + \frac{q_\alpha k_{\alpha'}}{k' \cdot k} \right) \right]}_{\equiv K_{\alpha\alpha'}^g} \right]$$

$$= \int dk \Delta_{\lambda\alpha}(k) \Delta_{\lambda\alpha'}(k') \left\{ K_{\alpha\alpha'}^f f(k) + K_{\alpha\alpha'}^g g(k) + L_{\alpha\alpha'} g(q) \right\}$$

Thus verified (2.27).

To show that (2.20) is an acceptable solution of (2.19). ✓

$$\Pi_{\mu\nu\lambda}(k, k', p) p_\lambda = \Pi_{\mu\nu}(k) - \Pi_{\mu\nu}(k')$$

$$\text{RHS} = -k^2 \left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right) b(k) - K_\mu K_\nu c(k) \quad K_\mu = k_\mu - \frac{\eta_\mu k^2}{k \cdot \eta}$$

$$= \left\{ k \rightarrow k' \right\}_{\mu \leftrightarrow \nu}$$

LHS = 15 terms.

$$\Pi_{\mu\nu\lambda}(k, k', p) = \delta_{\mu\nu} \left[\frac{k_\lambda}{k \cdot p} g(k) - \frac{k'_\lambda}{k' \cdot p} g(k') \right]$$

$$+ \delta_{\mu\lambda} \left[\frac{k'_\nu}{k' \cdot k} g(k') - \frac{p_\nu}{p \cdot k} g(p) \right] + \delta_{\lambda\nu} \left[\frac{p'_\mu}{p \cdot k'} g(p) - \frac{k'_\mu}{k \cdot k'} g(k') \right]$$

Keep just these terms on now. Then apply p_λ :

$$= \delta_{\mu\nu} \left[\frac{k \cdot p}{k \cdot p} g(k) - \frac{k' \cdot p}{k' \cdot p} g(k') \right]$$

$$+ p_\lambda \left[\frac{k'_\nu}{k' \cdot k} g(k') - \frac{p_\nu}{p \cdot k} g(p) \right] + p_\lambda \left[\frac{p'_\mu}{p \cdot k'} g(p) - \frac{k'_\mu}{k \cdot k'} g(k') \right]$$

$$= \delta_{\mu\nu} [g(k) - g(k')] + p_\mu k_\nu \left[\frac{g(k)}{k \cdot k'} \right] - \frac{p_\mu p_\nu}{p \cdot k} g(p)$$

$$= \delta_{\mu\nu} g(k) + \cancel{p_\mu k'_\nu g} - p_\mu k'_\nu \frac{g(k)}{(k \cdot k')} + \frac{p_\mu p'_\nu}{p \cdot k'} g(p) + \text{etc}$$

So

$$\boxed{\text{1st 3 terms} = \left[\delta_{\mu\nu} g(k) - p_\mu k'_\nu \frac{g(k)}{k \cdot k'} + \frac{p_\mu p'_\nu}{p \cdot k'} g(p) \right] \xrightarrow{\left\{ k \leftrightarrow k' \right\}_{\mu \leftrightarrow \nu}}}$$

Third set of 3 terms:

$$\left(\delta_{\mu\nu} \eta_\lambda \left[\frac{f(k') - f(k)}{p \cdot \eta} \right] + \delta_{\mu\lambda} \eta_\nu \left[\frac{f(p) - f(k')}{k \cdot \eta} \right] + \delta_{\lambda\nu} \eta'_\mu \left[\frac{f(k) - f(p)}{k' \cdot \eta} \right] \right) p_\lambda$$

$$= -\delta_{\mu\nu} \frac{(n \cdot p)}{(n \cdot p)} f(k^2) + p'_\mu \eta_\nu \frac{f(p)}{k \cdot \eta} + p'_\mu \eta'_\nu \frac{f(k^2)}{k' \cdot \eta}$$

$$3^{\text{rd}} \text{ 3 terms} = \left[-\sum_{\mu\mu'} f(k) + \eta_{\mu\mu'} \frac{f(p)}{k \cdot n} + p_{\mu} \eta_{\mu'} \frac{f(k)}{k' \cdot n} \right] - [\text{switch}]$$

Second set:

$$\begin{aligned} & \frac{g(k)}{k^2} (k)_{\mu} (k)_{\mu'} \left[\frac{k'_{\mu}}{k \cdot p} - \frac{p_{\mu}}{k \cdot k'} \right] \\ & + \frac{g(k')}{k'^2} (k')_{\lambda} (k')_{\mu} \left[\frac{p_{\lambda'}}{k \cdot k} - \frac{k'_{\lambda'}}{k' \cdot p} \right] \\ & + \frac{g(p)}{p^2} (p)_{\mu} (p)_{\mu'} \left[\frac{k_{\lambda}}{p \cdot k'} - \frac{k'_{\lambda}}{p \cdot k} \right] \end{aligned}$$

Apply p:

$$\frac{g(k)}{k^2} (k)_{\mu} (p \cdot k) \left[\frac{k'_{\mu}}{(p \cdot k)} - \frac{p_{\mu}}{(k \cdot k')} \right] + \frac{g(p)}{p^2} (p_{\mu}) (p_{\mu'}) \left[\frac{p \cdot k}{p \cdot k'} \right]$$

So

$$2^{\text{nd}} \text{ 3 terms} = \left[\frac{g(k)}{k^2} (k)_{\mu} (p \cdot k) \left[\frac{k'_{\mu}}{(p \cdot k)} - \frac{p_{\mu}}{(k \cdot k')} \right] + \frac{g(p)}{p^2} (p_{\mu}) (p_{\mu'}) \left[\frac{p \cdot k}{p \cdot k'} \right] \right] - [sw]$$

Fourth Set

$$\begin{aligned} & \eta_{\mu\mu'} \left\{ \frac{(k')_{\lambda}}{k \cdot n} \left[\frac{f(k')}{k' \cdot n} + \frac{f(p)}{p \cdot n} \right] - \frac{(k)_{\lambda}}{k' \cdot n} \left[\frac{f(k)}{k \cdot n} + \frac{f(p)}{p \cdot n} \right] \right\} \\ & + \eta_{\mu'\mu} \left\{ \frac{(p)_{\mu}}{k' \cdot n} \left[\frac{f(p)}{p \cdot n} + \frac{f(k)}{k \cdot n} \right] - \frac{(k')_{\mu}}{p \cdot n} \left[\frac{f(k')}{k' \cdot n} + \frac{f(k)}{k \cdot n} \right] \right\} \\ & + \eta_{\lambda\mu'} \left\{ \frac{(k)_{\mu'}}{p \cdot n} \left[\frac{f(k)}{k \cdot n} + \frac{f(k')}{k' \cdot n} \right] - \frac{(p)_{\mu'}}{k \cdot n} \left[\frac{f(p)}{p \cdot n} + \frac{f(k')}{k' \cdot n} \right] \right\} \end{aligned}$$

$$\Rightarrow -\eta_{\mu\mu'} \frac{(p \cdot k)}{(k' \cdot n)} \left[\frac{f(k)}{k \cdot n} + \frac{f(p)}{p \cdot n} \right] + \text{cancel} = 4^{\text{th}} \text{ set of 3 terms}$$

$$+ (p \cdot n) \eta_{\mu'} \left[\frac{(p_{\mu}) f(k)}{k \cdot n k' \cdot n} - \frac{(k')_{\mu} f(k)}{p \cdot n k \cdot n} \right] + (p \cdot n) \eta_{\mu} \left[\frac{k'_{\mu} f(k)}{p \cdot n k \cdot n} - \frac{p_{\mu} f(p)}{k \cdot n p \cdot n} \right] + [sw]$$

5th set

$$\eta_\mu \eta_{\mu'} \eta_\lambda \left\{ \frac{f(k)}{(k \cdot n)^2} \left(\frac{k \cdot k'}{p \cdot n} - \frac{k \cdot p}{k' \cdot n} \right) + \frac{f(k')}{(k' \cdot n)^2} \left(\frac{k' \cdot p}{k \cdot n} - \frac{k' \cdot k}{p \cdot n} \right) + \frac{f(p)}{(p \cdot n)^2} \left(\frac{p \cdot k}{k' \cdot n} - \frac{p \cdot k'}{k \cdot n} \right) \right\}$$

$$\Rightarrow \boxed{(p \cdot n) \eta_\mu \eta_{\mu'} \left[\frac{f(k)}{(k \cdot n)^2} \left(\frac{k \cdot k'}{p \cdot n} - \frac{k \cdot p}{k' \cdot n} \right) + \frac{f(p)}{(p \cdot n)^2} \frac{p \cdot k}{k' \cdot n} \right] + \text{switch}} \\ = 5^{\text{th}} \text{ set}$$

Now look again at RHS of ward.

$$\text{RHS} = -k^2 (\delta_{\mu\mu'} - \frac{k_\mu k_{\mu'}}{k^2}) b(k) - \left(k_\mu - \eta_\mu \frac{k^2}{k \cdot n} \right) \left(k_{\mu'} - \eta_{\mu'} \frac{k^2}{k \cdot n} \right) c(k) + \text{switch}.$$

We have made elimination to do! Omit switches from now on. Gather all $g_{\mu\mu'}$ terms:

$$g_{\mu\mu'} [g(k) - f(k)] = -k^2 g_{\mu\mu'} = g_{\mu\mu'} [-k^2 b(k)].$$

This requires

$$(g - f) = -k^2 b.$$

so $g_{\mu\mu'} \checkmark$

$$\text{Check: } g - f = -k^2 (b - c) - k^2 c = -k^2 b \checkmark.$$

Next, do $\eta_\mu \eta_{\mu'}$:

$$\eta_\mu \eta_{\mu'} \left[-\frac{(p \cdot k)}{(k' \cdot n)} \left[\frac{f(k)}{k \cdot n} + \frac{f(p)}{p \cdot n} \right] + (p \cdot n) \left\{ \frac{f(k)}{(k \cdot n)^2} \left(\frac{k \cdot k'}{p \cdot n} - \frac{k \cdot p}{k' \cdot n} \right) + \frac{f(p)}{(p \cdot n)^2} \frac{p \cdot k}{k' \cdot n} \right\} \right]$$

The $f(p)$ must go away

$$\text{coeff} = f(p) \left\{ \frac{-(p \cdot k)}{(p \cdot n)(k' \cdot n)} + \frac{(p \cdot n)(p \cdot k)}{(p \cdot n)(k' \cdot n)} \right\} + f(k) \left\{ \frac{-(p \cdot k)}{(k' \cdot n)(k \cdot n)} + \frac{(p \cdot n)}{(k \cdot n)^2} \left[\frac{k \cdot k'}{p \cdot n} - \frac{k \cdot p}{k' \cdot n} \right] \right\}$$

$$\text{coeff of } f(k) \text{ is: } \frac{f(k)}{(k \cdot n)} \left[\frac{-p \cdot k \cdot k \cdot n}{k' \cdot n \cdot k \cdot n} + \frac{k \cdot n}{k' \cdot n \cdot k \cdot n} - \frac{p \cdot n \cdot k \cdot p}{k \cdot n \cdot k' \cdot n} \right] \\ = \frac{f(k)}{(k \cdot n)^2 (k' \cdot n)} [-p \cdot k \cdot k \cdot n + k' \cdot n \cdot k \cdot k' - p \cdot n \cdot k \cdot p]$$

$$k+k'+p=0$$

$$N = (-p \cdot k \cdot k \cdot n + k' \cdot n \cdot k \cdot k' - p \cdot n \cdot k \cdot p)$$

$$= k \cdot p [-k \cdot n - p \cdot n] + k' \cdot n \cdot k \cdot k' = +k \cdot p \cdot n \cdot (-k-p) + k' \cdot n \cdot k \cdot k'$$

$$= n \cdot k' [k \cdot p + k \cdot k'] = n \cdot k' \cdot k \cdot [p+k'] = -n \cdot k' (k \cdot k) = -k^2 (n \cdot k')$$

$$\text{So: } \frac{f(k)}{(k \cdot n)^2 (k' \cdot n)} [-k^2 (n \cdot k')] = \frac{-k^2 f(k)}{(k \cdot n)^2}$$

$$\text{So check to see if } \frac{-k^2 f(k)}{(k \cdot n)^2} = \frac{-k^2 \cdot k^2}{(k \cdot n)^2} c(k)$$

$$\Rightarrow f = k^2 \cdot c \quad \checkmark \quad \text{so } \eta_\mu \eta_{\mu'} \checkmark$$

Next, do the $\eta_\mu \eta_{\mu'}$: well, must gather all terms + see what we have:

$$\begin{aligned} & \left[-p_\mu k_{\mu'} \frac{g(k)}{k \cdot k'} + \frac{p_\mu p_{\mu'}}{p \cdot k'} g(p) + \cancel{\eta_\mu p_{\mu'} \frac{f(p)}{k \cdot n}} + p_\mu \cancel{\eta_{\mu'} \frac{f(k)}{k' \cdot n}} \right. \\ & + k_\mu k_{\mu'} \frac{g(k)}{k^2} - k_{\mu'} p_\mu \frac{g(k)}{k^2} \frac{p \cdot k}{k' \cdot k} + \frac{g(p)}{p^2} p_\mu p_{\mu'} \left(\frac{p \cdot k}{p \cdot k'} \right) \\ & \left. + \eta_{\mu'} p_\mu \left(\frac{p \cdot n}{k \cdot n} \frac{f(k)}{k' \cdot n} \right) - \cancel{\eta_{\mu'} k_{\mu'} \frac{f(k)}{k \cdot n}} + \cancel{\eta_\mu k_{\mu'} \frac{f(k)}{k \cdot n}} - \cancel{\eta_\mu p_{\mu'} \frac{f(p)}{k \cdot n}} \right] \end{aligned}$$

This is all of it.

$$g(p) \left[\frac{p_\mu p_{\mu'}}{p \cdot k'} + \frac{p_\mu p_{\mu'}}{p^2} \left(\frac{p \cdot k}{p \cdot k'} \right) \right] = g(p) p_\mu p_{\mu'} \left[\frac{1}{p \cdot k'} + \frac{p \cdot k}{p^2 p \cdot k'} \right]$$

$$= \frac{g(p) p_\mu p_{\mu'}}{p \cdot k'} \left[1 + \frac{p \cdot k}{p^2} \right] = \frac{g(p) p_\mu p_{\mu'}}{p \cdot k' p^2} [p \cdot p + p \cdot k] = \left\{ -\frac{g(p) p_\mu p_{\mu'}}{p^2} \right\} \text{troubles}$$

$$\text{Ah yes, but } \left[\frac{g(p) p_\mu p_{\mu'}}{p^2} \right] - (\text{switch}) = (\quad) - (\text{same}) = 0 \quad \checkmark$$

Remaining terms are, involving n :

$$\eta_{\mu} p_{\mu} \left[\frac{f(k)}{k \cdot n} + \frac{p \cdot n}{k \cdot n} \frac{f(k)}{k \cdot n} \right] + \frac{f(k)}{k \cdot n} [\eta_{\mu} k_{\mu} - \eta_{\mu}' k_{\mu}']$$

$$\Rightarrow \frac{f(k)}{(k \cdot n)} [k_{\mu} + k_{\mu}'] \eta_{\mu}' \left[1 + \frac{p \cdot n}{k \cdot n} \right] - \eta_{\mu}' k_{\mu}' \frac{f(k)}{k \cdot n}$$

$$\eta_{\mu}' k_{\mu}' \left[\cancel{\frac{f(k)}{k \cdot n}} \frac{(k \cdot n)}{k \cdot n} - \frac{f(k)}{k \cdot n} \right] \quad \text{leaves} \quad \eta_{\mu}' k_{\mu} \left(\frac{f(k)}{k \cdot n} \right)$$

So all n -terms are:

$$\frac{f(k)}{k \cdot n} [\eta_{\mu} k_{\mu}' + \eta_{\mu}' k_{\mu}]$$

Should be:

$$\eta_{\mu} k_{\mu}' \left(\frac{k^2 c(k)}{k \cdot n} \right) + \eta_{\mu}' k_{\mu} (") \quad \text{so} \quad \frac{f(k)}{k \cdot n} = ? \frac{k^2 c}{k \cdot n} \checkmark$$

Only remaining terms are now:

$$\left[-p_{\mu} k_{\mu}' \frac{g(k)}{k \cdot k' k^2} + k_{\mu}' k_{\mu} \frac{g(k)}{k^2 k \cdot k'} - k_{\mu}' p_{\mu} \frac{g(k)}{k^2} \frac{p \cdot k}{k' \cdot k} \right]$$

$$= \frac{g(k)}{k \cdot k' k^2} \left[-p_{\mu} k_{\mu}' k^2 + k_{\mu}' k_{\mu} k \cdot k' - k_{\mu}' p_{\mu} p \cdot k \right]$$

$$[] = p_{\mu} k_{\mu}' \left[\frac{-k^2 - k \cdot p}{k \cdot k'} \right] + k_{\mu}' k_{\mu} k \cdot k' = k \cdot k' [p_{\mu} k_{\mu}' + k_{\mu}' k_{\mu}]$$

$$= k \cdot k' k_{\mu}' [-k_{\mu}] = -k_{\mu} k_{\mu}' (k \cdot k')$$

$$\text{So} \rightarrow \cancel{\frac{g(k) k_{\mu} k_{\mu}'}{k^2}} = ? \neq k_{\mu} k_{\mu}' c(k) \Rightarrow g = k^2 c(k) \text{ NO!}$$

$$\quad \quad \quad - \cancel{k_{\mu} k_{\mu}'} + b(k) \Rightarrow g = -k^2 (b - c) \checkmark \text{ yes.}$$

Conclusion: after a lot of work, I have shown that (2.20) with (2.21) and (2.22) does satisfy the Ward, (2.14).

All terms in (2.20) were needed! Could not drop some.
Any new terms must cancel.

$$(2.16) + (2.17)$$

About the d+e terms.

① First, let's check (2.16) and (2.17):

$$q_\mu q_\nu \overline{\Pi}_{\mu\nu} = -\frac{1}{2} \int dk (q_\mu i \overline{\Pi}_{\mu\alpha}^\circ(q, -k, -k') / \Delta_{\alpha\alpha'}(k) \Delta_{\alpha\alpha'}(k') [i \overline{\Pi}_{\alpha\alpha'}^\circ(k, k', -q) q_\nu]$$

Recall that $n \cdot \Delta = 0$ and $q \cdot \Pi = 0$ [for full n base]. Thus, the LHS is

$$q_\mu q_\nu \overline{\Pi}_{\mu\nu} = \underline{d \cdot q^2 + e \cdot (n \cdot q)^2 / n^2} \quad \checkmark \quad \text{Now for the RHS.}$$

$$q_\mu i \overline{\Pi}_{\mu\alpha}^\circ(q, -k, -k') = i \overline{\Pi}_{\alpha\mu}^\circ(-k, -k', q) q_\mu = \Pi_{\alpha\alpha'}^\circ(-k) - \Pi_{\alpha\alpha'}^\circ(-k')$$

This is the bare Ward. This interim is $\Pi_{\alpha\alpha'}^\circ(k) - \Pi_{\alpha\alpha'}^\circ(k')$

$$= -[k^2 \delta_{\alpha\alpha'} - k_\alpha k_{\alpha'}] + [k'^2 \delta_{\alpha\alpha'} - k'_\alpha k'_{\alpha'}] = \delta_{\alpha\alpha'} (k'^2 - k^2) + [k_\alpha k_{\alpha'} - k'_\alpha k'_{\alpha'}]$$

Meanwhile,

$$-i \overline{\Pi}_{\alpha\alpha'}^\circ(k, k', -q) q_\nu = -\Pi_{\alpha\alpha'}^\circ(k) + \Pi_{\alpha\alpha'}^\circ(k')$$

$$\text{But then } \Delta_{\alpha\alpha'}(k) \Pi_{\alpha\alpha'}^\circ(k) = \delta_{\alpha\alpha'} - \frac{k_\alpha n_{\alpha'}}{(k \cdot n)}$$

$$\Delta_{\alpha\alpha'}(k') \Pi_{\alpha\alpha'}^\circ(k') = \delta_{\alpha\alpha'} - \frac{k'_\alpha n_{\alpha'}}{(k' \cdot n)}$$

$$\text{So RHS} = \frac{1}{2} \int dk \left\{ \underbrace{\delta_{\alpha\alpha'} (k'^2 - k^2) + (k_\alpha k_{\alpha'} - k'_\alpha k'_{\alpha'})}_{\text{anti}} \right\} \left\{ \underbrace{\Delta_{\alpha\alpha'}(k') \left[-\delta_{\alpha\alpha'} + \frac{k_\alpha n_{\alpha'}}{(k \cdot n)} \right]}_{\text{anti}} + \Delta_{\alpha\alpha'}(k) \left[\delta_{\alpha\alpha'} - \frac{k'_\alpha n_{\alpha'}}{(k' \cdot n)} \right] \right\}$$

$$= \frac{1}{2} \int dk \left\{ \delta_{\alpha\alpha'} (k'^2 - k^2) + (k_\alpha k_{\alpha'} - k'_\alpha k'_{\alpha'}) \right\} \left\{ \Delta_{\alpha\alpha'}(k') - \Delta_{\alpha\alpha'}(k) \right\}$$

$$\text{Like } \int dk f^\dagger(k, k') [f(k) - f(k')] = \int dk \dots$$

$$= \frac{1}{2} \int dk \left[\delta_{\alpha\alpha'} (k'^2 - k^2) + (k_\alpha k_{\alpha'} - k'_\alpha k'_{\alpha'}) \right] \Delta_{\alpha\alpha'}(k')$$

$$= + \int dk \left[\delta_{\alpha\alpha'} (k'^2 - k^2) + (k_\alpha k_{\alpha'} - k'_\alpha k'_{\alpha'}) \right] \Delta_{\alpha\alpha'}(k')$$

Thus: (2.16) $\checkmark$

Solve (2.15) + (2.16) for $\Delta_{\lambda\lambda}$.

① First, examine:

$$I_1 \equiv \int dk' [k_\lambda k'_\lambda - k'_\lambda k_\lambda] \left\{ (\delta_{\lambda\lambda} - n_\lambda n'_\lambda) B' + N'_\lambda N'_\lambda C' \right\} \left(-\frac{1}{k'^2} \right) = \Delta_{\lambda\lambda}(k')$$

$$= \int dk' [(k^2 - k'^2 - a^2 + a'^2) B' + ((k \cdot N')^2 - (k' \cdot N')^2) C'] \left(-\frac{1}{k'^2} \right)$$

Now remove all k -stuff in favor of k' :

$$\text{coeff } B' = (q - k')^2 - k'^2 - (c - a')^2 + a'^2 = q^2 - 2qk' - c^2 + 2ca' \doteq q^2 - c^2$$

because odd terms vanish under $\int dk'$! Next,

$$\text{coeff } C' = [(q \cdot N') - (k' \cdot N')]^2 - (k' \cdot N')^2$$

$$= (q \cdot N')^2 - 2(q \cdot N')(k' \cdot N') \doteq (q \cdot N')^2 \text{ same reason.}$$

$N' = n - k'/n \cdot k' \rightarrow +N'$

Thus:

$$I_1 = \int dk' (k_\lambda - k'_\lambda) \Delta(k') = \int dk' [(q^2 - c^2) B' + (q \cdot N')^2 C'] / (-k'^2)$$

② Next:

$$\Delta_{\lambda\lambda}(k') = 3B' + (N' \cdot N') C'$$

So

$$I_2 \equiv \int dk' (a - a') [3B' + (N' \cdot N') C']$$

$$= \int dk' (\underbrace{c - 2a'}_{\text{odd}}) [3B' + (N' \cdot N') C'] = 0 \quad c \int dk' [3B' + N'^2 C']$$

So

$$I_2 \equiv \int dk' n \cdot (k - k') \Delta_{\lambda\lambda}(k') = \int dk' [(3c) B' + (N'^2) C'] / (-k'^2)$$

③ Next, $I_3 = \int dk' [(q - k')^2 - k'^2] \Delta_{\lambda\lambda}(k') \quad [q^2 - \underbrace{2qk'}_{\text{odd}}]$

$$\Rightarrow I_3 \equiv \int dk' (k^2 - k'^2) \Delta_{\lambda\lambda}(k') = \int dk' [(3q^2) B' + (N'^2 q^2) C'] / (-k'^2)$$

$$N' = n - k'/n \cdot k'$$

We represent:

$$I_1 = (q^2 - c^2, (q \cdot N')^2) \checkmark$$

$$I_2 = (3c, N'^2 c) \checkmark$$

$$I_3 = (3q^2, N'^2 q^2) \checkmark$$

$$q \cdot N' = c - b'/a'$$

Now (2.15, 16) say:

$$c^2(d+e) = I_2 c \checkmark$$

$$q^2 d + c^2 e = I_3 - I_1 \checkmark$$

$$\text{Then: } (c^2 - q^2)d = I_2 c - I_3 + I_1 \checkmark$$

$$= (3c^2 - 3q^2 + q^2 - c^2, N'^2 c^2 - N'^2 q^2 + (q \cdot N')^2)$$

$$= (2c^2 - 2q^2, N'^2(c^2 - q^2) + (q \cdot N')^2)$$

$$= (c^2 - q^2) \left(2, N'^2 + \frac{(q \cdot N')^2}{c^2 - q^2} \right) \checkmark$$

$$\Rightarrow d = \left(2, \frac{k'^2 - a'^2}{a'^2} + \frac{[c - b'/a']^2}{c^2 - q^2} \right)$$

$$= \left(2, \gamma' - 1 + \frac{[a'^2 c^2 - 2ca'b' + b'^2]}{a'^2(c^2 - q^2)} \right)$$

Conclusion: does not seem to fly. Their claimed answer is

$$d = \frac{2}{3} \int dk' [A' + 2B'] / (-k'^2) = \frac{2}{3} (A' + 2B')$$

$$\text{But, } c'(1-\gamma') = B' - A' \text{ so } B' = A' + (1-\gamma')c' \Rightarrow$$

$$\frac{2}{3} [A' + 2A' + 2(1-\gamma')c'] = \dots$$

At this point we have shown that

$$d = - \int \frac{dk'}{k'^2} \left[2B' + c' \left\{ N'^2 + \frac{(q \cdot N')^2}{c^2 - q^2} \right\} \right] \quad \text{assume correct}$$

Since the last integral is a can be made into for N'^2 only, a ~~can~~, we can say:

$$\int \frac{dk'}{k'^2} (q \cdot N')^2 = q^\mu q^\nu T_{\mu\nu}$$

$$\text{where } T_{\mu\nu} = \int \frac{dk'}{k'^2} N'_\mu N'_\nu = I_1 \eta_{\mu\nu} + I_2 g_{\mu\nu}$$

$$\text{Now } \eta_\mu T_{\mu\nu} = 0 = (I_1 + I_2) \eta_\nu \Rightarrow I_2 = -I_1 \quad \text{Thus}$$

$$T_{\mu\nu} = I_2 (g_{\mu\nu} - \eta_{\mu\nu})$$

$$\text{Next, } T_{\mu\mu} = \int \frac{dk'}{k'^2} (N')^2 = I_2 (4-1) = 3I_2 \Rightarrow I_2 = \frac{1}{3} \int \frac{dk'}{k'^2} (N')^2$$

Then

$$q^\mu q^\nu T_{\mu\nu} = q^\mu q^\nu I_2 (g_{\mu\nu} - \eta_{\mu\nu}) = I_2 (q^2 - c^2)$$

Thus

$$- \int \frac{dk'}{k'^2} \frac{(q \cdot N')^2}{c^2 - q^2} = I_2 = \int \frac{dk'}{k'^2} \left[\frac{1}{3} N'^2 \right]$$

Restate:

$$\int \frac{dk'}{k'^2} \left[\frac{(q \cdot N')^2}{q^2 - c^2} \right] = \int \frac{dk'}{k'^2} \left[\frac{N'^2}{3} \right]$$

$$\text{Now } (8'-1)C' = A' - B'$$

So,

$$d = - \int \frac{dk'}{k'^2} \left[2B' + c' \left\{ \frac{2}{3} (8'-1) \right\} \right]$$

$$= - \int \frac{dk'}{k'^2} \left[\frac{2}{3} B' + \frac{2}{3} (A' - B') \right] = - \frac{2}{3} \int \frac{dk'}{k'^2} \left[\frac{2}{3} B' + A' \right]$$

$$= - \frac{2}{3} \int \frac{dk'}{k'^2} [A' + 2B'] \quad (2.12) \quad \checkmark$$

Now go back to earlier:

$$I_1 = (q^2 - c^2, (q \cdot N')^2) = (q^2 - c^2, \frac{1}{3} N'^2 (q^2 - c^2))$$

$$= (q^2 - c^2) [1, \frac{1}{3} N'^2] = (q^2 - c^2) [1, (\gamma^2 - 1)/3]$$

$$I_2 = c [3, \gamma^2 - 1]$$

$$I_2 = c I_0$$

$$I_3 = q^2 [3, \gamma^2 - 1]$$

$$I_3 = q^2 I_0$$

$$I_1 = \left(\frac{q^2 - c^2}{3}\right) I_0$$

$$I_1 = \left(\frac{q^2 - c^2}{3}\right) [3, \gamma^2 - 1]$$

$$I_3 = 3I_1 + I_2$$

Thus we see that $I_3 - I_2 = 3I_1 \quad ! = I_0 \left[\frac{2}{3} c^2 - \frac{2}{3} q^2 \right]$

So $c^2(d+e) = I_2 c$
 $q^2 d + c^2 e = I_3 - I_1$

$$\Rightarrow (c^2 - q^2) d = c I_2 - I_3 + I_1 = c I_2 - 3I_1 - I_2 + I_1$$

$$\Rightarrow d = \frac{(-2)I_1 + (c-1)I_2}{c^2 - q^2} = -2 \left(\frac{q^2 - c^2}{3} \right) I_0 + (c-1) c I_0$$

$$q^2(d+e) = q^2 I_2 / c$$

$$q^2 d + c^2 e = I_3 - I_1$$

$$\Rightarrow d = I_0 \frac{\frac{2}{3}(c^2 - q^2)}{c^2 - q^2} = I_0 \left\{ -\frac{2}{3} q^2 + \frac{2}{3} c^2 + c^2 - c \right\}$$

$$(q^2 - c^2) e = \frac{q^2}{c} I_2 - I_3 + I_1$$

$$d = \frac{2}{3} I_0$$

$$(q^2 - c^2) e = \frac{q^2}{c} c I_0 + \left(\frac{q^2 - c^2}{3} \right) I_0 - q^2 I_0 = I_0 \left\{ q^2 + \frac{q^2}{3} - \frac{c^2}{3} - q^2 \right\}$$

$$(q^2 - c^2) e = \frac{I_0}{3} (q^2 - c^2) \Rightarrow e = \frac{1}{3} I_0$$

$$\Rightarrow \boxed{d = 2e} \quad \checkmark \quad (?..?)$$

$$(2.15) + (2.18)$$

$$\Pi_{\mu\nu}(q) = \frac{1}{2} \int d^4k \, i \Gamma_{\mu\lambda\lambda'}(q, k, k') \Delta_{\lambda\lambda'}(k) \Delta_{\lambda'\lambda}(k') \, i \Gamma_{\nu\sigma\sigma'}(k, k', -q)$$

save (2.11). Dot into $\eta_{\mu\nu}$:

$$\text{LHS} = \eta_{\mu\nu} \left[\Pi_{\mu\nu}(q) - \Pi_{\mu\nu}^0(q) + d \left(\frac{q_\mu q_\nu}{q^2} - \frac{Q_\mu Q_\nu}{q^2} \right) + e \frac{\eta_{\mu\nu}}{n^2} \right]$$

$$= ?$$

$$\eta_{\mu\nu} [-(q^2 \delta_{\mu\nu} - q_\mu q_\nu)] = -q^2 n^2 + (q \cdot n)^2 = \eta_{\mu\nu} \Pi_{\mu\nu}^0$$

$$\eta_{\mu\nu} \Pi_{\mu\nu} = b(q) [-q^2 n^2 + (q \cdot n)^2] - (n \cdot Q)^2 c(q)$$

$$\eta_{\mu\nu} d \left(\frac{q_\mu q_\nu}{q^2} - \frac{Q_\mu Q_\nu}{q^2} \right) = d \cdot \left[\frac{(n \cdot q)^2}{q^2} - \frac{(n \cdot Q)^2}{q^2} \right]$$

$$= \frac{d}{q^2} \left[(n \cdot q)^2 - (n \cdot [q - \frac{n q^2}{q \cdot n}])^2 \right]$$

$$(n \cdot q)^2 - [n \cdot q - \frac{n^2 q^2}{q \cdot n}]^2 = - \left[(n \cdot q)^2 + \frac{n^4 q^4}{(q \cdot n)^2} - 2 n^2 q^2 \right]$$

$$\text{So get } \frac{d}{q^2} \left\{ (n \cdot q)^2 - (n \cdot [q - \frac{n q^2}{q \cdot n}])^2 \right\} = \frac{n^4 q^4}{(q \cdot n)^2} + 2 n^2 q^2 = d n^2 \left[2 - \frac{n^2 q^2}{(q \cdot n)^2} \right]$$

Look closely at this d term

$$[d \cdot n^2 (2 - \delta_q)]$$

The e-term is: $e \frac{n^2 n^2}{n^2} = [e n^2]$ Remainder is thus:

Π^0

$$\text{LHS (other hand, e)} = b(q) [-q^2 n^2 + (q \cdot n)^2] - c(q) (n \cdot Q)^2 \neq [-q^2 n^2 + (q \cdot n)^2]$$

$$\text{Now: } n \cdot Q = n \cdot [q - \frac{n q^2}{q \cdot n}] = n \cdot q - \frac{n^2 q^2}{q \cdot n} = n \cdot q [1 - \delta_q]$$

$$\text{So } \boxed{n \cdot Q = n \cdot q (1 - \delta_q)}$$

$$\text{LHS} = b(q) (q \cdot n)^2 [1 - \delta_q] - c(q) (q \cdot n)^2 [1 - \delta_q]^2 \neq (q \cdot n)^2 [1 - \delta_q]$$

$$= (q \cdot n)^2 (1 - \delta_q) \left[b(q) - c(q) (1 - \delta_q) \right] = (q \cdot n)^2 (1 - \delta_q) (a - 1)$$

$$\text{But } c(1 - \delta) = b - a \text{ so } \cancel{b} - \cancel{b} + a + 1 = (-1 + a)$$

$$\text{So LHS}_{\text{rest}} = (q \cdot n)^2 \frac{(1-\delta_q)(a-1)}{\delta_q} \frac{n^2}{n^2} \delta_q$$

$$= \cancel{(q \cdot n)^2} (\delta_q^{-1} - 1) (a-1) \cancel{\frac{n^2}{n^2}} \cdot \left(\frac{\cancel{q^2 n^2}}{\cancel{q \cdot n^2}} \right) = \left[-q^2 n^2 (a-1) (1 - 1/\delta_q) \right]$$

So entire LHS is

$$\left(\text{LHS} = n^2 \left[-q^2 (a-1) (1 - 1/\delta_q) + d(2-\delta_q) + e \right] \right) \xrightarrow{\text{check}}$$

$$\text{RHS} = n_\mu n_\nu \quad \text{we need} \quad q = k = k' = 0$$

$$\begin{aligned} n_\mu i \Gamma_{\mu\lambda\lambda'}^0(q, -k, -k') &= n_\mu \left[(q+k)_\lambda \delta_{\mu\lambda} + (-k+k')_\mu \delta_{\lambda\lambda'} + (-k'-q)_\lambda \delta_{\mu\lambda'} \right] \\ &= \underbrace{n_\lambda (q+k)_\lambda}_{\text{kill}} + \underbrace{n_{\lambda'} (-k+k')_\lambda}_{\text{kill}} + n \cdot (-k'-q) \delta_{\lambda\lambda'} \\ &\quad \xrightarrow{\quad} n_\mu \delta_{\lambda\lambda'} (k'-k) \end{aligned}$$

$$= -\delta_{\lambda\lambda'} n \cdot (q+k')$$

$$\text{So RHS} = \underbrace{+\frac{1}{2} \int dk \cdot n \cdot (k'-k)}_{n \cdot (k'+2k)} \underbrace{\Delta_{\lambda\lambda'}(k) \Delta_{\lambda\lambda'}(k') i \Gamma_{\mu\mu\nu}^0(k, k', -q) n_\nu}_{\text{antisym.}}$$

Repeat:

$$\begin{aligned} n_\mu [i \Gamma_{\mu\lambda\lambda'}^0(q, -k, -k')] &= n_\mu \left[\cancel{(q+k)_\lambda} \delta_{\mu\lambda} + \delta_{\lambda\lambda'} (-k+k')_\mu + \cancel{\delta_{\lambda'\mu} (-k'-q)_\lambda} \right] \\ &= +n_\mu (k'-k)_\mu \delta_{\lambda\lambda'} \\ &= +n \cdot (k'-k) \delta_{\lambda\lambda'} \end{aligned}$$

recheck e-term:

$$\bullet \quad e n^2 = + \frac{1}{2} \int dk \cdot \underbrace{\eta_\mu [i \bar{\rho}_{\mu\alpha\alpha'}(q, -k, -k')]}_{n \cdot (k' - k) \delta_{\lambda\lambda'}} \Delta_{\lambda\alpha}(k) \Delta_{\lambda\alpha'}(k') [i \bar{\rho}_{\alpha\alpha'}(k, k', -q)] n_\nu$$

$$\Rightarrow e n^2 = + \frac{1}{2} \int dk \cdot n \cdot (k' - k) \Delta_{\lambda\alpha}(k) \Delta_{\lambda\alpha'}(k') [i \bar{\rho}_{\alpha\alpha'}(k, k', -q)] n_\nu$$

$\Rightarrow$ overall sign.

Recheck (2.15)

$$\begin{aligned} \text{LHS} &= \eta_\mu q_\nu \bar{\Pi}_{\mu\nu} = \eta_\mu q_\nu (\bar{\Pi}_{\mu\nu} - \bar{\Pi}_{\mu\nu}^0 + d \frac{q_\mu q_\nu}{q^2} + e \frac{\eta_\mu \eta_\nu}{n \cdot c}) \\ &= \eta_\mu q_\nu [\cancel{\bar{\Pi}_{\mu\nu}^0}] + d (n \cdot q) + e (n \cdot q) = [(n \cdot q)(e + d)] \end{aligned}$$

RHS

$$\begin{aligned} &= - \frac{1}{2} \int dk \cdot n \cdot (k' - k) \Delta_{\lambda\alpha}(k) \Delta_{\lambda\alpha'}(k') \underbrace{[i \bar{\rho}_{\alpha\alpha'}(k, k', -q)]}_{\downarrow} \underbrace{q_\nu}_{\downarrow} \\ &= \cancel{\frac{1}{2}} \int dk \cdot n \cdot (k' - k) \underbrace{\Delta_{\lambda\alpha}(k) \Delta_{\lambda\alpha'}(k')}_{\text{kill}} \bar{\Pi}_{\alpha\alpha'}(k) - \bar{\Pi}_{\alpha\alpha'}(k') \\ &\quad \rightarrow (\delta_{\lambda\alpha'} - \frac{k_\lambda n_{\alpha'}}{k \cdot n}) \end{aligned}$$

$$= \cancel{\frac{1}{2}} \int dk \cdot n \cdot (k' - k) \Delta_{\lambda\lambda}(k')$$

I seem to disagree again!! No, agree. The Ward makes explanation.

So I think sign is wrong in (2.18).

The base equation. (2.19)

① First let's derive B2 (2.19)

$$Q^2 = q^2(x-1)$$

$$\Pi_{\text{yu}}^0 = -3q^2 = (-q^2)(+3)$$

$$\Pi_{\text{yu}} = (-q^2)(2b+a)$$

$$\text{rest}_{\text{yu}} = d(2-x) + e$$

$$\Rightarrow \Pi_{\text{yu}} = (-q^2)(2b+a-3) + d(2-x) + e$$

The RHS is same as in (2.11), so they made a slight error with the d.
This is not significant however.

Also args of Π^0 are wrong!

$$(2.32) + (2.33) + (3.3) + (3.4) + (3.5)$$

$$\begin{cases} f = q^2 c \\ g = -q^2(b-c) \end{cases}$$

$$(f-g) = q^2 b$$

$$\begin{cases} b = \frac{1}{q^2} [f-g] \\ c = \frac{1}{q^2} [f] \end{cases}$$

$$f(k) = k^2 c(k) = k^2 \left(\frac{b-a}{1-\gamma} \right) = \frac{k^2}{(1-\gamma)} \left[\frac{1}{2_k \beta_k} - \frac{\gamma}{2_k \alpha_k} \right]$$

$$\text{so } f(k) = \frac{k^2}{2_k} \frac{(1/\beta - \gamma/\alpha)}{1-\gamma}$$

$$\begin{aligned} g &= -k^2 \left(b - \left\{ \frac{b-a}{1-\gamma} \right\} \right) = -k^2 \left[\frac{b-a}{1-\gamma} - \frac{b-a}{1-\gamma} \right] = -k^2 \left[\frac{a-b\gamma}{1-\gamma} \right] \\ &= -k^2 \left[\frac{\gamma}{2\alpha} - \frac{\gamma}{2\beta} \right] = -\frac{k^2 \gamma}{2(1-\gamma)} \left[1/\alpha - 1/\beta \right] \end{aligned}$$

So put (2.31) into (2.21):

$$c = \frac{B-A}{1-\gamma} = \frac{2}{1-\gamma} [\beta - \alpha]$$

$$-q^2 \left(\frac{\gamma_q}{2_1 \alpha_q} - 1 \right) (1 - 1/\gamma_q) + (2 - \gamma_q) d + e$$

$$= \int dk \cdot \left[\frac{1}{k^2} \left\{ \left(\delta_{\lambda\sigma} - \frac{n_\lambda n_\sigma}{n^2} \right) \sum_k \beta_k + \frac{N_\lambda^* N_\sigma^*}{n^2} \frac{\sum_k}{1-\gamma_k} (\beta_k - \alpha_k) \right\} \right]$$

($\delta_{\lambda\sigma}(k)$)

$$\left[\frac{1}{k'^2} \left\{ \left(\delta_{\lambda\sigma'} - \frac{n_\lambda n_{\sigma'}}{n^2} \right) \sum_{k'} \beta_{k'} + \frac{N_\lambda^{k'} N_{\sigma'}^{k'}}{n^2} \frac{\sum_{k'}}{1-\gamma_{k'}} (\beta_{k'} - \alpha_{k'}) \right\} \right]$$

$$\times \left[K_{\sigma\sigma'}^f \left(\frac{k^2}{2_k} \frac{(1/\beta_k - \gamma_k/\alpha_k)}{1-\gamma_k} \right) - K_{\sigma\sigma'}^g \left(\frac{k^2}{2_k} \frac{\gamma_k}{1-\gamma_k} \left(\frac{1}{\alpha_k} - \frac{1}{\beta_k} \right) \right) \right]$$

$$- \frac{1}{4} L_{\sigma\sigma'} \left(\frac{q^2}{2_1} \left(\frac{\gamma_1}{1-\gamma_1} \right) \left(\frac{1}{\alpha_1} - \frac{1}{\beta_1} \right) \right)$$

✓ hold

$$= \int dk \frac{2_n 2_{n'}}{k^2 (k')^2} \left\{ K_{\sigma\sigma'}^f \left(\frac{k^2}{2_k} \frac{(1/\beta - \gamma/\alpha)}{1-\gamma} \right) - K_{\sigma\sigma'}^g \left(\frac{k^2}{2_k} \frac{\gamma}{1-\gamma} \left(\frac{1}{\alpha} - \frac{1}{\beta} \right) \right) - L \left(\frac{q^2}{2_1} \left(\frac{\gamma_1}{1-\gamma_1} \right) \left(\frac{1}{\alpha_1} - \frac{1}{\beta_1} \right) \right) \right\}$$

where $L = L_{00'} \left[\left(\delta_{\lambda\lambda'} - \frac{n_\lambda n_{\lambda'}}{n^2} \right) + \frac{N_\lambda N_{\lambda'}}{n^2} \right] \left[\left(\delta_{\lambda\lambda'} - \frac{n_\lambda n_{\lambda'}}{n^2} \right) + \frac{N'_\lambda N'_{\lambda'}}{n^2} \right]$

where $L_{00'} = \frac{n \cdot (k-k') \cdot n \cdot k'}{n^2 q \cdot k} \left\{ \frac{q_a q_{a'}}{q^2} \right\}$

So multiply out:

Lead term is $L_{\lambda\lambda'} = \frac{n \cdot (k-k') \cdot n \cdot k'}{n^2 q \cdot k} \left\{ \frac{q_\lambda q_\lambda}{q^2} \right\} \approx 0 \checkmark$

Next we get

$\checkmark \textcircled{2} = \frac{n \cdot (k-k') \cdot n \cdot k'}{n^2 q \cdot k} \left\{ \frac{q_a q_{a'}}{q^2} \right\} \cdot \left[-\frac{n_a n_{a'}}{n^2} - \frac{n_{a'} n_a}{n^2} \right] =$

$= \frac{n \cdot (k-k') \cdot n \cdot k'}{n^2 q \cdot k} \left(\frac{q_a q_{a'}}{q^2} \right) \left[-2 \frac{n_a n_{a'}}{n^2} \right]$

$\checkmark \textcircled{3} = L_{00'} \cdot \left[\frac{N'_a N'_{a'}}{n^2} + \frac{N_{a'} N_a}{n^2} \right]$

$\checkmark \textcircled{4} = L_{00'} \left(+ \frac{n_a n_{a'}}{n^2} \right)$

$\textcircled{5} = 0 \quad \textcircled{6} = \frac{(N \cdot N')}{n^4} N_a N_{a'} L_{00'}$

So my overall result is

$L = L_{00'} \left[\delta_{\lambda\lambda'} - \frac{n_\lambda n_{\lambda'}}{n^2} + \frac{N_\lambda N_{\lambda'} + N'_\lambda N'_{\lambda'}}{n^2} + \frac{(N \cdot N') N_\lambda N_{\lambda'}}{n^4} \right]$

But I say $L_{00'} = \frac{n \cdot (k-k') \cdot n \cdot k'}{n^2 q \cdot k} \left\{ \frac{q_a q_{a'}}{q^2} \right\} \quad k+k'=q$

K is obvious & shows that (2.15)'s sign error is not known

But, look against $L = L_{00'}^{(h,k')} \times \underbrace{f_{00'}(h,k')}_{\text{symm-undr "switch"}}$

/go to back of 2

This is three integrals. We have:

$$\begin{aligned} & -q^2 \left(\frac{\gamma_q}{2_q \alpha_q} - 1 \right) (1 - 1/\gamma_q) + [(2 - \gamma_q)d + e] \\ &= - \frac{q^2}{2_q} \left(\frac{\gamma_q}{1 - \gamma_q} \right) \left(\frac{1}{\alpha_q} - \frac{1}{\beta_q} \right) \int dk \frac{2_k 2_{k'} L}{k^2 k'^2} \\ &+ - \int dk \frac{2_{k'}}{k'^2} \left\{ K^g \left(\frac{\gamma_k}{1 - \gamma_k} \right) \left(\frac{1}{\alpha_k} - \frac{1}{\beta_k} \right) + K^f \left(\frac{1}{\beta_k} + \frac{\gamma_k}{\alpha_k} \right) \frac{1}{(1 - \gamma_k)} \right\} \end{aligned}$$

Now put [] in RHS + divide by $(-q^2)$

$$\begin{aligned} & \left[\frac{\gamma_q - 2_q \alpha_q}{2_q \alpha_q} \right] (1 - 1/\gamma_q) \\ &= \frac{1}{2_q} \left(\frac{\gamma_q}{1 - \gamma_q} \right) \left(\frac{1}{\alpha_q} - \frac{1}{\beta_q} \right) \int dk \frac{2_k 2_{k'} L}{k^2 k'^2} \\ &+ \frac{1}{q^2} \int dk \frac{2_{k'}}{k'^2} \left\{ K^g \left(\frac{\gamma_k}{1 - \gamma_k} \right) \left(\frac{1}{\alpha_k} - \frac{1}{\beta_k} \right) + K^f \left(\frac{\gamma_k}{\alpha_k} - \frac{1}{\beta_k} \right) \left(\frac{1}{1 - \gamma_k} \right) \right\} \\ &+ \frac{1}{q^2} [(2 - \gamma_q)d + e] \end{aligned}$$

Redo LHS:

$$\text{LHS} = \frac{(\gamma_q - 1)}{2_q \alpha_q} - (1 - 1/\gamma_q) \quad \text{So (2.32) OK except for this one sign.}$$

Next, set $\alpha = \gamma$ and $\beta = 1$ in the above to get

$$\begin{aligned} \frac{(\gamma_q - 1)}{2_q \gamma_q} &= (1 - 1/\gamma_q) + (2_q)^{-1} \left(\frac{\gamma_q}{1 - \gamma_q} \right) \left(\frac{1}{\gamma_q} - 1 \right) \int dk \frac{2_k 2_{k'} L}{k^2 k'^2} \\ &+ \frac{1}{q^2} \int dk \frac{2_{k'}}{k'^2} \left\{ K^g \left(\frac{\gamma_k}{1 - \gamma_k} \right) \left(\frac{1}{\gamma_k} - 1 \right) + K^f \left(1 - 1 \right) \left(\frac{1}{1 - \gamma_k} \right) \right\} + \frac{1}{q^2} [(2 - \gamma_q)d + e] \end{aligned}$$

$$\frac{1}{2_q} (1 - 1/\gamma_q) = (1 - 1/\gamma_q) + \frac{1}{2_q} \int dk \frac{2_k 2_{k'} L}{k^2 k'^2}$$

$$+ \frac{1}{q^2} \int dk \frac{2_{k'}}{k'^2} K^g + \frac{1}{q^2} [(2 - \gamma_q)d + e]$$

go to back of p1

$$\text{So } L = L_{aa'}(k, k') f_{aa'}^S(k, k')$$

↑ picks out antisym part of L

L^S

$$L_{aa'}(k, k') = \frac{1}{2} \left[\frac{n \cdot (k - k') n \cdot k'}{n^2 q \cdot k} \left(\frac{q_a q_{a'}}{q^2} \right) + \frac{n \cdot (k' - k) n \cdot k}{n^2 q \cdot k'} \left(\frac{q_a q_{a'}}{q^2} \right) \right]$$

$$\frac{1}{2} \left[\frac{n \cdot (k - k') n \cdot k'}{n^2 q \cdot k} \left(\frac{q_a q_{a'}}{q^2} \right) - \frac{n \cdot (k' - k) n \cdot k}{n^2 q \cdot k'} \left(\frac{q_a q_{a'}}{q^2} \right) \right]$$

So we can replace

$$L_{aa'}(k, k') \rightarrow L_{aa'}^S(k, k') = \frac{1}{2} \left(\frac{q_a q_{a'}}{q^2} \right) \frac{1}{n^2} \left[\frac{n \cdot (k - k') n \cdot k'}{q \cdot k} - \frac{n \cdot (k' - k) n \cdot k}{q \cdot k'} \right]$$

$$a \quad L_{aa'}^S(k, k') = \frac{n \cdot (k - k')}{n^2} \left\{ \frac{q_a q_{a'}}{q^2} \right\} \cdot \left(\frac{1}{2} \left[\frac{n \cdot k'}{q \cdot k} - \frac{n \cdot k}{q \cdot k'} \right] \right)$$

Try One

$$\int dk \cdot h(k) h(k') L(k, k')$$

if you switch both

$$= \int dk \cdot \underbrace{h(k) h(k')}_{\text{sym}} L_{aa'}(k, k') f_{aa'}^S(k, k')$$

$$= \int dq \cdot h^S(k, k') L_{aa'}(k, k') f_{aa'}^S(k', k)$$

$$= \int dq \cdot h^S(k, k') L_{aa'}(k, k') f_{aa'}^S(k', k)$$

rename a, a'

Now switch $k \leftrightarrow k'$ since no effect in $q = k + k'$

$$= \int dq \cdot h^S(k', k) L_{aa'}(k', k) f_{aa'}^S(k, k')$$

$$= \int dk \cdot h^S(k, k') L_{aa'}(k', k) f_{aa'}^S(k, k')$$

This shows that you can make the switch for free in L , only because of its position under the integration. This then verifies (3.4) fully:

$$L_{aa'} = \frac{n \cdot (k - k') n \cdot k'}{n^2 q \cdot k} \left(\frac{q_a q_{a'}}{q^2} \right) = - \frac{n \cdot (k - k') n \cdot k}{n^2 q \cdot k'} \left(\frac{-q_a q_{a'}}{q^2} \right)$$

↑ (2.26)

So (3.4), (3.5) all consistent with

their being unaware of error in 2.8

$$(3.6) + (3.7) + (3.10) + (3.11) = \text{small sign error (2.11)}$$

● Pretend that $-L = k^c K / q^2 = (1 - 1/\alpha)$. Then (3.3) becomes:

$$-\frac{1}{2_\gamma} \cancel{(1 - 1/\alpha)} = \cancel{(1 - 1/\alpha)} \rightarrow -\frac{1}{2_\gamma} \int dk \frac{2_k 2_{k'}}{k^c k'^c} \cancel{(1 - 1/\alpha)} \\ + \frac{1}{q^2} \int dk \frac{2_{k'}}{(k')^c} \frac{q^2}{k^c} \cancel{(1 - 1/\alpha)}$$

$$\Rightarrow -\frac{1}{2_\gamma} = -1 - \frac{1}{2_\gamma} \int dk \frac{2_k 2_{k'}}{k^c (k')^c} + \int dk \frac{2_{k'}}{k^c (k')^c} \quad \text{This is (3.6)}$$

Thus solve for 2_γ :

$$(2_\gamma)^{-1} \left[-1 + \int dk \frac{2_k 2_{k'}}{k^c k'^c} \right] = \left[-1 + \int dk \frac{2_k 2_{k'}}{k^c k'^c} \right]$$

$$\Rightarrow \frac{1}{2_\gamma} = \frac{[-1 + \int dk 2_k 2_{k'} / k^c k'^c]}{[-1 + \int dk 2_k 2_{k'} / k^c k'^c]} \quad \checkmark \quad (3.7)$$

Now install $2_\gamma = (\Lambda^2 / q^c)^\alpha$ to get

$$\left(\frac{q^2}{\Lambda^2} \right)^\alpha = \frac{\left[\pm 1 + \int dk (\Lambda^2 / k^c)^\alpha / k^c k'^c \right]}{\left[\pm 1 + \int dk (\Lambda^2 / k^c)^\alpha (\Lambda^2 / k'^c)^\alpha / k^c k'^c \right]}$$

$$\alpha = 0 \quad \beta = \frac{2 - \beta}{2 - \beta} = 1 - \alpha$$

$$\text{Let } I_1 = (\Lambda^2)^\alpha \cdot \int d^4 k \cdot \frac{1}{(k^2)^1 (k'^2)^{1+\alpha}} = (\Lambda^2)^\alpha \cdot \lim_{\alpha \rightarrow 0} \frac{\Gamma(1-\alpha) \Gamma(1+\alpha)}{\Gamma(2+\alpha) \Gamma(1-\alpha)} \left(\frac{1}{q^c} \right)^{1+\alpha}$$

$$\beta = 0 \quad \alpha = \alpha = (\Lambda^2)^\alpha \frac{\Gamma(1-\alpha)}{\Gamma(1+\alpha)} \cdot \frac{\Gamma(\alpha)}{\Gamma(2-\alpha)} \left(\frac{1}{q^c} \right)^\alpha i\pi^2$$

$$= \frac{(\Lambda^2)^\alpha \cdot \Gamma(1-\alpha) \Gamma(\alpha)}{(1-\alpha) \Gamma(1-\alpha) \Gamma(\alpha+1)} \left(\frac{1}{q^c} \right)^\alpha i\pi^2 \quad \checkmark$$

$$\bullet \text{ Next: } I_2 = (\Lambda^2)^{2\alpha} \cdot \int d^4 h \frac{1}{k^{1+\alpha}} \frac{1}{(h'^2)^{1+\alpha}} = (\Lambda^2)^{2\alpha} \cdot i\pi^2 \cdot \frac{\Gamma(1-\alpha)^2}{\Gamma(1+\alpha)^2} \frac{\Gamma(2\alpha)}{\Gamma(1-2\alpha)} \left(\frac{1}{q^c} \right)^{2\alpha}$$

3.11 obvious

$$\frac{\Gamma(\frac{1}{2}) \Gamma(\frac{3}{2}) \Gamma(1)}{\Gamma(1) 2^{\frac{1}{2}} \Gamma^2(\frac{1}{2})} = \frac{\frac{1}{2} \Gamma(\frac{1}{2})^2 \cdot 1}{1 \cdot \frac{1}{2} \Gamma(\frac{1}{2})^2} = 1.$$

$$\dots (3.1) + (3.2) + (2.4) + (2.10)$$

Gauge theory in the Axial Gauge.

6.23.78.

- ① Remember that the path integral quantization for a gauge theory looks like this, after you drop off the infinite group factor:

$$W[J] = \int [dA] \underbrace{\Delta_f(A)}_{\det} \underbrace{\prod_{x,a} \delta[f_a(A)]}_{\text{delta.}} e^{iS[A,J]}$$

- ② the determinant is

$$\Delta_f(A) = \det[M_{ab}(x,y)] = \det\left[\frac{\nabla f_a(x)}{\nabla g_b(y)}\right]$$

$$= \det\left[\int d^4z \frac{\nabla f_a(x)}{\nabla A_\mu^c(z)} \cdot \frac{\nabla A_\mu^c(z)}{\nabla g_b(y)}\right]$$

where, in general,

$$\frac{\nabla A_\mu^c(z)}{\nabla g_b(y)} = \left[-i f_{bd}^c A_\mu^d(y) - \delta_b^c \frac{1}{g} \partial_\mu^\dagger \right] \delta^4(y-z)$$

You can see that, for ~~non~~abelian theory where $f \neq 0$, this is indep of A . In a usual linear gauge condition so is $\nabla f / \nabla A$; then you can conclude that $\Delta_f(A) \neq f(A)$, so ignore it.

For abelian theory, you write $\Delta_f[A] = \int [dc] [det] e^{i \int d^4x dy c_a^\dagger(x) M_{ab}(x,y) c_b(y)}$

and then you have ghosts.

- ③ In general, we have

$$M_{ab}(x,y) = \left[-i f_{bd}^c A_\mu^d(y) + \delta_b^c \frac{1}{g} \partial_\mu^\dagger \right] \cdot \frac{\nabla f_a(x)}{\nabla A_\mu^c(y)}$$

Suppose gauge condition is

$$f_a(x) = \eta^\mu A_\mu^a(x) = 0 \quad \forall a$$

$$\text{Then } \frac{\nabla f_a(x)}{\nabla A_\mu^c(y)} = ? \quad \delta f_a(x) = \eta^\mu \delta A_\mu^a(x) = \int d^4y \eta^\mu (\delta A_\mu^a(y)) \delta^4(x-y)$$

$$\Rightarrow \frac{\nabla f_a(x)}{\nabla A_\mu^c(y)} = \delta_a^c \eta^\mu \delta(x-y)$$

But then you get $A_\mu^\dagger n^\mu = 0$ so

In axial gauge $n^\mu A_\mu = 0$, $\Delta[A]$ is indep of A , no ghosts.

④ What now about the "delta"? My notes tell me that

$$\Delta[A] = -\frac{1}{2\alpha} f_\alpha(A) f_\alpha(A) \quad \alpha = \text{arb.}$$

In axial gauge this is then

$$= -\frac{1}{2\alpha} (n \cdot A^a)(n \cdot A^a).$$

⑤ So now what we got? For axial gauge QCD I see this:

$$\int [dA] e^{+i \int d^4x \left[\frac{1}{2} A_\mu^a (g_{\mu\nu} \square - \partial_\mu \partial_\nu) A_\nu^a - \frac{1}{2\alpha} (n \cdot A^a)(n \cdot A^a) \right]}$$

$$\text{So } \mathcal{L} = \frac{1}{2} A_\mu^a \left[\delta_a^b (g_{\mu\nu} \square - \partial_\mu \partial_\nu) - \frac{1}{2} n_\mu n_\nu \delta_a^b \right] A_\nu^b$$

$$\text{Again } \mathcal{L}_{\text{prop}}(A) = \frac{1}{2} A_\mu^a(x) \left[\delta_{ab} \left(g_{\mu\nu} \square_x - \partial_\mu \partial_\nu - \frac{1}{2} n_\mu n_\nu \right) \right] A_\nu^b$$

The thing in brackets is the inverse propagator. So

$$D_{\mu\nu}^{-1\ ab}(k) = \delta^{ab} \left(g_{\mu\nu} k^2 - k_\mu k_\nu - \frac{1}{2} n_\mu n_\nu \right)$$

⑥ What propagator goes with this?

$$D_{\mu\lambda}^{ac}(k) D_{\lambda\nu}^{cb}(k) = \delta^{ab} g_{\mu\nu}$$

Try most general sym. form:

$$D_{\mu\lambda}^{ac}(k) = \delta^{ac} \left[A g_{\mu\lambda} + B k_\mu k_\lambda + C n_\mu n_\lambda + D (n_\mu k_\lambda + n_\lambda k_\mu) \right]$$

S₀,

$$\begin{aligned}
 & [A g_{\mu\nu} + B k_{\mu} k_{\nu} + C \eta_{\mu\nu} + D (\eta_{\mu} k_{\nu} + \eta_{\nu} k_{\mu})] [g_{\lambda\nu} k^{\lambda} - k_{\lambda} k_{\nu} - \frac{1}{2} \eta_{\lambda\nu} n^{\lambda}] \\
 & = k^{\lambda} A g_{\mu\nu} - A k_{\mu} k_{\nu} - A \frac{1}{2} \eta_{\mu\nu} n^{\lambda} + B k^{\lambda} k_{\mu} k_{\nu} - B k_{\mu} k_{\nu} k^{\lambda} - B \frac{1}{2} k_{\mu} \eta_{\nu\lambda} (n^{\lambda} \cdot k) \\
 & + C k^{\lambda} \eta_{\mu\nu} n_{\lambda} - C \eta_{\mu\nu} k^{\lambda} (n^{\lambda} \cdot k) - C \frac{1}{2} \eta_{\mu\nu} n^{\lambda} n^{\lambda} + D k^{\lambda} (\eta_{\mu} k_{\nu} + \eta_{\nu} k_{\mu}) \\
 & - D k^{\lambda} \eta_{\mu} k_{\nu} - D (n^{\lambda} \cdot k) k_{\mu} k_{\nu} - \frac{1}{2} D \eta_{\mu\nu} (k^{\lambda} \cdot n_{\lambda}) - D \frac{1}{2} n^{\lambda} k_{\mu} n_{\nu} \\
 & = g_{\mu\nu} [A k^{\lambda}] + k_{\mu} k_{\nu} [-A + B k^{\lambda} - B k^{\lambda} - D (n^{\lambda} \cdot k)] = g_{\mu\nu} \\
 & + \eta_{\mu\nu} [-A/\alpha + C k^{\lambda} - C n^{\lambda}/\alpha - (D/\alpha) (k \cdot n)] \\
 & + k_{\mu} n_{\nu} [-\frac{B}{\alpha} (n^{\lambda} \cdot k) + D k^{\lambda} - D \frac{n^{\lambda}}{\alpha}] \\
 & + k_{\nu} \eta_{\mu} [-C (n^{\lambda} \cdot k) + D k^{\lambda} - D \frac{n^{\lambda}}{\alpha}]
 \end{aligned}$$

S₀,

$$\begin{aligned}
 & \checkmark k^{\lambda} A = 1 \quad \Rightarrow A = 1/k^{\lambda} \\
 & \checkmark -A - D (n^{\lambda} \cdot k) = 0 \quad \Rightarrow D = -1/(n^{\lambda} \cdot k) k^{\lambda} \\
 & \checkmark -A/\alpha + C k^{\lambda} - C n^{\lambda}/\alpha - (D/\alpha) k \cdot n = 0 \quad \text{redundant} \\
 & \checkmark -(B/\alpha) n^{\lambda} \cdot k + D k^{\lambda} - D n^{\lambda}/\alpha = 0 \\
 & \checkmark -C (n^{\lambda} \cdot k) = 0 \quad \Rightarrow C = 0
 \end{aligned}$$

$$+ \frac{B}{\alpha} \frac{n^{\lambda} \cdot k}{k^{\lambda}} = -D \left(\frac{n^{\lambda}}{\alpha} - k^{\lambda} \right) = + \frac{1}{n^{\lambda} \cdot k} \left(\frac{n^{\lambda}}{\alpha} - k^{\lambda} \right) \frac{1}{k^{\lambda}}$$

so

$$B = \frac{1}{(n^{\lambda} \cdot k)^2} \left(\frac{n^{\lambda}}{\alpha} - k^{\lambda} \right) \frac{1}{k^{\lambda}} = \frac{1}{(n^{\lambda} \cdot k)^2} \frac{(n^{\lambda} - \alpha k^{\lambda})}{k^{\lambda}} = \frac{1}{(n^{\lambda} \cdot k)^2} \frac{(n^{\lambda} - \alpha k^{\lambda})}{k^{\lambda}}$$

Ergo,

$$D_{\mu\nu}(k) = \frac{1}{k^2} g_{\mu\nu} + \frac{1}{(n^{\lambda} \cdot k)^2} \frac{(n^{\lambda} - \alpha k^{\lambda})}{k^{\lambda}} k_{\mu} k_{\nu} - \frac{1}{(n^{\lambda} \cdot k)} \frac{(k_{\mu} k_{\nu} + k_{\nu} k_{\mu})}{k^{\lambda}}$$

$$S_0 \quad D_{\mu\nu}(k) = k^2 \left[g_{\mu\nu} + \frac{(n^{\lambda} - \alpha k^{\lambda})}{(n^{\lambda} \cdot k)^2} k_{\mu} k_{\nu} - \frac{(k_{\mu} n_{\nu} + k_{\nu} n_{\mu})}{(n^{\lambda} \cdot k)} \right]$$

Agree Kummer except this factor $\uparrow$.

The proposed by B2 is

$$= \frac{1}{n^2} \left[g_{\mu\nu} - \frac{\eta_\mu \eta_\nu}{n^2} + \frac{N_\mu N_\nu}{n^2} \right]$$

$$= g_{\mu\nu} - \frac{\eta_\mu \eta_\nu}{n^2} + \left(\eta_\mu - \frac{k_\mu n^2}{k \cdot n} \right) \left(\eta_\nu - \frac{k_\nu n^2}{k \cdot n} \right) \frac{1}{n^2}$$

$$= g_{\mu\nu} - \frac{\eta_\mu \eta_\nu}{n^2} + \frac{\eta_\mu \eta_\nu}{n^2} - \frac{n^2}{k \cdot n} \left[\eta_\mu k_\nu + \eta_\nu k_\mu \right] + \frac{k_\mu k_\nu n^2}{(k \cdot n)^2}$$

$$= g_{\mu\nu} + \frac{n^2}{(k \cdot n)^2} k_\mu k_\nu - \frac{1}{(k \cdot n)} [\eta_\mu k_\nu + k_\mu \eta_\nu]$$

⇒ This is more a Kummers in limit $\alpha \rightarrow 0$.

Summary: in the axial gauge $n \cdot A = 0$, if we interpret the thing in the Lagrangian is "the coordinate space inverse propagator", we convert to momentum space + find

$$[D^{-1}]_{\mu\nu}^{ab}(k) = -\delta^{ab} \left[g_{\mu\nu} k^2 - k_\mu k_\nu - \frac{1}{2} \eta_\mu \eta_\nu \right]$$

where $\alpha =$ what you like. By direct calculation, we find next that

$$[D]_{\mu\nu}^{ab}(k) = -\frac{1}{k^2} \left[g_{\mu\nu} + \frac{n^2(1 - \alpha k^2/n^2)}{(n \cdot k)^2} k_\mu k_\nu - \frac{(k_\mu \eta_\nu + k_\nu \eta_\mu)}{(n \cdot k)} \right] \delta^{ab}$$

as the momentum space propagator. In the limit the $\alpha \rightarrow 0$, D^{-1} is undefined, but

$$[D]_{\mu\nu}^{ab}(k) \Big|_{\alpha=0} = -\frac{1}{k^2} \left[g_{\mu\nu} - \frac{\eta_\mu \eta_\nu}{n^2} + \frac{N_\mu N_\nu}{n^2} \right] \delta^{ab}$$

where $N_\mu \equiv \eta_\mu - k_\mu n^2/(k \cdot n)$. This appears in Ball & Zee (3.1).

Can we replace D' with something sensible at $\alpha=0$? Try this.
 Define Π by

$$[D]_{\mu\nu}[\Pi]_{\nu\lambda} = (g_{\mu\lambda} - \frac{k_\mu n_\lambda}{n \cdot k})$$

extra term.

Claim: This is solved by $\Pi_{\mu\nu} = -(k^2 g_{\mu\nu} - k_\mu k_\nu)$. Lets see:

$$+ \frac{1}{k^2} \left[g_{\mu\nu} + \frac{n^2}{(n \cdot k)^2} k_\mu k_\nu - \frac{(k_\mu n_\nu + k_\nu n_\mu)}{(n \cdot k)} \right] \left[k^2 g_{\nu\lambda} - k_\nu k_\lambda \right]$$

= $g_{\mu\lambda}$ + rest. Where

$$\text{rest} = \left(+ \frac{1}{k^2} \right) \left[-k_\mu k_\lambda + \frac{n^2 k^2}{(n \cdot k)^2} k_\mu k_\lambda - \frac{n^2}{(n \cdot k)} k_\mu k_\lambda k^2 \right.$$

$$\left. - \frac{k^2}{(n \cdot k)} (k_\mu n_\lambda + k_\lambda n_\mu) + k_\mu k_\lambda \frac{(n \cdot k)}{(n \cdot k)} + \frac{k^2}{n \cdot k} n_\mu k_\lambda \right] = - \frac{k_\mu n_\lambda}{n \cdot k} \checkmark$$

Summary: since $[D']$ is undefined at $\alpha=0$, we define a new inverse propagator by:

$$[D]_{\mu\nu}^{\text{ab}} [\Pi]_{\nu\lambda}^{\text{bc}} \equiv g_{\mu\lambda} - k_\mu n_\lambda / (n \cdot k)$$

Then Π so-defined yields

$$\Pi_{\mu\nu}^{\text{ab}} = -\delta^{\text{ab}} [k^2 g_{\mu\nu} - k_\mu k_\nu]$$

As in Black. (2.10). Notice this useful property. $k_\mu \Pi_{\mu\nu}^{\text{ab}} = 0$.

Notes

Most General form of Γ .

$\Gamma_{\mu\nu\lambda}(p, q, k)$

Why cyclic symmetric?

$$\Gamma(\hat{p} \hat{q} \hat{k}) = -\Gamma(\hat{p} \hat{k} \hat{q}) = +\Gamma(\hat{k} \hat{p} \hat{q})$$

→ follows from pairwise antisymmetry!

Write as

$\Gamma_{\mu\nu\lambda}(k, k', p)$

What are the possible tensors?

$$-p = k + k'$$

$$T_{\mu\nu\lambda}^{(1)} = -g_{\mu\nu}[k_\lambda - k'_\lambda] + g_{\mu\lambda}[k'_\nu - p_\nu] + g_{\nu\lambda}[p_\mu - k_\mu]$$

$$T_{\mu\nu\lambda}^{(2)} = g_{\mu\nu} n_\lambda + g_{\mu\lambda} n_\nu + g_{\nu\lambda} n_\mu$$

Where did I get these? In general use basis

$$g_{\mu\nu}[k_\lambda, k'_\lambda, p_\lambda, n_\lambda] = 4 \text{ terms, but } (k-k'), (k+k'), p = -(k+k') \text{ does not occur. so only 2 terms, then cyclic indices.}$$

Next:

$$T_{\mu\nu\lambda}^{(3)} = (k)_\mu (k')_\nu (n)_\lambda = \text{illegal since must be antisymmetric.}$$

$$[(k)_\mu (k)_\nu (k)_\lambda - (k')_\mu (k')_\nu (k')_\lambda]$$

Well, need more systematic way:

$$V_1 V_1 V_1: \begin{matrix} k_\mu k_\nu k_\lambda \\ k'_\mu k'_\nu k'_\lambda \\ p_\mu p_\nu p_\lambda \\ n_\mu n_\nu n_\lambda \end{matrix} = \begin{matrix} k_\mu k_\nu p_\lambda \\ (p+k)(p+k')(p+k) \\ n_\mu n_\nu p_\lambda \end{matrix} \neq \begin{matrix} k_\mu k'_\nu k'_\lambda \\ p^3, p^2 k, k^2 p, k^3 \\ n_\mu n_\nu k_\lambda \end{matrix} = V_1 V_1 V_2 \text{ class. } (V_1 = k)$$

Must do all together.

$(k k k)$ ✓	$(k k k')$ ✓	$(k k' k')$ ✓	$(k' k' k')$ ✓	$\left. \begin{matrix} k k' \\ k p \\ k n \\ k' p \\ n p \\ n k' \end{matrix} \right\} 16 \text{ terms.}$
$(k k p)$	$(k k p)$	$(k p p)$	$(p p p)$	
$(k k n)$ ✓	$(k n n)$ ✓	$(k n n)$ ✓	$(n n n)$ ✓	
$(k' k' p)$	$(k' p p)$	$(k' p p)$	$(p p p)$	
$(n n p)$	$(n p p)$	$(n p p)$	$(p p p)$	
$(n n k')$ ✓	$(n k' k')$ ✓	$(n k' k')$ ✓	$(k' k' k')$ ✓	

Get rid of p : → 9 terms.

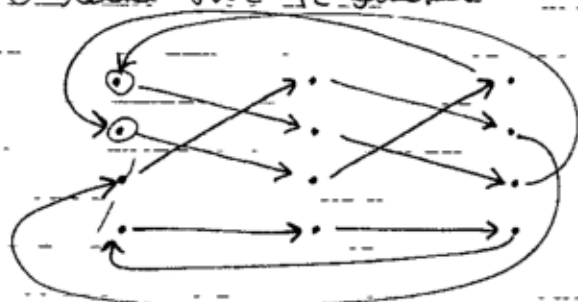
Go to next page

Most General $g_{\mu\nu}$ is:

$$g_{\mu\nu}(k, k', p, n)$$

$$= f_1 \cdot g_{\mu\nu}(k)_\lambda + f_2 \cdot g_{\mu\nu}(k')_\lambda + f_3 \cdot g_{\mu\nu}(p)_\lambda + f_4 \cdot g_{\mu\nu}(n)_\lambda + f_5 \cdot g_{\mu\nu}(k)_\lambda + f_6 \cdot g_{\mu\nu}(k')_\lambda + f_7 \cdot g_{\mu\nu}(p)_\lambda + f_8 \cdot g_{\mu\nu}(n)_\lambda + f_9 \cdot g_{\mu\nu}(k)_\lambda + f_{10} \cdot g_{\mu\nu}(k')_\lambda + f_{11} \cdot g_{\mu\nu}(p)_\lambda + f_{12} \cdot g_{\mu\nu}(n)_\lambda \rightarrow 1$$

Can I relate these 12 functions?



Can lower to 4, one from each set!

$$\left. \begin{array}{l} f_1 \cdot g_{\mu\nu}(k)_\lambda + \text{cyclic} \\ f_2 \cdot g_{\mu\nu}(k')_\lambda + \text{cyclic} \end{array} \right\} \rightarrow g_{\mu\nu} \left[(k)_\lambda f_1(k, k', n) - (k')_\lambda f_1(k', k, n) \right] + \text{cyclic}$$

$$f_3 \cdot g_{\mu\nu}(p)_\lambda + \text{cyclic}$$

$$f_4 \cdot g_{\mu\nu}(n)_\lambda + \text{cyclic} \rightarrow g_{\mu\nu} \left[(n)_\lambda f_4^A(k, k', n) \right] + \text{cyclic}$$

But $f_3 g_{\mu\nu}(-k - k')_\lambda + \text{cyclic}$, just adds to f_1, f_2 .

Also:

$$f_1 \cdot g_{\mu\nu} \left[f_1(k)_\lambda + f_2(k')_\lambda \right] + \text{cyclic}$$

$$= g_{\mu\nu} \left[f_1(k, k', n) (k)_\lambda + f_2(k, k', n) (k')_\lambda \right] + \text{cyclic}$$

We need antisym.

$$= -f_1(k, k', n) f_2(k, k', n) = -f_1(k', k, n)$$

$$= g_{\mu\nu} \left[f_1(k, k', n) (k')_\lambda + f_2(k', k, n) (k)_\lambda \right] + \text{cyclic}$$

$k_\mu k_\mu k_\lambda + \cancel{k_\mu k_\lambda k_\mu}$ all the same

X

$$\left. \begin{aligned} & k_\mu k_\mu (k'_\lambda) + (k'_\mu) k_\mu (k'_\lambda) + (k'_\mu) (k'_\lambda) k_\mu \\ & \swarrow \quad \searrow \quad \swarrow \quad \searrow \quad \swarrow \quad \searrow \\ & k_\mu k'_\mu k'_\lambda \quad k'_\mu k_\mu k'_\lambda \quad k'_\mu k'_\lambda k_\mu \end{aligned} \right\}$$

Regroup these as $(k_\mu k_\mu k'_\lambda - k'_\mu k_\mu k'_\lambda) = 2$

$$(k_\mu k'_\mu k_\lambda - k'_\mu k_\mu k'_\lambda) = 3$$

$$(k'_\mu k_\mu k_\lambda - k'_\mu k'_\lambda k_\mu) = 4$$

$(k'_\mu) k_\mu (k'_\lambda)$ combine with #1

$$(k_\mu k_\mu k_\lambda - k'_\mu k'_\mu k'_\lambda) \text{ new \#1.}$$

Next: $k_\mu k_\mu n_\lambda + \cancel{k_\mu k_\lambda n_\mu} + \cancel{k'_\mu k_\lambda n_\mu} + k'_\mu k_\lambda n_\mu$

$$\begin{array}{ccccc} & \downarrow & & \swarrow & \searrow \\ k_\mu k'_\mu n_\lambda & & k'_\lambda k'_\mu n_\mu & & k'_\mu k_\lambda n_\mu \end{array}$$

so

$$(k_\mu k_\mu n_\lambda - k'_\mu k'_\mu n_\lambda) \quad 5$$

$$(k_\lambda k_\mu n'_\mu - k'_\mu k'_\lambda n_\mu) \quad 6$$

$$(k_\mu k_\lambda n_\mu - k'_\lambda k'_\mu n'_\mu) \quad 7$$

Finally: $(n_\mu n'_\mu n_\lambda) \quad 8.$

So, ignoring p I see 8 allowed tensors. So go write P in most general form: $\swarrow$ go back to 1

But how do I build in k - p antisym.

Also $n_\mu n'_\mu k_\lambda$

Partial Conclusion:

Considering only those terms which contain g , the most general form that Γ can have [requiring cyclic sym + kk' antisym] is:

$$\Gamma_{\mu\nu\lambda}(k, k', p; n)$$

$$= g_{\mu\nu} \left\{ (k)_\lambda f_1(k, k', n) - (k')_\lambda f_1(k, k', n) \right\} + \text{cyclic} \\ + g_{\mu\nu} \left\{ (n)_\lambda f_2^A(k, k', n) \right\} + \text{cyclic}$$

where f_2^A must be antisymmetric.

Note: since this Γ is cycl sym, kk' antisym $\Rightarrow k'p$ antisym $\Rightarrow p, k$ antisym.

Result so far:

$$\begin{aligned}
 \Gamma_{\mu\alpha}(k, k', p; n) = & \left\{ F_1(k, k', n) \left[\underline{g_{\mu\alpha} k_\lambda} + g_{\mu\lambda} k'_\mu + \overline{g_{\mu\lambda}} (k_\mu + k'_\mu) \right] \right\} \\
 & + F_2^A(k, k', n) \left[\underline{g_{\mu\alpha} n_\lambda} + g_{\mu\lambda} n_\mu + g_{\mu\mu} n_\alpha \right] \\
 & + \left\{ F_3(k, k', n) \underline{k_\mu k_{\mu'}} k_\lambda - \text{switch} \right\} \\
 & + \left\{ F_4(k, k', n) k_\mu k_{\mu'} k'_\lambda - \text{switch} \right\} \\
 & + \left\{ F_5(k, k', n) k_\mu k'_{\mu'} k_\lambda - \text{switch} \right\} \\
 & + \left\{ F_6(k, k', n) \underline{k'_\mu k_{\mu'}} k_\lambda - \text{switch} \right\} \\
 & + \left\{ F_7(k, k', n) k_\mu k_{\mu'} n_\lambda - \text{sw} \right\} \\
 & + \left\{ F_8 k_\lambda k_\mu n_{\mu'} - \text{sw} \right\} \\
 & + \left\{ F_9 k_{\mu'} k_\lambda n_\mu - \text{sw} \right\} + \left\{ F_{10} n_\mu n_{\mu'} n_\lambda \right\}.
 \end{aligned}$$

So I am down to 10 functions.

$$\Gamma_{\mu\nu\lambda}(k, k', p; n)$$

$$\eta_\mu \eta_\nu \eta_\lambda \cdot f_3^A(k, k', n) + \text{cyclic.}$$

$$= \eta_\mu \eta_\nu \eta_\lambda \left[f_1^A(k, k') + f_2^A(k', p) + f_3^A(p, k) \right]$$

$$f(k, k', p)$$

$$\left\{ \frac{f(q_1)}{(q_1 \cdot n)^2} \left(\frac{-q_1 \cdot q_2}{(q_1 \cdot n + q_2 \cdot n)} + \frac{q_1 \cdot (q_1 + q_2)}{q_2 \cdot n} \right) \right\}$$

$$= \frac{f(q_1)}{(q_1 \cdot n)^2} \left[\frac{-(q_1 \cdot q_2)(n \cdot q_2) + [q_1^2 + (q_1 \cdot q_2)][q_1 \cdot n + q_2 \cdot n]}{n \cdot (q_1 + q_2) \cdot q_2 \cdot n} \right]$$

Only thing you can say is the sum must be totally antisym.

$$f^A(k, k', p) = \frac{f(k)}{(k \cdot n)^2} \left(\frac{k \cdot k'}{p \cdot n} - \frac{k \cdot p}{k' \cdot n} \right) + \text{must be anti in } k' \leftrightarrow p$$

So in $\Gamma_{\mu\nu\lambda}(k, k', p; n)$, the term bilinear in n must be of the form:

$$\Gamma_{\mu\nu\lambda}(k, k', p; n) = (\eta_\mu \eta_\nu \eta_\lambda) f_A^3(k, k') + \text{cyclic}$$

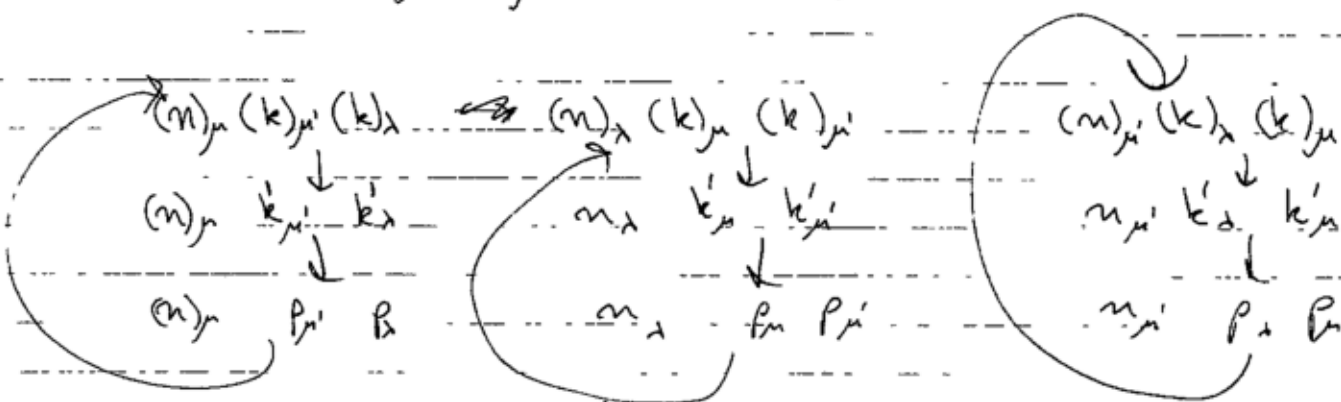
To analyze the bilinear term, replace $\eta_{\mu\nu} \rightarrow (\eta_\mu)_\rho (\eta_\nu)_\rho$ + get same result.

$$\Gamma_{\mu\nu\lambda}(k, k', p; n) = (\eta_\mu)_\lambda (\eta_\nu)_\lambda \left\{ (k)_\lambda f_4(k, k', n) - (k')_\lambda f_4(k', k, n) \right\} + \text{cyclic.}$$

The only problem left is :
 terms with no $q_{\mu\nu}$ and zero or one n 's.

like

$$(n)_\mu (k)_\mu' (k)_\lambda + \text{cyclic.}$$



$$S_0 \left\{ \begin{aligned} & f_5(k, k') (n)_\mu (k)_\mu' (k)_\lambda \\ & + f_6^{PA}(k, k') (n)_\lambda (k)_\mu (k)_\mu' \\ & + f_7(k, k') (n)_\mu' (k)_\lambda (k)_\mu \end{aligned} \right\} + \text{cyclic} \rightarrow \left\{ \begin{aligned} & f_5(k', p) n_\mu h_\mu' k'_\lambda \\ & f_6^{PA}(k', p) n_\lambda h'_\mu h'_\mu' \\ & f_7(k', p) n_\mu' h'_\lambda k'_\mu \end{aligned} \right.$$

$$\begin{aligned} & \left[f_5(k, k') \{ n_\mu k_\mu' k_\lambda \} + f_7(k', p) \{ s_\omega \} \right] \\ & = - \left[f_5(k', k) \{ n_\mu' k_\mu k'_\lambda \} + f_7(k, p) \{ n_\mu k_\mu' k_\lambda \} \right] \\ & \Rightarrow f_5(k, k') = - f_7(k, p) \\ & f_7(k', p) = - f_5(k', k) \end{aligned}$$

$$+ f_5(p, k) n_\mu p_\mu' p_\lambda \Rightarrow f_5(p, k') = - f_5(p, k) ?$$

$$f_6^{PA}(p, k) n_\lambda p_\mu p_\mu'$$

$$f_7(p, k) n_\mu' p_\lambda p_\mu$$

$$f_5(k, k') n_\mu k_\mu' k_\lambda + \text{cyclic } f_5(k', p) n_\mu' k_\lambda' k_\mu' + f_5(p, k) n_\lambda p_\mu p_\mu'$$

$$f_6(k, k') n_\lambda k_\mu k_\mu' + \text{cyclic } f_6(k', p) n_\mu k_\mu' k_\lambda' + f_6(p, k) n_\mu' p_\lambda p_\lambda'$$

$$f_7(k, k') n_\mu' k_\lambda k_\mu + \text{cyclic } f_7(k', p) n_\lambda k_\mu' k_\mu' + f_7(p, k) n_\mu p_\mu' p_\lambda$$

$$\begin{aligned} & f_8(k, k') n_\mu k_\mu' k_\lambda' + \text{cyclic } f_8(k', p) n_\mu' k_\lambda' p_\mu + f_8(p, k) n_\lambda p_\mu k_\mu' \\ & \textcircled{A} f_9(k, k') n_\lambda k_\mu k_\mu' + \text{cyclic } f_9(k', p) n_\mu k_\mu' p_\lambda + f_9(p, k) n_\mu' p_\lambda k_\mu \\ & f_{10}(k, k') n_\mu' k_\lambda k_\mu' + \text{cyclic } f_{10}(k', p) n_\lambda k_\mu' p_\mu' + f_{10}(p, k) n_\mu' p_\mu k_\lambda \end{aligned}$$

$$\text{--- } f_{11}(k, k') n_\mu k_\mu' k_\lambda' + \text{cyclic } f_{11}(k', p) n_\mu p_\lambda k_\mu' + f_{11}(p, k) n_\lambda k_\mu p_\mu' \text{---}$$

$$\text{--- } f_{12}(k, k') n_\lambda k_\mu' k_\mu' + \text{cyclic } f_{12}(k', p) n_\mu p_\mu' k_\lambda' + f_{12}(p, k) n_\mu k_\lambda p_\mu \text{---}$$

$$\text{--- } f_{13}(k, k') n_\mu k_\lambda k_\mu' + \text{cyclic } f_{13}(k', p) n_\lambda p_\mu k_\mu' + f_{13}(p, k) n_\mu k_\mu' p_\lambda \text{---}$$

Look at first pair

$$P \approx [f_5(k, k') n_\mu k_\mu' k_\lambda + f_5(k', p) n_\mu' k_\lambda' k_\mu']$$

$$= - [f_5(k', k) n_\mu' k_\mu' k_\lambda' + f_5(k, p) n_\mu k_\lambda k_\mu'] \quad \text{if there were no other terms.}$$

$$\Rightarrow \begin{aligned} -f_5(k, p) &= f_5(k, k') & \Rightarrow & f_5(k, k') + f_5(k, -k-k') = 0 \\ -f_5(k', k) &= f_5(k', p) & & f_5(k', k) + f_5(k', -k-k') = 0. \end{aligned}$$

$$[f_5(a, b) + f_5(a, -(a+b)) = 0] \leftarrow \text{This restricts } f_5 \text{ to some extent.}$$

But also

$$[f_5(p, k) n_\lambda p_\mu p_\mu'] = - [f_5(p, k') n_\lambda p_\mu' p_\mu']$$

$$\Rightarrow f_5(p, k) + f_5(p, p-k) = 0, \text{ same}$$

crossed all crossed
by preceding 6
rows.

Look at the 8, 9, 10 group.

• $-f_8(k', k) = f_{10}(k, k')$ so f_{10} not independent!

Also, $-f_8(k, p) = f_{10}(p, k) = -f_8(k, p)$ nothing new.

$$-f_8(p, k') = f_{10}(k', p) = -f_8(p, k') \quad \text{!!}$$

$$\text{So } \left[\begin{matrix} f_{10}(k, k') \\ [13] \end{matrix} = - \begin{matrix} f_8(k', k) \\ [11] \end{matrix} \right] \quad f$$

$$\text{Also, } -f_9(k', k') = f_9(k, k') \Rightarrow \left[\begin{matrix} f_9^A \\ [12] \end{matrix} \right]$$

So we can combine:

$$\left[f_8(k, k') \eta_\mu k_\mu k'_\lambda - f_8(k', k) \eta_\mu k'_\mu k_\lambda \right] + \text{cyclic.}$$

$$= \left[f_8(k, k') \eta_\mu k_\mu k'_\lambda - (\text{sw}) \right] + \text{cyclic.}$$

• Well, after removing last 3 rows I get:

$$\left\{ \begin{aligned} & \eta_\mu \left[f_5 k_\mu k'_\lambda + f_8 k'_\mu k_\lambda \right] \\ & + \eta_{\mu'} \left[f_7 k_\mu k'_\lambda + f_{10} k'_\mu k_\lambda \right] \\ & + \eta_\lambda \left[f_6 k_\mu k'_\mu + f_9^A k_\mu k'_\mu \right] \end{aligned} \right\} + \text{cyclic}$$

$$\left\{ \begin{aligned} & \eta_\mu k'_\mu \left[f_5 k_\lambda + f_8 k'_\lambda \right] \\ & + \eta_{\mu'} k'_\mu \left[f_7 k_\mu + f_{10} k'_\mu \right] \\ & + \eta_\lambda k_\mu \left[f_6 k'_\mu + f_9^A k'_\mu \right] \end{aligned} \right\} + \text{cyclic.}$$

Except for $f_{10} = -f_8$, & see no further simplification.

• No such terms appear in (2.20) at all.

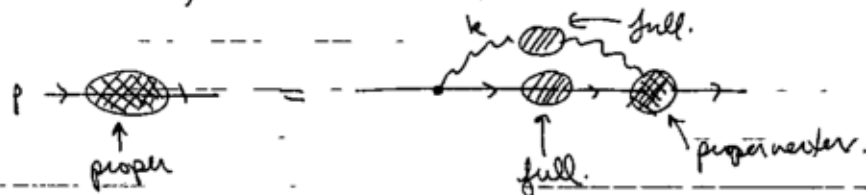
Ball Zacks

"Infrared Properties of the Glue Propagator in NAGT."

CALT 68-647 (22 pages).

I. Introduction. Conjecture: non-linear glue coupling $\Rightarrow$ singular IR behavior $\Rightarrow$ confinement. Quarks feel confinement but do not generate it. Also, solution will be non-perturbative.

In QED, since no massless electrons, photon pole is not shifted away from $k^2=0$, so $D(k) = 2/k^2$ near $k=0$. In QED, the Dyson reads:



/page 290 BD.

"Infrared" means p^2 near m^2 , controlled by small- k integration region. The Ward Identity says:

$$q_\mu \Gamma^\mu(p', p) = \Sigma(p) - \Sigma(p') \text{ as I proved it in QED notes.}$$

This says: $q^\mu \text{ (blob) }_\mu = \text{blob } p - \text{blob } p'$. Easiest to draw as

$$q_\mu \Gamma^\mu(p', p) = S_F^{-1}(p') - S_F^{-1}(p)$$

So knowing $S_F(p)$ tells you a little about Γ^μ , not all. [I'll say F_1 , not F_2 , claim]. So in the Dyson, eliminate the vertex!

Now what about QCD? Use same strategy. Combine Dyson + Ward to get integral equation for gluon prop. We should then look for $g = g[\mu]$, (renom.) as fcn of the renorm. point. If g is singular in some sense, $\square??$ then confinement.

In massless theory, you need a new way to justify: "Ward determines the entire relevant part of the vertex." Some vague argument that if Ward equates UV divergences, it must also equate IR ones too. Balance. In fact they will just assume that Ward is ok in this respect.

When we write $\Delta_{\mu\nu}^{ab} = \text{glue propagator}$, there will be only 2 invariant fctns. Each obeys a scaling law. (1.1), (1.2). Thus

$$A(\frac{q^2}{\Lambda^2}, g_0) = \sum_i (\frac{q^2}{\Lambda^2})^i A_{\text{ren}}(1, g_i, g(q^2))$$

As $q^2 \rightarrow 0$, assume $g[q^2] = \text{finite limit}$. Then Σ must contain any confinement type singularity.

The real goal is to find that

$$g^2(q^2) \rightarrow (q^2)^{-\alpha} \text{ as } q^2 \rightarrow 0 \quad \alpha > 0$$

so that the running coupler blows up, IR slavery, confinement

Ansatz #1: The singular behavior of A, B is contained in Σ , so

$$A(q^2/\Lambda^2, \delta, g_0) = Z(q^2/\Lambda^2) \alpha(\delta)$$

$$B(q^2/\Lambda^2, \delta, g_0) = Z(q^2/\Lambda^2) \beta(\delta)$$

$$\delta = q^2 n^2 / (q \cdot n)^2$$

where $n^\mu =$ axial gauge fixed vector, $q \cdot n = 0$. It will not
that the same Z controls g .

$$g^2(q^2) = Z(q^2/\Lambda^2) g_0^2$$

Ansatz #2: $(B^{-1} - \delta A^{-1})$ is UV finite. Maybe it is also IR finite.
Assume yes. This might even be zero. Assume yes.
Then $\alpha(\delta) = \delta \cdot \beta(\delta)$. Assume also that $\beta = 1$.

Ansatz #3: neglect 4-gluon

More comments on similarity UV to IR limits

II. Integral Equations Derived

- A. Notation. For reasons not yet known to me, the gluon propagator in the axial gauge can be written in terms of two invariant form.
The "invariants" are q^2/Λ^2 and δ . go in there since these unrenormalized
invar. forms are calculated very hard.

$$\Delta_{\mu\nu}^{ab}(q) = - \frac{\delta_{ab}}{q^2} \left\{ \left(\delta_{\mu\nu} - \frac{\eta_\mu \eta_\nu}{\eta^2} \right) B(q^2/\Lambda^2, \delta, q_0) \pm \frac{N_\mu N_\nu}{\eta^2} C(q^2/\Lambda^2, \delta, q_0) \right\}$$

where C replaces Kummer's A, and $N_\mu = \eta_\mu - q_\mu \eta^2 / (q \cdot \eta)$. The delta δ_{ab}
shows that gluons carry color like quark carries flavor [p quark].
This function is the same regardless of the gluon's flavor color.
Suppress a, b.

Next, the "unrenorm" propagator is

$$\Delta_{\mu\nu} \Pi_{\lambda\rho} = \left[\delta_{\mu\nu} - \frac{q_\mu q_\nu}{(q \cdot \eta)} \right] \leftarrow \text{I guess this is "axial gauge unit."}$$

Then solve for Π to get

$$\Pi_{\mu\nu}(q) = -q^2 \left(\delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \right) b - \cancel{Q_\mu Q_\nu c}$$

where $b = B^{-1}$, $c = (b - a)/(1 - \delta)$, $a = \delta/\Lambda^2$. So $c = [1/B - \delta/\Lambda^2]/(1 - \delta)$. This
is the baby that is UV finite. Note
Now the free unrenorm prop is

$$\Pi_{\mu\nu}^0(q) = - (q^2 \delta_{\mu\nu} - q_\mu q_\nu)$$

Define

$$\overline{\Pi}_{\mu\nu}(q) \equiv \Pi_{\mu\nu}(q) - \Pi_{\mu\nu}^0(q) + d \left(\frac{q_\mu q_\nu}{q^2} - \frac{Q_\mu Q_\nu}{q^2} \right) + e \frac{\eta_\mu \eta_\nu}{\eta^2}$$

where d, e are so far arbitrary const.

Next, for reasons I have not checked, the Dyson they use involves unrenorm
propagator! I guess it's OK

$$[\tilde{m}^{\mu\nu}]^{-1} = [\tilde{m}^{\mu\nu}] +$$



$$\overline{\Pi}_{\mu\nu}(q) = \frac{1}{2} \int g_0^2 d^4 k \, i \Gamma_{\mu\lambda}^0(q, -k, k') \Delta_{\lambda\alpha}(k) \Delta_{\alpha\lambda'}(k') i \Gamma_{\lambda'\nu}^0(k', -q)$$

↑ insert UV cutoff

OK.

Meanwhile, recall gauge theory Feynman rules:

$$i\Gamma_{\lambda_1\lambda_2\lambda_3}(q_1, q_2, q_3) = -g\lambda_2(q_1 - q_2)\lambda_3 + \text{cyclic}.$$

B. The Ward:

$$i\Gamma_{\sigma_1\sigma_2\sigma_3}(q_1, q_2, q_3)(q_3)^{\sigma_3} = \overline{\Pi}_{\sigma_1\sigma_2}(q_1) - \overline{\Pi}_{\sigma_1\sigma_2}(q_2). \quad \text{OK.}$$

Now compute:

$$\overline{\Pi}_{\mu\nu}(q) \eta_{\mu} q_{\nu}$$

Well: see added: ✓ OK. Now apply $\eta_{\mu} q_{\nu}$ to RHS of (2.11). See calc:

OK, after 30 min calc I have verified (2.15). I used:

(2.1) which shows $n \cdot \Delta = 0$ on either index

(2.4) which shows $\Delta\pi = 1$, vaguely.

(2.11) the Dyson.

(2.9) the definition of d and e .

Now let's accept (2.16) for the moment, I know how to check it.

So, d and e are not arbitrary, but are well-defined integrals of A and B as shown in (2.17).

The idea is to generate scalar equations by dotting things into the Dyson. Whenever you dot one side into $\vec{q}$, you can use Ward to remove Γ !

(2.18), (2.19) I could also derive, no sweat.

OK, end result of part (B) is that (2.18), (2.19) are recast

as of the form (2.28); two equations with two unknowns, A and B !! Solve 'em!

Calculations: (separate sheet)

$$\textcircled{1} \quad \eta_{\mu\nu} \bar{\Pi}_{\mu\nu}(q) = \eta_{\mu\nu} q_{\nu} \left[\Pi_{\mu\nu} - \Pi_{\mu\nu}^0 + d \left(\frac{q_{\mu} q_{\nu}}{q^2} - \frac{q_{\mu} q_{\nu}}{q^2} \right) + e \frac{\eta_{\mu\nu}}{n^2} \right]$$

$$\textcircled{2} \quad \Pi_{\mu\nu} \textcircled{3} = d \left[n \cdot q - \frac{(n \cdot q)(q \cdot q)}{q^2} \right] = d \left[(n \cdot q) - [n \cdot \{ q \text{ stop} \}] \right] = d(n \cdot q)$$

$$(n \cdot q) = (n \cdot q) - \frac{(n \cdot n) q^2}{(q \cdot n)}$$

$$(q \cdot q) = q^2 - \frac{(n \cdot q) q^2}{(n \cdot q)} = 0 \quad \text{i.e. } q \perp Q$$

$$\textcircled{4} = e \cdot n^2 \frac{(n \cdot q)}{n^2} = e(n \cdot q)$$

$$\textcircled{5} = \eta_{\mu\nu} q_{\nu} [q^2 \delta_{\mu\nu} - q_{\mu} q_{\nu}] = (n \cdot q) q^2 - (n \cdot q) q^2 = 0$$

Well $q_{\mu} \Pi_{\mu\nu}^0 = -q^2 q_{\nu} + q_{\nu} q^2 = 0$. Of course, this is conserved current or same thing.

$q_{\mu} \Pi_{\mu\nu} = 0$ also: each term is separately 0.

Then:

$$\boxed{\eta_{\mu\nu} \bar{\Pi}_{\mu\nu}(q) = (n \cdot q) [d + e]}$$

$$q_{\nu} \bar{\Pi}_{\mu\nu}(q) = \frac{1}{2} \int dk i \tilde{P}_{\mu\lambda\lambda'}^0(\dots) \Delta_{\lambda\alpha}(k) \Delta_{\lambda'\alpha'}(k') \underbrace{i \Pi_{\alpha\alpha'}(k, k' - q) q_{\nu}}_{= \Pi_{\alpha\alpha'}(k') - \Pi_{\alpha\alpha'}(k)}$$

What about $\eta_{\mu\lambda} \tilde{P}_{\mu\lambda\lambda'}^0(q, -k, -k') = \eta_{\mu} \cdot [(q+k)_{\lambda'} \delta_{\mu\lambda} + (-k+k')_{\mu} \delta_{\lambda\lambda'} + (-k'-q)_{\lambda} \delta_{\mu\lambda'}]$

$$= (q+k)_{\lambda'} n_{\lambda} + n \cdot (k'-k) \delta_{\lambda\lambda'} - (q+k')_{\lambda} n_{\lambda'}$$

Then $\eta_{\mu\nu} q_{\nu} \bar{\Pi}_{\mu\nu}(q) = \frac{1}{2} \int dk [(q+k)_{\lambda'} n_{\lambda} + n \cdot (k'-k) \delta_{\lambda\lambda'} - (q+k')_{\lambda} n_{\lambda'}]$

(*) $\times \Delta_{\lambda\alpha}(k) \Delta_{\lambda'\alpha'}(k) \cdot [\Pi_{\alpha\alpha'}(k') - \Pi_{\alpha\alpha'}(k)]$

But also: $\Delta_{\lambda\alpha}^{(k)} \Pi_{\alpha\alpha'}(k) = [\delta_{\lambda\alpha'} - \frac{k_{\lambda} n_{\alpha'}}{(k \cdot n)}]$

$$\Delta_{\lambda'\alpha'}^{(k')} \Pi_{\alpha\alpha'}(k') = [\delta_{\lambda'\alpha} - \frac{k'_{\lambda'} n_{\alpha}}{(k' \cdot n)}]$$

$$\begin{aligned}
 \text{So RHS} &= \frac{1}{2} \int dk \cdot \left[(q+k)_\lambda n_\lambda - (q+k')_\lambda n'_\lambda + n \cdot (k'-k) \delta_{\lambda\lambda'} \right] \\
 &\times \left\{ \underbrace{\Delta_{\lambda\alpha}(k)}_x \left[\delta_{\lambda\alpha'} - \frac{k'_\lambda n'_\alpha}{(k' \cdot n)} \right] - \underbrace{\Delta_{\lambda\alpha'}(k')}_x \left[\delta_{\lambda\alpha} - \frac{k_\lambda n_\alpha}{(k \cdot n)} \right] \right\} \\
 &= \frac{1}{2} \int dk \left[(q+k)_\lambda n_\lambda - (q+k')_\lambda n'_\lambda + n \cdot (k'-k) \delta_{\lambda\lambda'} \right] \\
 &\times \left\{ \frac{k_\lambda n'_\alpha}{(k \cdot n)} \Delta_{\lambda\alpha'}(k') - \frac{k'_\lambda n_\alpha}{(k' \cdot n)} \Delta_{\lambda\alpha}(k) \right\}
 \end{aligned}$$

Oh, by the way $\eta_{\mu\nu} \Delta^{\mu\nu} = 0$ just as $q_\mu \Pi^{\mu\nu} = 0$. Of the above 6 terms, this kills two. So

$$= \frac{1}{2} \int dk \left[(q+k)_\lambda n_\lambda k_\lambda n'_\alpha \Delta^{\lambda\alpha}(k) \right] \quad k' = q - k$$

But $n'_\alpha \Delta^{\lambda\alpha} = 0$.

Go back to (*) and see that $n_\lambda \Delta^{\lambda\alpha}$ kills one term, etc. so you get

$$\frac{1}{2} \int dk n \cdot (k'-k) \Delta_{\lambda\alpha}(k) \Delta_{\lambda\alpha'}(k') \left[\Pi^{\alpha\alpha'}(k') - \Pi^{\alpha\alpha'}(k) \right].$$

Now use quoted rule

$$\begin{aligned}
 &\frac{1}{2} \int dk n \cdot (k'-k) \left[\Delta_{\lambda\alpha}(k) \left\{ \delta_{\lambda\alpha} - \frac{k'_\lambda n'_\alpha}{(k' \cdot n)} \right\} - \Delta_{\lambda\alpha'}(k') \left\{ \delta_{\lambda\alpha'} - \frac{k_\lambda n_\alpha}{(k \cdot n)} \right\} \right] \\
 &= \frac{1}{2} \int dk n \cdot (k'-k) \left[\Delta_{\lambda\lambda}(k) - \Delta_{\lambda\lambda}(k') \right] \\
 &= \frac{1}{2} \int dk n \cdot (k'-k) \Delta_{\lambda\lambda}(k) = I_1 + I_2 \\
 &\quad - \frac{1}{2} \int dk n \cdot (k'-k) \Delta_{\lambda\lambda}(k') \quad (q-2k) \quad q-2[q-k'] \\
 &\quad \quad \quad = -q+2k'
 \end{aligned}$$

$$I_2 = -\frac{1}{2} \int dk n \cdot (q-2k) \Delta_{\lambda\lambda}(q-k).$$

$$= -\frac{1}{2} \int dk' n \cdot (k-k) \Delta_{\lambda\lambda}(k') =$$

$$= -\frac{1}{2} \int [-dk'] n \cdot (-q+2k') \Delta_{\lambda\lambda}(k') \quad \checkmark$$

$$= \frac{1}{2} \int dk n \cdot (-q+2k) \Delta_{\lambda\lambda}(k)$$

$$= +I_1. \text{ OK, endpoint!!}$$

$$\text{so } = \int dk n \cdot (k'-k) \Delta_{\lambda\lambda}(k).$$

find k' here,
order

Summary of Section II so far:

The Dyson equation ignoring the 4-gluon is (2.11). This equation involves both the gluon propagator (and its inverse) and the vertex. Note that this is a tensor equation. By dotting this into the four possible combinations of vector-pairs, you generate 4 scalar equations. These equations are: (2.15), (2.16), (2.18), (2.19). I regard these 4 scalar equations as equivalent to the one tensor equation. (?) From the first two equations you can solve for the numbers d and e which we can regard as constants, albeit infinite ones. Well there is an implicit cutoff so don't worry.

So in the second more dynamic pair of equations we know what d and e are. However, seems to me that d and e are functionally dependent on A and B . But we are going to identify d and e with quadratically divergent terms on the RHS of the second pair of equations. We will then drop d and e on the left, and ignore quadratic divergent terms on the right. The second pair of equations both have the form (2.28).

A key step in this is to assume a certain functional form for the vertex, as in (2.20). Certainly this thing respects the Ward identity. I presume that in some sense this is the most general form that does. This thing is of course a function of A and B . K and L are just groupings of "spinology".

Part C: The Integral Equation. Assume that $A = 2\alpha$, $B = 2\beta$, so that $A, B \leftarrow 2, \alpha, \beta$. Then (2.27) $\rightarrow$ (2.32).

III. Outline of the Solution. ✓

Antisymmetry of the Triple Glue Vertex.

- ① First, using McKeen, combine gluons 1,2 into a state of definite J, L, S, p :

$$|q_1, q_2, v_2\rangle_c \equiv \underbrace{|q_1, m_1; s_1, v_1\rangle_c}_{\substack{\text{mass} \\ \uparrow 4\text{-mom}}} \otimes \underbrace{|q_2, m_2; s_2, v_2\rangle_c}_{\substack{\text{spin} \\ \uparrow \text{canonical not helicity}}}$$

The operation is

$$|\tilde{p}, m [w, j] l, s\rangle = \sqrt{\frac{q}{2}} \sqrt{\frac{2l+1}{4\pi}} \sum_{l_3 s_3} C(l s j l_3 s_3 m) |\tilde{p}, l l_3 s s_3\rangle$$

$\tilde{p} = m(1, 0, 0, 0)$ j_3 effective mass
 $j = \text{effective spin}$

where $|\tilde{p}, l l_3 s s_3\rangle = \int d\Omega D_{l l_3}^l(\Omega) R_{l_3}^l |\tilde{p}, s s_3\rangle$. So you generate in this way combined states of total spin j .

- ② Now, we are going to force these combined states to couple into a single gluon. Thus:

$\leftarrow S\text{-matrix is L.I.}$

$$\text{---} \bigcirc \text{---} = \langle \tilde{p}, m [M_3, S_3] | S | \tilde{p}, m [w, j] l, s \rangle$$

Thus, this gluon 3 is at rest, mass $M_3 = w$. Spin is $S_3 = 1 \Rightarrow j = 1$. Thus, only those 2-gluon states with label $j = 1$ can contribute.

- ③ What about statistics? Ignoring gluon internal symmetry you would say (see sym. notes):

gluons identical $\Rightarrow L + S = \text{even}$.

Then

j	L	S
1	0	X
1	1	X, 1, X
1	2	X, 2
1	3	X

$(L, S) = (1, 1)$ and $(2, 2)$ are the only allowed possibilities.

This guarantees that: $P_c |JLS\rangle = (-1)^{2S_1} |JLS\rangle$, i.e., proper statistics.

- ④ But if $S_1 = 1$, bosons, I get that there should be no antisymmetry! A paradox. Maybe color plays a role like isospin.

$\rightarrow$ over

Yes color is the answer. In fact,

$$\Gamma_{\lambda_1 \lambda_2 \lambda_3}^{a_1 a_2 a_3}(q_1 q_2 q_3) = C_{a_1 a_2 a_3} f_{\lambda_1 \lambda_2 \lambda_3}(q_1 q_2 q_3)$$

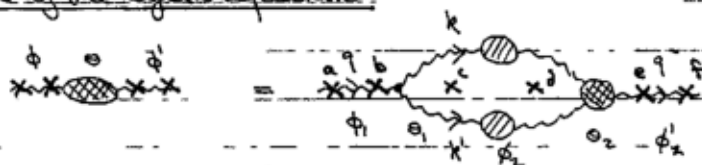
If you interchange two gluons completely, must be sym.

But if you interchange everything but the color label, f must be antisym.

Conclusion: In Ball Zach stuff we are in effect dealing with the function f with the color structure functions removed. Thus we really do expect this thing f to be antisymmetric if you switch two of the gluons, which here means you switch two momenta and their corresponding lorentz labels. There is no need to look into the details of the states.

How about $O(3)$ diagonalization of the Dyson equation?

Here is the picture:



For the moment, let's think of spacetime as $SO(4)$ and then the little group here is $SO(3)$. Put in some frames:

$$k^{(c)} = (E_k^c, \vec{k}_c) \quad q^{(c)} = (E_q^c, 0) = (\sqrt{q^2}, 0) \quad E_k^c = \frac{q^2 + k^2 - k'^2}{2q}$$

$$k'^{(c)} = (E_{k'}^c, -\vec{k}_c) \quad |\vec{k}|^2 = \frac{\Delta(q^2, k^2, k'^2)}{2q}$$

So $E_k^c, E_{k'}^c, E_q^c, |\vec{k}|^c$ are all functions of q^2, k^2, k'^2 , the invariants. Now define frame "c" to put $k^{(c)} = (E_k^c, 0, 0, +|\vec{k}|^c)$. Then rotation ϕ_2 does nothing.

Then

$$k^{(b)} = (E_k^c, |\vec{k}|^c \sin \theta_1, 0, |\vec{k}|^c \cos \theta_1), \text{ say.}$$

$$k^{(a)} = (E_k^c, k \sin \theta_1 \cos \phi_1, k \sin \theta_1 \sin \phi_1, 0, k \cos \theta_1) = k^{(a)}(g_1) \text{ loosely.}$$

However, $\tilde{\Pi}_{\mu\nu}(q)$ is really $\tilde{\Pi}_{\mu\nu}(q^2)$ + so has no dependence on ϕ, θ, ϕ' . Thus, let's also do this.

$$k^{(f)} = (E_k^c, k \sin \theta_2 \cos \phi_2', k \sin \theta_2 \sin \phi_2', 0, k \cos \theta_2) = k^{(f)}(g_2)$$

We can thus write $\Gamma^0 \rightarrow \tilde{\Gamma}_{\mu\nu}^0(g_1; q^2, k^2, k'^2) \quad g_2 = \tilde{g}_1 g$

$$\Gamma \rightarrow \tilde{\Gamma}_{\mu\alpha'\nu}(g_2; q^2, k^2, k'^2)$$

Also, $d^4k = k^3 dk \times \dots = dk_0 \cdot d^3k = dk_0 \cdot |\vec{k}|^2 d|\vec{k}| \cdot dg_1$

But $dk_0 |\vec{k}|^2 d|\vec{k}|$ is probably $dk^2 dk'^2 f(q^2, k^2, k'^2)$ must be, so we have

$$\tilde{\Pi}_{\mu\nu}[q^2] = \int dT \cdot \int dg_1 \tilde{\Gamma}_{\mu\nu}^0(g_1; T) \Delta_{\lambda\alpha}(k^2) \Delta_{\lambda'\alpha'}(k'^2) \tilde{\Gamma}_{\mu\alpha'\nu}(\tilde{g}_1 g; T)$$

So naively I can apply $\int dg_1 \tilde{\Gamma}_{\mu\nu}^0(g_1)$ to both sides. But left side is independent of g , so $m=m'=0$. Also, $j=0!!$ It all collapses! Why!?

(1) I don't know how to handle the indices μ, ν during diag.

(2) whole thing fails anyway at $t=q^2=0$ because group and its related functions will all be singular.

Save for a rainy day.

Attempt to use k^2, k'^2 as variables

Reduce to a double integral whose variables are $t = k^2$ and $t' = k'^2$??

$$q = k + k'$$

Suppose we say: $q = (\sqrt{q^2}, 0, 0, 0)$

$$k = (E_k, \vec{k}) \Rightarrow k^2 = E_k^2 + |\vec{k}|^2$$

$$k' = (E_{k'}, -\vec{k}) \Rightarrow k'^2 = E_{k'}^2 + |\vec{k}|^2$$

$$E_k + E_{k'} = \sqrt{q^2} \quad k^2 - k'^2 = E_k^2 - E_{k'}^2 = (E_k + E_{k'})(E_k - E_{k'})$$

$$\Rightarrow k^2 - k'^2 = \sqrt{q^2}(E_k - E_{k'}) \Rightarrow k^2 - k'^2 + q^2 = 2\sqrt{q^2}E_k$$

$$q^2 = \sqrt{q^2}(E_k + E_{k'}) \quad k'^2 + k^2 + q^2 = 2\sqrt{q^2}E_{k'}$$

$$\text{So } \boxed{\begin{aligned} E_k &= (k^2 - k'^2 + q^2) / 2\sqrt{q^2} \\ E_{k'} &= (k'^2 - k^2 + q^2) / 2\sqrt{q^2} \end{aligned}}$$

$$\text{Next: } |\vec{k}|^2 = k^2 - E_k^2 = k^2 - \frac{(k^2 - k'^2 + q^2)^2}{4q^2} = \frac{4q^2k^2 - (k^2 - k'^2 + q^2)^2}{4q^2}$$

$$= -\frac{\{(k^2 - k'^2 + q^2)^2 - 4q^2k^2\}}{4q^2} = -\frac{\{k^4 + k'^4 + q^4 - 2k^2k'^2 - 2k'^2q^2 - 2q^2k^2\}}{4q^2}$$

$$\text{So } \boxed{|\vec{k}|^2 = -\frac{\Delta(k^2, k'^2, q^2)}{4q^2}}$$

$$\text{Now let's relate: } \frac{dE_k}{dk^2} dk^2 \frac{d|\vec{k}|^2}{dk'^2} dk'^2 = J \cdot dk^2 \cdot dk'^2 = J \left(\frac{E_k, |\vec{k}|^2}{k^2, k'^2} \right) dk^2 \cdot dk'^2$$

$$\text{So } J = \begin{vmatrix} \frac{\partial E_k}{\partial k^2} & \frac{\partial E_k}{\partial k'^2} \\ \frac{\partial |\vec{k}|^2}{\partial k^2} & \frac{\partial |\vec{k}|^2}{\partial k'^2} \end{vmatrix}$$

$$\text{Now } \frac{\partial E_k}{\partial k^2} = \frac{1}{2\sqrt{q^2}} \quad \frac{\partial |\vec{k}|^2}{\partial k^2} = -\frac{1}{4q^2} [2k^2 - 2k'^2 - 2q^2]$$

$$\frac{\partial E_k}{\partial k'^2} = -\frac{1}{2\sqrt{q^2}} \quad \frac{\partial |\vec{k}|^2}{\partial k'^2} = -\frac{1}{4q^2} [2k'^2 - 2k^2 - 2q^2]$$

I seem to get $J = \frac{1}{2\sqrt{q^2}}$. Oh no.

$$dk^0 |\vec{k}|^2 d|\vec{k}| = \frac{dE_k |\vec{k}| d|\vec{k}|^2}{2} = \frac{|\vec{k}|}{2} dE_k d|\vec{k}|^2 \\ = \frac{|\vec{k}|}{2} \cdot \frac{1}{2\sqrt{q^2}} dk^2 dk'^2$$

But $|\vec{k}| = \frac{\sqrt{-\Delta(k^0, k^1, q^2)}}{2q}$ so $\left(\frac{\sqrt{-\Delta}}{8q^2} \right)$

Result:

$$d^4k = \left\{ \frac{\sqrt{-\Delta(k^0, k^1, q^2)}}{8q^2} d(k^0) d(k^1) \right\} d\Omega$$

We are now saying

$$\begin{cases} k^\mu = (E_k, \vec{k}) & q^\mu = (\sqrt{q^2}, \vec{0}) = (q, \vec{0}) \\ k'^\mu = (E_{k'}, -\vec{k}) \end{cases}$$

and

$$\begin{cases} \vec{k} = |\vec{k}| (\sin\theta \cos\phi, \sin\theta \sin\phi, \cos\theta) \\ d\Omega = d(\cos\theta) d\phi \end{cases}$$

Now where are vectors n^μ, N^μ etc

$$n^\mu = [\cos\alpha_n, 0, 0, \sin\alpha_n] \quad n^2 = 1$$

$\begin{matrix} \uparrow & \uparrow & \uparrow & \uparrow \\ w & x & y & z \end{matrix}$

Then

$$(q \cdot n) = q \cos\alpha_n \quad \text{Define: } \left(\alpha \equiv \cos\alpha_n = \frac{(q \cdot n)}{q} \text{ same as before} \right)$$

$$(k \cdot q) = q E_k = \frac{1}{2}(k^2 - k'^2 + q^2) = b \quad (q \cdot n) = q \alpha = c$$

$$(k' \cdot q) = q E_{k'} = \frac{1}{2}(k'^2 - k^2 + q^2) = b'$$

$$(k \cdot n) = E_k \cos \alpha_n + |k| \cos \theta \sin \alpha_n = \frac{1}{2q}(k^2 - k'^2 + q^2) \alpha + |k| \sqrt{1 - \alpha^2} \cos \theta = a$$

$$(k' \cdot n) = E_{k'} \cos \alpha_n - |k'| \cos \theta \sin \alpha_n = \frac{1}{2q}(k'^2 - k^2 + q^2) \alpha - |k'| \sqrt{1 - \alpha^2} \cos \theta = a'$$

Keep going on these dot products

$$(k \cdot N) = k \cdot \left[n - \frac{k}{n \cdot k} \right] = (k \cdot n) - \frac{k^2}{(k \cdot n)} = \left[a - \frac{k^2}{a} \right] \text{ and so on.}$$

Now when I expand L as before, I get

$$L = \frac{a(a' - a)}{q^2 b'} \left[q^2 - c^2 + \left(c - \frac{b}{a} \right)^2 + \left(c - \frac{b'}{a'} \right)^2 + \left(\frac{d}{aa'} - 1 \right) \left(c - \frac{b}{a} \right) \left(c - \frac{b'}{a'} \right) \right]$$

Now however

$$a - a' = \left[\frac{1}{q}(k^2 - k'^2) \alpha + 2|k| \sqrt{1 - \alpha^2} \cos \theta \right]$$

Well,

$$\begin{aligned} (k \cdot k') &= E_k E_{k'} - |k|^2 = k \cdot [q - k] = (k \cdot q) - k^2 \\ &= \frac{1}{2}(k^2 - k'^2 + q^2) - k^2 = \frac{1}{2}(q^2 - k^2 - k'^2) = d \end{aligned}$$

Make a table: Again, a and a' are only variables.

~~$(k \cdot q)$~~

$a = (n \cdot k) = \alpha E_k + \sqrt{1 - \alpha^2} k \cos \theta$	$a + a' = c$
$a' = (n \cdot k') = \alpha E_{k'} - \sqrt{1 - \alpha^2} k' \cos \theta$	$b + b' = q^2$
$b = (q \cdot k) = q E_k$	
$b' = (q \cdot k') = q E_{k'}$	
$c = (n \cdot q) = q \alpha$	
$d = (k \cdot k') = \frac{1}{2} [q^2 - k^2 - k'^2]$	
$E_k = \frac{1}{2q} (k^2 - k'^2 + q^2)$	$E_k + E_{k'} = q$
$E_{k'} = \frac{1}{2q} (k'^2 - k^2 + q^2)$	

Now let's look again at the entire L -term:

$$\int d^4k \frac{2(k^2/\Lambda^2)}{k^2} \frac{2(k'^2/\Lambda^2)}{k'^2} L$$

$$= 2\pi \cdot \int_0^\infty d(k^2) \int_0^\infty d(k'^2) \left[\frac{\sqrt{-\Delta(k^2, k'^2, q^2)}}{8q^2} \right] \frac{2(k^2/\Lambda^2)}{k^2} \frac{2(k'^2/\Lambda^2)}{k'^2} \times \int_{-1}^1 dy L(y)$$

where $y = \cos\theta$. Of all the basic dot products, only a, a' are functions of y , just as before, so same grouping is useful:

$$L = \frac{1}{q^2 b'} \left[A_0(a')^2 + A_1(a') + A_2(1) + A_3(a') + A_4(a')^{-2} + A_5(a')^{-1} \right]$$

with same coefficients as before:

$$A_0 = -2q^2$$

$$A_1 = c [3q^2 - 2(b-b')]$$

$$A_2 = c^2 [b - 3b' - q^2 + 2d] - 2(b^2 + bb' + b'^2)$$

$$A_3 = c^3 [b' - d] + c [3b'^2 + bb' + db - 2db'] + \frac{1}{c} [dbb']$$

$$A_4 = b' [c^2(d - b') - db]$$

$$A_5 = b [c(b - d) + \frac{1}{c} db']$$

But idea: why not maximize the symmetry:

$$\frac{a(a'-a)}{q^2 b'} = \frac{1}{2q^2} \left[\frac{a(a'-a)}{b'} + \frac{a'(a-a')}{b} \right] + (\text{antisym})$$

$$= \frac{1}{2q^2} (a'-a) \left[\frac{ab - a'b'}{bb'} \right] = \left[\frac{aa'q^2 - [a^2b + a'^2b']}{2bb'q^2} \right]!$$

yield nothing
do integral!

So I can now write:

$$L = \left[\frac{q^2(aa') - [ba^2 + b'a'^2]}{2q^2bb'} \right]$$

$$\times \left\{ \left[\frac{q^2}{2} + \frac{(dc^2 - bb')}{2aa'} + \frac{dbb'}{2a^2a'^2} - \frac{bc}{a} + \frac{b^2}{a^2} - \frac{dcb}{a^2a'} \right] + \left[k \leftrightarrow k' \right] \right\}$$

This L is explicitly symmetric; it is a product of two symmetric factors.

So full integral is:

$$\int dk dk' : f_1^s(k, k') \underbrace{f_2^s(k, k') f_3^s(k, k')}_{=L}$$

Now, we can simplify by using this effective L:

$$L_{\text{eff}} = 2 \left[\frac{q^2(aa') - [ba^2 + b'a'^2]}{2q^2bb'} \right] \left[\frac{q^2}{2} + \frac{(dc^2 - bb')}{2aa'} + \frac{dbb'}{2a^2a'^2} - \frac{bc}{a} + \frac{b^2}{a^2} - \frac{dcb}{a^2a'} \right]$$

Then we have made fullest use of the symmetry: next, we should group out this thing.

$$2q^2bb' L_{\text{eff}} = (q^2aa' - ba^2 - b'a'^2) \left(q^2 + \frac{(dc^2 - bb')}{aa'} + \frac{dbb'}{a^2a'^2} - \frac{2bc}{a} + \frac{2b^2}{a^2} - \frac{2dcb}{a^2a'} \right)$$

Yes, but when all this is reduced, we will get same result as that already recorded. Moreover, the integrals will be much the same:

$$a = A + By \quad da = B dy$$

$$A = \alpha(k^2 - k'^2 + q^2)/2q = \alpha E_k$$

$$B = \sqrt{1 - \alpha^2} |k|$$

$$\int_{-1}^1 dy \frac{1}{a} = \frac{1}{B} \int_{A-B}^{A+B} \frac{da}{a} = \frac{1}{B} \ln \left(\frac{A+B}{A-B} \right)$$

$$a' = A' - By$$

$$da' = -B dy$$

$$A' = \alpha(k'^2 - k^2 + q^2)/2q = \alpha E_{k'}$$

$$\int_{-1}^1 dy \frac{1}{a'} = -\frac{1}{B'} \int_{A'+B}^{A'-B} \frac{da'}{a'} = -\frac{1}{B'} \ln \left(\frac{A'-B}{A'+B} \right) = \frac{1}{B'} \ln \left(\frac{A'+B}{A'-B} \right)$$

$$\int_{-1}^1 dy (a')^2 = \int_{-1}^1 dy (A' - By)^2 = \int_{-1}^1 dy [A'^2 - 2A'B y + B^2 y^2]$$

$$= 2A'^2 - 2A'B \cdot 0 + B^2 \frac{2}{3} = 2A'^2 + \frac{2}{3} B^2$$

$$\int_{-1}^1 dy (a') = \int_{-1}^1 dy (A' - By) = 2A' \quad \int_{-1}^1 dy = 2.$$

Summarize:

$$\int_{-1}^1 dy (a')^2 = 2\alpha^2 E_k^2 + \frac{2}{3}(1-\alpha^2)|\vec{k}|^2$$

$$\int_{-1}^1 dy (a') = 2\alpha E_k$$

$$\int_{-1}^1 dy (1) = 2$$

$$\int_{-1}^1 dy \left(\frac{1}{a'}\right) = \frac{1}{\sqrt{1-\alpha^2}|\vec{k}|} \ln \left[\frac{\alpha E_k + \sqrt{1-\alpha^2}|\vec{k}|}{\alpha E_k - \sqrt{1-\alpha^2}|\vec{k}|} \right]$$

$$\int_{-1}^1 dy \left(\frac{1}{a}\right) = \frac{1}{\sqrt{1-\alpha^2}|\vec{k}|} \ln \left[\frac{\alpha E_k + \sqrt{1-\alpha^2}|\vec{k}|}{\alpha E_k - \sqrt{1-\alpha^2}|\vec{k}|} \right]$$

$$\int_{-1}^1 dy \frac{1}{2(a')^2} = \frac{2}{\alpha^2 E_k^2 - |\vec{k}|^2}$$

$$\int_{-1}^1 dy \frac{1}{2(a')^2} = \frac{1}{+B} \int_{A'-B}^{A'+B} da' \frac{1}{(a')^2} = \frac{1}{B} \int_{A'-B}^{A'+B} dx (x)^{-2} = \frac{2}{A'^2 - B^2}$$

$$= \frac{2}{\alpha^2 E_k^2 - (1-\alpha^2)|\vec{k}|^2} = \frac{2}{\alpha^2 E_k^2 - |\vec{k}|^2}$$

It is not at all clear that this makes anything simpler. In order to have a chance of doing those logs you need $|k|$ as a variable.
Now about:

$$s \equiv |k|/q$$

$$k^2 + k'^2 = \frac{1}{2} q^2 [1 + w^2 + 4s^2]$$

$$k^2 - k'^2 = \cancel{w^2} = q^2 w$$

$$w \equiv (E_k - E_{k'})/q$$

Then $E_k = \frac{1}{2} q (1+w)$

$$E_{k'} = \frac{1}{2} q (1-w)$$

$$k^2 = \frac{1}{4} q^2 [(1+w)^2 + 4s^2]$$

$$k'^2 = \frac{1}{4} q^2 [(1-w)^2 + 4s^2]$$

Jacobian: $dE_k d|k| \cdot |k|^2 = q dw \cdot q ds \cdot q^2 s^2 = q^4 s^2 ds dw$.

Look at our symbols:

$$b = \frac{1}{2} q^2 (1+w)$$

$$b' = \frac{1}{2} q^2 (1-w)$$

$$c = q\alpha$$

$$d = \frac{1}{4} q^2 [1 - w^2 - 4s^2]$$

$$(b - b') = q^2 w$$

Look at the integrals:

$$I_0 =$$

Explain page 5, version 3.

(of B2 paper)

base gauge

essentially correct.

g_0

$$A_0(q^2, \Lambda^2, g_0) \\ \hookrightarrow = A_0\left(\frac{q^2}{\Lambda^2}, g_0\right)$$

← do not try (yet) to take $q^2 \rightarrow 0$ + call that "L" because such a D_0 would $= \infty$ if confinement

The λ_1 -gauge

Choose an arbitrary $Z = 14,736$, eg. Let

$$g_1 = Z g_0$$

Treat as a RG transform Z . Then we expect

$$A_1 = Z^{-1} A_0$$

Define λ by: $A_1(q^2 = \lambda^2) = 1$. Then:

$$Z = A_0\left(\frac{\lambda^2}{\Lambda^2}, g_0\right) = Z(\lambda^2/\Lambda^2) \quad , \text{ drop the } g_0 \text{ for now.}$$

$$\text{Thus, } A_1 = Z^{-1}(\lambda^2/\Lambda^2) A_0\left(\frac{q^2}{\Lambda^2}, g_0\right) = A_1\left(\frac{\lambda^2}{\Lambda^2}, \frac{q^2}{\Lambda^2}, g_0\right)$$

Reformulate to get $A_1 = A_1\left(\frac{q^2}{\lambda^2}, \frac{\Lambda^2}{\lambda^2}, g_0\right)$ since dimensionless.

$$\text{One: } A_1\left(1, \frac{\Lambda^2}{\lambda^2}, g_0\right) = 1.$$

Now, reformulate again. Replace Λ^2/λ^2 with Z , then with g so

$$A_1 = A_1\left(\frac{q^2}{\lambda^2}, g\left(\frac{\Lambda^2}{\lambda^2}\right), g_0\right) \leftarrow \text{Jim says no, do this } A_1\left(\frac{q^2}{\lambda^2}, \frac{\Lambda^2}{\lambda^2}, g\left(\frac{\Lambda^2}{\lambda^2}\right)\right)$$

Suppress the g_0 + call (2) $g(\lambda)$: $A_1 = A_1\left(\frac{q^2}{\lambda^2}, g(\lambda)\right)$

Thus:

$$1) \quad A_0\left(\frac{q^2}{\Lambda^2}, g_0\right) = Z(\lambda^2/\Lambda^2) A_1\left(\frac{q^2}{\lambda^2}, g(\lambda)\right)$$

$$2) \quad g(\lambda) = Z(\lambda^2/\Lambda^2) g_0$$

② Repeat the above in more abstract terms:

a) start with a function $h(x)$

b) define $f_Z(x) \equiv h(x)/Z$ where $Z = \text{a number}$.

c) search for a value of x , $x = \lambda$, such that $f_Z(\lambda) = 1$. Then:

$$f_Z(\lambda) = \frac{h(\lambda)}{Z} = 1 \Rightarrow \begin{matrix} \lambda = \lambda(Z) \\ Z = Z(\lambda) \end{matrix} \updownarrow \text{ie, these are related.}$$

d) now replace the Z label on $f_Z(x) = f(x; Z)$ by λ to get $F(x; \lambda)$. Thus

$$F(x; \lambda) = h(x)/Z(\lambda)$$

You can now think of λ as the independent variable.

③ Now go back to B2: we found that

$$A(q^2/\Lambda^2, g_0) = Z(\lambda^2/\Lambda^2; g_0) A_{\text{REN}}(q^2/\lambda^2, g(\lambda^2); g_0)$$

$$\text{or } h(x) = Z(\lambda) F(x; \lambda) \text{ in abstract example.}$$

Since $\lambda = \text{free param}$, let $\lambda^2 = q^2$. Then

$$A(q^2/\Lambda^2; g_0) = Z(q^2/\Lambda^2; g_0) \underbrace{A_{\text{REN}}(1, g(q^2); g_0)}$$

but I thought this = 1 for first arg = 1.

Then I get $A = Z$ which is their big ansatz. What about γ ?

④ I should check to see if M as in B2 (1.6) AF thing is defined as place where $A_{\text{REN}} \equiv 1$ as I have guessed. This seems to agree with Abank. eqn (17).

⑤ Redo in axial gauge. We now have:

$$\delta = \frac{q^2 n^2}{(q \cdot n)^2}$$

$$\Delta_{\mu\nu} = \frac{1}{q^2} \left\{ B (g_{\mu\nu} - \eta_\mu \eta_\nu) + C N_\mu N_\nu \right\} = \frac{1}{\hat{q} \cdot \hat{n}}$$

These B, C are in the bare RG gauge, i.e., "unrenormalized". Now, let

$$\begin{aligned} \tilde{g} &= Z_g g_0 & \tilde{C} &= Z_C C & B &= B(q^2/\Lambda^2, \delta; g_0) \\ \tilde{B} &= Z_B B & \tilde{\Delta} &= Z_\Delta \Delta \end{aligned}$$

Now demand that

$$\tilde{B}(\frac{q^2}{\Lambda^2}, \Lambda^2, \delta; g_0, Z) = 1 \Rightarrow B(\frac{\lambda^2}{\Lambda^2}, \delta; g_0) = Z.$$

Thus I seem to get $Z = Z(\lambda^2/\Lambda^2; \delta; g_0)$. But maybe you can show that $\tilde{g}$ is indep of δ , therefore so is Z . Assume such is the case. Then seems to me we get

$$B(\frac{q^2}{\Lambda^2}, \delta; g_0) = Z(\lambda^2/\Lambda^2) \tilde{B}(\frac{q^2}{\lambda^2}, g(\lambda^2); \delta; g_0)$$

and again $\tilde{B}(1, xxx) = 1$ so $B = C = Z$ for any value of q^2 .

⑥ Normally, setting $\lambda^2 = q^2$ does not give $\tilde{B} = 1$:

$$\tilde{B}(q^2, \Lambda^2, \lambda^2, \delta, g_0): \quad \tilde{B}(q^2 = \lambda^2, \Lambda^2, \lambda^2, r, g_0) = 1$$

$$\text{Yes I guess it does.} \quad \tilde{B}(q^2, \Lambda^2, q^2, r, g_0) = 1 \checkmark$$

Back to math example:

$$F(x; \lambda) = h(x)/Z(\lambda)$$

$$\begin{aligned} F(x=\lambda; \lambda) &= 1 \Rightarrow h(x) = Z(\lambda) \cdot F(x; \lambda) \\ &= Z(x) F(x; x) \\ &= Z(x). \end{aligned}$$

But that is right! $Z(\lambda) = h(\lambda)$ all along. Thus, by construction we can replace our original fctn $h(x)$ with $Z(x)$.

⑦ Possible problem:

$$\tilde{C}_2(q^2) = \tilde{Z}^{-1} C(q^2)$$

$$\tilde{B}_2(q^2) = \tilde{Z}^{-1} B(q^2)$$

How do you know that $\exists$ one value of q^2 which sets both $\tilde{C}$ and $B=1$!

You might find value of q^2 which sets $\tilde{B}(q^2=\lambda^2)=1$, get your $Z(\lambda)$, then evaluate $\tilde{C}(q^2=\lambda^2)$. Then

$$\tilde{C}(q^2=\lambda^2, \delta, g_0, g(\lambda) \dots) = K(\delta, \lambda, g_0) \text{ then}$$

$$\tilde{\Delta}_{\mu\nu}$$

$$\tilde{\Delta}_{\mu\nu}(q^2=\lambda^2) = \frac{-1}{\lambda^2} [(\delta_{\mu\nu} - g_{\mu\nu}) \cdot 1 + N^\mu N^\nu K(\delta, \lambda, g_0)]$$

can't get rid of this

⑧ Another problem. Go back to where you tried this:

$$\tilde{B}(q^2=\lambda^2, \lambda^2, \delta, g_0, 2) = 1$$

But $\delta \hat{=} (\hat{q} \cdot \hat{n})$, so $\tilde{B} = \tilde{B}(q^2, \text{direction of } q)$. Thus,

if $\tilde{B}(q^2=\lambda^2)=1$ for q in some direction, then it will not be 1 in some other direction. Thus, the best you can do is

$$\tilde{B}(q^2=\lambda^2, \delta \dots) = \beta(\delta)$$

$$\tilde{C}(q^2=\lambda^2, \delta \dots) = \delta(\delta)$$

And maybe you could also require that $\beta(\delta)=1$ when $\hat{q} \parallel \hat{n}$, i.e., when $\delta=1$. Cannot also demand that $\delta \parallel 1=1$.

So we then get

$$\tilde{B}(q^2 = \lambda^2, \lambda, \Lambda^2, \dots) = \beta(\lambda) = Z(\lambda^2/\Lambda^2) B(q^2 = \lambda^2, \Lambda^2, g_0, \lambda)$$

and here you see how λ -dependence could cancel so Z indep of λ .

Follow through to get

$$B\left(\frac{q^2}{\Lambda^2}, \lambda, g_0\right) = Z(q^2/\Lambda^2) \underbrace{\tilde{B}(1, \lambda, g(q^2), g_0)}_{= \beta(\lambda)}$$

So there seems to be no question

$$B\left(\frac{q^2}{\Lambda^2}, \lambda, g_0\right) = Z(q^2/\Lambda^2) \beta(\lambda) \quad \text{etc.}$$

The real ansatz is that $\beta(\lambda) = 1$!

⑨ So far I have defined λ as the renormalization point for the function $\tilde{B}$. I.e.,

$$\tilde{B}[q^2 = \lambda^2; \dots, \lambda^2, \dots] = \text{some desired form}, \quad \beta(\lambda).$$

This is like saying that $\Gamma^{(2)}(p^2 = \lambda^2) =$

$$\Delta_F = \frac{A(p^2)}{p^2 - m^2}$$

$$\Delta_F(p^2 = 0) = \frac{A(0)}{-m^2} = -\frac{m^2}{A(0)} = -m^2$$

$$\Delta_F(p^2 = 0, \lambda, m^2) = -m^2$$

for any λ , we might require this be true. I.e., this will define $\lambda = \lambda(2)$.

This is Abarb. (20)

Tinclanis: ALEN really is renam'! so $\rightarrow$ const
as $\lambda \rightarrow \infty$, even though $\lambda = \text{free}$
parameter. Check.