

Dimensional Regulation & Axial Gauge

Hooft & Veltman
(1972)

Leibbrandt
(1975)

Konetschny
(1978)

Frankel & Meuldermans
(1976)

Phil Lucht notes 1979

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7.A.1

REGULARIZATION AND RENORMALIZATION OF GAUGE FIELDS

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Abstract: A new regularization and renormalization procedure is presented. It is particularly well suited for the treatment of gauge theories. The method works for theories that were known to be renormalizable as well as for Yang-Mills type theories. Overlapping divergencies are disentangled. The procedure respects unitarity, causality and allows shifts of integration variables. In non-anomalous cases also Ward identities are satisfied at all stages. It is transparent when anomalies, such as the Bell-Jackiw-Adler anomaly, may occur.

1. INTRODUCTION

Recently it has been shown [1] that it is possible to formulate renormalizable theories of charged massive vector bosons. The derived Feynman rules involve ghost particles, and in order to establish unitarity and causality of the S -matrix Ward identities are needed. The necessary combinatorial techniques were given in ref. [2], in the treatment of massless Yang-Mills fields. It was emphasized that these same techniques work also in the case of massive vector boson theories obtained from the massless theory by means of the Higgs-Kibble [3] mechanism. Stated somewhat differently: the manifestly renormalizable set** of Feynman rules involving ghosts may be transformed into a set of manifestly unitary and causal Feynman rules by means of Ward identities. Actually these manifestly unitary and causal Feynman rules are quite meaningless in view of the occurring divergencies, and a direct proof of unitarity and causality starting from the manifestly renormalizable rules is to be preferred. This is precisely the program carried through in refs. [1, 2].

However, even with a set of manifestly renormalizable rules one cannot be sure that a consistent theory results unless a suitable cut-off and subtraction procedure has been defined. In particular, since unitarity depends crucially on the validity of the Ward identities one must have a procedure that respects those Ward identities. In ref. [2] the existence of such a procedure was proven for diagrams containing at

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** i.e. renormalizable with respect to power counting.

Derive Eq (2)

Let $kx = \tilde{k}$ ✓
 $k^2 + m^2 = \tilde{m}^2$ ✓ $\Rightarrow du = \tilde{p}^2 + 2p\tilde{k} + \tilde{m}^2$ ✓

The inner integral is: $4(A8)_{\mu\nu} + 2k_\nu(A6)_\mu + 2k_\mu(A6)_\nu + k_\mu k_\nu(A5)$
 $= 2[(A7) + (k^2 + m^2)(A5)] \delta_{\mu\nu} - 4\delta_{\mu\nu} k_\alpha(A6)_\alpha$ ✓

But $(A6)_\mu = (-\tilde{k}_\mu)(A5)$ ✓, so write out:

$4(A8)_{\mu\nu} - 2(A7)\delta_{\mu\nu} + (A5) [k_\mu k_\nu - 2x k_\mu k_\nu - 2k_\mu x k_\nu - 2(k^2 + m^2)\delta_{\mu\nu}]$
 $+ 4x\delta_{\mu\nu} k^2$ ✓

Let $\frac{i\pi^{\frac{1}{2}}}{(\tilde{m}^2 - k^2)^{\alpha - \frac{1}{2}n}} \Gamma(\alpha) = F$ ✓. Then.

$\tilde{m}^2 - k^2 = k^2 x + m^2 - k^2 x$
 $= m^2 + k^2 x(1-x)$
 $= D$ ✓

$4F [\Gamma(\alpha - \frac{1}{2}n) \tilde{k}_\mu \tilde{k}_\nu + \Gamma(\alpha - 1 - \frac{1}{2}n) \frac{1}{2} \delta_{\mu\nu} (\tilde{m}^2 - k^2)]$ ✓ $\alpha = 2$

$- 2\delta_{\mu\nu} F [\Gamma(\alpha - \frac{1}{2}n) k^2 + \Gamma(\alpha - 1 - \frac{1}{2}n) \frac{1}{2} n (\tilde{m}^2 - k^2)]$ ✓

$+ F \Gamma(\alpha - \frac{1}{2}n) [k_\mu k_\nu - 4x k_\mu k_\nu - 2(k^2 + m^2)\delta_{\mu\nu}]$ ✓ $\Rightarrow -2(k^2 + m^2 - 2xk^2)\delta_{\mu\nu}$

$= 4F [\Gamma(\alpha - \frac{1}{2}n) x^2 k_\mu k_\nu + \Gamma(\alpha - 1 - \frac{1}{2}n) \frac{1}{2} \delta_{\mu\nu}]$ ✓

$- 2\delta_{\mu\nu} F [\Gamma(\alpha - \frac{1}{2}n) x^2 k^2 + \Gamma(\alpha - 1 - \frac{1}{2}n) \frac{1}{2} n]$ ✓

$+ F \Gamma(\alpha - \frac{1}{2}n) [k_\mu k_\nu (1-4x) - 2(k^2 + m^2)\delta_{\mu\nu}]$ ✓

$= 4F [\Gamma(\alpha - \frac{1}{2}n)]$

$= F k_\mu k_\nu \{ 4\Gamma(\alpha - \frac{1}{2}n) x^2 + (1-4x)\Gamma(\alpha - \frac{1}{2}n) \}$ ✓

$+ F \delta_{\mu\nu} \{ 2\Gamma(\alpha - 1 - \frac{1}{2}n) D - 2\Gamma(\alpha - \frac{1}{2}n) x^2 k^2 - 2\Gamma(\alpha - 1 - \frac{1}{2}n) \frac{1}{2} n D - 2\Gamma(\alpha - \frac{1}{2}n) (k^2 + m^2) \}$ ✓

$= F k_\mu k_\nu \Gamma(\alpha - \frac{1}{2}n) [1-4x+4x^2] (1-2x)^2$ ✓

$+ F \delta_{\mu\nu} \{ \Gamma(\alpha - 1 - \frac{1}{2}n) [2D - nD] + \Gamma(\alpha - \frac{1}{2}n) [-2x^2 k^2 - 2(k^2 + m^2)] \}$ ✓

$\frac{i\pi^{\frac{1}{2}}}{(D)^{\alpha - \frac{1}{2}n}} \Gamma(\alpha)$
 $\frac{1}{(D)^{\alpha - \frac{1}{2}n}} \Gamma(\alpha)$
 so $k_\mu k_\nu$ check not

The δ_{uv} term is

$$-Fk^2 \delta_{uv} \Gamma(2 - \frac{1}{2}n) [(1-2x) + (1-2x)^2]$$

↑
extra.

↑ connect.

$$1-2x = 1-2(1-y) =$$

$$1-x = y \quad 1-2+2y =$$

$$x = 1-y \quad = -1+1y$$

But, $\int_0^1 dx \frac{(1-2x)}{f[x(1-x)]} = - \int_0^1 dy \frac{(1-2y)}{f[y(1-y)]} \Rightarrow = 0. \checkmark$

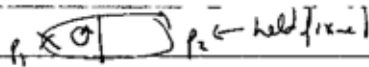
QED

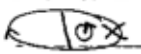
Examine (29)

① Overall power count is $\frac{8}{2} = 2$, like k^2 , quadratic.

② Hold p_2 fixed, then $d p_1$ is $\frac{4}{2}$, like $\frac{d k}{k}$, i.e. log

③ To isolate this log subgraph divergence, do the $d p_1$ integral, it will have a pole at $n=4$, subtract this pole as in (3a)! Symbolically this sub term is called.



④ There is also a  to be subtracted.

⑤ There is also  held $p_1 - p_2 = \text{fixed} \equiv p_3$

$$\int d p_3 d p_1 \frac{1}{(p_1^2 + m_1^2)^4 ((p_1 - p_3)^2 + m_3^2)^2 (p_3^2 + k^2)^2} \dots$$

So each subloop corresponds here to a log subgraph div + you must subtract as in (3a).

⑥ Why do you get $\frac{A}{n-4} + \text{not } \frac{A p_2^2 + B}{n-4}$? Because second integral can be at worst overall quadratic $= \frac{4}{2}$, cannot have extra $\sim$ get $\frac{6}{2}$.

Calculate Integral #1. The $\int dx$ is obvious from usual FP, I have checked it.

$$(A5) \quad \int d^4 p \frac{1}{(p^2 + 2k \cdot p + k^2 + m^2 - k'^2)^\alpha}$$

$$= \int d^4 p \frac{1}{[p^2 + (m^2 - k'^2)]^\alpha} = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})} \int_0^\infty \frac{dr r^{n-1}}{[r^2 + (m^2 - k'^2)]^\alpha} \quad \begin{matrix} \beta = n-1 \\ M^2 = (m^2 - k'^2) \end{matrix}$$

$$= \frac{1}{2} \frac{\Gamma(\frac{n}{2}) \Gamma(\alpha - \frac{1}{2}n)}{\Gamma(\alpha)} (m^2 - k'^2)^{\alpha - \frac{1}{2}n}$$

To explain the (i).

$$(A6) \quad Ak^2 = I_n k^4 = \int d^4 p \left[\frac{kp}{(p^2 + 2kp + m^2)^\alpha} \right] \quad I_n = Ak_n$$

Let $p' = p + k$ ← odd.

$$= \int d^4 p' \frac{k \cdot (p' - k)}{(p'^2 + (m^2 - k'^2))^\alpha} = -k^2 \times (A.5) = Ak^2$$

So $(A6)^n = (-k^4)(A5)$

$$(A7) = \int d^4 p' \frac{(p' - k)^2}{(p'^2 + (m^2 - k'^2))^\alpha} = k^2 (A5) + (A5)^{\text{with } n \rightarrow n+2 \uparrow \text{ not!}}$$

$$= \frac{k^2 i \pi^{\frac{n}{2}}}{(m^2 - k'^2)^{\alpha - \frac{1}{2}n}} \frac{\Gamma(\alpha - \frac{1}{2}n)}{\Gamma(\alpha)} + \frac{i \pi^{\frac{n}{2}} \pi}{(m^2 - k'^2)^{\alpha - \frac{1}{2}n}} \frac{\Gamma(\alpha - \frac{1}{2}n - 1)}{\Gamma(\alpha)}$$

$$= \frac{i \pi^{\frac{n}{2}}}{(m^2 - k'^2)^{\alpha - \frac{1}{2}n}} \frac{1}{\Gamma(\alpha)} \left\{ k^2 \Gamma(\alpha - \frac{1}{2}n) + \cancel{\pi} (m^2 - k'^2) \Gamma(\alpha - \frac{1}{2}n - 1) \right\}^{\frac{n}{2}}$$

$\frac{n}{2} \Gamma(\frac{n}{2})$

No: second term is $\frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})} \frac{1}{2} \frac{\Gamma(\frac{n}{2} + 1) \Gamma(\alpha - \frac{1}{2}n - 1)}{\Gamma(\alpha) (m^2 - k'^2)^{\alpha - \frac{1}{2}n - 1}}$ add $\frac{n}{2}$, not π .

$$(A8)_{\mu\nu} = Ak_{\mu}k_{\nu} + Bg_{\mu\nu}$$

$$(A8)_{\mu\nu} k_{\mu}k_{\nu} = Ak^4 + Bk^2 = \int \frac{d^4 p}{(2\pi)^4} (?)$$

$$(A8)_{\mu\nu} = Ak^2 + 4B =$$

$$\begin{aligned} & (k(p' - k))^2 \\ & (kp')^2 + k^4 - 2 \\ & (k \cdot p')^2 - k^4 - 2k^2 k \cdot p' \end{aligned}$$

$$p' = p + k$$

$$(A8)_{\mu\nu k p k\nu} = \int d^4 p \frac{(k \cdot p)^2}{(p^2 + 2k \cdot p + m^2)^2}$$

$$= \int d^4 p' \frac{[k \cdot p' - k \cdot k]^2}{[p'^2 + (m^2 - k^2)]^2} = +k^4 (A5) + \underbrace{\int d^4 p' \frac{(k \cdot p')^2}{[p'^2 + (m^2 - k^2)]^2}}_I$$

This is not a scalar integral.

$$(A8)_{\mu\mu} = (A7)$$

$$S_0 \quad I = \int d^4 p' \frac{k^2 r^2}{(r^2 + (m^2 - k^2))^2} \cdot \underbrace{\int_{-1}^1 dz (\sqrt{1-z^2})^{n-3}}_{2 \int_0^{\pi/2} d\theta (\sin\theta)^{n-2} (\cos\theta)^2} \cdot \frac{2 \pi^{\frac{n}{2}-1}}{\Gamma(\frac{n}{2})}$$

$$2 \cdot \frac{\pi^{\frac{n}{2}}}{2} B\left(\frac{n-1}{2}, \frac{3}{2}\right)$$

$$n-1 = n-2$$

$$n-1 = 2$$

$$n = n-1$$

$$n = 3$$

$$I = \frac{2 \pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})} \frac{\Gamma(\frac{n-1}{2}) \Gamma(\frac{3}{2})}{\Gamma(n)} k^2 \times \left[\frac{\frac{n}{2} \Gamma(\frac{n}{2}) \Gamma(\frac{3}{2})}{\Gamma(n)} \frac{\Gamma(\frac{3}{2})}{\Gamma(n)} \frac{1}{(m^2 - k^2)^{n-1}} \right]$$

$$I = \frac{\pi^{\frac{n}{2}} k^2}{\Gamma(\frac{n}{2})} \frac{\Gamma(\frac{n-1}{2}) \Gamma(\frac{3}{2})}{\Gamma(n)} \frac{1}{(m^2 - k^2)^{n-1}}$$

$$I = \frac{\pi^{\frac{n}{2}} k^2}{\Gamma(\frac{n}{2})} \frac{\Gamma(\frac{n-1}{2}) \Gamma(\frac{3}{2})}{\Gamma(n)} \frac{1}{(m^2 - k^2)^{n-1}}$$

$$S_0 \quad (A8)_{\mu\nu k p k\nu} = k^4 (A5) + I = A k^4 + B k^2$$

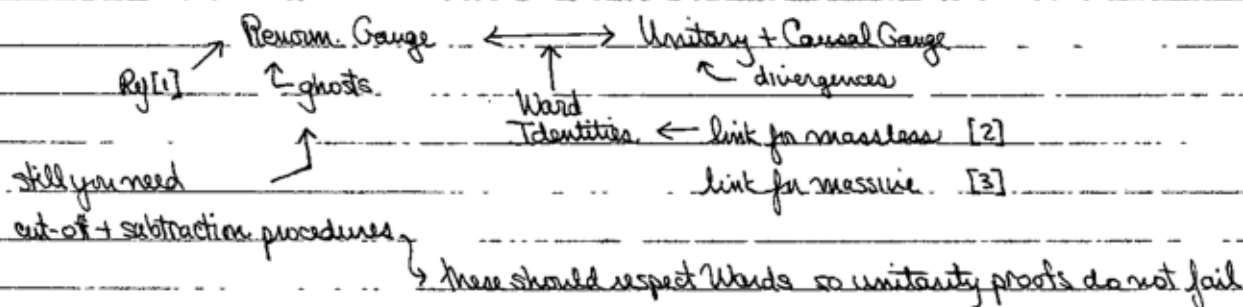
$$(A8)_{\mu\mu} = (A7) = A k^2 + 4B$$

$$S_0 \quad A = (A5) \quad B = \frac{i \pi^{\frac{n}{2}} \Gamma(\frac{n-1}{2}) \Gamma(\frac{3}{2})}{2 \Gamma(n) (m^2 - k^2)^{n-1}}$$

$$\Rightarrow (A8)_{\mu\nu} = (A5) k_\mu k_\nu + B g_{\mu\nu} = \frac{i \pi^{\frac{n}{2}}}{\Gamma(n) (m^2 - k^2)^{n-1}} \left[\Gamma(\frac{n-1}{2}) k_\mu k_\nu + \frac{1}{2} (m^2 - k^2) \Gamma(\frac{n-1}{2}) g_{\mu\nu} \right]$$

(Reg. and Ren. of Gauge Fields)
 t'Hooft + Veltman
 NP B44, 189-213 (1972) 24p.

1. Introduction. Recall this business of two kinds of gauges



[2] used a 5th dimension, OK for one-loop diagrams only!

Here instead we will go to complex n .

[1] Bollini, Giambagi, Gonzalez (1964) + Speer, book (1969) also consider continuation of the prop. exp.

2. Definitions. We start with an example: compute $\Pi_{\mu\nu}$ to order e^2 in scalar QED, but do so in n -dimensions. You get

$$\Pi_{\mu\nu} = e^2 \pi^{\frac{n}{2}} \Gamma(2 - \frac{1}{2}n) (k_\mu k_\nu - k^2 \delta_{\mu\nu}) \int_0^1 dx (1-2x)^{\frac{1}{2}n-2} D$$

As $n \rightarrow 4$, you see a pole. This is the divergence! Just subtract it out to get

$$\Pi_{\mu\nu}^{(Ren)}(k) = -ie^2 \pi^2 (k_\mu k_\nu - k^2 \delta_{\mu\nu}) \underbrace{\int_0^1 dx (1-2x)^2 \ln D(k, x)}_{\Pi(k)} + \text{const} \times (k_\mu k_\nu - k^2 \delta_{\mu\nu})$$

The const. cannot be found.

Now you get

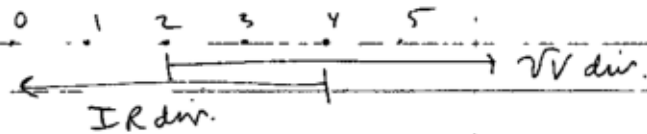
$$\Pi_{\mu\nu}^{(Ren)}(k) = (k_\mu k_\nu - k^2 \delta_{\mu\nu}) \Pi(k)$$

$$\Pi(k) = -ie^2 \pi^2 \int_0^1 dx (1-2x)^2 \ln D(k, x) + \text{const.}$$

Obviously, const affects $\Pi(0)$.

So easy to renormalize. Now how do you justify this?!

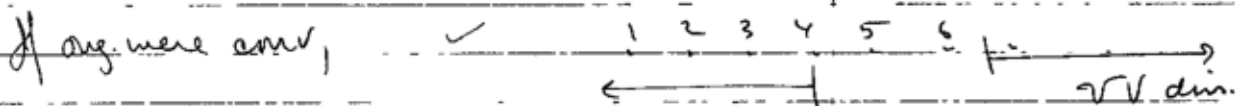
Justification Eq (6) is seen from (1) but Eq (6) diverges if $n \geq 2$ (VV) or if $n \leq 4$ (IR). So its OK for $2 < n < 4$ not OK anywhere!



So, continue by parts to lower the IR limit to $4-2\lambda$. For $\lambda=2$, you are then OK in $n \in (0, 2)$, then you continue off this interval.

The surface terms just give well-defined IR poles + keep them if you like.

You can choose λ and then n (say $\lambda=6$, n near -2) to get as much conv. as you need in both IR and VV.



OK, I am happy with this justification. The question asked is this: the original expression (1) diverges for $n > 2$; even when you write it as (6), it still diverges. Also, it has an IR divergence that somehow isn't really there. You do some derivatives to lower the (apparent) IR divergence off to the left in the n axis, so that you get a real region of convergence between some negative n and $n=2$. Then all you have to do is continue off this region to the neighborhood of $n=4$. For the one-loop case they do this explicitly. But to me, being used to analyticity, this is really no worry. Once I have computed result (2), I can see immediately how to continue the damn thing to n near 4. So I don't have to worry about the details of the "partial p " operation and formula (12).

In other words, I believe that any integral you do in n -dimensions will be an analytic function of n and thus you can take a limit as n goes to 4.

Subtracting out the pole at $n=4$ with its residue is equivalent to adding a counterterm in the lagrangian with the same pole effect. Fine, I am beginning to like this stuff.

3. Wards, Unitarity To show that the trivial Ward still holds in n -dimensions is just algebra using $p = p + P$ with $p \cdot P = 0$ etc. To show unitarity, Cutkosky's original arguments still go thru, they claim, so no problem with "cutting rules" for general n .

Section 4: Renormalization.

For one loop diagrams, you want to show that the residue of the pole which you are going to subtract out is a polynomial with real coefficients in the external momenta. The point here is that you want to avoid getting anything imaginary in the residue because then when you subtract, you will change the imaginary part in that order, but this is not allowed because the imaginary part is uniquely determined through unitarity by lower orders and cannot be changed by a subtraction term.

The one loop case is easy and you can quickly convince yourself that there is simply a real polynomial residue. By the way, a log divergence makes a pole at $n=4$ as does any divergence.

The two loop case is more interesting and brings in the overlap problem. Consider a two-loop diagram, eg, equation (24) with general exponents of (29) with obvious exponents. There is an overall integration, and there are three subintegrations. If any of the subs diverge, you will get a pole at $n=4$. You must subtract this out. So you make three subtractions as in figure 5, see equation (30). This is all quite clear.

In addition to these subgraph subtractions (which remove subgraph divergences), you also have to make an overall subtraction, ie, you must remove a pole which corresponds to the "overall" integration divergence. This is shown in (31). $\leftarrow N^0$

The problem they worry about is this: when you deal with a subgraph subtraction integral, you get a pole times a residual integral. In principle, that subgraph pole times the finite part of the residual integral can generate logs of external momenta in the obvious way. Such logs are unacceptable from point of view of being renormalizable (I don't know why yet). So you have to show that the original two-loop diagram exactly cancels any such logs!

The example (29) shows how this happens: (30) shows you the potential log generator from a subtraction graph. Eq. (31) shows you the original graph after it has been "partialled" for continuation. You can see that the second term has a potential log-generator which exactly cancels the subtraction (30), so you are OK. The other two terms in (31) will contain terms cancelling the other two subtraction potential log-generators.

So here we had the tough example of overlapping divergences, yet the proof of renormalizability went through OK. To renormalize this graph, just subtract out the subgraph poles [proof shows no logs left over]. The overall subtraction is not discussed, but you would do it too somehow.

Where does "overlap" come in here? In a non-overlapping situation, as shown in figure 6, you just subtract out the subgraph poles. In a subtraction term there is no residual integration, so you can never get $\log(k^2)$ stuff, so no problem.

Only in the overlapping divergences case do you have a residual integral multiplying your pole residue, so you have to worry!

5. Extention to Fermions Straightforward, as long as no γ_5 or ϵ tensors.

6. Limitations: Suppose you have QED with an axial vertex, which you recall is related to the Adler Bell Jackiv triangle graph anomaly, though I cant remember what this anomaly is. Ward violation? Anyway, when you have γ_5 you are screwed because it can be expressed in terms of the ϵ tensor which does not generalize to complex n , or at least they dont know how to do it. Nevertheless, they go onto compute the "anomaly" and get the right answer.

App A: dimensional integrals dictionary

App B: Theorem that in general shows cancellation of potential logs.

App c: trace comments for fermion case extention.

fin 1.23.79

Leibbrandt
(1975)

Introduction to the technique of dimensional regularization*

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The purpose of this review article is to explain and illustrate in detail the technique of dimensional regularization, which is a major mathematical tool in the renormalization program of gauge theories. The most important single feature of the new technique is the concept of analytic continuation in the number of space-time dimensions 2ω , where the regulating parameter ω is complex in general, and $\omega = 2$ corresponds to four-dimensional space-time. The technique of dimensional regularization preserves the local gauge symmetry of the underlying Lagrangian and thereby permits a consistent gauge-invariant treatment of divergent Feynman integrals to all orders in perturbation theory. The method can thus be applied—as demonstrated in this article—not only to Abelian gauge models, but more importantly to non-Abelian theories such as Yang-Mills fields and quantum gravity, to which the majority of conventional regularization procedures is inapplicable. We illustrate both the advantages and the limitation of dimensional regularization, as well as its extension to massless particles.

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I. INTRODUCTION

A. Ultraviolet infinities in quantum field theory

1. Preliminary remarks

The treatment of ultraviolet infinities continues to be one of the most challenging and tenacious problems in relativistic quantum field theory. Only in quantum electrodynamics has it been possible to eliminate these infinities consistently and in a physically meaningful manner by absorbing them into the bare, i.e., unobserved, charge and mass of the electron. The result is the successful renormalization program of Dyson (1949a,b) and Salam (1951a,b), in which the renormalized quantities, mass and charge, correspond precisely to the observed mass and charge of the electron. Although many questions remain to be answered for either the strong, gravitational, or weak interactions, there are indications now that perhaps one or even two of these interactions might be renormalized in the not-too-distant future. The reasons for this cautious optimism are (i) the recent progress in the unification of electromagnetic and weak interactions, (ii) the breakthrough in the renormalizability of certain non-

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† Alexander von Humboldt Fellow, on sabbatical leave at DESY and II. Institut für Theoretische Physik der Universität Hamburg, Hamburg, Germany.

1974), we conclude once again that the continuous dimension method yields the same result as other more conventional regularization procedures.

VIII. CONCLUDING REMARKS

In this review we have applied the technique of dimensional regularization to divergent Feynman integrals in the context of Abelian as well as non-Abelian gauge theories. The technique has, in our opinion, three distinct advantages. In the first place, it is simpler and more elegant than other, more conventional regularization methods such as the Pauli-Villars prescription. Secondly, dimensional regularization is powerful enough to handle efficiently and on the same footing both ultraviolet and genuine infrared divergences. Finally, and this is its most significant feature, the technique is eminently well suited for dealing with gauge theories, since it preserves the local gauge symmetry of the underlying Lagrangian. The preservation of this symmetry was demonstrated in Sec. III for the vacuum polarization tensor in quantum electrodynamics, and in Sec. VI by regularizing non-Abelian massless spin-two quantum gravity to lowest order in the gravitational coupling constant. It was specifically shown there that the sum of the graviton and fictitious-particle contributions to the graviton propagator satisfies Slavnov-Taylor identities [Eqs. (6.17) and (6.18)] and that the finite portion of this sum can be extracted in a manner which is consistent with these identities. In summary, dimensional regularization permits a consistent gauge-invariant treatment of divergent Feynman amplitudes to all orders in perturbation theory.

Although the concept of dimensional regularization is easy to grasp once the notion of analytic continuation is clearly understood, the method should not be applied indiscriminately to any model possessing gauge symmetry. Before embarking on an explicit calculation, it is best to ascertain first whether or not the underlying theory—be it Abelian or non-Abelian—(i) is massive, (ii) is massless, or (iii) contains, through Slavnov-Taylor identities or otherwise, factors of γ^5 . Let us briefly examine these three possibilities.

For massive theories the prescription of 't Hooft and Veltman (1972a), Bollini and Giambiagi (1972), and Ashmore (1972, 1973) works remarkably well, as demonstrated in Secs. III.A and V, and ambiguities can easily be avoided. For massless theories the prescription given in Sec. II.B.1 requires modification (Leibbrandt and Capper, 1974a) due to the appearance of infrared divergences connected specifically with massless particles (see Sec. IV.B). The modification involves basically a redefinition of the original 2ω -dimensional Gaussian integral which permits a consistent treatment of massless tadpoles as well as $\delta^4(0)$ terms (Sec. IV.C) and preserves, moreover, the crucial Slavnov-Taylor identities associated with the graviton self-energy loop (Sec. VI).

Finally, extreme care must be exercised whenever the theory contains anomalies characterized by $\epsilon_{\alpha\beta\mu\nu}$ or γ^5 . Since the latter has only been generalized successfully to even-dimensional spaces (Sec. III.C), the continuous dimension method could create ambiguity problems if these anomalies persist in the final Slavnov-Taylor identities. Although the

anomalies can sometimes be made to cancel by a judicious redefinition of the fundamental fermion fields, it is nevertheless desirable to continue the search for a γ^5 matrix valid in arbitrary dimensions, so that the technique of dimensional regularization may be applied unambiguously to an even greater variety of physical models than has hitherto been the case.

ACKNOWLEDGMENTS

The author is grateful to Professor T. D. Newton and Drs. D. A. Akyeampong, J. Strathdee, and G. 't Hooft for suggesting important improvements to the preliminary draft, to Professor S. B. Treiman for his advice and encouragement during the early stages of the project and to Professor M. Veltman, for several constructive criticisms of the final version of this review. It is also a pleasure to thank Professor Abdus Salam, the International Atomic Energy Agency and UNESCO for hospitality at the International Centre for Theoretical Physics, Trieste, where the manuscript was completed.

APPENDIX. USEFUL INTEGRATION FORMULAS

The following list of 2ω -dimensional integrals is divided into two categories: the integrals in the first category hold for massive particles ($m \neq 0$), whereas those in category B are valid for integrals associated with massless fields ($m = 0$).

A. Massive integrals

Formulas (A1)–(A6) below are taken from Appendix A of 't Hooft and Veltman (1972a). In transferring them we have, for the sake of consistency, replaced the complex variable n by 2ω and divided each integral by $(2\pi)^{2\omega}$. These integrals hold for $m^2 \neq 0$, ω arbitrary, and are particularly useful in connection with the discussion in Secs. III and V.

$$\int \frac{d^{2\omega}q}{(2\pi)^{2\omega}(q^2 + 2k \cdot q + m^2)^\alpha} = \frac{i}{(4\pi)^\omega (m^2 - k^2)^{\alpha-\omega}} \frac{\Gamma(\alpha - \omega)}{\Gamma(\alpha)}, \quad (\text{A1})$$

$$\int \frac{d^{2\omega}q q_\mu}{(2\pi)^{2\omega}(q^2 + 2k \cdot q + m^2)^\alpha} = \frac{i}{(4\pi)^\omega (m^2 - k^2)^{\alpha-\omega}} \frac{\Gamma(\alpha - \omega)}{\Gamma(\alpha)} (-k_\mu), \quad (\text{A2})$$

$$\int \frac{d^{2\omega}q q^2}{(2\pi)^{2\omega}(q^2 + 2k \cdot q + m^2)^\alpha} = \frac{i}{(4\pi)^\omega (m^2 - k^2)^{\alpha-\omega}} \frac{1}{\Gamma(\alpha)} \times \{\Gamma(\alpha - \omega)k^2 + \Gamma(\alpha - 1 - \omega)\omega(m^2 - k^2)\}, \quad (\text{A3})$$

$$\int \frac{d^{2\omega}q q_\mu q_\nu}{(2\pi)^{2\omega}(q^2 + 2k \cdot q + m^2)^\alpha} = \frac{i}{(4\pi)^\omega (m^2 - k^2)^{\alpha-\omega}} \frac{1}{\Gamma(\alpha)} \times \{\Gamma(\alpha - \omega)k_\mu k_\nu + \Gamma(\alpha - 1 - \omega)\frac{1}{2}\delta_{\mu\nu}(m^2 - k^2)\}, \quad (\text{A4})$$

Intro. to Technique of Dim. Reg.
George Liebbrandt, Canada
RMP 47/4 (1975) 28 large pages.

I. INTRODUCTION.

A. UV ω 's in QFT.

- (a) Pauli-Villars reg means put in auxiliary massive fields, a purely formal deal
- (b) analyticity in exponents of propagator away from 1
- (c) Speer's book. You take each propagator + replace exponent by Δ . Then use expo reg to define prop for complex Δ . This is then a "generalized Feynman amplitude". Then apply "evaluator" operator which removes the poles. Speer proved that this method is equivalent to the BPH counterterm method. But, Speer's "analytic continuation" is not Ward-preserving, a weakness. But was connected to BPH?

→ good set of Renorm. refs is given here! A short history also.

note: 't Hooft's proof that gauge th's are renorm. hinges on this dim reg stuff to preserve the Wards. 1971 was 't Hooft's big year.

outline: ✓ I. Intro.

$$\underline{d = (2\omega)}$$

✓ II. Dim. Reg. method.

III. QED examples

IV. Massless fields case (i.e., IR problems!)

V. Yang-Mills case, example

VI. Quantum Gravity

VII. Higher Order examples.

II. The TECHNIQUE of Dim. REG.

Main technique Follows Ashmore 1972. Expo all props using $\int d^d x$. Then do the $d^4 k$'s using formula. Then the d 's → Γ 's which have poles at $\omega=2$, etc. All clearly stated! But watch out for: γ^5 , $\epsilon^{\mu\nu\lambda\sigma}$; IR singularities.

C. Other methods

1. 't H - Velt summary ✓

2. 't H's $n=5$ one loop method + Brouwer method. — no invariance

end II.

III Examples from QED

A. Σ . They use usual $\int dx$, but must recall to use

$$h(\delta^\mu \delta^\nu) = 2^\omega \delta^{\mu\nu} \quad \{\delta_\mu, \delta_\nu\} = 2 \delta_{\mu\nu}$$

$$\delta_\mu \not{p} \delta^\mu = 2(1-\omega) \not{p}$$

$$\delta_\mu \not{p} \not{q} \delta^\mu = 4 p \cdot q + 2(\omega-2) \not{p} \not{q}$$

def: "symmetric integration" ~ means odd fcts $\rightarrow 0$.

The exact result $\Sigma'(p, 2\omega)$ is given in terms of $F(a, b; c; z)$. You then expand in additional way

$$\Sigma = A + (i \not{p} + m) B + (i \not{p} + m)^2 \Sigma_F$$

[You separately regulate IR with λ cutoff, say].

$$\text{Now: } P(2-\omega) = \frac{1}{2-\omega} + \psi(1) + \frac{1}{2} (2-\omega) \left[\frac{\pi^2}{3} + \psi^2(1) - \psi'(1) \right] + \dots$$

A and B contain $P(2-\omega)$ + so have poles; $\Sigma_F = \text{finite}$. Somehow, Σ_F is the interesting piece, but end of example.

B. II. Same method, exact result is $\Pi_{\mu\nu} = (p^2 g_{\mu\nu} - p_\mu p_\nu) \Pi$

$$\text{where } \Pi = \frac{e^2 P(2-\omega) (m^2)^{\omega-2}}{3(2\pi)^\omega} F(2-\omega, 2; \frac{5}{2}; -\frac{p^2}{4m^2})$$

Then you make $\Pi_F^{\text{reg}} = \Pi - \Pi(0)$ and pole is cancelled. Note the you have g.i.
Hw.

C. Anomalies: $\int d^4x \text{Tr} T_{\text{anom}} = \text{dets} \times \text{stuff}$.

To avoid RHS $\neq 0$, choose groups with $d=0$ like $SU(2)$ (not $SU(3)$). Then build your theory with such "safe" groups.

Again, I overlook this subject now.

(13 pages of the 28 now read + absorbed, no need to copy dots.

IV. The Massless Fields Problems

What is $I = \int \frac{d^2 k}{(2\pi)^2} \cdot \frac{1}{k^2}$? = 0 in pole

a) One idea is put in mass first to get $(k^2 + m^2)^{-w-2} \sim \frac{\Gamma(2-w)(m^2)^{w-2}}{(4\pi)^w}$ $w = \frac{d}{2}$

Assume $\text{Re}(w) > 2$. Then $\lim_{m^2 \rightarrow 0} = 0$.

True on $\text{Re}(w) > 2$ axis, so maybe that is the answer $\forall w$?

On the other hand, formula yields

$$\frac{2\pi^w}{(2\pi)^{2w}} \frac{1}{\Gamma(w)} \int_0^\infty k^{2w-1-2} dk$$

diverges $\forall w$

c) or, you can say $\frac{1}{k^2} = \int_0^\infty dx e^{-xk^2}$ to get

$$\int \frac{d^2 k}{(2\pi)^2} \int_0^\infty dx e^{-xk^2} \sim \int_0^\infty dx \cdot \frac{1}{(2\pi)^{2w}} \left(\frac{\pi}{x}\right)^w = \frac{1}{(4\pi)^w} \int_0^\infty dx x^{-w} = \infty \quad \forall w.$$

Same problem

Idea: Write $\int \frac{d^2 k}{(2\pi)^2} \exp[-xk^2 + 2b \cdot k] = \frac{1}{(4\pi)^w} x^{-w} \exp\left[\frac{b^2}{x}\right] \exp[-x f(w)]$

Before $f(w) \equiv 0$ but now we let $f(w)$ be some other fct which vanishes when $2w = \text{integers}$. To, we make a different continuation of the integral.

$f(w)$ = "continuity fct" of poles + Residues = list of properties. The $f(w) > 0$ then converges your x -integral so that

$$I(\text{above}) = \frac{\Gamma(1-w)}{(4\pi)^w} [f(w)]^{w-1}$$

Expand $f(w)$ about $w=2$. But $f(w) \neq 0$ up to some finite order, so no effect. Then result is

$$I = -\frac{4i\pi^3}{(4\pi)^w} \left[(2-w) + (2-w)^2 \chi(2) + \dots \right]$$

So $I = 0$ at $w \rightarrow 2$. I is "regularized" to zero. $\checkmark$

① What about $\int \frac{d^w k}{(2\pi)^w} = J$?

Same "prescription" yields $J=0$ as $w \rightarrow 2$, though this is formally $\delta^w(0)$.

③ Using their $\epsilon(w)$ idea, authors claim to know that

$$\int \frac{d^w k}{(2\pi)^w} \cdot (k^2)^{\beta-1} = \begin{cases} 0 & \text{for } w=2 \text{ and } \beta=0 \text{ (tadpole)} \\ 0 & \text{for any } w \text{ and } \beta=1,2,3,\dots \end{cases}$$

The "t'Hooft-Veltman conjecture" says you can consistently set this $=0$ for any w, β .

OK, so I have read this section. I don't really know what to think about it. They are really saying that any power integral gets regularized to zero in a consistent fashion. The basic cutoff problem with these integrals is that you have trouble making the analytic continuation because there is no domain of convergence. Maybe I will look into their papers to see further comments.

I don't know if this will help out effort much because we always are dealing with $n.k$ stuff.

The list of "IR integrals" at the end of the paper does not seem to have this IR problem, ie, no domain of convergence. Eg, they give:

$$\int \frac{d^w q}{(2\pi)^w} \cdot \frac{1}{q^2 (q-p)^2} = \frac{(p^2)^{w-2}}{(4\pi)^w} \Gamma(2-w) B(w-1, w-1)$$

which is what I got also, + is one of our integrals of interest.

V. Sample Calc using dim reg Calculate $g-2$ in the GG model. $SO(3)$ Calc was done by Primack + Quinn 1972. It's just that they use the DReg somewhere along the line. Other refs given.

VI. QUANTUM GRAVITY A) Pure Quant. Gravity.

↙ physical graviton field

$$\tilde{g}^{\alpha\beta} = \sqrt{-g} g^{\alpha\beta} = \text{the "field"} = \delta^{\alpha\beta} + \kappa \phi^{\alpha\beta}$$

$$\mathcal{L} = \mathcal{L}(\tilde{g}^{\alpha\beta}, \text{derivs of same, and } w!) \quad \leftarrow = \sqrt{32\pi G}$$

group is I guess Poincaré, non-abelian so there will be ghosts

Is there is a FP det $\Delta[\tilde{g}^{\mu\nu}]$ which gets replaced by vector ghosts (fermions) ξ and η . You quantize in the usual way.

Note: comment about 10,000 term verification, tell Tim ✓

So, they discuss the graviton self-energy $\mu\mu' \otimes \nu\nu'$ in lowest order, which involves lots of graphs including those with the fermions vector ghosts ξ and η . Result is that yes, the corresponding Ward Identity (differential) is satisfied for general ω . Again, the poles are identified + then subtracted out.

Compare: $\text{self-energy diagram} = \bar{D}_{\mu\mu'}(q^2) \Pi_{\nu\nu'}(q^2) \bar{D}_{\nu\nu'}(q^2) = \bar{D}_{\mu\nu}^{\text{conn}}(q^2)$

$\text{self-energy diagram} = \Pi_{\mu\nu}(q^2) = \text{connected graphs in SE.}$

In gravity it is $\bar{D}_{\mu\alpha\beta} T_{\alpha\beta\gamma\delta} \bar{D}_{\gamma\delta\mu} = Q_{\mu\nu}$

In QCD, gauge invariance $\Rightarrow q^\mu \Pi_{\mu\nu} = 0$ (a Ward)

In QCD, result is $p_\mu p_\nu Q_{\mu\nu}(p^2) = 0$

In $T_{\alpha\beta\gamma\delta}$ there are 5 scalar invariants, only 2 is QCD

B) Corrections The photon and neutrino correction to the graviton self energy are discussed

C) Note: counterterms needed to cancel one loop divergences are not of the same form as terms in original Einstein-Hilbert Lagrangian ~ theory seems to be non-renormalizable! Even 't Hooft has looked into this matter.

VII. Higher Order Diagrams

A) Multi-loop Integrals - you must use same dk : ω for each loop if you hope to maintain gauge invariance. Discussion proceeds for a general Feynman integral, taken from Abus+Lee review. Parts also taken from 't Hooft & Veltman. Gauge invar. is preserved to all orders, I will consult later for details.

B) Graviton Vertex: Another example of an ω -val.

VIII Conclusion A good summary, see copy

Konetschny
(1977)

Mass Shell Behaviour of Quantum Chromodynamics in the Axial Gauge^(*).

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(ricevuto il 22 Dicembre 1977).

Summary. — It is shown that, in the noncovariant axial gauge, quark and gluon mass shell renormalization constants (if suitably defined) obey the classic Ward identity of quantum electrodynamics $Z_1 = Z_2$. Similar relations hold for the pure gluon sector. The validity of such identities in covariant gauges has been proven recently in the literature. The derivation in the axial gauge is considerably simpler than in covariant gauges.

1. — Introduction.

The investigation of non-Abelian Yang-Mills theories is by far not a closed subject. Whereas the ultraviolet (UV) divergences have been shown to be tractable ⁽¹⁾ (leading to a renormalizable theory), the same cannot yet be said of the infra-red (IR) singularities. There are, however, good indications ⁽²⁾

^(*) To speed up publication, the author of this paper has agreed to not receive the proofs for correction.

⁽¹⁾ G. 't HOOFT: *Nucl. Phys.*, **33 B**, 173 (1971); **35 B**, 167 (1971); for a review, cf. G. COSTA and M. TONIN: *Riv. Nuovo Cimento*, **5**, 29 (1975).

⁽²⁾ YORK PENG YAO: *Phys. Rev. Lett.*, **36**, 653 (1976); TH. APPELQUIST, J. CARAZZONE, H. KLUBERG-STERN and M. ROTH: *Phys. Rev. Lett.*, **36**, 768 (1976); L. TYBURSKI: *Phys. Rev. Lett.*, **37**, 319 (1976); *Nucl. Phys.*, **116 B**, 291 (1976); J. FRENKEL, R. MEULDERMANS, I. MOHAMMAD and J. C. TAYLOR: *Phys. Lett.*, **64 B**, 211 (1976); *Nucl. Phys.*, **121 B**, 53 (1977); A. GARCIA ALVAREZ: *Nucl. Phys.*, **120 B**, 355 (1977); T. KINOSHITA and A. UKAWA: *Phys. Rev. D*, **13**, 1573 (1976); **15**, 1596 (1977); G. STERMAN: *Phys. Rev. D*, **14**, 2123 (1976).

that in perturbation theory (which by itself might not be the appropriate framework to tackle the question of IR behaviour) the divergences from real and virtual soft vector quanta mutually cancel for quantum chromodynamics (QCD, the theory of massive quarks interacting with massless gluons) in much the same way as in quantum electrodynamics ⁽³⁾ (QED)—at least in the non-forward direction ⁽⁴⁾. The infra-red finite transition rates thus obtained still involve the unrenormalized coupling constant, and the next task is to understand how renormalization affects the final result.

In this context it has been shown recently by CVITANOVIĆ ⁽⁵⁾ that in covariant gauges of QCD the *mass shell* quark renormalization constants obey the classic QED Ward identity $Z_1 = Z_2$, and similarly for the gluonic Z 's. The remarkable feature of these Ward identities is that they call for an interplay ⁽⁶⁾ of UV and IR singularities. This requirement is met by the dimensional regularization ⁽⁷⁾ scheme where integrals without any internal mass scale are set equal to zero ⁽⁸⁾.

Covariant gauges of non-Abelian Yang-Mills theories necessitate the introduction of fictitious fields into the Lagrangian to maintain unitarity. These 'Faddeev-Popov ghosts' ⁽⁹⁾ complicate the Slavnov-Taylor ⁽¹⁰⁾ identities which express the gauge invariance of the theory. Ghost-free gauges circumventing these complications can be constructed ^(11,12) and shown to be renormalizable as well as unitary ^(12,13). The price to be paid in this 'axial gauge' ⁽¹⁴⁾ is

⁽³⁾ F. BLOCH and A. NORDSIECK: *Phys. Rev.*, **52**, 54 (1937); J. M. JAUCH and F. ROHLICH: *The Theory of Photons and Electrons* (Reading, Mass., 1955).

⁽⁴⁾ The forward direction is more complicated than in QED: C. T. SACHRAJDA: *Phys. Rev. D*, **14**, 1072 (1976); L. MATSSON and R. MEULDERMANS: *Phys. Lett.*, **70 B**, 309 (1977).

⁽⁵⁾ P. CVITANOVIĆ: *Nucl. Phys.*, **130 B**, 114 (1977).

⁽⁶⁾ G. 't HOOFT and M. VELTMAN: *Nucl. Phys.*, **44 B**, 189 (1972).

⁽⁷⁾ G. LEIBBRANDT: *Rev. Mod. Phys.*, **47**, 849 (1975).

⁽⁸⁾ L. D. FADDEEV and V. N. POPOV: *Phys. Lett.*, **25 B**, 29 (1967).

⁽⁹⁾ A. SLAVNOV: *Theor. Math. Phys.*, **10**, 99 (1972); J. C. TAYLOR: *Nucl. Phys.*, **33 B**, 436 (1971).

⁽¹⁰⁾ W. KUMMER: *Acta Phys. Austr.*, **41**, 315 (1975).

⁽¹¹⁾ J. FRENKEL: *Phys. Rev. D*, **13**, 2325 (1976).

⁽¹²⁾ W. KONETSCHNY and W. KUMMER: *Nucl. Phys.*, **100 B**, 106 (1975); **108 B**, 397 (1976).

⁽¹³⁾ A. SUGAMOTO, H. YAMAMOTO and N. NAKAGAWA: University of Tokyo preprint UT-291 (August 1977).

⁽¹⁴⁾ R. L. AENOWITT and S. I. FICKLER: *Phys. Rev.*, **127**, 1821 (1962); W. KUMMER: *Acta Phys. Austr.*, **14**, 149 (1961); J. SCHWINGER: *Phys. Rev.*, **130**, 402 (1963); Y. P. YAO: *Journ. Math. Phys.*, **5**, 1319 (1964); E. S. FRADKIN and I. V. TYUTIN: *Phys. Rev. D*, **2**, 2841 (1970); A. CHAKRABARTI and C. DARZEN: *Phys. Rev. D*, **9**, 2484 (1974); E. TOMBOULIS: *Phys. Rev. D*, **8**, 2736 (1973); R. DELBOURGO, A. SALAM and J. STRATHDEE: *Nuovo Cimento*, **23 A**, 237 (1974); J. M. CORNWALL: *Phys. Rev. D*, **10**, 500 (1974); R. J. CREWSTER: *Weak and Electromagnetic Interactions at High Energies*, edited by M. LÉVY et al. (New York, N. Y., 1976), p. 345.

Section 2: Author writes a most general kinematic form for the electron self-energy (axial gauge QCD) and then after doing mass renorm goes on to define a Z_2 . Meanwhile, the quark-quark-gluon vertex has a most general kinematic form shown in (2.6), and the first function GAMMA1 is used to define Z_1 . Then apply the Ward Identity (called Lee identity because this is QCD, but it looks like usual QED Ward-Tak identity), and conclude that $Z_1 = Z_2$ almost.

Aside: remember that SIGMA-SIGMA(sh)(the mass renormed electron self-energy) still has divergences which show up as poles at $n=4$ in the dim reg, ie, log div's. In the dime reg, this log divergence appears as const/pole (because in general residue must be polynomial in ext momenta and in log case that poly is const, see tH-V). Thus, when you add a counterterm like $(1-Z_2)\text{prop}$ on whatever to cancel this log divergence pole, that counterterm must have const/pole also. Thus, Z_2 must be independent of p . This part of Z_2 is called Z_{2c} , and discussion here is on top of page 470. So, the fact that you conclude that $Z_{1c}=Z_{2c}$ is enough to prove renormalizability for this sector of the theory, since it is these Z_{1c} which you put in the counterterms.

Section 3: Here author explicitly verifies the little QED-like Ward identity to lowest order in QCD perturbation theory. The graphs are obvious, the actual integrals are done in Appendix B using rules of Appendix A (Feynman rules). Much use is made of "regularizing" power integrals to zero. A power integral is also an integral without a scale, by the way. So this section is OK, except I don't quite see how a detail in the appendix B works, see top of page 478.

Notice, by the way, that the Ward identity is fully verified. Author does not simply show that both sides have the same divergence (egm that both sides have the same $\log \Lambda$)(or both sides have the same $n=4$ pole residue.) He shows completely the equality, and avoids doing the last pair of integrals.

Section 4: So much for the quark-gluon QED-like Ward identity. Next, author considers the Ward which we are interested in, relating the 3-vertex to the gluon propagator. I recall from other papers that the gluon-Ward in covariant gauge says $Z_3/Z_4=Z_3'/Z_4'$ where: Z_3 relates to the gluon propagator, and Z_4 goes with the vertex; and where the primed Z's refer to the same objects for the ghosts. As author notes, you might expect that in the axial gauge where ghosts decouple ($Z_3'=Z_4' = 1$) you might expect that $Z_3=Z_4$. He now goes on to show that this is in fact true.

How? He writes down a general form for the vertex dotted between two special

polarization vectors (thus, he avoids writing down a most-general tensor form for the vertex). Then the Ward becomes (4.2). He also writes down a most general form for the inverse propagator Π as in (4.4). In this propagator Z_3 is identified, and in the vertex Z_4 is also identified. The ward then says that $Z_3=Z_4$, not just for the constant parts.

What about verifying this Ward? For terms only involving quark-loops, it is easily verified. But they are unable to do it for the pure glue loops because "there are too many terms". We have done this verification only for the divergent part, ie, we showed Ward OK to log LAMBDA terms only, or, $n=4$ pole residue only. We never dealt with all those lower terms. So here is perhaps an open problem that we can utilize reduce to do.

Comments on Section 2.

1. In the axial gauge, the presence of the vector n^u causes the general forms for the electron self energy and the electron-electron-photon vertex to be more complicated. (By electron I mean quark, by photon I mean gluon, this is QCD). Somehow it is still possible to define a Z_1 and Z_2 in analogy with QED, and you suspect that you should get a Ward result $Z_1=Z_2$.
2. The author Konetchny goes ahead to define Z_1 in an obvious way and Z_2 in a way that I believe but which I have not really proved is right. With his Z 's, the author finds that you do not get simply $Z_1=Z_2$ but that there is an extra gauge term as shown in (2.9). It is then argued that a true Z used as a counterterm in renorm theory cannot be a function of p because that means non-local. So to get the Z 's you would actually use to renormalize, you are supposed to take the constant part of Z_1 and Z_2 , ie, p and n independent. (first implies second). But it is obvious that these constants are equal, so at least we have that much.
3. Interesting point is made here. I am now used to p -dependent Z 's certainly through the ratio γ ; but renormalization Z 's really have to be p independent! Having that extra vector n^u really complicates the pie, but the reward is simple QED-like Ward identities.
4. Mass renormalization is easy to understand and consistent with my own notes. You put in the physical mass m , then you add a counterterm δm so that the mass does not get shifted when propagator dresses. I note with some interest that, because of the $n.p$ thing, this mass counterterm is actually momentum dependent! You are then allowed to subtract momentum dependent things from your self-energy in axial gauge. I await with interest to see how the gluon self-energy case works.

Section 3: By working out a few feynman graphs, the ward identity (2.9) is verified for the quark self-energy. The constant parts are thus also equal. This section involves appendices A,B, and C, in particular, integrals are all done in the dimreg method which I should now study a little. I think the equality of the Z 's is needed for the proof of renormalizability and in the axial gauge this has been a question.

[Konetschny (Wien)]

(rec'd 22 dec 1977)

Mass Shell Behavior of QCD, Axial gauge

[N.C. 44A, 465 (1978)]

1. Introduction. This paper considers gluons and quarks in axial gauge QCD. For a review of what has been done already, see Refs page. In particular, C. (5) showed that the classic Ward $Z_1 = Z_2$ obtains, but this was done in covariant gauge. This paper will duplicate that result in the axial gauge, & will write most general forms for propagators, etc. See below for summary.

2. To show that $Z_1 = Z_2$

① $\Sigma(p, \eta) = \alpha \cdot 1 + \beta \not{p} + \gamma \not{\eta} + \epsilon [\not{p}, \not{\eta}]$ general form in axial gauge (1t) ✓

Let $\delta m = \bar{u}(p) \Sigma u(p)$ be mass counterterm. ② Pause & look at self notes on electron prop in QED. There, we find: (changing sign of δm)

$$D(n) = \frac{1}{\not{p} - m - \Sigma'(1, 2, \dots, n) + \delta m}$$

where $\Sigma'(1, 2, \dots, n)$ is self-energy to order g^n . Near $\not{p} = m$ (mass shell) we get

$$\Sigma' = A + B(\not{p} - m)$$

The mass renorm condition is then $A = +\delta m$. But, $A = \bar{u} \Sigma u$ & since $\not{p}u = mu$, and we use $\bar{u}u = 1$ norm. So we get

$$\delta m = +\bar{u} \Sigma u \quad \text{for my own QED deal.}$$

③ Now let's look here in axial QCD. The definition of mass is that $\bar{u} Du$ has a pole at $\not{p}^2 = m^2$, which I think you can show means $\bar{u} \Sigma' u = 0$ at $\not{p}^2 = m^2$. Thus you want $\bar{u} [\delta m - \Sigma'] u = 0 \Rightarrow \delta m = \bar{u} \Sigma u$ even when Σ contains η .

$$\text{So } \delta m = \bar{u} \Sigma u = \alpha + \beta m + \gamma \frac{\not{p} \not{p}}{m} \quad \checkmark$$

$$\Rightarrow [\Sigma' - \delta m] = \Sigma' = \beta(\not{p} - m) + \gamma(\not{\eta} - \frac{\not{p} \not{p}}{m}) + \epsilon [\not{p}, \not{\eta}]$$

and $\bar{u} \Sigma' u = 0$. Thus, quark mass at pole is m .

By fiddling with δ^3 , you can rewrite general Σ' as

$$\Sigma' = \beta(\not{p} - m) + d(\not{p} - m)\not{x} + e\not{x}(\not{p} - m).$$

Since $\Sigma' = E'$, d, e are dimless + thus VT finite (ie, cannot have $d \sim \Lambda^2$ I guess).

④ Now what is Z_2 ? In my own notes I get $Z_2 =$ residue on shell, so maybe ... $\rightarrow$ go to (A)

$\rightarrow$ return from (B)

I have gone off + derived some of their equations. The point is that, in the axial gauge, you tend to get

$$\Sigma' \approx (\not{p} - m) \cdot (a + b\not{x}) \quad \text{as } \not{p} \rightarrow m$$

and this is not the same as $\Sigma' \propto \text{const} \cdot (\not{p} - m)$. This detail makes an improvement in some ref. (11) argument that I don't follow.

⑤ Now write form for the vertex:

$$\Gamma_{\mu}^{abc} = C^{abc} g_0 \cdot [\Gamma_1 \gamma_\mu + \Gamma_2 \eta_\mu + \Gamma_3 \not{q}_\mu \not{n}^\nu]$$

But $\Gamma_2 = \frac{n \cdot p}{n^2} \tilde{\Gamma}_2$
 $\Gamma_3 = \frac{n \cdot p}{n^2} \tilde{\Gamma}_3$
 $\Rightarrow \tilde{\Gamma}_2, \tilde{\Gamma}_3 = \text{dimless, therefore VT finite}$

Write $\tilde{\Gamma}_1 \equiv \tilde{Z}_1$. Then use axial gauge ward to get

$$\tilde{Z}_2 = \tilde{Z}_1 + \underbrace{\Gamma_2 \frac{p \cdot n}{n^2}}_{\text{extra piece.}}$$

3. Check $\tilde{Z}_1 = \tilde{Z}_2$ to order g^2 . To our order,

$$\tilde{Z}_2 \approx 1 - ig^2 W_3 B$$

$$\tilde{Z}_1 \approx 1 - L$$

$$L = \Gamma_1 - 1$$

$\nwarrow$ coeff of γ_μ

Compare (B.8) with (B.12)¹³ γ_μ part to get $\tilde{Z}_1 = \tilde{Z}_2 + \text{piece}$

Big deal. See App B + C.

derive (2.1),

(A)

[2.1] $\Sigma(p, n) = \alpha + \beta p + \gamma n + \epsilon [p, n]$ where $n^2 = 1$.

Why is this most general form? What discrete tensors can you have?

$1, \gamma^5, \gamma^\mu$ $p, n, \{p, n\} = 2p \cdot n \frac{1}{2}$ include in α .
 $\uparrow$ points, viz. index

$\sigma^{\mu\nu} p_\mu n_\nu = [p, n]$ So check it.

[2.2] $\bar{u} \Sigma u = \bar{u} u \alpha + \beta \bar{u} p u + \gamma \bar{u} n u + \epsilon \bar{u} [p, n] u$
 $= 1 \cdot \alpha + \beta \cdot m + \gamma \frac{n \cdot p}{m} + 0$

$\Rightarrow$

$\Sigma - \bar{u} \Sigma u = \Sigma - \delta m = \Sigma' = \beta(p-m) + \gamma(n - \frac{n \cdot p}{m}) + \epsilon [p, n]$

[2.2a]

$\{p-m, n\}_+ = -2m n + 2 \frac{p \cdot n}{m} m = 2m \left[-n + \frac{p \cdot n}{n} \right]$

So $\left(n - \frac{p \cdot n}{m} \right) = -\frac{1}{2m} [p-m, n]_+$

[2.2b] $[p, n] = [p-m, n]$ obvious.

$\Rightarrow d+e = -\gamma/m$ ✓
 $d-e = 2\epsilon$ ✓

[2.3] Thus,

$\Sigma' = \beta(p-m) + \gamma \left(-\frac{1}{2m} \right) \{p-m, n\}_+ + \epsilon [p-m, n]$
 $= \beta(p-m) + \underbrace{\left[-\gamma/2m + \epsilon \right]}_d (p-m) n + \underbrace{\left[-\gamma/2m - \epsilon \right]}_e n (p-m)$

[2.3a] Define Σ_2 by $\bar{D} = \Sigma_2(p-m) + \bar{D}(p=m)$

I.e., I would say you should let Σ_2 be the full coeff of $(p-m)$ in $\bar{D}$. Actually, they define it differently. I would say,

$$\bar{D} = \bar{L}_2^{-1}(\not{p}-m) + \bar{D}(\not{p}=m)$$

$$\Rightarrow \not{p}-m + \Sigma' = \bar{L}_2^{-1}(\not{p}-m) + \bar{D}(\not{p}=m)$$

$$\Rightarrow (\bar{L}_2^{-1}-1)(\not{p}-m) = \Sigma'(\not{p}) + \text{const in } \not{p} \uparrow$$

$$\Rightarrow (\bar{L}_2^{-1}-1)\delta^\mu = \frac{\partial \Sigma'(\not{p})}{\partial p^\mu} \Rightarrow (\bar{L}_2^{-1}-1) \frac{p^\mu}{m} = \bar{u} \frac{\partial \Sigma'}{\partial p^\mu} u$$

$$\Rightarrow \boxed{\bar{L}_2^{-1}-1 = \frac{p^\mu}{m} \bar{u}(\not{p}) \frac{\partial \Sigma'}{\partial p^\mu} u(\not{p}) = \text{their } \bar{L}_2^{-1}}$$

$$\text{Now, } \bar{D} - \bar{L}_2^{-1}(\not{p}-m) = \bar{D}(\not{p}=m) = \Sigma'(\not{p}=m) \quad \cancel{\Sigma'(\not{p})}$$

$$\text{So } \not{p}-m + \Sigma' - \bar{L}_2^{-1}(\not{p}-m) = \Sigma'(\not{p}=m)$$

$$\Rightarrow -(\not{p}-m)(\bar{L}_2^{-1}-1) + \Sigma' = \Sigma'(\not{p}=m)$$

$$\Rightarrow \Sigma' - (\not{p}-m) \bar{L}_2^{-1} = \Sigma'(\not{p}=m) = \Sigma''$$

They are consistent.

(2.4) Firstly, take it as shown + see what it gives: I have derived these facts:

$$(\not{p}-m) \not{x} (\not{p}-m) = -(\not{p}^2 - m^2) \not{x} + 2\not{p} \cdot \not{x} (\not{p}-m)$$

$$(\not{p}-m) \not{p} (\not{p}-m) = 2m^2 (\not{p}-m) \checkmark$$

$$(\not{p}-m) [\not{p}, \not{x}] (\not{p}-m) = -(\not{p}^2 - m^2) [\not{p}, \not{x}]$$

Insert these into (2.4) to get:

$$\Sigma'' = -m(d+e) \not{x} \pm \frac{1}{2}(d-e) [\not{p}, \not{x}]$$

Now let's see what my way gives:

$$\Sigma'' = \Sigma' - \text{the } \beta \text{ term} = \delta \left(\not{x} - \frac{\not{p} \cdot \not{x}}{m} \right) + e [\not{p}, \not{x}]$$

2.3a Define $\tilde{L}_2^{-1}$ as they do. In QED this would be:

$$D = \frac{1}{\not{p} - m + \Sigma'(\not{p})} = \frac{1}{(\not{p} - m) + B(\not{p} - m)} = \frac{(1+B)^{-1}}{(\not{p} - m)}$$

$$\Rightarrow \tilde{L}_2 = (1+B)^{-1} \Rightarrow \tilde{L}_2^{-1} = (1+B)$$

$$\Rightarrow (\tilde{L}_2^{-1} - 1) = B = ?$$

$$\Sigma' = A + B(\not{p} - m) - \delta m = B(\not{p} - m)$$

$$\frac{\partial \Sigma'}{\partial \not{p}} = B \cdot \gamma^\mu \quad \text{as } \not{p} \rightarrow m.$$

$$\Rightarrow \not{p}^\mu \frac{\partial \Sigma'}{\partial \not{p}} = B \not{p} \Rightarrow B = \frac{1}{m} \bar{u} \not{p}^\mu \frac{\partial \Sigma'}{\partial \not{p}} u$$

Now in QED I would say:

$$\tilde{L}_2^{-1} - 1 = \frac{1}{m} \bar{u}(\not{p}) \not{p}^\mu \frac{\partial \Sigma'}{\partial \not{p}} u(\not{p}) \quad \text{where } \Sigma' = B(\not{p} - m).$$

They omit the 1, but use same definition. So

$$(\tilde{L}_2^{-1})^{\text{they}} = (\tilde{L}_2^{-1})^{\text{me}} - 1 = \frac{1}{m} \bar{u}(\not{p}) \not{p}^\mu \frac{\partial \Sigma'}{\partial \not{p}} u(\not{p}) \cdot |^P$$

Now:

$$\begin{aligned} \not{p}^\mu \frac{\partial \Sigma'}{\partial \not{p}} &= \not{p}^\mu \frac{\partial}{\partial \not{p}} \left[\beta(\not{p} - m) + d(\not{p} - m) \not{\chi} + e \not{\chi}(\not{p} - m) \right] \\ &= \beta(\cancel{\not{p}} - m) + d(\cancel{\not{p}} - m) \not{\chi} + e \not{\chi}(\cancel{\not{p}} - m) \quad (\text{drop } \alpha(\not{p} - m)) \\ &= \beta \not{p} + d \not{p} \not{\chi} + e \not{\chi} \not{p} \checkmark \end{aligned}$$

$$\begin{aligned} \Rightarrow \bar{u}(\not{p}) \not{p}^\mu \frac{\partial \Sigma'}{\partial \not{p}} u(\not{p}) &\stackrel{\text{exact}}{=} \beta m + d \not{\chi} \frac{\not{p}^\mu}{\not{p}} + e m \frac{\not{p}^\mu}{\not{p}} \\ &= \beta m + (d+e) m \cdot \not{p} \checkmark \end{aligned}$$

Thus, $\boxed{[\tilde{L}_2^{-1}]^{\text{they}} = \beta + (d+e) \frac{\not{p} \cdot \not{p}}{m}} \checkmark$

Now define: $\Sigma'' \equiv \Sigma' - \bar{\ell}_2^{-1}(\phi-m)$. I.e., pull out all $(\phi-m)$.

$$\Sigma'' = \cancel{\beta(\phi-m)} + d(\phi-m)\chi + e\chi(\phi-m) - \left[\beta + (d+e)\frac{n\cdot p}{m} \right] (\phi-m)$$

$$= d(\phi-m)\chi + e\chi(\phi-m) + (-d-e)\frac{n\cdot p}{m}(\phi-m)$$

= ?

Wait

$$\begin{aligned} \Sigma'(\phi, n) &= \beta(\phi-m) + \gamma\left(\chi - \frac{p\cdot n}{m}\right) + \epsilon[\phi, \chi] \\ &= \beta(\phi-m) - m(d+e)\left(\chi - \frac{p\cdot n}{m}\right) + \frac{1}{2}(d-e)[\phi, \chi] \\ &= \beta(\phi-m) + d(\phi-m)\chi + e\chi(\phi-m), \end{aligned}$$

$$\hookrightarrow \Sigma'' = \underbrace{-(d+e)\frac{n\cdot p}{m}(\phi-m)}_{\text{from m}} - m(d+e)\chi + m(d+e)\frac{p\cdot n}{m} + \frac{1}{2}(d-e)[\phi, \chi]$$

Pause to check [2.46]

$$\begin{aligned} \frac{p^\mu}{m} \bar{u}(p) \frac{\partial \Sigma''}{\partial p^\mu} u(p) &= \bar{\ell}_2^{-1} - \frac{p^\mu}{m} \bar{u}(p) \bar{\ell}_2^{+} \frac{\partial}{\partial p^\mu} u(p) = 0 \checkmark \\ &= 0 \text{ acc. to } \Sigma'' \equiv \Sigma' - (\phi-m) \bar{\ell}_2^{-1}. \end{aligned}$$

Now back to the above.

$$\Sigma'' = \underbrace{-m(d+e)\chi + \frac{1}{2}(d-e)[\phi, \chi]}_{\text{former result}} + \underbrace{(d+e)\left[2p\cdot n - \frac{p\cdot n}{m}\phi\right]}_{\text{extra piece}}$$

Something wrong!

[Thus, ~~this~~ (my expression for z_2) + (their def of Σ'') yields a result for Σ'' which conflicts with (2.4).

(2.5) Says: $(p-m)z_2^{-1} + \Sigma'' = \Sigma'$. This is OK, assuming (2.4).

(B)



ext



$$= -g^3 W_1(t^a)_{ij} \left[\int \frac{d^4 k}{(2\pi)^4} \frac{1}{k^2} \gamma^\nu P \gamma_\mu P \gamma_\nu \leftarrow \text{1st line (B.10) (Feynman part). } P \not{k} = -1 \right. \\ \left. + \int \frac{d^4 k}{(2\pi)^4} \frac{1}{k^2} \left(\cancel{k} \cancel{P} \cancel{\gamma} \cancel{P} \cancel{k} \right)^{-1} \frac{1}{(\hat{n} \cdot k)^2} \right] = 0 \text{ by angular} \\ \left. - \int \frac{d^4 k}{(2\pi)^4} \frac{1}{k^2} \left(\cancel{k} \cancel{P} \gamma_\mu P \cancel{k} \right)^{-1} \frac{1}{(\hat{n} \cdot k)} \right. \\ \left. + \frac{\cancel{k} P \gamma_\mu \cancel{P} \cancel{k}}{(\hat{n} \cdot k)} \right] \text{ last term in (B.10).}$$

Appendix B revisited [I now have read 'H-V and Leib']

(B.1) says $\Sigma_1^{(2)} = I^{(F)} + I^{(T)}$
 using $g_{\mu\nu} \in D$ only $\leftarrow$ the rest.

Do with d^4k to get

$$\begin{cases} \bar{u} I^{(F)} u = \frac{i m^{1-\epsilon}}{(2\pi)^n} \Gamma\left(\frac{\epsilon}{2}\right) \frac{3-\epsilon}{1-\epsilon} & \epsilon = 4-n \\ \bar{u} I^{(T)} u = 0 \end{cases} \quad \leftarrow \text{after } \Sigma \text{ pulled out.}$$

Since we want to be verifying (2.9), we will need, (as on p. 468),

$$\frac{p^\mu}{m} \bar{u}(p) \frac{\partial \Sigma^1}{\partial p^\mu} u(p) = B^{(F)} + B^{(T)}$$

$\uparrow$ given in (B.7).

Look at (B.4), $(\delta \cdot \tilde{n}) \left[\frac{1}{k^2 k \cdot n} \right] = \delta \cdot \tilde{n} \left[\frac{1}{k^2 k \cdot n} \right] + \frac{1}{2} \frac{n^2}{k^2} \frac{\partial}{\partial n^\mu} \left[\frac{1}{k \cdot n} \right]$
 $= \delta \cdot \tilde{n} \left[\frac{1}{k^2 k \cdot n} \right] + \frac{1}{2} \frac{n^2}{k^2} \frac{(\delta \cdot k)}{(n \cdot k)^2} - \frac{1}{(n \cdot n)} k^\mu$
 consider the last.

$$I_{\text{mistern}}^{(T)} = + \int \frac{d^4k}{(2\pi)^4} \delta^\mu k_\mu \frac{1}{[\dots]} + \frac{1}{2} \frac{n^2}{k^2} \frac{\partial}{\partial n^\mu} \frac{k_\mu}{(n \cdot k)^2}$$

Then $B_{\text{item}}^{(T)} = \left(\frac{p^\mu}{m} \bar{u}(p) \left[\int \frac{d^4k}{(2\pi)^4} \delta^\mu \frac{\partial}{\partial p^\mu} \left[\frac{1}{k \cdot n} \right] \delta^\nu \right] u(p) \right) \frac{k_\mu k_\nu}{k^2 (n \cdot k)^2}$
 $\leftarrow = -1$
 $= \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 (n \cdot k)^2} \rightarrow 0$ for unknown reason.

The final result is

$$\Sigma_2^{-1(s)} = B^{(F)} + B^{(T)}$$

$\rightarrow$ as in (B.8)
 $\rightarrow$ as in (B.7)

(over)

Now,

$$(B.10) = W_1 \cdot \text{fign}(\text{Fig 1}) + W_2 \cdot \text{fign}(\text{Fig 2}) + (\text{background parts})$$

Now look at (2.8). We just calc'd the RHS above, LHS is now:

$$\bar{u} \gamma^\mu u + \bar{u} \gamma^\mu u = (B.11) + (B.12)$$

So the Ward says, roughly

$$(B.7) + (B.8) \stackrel{!}{=} (B.11) + (B.12)$$

Refs

- (1) UV div. of QCD tractable
- (2) for QCD, pert. theory yields the usual real+virtual cancellation
- (3) Block Nordsieck QED ref.
- (4) Sourced direction cancellation detail.
- (5) Cvitanović paper. Shows $Z_1 = Z_2$ in covariant gauge. Shows UV/IR sing. interplay.
- (6) t'Hooft Veltman dim. reg.
- (7) about $S d^4 k / K^4 = 0$ idea.
- (8) Fadeev-Popov (ghosts)
- (9) Slavnov-Taylor Identities (with ghosts); equality of Z^5 ;
- (10) axial QCD; $q \cdot \Pi = 0$; form of $\Pi_{\mu\nu}$.
(Kummer)
- (11) axial QCD
- (12) QCD is renorm. in axial; precise value of n -k pole; use of $\epsilon \cdot p = \epsilon \cdot n = 0$; $q \cdot \Pi = 0$;
- (13) QCD is renorm. in axial.
- (14) axial gauge refs.
- (15) Lee identities (Baner)
- (16) Brell on metric + Z 's
- (17) author + Kummer: quark self energy, axial; assume $Z' \propto (\not{p} - m) \cdot \text{const}$;
- (18) Pagels: general form for Σ .
- (19) idea that $Z \propto (\not{p} - m)(a + b \not{x})$; $m p = 0$, 1 usage;

(20)

(21)

(22)

(23) double poles in feyn.int's.

(24) GR integrals

Frankel & Meuldermans
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INFRARED BEHAVIOUR OF SELF-ENERGY FUNCTIONS IN THE AXIAL GAUGE

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We discuss, in the axial gauge, the infrared singularities of the self-energy functions which determine the exponential of leading infrared divergencies in massless Yang-Mills theories.

Recently, Cornwall and Tiktopoulos [1] have extensively discussed, in the Feynman gauge, the exponentiation of leading infrared divergencies in massless Yang-Mills theories. Furthermore, as shown in ref. [2], in non-covariant gauges these divergencies occur only in the external line self-energy. This fact, together with the gauge invariance of the S -matrix, lead to a simple proof of the exponentiation. In order to ensure gauge invariance, use has been made of the dimensional regularization scheme [3] for both ultra-violet and infrared divergences [4].

In this note, we would like to present in the axial gauge, a more detailed discussion of the infrared divergences which occur in the self-energy function. In order to isolate these divergences in a systematic way we proceed as follows. First, the dimension n of space-time is chosen to be less than 4 in order to regulate the U.V. divergencies, with the external momentum off mass-shell. In this case no infrared singularities are present. By subtracting off the U.V. divergences we obtain a function which is regular at $n = 4$, which is then analytically continued to $n > 4$ to regulate the infrared divergences. These singularities will manifest themselves as the external momentum is finally put on shell, as poles at $n = 4$, the leading one exhibiting a double pole structure in lowest order of perturbation theory.

In order to illustrate these facts, we consider field theories described by the Lagrangian given below, in which massless vector particles interact with themselves and with massless fermions:

$$L = -\frac{1}{4} F_{\mu\nu}^a F_{\mu\nu}^a - \bar{\psi} \mathcal{D}_\mu \gamma_\mu \psi - L_g, \quad (1)$$

where

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g f^{abc} A_\mu^b A_\nu^c \quad (2a)$$

is the gauge covariant field strength tensor and

$$\mathcal{D}_\mu = \partial_\mu - i g t^a A_\mu^a \quad (2b)$$

is the covariant derivative. The term L_g specifies the axial gauge [5]:

$$L_g = \lim_{\alpha \rightarrow 0} \frac{1}{2\alpha} (\Lambda_\mu A_\mu^a)^2 \quad (3)$$

Λ_μ being an arbitrary four-vector, with $\Lambda^2 \neq 0$. In this case, the vector boson propagator becomes, in momentum space:

$$D_{\mu\nu}(Q, \Lambda) = \frac{1}{Q^2 - i\epsilon} \left[\delta_{\mu\nu} - \frac{Q_\mu \Lambda_\nu + Q_\nu \Lambda_\mu}{Q \cdot \Lambda} + \frac{\Lambda^2 Q_\mu Q_\nu}{(Q \cdot \Lambda)^2} \right] \quad (4)$$

where the denominators $Q \cdot \Lambda$ and $(Q \cdot \Lambda)^2$ are defined by the principal value prescription.

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Equations on the first page just define the theory: QCD in axial gauge. Notice the idea that you really do have to take the parameter to zero to get axial gauge, as Chiu noted.

Second page claims the self energy is $A\epsilon + B\epsilon$, although I don't see why you don't also have a term $C\epsilon$ for example, as I found in the massive case with my Kon fiddlings.

In lowest order, they set about computing this electron self-energy. The result is (10) where they use the dim reg deal. Fine. By the way, they use massless electrons (quarks).

Look at their result (10). The poles at $n=4$ represent the UV divergence, which here is $\log$ (mass renorm is already performed, hence no 1 term in (5)). I'm a little confused about how the mass renorm enters (they don't mention it but Kon did). Maybe in this low order calc there is nothing to subtract.

Anyway, look at (10). You clearly see the UV poles at $n=4$. This thing is binned off mass shell so there are no IR sings. If you let $p^2 = 0$, you extra sings because $n.LT.4$ will cause divide by zero.

So holding off mass-shell, just subtract out the UV poles as in (11). Then you are supposed to go to the mass shell and observe the sudden appearance of some new poles at $n=4$. These must be the IR sings since you already removed the UV sings.

This sounds nice, but not enough detail is given to let me follow the calc. In particular, I don't think that (12) can be proved from what is given in (10). In (10), if you continue to $n.GT.4$ and let $p^2=0$, the first term in Σ dies, and fails to cancel the UV poles subtracted off, that I see. But I don't see where those IR poles are coming from, unless it is in "f" and "finite" which they don't state.

Notice that (12) really gives the behavior of the self-energy near the mass shell, since second term is order p^2 . This is I guess what you need to compute Z_2 related to N_F and you see the presence of double poles at $n=4$ which are associated with the IR sings.

So their point is to show that there are double poles at $n=4$ which are related to the IR sing of the massless quark self energy.

Next, they consider the gauge or gluon self energy in lowest order. The "leaf" graph gets regularized away and you are left with just our "Born Loop" and the "Quark Loop" graphs. They then repeat the above analysis: keep off shell and find the UV poles, then subtract out these poles. Then go on shell and discover that you now have some double-poles at $n=4$ which are connected to the IR sings. Still the details elude me. I don't even know if they are doing Euclidean or Minkowski. The big result is then that Z_3 has double IR poles.

So really this paper should be titled: discussion of the double IR poles in quark and gluon self energy to lowest order perturbation theory. The significance of these double-poles is that Cornwall and Tiktop showed that these poles play some critical role in proving the real/virtual IR exponentiation/cancellation similar to the YFZ work in QED of 1961. Of course that assumes gluons are not contained, if they can be emitted as real particles.... hmmm.