

Gauge Theory Notes I , II

Notes (1979)

Ward Identity
Slavnov

Group

$[dA]$ $\det M$

New Notes
(1979)

1. Basic fact sheets.
2. Origins of the Fp determinant (FPObject = $\det M$ on gauge surface)
3. Comparison of the FP Object vs the FP Determinant.
4. Extraction of group volume (Fadeev-Popov trick)
5. Finite A gauge transformation.
6. SEcond-order term in A-transformation (but I never used this).
7. Euler-Lagrange equations for Yang Mills Theory
8. Rules for Space-Matrix Manipulation (also color-matrix stuff).
9. Discussion of kinds of gague transformations: global, local, ultra-local (constrained).
10. Renorm parts of YMT.
11. Old Color-Matrix notation.
12. About Ghosts.
13. Feynman rules for gauge theories.

A short summary of gauge theory notes as of Apr 28, 1979.

- ① By combining requirements of global and local invariance under finite group transformations you learn how $\vec{A}_\mu$ must transform:

$$\boxed{D'_\mu \equiv (\partial_\mu - ig \vec{G} \cdot \vec{A}'_\mu)} \quad \boxed{\mathcal{L}(\phi, D'_\mu \phi) = \mathcal{L}(\phi, D_\mu \phi)} \quad \text{gauge invariance}$$

$$\Rightarrow \boxed{\vec{G} \cdot \vec{A}'_\mu = [\vec{U} \vec{G} \vec{U}^\dagger] \cdot \vec{A}_\mu + \frac{1}{ig} [\partial_\mu \vec{U}] \vec{U}^\dagger} \leftarrow \text{how } \vec{A}_\mu \text{ xforms}$$

Here, $\vec{G}$ are the 8 generator matrices of dimension to match the matter fields ϕ . I.e., rep σ .

$$\phi' = \underline{U} \phi = \underline{e}^{-i\vec{\Theta} \cdot \vec{G}} \phi \Rightarrow \delta \phi = -i\vec{\Theta} \cdot \vec{G} \phi$$

Perhaps denote matter field vectors by under tiddle:

$$\boxed{\underline{\phi}' = \underline{U} \underline{\phi} \Rightarrow \delta \underline{\phi} = -i\vec{\Theta} \cdot \vec{G} \underline{\phi}} \leftarrow \text{transform of a matter vector.}$$

- ② Assume this Lie Algebra:

$$\boxed{[G_a, G_b] = i C_{abc} G_c} \quad \begin{array}{l} \text{T.A., like } \epsilon. \\ \text{structure const's} \end{array}$$

Define cross product as:

$$\boxed{(\vec{A} \times \vec{B})_a = C_{abc} A^b B^c} \quad \begin{array}{l} \rightarrow \text{meaning of } \times \\ \text{just like } \epsilon. \end{array}$$

Then the infinitesimal of the above $\vec{A}'_\mu \leftrightarrow \vec{A}_\mu$ rule is

$$\boxed{\delta \vec{A}_\mu = \vec{\Theta} \times \vec{A}_\mu - \frac{1}{g} \partial_\mu \vec{\Theta}} \quad \leftarrow \text{xform of adjoint vect + piece}$$

$$\delta A^a_\mu = -\frac{1}{g} D^a_{\mu b} \Theta_b$$

- ③ An alternative way to learn how $\vec{A}_\mu$ transforms is to say:

$$(D'_\mu \phi) = \underline{U} [D_\mu \phi] \quad \text{finite}$$

$$\delta (D_\mu \phi) = -i\vec{\Theta} \cdot \vec{G} [D_\mu \phi] \quad \text{infesimal}$$

I.e., you want $[D_\mu \phi]$ to behave like any other vector ϕ . But $\partial_\mu \phi$ does not work right, so you fix it up.

$$\boxed{\vec{F}_{\mu\nu} \equiv (\partial_\mu \vec{A}_\nu - \partial_\nu \vec{A}_\mu) + g (\vec{A}_\mu \times \vec{A}_\nu) \Rightarrow \delta \vec{F}_{\mu\nu} = \vec{\Theta} \times \vec{F}_{\mu\nu}} \quad \begin{array}{l} \text{field tensor.} \\ \Rightarrow \vec{F}_{\mu\nu} \cdot \vec{F}_{\mu\nu} = \text{scalar!} \end{array}$$

Other facts.

$$1. \quad A_a^\mu [g] = A_a^\mu(x) + C_{aef} \Theta_e(x) A_f^\mu(x) - \frac{1}{g} (\partial^\mu \Theta_a(x)) + \text{order}(\Theta^2)$$

$$2. \quad \left(\frac{\nabla A_a^\mu(x)}{\nabla A_b^\nu(y)} \right) = [\delta_{ab} - C_{abe} \Theta_e(x)] \delta_{\mu\nu} \delta(x-y) + \text{order}(\Theta^2)$$

$$3. \quad D_\mu^{(adj)} [A]_{ab} = \delta_{ab} \partial_\mu - g C_{abe} A_{e,\mu}$$

$$4. \quad \left(\frac{\nabla D_{ab}^\mu [A]}{\nabla A_c^\nu(y)} \right) = -g C_{abc} \delta_{\mu\nu} \delta(x-y)$$

How to define the FP determinant in discrete spacetime in a clear manner!

Etc!!

3.17.79

Idea: when we write $A^g(x) = \text{"object"}$, we must understand that "object" is really a functional of the gauge function g . Why? You know that $A^g(x)$ involves derivatives of the gauge function, evaluated at x . Thus, you need a knowledge of the gauge function $g(y)$ for y in the neighborhood of x , not just at x . That is why $A^g(x)$ is a functional of the function g . In principle, you might want to imagine that $A^g(x)$ depends on the value $g(y)$ takes at all y , not just y near x . Certainly if you try to write down things in discrete spacetime, you will have to do something like this to get the idea of derivatives!

With this in mind, suppose you have a gauge condition $f(A^g(x)) = a$. Suppose we write this in a notation which shows the "functionalness": $f(A^g(x)) = F(g/A(x))$ where anything appearing before the slash indicates a functional dependence.

Now suppose you have two spacetime points x_1 and x_2 . Then you have two gauge conditions. Instead of showing functional dependence, I will show explicit dependence:

$$\begin{aligned} F(\tilde{g}_1, \tilde{g}_2; A(x_1)) &= a & \text{or } F(\tilde{g}/A(x_1)) &= a \\ F(\tilde{g}_1, \tilde{g}_2; A(x_2)) &= a & \text{or } F(\tilde{g}/A(x_2)) &= a \end{aligned}$$

This means that if you know the two numbers $A(x_1)$ and $A(x_2)$, then you could presumably solve these two gauge equations for the solution function $g = (\tilde{g}_1, \tilde{g}_2)$.

Another thing we can do is define two variables:

$$\begin{aligned} F_1 &= F(\tilde{g}_1, \tilde{g}_2; A(x_1)) \\ F_2 &= F(\tilde{g}_1, \tilde{g}_2; A(x_2)) \end{aligned}$$

Of course for the solution function $\tilde{g} = (\tilde{g}_1, \tilde{g}_2)$ these guys are each equal to a . But we can define these functions for any function g . Then we can do a change of variables like so:

$$dF_1 dF_2 = J \cdot dg_1 dg_2$$

where

$$J = \begin{vmatrix} \frac{\partial F_1}{\partial g_1} & \frac{\partial F_1}{\partial g_2} \\ \frac{\partial F_2}{\partial g_1} & \frac{\partial F_2}{\partial g_2} \end{vmatrix} = J[F, A, g]$$

Here I emphasize that the Jacobian is a functional of g (ie, depends on the function g at all its points), of A (depends on A at all x_1), and also of F , depends on all values of its x_1 argument. Of course if you want, you can evaluate the functional $J[F,A,g]$ at the function $g = \tilde{g}$.

What is usually done, I think, is to go back to the gauge condition $f(A^g(x))=0$ and insist that the solution function be $\tilde{g} = (\tilde{g}_1, \tilde{g}_2) = (e, e)$. Probably you can solve this gauge equation to get: $\tilde{A} = A[f, g]$. Ie, the function $\tilde{A}$ is a functional of the functions f and g . Or, you could solve the other way (as we did above) to say $\tilde{g} = g[f, A]$.

What does this say? You are given the function A (ie, you know $A(x_1)$ and $A(x_2)$). You are also given the form of the gauge condition f . Then you can solve for the gauge function $\tilde{g}$ which solves it. In the other direction, $\tilde{A} = A[f, g]$ says that given some functions f and g , the field must be $\tilde{A}$ to satisfy the gauge condition. Presumably there is not a unique $\tilde{A}$, but there is a class of solutions so the gauge condition is really a constraint.

Usually, one fixes the gauge function $g = e$ for each x , and this implies some fields $\tilde{A} = A[f, g=e]$.

Now, since "object" = $A^g(x)$ is a functional of g , we mean that it depends on all $g(y)$. Thus, it depends (in our example) on $g(x_1)$ and $g(x_2)$ which we have called g_1 and g_2 .

So, going to a finite number of spacetime points n , and still only one color, we have:

$$\begin{aligned} \prod_{i=1}^n (dF(g|A(x_i))) &= \left| \frac{\partial F(g|A(x_i))}{\partial g(x_j)} \right| \prod_{i=1}^n (dg(x_i)) \\ &= \left| \frac{\partial f^g(A(x_i))}{\partial g(x_j)} \right| \\ &= \det M(x_i, x_j) \end{aligned}$$

$$\text{where } M(x_i, x_j) = \partial f^g(A(x_i)) / \partial g(x_j)$$

As you let the number of points go to infinity, the object $M(x,y)$ becomes a functional derivative in the sense of my ∇ symbol. Moreover, you usually end up evaluating $\det M = J[f, A, g]$ at $g = \tilde{g}$ where the gauge condition is satisfied, ie, "on the gauge surface". Also, usually you choose $\tilde{g} = e$ so $\Theta = 0$ for parameters.

My key error before was failing to realize that $f(A^g(x))$, and for that matter even $A^g(x)$ itself, is a functional of the function g , not just a function of g at the point x . I used to say things like this: "consider $f(A^g(x)) = a$. For each point x find a g that works, thus $f(A^{g(x)}(x)) = a$." Ie, I was thinking that each point x acted by itself and points were not mixed. Thus, I was unable to come up with a jacobian!

In fact, you have a separate gauge condition at each point x , and each of these (infinite number of) gauge conditions involves $g(y)$ at all points y . Thus, each f is a function of all the values of $g(y)$ and you can define a transformation of variables from the $g(x_i)$ to the $f(x_j)$ using the Jacobian.

I am writing all these words because I never want to be confused by this again. This is the fourth or fifth/ time I have "done" this FP deal, but this is the first time I have really understood how it really works in the spacetime sector. The color part is easy.

In fact, here is the generalization to color:

$$f_\alpha(A^g(x)) = a_\alpha \quad \alpha = 1, 2, \dots, N$$

$$\underbrace{\prod_{i=1}^n \prod_{\alpha=1}^N \left\{ df_\alpha(A^g(x_i)) \right\}}_{\equiv [df(A^g)]} = J[f, A, g] \underbrace{\prod_{i=1}^n \prod_{\alpha=1}^N \left\{ dg_\alpha(x_i) \right\}}_{\equiv [dg]}$$

where $J = \det M_{\alpha\beta}(x_i, x_j)$

where

$$M_{\alpha\beta}(x_i, x_j) = \frac{\partial f_\alpha(A^g(x_i))}{\partial g_\beta(x_j)}$$

So here is the change of variables and its inverse:

$$[df(A^g)] = J[f, A, g] [dg] \quad (1)$$

$$\delta_1 (f(A^g) - f(A^{g'})) = J^{-1}[f, A, g] \delta_1 (g - g') \quad (2)$$

To show the validity of this (2), apply (1) in the variable g to both sides then integrate over $df(A^g)$ on left, which is dg on the right. Or, apply (1) in variable g' and integrate over g' . The primes on each side of (2) must match, but are arbitrary.

In (2), replace the g with g' inside J^{-1} (for free since delta). Then integrate both sides on dg to get:

$$J^{-1}[f, A, g'] = \int [dg] \delta_1 (f(A^g) - f(A^{g'}))$$

This is true for any g' . Now, however, assume that the gauge condition said $f(A^e(x)) = a$. Then we can set $g' = \text{identity}$ on both sides. The left side is no longer a functional of g , but it gains function dependence on a (not functional, a is a constant independent of x). Thus we have:

$$J^{-1}[f, A | a] = \int [dg] \delta_1 (f(A^g) - a)$$

or

$$1 = \frac{J[f, A | a] \cdot \int [dg] \delta_1 (f(A^g) - a)}{\Delta_{f,a}[A]} \quad (3)$$

where

$$J[f, A | a] = \det M_{\alpha\beta}(x, y)$$

$$M_{\alpha\beta}(x, y) = \left(\frac{\nabla f_{\alpha}(A^{[g]}(x))}{\nabla g_{\beta}(y)} \right)_{g=e} \quad (4)$$

If we like we can write M as a product of derivatives as follows:

$$M_{\alpha\beta}(x,y) = \sum_y \int dz \cdot \left[\frac{\nabla f_{\alpha}(A(x))}{\nabla A_y(z)} \right] \cdot \left(\frac{\nabla A_y^{[g]}(z)}{\nabla g_{\beta}(y)} \right)^{g=e}$$

In the first factor we have gone ahead and set $g=e$ so you don't see any A^g objects there. But I can't simplify the second factor without forgetting what I am doing! You cannot set $g=e$ first then do the functional derivative!

The second factor may be computed without knowing the gauge function. But first a few comments: our gauge condition is $f(A) = a$, but that doesn't tell you anything about the derivative appearing above as the first factor. It is not zero! If you ever get confused on this, put back the previous form with $g=e$ tacked on the outside.

First let's consider another question: does M and thus $J=\det M$ really "know" about the value of a ? Looking at the above, you would think not because df/dA knows nothing about a . Also, looking at the integral representation for J , if you hold your fields A fixed and vary a a little, I don't see how J^{-1} can change. Of course if you consider $f(A^g(x))=a$ and you keep A fixed and vary a , (also keeping f fixed), then you are varying the solution $\tilde{g}$ away from its original value e . But I don't think equation (3) cares what the hit value is. For the original a , maybe the integral in (3) "hit" at $g=e$, and for a new "a" it hits at some other g . The result should be independent of a ! This may be helpful in deriving identities!

So, let's return to our idea of writing the second term above in general case. First let me do a few changes: 1) put in Lorentz μ index on A field 2) change to a,b,c color indices; 3) change from g to parameter Θ . Thus the above reads:

$$M_{ab}(x,y) = \sum_{c,\mu} \int dz \cdot \left[\frac{\nabla f_a(A(x))}{\nabla A_{c,\mu}(z)} \right] \cdot \left(\frac{\nabla A_{c,\mu}^{[g]}(z)}{\nabla \Theta_b(y)} \right)^{\Theta=e}$$

For Yang-Mills theory we know what A^B looks like near the identity, so we can compute the desired derivative:

$$A_c^{\mu [y]}(z) = A_c^{\mu}(z) - \frac{1}{g} D_{cd}^{\mu}(A(z)) \Theta_d(z) + \text{order}(\Theta^2)$$

$$\Rightarrow \left(\frac{\nabla A_c^{\mu}(z)}{\nabla \Theta_b(y)} \right)^{\Theta=0} = - \frac{1}{g} D_{cb}^{(\mu)} \delta(z-y)$$

Here I have used a theorem (see later) that says: functional derivatives pass right through regular derivatives. (Remember there is a regular derivative in D).

Now install this result in our general formula:

$$M_{ab}(x,y) = - \frac{1}{g} \sum_{\alpha, \mu} \int dz \cdot \left[\frac{\nabla f_{\alpha}(A(x))}{\nabla A_c^{\mu}(z)} \right] D_{cb}^{\mu}[A(z)] \delta(z-y)$$

Aside: I could if I wanted at this point parts the operator D onto the $[\]$, switch its color labels, and thus get an overall minus. This would free up the delta and get rid of the integral. But not profitable because you end up just reversing the process to get what you later want. (see next page)

Example: the covariant gauge

$$f_a(A(x)) = \frac{1}{\Lambda^2} \partial_{\mu} A_a^{\mu}(x) \Rightarrow \left[\frac{\nabla f_a(A(x))}{\nabla A_c^{\mu}(z)} \right] = \partial_{\mu}^{(x)} \delta(x-z) \delta_{a,c}$$

Here again we use fact that functional derivative reaches right through a regular derivative. If you now insert this into the above, what happens? Convert d_x to d_z on the delta, getting a minus sign. But then parts d_z over to the right side getting a compensating sign. Then the freed delta sets $z=x$. Obvious result is then:

$$M_{ab}(x,y) = - \frac{1}{g \Lambda^2} \partial_{\mu}^{(x)} [D_{ab}^{\mu}[A(x)] \delta(x-y)]$$

for gauge $\frac{1}{\Lambda^2} \partial_{\mu} A_a^{\mu}(x) = a_a$

$$\left[\underline{M} \right] = - \frac{1}{\Lambda^2 g} [\underline{\partial_{\mu}}] [\underline{D^{\mu}}]$$

If M is evaluated on the gauge surface (as we have claimed) you could actually make use of the factor that $d_{\mu} A^{\mu} = a$ inside your expression for M .

Recall how this $\det M$ gets into the generating functional. It gets in there through explicit use of equation (3) above, so $\det M$ as it appears there is evaluated on the gauge surface. Even if you later "replace" the gauge delta with something else, your J is still evaluated on the gauge surface.

Example 2: axial gauge.

$$f_a(A(x)) = \eta_\mu A_a^\mu(x) = a_a.$$

$$\left[\frac{\nabla f_a(A(x))}{\nabla A_c^\mu(z)} \right] = \eta_\mu \delta(x-z) \delta_{a,c}$$

$$\begin{aligned} \Rightarrow M_{ab}(x,y) &= -\frac{1}{g} \eta_\mu D_{ab}^\mu[A(x)] \delta(x-y) \\ &= -\frac{1}{g} \eta_\mu \left[\overset{\delta_{ab}}{\partial^\mu} - ig (G^c)_{ab} A_c^\mu(x) \right] \delta(x-y) \\ &= -\frac{1}{g} \left[\eta_\mu \partial^\mu - ig (G^c)_{ab} a_c \right] \delta(x-y). \end{aligned}$$

Since I have been talking pure Yang-Mills theory, these G matrices must be the adjoint representation generators, related to the C_{Abc} . Later I will have to see what happens when you have matter fields also.

The point here is that in axial gauge M and thus $\det M$ which appears in the FP quantization (and is on gauge surface) is independent of A and acts as an overall constant. Hence, no ghosts!

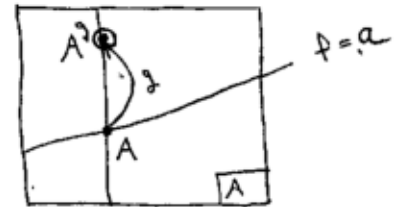
A Review of Definition of FP Object (the invariant thing) and Comparison to FP Determinant

1. First, as summarized in the preceding pages, starting with a function $f(A^g)$ we are able to define a change of variables and a Jacobian for this change. Roughly, the idea was to say $f_i = f(A^g(x_i)) =$ a functional of g , so you compute df_i product against $dg(x_i)$ product and you get a Jacobian:

$$\left[\prod_i df(A^g(x_i)) \right] = J_f[A^g] \cdot \left[\prod_i dg(x_i) \right]$$

$$J_f[A^g] = \det M_{ab}(x, y; f, A^g)$$

FP determinant $\rightarrow M_{ab}(x, y; f, A^g) = \left(\frac{\nabla f_a(A^g(x))}{\nabla \Theta_b(y)} \right)$



Notice that this Jacobian is a functional of A^g which is some point in the A -plane. If we arbitrarily say that A is a point on the gauge surface, then A^g in general is a point not on the gauge surface, but is off somewhere on the orbit passing through A . This determinant $J_f[A^g]$ is not the FP determinant. That will be defined later.

In the preceding notes we also showed that the above variable change implies a sort of integral representation for J :

$$J_f^{-1}[A^g] = \int [dg] \delta(f(A^g) - f(A^g))$$

2. Here is the definition of the FP determinant:

$$\boxed{\Delta_f^{-1}[A] \equiv \int [dg] \delta(f(A^g) - a) = \Delta_f^{-1}[A^g]} \quad \begin{array}{l} \text{"FP object"} \\ \leftarrow \end{array}$$

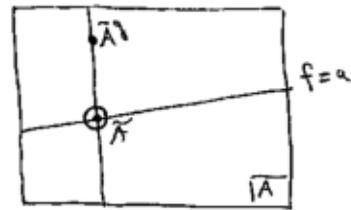
Here the symbol a can be any function of spacetime. WE use it above to specify the chosen gauge surface $f(A^g) = a$. It is also a color vector. Sometimes it is chosen zero, sometimes it is taken to be a constant. The main point is that it is independent of A ! From that fact, you see at once by Haar invariance that the FP Det is a constant over any given orbit, hence the rightmost equality above.

3. Clearly, based on the above discussion, you can see that the FP determinant equals the Jacobian thing when the jacobian thing is evaluated at an A on the gauge surface!

$$\Delta_f[\tilde{A}^3] = \Delta_f[\tilde{A}] = J_f[\tilde{A}]$$

$$f(\tilde{A}) = a$$

Thus, the FP "object" anywhere on the orbit shown is equal to the Jacobian determinant only at the intersection point.



There is no simple relation between the FP object and the Jacobian evaluated at some point not on the gauge surface! (red dot on last page). That is because the Jacobian det has that extra term in its dg' integral representation which the FP thing~~s~~ does not have.

Conclusion: The "FP object" is constant over an orbit. It is equal to the Jacobian determinant only when this Jacobian thing is evaluated at the orbit point which lies on the gauge surface. The FP object has no connection to the Jacobian evaluated at some other point on the orbit.

Doubts about the Fundamentality of the FP Determinant.

I have established that there are two objects and that they are different. The first object is the Jacobian determinant, the second is the FP object. To review, the FP object is constant on orbits, the Jac det thing is not. These two objects agree at the intersection of the gauge surface, but not elsewhere. See other papers on this.

Here is my new worry and concern:

1) In the rather correct-looking Lagrange multiplier quantization procedure of Franklin and Tyutin, it suddenly appears that the object which appears inside the path integral is in fact the Jacobian object and not the FP object. In particular, see their equation (2.37). Notice that there is no gauge surface delta, rather there is the usual "gauge fixing term" in the Lagrangian. The significance is this: you do integrate over fields that are not on the gauge surface here, although their contribution is exponentially damped as you leave the gauge surface (that is what the gauge fixing term effectively does). Therefore, during the portions of that integration where you are not on the gauge surface, the Jac det object which appears in there is definitely not equal to the FP object which FT define in (2.43). It was my impression that the FP object appeared and not the Jac det object, but apparently I am wrong.!

2) This makes me go back and examine what I think I know about "exponentiation of the delta". First I looked at my own "version 3" of this. I now think my argument was invalid for the following reason: here is what I did:

$$\begin{aligned} & \int [da] e^{iS[A]} \left[\underbrace{\int da G(a)}_1 \left[\underbrace{\Delta_{f,a}[A] \int dg \delta[f(A^g) - a]}_1 \right] \right] \\ &= \int [da] e^{iS[A]} \Delta_{f,a}[A] \int dg G[f(A^g)] \quad ? \\ &= \left(\int dg \right) \left[\int [da] e^{iS[A]} G[f(A)] \Delta f[A] \right]. \end{aligned}$$

↓ by same steps as on "extraction" page.

The idea was that $G(a)$ be a decaying exponential so the path integral $[da] G(a)$ be normalizable, and then at the end you knock off the dg integration. The result looks good, but I think I have screwed up: the FP object as defined above really depends on "a", which I have gone back and labelled in red: Thus, you can't move that da integral to the right, so the derivation is just plain wrong.

3) Next, I looked back at Abers and Lee and learned that they only do exponentiation in the sense that $q \rightarrow 0$. See their page 87. They never claim any validity except as $q = 0$. Only I claimed validity based on my own false derivation as above.

4) I now understand the correct way to do this stuff. Here is the sequence of events:

- A. Do the extraction of the group integral, see single sheet in these notes. This step makes use only of the invariance of the FP "object" and of the measure dA under regular, unconstrained gauge transformations. You are left then with the FP object and a gauge surface delta function inside your path integral.
- B. The next step is to now replace the FP object by the FP determinant. After this replacement is made, you next show that the product $[dA] \det M[A]$ is invariant under constrained gauge transformations. This fact then establishes that the generating functional is independent of parameter a. Then you can go ahead and effect "exponentiation of the gauge delta". This is just how Lee Zinn-Justin-4 do it, with the result Eq. 11. Your $\det M$ then becomes the ghosts and you are all set.
- C. Recall that Fadkin and Tyutin obtain the identical result by doing a Lagrange multiplier trick. Just another way to do it.

Extraction of Group Volume,

$$\int [dA] e^{iS[A]} = \int [dA] e^{iS[A]} \overbrace{\left\{ \mathcal{J}_{f,a}[A] \int [dg] \delta[f(A^g) - a] \right\}}^{=1}$$

↑ as defined on other sheet ["FP object"]

$$= \int [dg] \underbrace{\int [dA] e^{iS[A]} \mathcal{J}_{f,a}[A] \delta[f(A^g) - a]}_{\text{let } A \rightarrow A^{(g')}} = 1$$

But $[dA^{g'}] = [dA]$ / ~~not~~ unconstrained GT. ✓

$S[A^{g'}] = S[A]$ / invariance of basic Lagrangians ✓

$\mathcal{J}_{f,a}[A^{g'}] = \mathcal{J}_{f,a}[A]$ / ~~not~~ always true ✓

Thus, we get

$$= \underbrace{\int [dg]}_{\text{this is now a multiplic. fact.}} \int [dA] e^{iS[A]} \mathcal{J}_{f,a}[A] \delta[f(A) - a]$$

Conclusion: If basic $S[A]$ action is invariant under gauge x-forms, and if $([dA] \mathcal{J}_{f,a}[A])$ is also invariant, then you get:

$$\left\{ \frac{\int [dA] e^{iS[A]}}{\int [dg]} \right\} = \int [dA] \mathcal{J}_{f,a}[A] \underbrace{\delta[f(A) - a]}_{\text{only } A \ni f(A)=a \text{ are contributing!}} e^{iS[A]}$$

Footnote: The finite transform for A .

○ We know that

$$\vec{G} \cdot \vec{A}'_\mu = (\vec{V} \vec{G} \vec{V}^{-1}) \cdot \vec{A}_\mu + \frac{1}{ig} [\partial_\mu \vec{V}^\sigma] \vec{V}^{-1}$$

where $\vec{V}^\sigma = \exp(-i\vec{\Theta} \cdot \vec{G}^\sigma)$ in the chosen rep σ . So

$$G_b^\sigma G_a^\sigma A'_{\mu,a} = G_b^\sigma \vec{V}^\sigma G_a^\sigma \vec{V}^{-1} A_{\mu,a} + \frac{1}{ig} G_b^\sigma [\partial_\mu \vec{V}^\sigma] \vec{V}^{-1}$$

where we left-multiplied by G_b^σ . Now do trace to get

$$\begin{aligned} F^\sigma \delta_{ba} A'_{\mu,a} &= \text{trace} [G_b^\sigma \vec{V}^\sigma G_a^\sigma \vec{V}^{-1}] A_{\mu,a} + \frac{1}{ig} \text{tr} [G_b^\sigma [\partial_\mu \vec{V}^\sigma] \vec{V}^{-1}] \\ &= F^\sigma A'_{\mu,b} \end{aligned}$$

○ Thus,

$$A'_{\mu,a} = \underbrace{\frac{1}{F^\sigma} \text{trace} [G_a^\sigma \vec{V} G_b^\sigma \vec{V}^{-1}]}_{\text{must be } (e^{-i\vec{\Theta} \cdot \vec{G}})_{ab} = (V^{\text{adj}})_{ab}} A_{\mu,b} + \frac{1}{ig} \frac{1}{F^\sigma} \text{tr} [G_a^\sigma [\partial_\mu \vec{V}^\sigma] \vec{V}^{-1}]$$

As a check on this, let $\vec{\Theta} = 0$. Then $\delta_{ab} = \frac{1}{F^\sigma} \text{tr} [\vec{G}_a^\sigma \vec{G}_b^\sigma] \checkmark$

$$S \quad A_a'^\mu = U_{ab}^{\text{adj}} A_b^\mu + \frac{1}{ig} \frac{1}{F^\sigma} \text{tr} [G_a^\sigma [\partial_\mu \vec{V}^\sigma] (\vec{V}^\sigma)^{-1}]$$

Conclude:

$$\begin{aligned} A_a'^\mu &= U_{ab}^{\text{adj}} A_b^\mu + \frac{1}{ig} (F^\sigma)^{-1} \text{trace} [G_a^\sigma (\partial_\mu \vec{V}^\sigma) (\vec{V}^\sigma)^{-1}] \\ &\approx A_a^\mu + (\vec{\Theta} \times \vec{A}_\mu)_a - \frac{1}{g} \partial_\mu (\vec{\Theta})_a \end{aligned}$$

○ is A' indep of σ ?? otherwise won't remain local gauge for several fields.

Second-Order Transformation of Gauge Field

① To all orders in gauge-function we know that

$$(A')^\mu_a = U^{(adj)}_{ab} A^\mu_b + \frac{1}{ig} \frac{1}{F^\sigma} \text{trace}^{(\sigma)} [G_a(\partial_\mu U)(U^\dagger)]$$

This result arises when you require that

$$[D_\mu(A')\phi] = \underline{\underline{U}} [D_\mu(A)\phi] \quad \text{where} \quad \phi' = U\phi.$$

In doing this, you identify

$$\underline{\underline{[U^{(adj)}]}}_{ab} = \frac{1}{F^\sigma} \text{tr}^{(\sigma)} [G_a U G_b U^\dagger] \quad \text{for any rep } \sigma.$$

To check this last result, go to lowest order to learn that

$$\underline{\underline{[G_c^{(adj)}]}}_{ab} = -i C_{cab}$$

② Now do second order calc. I get

$$1) \quad [U^{(adj)}]_{ab} = \delta_{ab} - i \vec{\Theta} \cdot [\vec{G}^{(adj)}]_{ab} - \frac{1}{2} \Theta_c \Theta_d ([G_c^{(adj)}][G_d^{(adj)}])_{ab}$$

$$2) \quad \frac{1}{F^\sigma} \text{tr}^{(\sigma)} [\vec{G}_a(\partial_\mu U)U^\dagger] = -i(\partial_\mu \vec{\Theta}) + \frac{1}{2} i [(\partial_\mu \vec{\Theta}) \times \vec{\Theta}]$$

Put pieces together to get

$$\begin{aligned} \vec{A}'_\mu = & \vec{A}_\mu - i \Theta_a [\vec{G}_a \vec{A}_\mu] - \frac{1}{2} \Theta_a \Theta_b [\vec{G}_a \vec{G}_b \vec{A}_\mu] \\ & - \frac{1}{g} (\partial_\mu \vec{\Theta}) + \frac{1}{2g} (\partial_\mu \vec{\Theta}) \times \vec{\Theta} \end{aligned}$$

$\frac{1}{2} \{G_a, G_b\}$
= some h.

= new gauge field to second order in Θ_a

Euler-Lagrange Equations of Motion for Yang-Mills field theory.

1. The result appears in the Fradkin/Tyutin paper, here is the derivation.

Euler-Lagrange says: $\partial_\alpha \left[\frac{\partial \mathcal{L}}{\partial (\partial_\alpha A_\beta^a)} \right] = \frac{\partial \mathcal{L}}{\partial (A_\beta^a)}$ where these are partial derivatives, not functional.

But $\mathcal{L} = -\frac{1}{4} F_{\mu\nu}^a F_{\mu\nu}^a$.

Compute: $\frac{\partial \mathcal{L}}{\partial F_{\mu\nu}^a} = -F_{\mu\nu}^a$ /note $\frac{\partial F_{\alpha\beta}^c}{\partial F_{\mu\nu}^a} = \delta^{ac} [\delta_{\alpha\mu} \delta_{\beta\nu} - \text{reverse}]$.

Compute:

$$\frac{\partial F_{\mu\nu}^a}{\partial (\partial_\alpha A_\beta^d)} = \delta^{ad} [\delta_{\alpha\mu} \delta_{\beta\nu} - \delta_{\alpha\nu} \delta_{\beta\mu}]$$

Therefore, $\frac{\partial \mathcal{L}}{\partial (\partial_\alpha A_\beta^d)} = -2 F_{\alpha\beta}^d$ (1) $\Rightarrow \Pi_\beta^d = -2 F_{0\beta}^d$

Next, compute

$$\frac{\partial F_{\mu\nu}^a}{\partial (A_\beta^d)} = g C^{abd} [\delta_{\nu\beta} A_\mu^b - \delta_{\mu\beta} A_\nu^b]$$

Then: $\frac{\partial \mathcal{L}}{\partial (A_\beta^d)} = -2g C^{dab} F_{\mu\beta}^a A_\mu^b$ (2)

So put (1) and (2) together + regroup to get

$$[\partial^\alpha \delta^{ad} - g C^{dab} A_\alpha^b] F_{\alpha\beta}^a = 0$$

$$D_{d\alpha}^a [A] F_{\alpha\beta}^a = 0.$$

So result is

$$D_{ab}^\mu [A] F_{\mu\nu}^b = 0$$

Rules for Space Matrix Manipulation

1. $\partial_\mu^{(x)} \delta(x-y) = [\underline{\partial}_\mu]_{xy}$
2. $A(x) \delta(x-y) = [\underline{A}]_{xy} = [\underline{A}]_{yx}$ where A is just a function, no derivs.
3. $\vec{M}(x) \delta(x-y) = [\underline{M}]_{xy}$ for any operator $\vec{M}$!
4. Let M_1 and M_2 be any two operators. Then:

$$\int dy [\underline{M}_1]_{xy} [\underline{M}_2]_{yz} = [\underline{M}_1 \underline{M}_2]_{xz} = \vec{M}_1(x) [\vec{M}_2(x) \delta(x-z)]$$
5. $\delta(x-y) = [\underline{1}]_{xy}$.
6. Implicit sums on repeated indices:

$$[\underline{M}_1]_{xy} [\underline{M}_2]_{yz} = [\underline{M}_1 \underline{M}_2]_{xz}$$

Might as well make this a matrix equation: $[\underline{M}_1][\underline{M}_2] = [\underline{M}_1 \underline{M}_2]$

7. $A(x) = \int dy [\underline{A}]_{xy} = \int dy [\underline{A}]_{yx} = [\underline{A}]_x = \text{a vector.}$
8. Let $f(x) = \int dy M(x,y) h(y) \Rightarrow [\underline{f}]_x = \int dy [\underline{M}]_{xy} [\underline{h}]_y$
 $\Rightarrow [\underline{f}] = [\underline{M}][\underline{h}]$

If $M(x,y) = \vec{M}(x) \delta(x-y)$, then

$$\begin{aligned} f(x) &= \int dy M(x,y) h(y) = \int dy [\vec{M}(x) \delta(x-y)] h(y) = \int dy \delta(x-y) \vec{M}(y) h(y) \\ &= \vec{M}(x) h(x). \end{aligned}$$

9. Purely function matrices are notation-invariant.

$$[\underline{f}_1][\underline{f}_2] = [\underline{f}_2][\underline{f}_1] = [\underline{f}_1 \circ \underline{f}_2] = [\underline{f}_2 \circ \underline{f}_1] = [\underline{f}_1 \underline{f}_2] = [\underline{f}_2 \underline{f}_1]$$

$$[\underline{f}_i]_{xy} = f_i(x) \delta(x-y)$$

↑
no operator.

Discussion of an Ambigjity in the SpaceMatrix Notation.

1. Generally speaking, the product of two space matrices involves a composition of operators. Here is an example:

$$[\underline{m}_1] = [\underline{\partial}^M + \underline{A}^M]$$

$$[\underline{m}_2] = [\underline{\partial}_\mu + \underline{B}_\mu]$$

$$\begin{aligned} ([\underline{m}_1][\underline{m}_2])_{xy} &= \int dz [\underline{m}_1]_{xz} [\underline{m}_2]_{zy} \\ &= \int dz \{ [\partial_x^M + A^M(x)] \delta(x-z) \} \{ [\partial_\mu^{(z)} + B_\mu^{(z)}] \delta(z-y) \} \\ &= \int dz \delta(x-z) [\partial_z^M + A_z^M] ([\partial_\mu^{(z)} + B_\mu^{(z)}] \delta(z-y)) \\ &= [\partial_x^M + A^M(x)] ([\partial_\mu^x + B_\mu(x)] \delta(x-y)) \\ &= [\partial_x^M + A^M(x)] \circ [\partial_\mu^x + B_\mu(x)] \delta(x-y) \\ &= [\underline{m}_1 \circ \underline{m}_2]_{xy} \end{aligned}$$

Two lines back I have put in a "composition operator". What this means is that you take the operator on the right and act on what is to the right of it (here a delta), and then you act on that result with the operator on the left. This is just the usual thing you mean when you compose operators. Eg, in the above case the explicit result is:

$$\begin{aligned} &= [\partial_x^M + A^M(x)] (\partial_\mu^x \delta(x-y) + B_\mu(x) \delta(x-y)) \\ &= \partial_x^2 \delta(x-y) + A^M(x) \partial_\mu^x \delta(x-y) + [\partial_x^M B_\mu(x)] \delta(x-y) + B_\mu(x) \partial_x^M \delta(x-y) \\ &\quad + A^M(x) B_\mu(x) \delta(x-y). \end{aligned}$$

2. So why bother with a special notation? Consider this simpler example:

$$\left. \begin{aligned} [\underline{m}_1] &= \underline{\partial}^M \\ [\underline{m}_2] &= B_\mu(x) \end{aligned} \right\} \Rightarrow ([\underline{m}_1][\underline{m}_2])_{xy} = (\partial_x^M \circ B_\mu(x)) \delta(x-y)$$

$$= (\partial_x^M B_\mu) \delta(x-y) + B_\mu(x) \partial_x^M \delta(x-y)$$

$$\text{Thus, } \partial_x^M \circ B_\mu \delta(x-y) = \partial_x^M [B_\mu(x) \delta(x-y)]$$

$$\neq (\partial_x^M B_\mu) \delta(x-y)$$

You could always keep the distinction clear by putting in correct parentheses, but the problem is that this is usually not done. For example, in the covariant gauge when you compute M here is what you really get:

$$\begin{aligned} M_{ab}(x,y) &= -\frac{1}{g} \partial_\mu^{(*)} D_{ab}^\mu(x) \delta(x-y) \\ &\neq -\frac{1}{g} (\partial_\mu^* D_{ab}^\mu(x)) \delta(x-y). \end{aligned}$$

3. In spacematrix notation, the above equality is easily stated because recall that it is operator composition that goes with matrix multiplication. Thus:

$$[\underline{M}_{ab}] = -\frac{1}{g} [\underline{\partial}_\mu \circ \underline{D}_{ab}^\mu] = -\frac{1}{g} [\underline{\partial}_\mu] [\underline{D}_{ab}^\mu].$$

4. Therefore, we have now a precise enough notation to tell these objects apart:

$$(a) \quad [\underline{\partial}_\mu \underline{\theta}]_{xy} = (\partial_\mu^* \theta(x)) \delta(x-y)$$

$$[\underline{\partial}_\mu \underline{\theta}] \neq [\underline{\partial}_\mu] [\underline{\theta}].$$

$$(b) \quad [\underline{\partial}_\mu \circ \underline{\theta}]_{xy} = \partial_\mu^* [\theta(x) \delta(x-y)]$$

$$[\underline{\partial}_\mu \circ \underline{\theta}] = [\underline{\partial}_\mu] [\underline{\theta}]$$

In fact, here is a little identity involving these look-alike objects:

$$[\underline{\partial}_\mu \circ \underline{\theta}] = [\underline{\theta} \circ \underline{\partial}_\mu] + [\underline{\partial}_\mu \underline{\theta}]$$

Thus, we can do an order reversal:

$$[\partial_\mu][\theta_a] = [\theta_a][\partial_\mu] + [\partial_\mu \theta_a] \quad (1)$$

Here is an application of (1)

$$[\underline{D}_{bc}^\mu][\theta_a] = [\theta_a][\underline{D}_{bc}^\mu] + \delta_{bc} [\partial_\mu \theta_a]$$

Color Matrix Notation

① I use double-curved underline to indicate a color matrix.

Example : $[\underline{\underline{C}}^a]_{bc} = C^a_{bc} = C_{abc}$

② I combine this with double-straight underline :

Example $[\underline{\underline{D}}^M_{ab}]_{xy} = [\vec{D}^M_{ab}(x) \delta(x-y)]$
 $= [\underline{\underline{D}}^M]_{xy}_{ab}$

③ Color matrices of course commute with space matrices

Example : $[\underline{\underline{C}}^c][\underline{\underline{O}}_c] = [\underline{\underline{O}}_c][\underline{\underline{C}}^c]$

④ I use "tr" for a space-trace, but "Tr" for a total trace.

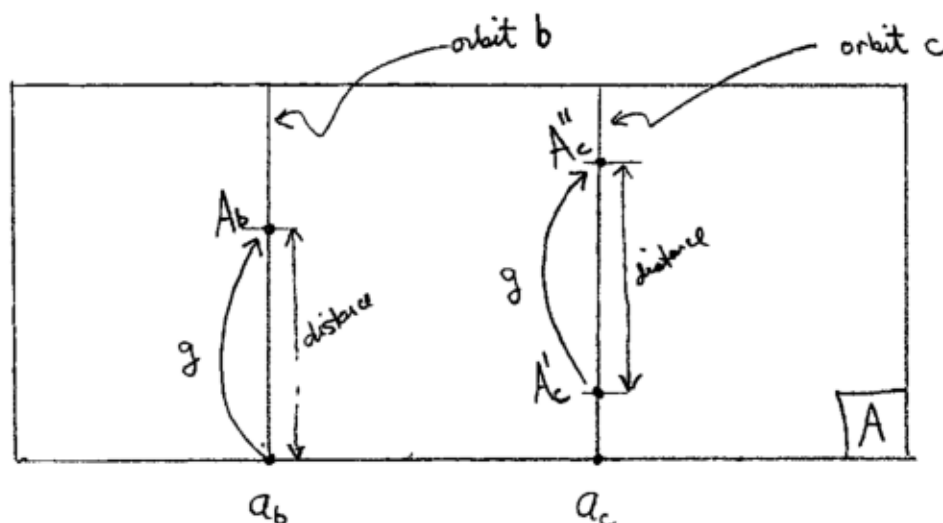
Example : $\text{Tr} \{ [\underline{\underline{D}}^M_{ab}][\underline{\underline{D}}^M_{ba}] \} = \text{tr} \{ [\underline{\underline{D}}^M_{ab}][\underline{\underline{D}}^M_{ba}] \}$

⑤ Full transpose idea :

$$M^T_{ab}(x,y) = M_{ba}(y,x).$$

A Discourse on Global, Local, and Ultra-local gauge transformations, and their relation to the notion of a Gauge Surface

1. Orbits: let Rectangle be the space of all fields. Each field A is represented by a single point in the rectangle. In this discourse we are never going to talk about the value of a field function $A(x)$ at some spacetime point,; the point in the rectangle represents the function as a whole, ie, objects defined with Rectangle as a domain will be functionals of A . We should still regard each point A as a vector in some color space, however. By saying this, we imply there is a "gauge group"; ie, there is a way to take any field vector A in the Rectangle and turn it into some new field A' via some kind of gauge transformation which is related to a gauge group which we assume has been chosen. Once there is a gauge group and the possibility of gauge transformations; you can see that the Rectangle can be visualized as the union of a bunch of orbits, where an orbit is the locus of all fields you can get to from a given field. Below is an illustration of the Rectangle and a few orbits. The space is deformed until the orbits are vertical lines (or wafers, if you want more dimensions). Furthermore, we make a graphical simplification: if you start at some field A_c on the orbit labelled by "c" and you go to a new field vector $A'_c = U(g)A_c$ via some transformation labeled by group element g (or its parameters), then the distance you move along the orbit is the same for all orbits, for given g . Obviously this is not really true, you can't say this no matter how you distort the Rectangles "surface metric" or whatever. But I will do it anyway because it helps illustrate some points below. As an aside, it should be mentioned that, like A , the symbol g represents a vector of parameters each of which is a function of spacetime, but we ignore the spacetime aspect here: everything below is really a functional of arguments, not a function. No need ever to expose the spacetime arguments. One final detail: I may want to have a sort of "reference" field for each orbit. Lets take these to be the points along the bottom of the rectangle. I will call the reference point on orbit c by the name a_c . Any field on orbit c can be gotten like so: $A_c = U(g) a_c$.



2. Global Gauge Transformation. Under any kind of gauge transformation (global, local, or whatever), each field point in a given orbit gets mapped into some other field point in the same orbit. That's the definition of orbit: no gauge transformation no matter how fancy can take a field point over into some different orbit.

Now, by "global" gauge transformation we mean simply that the gauge transformation in each orbit is labelled by a global, overall group label g . I.e., the group transformation on the fields $U(g)$ is going to be the same in each orbit. The preceding drawing shows the effect of such a global gauge transformation on two points called a_b and A_c' . The effect is that each point is "raised" by 3 cm on the graph. (Presumably the graph is spacelike so if you raise through the top you reappear on the bottom)

Suppose for some reason we were interested in a particular locus of field points ~~that is interesting~~. The effect of a global gauge transformation on such a locus would be simply to translate the locus (the figure) vertically without distortion. The global transformation is given simply by: $A_c' = U(g) A_c$ where A_c is a field point in orbit c , and where A_c' is the new point. This applies to each orbit, and the point is that g is the same for all orbits.

3. Local gauge transformation. This is the same thing as the above except that now we allow the gauge parameter g to be different for each orbit. Thus, we have $A_c' = U(g_c) A_c$. Graphically this means that if you do a local gauge transformation on a figure, the figure gets vertically shifted and also distorted because the amount of vertical shift varies as you go from one orbit to the next.

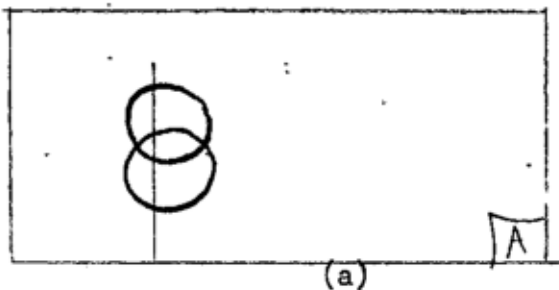
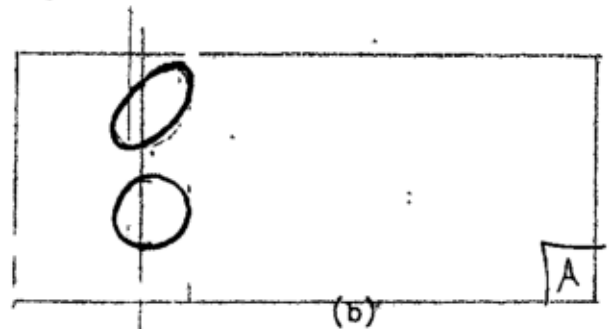


Fig 2: (a) shows the effect of a global gauge transformation on the green figure; the result is the purple figure, no distortion.



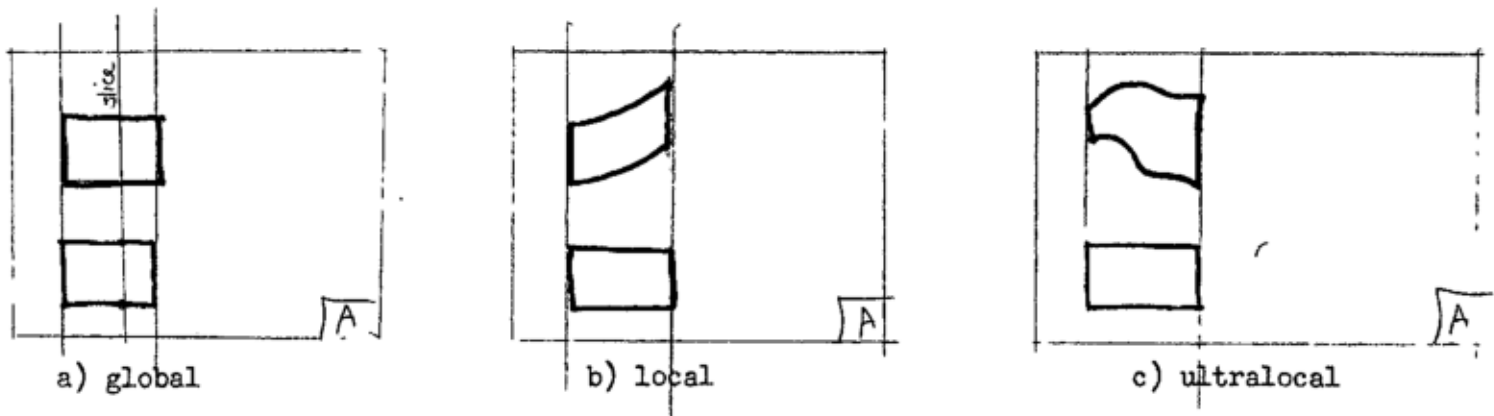
(b) Here is effect of local gauge transformation on same green figure. Orbits on the right "upshift" more than on left, so figure distorts. But distance from top to bottom along same orbit of each figure is same, by our graph convention.

4. Ultra-local Gauge Transformation. Now imagine that someone hands you a function $H(A_c)$ ie, a function of the points on orbit c . Suppose you do a gauge transformation where, for each orbit, you choose $g = H(A_c)$. Then you are saying:

$$A'_c = U(g=H(A_c)) A_c$$

I have called this ultralocal because: even within a given orbit, the value of g which tells you how much to transform is different for each point in the orbit. Ie, if you want to see where point A_c goes (of course it still ends up in the same orbit c somewhere), you have to first go and compute $g = H(A_c)$. Then you use that g to get A'_c .

What does an ultra-local transformation do to a figure? The figure below gives a better comparison of the three kinds of gauge transformations:



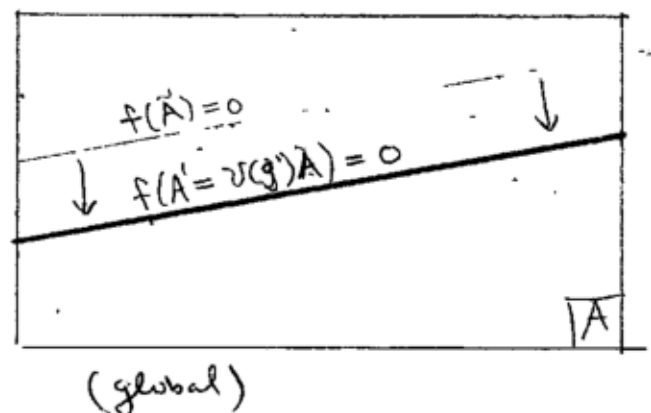
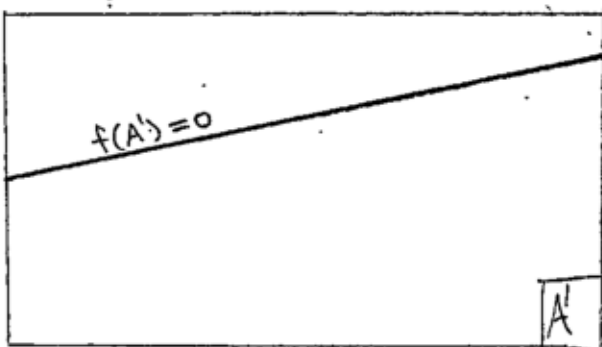
In the global case, each "slice" of the figure is raised by the same amount. In the local case, each slice is raised by a different amount, but the distance between points on each slice is preserved. ("Slice" refers to orbital portion of the figure"). In the third case, the whole idea of "slice-being-raised" breaks down. Or you could say: each slice is raised and stretched or shrunk. Vertical distance is no longer preserved because the effective g for the two different points on ^{a given} each slice is not the same. I made the earlier graphical convention just so I could illustrate these three cases as above.

5. Notion of a Gauge Surface. Consider a functional $f[A]$ defined over the rectangle. Imagine that for each point A , f is some real number, so you can imagine $f[A]$ as a surface hovering above the Rectangle, possibly intersecting it. ~~For example~~ Actually, $f[A]$ is really a vector-valued function in color space, so each point on this hovering surface is really a vector, but cannot visualize such a thing so imagine just each $f[A]$ as a real number.

Lets say that the A -rectangle lies in the x,y plane and that the f axis is the z -axis. Then points A for which $f=0$ are actually points in the Rectangle. For some arbitrary functional $f[A]$, you may or may not get some "figure of intersection" with the A -plane. For carefully chosen functionals f , the hovering surface f might intersect the A -plane rectangle in a curve which intersects each orbit exactly once. One might wonder, for some given f , is this a likely thing to happen? Consider $f[A_c]$ where A_c is some point on the orbit c . Assume that $f[A_c] \neq 0$. It does seem vaguely likely to me that as you move A_c around on the orbit, somewhere you will get $f=0$. In fact, it even seems likely that it will happen more than once on the orbit. But lets leave this for a later discussion. For now assume that $f[A] = 0$ describes a curve in the Rectangle which does in fact intersect each orbit exactly once. Such a surface is called a gauge surface. It serves as a sort of backbone through the rectangle, sort of a thruway. You can get to any place in the Rectangle by doing a gauge transformation off the gauge surface.

6. Gauge surfaces under gauge transformations. Suppose we have some gauge surface defined by the points A which satisfy $f[A] = 0$. But suppose we now do a gauge transformation on the fields so we end up with new, transformed fields A' . Suppose we now insist that $f[A'] = 0$. In the A' -plane, this surface is the same as it was before, but you might wonder what this surface looks like in the A -plane.

First, consider global transformation. Then you are interested in the surface $A_c = U(g)^{-1} \tilde{A}_c$ where $\tilde{A}_c$ is the point on orbit c which lies on the gauge surface, ie, $f[\tilde{A}_c] = 0$. Ie, you have to map backwards to see what the surface looks like in A -space. As you can see, it is just the surface lowered by a constant amount, because this is a global gauge transformation. Thus, we have:



In other words, we are doing a gauge transformation on a gauge surface. When you set $f[A] = 0$ you got one surface, when you set $f[U(g)A] = f[A^G] = 0$ you get a different surface. For a global transformation, the two surfaces are "parallel". But for either the local or ultralocal transformations, the new surface is a completely new surface, bearing no resemblance to the original surface.

Exactly how do you find the new surface, given the original one? Well, if you know the original surface, then you know all points $\tilde{A}$ which lie on the surface. Then to get the new surface for global case, you just examine the transformation:

$$A = U(g)^{-1} \tilde{A}$$

Thus, you just plug in and you've got your new surface. The same goes for a local transformation where g becomes g_c . You know g_c for each orbit, plug in and you're done.

However, for the ultralocal case it's not quite so easy. You have:

$$A = U[g[A]]^{-1} \tilde{A}$$

Here, for each $\tilde{A}$ you have to solve a non-linear equation of some sort to get A .

7. Jacobian going from fields A to fields A' . Recall that each field is a color vector, so there is a color matrix connecting A' to A , for a given gauge transformation. (think of a particular spacetime point if you like). But also, since A is a gauge field (a Yang Mills field), there is the "extra piece" in the transformation law. Specially,

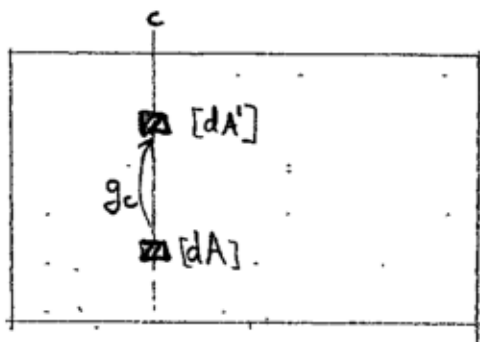
$$A'_a = U_{ab}^{adj}(g) A_b + \frac{1}{ig} (F_{adj})^a_{bc} [G_c(2\pi v) \tilde{v}^b] \quad (1)$$

If you want to convert some integration from dA to dA' , you have to do a Jacobian which schematically is $J = \det(dA'/dA)$. If you are dealing with a global GT, then $A'(x)$ is just a function of $A(x)$ and you see that $J=1$ trivially (ie, U is a special matrix since gauge group is like $SU(n)$, so $\det U=1$; notice that the "extra piece" does not contain A at all and thus does not contribute to this Jacobian in the global case.)

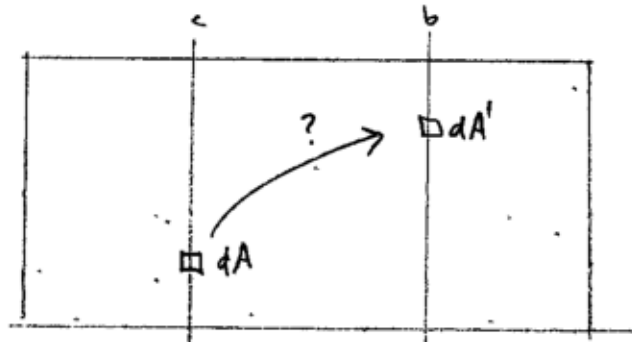
This then leads to the usual result that $[dA'] = [dA]$ in the sense of path integrals, for global transformations. What about locals? Again, for each orbit you have some g_c , so for all the A 's on that orbit, you transform with g_c . Now to compute the full J , you want to consider all A points in the Rectangle. However, you know that a GT cannot take some point A_c off the orbit of A_c . Thus, when you talk dA'/dA , you know you are talking only dA'_c/dA_c . You might say: why can't you talk about varying dA_c off the orbit c ? Well, I think it is this: we are going to do a change of variables from A to A' , but we are going by fiat to only do this so that A' is a

function of A 's on same orbit. In fact that must be because that is what the above expression says: . We are talking about $dA(A)$, ie, the differential at some point in the Rectangle, a little volume (area) at some point A . This point A lies on some orbit, so call it A_c . The new volume of interest, dA' , lies on the same orbit. Somehow it seems to me that, even though g_c does vary with the fields A if you move horizontally in the A -plane (ie, if you move across the orbits) g_c does not vary if you move vertically. By the way, I also imagine g_c as changing smoothly as you change to adjacent orbits.

Thus, when you compare dA and dA' in the figure below, the g in the transformation ~~xxx~~ $U(g_c)$ is really a constant. Thus the Jacobian, which in principle might have been a function of position in the A -plane, is again just unity for a local GT as well as for a global one. If you do an integration using dA , yes the dA might run over the entire Rectangle, and so do the dA' . But for each particular dA , dA' lies on the same orbit. So you never talk about a Jacobian relating $d(\text{volumes})$ on different orbits. Ie, you deal with figure (a) below, but never with figure (b).



yes, this is it



this is just nonsense.

I think this is all right, but maybe later I'll change my mind.

Now finally consider the ultralocal gauge transformation case. Now $g = g(A_c)$ and is a function of your position on a given orbit. Now all the g 's appearing in formula (1) on preceding page are really $g(A)$. Thus, when you compute dA'/dA , you have to include the fact that g depends on A . In fact, since the "extra term" involves derivatives of g , it will involve derivatives of A , and this means that in the ultralocal case A' is a functional of A , so you have to compute $J = \nabla A' / \nabla A$ and you now have to do determinant in the (x,y) spatime sense as well as color. It is very easy to see how, in the ultralocal case, the Jacobian J might not be 1, and in fact might be a functional of A , your location in the Rectangle.

By the way, this "ultrlocal" gauge transformation idea is, I think, what Lee and Zinn Justin and Slavnov are talking about when they speak of a "constrained" or "nonlinear" gauge transformation. They have a function $g = H(A)$ in mind, so on each orbit $g = H(A_c)$ and you have exactly this ultralocal case. So, $[dA]$ is not invariant for an ultralocal gauge transformation, but it is invariant for a global or local one!

Renorm Parts of YMTh Exposed!



① (covariant gauge)

$$D = 4 - E_{\text{g}}$$



$$D = 2$$



$$D = 2$$



$$D = 1$$



$$D = 1$$



$$D = 0$$



$$D = 0$$



⑬ Matrix Notation.

Let $\underline{\underline{A}}^M(x) \equiv A_c^M(x) \underline{\underline{G}}^c = \text{gauge matrix} = \vec{A}^M \cdot \vec{\underline{\underline{G}}}$

$\underline{\underline{C}}(x) = C_a(x) \underline{\underline{G}}_a = \text{ghost matrix} = \vec{C} \cdot \vec{\underline{\underline{G}}}$

Assume that $\text{tr}[G^a G^b] = k \delta^{ab}$ as usual [could take $k = \frac{1}{2}$].

a) consider:

$$[A^M, A^N] = A_a^M A_b^N [G^a, G^b] = A_a^M A_b^N i C_{abc} G_c$$

$$= i G_c \cdot C_{cab} A_a^M A_b^N = i \vec{G} \cdot \vec{A}^M \times \vec{A}^N$$

Then: $F_{\mu\nu} \equiv \vec{F}_{\mu\nu} \cdot \vec{G} = (\partial_\mu A_\nu - \partial_\nu A_\mu) + g(-i)[A_\mu, A_\nu]$

So

$$\boxed{F_{\mu\nu} \equiv \vec{G} \cdot \vec{F}_{\mu\nu} = (\partial_\mu A_\nu - \partial_\nu A_\mu) - ig[A_\mu, A_\nu]}$$

b) next:

$$\text{tr}[F_{\mu\nu} F^{\mu\nu}] = F_{\mu\nu}^a F^{\mu\nu b} \text{tr}[G_a G_b] = k \vec{F}_{\mu\nu} \cdot \vec{F}^{\mu\nu}$$

Thus,

$$\mathcal{L} = -\frac{1}{4} \vec{F}_{\mu\nu} \cdot \vec{F}^{\mu\nu} = \boxed{-\frac{1}{4k} \text{tr}[F_{\mu\nu} F^{\mu\nu}] = \mathcal{L}_{\text{gauge}}}$$

c) gauge fix is:

$$\mathcal{L}_{\text{fix}} = -\frac{1}{2\alpha} (\partial_\mu \vec{A}^\mu) \cdot (\partial^\mu \vec{A}_\mu)$$

$$= +\frac{1}{2\alpha} A^\mu \partial_\mu \partial^\nu A_\nu = +\frac{1}{2} A^\mu \left\{ \frac{1}{2} \partial_\mu \partial^\mu \right\} A^\nu$$

But, $\text{tr}[(\partial_\mu A^\mu)^2] = \text{tr}[(\partial_\mu A^\mu)(\partial_\nu A^\nu)] = (\partial_\mu A_a^\mu)(\partial_\nu A_b^\nu) \cdot k \delta_{ab}$

$$= k (\partial_\mu \vec{A}^\mu) \cdot (\partial_\nu \vec{A}^\nu)$$

So

$$\boxed{\mathcal{L}_{\text{fix}} = -\frac{1}{2\alpha k} \text{tr}[(\partial_\mu A^\mu)^2]}$$

d) ghosts: $\text{tr}[\partial_\mu \underline{\underline{C}}^\dagger \partial^\mu \underline{\underline{C}}] = k \partial_\mu \vec{C}^\dagger \cdot \partial^\mu \vec{C}$

$$\begin{aligned}
 \text{tr} [\partial_\mu C^\dagger [A_\mu, C]] &= \partial_\mu C_a^\dagger A_\mu^c C_b \text{tr} [G_a \underbrace{[G_c, G_b]}_{i C_{cbd} G_d}] \\
 &= -ik C_{abc} [\partial_\mu C_a^\dagger] [C_b] [A_c^\dagger] \underbrace{i C_{cbd} \text{tr} [G_a G_d]}_{k \delta_{ad}} \\
 &= ik C_{cba} = -ik C_{abc}
 \end{aligned}$$

Thus I get:

$$\begin{aligned}
 \mathcal{L}_{\text{ghost}} &= \frac{1}{k} \text{tr} [\partial_\mu C^\dagger \partial^\mu C] - g \left(\frac{1}{ik} \right) \text{tr} [\partial_\mu C^\dagger [A_\mu, C]] \\
 \Rightarrow \mathcal{L}_{\text{ghost}} &= \frac{1}{k} \text{tr} \{ \partial_\mu C^\dagger \partial^\mu C - ig (\partial_\mu C^\dagger) [A_\mu, C] \}
 \end{aligned}$$

Compare Gross & Wilczek: They use

$$\begin{aligned}
 k &= \frac{1}{2} & A_\mu &= B_\mu \\
 C &= \phi & g^{\text{GW}} &= ig^{\text{me}}
 \end{aligned}$$

Better:

me	GW
A_μ	B_μ
C	ϕ
G_a	t_a
C_{abc}	C_{abc}
k	$\frac{1}{2}$
g	$-ig^{\text{GW}}$

me	FP
$+ig$	$\in$
$\mathcal{V}$	Ω

Other matrix usages.

$$\begin{aligned}
 [D_\mu]_{ab} &= (\delta_{ab} \partial_\mu - ig (G_c)_{ab} A_\mu^c) \\
 \underline{D}_\mu &= \underline{1} \partial_\mu - ig \underline{\vec{G}} \cdot \underline{\vec{A}}_\mu
 \end{aligned}
 \left. \vphantom{\begin{aligned} [D_\mu]_{ab} &= (\delta_{ab} \partial_\mu - ig (G_c)_{ab} A_\mu^c) \\ \underline{D}_\mu &= \underline{1} \partial_\mu - ig \underline{\vec{G}} \cdot \underline{\vec{A}}_\mu \end{aligned}} \right\} \rightarrow \underline{D}_\mu = [\underline{1} \partial_\mu - ig \underline{A}_\mu]$$

General rule:

$$\begin{aligned}
 [\underline{A}, \underline{B}] &= i \underline{\vec{G}} \cdot (\underline{\vec{A}} \times \underline{\vec{B}}) \\
 \text{tr} \{ [\underline{A}, \underline{B}] \underline{C} \} &= i (\underline{\vec{A}} \times \underline{\vec{B}}) \cdot \underline{\vec{C}}
 \end{aligned}$$

Then: $\delta \vec{A}_\mu = \vec{\Theta} \times \vec{A}_\mu - \frac{1}{g} \partial_\mu \vec{\Theta}$

$\Rightarrow \delta A_\mu = -i [\Theta, A_\mu] - \frac{1}{g} \partial_\mu \Theta$ where $\underline{\underline{\Theta}} = \vec{\Theta} \cdot \underline{\underline{G}}$

Full rule is:

$$A'_\mu = U A_\mu U^{-1} + \frac{1}{ig} [\partial_\mu U] U^{-1}$$

where $U = \exp(-i \vec{\Theta} \cdot \underline{\underline{G}}) = \exp(-i \underline{\underline{\Theta}})$

Preliminary Ghost Stuff: how to do detM without ever using ghost fields.

1. Here is our starting point:

$$\begin{aligned} [M] &= \left(\frac{1}{g\sqrt{2}} \right) \left[[\partial^2] \frac{1}{2} - g \oint [\partial_\mu] [A_\mu^\dagger] \right] \\ &\doteq \left\{ \frac{1}{2} - g \oint [\partial^2]^{-1} [\partial_\mu] [A_\mu^\dagger] \right\} \equiv \left\{ \frac{1}{2} - \frac{L}{2} \right\} \end{aligned}$$

Here we have dropped a finite constant and the d^2 operator. These things just make overall, infinite constants which play no role. These constants could be regulated to be finite by putting finite number of points in spacetime or something. Or by setting delta function equal to a gaussian.

2. Next, we compute detM in this way:

$$\det M = e^{\hbar \text{tr} M} = e^{\hbar \text{tr} (1-L)} = e^{-\sum \frac{1}{n} \hbar \left[\left(\frac{L}{2} \right)^n \right]}$$

Now, even though we have no ghost fields per se, we can still identify the inverse laplacian there as the coordinate space ghost propagator. I will force this propagator to have same normalization as my "feynman rules" ghost propagator. Thus:

$$\underline{D}_{ab} = \delta_{ab} (\partial^2)^{-1} \quad \text{or} \quad D_{ab}(x,y) = \delta_{ab} (\partial_x^2)^{-1} \delta(x-y).$$

$$\Rightarrow \left[\frac{L}{2} \right] = + g \oint \left[\frac{C}{2} \right] \left[\frac{D}{2} \right] [\partial_\mu] [A_\mu^\dagger]$$

$$\begin{aligned} \Rightarrow L_{a_1 a_2}(x_1, y_1) &= \int (+g) C_{a_1 b_1}^f D_{b_1 a_2}^g(x_1, z_1) [\partial_\mu^{(z_1)} \delta(z_1 - y_1)] A_\mu^\dagger(y_1) dz_1 \\ &= -g C_{a_1 b_1}^{f_1} [\partial_{\mu_1}^{(y_1)} D_{b_1 a_2}^g(x_1, y_1)] A_{f_1}^{\mu_1}(y_1) \end{aligned}$$

+ b. c.

Feynman Rules for Gauge Theory with Correct Normalizations.

(see page 10 for read off.)

1. The Bare Propagator. Let us assume the alpha-landau gauge here. We then wish to combine the usual $F_{\mu\nu}$ lagrangian with the gauge fixing term. Here is how it goes:

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}^a F_{\mu\nu}^a - \frac{1}{2\alpha} (\partial_\mu A_\mu^a)^2 + A_\mu^a J_\mu^a$$

$$F_{\mu\nu}^a(g=0) = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a$$

Suppress color labels for a while. Then:

$$\mathcal{L} = -\frac{1}{4} (\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial_\mu A_\nu - \partial_\nu A_\mu) - \frac{1}{2\alpha} (\partial_\mu A_\mu)(\partial_\nu A_\nu)$$

Do parts to get

$$\mathcal{L} = +\frac{1}{2} A_\nu \partial_\mu (\partial_\mu A_\nu - \partial_\nu A_\mu) + \frac{1}{2\alpha} (A_\mu \partial_\mu)(\partial_\nu A_\nu)$$

$$\mathcal{L} = \frac{1}{2} A_\mu [\Box^2 g_{\mu\nu} - (1-\frac{1}{\alpha}) \partial_\mu \partial_\nu] A_\nu$$

Reinsert color to get

$$\mathcal{L} = +\frac{1}{2} A_\mu^a \left\{ \delta^{ab} (\Box^2 g_{\mu\nu} - (1-\frac{1}{\alpha}) \partial_\mu \partial_\nu) \right\} A_\nu^b + A_\mu^a J_\mu^a$$

So let

$$\vec{\mathcal{O}}_{\mu\nu}^{ab}(x) \equiv \delta^{ab} (\Box_{(x)}^2 g_{\mu\nu} - (1-\frac{1}{\alpha}) \partial_\mu^{(x)} \partial_\nu^{(x)})$$

Then:

$$\mathcal{L} = +\frac{1}{2} A_\mu^a \vec{\mathcal{O}}_{\mu\nu}^{ab} A_\nu^b + A_\mu^a J_\mu^a$$

$$\text{Now: } \left[\int [dA] e^{+i \int d^4x \mathcal{L}} = e^{\frac{-1}{i} \int d^4x d^4y J_\mu^a(x) \vec{\mathcal{O}}_{\mu\nu}^{ab}(x,y) J_\nu^b(y)} \right]$$

see matrix rules.

Thus,

$$[i D_{\mu\nu}^{ab}(x,y)] = \frac{(-i)^2}{W} \frac{\delta^2 W}{\delta J_\mu^a(x) \delta J_\nu^b(y)} = (-i)^2 (-\frac{1}{2}i)(2) \vec{\mathcal{O}}_{\mu\nu}^{ab}(x,y)$$

⇒

$$[i D_{\mu\nu}^{ab}(x,y)] = \langle 0 | T(A_\mu^a(x) A_\nu^b(y)) | 0 \rangle = G_{\mu\nu}^{(2)ab}(x,y)$$

$$= i \tilde{D}_{\mu\nu}^{ab}(x,y) = (-i)^2 \delta^2 / \delta \delta \delta \delta$$

where

$$\tilde{D}_{\mu\nu}^{ab}(x) \tilde{D}_{\nu\lambda}^{bc}(x,y) = + \delta^{ac} \delta_{\mu\lambda} \delta(x-y)$$

To make more sense out of this, go into fourier transform. To do so, use standard transform conventions in notes, assume D is function of x-y. Then:

$$D_{\nu\lambda}^{bc}(x,y) = \int \frac{d^4 k}{(2\pi)^4} e^{-ik \cdot (x-y)} D_{\nu\lambda}^{bc}(k) \quad \partial_\mu \rightarrow -ik_\mu$$

$$\Rightarrow \left(\delta^{ab} \left[-k^2 g_{\mu\nu} + (1-\alpha) k_\mu k_\nu \right] \right) D_{\nu\lambda}^{bc}(k) = \delta^{ac} \delta_{\mu\lambda}$$

$$\Rightarrow \boxed{D_{\nu\lambda}^{bc}(k) = -\frac{\delta^{bc}}{k^2} \left[g_{\nu\lambda} - (1-\alpha) \frac{k_\nu k_\lambda}{k^2} \right]}$$

multiply to check



Therefore, here is the correctly normalized Feynman rule for the bare propagator. :

$$G_{\mu\nu}^{(2)ab}(k) = \text{diagram} = [i D_{\mu\nu}^{ab}(k)]$$

$$= -\frac{i \delta^{ab}}{k^2} \left[g_{\mu\nu} - (1-\alpha) \frac{k_\mu k_\nu}{k^2} \right]$$

As a check, set alpha=1 for Feynman gauge. Ignore color, and you duplicate photon Feynman rule as in Bjorken and Drell, including the sign. (see page 382 volume 2)

2. The Four Glue Vertex. Let's consider only this particular interaction term.

In general you can say this:

$$W[J] = e^{+i \int dx \mathcal{L}_{int} [\frac{1}{i} \frac{\delta}{\delta J}, \frac{1}{i} \frac{\delta}{\delta J}]} \frac{\int dA d\bar{c} d\bar{c} e^{+i \int dx \mathcal{L}}}{(e^{-\frac{i}{2} \int dx dy \bar{J} \cdot D \cdot J} \times (\text{similar ghost factor}))}.$$

Here I have included consideration of the ghosts. But for the moment we are not interested in taking derivatives wrt the ghost source j , so set that factor always = unity. Then we are left with this:

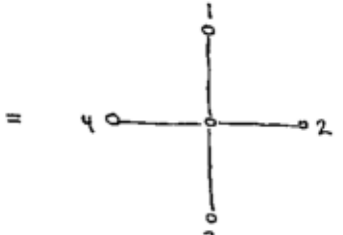
$$W[J] = \exp i \left\{ \int dx \mathcal{L}_{int} \left(\frac{1}{i} \frac{\delta}{\delta J} \right) \right\} e^{-\frac{i}{2} \int dx dy J_\mu^a(x) D_{\mu\nu}^{ab}(x,y) J_\nu^b(y)}$$

where
$$\mathcal{L}_{int}(A) = -\frac{1}{4} g^2 C^{abc} C^{ab'c'} A_\mu^b A_\nu^c A_\mu^{b'} A_\nu^{c'}$$

$$= -\frac{1}{4} g^2 C_{a_1 a_2 a} C_{a_3 a_4 a} g_{\mu_1 \mu_3} g_{\mu_2 \mu_4} A_{\mu_1}^{a_1} A_{\mu_2}^{a_2} A_{\mu_3}^{a_3} A_{\mu_4}^{a_4}$$

This may not be the most efficient way to examine the Feynman rule, but let's examine the 4-point Greens function like so:

$$G_{\nu_1 \nu_2 \nu_3 \nu_4}^{(4) b_1 b_2 b_3 b_4}(x_1, x_2, x_3, x_4) = \frac{(-i)^4 \delta^4 W[J]}{\delta J_{\nu_1}^{b_1}(x_1) \delta J_{\nu_2}^{b_2}(x_2) \delta J_{\nu_3}^{b_3}(x_3) \delta J_{\nu_4}^{b_4}(x_4)}$$



$$= \int dx \cdot [i D_{\mu_1 \nu_1}^{a_1 b_1}(x_1, x)] [i D_{\mu_2 \nu_2}^{a_2 b_2}(x_2, x)] [i D_{\mu_3 \nu_3}^{a_3 b_3}(x_3, x)] \times [i D_{\mu_4 \nu_4}^{a_4 b_4}(x_4, x)] \cdot (\text{Feynman rule for vertex})$$

In other words, we already know the Feynman rule for a glue propagator: put in iD for each line. Thus, when you consider this lowest order graph, what ever is "left over" in the parenthesis must be the Feynman rule for the vertex. So all we have to do is see what goes in there:

$$G^{(4)} = +i \int dx \left[-\frac{1}{4} g^2 C_{a_1 a_2 a} C_{a_3 a_4 a} g_{\mu_1 \mu_3} g_{\mu_2 \mu_4} \right] (-i)^4 (-i)^4$$

$$\times \partial_{\nu_1}^{b_1}(x_1) \partial_{\nu_2}^{b_2}(x_2) \partial_{\nu_3}^{b_3}(x_3) \partial_{\nu_4}^{b_4}(x_4) \partial_{\mu_1}^{a_1}(x) \partial_{\mu_2}^{a_2}(x) \partial_{\mu_3}^{a_3}(x) \partial_{\mu_4}^{a_4}(x)$$

$$\times \left(\frac{1}{4!} \right) \left[-\frac{i}{2} \int dy dz J_\mu^a(y) D_{\mu\nu}^{ab}(y,z) J_\nu^b(z) \right]^4$$

Now the first thing to do here is let the "external" functional derivatives come in and do their thing. You r 4 external derivatives act on a product of four factors. Consider the $d(x_1)$ derivative. It can act in 4 ways, ie, it can hit into any of the four factors. Then the $d(x_2)$ can hit on any of the three remaining factors, and so on. In this way you generate a $4!$ which cancels the one shown from the exponential.

Now consider $d(x_1)$ hitting just one of the factors. It can hit either J inside the factor. In effect, this makes a factor of 2 which cancels the two in the $-i/2$. So, the net result is this, after the external pincers have acted:

$$G^{(4)} = +i \left[-\frac{1}{4} g^2 C_{a_1 a_2 a} C_{a_3 a_4 a} g_{\mu_1 \mu_3} g_{\mu_2 \mu_4} \right] (-i)^4 \int dx$$

$$\cdot \partial_{\mu_1}^{a_1}(x) \partial_{\mu_2}^{a_2}(x) \partial_{\mu_3}^{a_3}(x) \partial_{\mu_4}^{a_4}(x)$$

$$\cdot \prod_{i=1}^4 \left[-i \int dy J_{\mu}^a(y) D_{\mu\nu}^{ab}(y, x_i) \right]$$

convert $(-i)$ to $(+i)$ in each prop.

Now consider the action of the "internal" functional derivatives. Again, each of the derivatives shown comes in one at a time. The first one can hit one of four factors, the second hits one of the remaining three, and so on. Thus, there will be $4!$ terms corresponding to all these different hits. If you know one term, you get all the others by just summing permutations. So let's pick as our one term the one where the a_1 hits the b_1 things and so on. Then we get:

$$G^{(4)} = \int dx \left[i D_{\mu\nu}^{a_1 b_1}(x, x_1) \right] [2] [3] [4] \cdot \left(+i \left(-\frac{1}{4} g^2 \right) C_{a_1 a_2 a} C_{a_3 a_4 a} g_{\mu_1 \mu_3} g_{\mu_2 \mu_4} \right)$$

In this term, a_1 got tied to b_1 , a_2 was tied to b_2 and so on. On some other term perhaps b_1 ties to a_2 and b_2 ties to a_1 . In that term you would rename the summation variables a_1 and a_2 ; the effect would be to permute the object in parenthesis. Thus, here is my tentative conclusion:

$$\text{Rule} = +i \left(-\frac{1}{4} g^2 \right) \sum_P C_{a_1 a_2 a} C_{a_3 a_4 a} g_{\mu_1 \mu_3} g_{\mu_2 \mu_4}$$

There are $24 = 4!$ terms here. The usual thing people do is classify these terms into three groups of 8 terms as follows:

(12, 34) stands for 8 terms you get by doing selected perms.

$$\begin{aligned}
 &= i(-\frac{1}{4}g^2) C_{a_1 a_2 a} C_{a_3 a_4 a} \times \left[\underset{\substack{\uparrow \\ (1 \leftrightarrow 2)}}{g_{\mu_1 \mu_3}} g_{\mu_2 \mu_4} - \underset{\substack{\uparrow \\ (1 \leftrightarrow 2)}}{g_{\mu_2 \mu_3}} g_{\mu_1 \mu_4} \right. \\
 &\quad \left. - \underset{\substack{\uparrow \\ (3 \leftrightarrow 4)}}{g_{\mu_1 \mu_4}} g_{\mu_2 \mu_3} + \underset{\substack{\uparrow \\ (1 \leftrightarrow 2) \text{ and } (3 \leftrightarrow 4)}}{g_{\mu_2 \mu_4}} g_{\mu_1 \mu_3} + g_{\mu_3 \mu_1} g_{\mu_4 \mu_2} + 3 \text{ more} \right] \\
 &= -i g^2 C_{a_1 a_2 a} C_{a_3 a_4 a} \left[g_{\mu_1 \mu_3} g_{\mu_2 \mu_4} - g_{\mu_2 \mu_3} g_{\mu_1 \mu_4} \right]
 \end{aligned}$$

Thus the 8 terms in this group make the above pair of terms four times, cancelling the factor of 1/4th. So we end up with this statement of the rule:

$$\text{Rule} = -i g^2 \left\{ (12, 34) + (13, 24) + (14, 23) \right\}$$

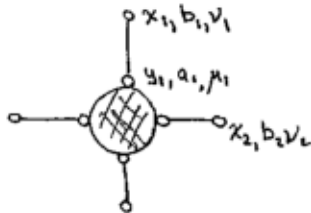
$$\text{where } (12, 34) = C_{a_1 a_2 a} C_{a_3 a_4 a} \left[g_{\mu_1 \mu_3} g_{\mu_2 \mu_4} - g_{\mu_2 \mu_3} g_{\mu_1 \mu_4} \right]$$

Notice that (12, 34) = (12, 43) because you change the sign of both factors. Thus, it is just the pairings you need to remember, no signs. In the first term group, 12 are together, in the second its 13, in the third its 14.

This rule agrees exactly with what I got last time I did this. And last time I checked it against various other sources so I'm happy with it.

2.5 The Four Glue Vertex, More Detail.

I think it would be good to change the definition of the "4-vertex rule" so that it is defined to be the thing which appears in here:



$$G_{v_1 v_2 v_3 v_4}^{(4) b_1 b_2 b_3 b_4}(x_1, x_2, x_3, x_4) = (-i)^4 \delta^4 W.$$

$$= \int dy_1 dy_2 dy_3 dy_4 [i D_{\mu_1 v_1}^{a_1 b_1}(x_1, y_1)] [i D_{\mu_2 v_2}^{a_2 b_2}(x_2, y_2)] [\#3] [\#4] \left(\Gamma_{\mu_1 \mu_2 \mu_3 \mu_4}^{(4) a_1 a_2 a_3 a_4} \{y_1, y_2, y_3, y_4\} \right)$$

If other terms were added, this would really be the definition of the proper 4-point function. Here of course these are all bare propagators and this vertex will be the "coordinate space feynman rule". Of course we already know the answer from preceding pages. Result is this:

← coordinate space

$$\text{Bare } \Gamma_{\mu_1 \mu_2 \mu_3 \mu_4}^{(4) a_1 a_2 a_3 a_4}(y_1, y_2, y_3, y_4) = -ig^2 \delta(y_1 - y_4) \delta(y_2 - y_3) \delta(y_3 - y_4) \{ (12, 34) + \text{etc.} \}$$

Here y_4 plays the role of variable called x before. You can see that this duplicates earlier result when inserted there. Notice that this thing is really a function of three three difference coordinates, as is always the case for a translation invariant function.

Now lets fourier transform all of the above. We get:

$$G_{v_1 v_2 v_3 v_4}^{(4) b_1 b_2 b_3 b_4}(p_1, p_2, p_3, -) = [i D_{\mu_1 v_1}^{a_1 b_1}(p_1)] [2] [3] [4] \Gamma_{\mu_1 \mu_2 \mu_3 \mu_4}^{(4) a_1 a_2 a_3 a_4}(p_1, p_2, p_3, -)$$

Here the twopi fourth delta overall has already been removed, so these guys are only functions of the three momenta shown, the fourth being determined by conservation. The object here is fourier of the above like so:

$$\begin{aligned} \Gamma(p_1, p_2, p_3, -) &= \int d\xi_1 d\xi_2 d\xi_3 e^{+ip_1 \xi_1} e^{ip_2 \xi_2} e^{ip_3 \xi_3} \Gamma(\xi_1, \xi_2, \xi_3, 0) \\ &= -ig^2 \{ (12, 34) + \text{etc.} \}. \end{aligned}$$

← momentum space.

$$\Rightarrow \text{Bare } \Gamma_{\mu_1 \mu_2 \mu_3 \mu_4}^{(4) a_1 a_2 a_3 a_4}(p_1, p_2, p_3, -) = -ig^2 \{ (12, 34) + \text{etc.} \}$$

3. The Three Glue Vertex. As before, we have $W[J]$ given by

$$\int dx \mathcal{L}_{int}(A) = -g C^{abc} (\partial_\mu A_\nu^a) A_\mu^b A_\nu^c$$

and now we focus on the triple glue interaction term which is:

$$W[J] = e^{i \int dx \mathcal{L}_{int}(\frac{1}{i} \frac{\delta}{\delta J})} e^{-\frac{i}{2} \int dx dy \mathcal{T}_\mu^a(x) D_{\mu\nu}^{ab}(x,y) \mathcal{T}_\nu^b(y)}$$

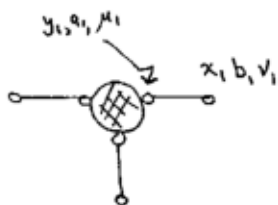
We began with $-\frac{1}{4}$, there were two cross terms, then we tensor-doubled so -1 . By just fiddling we can rewrite this same object as follows:

$$i \int dx \mathcal{L}_{int} = i \int dy_1 dy_2 dy_3 \left[+g C^{a_1 a_2 a_3} g_{\mu_1 \mu_3} [\partial_{\mu_2}^{(y_1)} \delta(y_3 - y_1)] \delta(y_3 - y_2) \right] \\ \times A_{\mu_1}^{a_1}(y_1) A_{\mu_2}^{a_2}(y_2) A_{\mu_3}^{a_3}(y_3)$$

Now we search for the following object:

$$G_{\nu_1 \nu_2 \nu_3}^{(3) b_1 b_2 b_3}(x_1, x_2, x_3) = (-i)^3 \frac{\delta^3 W[J]}{\delta \mathcal{T}_{\nu_1}^{b_1}(x_1) \delta \mathcal{T}_{\nu_2}^{b_2}(x_2) \delta \mathcal{T}_{\nu_3}^{b_3}(x_3)}$$

$$= \int dy_1 dy_2 dy_3 [i D_{\mu_1 \nu_1}^{a_1 b_1}(x_1, y_1)] [\#2] [\#3] \Gamma_{\mu_1 \mu_2 \mu_3}^{(3) a_1 a_2 a_3}(y_1, y_2, y_3)$$



$$= (-i)^6 \int dy_1 dy_2 dy_3 \left[i g C^{a_1 a_2 a_3} g_{\mu_1 \mu_3} \delta(y_3 - y_2) [\partial_{\mu_2}^{(y_1)} \delta(y_3 - y_1)] \right] \\ \times \partial_{\nu_1}^{b_1}(x_1) \partial_{\nu_2}^{b_2}(x_2) \partial_{\nu_3}^{b_3}(x_3) \partial_{\mu_1}^{a_1}(y_1) \partial_{\mu_2}^{a_2}(y_2) \partial_{\mu_3}^{a_3}(y_3) \\ \times \left(\frac{1}{3!} \right) \left[-\frac{i}{2} \int dx dy \mathcal{T}_\mu^a(x) D_{\mu\nu}^{ab}(x,y) \mathcal{T}_\nu^b(y) \right]^3$$

Remember what you are doing: computing $G^{(3)}$ by taking three external derivatives on the $W[J]$ given above. No steps are left out here. Now exactly as before we can apply the external (say) derivatives to kill the factorial and get:

$$(-i)^6 \int dy_1 dy_2 dy_3 \left[i g C^{a_1 a_2 a_3} g_{\mu_1 \mu_3} \delta(y_3 - y_2) \partial_{\mu_2}^{(y_1)} \delta(y_3 - y_1) \right]$$

$$\times \partial_{\mu_1}^{a_1}(y_1) \partial_{\mu_2}^{a_2}(y_2) \partial_{\mu_3}^{a_3}(y_3) \cdot \left[-i \int dx dy \mathcal{T}_\mu^a(x) D_{\mu \nu_1}^{a b_1}(x, y_1) \right] [\#2] [\#3]$$

Now we take the internal derivatives. They can go in all possible ways, here is one of the ways:

$$\int dy_1 dy_2 dy_3 [i D_{\mu_1 \nu_1}^{a_1 b_1}(x_1, y_1)] [i D_{\mu_2 \nu_2}^{a_2 b_2}(x_2, y_2)] [i D_{\mu_3 \nu_3}^{a_3 b_3}(x_3, y_3)] \\ \times \left\{ i g C^{a_1 a_2 a_3} g_{\mu_1 \mu_3} \delta(y_3 - y_2) [\partial_{\mu_2}^{(y_1)} \delta(y_3 - y_1)] \right\}$$

As before, there are going to be now 6 terms, one of which you see. If a_2 had gone onto b_1 , we could just rename things etc etc. Same workings as before. Notice that groups of labels all move together. So our result is this coordinate-space result:

← coordinate space.

$$\Gamma_{\mu_1 \mu_2 \mu_3}^{(3) a_1 a_2 a_3}(y_1, y_2, y_3) = + i g \sum_P C^{a_1 a_2 a_3} g_{\mu_1 \mu_3} \delta(y_3 - y_2) [\partial_{\mu_2}^{(y_1)} \delta(y_3 - y_1)]$$

There are various ways to write out these 6 terms, but lets instead go first to momentum space. We have;

$$\Gamma_{\mu_1 \mu_2 \mu_3}^{(3) a_1 a_2 a_3} = i g \sum_P C^{a_1 a_2 a_3} g_{\mu_1 \mu_3} \delta(-p_2) \partial_{\mu_2}^{(p_1)}(-p_1)$$

where

$$p_1 = y_1 - y_3$$

$$p_2 = y_2 - y_3$$

$$= - i g \sum_P C^{a_1 a_2 a_3} g_{\mu_1 \mu_3} \delta(p_2) \partial_{\mu_2}^{(p_1)}(p_1)$$

$$\Rightarrow \Gamma^{(3)}(p_1, p_2, -) = \int d p_1 d p_2 e^{+i p_1 p_1} e^{+i p_2 p_2} \quad \text{partic. num.} \quad \partial_{\mu} \rightarrow -i p_{\mu}$$

$$= - i g \sum_P C^{a_1 a_2 a_3} g_{\mu_1 \mu_3} \cdot [-i p_1]_{\mu_2}$$

$$\Rightarrow \Gamma_{\mu_1 \mu_2 \mu_3}^{(3) a_1 a_2 a_3}(p_1, p_2, -) = - g \sum_P C^{a_1 a_2 a_3} g_{\mu_1 \mu_3} (p_1)_{\mu_2}$$

$$= + g \sum_P C^{a_1 a_2 a_3} g_{\mu_1 \mu_2} (p_1)_{\mu_3}$$

where we did $1 \leftrightarrow 2$.

Writing out the terms we get:

← agrees with earlier.

$$= +g \sum_P C^{a_1 a_2 a_3} g_{\mu_1 \mu_2} (P_1)_{\mu_3}$$

← momentum space.

$$\Gamma_{\mu_1 \mu_2 \mu_3}^{(3) a_1 a_2 a_3} (P_1, P_2, -) = +g C^{a_1 a_2 a_3} \left\{ g_{\mu_1 \mu_2} [(P_1)_{\mu_3} - (P_2)_{\mu_3}] + \text{cyclic.} \right\}$$

Notice that, as was not the case with the 4-gluon vertex, all the terms in the 3-gluon have a common color factor which can be extracted as shown. Eg, the cyclic terms all have same C factor.

4. General Method to "READ OFF" the rules. Now that I have worked a few examples, it is clear that you don't have to redo all this work everytime you want to read a feynman rule from a lagrangian.

a) propagators: First, examine the lagrangian to find the operator in $L = +\frac{1}{2} \phi O \phi$. Here I am calling the operator O as in the example. The coordinate-space propagator, D , is precisely the "inverse" of this operator, ie, its Greens function. It is usually easiest to convert O to momentum space using the rule $d_u = -ik_u$. Then get its inverse in momentum space which is not very hard. Using this method, I could immediately get the propagator just by a read-off, but it is done in detail in preceding pages.

b) vertices: First you write down your interaction Lagrangian vertex of interest. You then multiply by +i (caused because it is +i L_{int} as in detail). Then replace each derivative with -ip where the p refers to the field acted upon. At once you arrive at the momentum-space feynman rule. To get the coordinate space rule it's not quite so easy, plus I don't care so much about that so forget it. I forget to say that after you multiply by plus i, and replace derivatives by -ip, you then have to sum over all permutations of the field labels. This, you recall, arises from the d/dJ's going in in the last step in all possible ways.

Example: 3-gluon

$$\mathcal{L} = -g C^{a_1 a_2 a_3} g_{\mu_1 \mu_3} [\partial_{\mu_2} A_{\mu_1}^{a_1}] A_{\mu_2}^{a_2} A_{\mu_3}^{a_3}$$

$$\Rightarrow \text{Rule} = (+i)(-g) \sum_P C^{a_1 a_2 a_3} g_{\mu_1 \mu_3} [-i(P)_{\mu_2}]$$

You can see that it does work. Of course you usually procede to write out the permutation sum in some nice way. Notice that you must always start off by writing your L_{int} in such a way that each field gets its own set of labels. Ie, you insert g_{uv} 's as needed, etc. The derivatives can just sit there acting on their chosen field. Recall also that "Rule" refers to the lowest order term in the proper vertex with no additional factors! (as was noted above).

Example: 4-gluon

$$\mathcal{L} = -\frac{1}{4} g^2 C^{a_1 a_2 a} C^{a_3 a_4 a} g_{\mu_1 \mu_3} g_{\mu_2 \mu_4} A_{\mu_1}^{a_1} A_{\mu_2}^{a_2} A_{\mu_3}^{a_3} A_{\mu_4}^{a_4}$$

$$\Rightarrow \text{Rule} = (+i)(-\frac{1}{4} g^2) \sum_P C^{a_1 a_2 a} C^{a_3 a_4 a} g_{\mu_1 \mu_3} g_{\mu_2 \mu_4}$$

5. Ghosts. As an application of the above method of "reading off feynman rules" lets do the ghosts in the covariant gauge. The general idea is this:

$$\det[M] = \int d\bar{c} dc \ e^{+i \int dx \ \bar{c}_a^{\dagger}(x) \tilde{M}_{ab}(x; \phi) c_b(x)}$$

and in covariant alpha-landau gauge we have:

$$[M] = \left(-\frac{1}{g\sqrt{\alpha}}\right) \partial_{\mu} \underline{D}^{\mu} = \left(-\frac{1}{g\sqrt{\alpha}}\right) \left\{ \square^2 \underline{1} - g \underline{C}^c [\partial_{\mu}] [\underline{A}_c^{\mu}] \right\}$$

$$\Rightarrow \tilde{M}_{ab}(x) = \left[\square^2 \delta_{ab} - g C_{abc} \partial_{\mu} (A_c^{\mu}) \right] \times \left(\frac{-1}{g\sqrt{\alpha}} \right)$$

As usual in spacematrix notation, the d_{μ} here acts on everything to its right, not just the A^{μ} sitting there. Lets now do a parts on each term and drop the outside factor. These factors have no significance, since they just make an overall constant which cannot affect any rules, and will be cancelled in the divided by W stage anyway. Thus we have:

$$\det M = \int d\bar{c} dc \ e^{+i \int dx \ [-\partial_{\mu} \bar{c}_a^{\dagger} \partial^{\mu} c_a + g C_{abc} (\partial_{\mu} \bar{c}_a^{\dagger}) A_c^{\mu} c_b]}$$

$$\Rightarrow \underline{\mathcal{L}(c^{\dagger}, c)} = + \bar{c}_a^{\dagger} \partial^2 c_a + g C_{abc} (\partial_{\mu} \bar{c}_a^{\dagger}) A_c^{\mu} c_b \quad /$$

Now we can read off the rules. On e detail is that the propagator here does not have a $\frac{1}{2}$ factor. This compensates for the facto that the d/dJ's now can only go in one way instead of two ways to pick off the propagator. So we find:

$$\mathcal{O}_{ab} = \delta_{ab} \partial^2 \Rightarrow \tilde{\mathcal{O}}_{ab} = -\delta_{ab} k^2 \Rightarrow D_{ab} = -\delta_{ab} \frac{1}{k^2}$$

$$\Rightarrow \boxed{(\text{-----}) = i D_{ab}(k^2) = -i \delta_{ab} \frac{1}{k^2} \quad /}$$

$$\boxed{\begin{array}{c} a \rightarrow \bullet \text{---} b \\ \quad \downarrow \\ \quad c, \mu \end{array}} = + g C_{abc} P^{\mu} = \text{BARE } \gamma_{abc}^{\mu}(p_a, p_b, -) \quad /$$

Ward Identity

Slavnov

Comment: This section was done as I was reading the Slavnov and Lee Zinn Justin sequence. The Ward Identities are needed to do renormalization of YMT. All this stuff is in the covariant gauge because Slavnov's work was.

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The Inverse Gauge Propagator.

This is very easy, I just want to have it on a separate sheet. We have:

$$D_{\mu\nu}(k) = -\frac{1}{k^2} \left\{ A(k^2) P_{\mu\nu}^T + \alpha P_{\mu\nu}^L \right\} \quad \text{as in page 7.}$$

$$D_{\nu\lambda}^{-1}(k) = -k^2 \{ B P_{\nu\lambda}^T + C P_{\nu\lambda}^L \}$$

$$\begin{aligned} D_{\mu\nu} D_{\nu\lambda}^{-1} &= \delta_{\mu\lambda} = AB P_{\mu\lambda}^T + \alpha C P_{\mu\lambda}^L \\ &= AB (\delta_{\mu\lambda} - P_{\mu\lambda}^L) + \alpha C P_{\mu\lambda}^L \\ &= \underbrace{AB}_{\text{"}} \delta_{\mu\lambda} + \underbrace{P_{\mu\lambda}^L}_{\text{"}} (\alpha C - \underbrace{AB}_{\text{"}}) \end{aligned}$$

$$\int_0$$

$$D_{\mu\nu}(k) = -\frac{1}{k^2} \left\{ A(k^2) P_{\mu\nu}^T + \alpha P_{\mu\nu}^L \right\}$$

$$\bar{D}_{\mu\nu}^{-1}(k) = -k^2 \left\{ \frac{1}{A(k^2)} P_{\mu\nu}^T + \frac{1}{\alpha} P_{\mu\nu}^L \right\}$$

$$P_{\mu\nu}^T(k) \equiv \left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right)$$

$$P_{\mu\nu}^L(k) \equiv \frac{k_\mu k_\nu}{k^2}$$

$$(P^T)^2 = P^T$$

$$(p^L)^2 = p^L$$

$$P^L P^T = P^T P^L = 0$$

see page 11 below.

1. First, these facts are self-evident:

$$\begin{cases} W[\gamma, j, j^+] = \int d\phi \, d\phi^\dagger d\phi \, e^{iS} \\ S = S_0 + \int dx dy \, c_a^\dagger(x) M_{ab}(x, y; \phi) c_b(y) + \int dx [c_a^\dagger(x) j_a^+(x) + c_a(x) j_a(x)] \\ S_0 = S(\phi) - \frac{1}{2} F^2 + \int dx \, \delta_i(x) \phi_i(x) \end{cases}$$

$$\Rightarrow \left\langle 0 | T (c_a(x) c_b^\dagger(y)) | 0 \right\rangle^\gamma \equiv i D_{ab}^\gamma(x, y)^\gamma = \left[\frac{(-i)^2 \delta^2 W}{W \cdot \delta j_a(x) \delta j_b^\dagger(y)} \right]_{j=j^+=0}^{\gamma}$$

This object is of course the dressed ghost propagator. As defined here, the sources of the other fields has been left on.

2. On the other hand, we can go backwards and "do" the ghost-field integrations to get:

$$W[\gamma, j, j^+] = \int d\phi \, \det M(\phi) \, e^{iS_0(\phi)} \, e^{-i \int dx dy \, j_b^\dagger(y) \tilde{M}_{ba}^{-1}(y, x) j_a(x)}$$

where we have used this gaussian integral:

$$\int d\phi \, d\phi^\dagger \exp i \int dx [c^\dagger \tilde{M} c + j c + j^\dagger c^\dagger] = (\det M) e^{-i \int dx \, j^\dagger \tilde{M}^{-1} j}$$

Now we can take the second derivative with respect to j to get another expression for the ghost propagator:

$$\begin{aligned} i D_{ab}^\gamma(x, y)^\gamma &= \left[W[\gamma]^{-1} \cdot \int d\phi \, \det M(\phi) \, \tilde{M}_{ba}^{-1}(y, x; \phi) \, e^{iS_0(\phi)} \cdot (-i)^3 \right] \\ &= W[\gamma]^{-1} i \tilde{M}_{ba}^{-1}(y, x; \frac{\delta}{\delta \gamma}) \cdot W[\gamma] \end{aligned}$$

where $W[\gamma] \equiv W[\gamma, 0, 0]$,

$$= \int \left\langle 0 | T (\tilde{M}_{ba}^{-1}(y, x; \phi)) | 0 \right\rangle$$

$$= \text{diagram: a circle with a cross inside, connected to two external lines labeled } ax \text{ and } by$$

Analysis of Ward Identity.

- ① For starters, assume only ghosts + gauge fields, assume covariant gauge.
Then the L25-4 Ward reads:

$$\left\{ -\frac{\partial^\mu}{\sqrt{2}} \left[\frac{1}{i} \frac{\delta}{\delta J_{\mu c}(x)} \right] - \frac{1}{g} \int dy J_{a\mu}(y) \left[\delta_{ab} \partial_\mu - g C_{abd} \left[\frac{1}{i} \frac{\delta}{\delta J_{d\mu}(y)} \right] \right] M_{bc}^{-1}(y, x; \frac{1}{i} \frac{\delta}{\delta J}) \right\} W[J] = 0$$

/covariant gauge, pure Yang Mills

c, x häng.

2. Suppose you take no derivatives, then set $J=0$. The entire second term vanishes since it is linear in J . The first term gives you the VEV of a gauge field all by itself, and this must vanish by symmetry I think. Only scalar fields can have VEV. Thus, you get $0=0$ which is fine.

3. Now take a single gauge derivative: First, repeat the above in D -notation:

$$F_a(A) = \frac{1}{\sqrt{2}} \partial_\mu A_a^\mu \quad \text{to get} \quad -\frac{1}{2\alpha} (\partial_\mu A^\mu)^2$$

$$\Rightarrow \left[-\frac{1}{\sqrt{2}} \partial^\mu \left[\frac{1}{i} \frac{\nabla}{\nabla_{J_{\mu c}(x)}} \right] W[J] \right. \\ \left. = + \frac{1}{g} \int dy J_a^\mu(y) \vec{D}_{ab}^\mu \left(\frac{1}{i} \frac{\delta}{\delta J} \right) M_{bc}^{-1}(y, x; \frac{1}{i} \frac{\delta}{\delta J}) W[J] \right]$$

Apply $\frac{1}{i} \frac{\partial}{\partial J_{\mu d}(y)}$ to get

$$-\frac{1}{\sqrt{2}} \partial^\mu \left[\frac{1}{(i)^2} \frac{\delta^2 W[J]}{\delta J_c^\mu(x) \delta J_d^\nu(y)} \right] \\ = \frac{1}{i} \frac{1}{g} \vec{D}_{db}^\nu(y; \frac{1}{i} \frac{\delta}{\delta J}) M_{bc}^{-1}(y, x; \frac{1}{i} \frac{\delta}{\delta J}) W[J]$$

$$\underbrace{\frac{1}{\sqrt{2}} = \left(-\frac{1}{g\sqrt{2}} \right) \partial_\mu \frac{D^\mu}{\approx}}_{\downarrow}$$

Now apply $\frac{\partial_\nu}{\sqrt{2}}$ to both sides.

$$RHS = \frac{1}{i} \frac{1}{g\sqrt{2}} \partial_\nu \vec{D}_{db}^\nu(y) M_{bc}^{-1}(y, x) W = -\frac{1}{i} \delta_{dc} \delta(y-x) W.$$

Thus we get:

$$\bigcirc \quad -\frac{1}{\alpha} \partial_\nu^{(y)} \partial_\mu^{(x)} \left[\frac{(i)^{+2} \delta^2 W[\mathcal{T}]}{\delta \mathcal{T}_c^\mu(x) \delta \mathcal{T}_d^\nu(y)} \right] = -\frac{\delta_{dc}}{i} \delta(y-x) W[\mathcal{T}].$$

Divide through by $W[\mathcal{T}]$. $[] = G_{cd}^{(2)\mu\nu}(x,y) = i D_{cd}^{\mu\nu}(x,y)$

So result is

$$\frac{i}{\alpha} \partial_\mu^{(x)} \partial_\nu^{(y)} D_{cd}^{\mu\nu}(x,y) = \frac{\delta_{cd}}{i} \delta(x-y)$$

Rewrite:

$$\boxed{\frac{i}{\alpha} \partial_\mu^{(x)} \partial_\nu^{(y)} [i D_{ab}^{\mu\nu}(x,y)] = \delta_{ab} \delta(x-y)}$$

③ Fourier this at once:

$$[i D_{ab}^{\mu\nu}(x,y)] = \int \frac{d^4 p}{(2\pi)^4} e^{-ip \cdot (x-y)} [i D_{ab}^{\mu\nu}(p)]$$

$$\bigcirc \quad \delta(x-y) = \int \frac{d^4 p}{(2\pi)^4} e^{-ip \cdot (x-y)}$$

$$\Rightarrow \frac{1}{\alpha} (-ik_\mu)(+ik_\nu) [i D_{ab}^{\mu\nu}(k)] = \delta_{ab}$$

$$\Rightarrow \boxed{\frac{i}{\alpha} k_\mu k_\nu [i D_{ab}^{\mu\nu}(k)] = \delta_{ab}}$$

④ What is the base prop? If there were no interaction we'd have

$$\begin{aligned} W_0[\mathcal{T}] &= \int [dA] \exp i \left\{ -\frac{1}{2} A_a^\mu [\mathcal{O}_{ab}^{\mu\nu}] A_b^\nu + \mathcal{T}_a^\mu A_a^\mu \right\} dx \\ &= e^{-\frac{i}{2} \int dx dy \mathcal{T}_a^\mu(x) \mathcal{O}_{ab}^{\mu\nu}(x,y) \mathcal{T}_b^\nu(y)} \end{aligned}$$

$\Rightarrow$ see notes on this. Result is

$$\frac{i}{\alpha} k_\mu k_\nu [i D_{ab}^{\mu\nu}(k)] = \frac{\delta_{ab}}{k^2} \left[\frac{\alpha}{k^2} \frac{k^4}{k^2} \right] = \delta_{ab} \checkmark$$

Thus, our first Ward Identity makes a statement about a piece of the gauge propagator. It says that the longitudinal part of the (unrenormalized but dressed) is the same as the long part of the bare propagator: "the long part is not renormalized at all".

This is not as obvious as you would think. In a theory with only gluons and no ghosts you get this result quickly from the self-energy Dyson. Here you would have to include the ghosts in such an analysis. Even in the pure gauge thing you have to assume that the self-energy is purely transverse. I'm not sure why that should be obvious here. But the result stands: it comes right from the generalized Ward so it has to be correct. By the way, I don't think the gauge self-energy is transverse! Eg, in axial gauge I know it is not because there is a nPI equation, etc.

2. Ghost propagator insertion. Let us now rewrite the general Ward replacing the object M^{-1} with the ghost propagator according to separate sheet. Just insert WW^{-1} and use that nice result. You get:

$$-\frac{1}{\sqrt{\alpha}} \partial_\mu^{(*)} \left[\frac{(-i) \delta W(\tau)}{\delta J_\mu^c(x)} \right] = \frac{1}{ig} \int dy J_a^\mu(y) \vec{D}_{ab}(y, \partial_\tau) \left\{ \frac{W[\tau]}{W[\tau]} \cdot \frac{(-i)^2 \delta^2 W[\tau]}{\delta J_c(x) \delta J_b^+(y)} \right\}$$

I have not bothered to show the ghost sources which must be there in the rightmost factor in order for you to differentiate. Once this second derivative is computed, the j and j^+ have been set to zero. Thus, in this entire equation $j=j^+=0$ so you can do no further little j derivatives.

Now let's apply $(-1)d/dJ$ to both sides again, then set $J=0$:

$$-\frac{1}{\sqrt{\alpha}} \partial_\mu^{(*)} \left[\frac{(-i)^2 \delta^2 W[\tau]}{\delta J_\mu^c(x) \delta J_\nu^d(y)} \right] = -\frac{1}{g} \vec{D}_{ab}(y, (-i) \frac{\partial}{\partial \tau}) \left\{ \frac{W[\tau]}{W[\tau]} \cdot \frac{(-i)^2 \delta^2 W[\tau]}{\delta J_c(x) \delta J_b^+(y)} \right\}$$

Except for the little transversality restriction of renormalization deal above, I think now there is no purpose to keeping the pieces of D together, so let's go back to the start and write things out in detail:

$$\begin{aligned} -\frac{1}{\sqrt{\alpha}} \partial_\mu^{(*)} \left[\frac{(-i) \delta W}{\delta J_\mu^c(x)} \right] &= \frac{1}{ig} \int dy J_a^\mu(y) \partial_\mu^{(*)} \left[\frac{(-i)^2 \delta^2 W}{\delta J_c(x) \delta J_b^+(y)} \right] \\ &+ i \int dy J_a^\mu(y) C_{ab}^f \left[\frac{(-i)^3 \delta^3 W}{\delta J_a^\mu(y) \delta J_c(x) \delta J_b^+(y)} \right]. \end{aligned}$$

3. Simple Graph Notations.

solid line = ghost (bar indicates the c^+ field)

squiggle line = gauge particle (maybe any other particle)

ball = $W[J]$

attachment = $(-i) d/dJ$

standard label order": Lorentz, color, space

In this notation, the original Ward reads as follows:

$$-\frac{1}{\sqrt{\alpha}} \partial_\mu^{(x)} \text{ (ghost line with ball) }^{\mu c x} = \int dy [i \bar{\mathcal{D}}_a^\mu(y)] \left\{ -\frac{1}{g} \partial_\mu^{(y)} \text{ (ghost line) }_{a y}^{c x} + C_{ab}^f \text{ (ghost line) }_{b y}^{c x} \right\}$$

Ward Identity

If no SSB, obvious how satisfied if you let $J=0$. Now take one derivative to get:

$$\begin{aligned} -\frac{1}{\sqrt{\alpha}} \partial_\mu^{(x)} \text{ (ghost line with ball) }_{\nu d z}^{\mu c x} &= \int dy [i \bar{\mathcal{D}}_a^\mu(y)] \left\{ -\frac{1}{g} \partial_\mu^{(y)} \text{ (ghost line) }_{a y}^{c x} \nu d z + C_{ab}^f \text{ (ghost line) }_{b y}^{c x} \nu d z \right\} \\ &+ \left\{ -\frac{1}{g} \partial_\nu^{(z)} \text{ (ghost line) }_{d z}^{c x} + C_{db}^f \text{ (ghost line) }_{b z}^{c x} \nu f z \right\} \end{aligned}$$

If we let $J=0$ and divide through by W (first), we get a relation between some objects:

$$-\frac{1}{\sqrt{\alpha}} \partial_\mu^{(x)} \left[\text{ghost line with ball} \right]_{\nu d z}^{\mu c x} = -\frac{1}{g} \partial_\nu^{(z)} \left[\text{ghost line with ball} \right]_{d z}^{c x} + C_{db}^f \left[\text{ghost line with ball} \right]_{b z}^{c x} \nu f z$$

first-level Ward.

More Detail on the Level-1 Ward Identity. As noted on the last page, the identity reads:

$$-\frac{1}{\sqrt{\alpha}} \partial_{(x_1)}^{\mu_1} [i D_{\mu_1 \mu_2}^{a_1 a_2}(x_1, x_2)] = -\frac{1}{g} \partial_{(x_2)}^{\mu_2} [i D^{g, a_1 a_2}(x_1, x_2)] + C_{a_2 b_2 c_2} G_{\mu_2}^{a_1 c_2 b_2}(x_1, x_2, x_2)$$

Repeat:

$$-\frac{i}{\sqrt{\alpha}} \delta_{a_1 a_2} \partial_{(x_1)}^{\mu_1} D_{\mu_1 \mu_2}(x_1, x_2) = -\frac{i}{g} \partial_{(x_2)}^{\mu_2} D^g(x_1, x_2) \delta_{a_1 a_2} + C_{a_2 b_2 c_2} G_{\mu_2}^{a_1 c_2 b_2}(x_1, x_2, x_2)$$

First-level Ward Identity, Coordinate Space Representation

The Fourier on this is straightforward and gives:

$$-\frac{i}{\sqrt{\alpha}} \delta_{a_1 a_2} [-i k_1^{\mu_1}] D_{\mu_1 \mu_2}(k_1) = -\frac{i}{g} \delta_{a_1 a_2} [+i k_1^{\mu_2}] D^g(k_1) + C_{a_2 b_2 c_2} \int \frac{dk_2'}{(2\pi)^4} G_{\mu_2}^{a_1 c_2 b_2}(k_1, k_2', -)$$

$$-\frac{1}{\sqrt{\alpha}} \delta_{a_1 a_2} k_1^{\mu_1} D_{\mu_1 \mu_2}(k_1) = +\frac{1}{g} \delta_{a_1 a_2} k_1^{\mu_2} D^g(k_1) + C_{a_2 b_2 c_2} \int \frac{dk_2'}{(2\pi)^4} G_{\mu_2}^{a_1 c_2 b_2}(k_1, k_2', -)$$

First-level Ward Identity, Momentum Space Representation.

When we later apply another k to this thing and use the first-level M identity, we learn something about the longitudinal part of the gauge propagator. Namely, we learn that:

$$k_1^{\mu_1} D_{\mu_1 \mu_2}(k_1) = -\alpha \left(\frac{k_1^{\mu_2}}{k_1^2} \right)$$

$$+\frac{1}{g} \delta_{ad} k^\mu D^g(k) = +\sqrt{\alpha} \delta_{ad} \left(\frac{k^\mu}{k^2} \right) - C_{bcd} \int \frac{dk'}{(2\pi)^4} G_{\mu}^{acb}(k, k', -)$$

First-level Ward Identity, Momentum Space

If I dot this thing into k^u , I get the First-level M-identity, but this above is a more powerful result!

4. Triple Gauge derivative. Now lets apply a second d/dJ to our once-diff'd Ward and see what happens:

$$-\frac{1}{\sqrt{2}} \partial_\mu^{(x)} \text{ (diagram)} = \int dy [i J_a^\Lambda(y)] \left\{ -\frac{1}{g} \partial_\mu^{(y)} \text{ (diagram)} + C_{ab}^f \text{ (diagram)} \right\} \\ + \left\{ -\frac{1}{g} \partial_\lambda^{(w)} \text{ (diagram)} + C_{eb}^f \text{ (diagram)} \right\} + \left\{ -\frac{1}{g} \partial_\nu^{(z)} \text{ (diagram)} + C_{db}^f \text{ (diagram)} \right\}$$

Now set $J=0$ and write result compactly as:

$$-\frac{1}{\sqrt{2}} \partial_\mu^{(x)} \text{ (diagram)} = \left\{ -\frac{1}{g} \partial_\nu^{(z)} \text{ (diagram)} + C_{db}^f \text{ (diagram)} \right\} + \left\{ (\nu dz) \leftrightarrow (\lambda ew) \right\}$$

Rename:

$$-\frac{1}{\sqrt{2}} \partial_{\mu_1}^{(x_1)} \text{ (diagram)} = \left\{ -\frac{1}{g} \partial_{\mu_3}^{(x_3)} \text{ (diagram)} + C_{a_3 b}^f \text{ (diagram)} \right\} \\ + \left\{ 2 \leftrightarrow 3 \right\}.$$

second-level ward.

Note how the 2,3 symmetry is automatic. As was always claimed, the Ward relates various kinds of greens functions. Lets duplicate here the previous one:

$$-\frac{1}{\sqrt{2}} \partial_{\mu_1}^{(x_1)} \text{ (diagram)} = -\frac{1}{g} \partial_{\mu_2}^{(x_2)} \text{ (diagram)} + C_{a_2 b}^f \text{ (diagram)}$$

first-level ward.

You can see how these things are going to look. There is a tendency of the Ward to turn gluons into ghosts, and to mix in the $n+1$ point function with the n pointers. Rather messy. I suspect you have to use the Dysons to get these into standard form??

Review of the lowest level Ward Identities in good notation.

(0) Level Zero

$$-\frac{1}{\sqrt{\alpha}} \partial_{(x_1)}^{\mu_1} \begin{array}{c} \mu_1 a_1 x_1 \\ \circ \end{array} = \int dx_2 [i \mathcal{V}_{a_2}^{\mu_2}(x_2)]$$

$$\times \left\{ -\frac{1}{g} \partial_{(x_2)}^{\mu_2} \begin{array}{c} a_1 x_1 \\ \circ \\ a_2 x_2 \end{array} + C_{a_2 b_2 c_2} \begin{array}{c} a_1 x_1 \\ \circ \\ b_2 x_2 \quad \mu_2 c_2 x_2 \end{array} \right\}$$

(1) Level One

$$-\frac{1}{\sqrt{\alpha}} \partial_{(x_1)}^{\mu_1} \begin{array}{c} \mu_1 a_1 x_1 \\ \circ \quad \mu_2 a_2 x_2 \end{array} = \left\{ -\frac{1}{g} \partial_{(x_2)}^{\mu_2} \begin{array}{c} a_1 x_1 \\ \circ \\ a_2 x_2 \end{array} + C_{a_2 b_2 c_2} \begin{array}{c} a_1 x_1 \\ \circ \\ b_2 x_2 \quad \mu_2 c_2 x_2 \end{array} \right\}$$

$$+ \int dx_3 [i \mathcal{V}_{a_3}^{\mu_3}(x_3)] \left\{ -\frac{1}{g} \partial_{(x_3)}^{\mu_3} \begin{array}{c} a_1 x_1 \\ \circ \quad \mu_2 a_2 x_2 \\ a_3 x_3 \end{array} + C_{a_3 b_3 c_3} \begin{array}{c} a_1 x_1 \\ \circ \quad \mu_2 a_2 x_2 \\ b_3 x_3 \quad \mu_3 c_3 x_3 \end{array} \right\}$$

(2) Level Two

$$-\frac{1}{\sqrt{\alpha}} \partial_{(x_1)}^{\mu_1} \begin{array}{c} \mu_1 a_1 x_1 \\ \circ \quad \mu_2 a_2 x_2 \\ \mu_3 a_3 x_3 \end{array} = \left\{ -\frac{1}{g} \partial_{(x_2)}^{\mu_2} \begin{array}{c} a_1 x_1 \\ \circ \quad \mu_3 a_3 x_3 \\ a_2 x_2 \end{array} + C_{a_2 b_2 c_2} \begin{array}{c} a_1 x_1 \\ \circ \quad \mu_3 a_3 x_3 \\ b_2 x_2 \quad \mu_2 c_2 x_2 \end{array} \right\}$$

$$+ \{2 \leftrightarrow 3\}$$

$$+ \int dx_4 [i \mathcal{V}_{a_4}^{\mu_4}(x_4)] \left\{ -\frac{1}{g} \partial_{(x_4)}^{\mu_4} \begin{array}{c} a_1 x_1 \\ \circ \quad \mu_3 a_3 x_3 \\ a_4 x_4 \end{array} + C_{a_4 b_4 c_4} \begin{array}{c} a_1 x_1 \\ \circ \quad \mu_3 a_3 x_3 \\ b_4 x_4 \quad \mu_4 c_4 x_4 \end{array} \right\}$$

The idea is this: each time index the subscript on the J integration labels. Also, push down former gauge lines to make room for the new ones. One more for fun:

(2) Level Three

$$\begin{aligned}
 & -\frac{1}{\sqrt{\alpha}} \partial_{(x_1)}^{\mu_1} \left(\text{diagram with } \mu_1 a_1 x_1, \mu_4 a_4 x_4, \mu_3 a_3 x_3, \mu_2 a_2 x_2 \right) = \left\{ -\frac{1}{g} \partial_{(x_1)}^{\mu_2} \left(\text{diagram with } a_1 x_1, \mu_4 a_4 x_4, \mu_3 a_3 x_3, a_2 x_2 \right) + C_{a_2 b_2 c_2} \left(\text{diagram with } a_1 x_1, \mu_4 a_4 x_4, \mu_3 a_3 x_3, b_2 x_2, \mu_2 c_2 x_2 \right) \right\} \\
 & + \{ 2 \leftrightarrow 3 \} + \{ 2 \leftrightarrow 4 \} \\
 & + \int dx_5 [i T_{a_5}^{\mu_5}(x_5)] \left\{ -\frac{1}{g} \partial_{(x_5)}^{\mu_5} \left(\text{diagram with } a_1 x_1, \mu_4 a_4 x_4, \mu_3 a_3 x_3, \mu_2 a_2 x_2, a_5 x_5 \right) + C_{a_5 b_5 c_5} \left(\text{diagram with } a_1 x_1, \mu_4 a_4 x_4, \mu_3 a_3 x_3, \mu_2 a_2 x_2, b_5 x_5, \mu_5 c_5 x_5 \right) \right\}
 \end{aligned}$$

Note how the symmetrization comes out at each level: you need six terms to be completely symmetric in gluons 2,3,4, but the terms in curly braces are already 3-4 symmetric so only need to add the two terms shown. I think I can see the level n result:

(N) Level (n-1)

$$\begin{aligned}
 & -\frac{1}{\sqrt{\alpha}} \partial_{(x_1)}^{\mu_1} \left(\text{diagram with } \mu_1 a_1 x_1, \mu_n a_n x_n, \mu_3 a_3 x_3, \mu_2 a_2 x_2 \right) = \left\{ -\frac{1}{g} \partial_{(x_1)}^{\mu_2} \left(\text{diagram with } a_1 x_1, \mu_n a_n x_n, \mu_4 a_4 x_4, \mu_3 a_3 x_3, a_2 x_2 \right) + C_{a_2 b_2 c_2} \left(\text{diagram with } a_1 x_1, \mu_n a_n x_n, \mu_3 a_3 x_3, b_2 x_2, \mu_2 c_2 x_2 \right) \right\} \\
 & + \{ 2 \leftrightarrow 3 \} + \{ 2 \leftrightarrow 4 \} + \{ 2 \leftrightarrow 5 \} + \dots + \{ 2 \leftrightarrow n \} \\
 & + \int dx_{n+1} [i T_{a_{n+1}}^{\mu_{n+1}}(x_{n+1})] \left\{ \text{no need to write out} \right\}.
 \end{aligned}$$

Review of the M identities.

(1) Level One M

$$\partial_{(x_1)}^{\mu_2} \left\{ -\frac{1}{g} \partial_{(x_2)}^{\mu_2} \begin{array}{c} a_1 x_1 \\ | \\ \bigcirc \\ | \\ a_2 x_2 \end{array} + C_{a_2 b_2 c_2} \begin{array}{c} a_1 x_1 \\ | \\ \bigcirc \\ | \\ b_2 x_2 \end{array} \mu_2 c_2 x_2 \right\} = i\sqrt{\alpha} \delta(x_1 - x_2) \delta_{a_1 a_2} \bigcirc$$

(2) Level Two M

$$\partial_{(x_2)}^{\mu_2} \left\{ -\frac{1}{g} \partial_{(x_2)}^{\mu_2} \begin{array}{c} a_1 x_1 \\ | \\ \bigcirc \\ | \\ a_2 x_2 \end{array} \mu_3 a_3 x_3 + C_{a_2 b_2 c_2} \begin{array}{c} a_1 x_1 \\ | \\ \bigcirc \\ | \\ b_2 x_2 \end{array} \mu_3 a_3 x_3 \mu_2 c_2 x_2 \right\} = i\sqrt{\alpha} \delta(x_1 - x_2) \delta_{a_1 a_2} \begin{array}{c} \mu_3 a_3 x_3 \\ | \\ \bigcirc \end{array}$$

These are obtained by simply injecting. General one is this:

(n) Level (n-1)

$$\partial_{(x_2)}^{\mu_2} \left\{ -\frac{1}{g} \partial_{(x_2)}^{\mu_2} \begin{array}{c} a_1 x_1 \\ | \\ \bigcirc \\ | \\ a_2 x_2 \end{array} \mu_n a_n x_n \mu_3 a_3 x_3 + C_{a_2 b_2 c_2} \begin{array}{c} a_1 x_1 \\ | \\ \bigcirc \\ | \\ b_2 x_2 \end{array} \mu_n a_n x_n \mu_3 a_3 x_3 \mu_2 c_2 x_2 \right\} = i\sqrt{\alpha} \delta(x_1 - x_2) \delta_{a_1 a_2} \begin{array}{c} \mu_n a_n x_n \\ | \\ \bigcirc \\ | \\ \mu_3 a_3 x_3 \end{array}$$

Higher Derivatives of Ward Identities making use of the M identities.

Comment: Now is the time to introduce ultracompact notation so things can be written in finite amount of time. I'll bet I will find something like this in the literature.

$$1. \quad \partial_1 \equiv \partial_{(x_1)}^{\mu_1}$$

$$2. \quad \text{Diagram 1} = \text{Diagram 2} \equiv G_{123\dots n}$$

Diagram 1: A circle with n external lines. The first line is labeled with μ_1, a_1, x_1 . The second line is labeled with μ_2, a_2, x_2 . The third line is labeled with μ_3, a_3, x_3 . The fourth line is labeled with μ_n, a_n, x_n .

Diagram 2: A circle with n external lines. The first line is labeled with μ_1, a_1, x_1 . The second line is labeled with μ_2, a_2, x_2 . The third line is labeled with μ_3, a_3, x_3 . The fourth line is labeled with μ_n, a_n, x_n .

$$3. \quad F_{1234\dots n} \equiv \left\{ -\frac{1}{g} \partial_{(x_2)}^{\mu_2} \text{Diagram 3} + C_{a_2 b_2} \text{Diagram 4} \right\} \times (-\sqrt{\alpha})$$

Diagram 3: A circle with n external lines. The first line is labeled with μ_1, a_1, x_1 . The second line is labeled with μ_2, a_2, x_2 . The third line is labeled with μ_3, a_3, x_3 . The fourth line is labeled with μ_n, a_n, x_n .

Diagram 4: A circle with n external lines. The first line is labeled with μ_1, a_1, x_1 . The second line is labeled with μ_2, a_2, x_2 . The third line is labeled with μ_3, a_3, x_3 . The fourth line is labeled with μ_n, a_n, x_n .

$$4. \quad H_{1234\dots n} \equiv i\sqrt{\alpha} \delta(x_1 - x_2) \delta_{a_1 a_2} G_{345\dots n} \times (-\sqrt{\alpha})$$

$$\left(\begin{array}{l} H_{12} = -i\alpha \delta(x_1 - x_2) \delta_{a_1 a_2} \cdot 1 \\ H_{123} = 0 \end{array} \right)$$

5. P_{ij} exchanges $i \leftrightarrow j$ in quantity lying to the right.

In this compact notation we can write the level n-1 identities like so:

Level (n-1) Ward Identity

$$\partial_1 G_{12\dots n} = [1 + P_{23} + P_{24} + \dots + P_{2n}] F_{12\dots n}$$

Level (n-1) M Identity

$$\partial_2 F_{12\dots n} = H_{12\dots n}$$

The compaction factor is rather impressive, I think. Now I can start applying derivatives on the first Identity and make use of the second like so:

$$\partial_1 G_{12..n} = [1 + P_{23} + P_{24} + P_{25} + \dots + P_{2n}] F_{12..n}$$

$$\partial_2 \partial_1 G_{12..n} = \partial_2 [P_{23} + P_{24} + P_{25} + \dots + P_{2n}] F_{12..n} + H_{12..n}$$

$$\partial_3 \partial_2 \partial_1 G_{12..n} = \partial_3 \partial_2 [P_{24} + P_{25} + \dots + P_{2n}] F_{12..n} + [1 + P_{23}] \partial_3 H_{12..n}$$

$$\partial_4 \partial_3 \partial_2 \partial_1 G_{12..n} = \partial_4 \partial_3 \partial_2 [P_{25} + \dots + P_{2n}] F_{12..n} + [1 + P_{23} + P_{24}] \partial_4 \partial_3 H_{12..n}$$

These things are trivial to show, using compact identities on last page. As you move down you see that, in effect, a term is transported from the F listing to the H listing via the M identity. Here all terms except one are transferred:

$$\begin{aligned} \partial_1 \partial_2 \dots \partial_{(n-1)} G_{12..n} &= \partial_2 \partial_3 \dots \partial_{(n-1)} P_{2n} F_{12..n} \\ &+ [1 + P_{23} + P_{24} + \dots + P_{2,(n-1)}] \partial_3 \partial_4 \dots \partial_{(n-1)} H_{12..n} \end{aligned}$$

And finally, here is the full result with all terms transported over:

$$\partial_1 \partial_2 \dots \partial_n G_{12..n} = [1 + P_{23} + P_{24} + \dots + P_{2n}] \partial_3 \partial_4 \dots \partial_n H_{12..n}$$

Example 1: Write the preceding result in case $n=2$:

$$\partial_1 \partial_2 G_{12} = H_{12} = -i\alpha \delta(x_1 - x_2) \delta_{a_1 a_2} \cdot 1$$

This agrees with the same result found earlier, see page 7 red box (longitudinal prop deal).

Example 2: Try case $n=3$

$$\partial_1 \partial_2 \partial_3 G_{123} = [1 + P_{23}] \partial_3 \widehat{H_{123}} = 0$$

Example 3: try the case $n=4$

$$\partial_1 \partial_2 \partial_3 \partial_4 G_{1234} = [1 + P_{23} + P_{24}] \partial_3 \partial_4 H_{1234}$$

I can see that this will lead to the famous Ward relating the proper 4-point to lower objects. Return later to this.

Example 4: Take only two derivatives on a 3-point function to get:

$$\begin{aligned} \partial_1 \partial_2 G_{123} &= \partial_2 P_{23} F_{123} + H_{123} \\ &= \partial_2 F_{132} \end{aligned}$$

This result appears on top of page 9 and is the first step on the long journey of deriving the Slavnov Identity.

Comment: You can see that, unless the full complement of derivatives is applied, you are going to have some F terms, which is to say, you are going to involve ghosts.

Fourier Transformation of the Above Results.

I think it will be simpler to not expose the overall momentum conservation delta. Then we will use this direct fourier transformation:

$$f(x_1, x_2, \dots, x_n) = (2\pi)^{-4n} \int dk_1 \dots dk_n e^{-ik_1 x_1} \dots e^{-ik_n x_n} F(k_1, k_2, \dots, k_n)$$

With this fourier convention, the rule for momentum space rep of a derivative is clearly:

$$\partial_{x_1}^{\mu_1} \rightarrow [-i k_1^{\mu_1}]$$

Now lets examine the transforms of our basic objects. The object $F_{12\dots n}$ is a little tricky because two lines have the same space label x_2 . Consider the two terms in F as shown on page 5d. Expand them like this:

$$F_1(x_1, x_2, x_3 \dots x_n) = \left[(2\pi)^{-4n} \int dk_1 e^{-ik_1 x_1} \dots \int dk_n e^{-ik_n x_n} \right] F_1(k_1, k_2, k_3 \dots k_n)$$

$$F_2(x_1, x_2, \overset{\downarrow}{x_2}, x_3, \dots, x_n) = \left[\overset{\downarrow}{\int} \right] \int \frac{dk'_2}{(2\pi)^4} e^{-ik'_2 x_2} F_2(k_1, k_2, k'_2, k_3, \dots, k_n)$$

The second term really is an $(n+1)$ point function and has that many arguments. If just has an extra integration. So I might define fourier transformed F like this:

$$\tilde{F}_{12\dots n} = \left\{ + \frac{1}{g} i k_2^{\mu_2} \begin{array}{c} a_1 k_1 \\ \swarrow \mu_n a_n k_n \\ \circlearrowleft \\ \searrow \mu_3 a_3 k_3 \\ a_2 k_2 \end{array} + c_2 b_2 c_2 \int \frac{dk'_2}{(2\pi)^4} e^{-ik'_2 x_2} \begin{array}{c} a_1 k_1 \\ \swarrow \mu_n a_n k_n \\ \circlearrowleft \\ \searrow \mu_3 a_3 k_3 \\ b_2 k_2 \quad \mu_2 c_2 k'_2 \end{array} \right\}$$

$$\tilde{G}_{12\dots n} = \begin{array}{c} \mu_1 a_1 k_1 \\ \swarrow \\ \circlearrowleft \\ \searrow \mu_n a_n k_n \end{array}$$

All these results are obtained just by getting the coefficient of the n -fold integration.

$$\tilde{H}_{12\dots n} = (2\pi)^4 \delta(k_1 + k_2) \tilde{G}_{34\dots n} \times (-i\alpha \delta_{a_1 a_2})$$

$$(\tilde{H}_{12} = (2\pi)^4 \delta(k_1 + k_2) (-i\alpha \delta_{a_1 a_2}))$$

$$(\tilde{H}_{123} = 0)$$

Now it is completely trivial to express all our previous identities in terms of fourier objects. Just put twiddles over things and replace derivatives according to the rule, that is all there is to it.

Example 1: two derivatives on propagator:

$$\boxed{(-i)^2 k_1 k_2 \tilde{G}_{12} = \tilde{H}_{12}}$$

$$-k_1^{\mu_1} k_2^{\mu_2} [(2\pi)^4 \delta(k_1+k_2) i D_{\mu_1 \mu_2}^{a_1 a_2}(k_1)] = (2\pi)^4 \delta(k_1+k_2) [-i \alpha \delta_{a_1 a_2}]$$

$$\Rightarrow \boxed{k_1^{\mu_1} k_1^{\mu_2} D_{\mu_1 \mu_2}(k_1) = -\alpha}$$

Here I have just elaborated the identity.

Example 2: single derivative on propagator.

$$-i k_1 \tilde{G}_{12} = \tilde{F}_{12}$$

This is what i have called the First-Level Ward identity. However, we have information about $k_1 G_{12}$ from an entirely different source: tensor forms. Write:

$$-i k_1^{\mu_1} (2\pi)^4 \delta(k_1+k_2) i D_{\mu_1 \mu_2}^{a_1 a_2}(k_1) = -i k_1 \tilde{G}_{12}$$

$$= -i (2\pi)^4 \delta(k_1+k_2) i \delta_{a_1 a_2} [k_1^{\mu_1} D_{\mu_1 \mu_2}(k_1)]$$

But $D = D^T + D^L$. We know that $k_1^{\mu_1} k_1^{\mu_2} D_{\mu_1 \mu_2}^L(k_1) = -\alpha$

$$\Rightarrow D_{\mu_1 \mu_2}^L(k_1) = -\alpha \frac{k_1^{\mu_1} k_1^{\mu_2}}{k_1^2}$$

$$\text{Then } k_1^{\mu_1} D = k_1^{\mu_1} D^L = -\alpha \left(\frac{k_1^{\mu_2}}{k_1^2} \right) \Rightarrow -i k_1 \tilde{G}_{12} = (2\pi)^4 \delta(k_1+k_2) \delta_{a_1 a_2} \left[-\alpha \left(\frac{k_1^{\mu_2}}{k_1^2} \right) \right]$$

So

$$k_1^{\mu_1} D_{\mu_1 \mu_2}(k_1) = -\alpha \left(\frac{k_1^{\mu_2}}{k_1^2} \right)$$

$$\boxed{-i k_1 \tilde{G}_{12} = -\alpha \left(\frac{k_1^{\mu_2}}{k_1^2} \right) (2\pi)^4 \delta(k_1+k_2) \delta_{a_1 a_2}}$$

5. The M^{-1} identity. We have been in effect replacing M^{-1} by the ghost propagator. But we must keep in mind that M^{-1} is also the inverse of M and this implies something about the ghost propagator. Namely, we know this:

$$\underbrace{\left(-\frac{1}{g\sqrt{\alpha}}\right) \partial_\mu^{(y)} D_{ab}^\mu(y, \partial_\tau) M_{bc}^{-1}(y, x; \partial_\tau)}_{\tilde{M}_{ab}(y, \partial_\tau)} = \delta(y-x) \delta_{ac}$$

$$\Rightarrow \left(-\frac{1}{ig\sqrt{\alpha}}\right) \partial_\mu^{(y)} D_{ab}^\mu(y, \partial_\tau) W[\tau] \underbrace{\left(W[\tau] i M_{bc}^{-1}(y, x; \partial_\tau) W[\tau]\right)}_{i D_{cb}^g(x, y)^\tau} = \delta(y-x) \delta_{ac} W[\tau]$$

$$i D_{cb}^g(x, y)^\tau = \left[\frac{(-i)^2 \delta^2 W}{W \cdot \delta j_c(x) \delta j_b^+(y)} \right]$$

$$\Rightarrow \left(-\frac{1}{ig\sqrt{\alpha}}\right) \partial_\mu^{(y)} \left[\delta_{ab} \partial_\mu^{(y)} - g C_{ab}^f (-i) \delta j_\mu^f \right] \left[\frac{(-i)^2 \delta^2 W}{\delta j_c(x) \delta j_b^+(y)} \right] = \delta(y-x) \delta_{ac} W.$$

$$\Rightarrow -\frac{1}{g} \square_y^2 \left[\frac{(-i)^2 \delta^2 W}{\delta j_c \delta j_a^+} \right] + \partial_\mu^{(y)} C_{ab}^f \left[\frac{(-i)^3 \delta^3 W}{\delta \partial_\mu^f \delta j_c(x) \delta j_b^+(y)} \right] = i\sqrt{\alpha} \delta(y-x) \delta_{ac} W.$$

$$\Rightarrow \left[-\frac{1}{g} \partial_{(y)}^2 \text{ (circle with } \overset{ax}{\uparrow} \text{ and } \underset{ay}{\downarrow}) } + C_{ab}^f \partial_{(y)}^\mu \text{ (circle with } \overset{ax}{\uparrow} \text{ and } \underset{by}{\downarrow} \text{, wavy line } \mu y \text{)} \right] = i\sqrt{\alpha} \delta(y-x) \delta_{ac} \quad \bigcirc$$

Of course you can differentiate this to get a family of identities. These identities can then be combined with the raw Ward identities to get simpler results, I bet.

6. Combine Lowest Ward with M-identity. Take the lowest Ward and differentiate it:

$$-\frac{1}{i\alpha} \partial_{\mu_2}^{(x_2)} \partial_{\mu_1}^{(x_1)} \text{ (diagram: circle with } \mu_1 a_1 x_1 \text{ and } \mu_2 a_2 x_2 \text{)} = \underbrace{-\frac{1}{g} \Box_{(x_2)} \text{ (diagram: circle with } a_1 x_1 \text{ and } a_2 x_2 \text{)} + C_{a_2 b}^f \partial_{\mu_2}^{(x_1)} \text{ (diagram: circle with } a_1 x_1 \text{ and } b x_2 \text{)}}_{= i\alpha \delta(x_1 - x_2) \delta_{a_1 a_2}}$$

$$\Rightarrow \partial_{\mu_1}^{(x_1)} \partial_{\mu_2}^{(x_2)} \text{ (diagram: circle with } \mu_1 a_1 x_1 \text{ and } \mu_2 a_2 x_2 \text{)} = -i\alpha \delta(x_1 - x_2) \delta_{a_1 a_2}$$

As before, we can fourier transform in obvious way. Rule is that $d_u = -ik_u$ for incoming momentum, so we get:

$$k_{\mu_1} k_{\mu_2} \text{ (diagram: circle with } \mu_1 a_1 k \text{ and } \mu_2 a_2 (-k) \text{)} = -i\alpha \delta_{a_1 a_2}$$

In momentum space we can express the gluon propagator like so:

$$\text{ (diagram: circle with } \mu_1 a_1 k \text{ and } \mu_2 a_2 (-k) \text{)} = i D_{a_1 a_2}^{\mu_1 \mu_2}(k) = -i \frac{\delta_{a_1 a_2}}{k^2} \left\{ A(k^2) \left(\delta_{\mu_1 \mu_2} - \frac{k_{\mu_1} k_{\mu_2}}{k^2} \right) + B(k^2) \left(\frac{k_{\mu_1} k_{\mu_2}}{k^2} \right) \right\}$$

Ie, there is only one possible color tensor, and two possible Lorentz tensors, so there are two scalar functions possible in the propagator. The above identity then tells us that:

$$-i \delta_{a_1 a_2} B(k^2) = -i \delta_{a_1 a_2} \alpha \Rightarrow B(k^2) = \alpha$$

But we know that $A(k^2) = 1$ and $B(k^2) = \alpha$.

Thus, our Ward identity tells us that, although we know nothing about $A(k^2)$, the object B does not get shifted at all from its bare value!

7. Restate the M identity and ascend one level. It is this:

$$-\frac{1}{g} \partial_{(x_2)}^2 \text{ (diagram: circle with } a_1 x_1 \text{ on top, } a_2 x_2 \text{ on bottom)} + C_{a_2 b_2 c_2} \partial_{(x_2)}^{\mu_2} \text{ (diagram: circle with } a_1 x_1 \text{ on top, } b_2 x_2 \text{ on bottom-left, } \mu_2 c_2 x_2 \text{ on bottom-right)} = i\sqrt{\alpha} \delta(x_1 - x_2) \delta_{a_1 a_2} \text{ (diagram: circle with } \mu_3 a_3 x_3 \text{ on right)}$$

Therefore, apply a derivative to get:

$$-\frac{1}{g} \partial_{(x_2)}^2 \text{ (diagram: circle with } a_1 x_1 \text{ on top, } a_2 x_2 \text{ on bottom, } \mu_3 a_3 x_3 \text{ on right)} + C_{a_2 b_2 c_2} \partial_{(x_2)}^{\mu_2} \text{ (diagram: circle with } a_1 x_1 \text{ on top, } b_2 x_2 \text{ on bottom-left, } \mu_2 c_2 x_2 \text{ on bottom-right, } \mu_3 a_3 x_3 \text{ on right)} = i\sqrt{\alpha} \delta(x_1 - x_2) \delta_{a_1 a_2} \text{ (diagram: circle with } \mu_3 a_3 x_3 \text{ on right)}$$

Now go to $J=0$ with these and divide by W to get:

$$-\frac{1}{g} \partial_{(x_2)}^2 \text{ (diagram: circle with } a_1 x_1 \text{ on top, } a_2 x_2 \text{ on bottom, } \mu_3 a_3 x_3 \text{ on right)} + C_{a_2 b_2 c_2} \partial_{(x_2)}^{\mu_2} \text{ (diagram: circle with } a_1 x_1 \text{ on top, } b_2 x_2 \text{ on bottom-left, } \mu_2 c_2 x_2 \text{ on bottom-right, } \mu_3 a_3 x_3 \text{ on right)} = 0$$

Second-level M-identity.

This identity makes a direct relation between the two greens functions shown.

8. Triple Gauge Identity. Now rewrite the triple gauge identity with 2,3 switched:

$$-\frac{1}{\sqrt{\alpha}} \partial_{\mu_1}^{(x_1)} \text{ (diagram: circle with } \mu_1 a_1 x_1 \text{ on top-left, } \mu_3 a_3 x_3 \text{ on top-right, } \mu_2 a_2 x_2 \text{ on bottom)} = \left\{ -\frac{1}{g} \partial_{(x_2)}^{\mu_2} \text{ (diagram: circle with } a_1 x_1 \text{ on top, } a_2 x_2 \text{ on bottom, } \mu_3 a_3 x_3 \text{ on right)} + C_{a_2 b_2 c_2} \partial_{(x_2)}^{\mu_2} \text{ (diagram: circle with } a_1 x_1 \text{ on top, } b_2 x_2 \text{ on bottom-left, } \mu_2 c_2 x_2 \text{ on bottom-right, } \mu_3 a_3 x_3 \text{ on right)} \right\} + \left\{ -\frac{1}{g} \partial_{(x_3)}^{\mu_3} \text{ (diagram: circle with } a_1 x_1 \text{ on top, } a_3 x_3 \text{ on bottom, } \mu_2 a_2 x_2 \text{ on right)} + C_{a_3 b_3 c_3} \partial_{(x_3)}^{\mu_3} \text{ (diagram: circle with } a_1 x_1 \text{ on top, } b_3 x_3 \text{ on bottom-left, } \mu_3 c_3 x_3 \text{ on bottom-right, } \mu_2 a_2 x_2 \text{ on right)} \right\}$$

Apply $d_{u_2}^{(x_2)}$ to both sides. The first curly bracket will then vanish by the M identity above leaving the following result:

Summary of M-identities of the first two levels:

1. First, here they are in coordinate space:

$$-\frac{1}{g} \square_{(x_2)}^2 \left[i D_{a_1 a_2}^{gh.}(x_1, x_2) \right] + C_{a_2 b_2 c_2} \partial_{(x_2)}^{\mu_2} \left[G_{\mu_2}^{a_1 c_2 b_2}(x_1, x_2, x_2) \right] = i \sqrt{\alpha} \delta(x_1 - x_2) \delta_{a_1 a_2}$$

first level

$$-\frac{1}{g} \square_{(x_2)}^2 \left[G_{\mu_3}^{a_1 a_2 a_3}(x_1, x_2, x_3) \right] + C_{a_2 b_2 c_2} \partial_{(x_2)}^{\mu_2} \left[G_{\mu_2 \mu_3}^{a_1 c_2 b_2 a_3}(x_1, x_2, x_2, x_3) \right] = 0$$

second level

2. Fourier transformation requires some care because space indices are pinned at same value. For example,

$$\delta(x_1 - x_2) = \int \frac{dp_1}{(2\pi)^4} e^{-i k_1 \cdot (x_1 - x_2)} \{ 1 \} \quad / \text{no problem.}$$

$$\begin{aligned} \partial_{(x_2)}^{\mu_2} \left[G_{\mu_2}^{a_1 c_2 b_2}(x_1, x_2, x_2) \right] &= \partial_{(x_2)}^{\mu_2} \int \frac{dp_1 dp_2}{(2\pi)^4 (2\pi)^4} e^{-i p_1 \cdot (x_1 - x_2)} \tilde{G}_{\mu_2}^{a_1 c_2 b_2}(p_1, p_2, -) \\ &= \int \frac{dp_1}{(2\pi)^4} e^{-i p_1 \cdot (x_1 - x_1)} \left\{ [+ i p_1^{\mu_2}] \int \frac{dp_2}{(2\pi)^4} \tilde{G}_{\mu_2}^{a_1 c_2 b_2}(p_1, p_2, -) \right\}. \end{aligned}$$

Thus,

$$+\frac{1}{g} k_1^2 D_{a_1 a_2}^{gh.}(k_1) + C_{a_2 b_2 c_2} (k_1)^{\mu_2} \int \frac{dk_2}{(2\pi)^4} \tilde{G}_{\mu_2}^{a_1 c_2 b_2}(k_1, k_2, -) = \sqrt{\alpha} \delta_{a_1 a_2}$$

first level M identity

$$+\frac{1}{g} k_2^2 G_{\mu_3}^{a_1 a_2 a_3}(k_1, k_2, -) + C_{a_2 b_2 c_2} \int \frac{dk_3}{(2\pi)^4} [-i(k_2 + k_3)^{\mu_2}] e^{-i k_3 \cdot (x_2 - x_3)} \tilde{G}_{\mu_2 \mu_3}^{a_1 c_2 b_2 a_3}(k_1, k_2, k_3, -)$$

This last thing is garbage. I have to switch 2,3 in the second level M-identity above first, then fourier transform:

$$G_{\mu_2 \mu_3}^{a_1 c_2 b_2 a_3}(k_1, k_2, k_3, -)$$

$$-\frac{1}{g} \Box_{(x_3)}^2 G_{\mu_2}^{a_1 a_2 a_3}(x_1, x_2, x_3) + C_{a_3 b_3 c_3} \partial_{(x_3)}^{\mu_3} G_{\mu_2 \mu_3}^{a_1 a_2 c_3 b_3}(x_1, x_2, x_3, x_3) = 0$$

second level M identity

$$+\frac{1}{g} k_3^2 G_{\mu_2}^{a_1 a_2 a_3}(k_1, k_2, -) + C_{a_3 b_3 c_3} [i(p_1 + p_2)]^{\mu_3} \int \frac{dp_3}{(2\pi)^4} G_{\mu_2 \mu_3}^{a_1 a_2 c_3 b_3}(p_1, p_2, p_3, -) = 0.$$

where $k_1 + k_2 + k_3 = 0$. Rewrite as,

$$+\frac{1}{g} k_3^2 G_{\mu_2}^{a_1 a_2 a_3}(k_1, k_2, -) + C_{a_3 b_3 c_3} [-i k_3^{\mu_3}] \int \frac{dk'_3}{(2\pi)^4} G_{\mu_2 \mu_3}^{a_1 a_2 c_3 b_3}(k_1, k_2, k'_3, -) = 0$$

second level M identity

In the integration on the right here, k_1 and k_2 are held fixed, the other two momenta vary. Thus, we can only extract one ghost and one gauge propagator thus:

$$\frac{1}{g} k_3^2 (i)^3 D^g(k_1) D_{\mu_2 \mu_2}(k_2) D^g(k_3) \gamma_{\lambda_2}^{a_1 a_2 a_3}(k_1, k_2, -)$$

$$= C_{a_3 b_3 c_3} [i k_3^{\mu_3}] (i)^2 D^g(k_1) D_{\mu_2 \mu_2}(k_2) \int \frac{dk'_3}{(2\pi)^4} G_{\lambda_2 \mu_3}^{a_1 a_2 c_3 b_3}(k_1, k_2, k'_3, -)$$

As long as there is some longitudinal component on the gluon propagator, its inverse exists. Thus we can cancel off the D's to get this:

$$\frac{1}{g} k_3^2 D^g(k_3) \gamma_{\lambda_2}^{a_1 a_2 a_3}(k_1, k_2, -) = k_3^{\mu_3} C_{a_3 b_3 c_3} \int \frac{dk'_3}{(2\pi)^4} G_{\lambda_2 \mu_3}^{a_1 a_2 c_3 b_3}(k_1, k_2, k'_3, +)$$

Divide through to get

$$\gamma_{\lambda_2}^{a_1 a_2 a_3}(k_1, k_2, -) = i k_3^{\mu_3} \cdot \left\{ \frac{g}{k_3^2 i D^g(k_3)} C_{a_3 b_3 c_3} \int \frac{dk'_3}{(2\pi)^4} G_{\lambda_2 \mu_3}^{a_1 a_2 c_3 b_3}(k_1, k_2, k'_3, -) \right\}$$

This is the object defined in Slavnov equation (27), or it had better be!

$$\equiv \gamma_{(\mu_3) \lambda_2}^{a_1 a_2 a_3}(k_1, k_2, -)$$

$$-\frac{1}{\sqrt{\alpha}} \partial_{\mu_1}^{(x_1)} \partial_{\mu_2}^{(x_2)} \text{ (diagram)} = -\frac{1}{g} \partial_{(x_2)}^{\mu_2} \partial_{(x_3)}^{\mu_3} \text{ (diagram)} + C_{a_3 b_3 c_3} \partial_{(x_2)}^{\mu_2} \text{ (diagram)}$$

However I am now stuck. I have no obvious way to eliminate the 2-gluon/2-ghost greens function on the right; some kind of Dyson equation is needed here with the simplification that two of the particles have the same x_3 . The elimination of this last term is accomplished by Lee and Zinn Justing by their mysterious and unexplained equation (4.10). Since I don't see how this goes, I have to still just accept their result (4.15). Hopefully all these things will be more obvious in the proper-vertex formulation.

However, I can apply one more derivative to the above to get:

$$-\frac{1}{\sqrt{\alpha}} \partial_{(x_1)}^{\mu_1} \partial_{(x_2)}^{\mu_2} \partial_{(x_3)}^{\mu_3} \text{ (diagram)} = 0$$

$$\Rightarrow p_1^{\mu_1} p_2^{\mu_2} p_3^{\mu_3} G(\mu_1 a_1 p_1, \mu_2 a_2 p_2, \mu_3 a_3 p_3) = 0.$$

Analogous to the propagator result. Not sure what this does for you, keep for later.

Retry: lets go fourier transform the above result and see what the effect is of the $x_4 = x_3$. I get:

$$+\frac{1}{\sqrt{\alpha}} k_1^{\mu_1} k_2^{\mu_2} \tilde{G}_{\mu_1 \mu_2 \mu_3}^{a_1 a_2 a_3}(k_1, k_2, -) = +\frac{1}{g} k_2^{\mu_2} k_3^{\mu_3} \tilde{G}_{\mu_2}^{a_1 a_2 a_3}(k_1, k_2, -) \quad (*)$$

$$+ C_{a_3 b_3 c_3} [-i k_2^{\mu_2}] \int \frac{dk'_3}{(2\pi)^4} \tilde{G}_{\mu_2 \mu_3}^{a_1 a_2 c_3 b_3}(k_1, k_2, k'_3, -)$$

While were here, lets write out the M-identity in k-space which is relevant here:

$$+\frac{1}{g} k_3^{\mu_3} \tilde{G}_{\mu_2}^{a_1 a_2 a_3}(k_1, k_2, -) + C_{a_3 b_3 c_3} [-i k_3^{\mu_3}] \int \frac{dk'_3}{(2\pi)^4} \tilde{G}_{\mu_2 \mu_3}^{a_1 a_2 c_3 b_3}(k_1, k_2, k'_3, -).$$

I still don't see how to get rid of the last 4-particle object in there. Lets go ahead and reduce to proper objects with the 3-point functions and see where this leads:

$$\tilde{G}_{\mu_1 \mu_2 \mu_3}^{a_1 a_2 a_3}(k_1, k_2, -) = \underbrace{\tilde{G}_{\mu_1 \lambda_1}^{a_1 d_1}(k_1)}_{= iD} \tilde{G}_{\mu_2 \lambda_2}^{a_2 d_2}(k_2) \tilde{G}_{\mu_3 \lambda_3}^{a_3 d_3}(k_3) \tilde{\Gamma}_{\lambda_1 \lambda_2 \lambda_3}^{d_1 d_2 d_3}(k_1, k_2, -)$$

Now this object appears dotted into some k's, and we already know from a lower Ward identity that :

$$k_1^{\mu_1} \tilde{G}_{\mu_1 \lambda_1}^{a_1 d_1}(k_1) = i k_1^{\mu_1} D_{\mu_1 \lambda_1}^{a_1 d_1}(k_1) = \frac{-i \delta_{a_1 d_1}}{k_1^2} \left\{ \alpha k_1^{\lambda_1} \right\} = -i \alpha \delta_{a_1 d_1} \left(\frac{k_1^{\lambda_1}}{k_1^2} \right).$$

Thus:

$$\text{LHS} = + \frac{1}{\sqrt{N}} [-i \alpha \delta_{a_1 d_1}] [-i \alpha \delta_{a_2 d_2}] \left(\frac{k_1^{\lambda_1}}{k_1^2} \right) \left(\frac{k_2^{\lambda_2}}{k_2^2} \right) i D_{\mu_3 \lambda_3}^{a_3 d_3}(k_3) \tilde{\Gamma}_{\lambda_1 \lambda_2 \lambda_3}^{d_1 d_2 d_3}(k_1, k_2, -).$$

But of course we know that the propagator in toto is diagonal in color, so we get:

$$(*) \text{ LHS} = -i \alpha^{\frac{3}{2}} D_{\mu_3 \lambda_3}(k_3) \left(\frac{k_1^{\lambda_1}}{k_1^2} \right) \left(\frac{k_2^{\lambda_2}}{k_2^2} \right) \tilde{\Gamma}_{\lambda_1 \lambda_2 \lambda_3}^{a_1 a_2 a_3}(k_1, k_2, -)$$

Similarly we can examine the first term on the RHS of (*). Let G(k) be the scalar ghost propagator (=iD), which we know is color diagonal also. Then we get:

$$(*) \text{ RHS}^{(1)} = \left[-i \alpha \frac{k_2^{\lambda_2}}{k_2^2} \right] k_3^{\mu_3} G(k_1) G(k_3) \tilde{\gamma}_{\lambda_2}^{a_1 a_2 a_3}(k_1, k_2, -)$$

Status: although I have something that looks like the left side of Slavnov's identity, the right side doesn't look right. Here I show two ghost propagators, whereas Slavnov only has one. Plus I still have that extra term with 2-gluon/2-ghost. Here then is the closest I am able to get:

$$\begin{aligned} & -i \alpha^{\frac{3}{2}} D_{\mu_3 \lambda_3}(k_3) \left[\frac{k_1^{\lambda_1}}{k_1^2} \right] \left[\frac{k_2^{\lambda_2}}{k_2^2} \right] \tilde{\Gamma}_{\lambda_1 \lambda_2 \lambda_3}^{a_1 a_2 a_3}(k_1, k_2, -) \\ & = (-i \alpha) \left[\frac{k_2^{\lambda_2}}{k_2^2} \right] k_3^{\mu_3} G(k_3) G(k_1) \tilde{\gamma}_{\lambda_2}^{a_1 a_2 a_3}(k_1, k_2, -) \\ & + C_{a_3 b_3 c_3} [-i k_2^{\mu_2}] \int \frac{d^4 k_3'}{(2\pi)^4} \tilde{G}_{\mu_2 \mu_3}^{a_1 a_2 c_3 b_3}(k_1, k_2, k_3', -) \end{aligned}$$

But never say die. Suppose I come in and apply a transverse projection operator:

$$P_{\mu\nu}^T(q) \equiv \left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \right)$$

$$P_{\mu\nu}^L(q) \equiv \frac{q_\mu q_\nu}{q^2}.$$

$$P_{\mu\nu}^T P_{\nu\lambda}^T = \left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \right) \left(g_{\nu\lambda} - \frac{q_\nu q_\lambda}{q^2} \right) = \left(g_{\mu\lambda} - \frac{q_\mu q_\lambda}{q^2} \right) = P_{\mu\lambda}^T \Rightarrow (P^T)^2 = (P^T) \checkmark$$

$$P_{\mu\nu}^L P_{\nu\lambda}^L = \frac{q_\mu q_\nu}{q^2} \frac{q_\nu q_\lambda}{q^2} = \frac{q_\mu q_\lambda}{q^2} = P_{\mu\lambda}^L \Rightarrow (P^L)^2 = (P^L) \checkmark.$$

$$P_{\mu\nu}^T D_{\nu\lambda}(q) = P_{\mu\nu}^T(q) \left[A \cdot P_{\nu\lambda}^T(q) + B P_{\nu\lambda}^L(q) \right] = A P_{\mu\lambda}^T(q) = D_{\mu\lambda}^T.$$

So let's apply a transverse projection operator from the left in k_3 .

$$P_{\mu_4\mu_3}^T(k_3) D_{\mu_3\lambda_3}(k_3) = D_{\mu_4\lambda_3}^T(k_3)$$

$$P_{\mu_4\mu_3}^T(k_3) k_3^{\mu_3} = \left(g_{\mu_4\mu_3} - \frac{(k_3)_{\mu_4} (k_3)_{\mu_3}}{k_3^2} \right) k_3^{\mu_3} = 0$$

So the second term dies and it is the third term that is significant. We now have:

$$\begin{aligned} & -i \alpha^{\frac{3}{2}} D_{\mu_4\lambda_3}^T(k_3) \left(\frac{k_1^{\lambda_1}}{k_1^2} \right) \left(\frac{k_2^{\lambda_2}}{k_2^2} \right) \Gamma_{\lambda_1\lambda_2\lambda_3}^{a_1a_2a_3}(k_1, k_2, -) \\ & = C_{a_3b_3c_3} [-i k_2^{\mu_2}] \left[\delta_{\mu_4\mu_3} - \frac{k_3^{\mu_4} k_3^{\mu_3}}{k_3^2} \right] \int \frac{dk_3'}{(2\pi)^4} \tilde{G}_{\mu_2\mu_3}^{a_1a_2c_3b_3}(k_1, k_2, k_3', -) \end{aligned}$$

Now the left side here is, apart from constant, precisely the LHS of Slavnov's identity.

Next, let's use the M-identity second level on the second piece of the RHS:

$$RHS^{(a)} = + (k_2^{\mu_2}) k_3^{\mu_4} \frac{1}{g} G_{\mu_2\mu_3}^{a_1a_2a_3}(k_1, k_2, -)$$

$$RHS^{(a)} = C_{a_3b_3c_3} [-i k_2^{\mu_2}] \int \frac{dk_3'}{(2\pi)^4} \tilde{G}_{\mu_2\mu_3}^{a_1a_2c_3b_3}(k_1, k_2, k_3', -).$$

Now go back to page 8c where I have defined the Slavnov object:

$$\gamma_{\mu_3 \lambda_2}^{a_1 a_2 a_3}(k_1, k_2, -) = \frac{g}{k_3^2 i D^g(k_3)} C_{a_3 b_3 c_3} \int \frac{dk'_3}{(2\pi)^4} \overset{m}{G}_{\lambda_2 \mu_3}^{a_1 a_2 c_3 b_3}(k_1, k_2, k'_3, -)$$

where

$$\gamma_{\lambda_2}^{a_1 a_2 a_3}(k_1, k_2, -) = i k_3^{\mu_3} \gamma_{\mu_3 \lambda_2}^{a_1 a_2 a_3}(k_1, k_2, -). \quad (29SI).$$

The squiggle over the G here means that one ghost and one gauge propagator iD, iD have been removed. Lets now put them back in to get:

$$- D^g(k_1) D_{\mu_2 \lambda_2}(k_2) \gamma_{\mu_3 \lambda_2}^{a_1 a_2 a_3} = \frac{g}{k_3^2 i D^g(k_3)} C_{a_3 b_3 c_3} \int \frac{dk'_3}{(2\pi)^4} G_{\mu_2 \mu_3}^{a_1 a_2 c_3 b_3}(k_1, k_2, k'_3, -)$$

Thus we get

$$C_{a_3 b_3 c_3} \int \frac{dk'_3}{(2\pi)^4} G_{\mu_2 \mu_3}^{a_1 a_2 c_3 b_3}(k_1, k_2, k'_3, -) = \frac{1}{ig} k_3^2 D^g(k_1) i D^g(k_3) D_{\mu_2 \lambda_2}(k_2) \gamma_{\mu_3 \lambda_2}^{a_1 a_2 a_3}(k_1, k_2, -)$$

This at last allows me to replace the 4-point function G with Slavnov's 3-point thing with the extra lorentz index. If I now go insert this into my Ward identity on page 11, the $k_2^{\mu_2}$ will hit the gauge propagator like so:

$$i k_2^{\mu_2} D_{\mu_2 \lambda_2}(k_2) = -i \alpha \left(\frac{k_2^{\lambda_2}}{k_2^2} \right) \quad / \text{see on page 10.}$$

Therefore, my Ward identity now reads:

$$\begin{aligned} & -i \alpha^{\frac{3}{2}} D_{\mu_4 \lambda_3}^{h_1}(k_3) \left(\frac{k_2^{\lambda_2}}{k_2^2} \right) \left(\frac{k_1^{\lambda_1}}{k_1^2} \right) \Gamma_{\lambda_1 \lambda_2 \lambda_3}^{a_1 a_2 a_3}(k_1, k_2, -) \\ &= \frac{\alpha}{g} k_3^2 D^g(k_3) \left[\frac{k_2^{\lambda_2}}{k_2^2} \right] D^g(k_1) \underbrace{P_{\mu_4 \mu_3}^T(k_3) \gamma_{\mu_3 \lambda_2}^{a_1 a_2 a_3}(k_1, k_2, -)}_{\equiv \gamma_{\mu_4 \lambda_2}^{a_1 a_2 a_3, h_1}} \end{aligned}$$

Let

$$\begin{aligned} -k_1 &= p+k \\ k_2 &= k \\ k_3 &= p \end{aligned}$$

It looks very good except I seem to have an extra ghost propagator in there. But, let's now go back and use the level-one M identity like so:

$$\frac{1}{g} k_3^2 D^g(k_3) \delta_{a_1 a_4} + C_{a_2 b_2 c_2} (k_3)^{\mu_2} \int \frac{dk_2'}{(2\pi)^4} G_{\mu_2}^{a_1 c_2 b_2}(k_3, k_2', -) = \int d\alpha \delta_{a_1 a_4}.$$

Put this into the Ward to get:

$$\begin{aligned} \text{RHS} &= \alpha^{\frac{3}{2}} \left(\frac{k_2^{\lambda_2}}{k_2^2} \right) D^g(k_1) \gamma_{\mu_4 \lambda_2}^{a_1 a_2 a_3, h} \\ &\rightarrow \alpha C_{a_4 b_2 c_2} (k_3)^{\mu_2} \left[\int \frac{dk_2'}{(2\pi)^4} G_{\mu_2}^{a_1 c_2 b_2}(k_3, k_2', -) \right] \left(\frac{k_2^{\lambda_2}}{k_2^2} \right) D^g(k_1) \gamma_{\mu_4 \lambda_2}^{a_1 a_2 a_3, h}. \end{aligned}$$

If I could show that this last term vanishes by color or something, then my result would be:

$$D_{\mu_4 \lambda_3}^h(k_3) \left(\frac{k_2^{\lambda_2}}{k_2^2} \right) \left(\frac{k_1^{\lambda_1}}{k_1^2} \right) \Gamma_{\lambda_1 \lambda_2 \lambda_3}^{a_1 a_2 a_3}(k_1, k_2, -) = \left(\frac{k_2^{\lambda_2}}{k_2^2} \right) i D^g(k_1) \gamma_{\mu_4 \lambda_2}^{a_1 a_2 a_3, h}(k_1, k_2, -)$$

And this is precisely Slavnov's result!!!!

Slavnov Debug Sequence.

1. Lets quickly review the steps which lead to the identity of interest here. I started with the Second-Level Ward Identity shown on page 5, copied over on page 8. I took a derivative on this thing and thereby obtained result on top of page 9; I made use of the SEcon-Level M identity to kill off two of the four terms. This identity is shown on page 8 in corrd space rep. Next, I converted the intermediate result -- top page 9-- to k-space. Then I took the two 3-point functions and expanded them to proper vertices. On the LHS this meant the generation of 3 gauge propagators. But the two k^u 's chopped two of them down and left the third. On the RHS, the single gauge prop got chopped, and there were two left over ghost propagators. Thus I ended up with result on bottom of page 10. Next I simply applied a k_3 transverse projector to both sides. The first term on the RHS was thereby killed off, result shown on page 11.

At this point, I had the correct LHS of Slavnov's identity, but I did not have the correct RHS. I was stuck with a 2-gluon/2-gauge piece on the right which Slavnov did not seem to have.

Next, I played with the Second-Level M-identity to try and create Slavnov's fancy ghost vertex with the two lorentiz indices. This identity relates the 3 to the 4. I expanded the 3-vertex down to a proper vertex on page 8c and then finally I came up with a definition for the Slavnov object, duplicated on page 12. Thus, I was finally able to get rid of by 4-point function in favor of this Slavnov object, hence result on bottom page 12. Finally I used the first-level M identity to replace the extra ghost propagator, and I finally obtained the Slavnov Identity plus an extra term.

So there is my summary.

Gamma-slash. Lets define a new Slavnov object like so:

$$\gamma_{\mu_3 \lambda_2}^{a_1 a_2 a_3}(k_1, k_2, -) \equiv \frac{g}{k_3^2 i D^g(k_3)} C_{a_3 b_3 c_3} \int \frac{dk'_3}{(2\pi)^4} G_{\lambda_2 \mu_3}^{a_1 a_2 c_3 b_3}(k_1, k_2, k'_3, -) \quad (14.1)$$

$$\gamma_{\mu_3 \lambda_2}^{a_1 a_2 a_3}(k_1, k_2, -) \equiv \gamma_{\mu_3 \lambda_2}^{a_1 a_2 a_3}(k_1, k_1, -) + P_{\mu_3 \lambda_2}^T(k_3) F^{a_1 a_2 a_3}(k_1, k_2, -) \quad (14.2)$$

I have copied my earlier definition of gamma from page 12 top, and have defined gamma-slash in terms of a scalar function F, as yet undetermined. Since the extra piece in gamma-slash is transverse, we maintain the desired considiotn that:

$$\gamma_{\lambda_2}^{a_1 a_2 a_3}(k_1, k_2, -) = i k_3^{\mu_3} \gamma_{\mu_3 \lambda_2}^{a_1 a_2 a_3}(k_1, k_2, -), \quad (14.3)$$

Let us now eliminat gamma in favor of gamma-slash in our final Slavnov Identity result, making use of (14.2) above. For our RHS we then get:

$$\begin{aligned} \text{RHS} = & \alpha^{\frac{3}{2}} \left(\frac{k_2^{\lambda_2}}{k_2^2} \right) D^g(k_1) \gamma_{\mu_4 \lambda_2}^{a_1 a_2 a_3, h} \\ & + \alpha^{\frac{3}{2}} \left(\frac{k_2^{\lambda_2}}{k_2^2} \right) D^g(k_1) F_{\mu_4 \lambda_2}^{a_1 a_2 a_3, h} \\ & - \alpha \left(\frac{k_2^{\lambda_2}}{k_2^2} \right) D^g(k_1) \gamma_{\mu_4 \lambda_2}^{a_4 a_2 a_3, h} \cdot \left(C_{a_4 b_2 c_2} k_3^{\mu_2} \int \frac{dk'_2}{(2\pi)^4} G_{\mu_2}^{a_1 c_2 b_2}(k_3, k'_2, -) \right) \\ & P_{\mu_4 \lambda_2}^T(k_3) F^{a_1 a_2 a_3} \end{aligned}$$

The first term is the one we want to get the Slavnov identity; the last two two terms we want to arrange to cancel. I will try now to make them cancel. Factor out alpha and $\frac{1}{k_2^2} D(k_1)$. Define the parenthetical grouping to be $H^{a_4 a_1}(k_3)$. Probably this is proportional to $d_{a_4 a_1}$ but does not mattedr for now. Then we have:

$$\sqrt{\alpha} k_2^{\lambda_2 \mu_3} P_{\mu_4 \lambda_2}^T(k_3) F^{a_1 a_2 a_3} - k_2^{\lambda_2} P_{\mu_4 \mu_3}^T(k_3) \gamma_{\mu_3 \lambda_2}^{a_4 a_2 a_3} \cdot H^{a_4 a_1}(k_3) = 0 ?$$

$$P_{\mu_4 \mu_3}^T(k_3) \left\{ \sqrt{\alpha} F^{a_1 a_2 a_3} k_2^{\mu_3} - H^{a_4 a_1}(k_3) k_2^{\lambda_2} \gamma_{\mu_3 \lambda_2}^{a_4 a_2 a_3} \right\} = 0 ?$$

Now expand the underlined vector into its tensor components:

$$k_2^{\lambda_2} \gamma_{\mu_3 \lambda_2}^{a_4 a_2 a_3} = A^{a_4 a_2 a_3} (k_2)^{\mu_3} + B^{a_4 a_2 a_3} (k_3)^{\mu_3}. \quad (15.1)$$

The transverse projector will kill the B term here and we'll get:

$$P_{\mu_4 \mu_3}^T(k_3) \left\{ (k_2)^{\mu_3} \left(\sqrt{\alpha} F^{a_1 a_2 a_3} - H^{a_4 a_1}(k_3) A^{a_4 a_2 a_3} \right) \right\} = 0 \quad ?$$

Clearly this will be satisfied if we choose F like so:

$$\boxed{F^{a_1 a_2 a_3} = \frac{1}{\sqrt{\alpha}} H^{a_4 a_1}(k_3) A^{a_4 a_2 a_3}} \quad (15.2)$$

So, without going another step, I now see that it is possible to define gamma-slash such that the Slavnov identity works out! I will continue and attempt to get an explicit expression for F and then for gamma-slash. First, solve (15.1) for A as follows:

$$\begin{aligned} P_{\mu_5 \mu_3}^T(k_3) (k_2^{\lambda_2} \gamma_{\mu_3 \lambda_2}^{a_4 a_2 a_3}) &= A^{a_4 a_2 a_3} P_{\mu_5 \mu_3}^T(k_3) (k_2)^{\mu_3} \\ &= A^{a_4 a_2 a_3} \left[k_2^{\mu_5} - \frac{k_2 \cdot k_3}{k_3^2} k_3^{\mu_5} \right]. \end{aligned}$$

$$\Rightarrow k_2^{\mu_5} P_{\mu_5 \mu_3}^T(k_3) k_2^{\lambda_2} \gamma_{\mu_3 \lambda_2}^{a_4 a_2 a_3} = A^{a_4 a_2 a_3} \left(k_2^2 - \frac{(k_2 \cdot k_3)^2}{k_3^2} \right) \frac{k_3^2}{k_3^2}$$

$$\Rightarrow \boxed{A^{a_4 a_2 a_3} = \left(k_3^2 k_2^{\mu_5} P_{\mu_5 \mu_3}^T(k_3) k_2^{\lambda_2} \gamma_{\mu_3 \lambda_2}^{a_4 a_2 a_3} \right) / (k_2^2 k_3^2 - (k_2 \cdot k_3)^2)}$$

Then

$$\begin{aligned} P_{\mu_3 \lambda_2}^T(k_3) F^{a_1 a_2 a_3} &= \frac{1}{\sqrt{\alpha}} H^{a_4 a_1}(k_3) P_{\mu_3 \lambda_2}^T(k_3) k_2^{\mu_5} P_{\mu_5 \mu_3}^T(k_3) k_2^{\lambda_2} \gamma_{\mu_3 \lambda_2}^{a_4 a_2 a_3} \\ &\quad \times \left(\frac{k_3^2}{k_2^2 k_3^2 - (k_2 \cdot k_3)^2} \right). \end{aligned}$$

We can now combine the k_2^{u5} and the P^T and the gamma to get another result for A:

$$T^{-1} \equiv k_2^2 k_3^2 - (k_2 \cdot k_3)^2$$

$$A^{a_1 a_2 a_3} = (k_3^2 T) k_2^\lambda \left[k_2^\sigma \gamma_{\sigma\lambda}^{a_1 a_2 a_3} + i \frac{k_2 \cdot k_3}{k_3^2} \gamma_{\cdot\lambda}^{a_1 a_2 a_3} \right]$$

From the first-level M-identity we know H,

$$H^{a_1 a_1}(k_3) = \delta^{a_1 a_1} \left[\sqrt{\alpha} - \frac{1}{g} k_3^2 D^g(k_3) \right]$$

thus we may conclude that F is given by:

$$F^{a_1 a_2 a_3} = \frac{1}{\sqrt{\alpha}} \left[\sqrt{\alpha} - \frac{1}{g} k_3^2 D^g(k_3) \right] (k_3^2 T) \left\{ k_2^\lambda k_2^\sigma \gamma_{\sigma\lambda}^{a_1 a_2 a_3} + i \frac{k_2 \cdot k_3}{k_3^2} \gamma_{\cdot\lambda}^{a_1 a_2 a_3} k_2^\lambda \right\}$$

I would prefer to see the Slavnov object combined with some propagators like so:

$$k_3^2 \gamma_{\sigma\lambda}^{a_1 a_2 a_3} = D^g(k_1)^{-1} D^g(k_3)^{-1} D_{\lambda\mu_2}^{-1}(k_2) \left[k_3^2 D^g(k_1) D^g(k_3) D_{\mu_2\lambda_2}(k_2) \gamma_{\sigma\lambda_2}^{a_1 a_2 a_3} \right]$$

But,

$$ig C_{a_3 b_3 c_3} \int \frac{dk'_3}{(2\pi)^4} G_{\cdot\mu_2\sigma}^{a_1 a_2 c_3 b_3}(k_1, k_2, k'_3, -)$$

$$k_2^\lambda D_{\lambda\mu_2}^{-1}(k_2) = -\frac{1}{\alpha} k_2^2 k_2^{\mu_2}$$

Thus, the first term in F above is

$$F_{(1)}^{a_1 a_2 a_3} = -\alpha^{-\frac{3}{2}} k_2^2 \quad T \quad \left[\sqrt{\alpha} - \frac{1}{g} D^g(k_3) \right] ig C_{a_3 b_3 c_3}$$

$$k_2^\sigma k_2^{\mu_2} \left[\int \frac{dk'_3}{(2\pi)^4} G_{\cdot\mu_2\sigma}^{a_1 a_2 c_3 b_3}(k_1, k_2, k'_3, -) \right] \cdot D^g(k_1)^{-1} D^g(k_3)^{-1}$$

So let's define a shorthand for this nasty 4-point function integral:

$$Q_{\mu_2 \sigma}^{a_1 a_2 a_3}(k_1, k_2, -) \equiv C^{a_3 b_3 c_3} \int \frac{dk'_3}{(2\pi)^4} G_{\mu_2 \sigma}^{a_1 a_2 c_3 b_3}(k_1, k_2, k'_3, -)$$

$$\Rightarrow \boxed{F_{(1)}^{a_1 a_2 a_3} = -ig \alpha^{-\frac{3}{2}} T \left[\sqrt{\alpha} - \frac{1}{g} D^3(k_3) \right] D^3(k_1)^{-1} D^3(k_2)^{-1} k_2^2 k_2^{\mu_2} k_2^{\sigma} Q_{\mu_2 \sigma}^{a_1 a_2 a_3}(k_1, k_2, -)}$$

$$\boxed{F_{(2)}^{a_1 a_2 a_3} = +i \alpha^{-\frac{1}{2}} T \left[\sqrt{\alpha} - \frac{1}{g} D^3(k_3) \right] (k_2 \cdot k_3) k_2^{\lambda} \gamma_{\lambda}^{a_1 a_2 a_3}}$$

Now take equation (14.2) and multiply by the various propagator factors to get:

$$\frac{1}{ig} k_3^2 D^3(k_1) D^3(k_3) D_{\mu_2 \lambda_2}(k_2) \gamma_{\mu_3 \lambda_2}^{a_1 a_2 a_3} = Q_{\mu_2 \mu_3}^{a_1 a_2 a_3}(k_1, k_2, -)$$

$$+ \frac{1}{ig} k_3^2 D^3(k_1) D^3(k_3) D_{\mu_2 \lambda_2}(k_2) P_{\mu_3 \lambda_2}^T(k_3) \{ F_{(1)}^{a_1 a_2 a_3} + F_{(2)}^{a_1 a_2 a_3} \}.$$

End: OK, I HAVE HAD ENOUGH. The fact is that the object you add, which I have called F , is rather complicated. It causes the definition of gamma-slash to be complicated. This complication is reflected in Lee ZJ's unexplained equation (4.10) which I now realize is in fact a definition of the object I call gamma-slashh (they call gamma with the two subscripts).

The expression for gamma-slash is not really significant for the implementation of the Slavnov identity. Just knowing that the definition exists is sufficient. Here then is the final result:

$$D_{\mu_4 \lambda_3}^T(k_3) \left(\frac{k_2^{\lambda_2}}{k_2^2} \right) \left(\frac{k_1^{\lambda_1}}{k_1^2} \right) \gamma_{\lambda_1 \lambda_2 \lambda_3}^{a_1 a_2 a_3}(k_1, k_2, -) = \left(\frac{k_2^{\lambda}}{k_2^2} \right) i D^3(k_1) \gamma_{\mu_4 \lambda_2}^{a_1 a_2 a_3, T}(k_1, k_2, -)$$

$$\text{where } i k_3^{\mu_3} \gamma_{\mu_3 \lambda_2}^{a_1 a_2 a_3}(k_1, k_2, -) = \gamma_{\lambda_2}^{a_1 a_2 a_3}(k_1, k_2, -)$$

and γ has a very messy definition.

Slavnov Identity!!!
(see Eq. (25) Slav.)

The 4-gluon Ward identity.

This is already stated on page 5f. We may convert to fourier notation to get:

$$(-i)^4 k_1 k_2 k_3 k_4 \tilde{G}_{1234} = (-i)^2 k_3 k_4 \tilde{H}_{1234} + (2 \leftrightarrow 3) + (2 \leftrightarrow 4)$$

Here we have chosen to put on the "full complement" of derivatives in order to not involve ghosts. We will find that the 4-gluon connects to the 3-gluon and 2-gluon.

Our first task is to decompose the object G_{1234} into its 1pI components. Recall that "connected" in the sense of generating functions means "no concomitant vacuum bubbles". Graphs that are disconnected in the S-matrix sense are still present in G_{1234} . Thus, graphically and as an equation we have:

$$\begin{aligned} \text{Diagram 1} &= \left\{ \text{Diagram 2} + (2 \leftrightarrow 3) + (2 \leftrightarrow 4) \right\} + \text{Diagram 3} \\ &+ \left\{ \text{Diagram 4} + (2 \leftrightarrow 3) + (2 \leftrightarrow 4) \right\} \end{aligned}$$

$$\begin{aligned} \tilde{G}_{1234} &= \left\{ \tilde{G}_{12} \tilde{G}_{34} + (2,3) + (2,4) \right\} + \tilde{G}_{11'} \tilde{G}_{22'} \tilde{G}_{33'} \tilde{G}_{44'} \tilde{\Gamma}_{1'2'3'4'} \\ &+ \left\{ \tilde{G}_{11'} \tilde{G}_{22'} \tilde{G}_{33'} \tilde{G}_{44'} \left[\tilde{\Gamma}_{1'3'5'} \tilde{G}_{55'} \tilde{\Gamma}_{5'2'4'} + (2,3) + (2,4) \right] \right\} \end{aligned}$$

Notation: the twiddle as usual means k-space object. Since momdeltas are still inside objects, repeated indices include momentum integrations with PI factors, as well as Lorentz and color index sums.

Our next task is to apply the four k's to the above expression. As was shown on 5h, when a k_1 hits a G_{11} , you get certain factors plus a two pi fourth momentum delta. In the second two terms above, these deltas remove all k-integrations, but in the first bracket you will have some deltas left over. Thus we have:

$$\begin{aligned} (-i)^4 k_1 k_2 k_3 k_4 \tilde{G}_{12} \tilde{G}_{34} &= (-i)^2 k_2^{\mu_2} k_4^{\mu_4} \cdot [-i k_1 \tilde{G}_{12}] [-i k_3 \tilde{G}_{34}] \\ &= (-i)^2 k_2^{\mu_2} k_4^{\mu_4} \left[-\alpha \left(\frac{k_1^{\mu_2}}{k_1^2} \right) (2\pi)^4 \delta(k_1 + k_2) \delta_{a_1 a_2} \right] \\ &\quad \times \left[-\alpha \left(\frac{k_3^{\mu_4}}{k_3^2} \right) (2\pi)^4 \delta(k_3 + k_4) \delta_{a_3 a_4} \right] \end{aligned}$$

So, $(-i)^4 k_1 k_2 k_3 k_4 \tilde{G}_{12} \tilde{G}_{34} = (-i)^2 (-\alpha)^2 \delta_{a_1 a_2} \delta_{a_3 a_4} (2\pi)^8 \delta(k_1 + k_2) \delta(k_3 + k_4)$

This is the first term in the first bracket on the LHS. Now before doing the rest of the LHS, let's examine the first term on the RHS of our Ward identity:

$$(-i)^2 k_3 k_4 \overbrace{(2\pi)^4 \delta(k_1 + k_2) \tilde{G}_{34}}^{\tilde{H}_{1234}} (-i\alpha \delta_{a_1 a_2})$$

But $(-i)^2 k_3 k_4 \tilde{G}_{34} = \tilde{H}_{34} = (2\pi)^4 \delta(k_3 + k_4) (-i\alpha \delta_{a_3 a_4})$ /page 5g,h.

So we get

$$= (-i\alpha)^2 \delta_{a_1 a_2} \delta_{a_3 a_4} (2\pi)^8 \delta(k_1 + k_2) \delta(k_3 + k_4).$$

Thus, the first term on the right balances the first term on the left. So we see that the purpose of all the terms on the right of this Ward identity is to balance the disconnected terms on the left. In other words, we just showed this:

$$(-i)^4 k_1 k_2 k_3 k_4 \tilde{G}_{12} \tilde{G}_{34} = (-i)^2 k_3 k_4 \tilde{H}_{1234}$$

These terms having cancelled off, we may now state the residual Ward Identity as:

$$(-\alpha)^4 \frac{k_1^{\mu_1}}{k_1^2} \frac{k_2^{\mu_2}}{k_2^2} \frac{k_3^{\mu_3}}{k_3^2} \frac{k_4^{\mu_4}}{k_4^2} \left[\tilde{\Gamma}_{1234} + \left\{ \tilde{\Gamma}_{135} \tilde{G}_{55'} \tilde{\Gamma}_{5'24} + (2 \leftrightarrow 3) + (2 \leftrightarrow 4) \right\} \right] = 0$$

In the groupings there are two integrations but three delta groups, one being the overall. Pulling all this stuff we get:

$$\frac{\mu_1 \mu_2 \mu_3 \mu_4}{k_1 k_2 k_3 k_4} \left[\int_{\mu_1 \mu_2 \mu_3 \mu_4}^{a_1 a_2 a_3 a_4} (k_1, k_2, k_3, -) + \left\{ \int_{\mu_1 \mu_3 \mu_5}^{a_1 a_3 a_5} (k_1, k_3, -) i D_{\mu_5 \mu_5'}(k_5) \int_{\mu_5 \mu_4 \mu_4}^{a_5 a_2 a_4} (k_5, k_2, -) + (2 \leftrightarrow 3) + (2 \leftrightarrow 4) \right\} \right] = 0$$

This result appears in LZJ-1. It relates the 4-proper, the 3-proper, and the 2-gluon. As already noted, no ghosts. Much simpler to derive than the Slavnov!!

Group

Group Summary Page

[valid for any gauge-theory Lie group].

1. For any unitary rep. σ :

$$G_a^{(\sigma)} = \text{hermitean}$$

$$C_{abc} = \text{T.A.} + \text{real}$$

$$d_{abc} = \text{T.S.} + \text{real}$$

$$U^{(\sigma)} = \exp(-i\Theta_a G_a^{(\sigma)})$$

$$[G_a^{(\sigma)}, G_b^{(\sigma)}] = i C_{abc} G_c^{(\sigma)}$$

$$\text{tr}^{(\sigma)}(G_a G_b) = F^{(\sigma)} \delta_{ab}, \quad F^{(\sigma)} = \text{real} \quad (= g_{ab}^{(\sigma)})$$

$$\text{tr}^{(\sigma)}(G_a [G_b, G_c]) = i F^{(\sigma)} C_{abc}$$

$$\text{tr}^{(\sigma)}(G_a \{G_b, G_c\}) = D^{(\sigma)} d_{abc}, \quad D^{(\sigma)} = \text{real.}$$

$$\text{tr}^{(\sigma)}(G_a G_b G_c) = \frac{i}{2} F^{(\sigma)} C_{abc} + \frac{1}{2} D^{(\sigma)} d_{abc}$$

$$C^{(\sigma)} = \left[\frac{N}{n_\sigma} F^{(\sigma)} \right] \mathbb{1}^{(\sigma)} \quad \text{if } \sigma = \text{irreducible (Shur).} \quad \left(C^{(\sigma)} \equiv \sum_{a=1}^N [G_a^{(\sigma)}]^2 \right)$$

$$\sum_{l \in n \otimes m} F^{(l)} = n F^{(n)} + m F^{(m)}$$

$$F^{(m^*)} = F^{(m)}$$

$$\sum_{l \in n \otimes m} D^{(l)} = n D^{(n)} + m D^{(m)}$$

$$D^{(m^*)} = -D^{(m)} \quad \left. \vphantom{\sum_{l \in n \otimes m} D^{(l)}} \right\} \text{build-up values.}$$

$$[U^{(\text{adj})}]_{ab} = \frac{1}{F^{(\sigma)}} \text{tr}^{(\sigma)}(G_a U G_b U^\dagger)$$

2. For adjoint rep, $n_{\text{adj}} = N$.

$$[G_a^{(\text{adj})}]_{bc} = -i C_{abc} \Rightarrow D^{(\text{adj})} = 0$$

$$F^{(\text{adj})} = k.$$

$$\text{tr}^{(\text{adj})}(G_a G_b) = C_{axy} C_{bxy} = -C_{xay} C_{ybx} = k \delta_{a,b}$$

$$i \text{tr}^{(\text{adj})}(G_a G_b G_c) = C_{xay} C_{ybz} C_{zcx} = -\frac{1}{2} k C_{abc}$$

A comment about $\text{tr}(G_i G_j)$ for Lie Group.

1. Assume that you have some generators of some dimension N (not necessarily the adjoint rep, no special rep) and they satisfy a Lie Algebra like so:

$$[G_i, G_j] = i c_{ijk} G_k$$

Assume that the constants c_{ijk} are real and form a totally antisymmetric tensor. I will postpone till later the question of how you arrange to do this. Probably for any Lie Group you can shuffle the generators until this is true. Assume it's true. Because the finite transformations $U = \exp(-i\theta \cdot G)$ are supposed unitary, the generator matrices in any rep must be Hermitian.

2. Recall that a Lie Algebra is invariant under a similarity on the generators. I.e., after you do a similarity, the structure constants are still the same as the above. (Not used)

3. Now define $g_{ij} = \text{trace}(G_i G_j)$. Obviously g_{ij} is symmetric. And because the generators are Hermitian, g_{ij} happens to be real. Recall in proving this that $\text{tr}(ABC) = \text{tr}(C^T B^T A^T)$.

4. Suppose the tensor g_{ij} is not diagonal. Since it is a real, symmetric matrix, it can be diagonalized by a real, orthogonal transformation. Thus we write:

$$g^d_{ij} = R^T_{ih} \text{tr}(G_h G_k) R_{kj} = \text{tr}\{R^T_{ih} \underline{G_h} R^T_{jk} \underline{G_k}\}$$

Define: $\tilde{G}_i \equiv R^T_{ij} G_j$. Then $g^d_{ij} = \text{tr}\{\tilde{G}_i \tilde{G}_j\}$

Now
$$\begin{aligned} [\tilde{G}_i, \tilde{G}_j] &= R^T_{ii'} R^T_{jj'} [G_{i'}, G_{j'}] = R^T_{ii'} R^T_{jj'} i c_{i'j'k'} G_{k'} \\ &= i \underbrace{[c_{i'j'k'} R^T_{ii'} R^T_{jj'} R_{kk}]}_{\equiv \tilde{c}_{ijk}} \tilde{G}_k = R^T_{kk'} \tilde{G}_{k'} \end{aligned}$$

Thus, we have come up with new generators (twiddled ones) and new structure constants. Notice that the new c 's are still real and totally antisymmetric. In our new Lie Algebra, the trace of two generators is at least a diagonal real matrix. So drop twiddles from now on.

5. We now want to show that this diagonal matrix is proportional to the identity. Consider:

$$d_{ijk} = -i \text{Tr} [G_i [G_j, G_k]] = C_{ijk} g_{ij}^d$$

Let $g_{ij}^d = F_i \delta_{ij} \Rightarrow$

$$d_{ijk} = F_i C_{jki}$$

But, $d_{ijk} = -i \text{Tr} [G_i G_j G_k - G_i G_k G_j] = -i \text{Tr} [-G_j G_i G_k + G_j G_k G_i]$
 $= -d_{jik}$

So $d_{ijk} = F_i C_{ijk}$

"
 $-d_{jik} = -F_j C_{jik} = F_j C_{ijk}$

For given $i \neq j$, as long as there is at least one k with $C_{ijk} \neq 0$, you get $F_i = F_j$.

$$\Rightarrow (F_i - F_j) C_{ijk} = 0.$$

Thus, assume such C_{ijk} exists. Then, $g_{ij}^d = F \delta_{ij}$.

To summarize, if we assume that:

- a) Lie Algebra can be cast into form where the constants c_{ijk} are real and antisymmetric
- b) For any i, j with $i \neq j$ there exists some k such that $c_{ijk} \neq 0$.

Then we can produce new structure constants and new generators such that

- 1) the new structure constants are still real and antisymmetric.
- 2) the new generators are still Hermitian (sum of Herms with real coeff = Herm!)
- 3) the object $\text{tr}(G_i G_j) = F \delta_{ij}$ where F is some constant.

6. Now, let's focus on a particular representation. For that representation, you could if you liked rescale the generators to make F be whatever you like. However, this would require scaling the structure constants. This is OK and does not affect their properties of reality and antisym and Jacobi. However, suppose you do this and you get F to be what you want for some rep. Then the structure constants are fixed and you are stuck with constants F for other reps which can no longer be adjusted!

7. For the adjoint representation, the matrix we called g_{ij} is probably the metric tensor of the Lie Algebra (Killing form). We are claiming that we can find a basis in which it is proportional to the identity matrix. Obviously, group must be semi-simple for this to work.

$$[G_i, G_j] = i C_{ijk} G_k \quad C_{ijk} \text{ are real antisym} \quad \sigma = \text{rep label.}$$

$$\text{Then } \text{tr}[G_i G_j] = F \delta_{ij} \quad \text{and} \quad \text{tr}[G_i [G_j, G_k]] = i F C_{ijk}.$$

Conclusion: Lie Algebra can always be arranged so that the following is true:

$$\begin{aligned}
 [G_a^\nu, G_b^\nu] &= i c_{abc} G_c^\nu && \text{where } c \text{ are real and antisymmetric} \\
 \text{trace}(G_a^\nu G_b^\nu) &= F^\nu \delta_{ab} \\
 \text{trace}(G_a^\nu [G_b^\nu, G_c^\nu]) &= i F^\nu c_{abc} && F^\nu = \text{real if } G_a \text{ Herm.}
 \end{aligned}$$

Here ν is a representation label and I imagine the generators to be always Hermitian so the group transformations are unitary. In the special case that the generator matrices of some rep happen to be pure imaginary (and hence antisymmetric to be Hermitian), the last result above simplifies to this:

$$\text{trace}(G_a^{\text{po}} G_b^{\text{po}} G_c^{\text{po}}) = i \frac{1}{2} F^{\text{po}} c_{abc}$$

This is the case for the adjoint representation wherein one has: $(G_a)_{bc} = -i c_{abc}$

add one.

1. In general, suppose you consider this trace:

$$\text{trace} (G_a^\sigma \{G_b^\sigma, G_c^\sigma\}) = D_{abc}^\sigma \quad [D_{abc}^\sigma \text{ is real if } G_a^\sigma \text{ are Herm.}]$$

At the moment, it seems to me that D is a representation-dependent, totally symmetric tensor. You can show its total symmetry by playing with the cyclicity of trace and nothing more. With the commutator in there, you could reduce the thing and get a tensor that was rep-independent, namely, the structure constants. But here you seem to get a different symmetric tensor for each representation. Maybe the tensor D is the same for all representations, but for now I will assume it is different.

I have a feeling that in $SU(2)$ the D tensor is zero for all reps; for $SU(3)$ I know it is not zero for the 3 rep; it is the d_{ijk} thing.

2. Given the above result and the other one with the commutator, you can conclude that:

$$\text{trace} [G_a^\sigma G_b^\sigma G_c^\sigma] = \frac{1}{2} i F^\sigma \underbrace{C_{abc}}_{\text{usual}} + \frac{1}{2} \underbrace{D_{abc}}_{\text{totally sym}}$$

3. Another misc. result: define the Casimir to be the sum of the squares of the gens in any particular rep. Then let N = number of generators in group. You have:

$$\text{trace} \left[\sum_{i=1}^N (G_i^\sigma)^2 \right] = \text{trace} [(CAS)^\sigma] = F^\sigma \cdot N$$

$\Rightarrow$

$$F^\sigma = \frac{1}{N} \text{tr} [G^\sigma] \quad \text{where} \quad G^\sigma = \sum_{a=1}^N (G_a^\sigma)^2$$

$$\text{For } SU(2), \quad G^\sigma = J^2(j) = j(j+1) \mathbf{1}^{(j)} \Rightarrow F^j = \frac{1}{3} \cdot j(j+1) \cdot (2j+1)$$

$$\text{For VIR, } \underline{G^\sigma} = T^\sigma \cdot \underline{1} \text{ by Shur. Thm,}$$

$$T^n = \frac{N}{n} F^n \quad \text{where } n = \dim \text{ of rep} \\ N = \# \text{ gen's.}$$

A Trace Theorem for the Rotation Group Lie Algebra.

1. First, recall the fact that the Clebsch-Gordon coefficients bring the direct product representation into block form. This also applies therefore to the direct-product generators. Thus we say:

$$(\underline{J_x})^{j_1 \otimes j_2} = \underline{C}^{j_1 j_2} \underline{F_x} (\underline{C}^{j_1 j_2})^\dagger$$

$$\text{OR, } [(\underline{J_x})^{j_1 \otimes j_2}]_{m_1 m_2, m'_1 m'_2} = \sum_{\substack{JM \\ JM'}} C_{m_1 m_2, JM}^{j_1 j_2} [\delta_{J, J'} (\underline{J_x})_{MM'}^J] C_{J' M', m'_1 m'_2}^{j_1 j_2 \dagger}$$

$$\text{where } C_{m_1 m_2, JM}^{j_1 j_2} = \langle j_1 j_2 m_1 m_2 | JM \rangle = \text{unitary matrix.}$$

Now consider an arbitrary product of direct product generators put into a trace. Eg,

$$\begin{aligned} \text{trace } [\underline{J_x}^{j_1 \otimes j_2} \underline{J_y}^{j_1 \otimes j_2}] &= \text{trace } [\underline{C} \underline{F_x} \underline{C}^{-1} \underline{C} \underline{F_y} \underline{C}^{-1}] = \text{trace } [\underline{F_x} \underline{F_y}] \\ &= \sum_{\substack{JM \\ JM'}} [\delta_{J, J'} (\underline{J_x})_{MM'}^J] [\delta_{J', J} (\underline{J_y})_{M'M}^J] = \sum_J \text{trace } [\underline{J_x}^J \underline{J_y}^J] \end{aligned}$$

If we now label the space of the operators on the trace, we get this result:

$$\boxed{\text{trace}^{(j_1 \otimes j_2)} [\underline{J_a} \underline{J_b} \dots] = \sum_{j=1, j_1, j_2}^{j_1+j_2} \text{trace}^{(j)} [\underline{J_a} \underline{J_b} \dots]} \quad \text{I}$$

2. As the second part of the theorem, we can write out the direct product trace as follows:

$$\begin{aligned} \text{tr} [\underline{J_a}^{j_1 \otimes j_2} \underline{J_b}^{j_1 \otimes j_2} \underline{J_c}^{j_1 \otimes j_2}] &= \\ \text{tr} [(\underline{J_a} \times 1 + 1 \times \underline{J_a})(\underline{J_b} \times 1 + 1 \times \underline{J_b})(\underline{J_c} \times 1 + 1 \times \underline{J_c})] &= \text{tr}^{(j_1)}(\underline{J_a} \underline{J_b} \underline{J_c}) \cdot d^{j_2}(1) \quad \text{since } d(\underline{J_c})=0. \\ &+ \text{tr}^{(j_2)}(\underline{J_a} \underline{J_b} \underline{J_c}) \cdot d^{j_1}(1) + 0^3. \end{aligned}$$

If you trace only three objects, only two of the 8 terms survive because the $\underline{J_i}$ are traceless in any rep because any group rep is assumed to be unimodular. For four or more objects, you will pick up lots of cross terms in an obvious way. So keep to 3:

$$\boxed{\begin{aligned} \text{trace}^{(j_1 \otimes j_2)} [\underline{J_a} \underline{J_b} \underline{J_c}] &= (2j_1+1) \text{trace}^{(j_2)} [\underline{J_a} \underline{J_b} \underline{J_c}] \\ &+ (2j_2+1) \text{trace}^{(j_1)} [\underline{J_a} \underline{J_b} \underline{J_c}] \end{aligned}} \quad \text{II}$$

3. As a test of the theorem, consider the following:

$$\text{trace}^{(j)}(T_a [T_b, T_c]) = i F^j_{abc} \quad \text{where} \quad F^j = \frac{1}{3} j(j+1)(2j+1).$$

Thus both parts of the theorem suggest that:

$$(2j_1+1) i_{abc} F^{j_2} + (2j_2+1) i_{abc} F^{j_1} = \sum_{j=j_1-j_2}^{j_1+j_2} i F^j_{abc}$$

$$\Rightarrow (2j_1+1) F^{j_2} + (2j_2+1) F^{j_1} = \sum_j F^j$$

To see if this is really true, do the j-sum using notes in Series section of Math Binder:

$$\sum (3F^j) = \sum j(j+1)(2j+1) = \sum [2j^3 + 3j^2 + j]$$

$$(A, B, C, D) = (2, 3, 1, 0) \Rightarrow b = 3(2j_1+1) \text{ and } d = j_1(2j_1+1)(j_1+1)$$

$$\Rightarrow \text{sum} = (2j_1+1)(2j_2+1) [j_1(j_1+1) + j_2(j_2+1)] \quad \checkmark$$

add 1: generalization of theorem to $SU(3)$ and beyond.

1. For any group, there is always going to be a unitary Clebsh matrix which brings your direct product to block form. The reps are now labelled conventionally by their dimensionality. Generators are always traceless and Hermitian. The "rows" within a rep will require more than one label, I think its two for $SU(3)$, eg. Part I of the theorem should go through with no problems whatsoever. Of course the sum is only over those reps in the direct product of your starting reps. The second part of the theorem is extremely general, applies for any operators (traceless) in direct product of any two spaces. Thus here is my theorem:

$$I. \quad \text{tr}^{(n \otimes m)} [G_a G_b \dots] = \sum_{l \in n \otimes m} h^{(l)} [G_a G_b \dots]$$

$$II. \quad \text{tr}^{(n \otimes m)} [G_a G_b G_c] = n h^{(m)} [G_a G_b G_c] + m h^{(n)} [G_a G_b G_c]$$

2. Application !A! Consider the commutator-trace relation and see what you get:

$$\sum_{l \in n \otimes m} F^l = n F^m + m F^n$$

Apply to $m \otimes 1$ to get $F^n = 1 \cdot F^n + n \cdot F^1 \Rightarrow F^1 = 0$ (singlet) ✓

Apply to $3 \otimes 3^*$ to get $F^8 = 3 F^3 + 3 F^{3^*}$

But, N.B. $G_a^{(3^*)} = -[G_a^{(3)}]^*$ says Williams } so $(F^{3^*} = F^3)$ if you work it out.

Thus, $F^8 = 6 F^3$. Is this really true?

For Gell-Mann, $\text{tr}(F_i F_i) = \frac{1}{4} \text{tr}(\lambda_i^2) = \frac{1}{4} \cdot 2 = \frac{1}{2} \Rightarrow F^3 = \frac{1}{2}$.

This suggests $F^8 = 3$. That is Ball/Luekt 3! & but its right.

What this says is that we have a "building-up" principle here: once you scale your overall algebra (and thus fix F^2), all the other F 's for the other representations can be determined by applying the above iterative formula. I'll just bet this is true.

add 2: application #2

3. Now consider the D-tensors and see what there is to see:

I have argued that $\text{tr} [G_a^n \{G_b^n, G_c^n\}_+] = D_{abc}^n = \text{T.Sym.}, \underline{\text{real}}.$

Theorem then yields:

$$\sum_{l \in n \otimes m} D_{abc}^l = n D_{abc}^m + m D_{abc}^n.$$

As usual, $D_{abc}^1 = 0$. Now do $3 \otimes 3^*$

$$\begin{aligned} D_{abc}^8 &= 3 D_{abc}^3 + 3 D_{abc}^{3^*} \\ &= 3 D^3 - 3 D^3 \\ &= 0. \end{aligned}$$

$$\text{but } \left(\begin{aligned} D^{(3^*)} &= D[-G^{(3)^*}] \\ &= -D^{(3)} \end{aligned} \right)$$

This agrees with what I know: for the regular rep where G's are given by structure constants, I know that $\text{tr}(G_a G_b G_c) = \text{totally antisym, proportional to } C_{abc} \text{ with no D part.}$ This result generalized quickly to any $SU(n)$. The adjoint rep or regular rep is always the n^2-1 rep which appears in $n \times n^* = (n^2-1)+1$. You will always find that

$$D_{abc}(\text{adjoint rep}) = 0 \quad \text{any group}$$

4. More application: there is no doubt a building up principle here also. For example, look at this case:

$$3 \otimes 3 = 6 \oplus 3^* \Rightarrow D_{abc}^6 + D_{abc}^{3^*} = 3 D^3 + 3 D^3$$

$$\Rightarrow D_{abc}^6 = 7 D_{abc}^3$$

$$\text{Gene says } \left| D_{abc}^3 = \frac{1}{2} d_{abc} \right|$$

I'll bet my hat that you can "build up" your D's in this way for all the UIR's. This means that all D tensors are proportional to the one for the fundamental representation! Thus we can remove the tensors and claim this:

$$D_{abc}^n = D^n \cdot d_{abc} \quad \text{totally sym tensor.}$$

one number for each rep

$$\sum_{l \in n \otimes m} D^l = n D^m + m D^n \quad \text{to build up.}$$

$$D^3 = \frac{1}{2} \quad \text{in } SU(3).$$

add 3 : still more

5. For $SU(2)$, we know by playing with the Pauli matrices that $D^{j=\frac{1}{2}} \text{ rep} = 0$. Thus, as you build up to the higher reps, you keep getting zero. There is no symmetric tensor to worry about in $SU(2)$, and there is only one in $SU(3)$ or any higher group (I think).

6. Therefore, we arrive at some nice results:

$$\text{tr}^{(n)}(G_a [G_b, G_c]) = i F^{(n)}_{abc} \quad \text{where}$$

$$\text{tr}^{(n)}(G_a \{G_b, G_c\}) = D^{(n)}_{abc}$$

$$\Rightarrow \text{tr}^{(n)}(G_a G_b G_c) = \frac{i}{2} F^{(n)}_{abc} + \frac{1}{2} D^{(n)}_{abc}$$

<u>$SU(3)$</u>	<u>n</u>	<u>$F^{(n)}$</u>	<u>$D^{(n)}$</u>
	1	0	0
	3	$\frac{1}{2}$	$\frac{1}{2}$
	3*	$\frac{1}{2}$	$-\frac{1}{2}$
	8	3	0

For $SU(2)$ of course all the D^j are zero, and $F^j = (1/3)j(j+1)(2j+1)$ as found on earlier sheet. I have checked the lowest few to verify that the building up principle does in fact build up the F^j correctly!!!

A relation between the adjoint representation and an arbitrary representation.

1. Recall from SU(2) theory that a "vector operator" can be defined in two equivalent ways, one is global and one infinitesimal:

$$[J_i^{(j)}, V_j^{(j)}] = i \epsilon_{ijk} V_k^{(j)} \quad \leftrightarrow \quad \mathbb{R}^{-1} V_i \mathbb{R} = R_{ij}^{(j)} V_j$$

Generalize to arbitrary group

$$[G_a^\sigma, V_b^\sigma] = i C_{abc} V_c^\sigma \quad \leftrightarrow \quad (U^\sigma)^{-1} V_a^\sigma (U^\sigma) = [U^{(adj)}]_{ab} V_b^\sigma$$

This, then, is the basic true fact about an "adjoint-vector operator" of any kind.

2. In particular, the generators themselves (in any representation you choose) comprise an "adjoint-vector operator". Thus we can say:

Definition of adjoint-vector operator in representation σ :

$$(1) \quad [G_a^\sigma, V_b^\sigma] = i C_{abc} V_c^\sigma$$

$$(2) \quad (U^\sigma)^{-1} V_a^\sigma U^\sigma = U_{ab}^{(adj)} V_b^\sigma$$

$$(U^\sigma)^{-1} G_a^\sigma U^\sigma = U_{ab}^{(adj)} G_b^\sigma$$

Now mult right by G_c^σ , take trace to get

$$U_{ab}^{(adj)} \text{Tr}^\sigma [G_a^\sigma G_b^\sigma G_c^\sigma] = \text{Tr}^\sigma [U_{ab}^{(adj)} G_a^\sigma G_b^\sigma G_c^\sigma] = \text{Tr}^\sigma [U_{ac}^{(adj)} G_a^\sigma G_b^\sigma G_c^\sigma] = \text{Tr}^\sigma [U_{ac}^{(adj)} G_a^\sigma G_b^\sigma G_c^\sigma]$$

$$\Rightarrow \boxed{U_{ab}^{(adj)} = \frac{1}{\text{Tr}^\sigma} \text{Tr}^\sigma [G_a^\sigma G_b^\sigma G_c^\sigma]}$$

This is a very powerful statement because:

- 1) it relates the adjoint rep to an arbitrary rep
- 2) it is a global statement, not just infinitesimal.

first order says: $i (G_a^{(adj)})_{bc} = \frac{1}{i \text{Tr}^\sigma} \text{Tr}^\sigma [G_a^\sigma [G_b^\sigma, G_c^\sigma]] = C_{abc}$

3. I was wondering what you could learn from the second-order comparison of this last equation. After tedious calculation, here is what I got:

$$\sum_{c,d} \Theta_c \Theta_d \left\{ h_{ab,cd} - i \left(\frac{D}{F^2} \right) [C_{dbe} d_{ace} + C_{bce} d_{dae}] \right\} \\ \equiv Q_{ab,cd}$$

where $h_{ab,cd} \equiv \epsilon_{abe} C_{cde}$

Upon examination, you find that both tensors h and Q are antisym under exchange of c,d : thus, you learn nothing at all here! This second-order result is automatically true for any tensors c_{ijk} and d_{ijk} as long as they have the right symmetry. It is amusing to watch how this equation elaborately arranges to be true without yielding up any information! At first I thought it was going to give some relation between the d -tensor and the c -tensor, but no deal. Probably these tensors are independent objects!

And, no doubt higher order terms will be zero (ie, equation will work) again by symmetry: you will in third order have a symmetric rank-3 tensor $\Theta_a \Theta_b \Theta_c$ times something which has each term antisym in some pair so you will get zero.

Of course the left-hand side is not zero.

4. I add this merely as an aside: The Jacobi Identity says this:

$$\left. \begin{aligned} H_{abcd} &= C_{abx} C_{cdx} + C_{bcx} C_{adx} + C_{cax} C_{bdx} \\ &= h_{ab,cd} + h_{bc,ad} + h_{ca,bd} \end{aligned} \right\} = 0$$

I don't know of any way to express the tensor $h_{ab,cd}$ in terms of other objects. I used to think you could write it in the terms of $g_{ab} g_{cd}$ etc, but now I don't think that is correct. Only for $SU(2)$ do you get that simple relation. Note that the object is not a direct-product representation of the Casimir in the adjoint rep. That's because you have to remember that $G^{axb} = G^a x 1 + 1 x G^b$ and you don't get $G^a x G^b$ in this way. Once I tried to get an expression for h in this way, but I got nothing.

But, if you sum on two indices you get [agrees w. Wybourne] $i C_{ab} = C_{ab}^{(wr)}$

$$C_{axy} C_{bxy} = g_{ab} = F^{(adj)} \delta_{ab} = \text{trace}^{(adj)} [G_a G_b]$$

Prediction for SU(3) based on these notes

① Consider $F^a = \frac{1}{8} \text{tr}[G^a]$ in the adjoint rep, we have ... well, this really says:

$$\text{VIR} \Rightarrow G^a = T^a \cdot \underline{1}^n \text{ so } d[\underline{1}^n] = n.$$

Thus,

$$F^a = \frac{1}{N} \cdot n T^a \Rightarrow \boxed{T^a = \left(\frac{N}{n}\right) F^a}$$

② Using my building up, I find that $F^8 = 3$. Thus, I expect $T^8 = \frac{8}{3} \cdot 3$

$$= 8. \text{ So } \sum_{i=1}^8 (G_i)^2 = 3 \cdot \underline{1}.$$

$$\text{Where as, in the } 3 \quad T^3 = \frac{8}{3} \cdot \frac{1}{2} = \frac{4}{3}$$

So I predict: $T^8 = 3 \quad T^3 = \frac{4}{3}$

③ Check with Cas.

$$F_i^2 = \frac{1}{2} \lambda_i$$

$$(17.67) \text{ tells you } T = \frac{1}{3}(p^2 + pq + q^2) + (p+q).$$

$$\text{So } \bar{3} = (p, q) = (1, 0) \Rightarrow \frac{1}{3}(1) + 1 = \frac{4}{3} \checkmark$$

$$8 = (1, 1) \Rightarrow \frac{1}{3}(1+1+1) + (1+1) = 3 \checkmark$$

Thus, for SU(3) we get

$$F^{(p,q)} = \frac{n}{8} T^n$$

$$\Rightarrow F^{(p,q)} = \frac{1}{8} \cdot \left[\frac{1}{2}(p+1)(q+1)(p+q+2) \right] \left[\frac{1}{3}(p^2 + pq + q^2) + (p+q) \right] / \text{SU}(3)$$

Properties of the Fundamental Representation of $SU(n_f)$ ← only section. so restricted!

1. Consider a Lie Algebra for which N = number of generators, and n_f = dimension of one of the fundamental representations. Consider only those algebras where:

$$n_f^2 - 1 = N$$

I think this includes all the $SU(m)$ algebras. Maybe more. Now, examine the fundamental representation which consists of $n_f \times n_f$ matrices. Assume these are Hermitian and traceless. [Yes, that means group rep is special and unitary, so I am talking about general $SU(m)$ case]. Consider this:

$$\{G_a, G_b\} = H_{abc} G_c + Q_{ab} \underline{1}$$

where we are talking only about the fund rep. Easy to show that the anticom is also Hermitian, thus it has only n^2 real parameters, not $2n^2$. This means you can write it as linear combination of the n^2 basic matrices, $n^2 - 1$ of which are the generators which are traceless, the other is the identity matrix. So far we know nothing about H_{abc} and Q_{ab} . However, take trace to get:

$$2 \cdot F^a \delta_{ab} = Q_{ab} \cdot n_f \Rightarrow \boxed{Q_{ab} = \left(\frac{2F^a}{n_f} \right) \delta_{ab}}$$

Next, left mult by G_d then do trace

$$D^a d_{dab} = H_{abc} F^a \delta_{cd} = F^a H_{abd}$$

$$\Rightarrow H_{abd} = (D^a / F^a) d_{abd}$$

$$\boxed{H_{abc} = \left(\frac{D^a}{F^a} \right) d_{abc}}$$

So:

$$\{G_a, G_b\} = \left(\frac{D^a}{F^a} \right) d_{abc} G_c + \left(\frac{2F^a}{n_f} \right) \delta_{ab}$$

where n_f = fund rep of $SU(m) = SU(n_f)$.

For $SU(3)$, $D=F=1/2$, $n_f=3$ so $\{G_a, G_b\} = d_{abc} G_c + \frac{1}{3} \delta_{ab}$

But $G_a = \frac{1}{2} \lambda_a$, so $\{\lambda_a, \lambda_b\} = 2d_{abc} + \frac{4}{3} \delta_{ab}$ ✓

For $SU(2)$, $D=0$, $F=\frac{1}{2} \Rightarrow \{G_a, G_b\} = \left(\frac{2 \cdot \frac{1}{2}}{2} \right) \delta_{ab} = \frac{1}{2} \delta_{ab}$

But $G_a = \frac{1}{2} \sigma_a$, so $\{\sigma_a, \sigma_b\} = 2\delta_{ab}$ ✓

2. Now, still in Sindarog,

$$[G_a, \{G_b, G_c\}] = \{[G_a, G_b], G_c\} + \{[G_a, G_c], G_b\}$$

/ general counsel.

Now left mult by Gd and use find rep facts.

$$\begin{aligned} G_d [G_a, \frac{D}{F^n} d_{bce} G_e] &= G_d i_{Cabe} \{G_e, G_c\} + n i_{Cace} \{G_e, G_b\} \\ &= \cancel{\frac{D}{F^n}} d_{bce} G_d i_{Caef} G_f = i_{Cabe} G_d \left(\cancel{\frac{D}{F}} d_{ecf} G_f + *1 \right) \\ &\quad + i_{Cace} G_d \left(\cancel{\frac{D}{F}} d_{ebf} G_f + *1 \right) \end{aligned}$$

Take der, cancel F^{nf} , get

$$d_{bce} c_{aef} \delta_{df} = c_{abe} d_{ecf} \delta_{df} + c_{ace} d_{ebf} \delta_{df}$$

$$\Rightarrow d_{bee} C_{aed} = C_{abe} d_{ecd} + C_{ace} d_{ebd}$$

$$\Rightarrow d_{bce} C_{dae} = d_{ede} C_{abe} + d_{bde} C_{ace}$$

$$\Rightarrow d_{\text{bee}} \underline{\text{Cade}} + d_{\text{cde}} \underline{\text{Cabe}} + d_{\text{bde}} \underline{\text{Cace}} = 0$$

$$\Rightarrow \underbrace{C_{abe} d_{cde}} + \underbrace{C_{ace} d_{bde}} + \underbrace{C_{ade} d_{bce}} = 0$$

This, then, is a Jacobi-like identity which applies to any $SU(m)$ group.

$$q_{ab, cd} + q_{ac, bd} + q_{ad, bc} = 0$$

I'm not sure of the symmetry shown with the γ added. Anyway, this identity appears as (17.61) in Gas. Now the idea is that you can create a new "adjoint-vector" operator:

$$D_a = \text{const. } d_{abc} \vec{G}_b \vec{G}_c \Rightarrow \underbrace{[G_a, D_b] = i C_{abc} D_c}_{\text{by means of the above identity.}} \quad (\text{not checked})$$

Thus in any representation there are two vector operators you can construct; one is the generators, the other is this combination of the generators. Thus, when you do Wigner Eckart you get two coefficients instead of one when you consider the matrix element of a physics adjoint-vector operator. These are the famous d and f coupling, and it seems that you will have them for any $SU(m)$ group. I don't want to get off on Wigner Eckart right now.

Probably this is the same as the adjoint appearing twice when you take $\text{adj} \times \text{adj}$.

$$[dA] \cdot \det M$$

Contents of this section on Invariance of the Product $[-d\tilde{A}] \det M[A]$. (gauge fields only)

1. Basic set ups. To show that $dJ = \text{Tr}(D (dg/dA))$ and $d[\ln(\det M)] = \text{Tr}(dM M^{-1})$ and how you want these two sum to zero to show desired invariance of product result. Third page on earlier paradoxes. Basically, $\det M$ differs from the invariant FP object. Also, $\det M$ is not invariant by itself even under free gauge transformations.

2. Proof of Theorem in the Covariant Gauge.

- A. expression for $d[\ln(\det M)]$ obtained
- B. expression for dJ obtained
- C. verification that they add to zero (Jacobi required here, θ factored out)
- D. a start at showing the application which appears in Fradkin and Tyutin

3. Proof of Theorem for General Gauge

- A. expression for dJ obtained, all-matrix notation
- B. same expression obtained for $d[\ln(\det M)]$, except minus sign so they add to zero.

4. Summary and Recap of General Gauge Theorem Proof. By equation:

1. General notation

- (1.1) shows what T means (complete Transpose)
- (1.2) definition of S
- (1.3) definition of R
- (1.4) general expression for M
- (1.5) relation D to D^T

2. Calculation of dJ

- (2.1) the condition of constraint
- (2.2) differentiation of (2.1) with respect to field A
- (2.3) definition of scalar operator F
- (2.4) solve (2.2) to get expression for S or (dg/dA) imposed by constraint
- (2.5) dJ final result
- (2.6) general formula for dJ , used to get (2.5)

3. Calculation of $d[\ln(\det M)]$

- (3.1) expression for dM by direct variation
- (3.2) crucial Jacobi/[Dg] Identity
- (3.3) result for dM with identity inserted
- (3.4) general formula for $d[\ln(\det M)]$
- (3.5) explicit final result for $d[\ln(\det M)]$, cancels with (2.5) above
- (3.6) final conclusion of invariance.

2. Field Jacobian for infinitesimal, constrained gauge transformation. Here are two general facts, valid in any gauge and for any gauge transformation:

$$\begin{cases} A_{\mu,a}^g \approx A_{\mu,a} - \frac{1}{g} D_{ab}^\mu \Theta_b \\ \left[\frac{\nabla D_{ab}^\mu}{\nabla A_{c,\nu}} \right] = -g C_{abc} \delta_{\mu\nu} [\underline{1}] \end{cases}$$

Therefore you find that

$$\left[\frac{\nabla A_{\mu,a}^g}{\nabla A_{c,\nu}} \right] = \delta_{\mu\nu} \delta_{ac} [\underline{1}] - \left\{ C_{acb} \delta_{\mu\nu} [\underline{\Theta_b}] + \frac{1}{g} [\underline{D_{ab}^\mu}] \left[\frac{\nabla \Theta_b}{\nabla A_{c,\nu}} \right] \right\}$$

This is of course through first order in the parameters, we are doing an infinitesimal transformation here. The Jacobian for going from dA^g to dA as integration variables in some functional integral is given by the full determinant of this object above. Ie, you want to tie together the space indices, Lorentz indices, and color indices. Consider for the moment that the object on the left above is a matrix H_{ij} in a general sense. Then consider:

$$H = 1 - Q, \quad Q \text{ small.}$$

$$\det H = e^{\text{tr} \ln H} = e^{\text{tr} \ln(1-Q)} = e^{\text{tr}(-Q)} = e^{-\text{tr} Q} \approx 1 - \text{tr} Q.$$

Therefore, the deviation of our desired Jacobian from unity — computed to first order in the group parameters — is given by minus the trace of the quantity in curly brackets above. When you trace, you tie a to c which kills the first term because structure constants are antisymmetric. Thus we have this general result:

$$[dA^g] = J \cdot [dA]$$

$$J = 1 - \frac{1}{g} \text{tr} \left\{ [\underline{D_{ca}^\nu}] \left[\frac{\nabla \Theta_a}{\nabla A_{c,\nu}} \right] \right\}$$

Qualifications:

- 1) infinitesimal gauge trans only
- 2) constrained or unconstrained
- 3) valid in any gauge
- 4) obviously $J=1$ if no constraint.

1. Using work already done, here is what I know:

$$a) \quad [dA^g] = J [dA] \quad J = 1 - \frac{1}{g} \text{tr} \left[[\underline{D}_{ca}^v] \left[\frac{\nabla \theta_a}{\nabla A_c} \right] \right]$$

$$b) \quad \frac{\delta(\det M)}{\det M} = \text{tr} \left[[\underline{\delta m}_{ab}] [\underline{\tilde{m}}_{ba}^{-1}] \right] = \delta(\ln(\det M))$$

Now use these facts to get the first order expected shift in the product of the two Jacobian objects:

$$\begin{aligned} [dA^g] \det M[A^g] &= [dA] J \cdot [\det M + \delta(\det M)] \\ &= [dA] J \cdot \det M[A] (1 + \delta(\ln(\det M))) \\ &= [dA] \det M[A] \cdot \left[1 - \frac{1}{g} \text{tr} \left[D \frac{\nabla \theta}{\nabla A} \right] \right] \left[1 + \text{tr} \left[(\delta m) \tilde{m}^{-1} \right] \right] \end{aligned}$$

Since each of these traces is first order in the parameters, the first-order condition for invariance is this:

$$-\frac{1}{g} \text{tr} \left(D \cdot \frac{\nabla \theta}{\nabla A} \right) + \text{tr} \left([\delta m] \tilde{m}^{-1} \right) = 0 \quad , \text{ie,}$$

$$\boxed{\delta J + \delta(\ln(\det M)) = 0}$$

$$\Rightarrow \text{tr} \left\{ [\underline{\delta m}_{ab}] [\underline{\tilde{m}}_{ba}^{-1}] - \frac{1}{g} [\underline{D}_{ab}^u] \left[\frac{\nabla \theta_b}{\nabla A_a} \right] \right\} = 0$$

$$\Leftrightarrow [dA] \det M[A] = \text{invariant in first order.}$$

This applies to constrained or nonconstrained, infinitesimal, any gauge you want, so far not specified.

Comments: It is very obvious that if you have an unconstrained transformation, the dA jacobian change vanishes because $\Theta = \text{constants}$, independent of A . Thus, the second term in that trace vanishes.

I am not at all convinced that the first term even should vanish. Do I expect it to vanish for an unconstrained gauge transformation, or not? Remember, this is not the FP object, it is the Jacobian object. A priori, I have no idea how this object changes as you gauge-transform off the gauge surface.

Have I ever before claimed that $\det M[A]$ should by itself be invariant as you move off the gauge surface in an unconstrained transformation??? The only place I may have is in the business of "extracting the group volume".

But look at that: as in Lee, you extract the group volume while maintaining the gauge surface delta function. I.e., you always have a fat $d(f-a)$ sitting there. Thus, in the extraction operation, you have the FP object, not the $\det M$ sitting in there. I.e., you never put in $\det M[A]$ until after you have performed the extraction!!! Thus, during the extraction process, you never dealt with invariance of the object $\det M$!!!

So, it is probably not invariant !

This then answers part of an old question: I was never able to show that $\text{trace}[dm m^{-1}]$ vanishes for an unconstrained transformation. That always bothered me. Now I see that it is not zero, and is not expected to be zero! That is why I was never able to prove it!!!

A.

Calculation of variation in the Jacobian object for constrained gauge transformation.

Covariant gauge only, A-fields only.

1. Here is what we want to compute, and here is also what we know to start with:

$$\frac{\delta(\det M)}{(\det M)} = \text{trace} [\bar{M}^{-1} \delta M]$$

$$[M_{ab}] = -\frac{1}{g} [\partial_\mu] [D_{ab}^\mu]$$

OR,

$$M_{ab}(x, y) = -\frac{1}{g} \partial_\mu^\star \left[D_{ab}^\mu[A(x)] \delta(x-y) \right]$$

$$\Rightarrow \delta M_{ab}(x, y) = \int dz \left[\frac{\nabla M_{ab}(x, y)}{\nabla_c^\nu(z)} \right] \delta A_c^\nu(z)$$

But,

$$\frac{\nabla M_{ab}(x, y)}{\nabla A_c^\nu(z)} = -\frac{1}{g} \partial_\mu^\star \left[\left(\frac{\nabla D_{ab}^\mu[A(x)]}{\nabla A_c^\nu(z)} \right) \delta(x-y) \right]$$

$$= -\frac{1}{g} \partial_\mu^{(\star)} \left[-g C_{abc} \delta_{\mu\nu} \delta(x-z) \delta(x-y) \right]$$

$$\boxed{\frac{\nabla M_{ab}(x, y)}{\nabla A_c^\nu(z)} = C_{abc} \partial_\nu^{(\star)} [\delta(x-z) \delta(x-y)]}$$

And,

$$\delta A_c^\nu(z) = -\frac{1}{g} [D_{cd}^\nu(z) \Theta_d(z)]$$

Thus,

$$\begin{aligned} \delta M_{ab}(x, y) &= \int dz \cdot C_{abc} \partial_\nu^{(\star)} [\delta(x-z) \delta(x-y)] \left(-\frac{1}{g} \right) (D_{cd}^\nu(z) \Theta_d(z)) \\ &= -\frac{1}{g} C_{abc} \partial_\nu^{(\star)} \left[\delta(x-y) (D_{cd}^\nu(x) \Theta_d(x)) \right] \end{aligned}$$

$$= -\frac{1}{g} C_{abc} \left\{ (D_{cd}^\nu(x) \Theta_d(x)) [\partial_\nu^{(\star)} \delta(x-y)] + \left(\partial_\nu^{(\star)} [D_{cd}^\nu(x) \Theta_d(x)] \right) \delta(x-y) \right\}$$

$$\boxed{\delta M_{ab}(x, y) = -\frac{1}{g} C_{abc} \partial_\nu^{(\star)} \left[(D_{cd}^\nu(x) \Theta_d(x)) \delta(x-y) \right]}$$

Thus, we get the following formula for $\det M$'s variation:

$$\begin{aligned}
 \text{tr} [\delta m (m')^{-1}] &= \int dx dy [\delta m_{ab}(x, y)] [m_{ba}^{-1}(y, x)] \\
 &= \int dx dy \left[-\frac{1}{g} C_{abc} \partial_\nu^{(x)} \{ [D_{cd}^\nu(x) \Theta_d(x)] \delta(x-y) \} \right] [m_{ba}^{-1}(y, x)] \\
 &= \int dx dy \left(+\frac{1}{g} C_{abc} \right) [D_{cd}^\nu(x) \Theta_d(x)] \delta(x-y) \partial_\nu^{(x)} m_{ba}^{-1}(y, x) \\
 &= \int dx dy \delta(x-y) \left\{ \left[\partial_\nu^{(x)} m_{ba}^{-1}(y, x) \right] \left(\frac{1}{g} C_{abc} \right) [D_{cd}^\nu(x) \Theta_d(x)] \right\} \\
 \frac{1}{g} C_{abc} \text{TR} \left[\left(\partial_\nu^{(x)} m_{ba}^{-1}(y, x) \right) (D_{cd}^\nu(x) \Theta_d(x)) \right] &= \text{tr} [\delta m (m')^{-1}]
 \end{aligned}$$

Repeat:

$$\frac{\delta(\det M)}{\det M} = \frac{1}{g} C_{abc} \text{TR} \left[\left(\partial_\mu^{(x)} m_{ba}^{-1}(y, x) \right) (D_{cd}^\mu(x) \Theta_d(x)) \right]$$

where $\text{TR} [f(x, y)] \equiv \int dx dy \delta(x-y) f(x, y).$

This result is very close to Frad/Tyutin thing. This use of symbol "TR" is helpful because it lets you avoid unpleasant bar notation to indicate derivative of M^{-1} . This really is the simplest way to say it: ie, this is what any other notation would really mean.

$$= -\frac{1}{g} \text{Tr} \left\{ [D] [m^{-1}] [C] [D \Theta]_c \right\}$$

look ok

I am hopeful that hoped-for result may be true for all Θ , in some sense. Therefore I am motivated to "parts off" as suggested by FT to free up the parameters so they have not derivatives acting on them. Thus I get:

$$- \frac{1}{g} C_{abc} \int dx dy \left[\delta(x-y) \left(\partial_\mu^{(x)} M_{ba}^{-1}(y,x) \right) \right] \overleftarrow{D}_{dc}^\mu(x) \Theta_d(x)$$

where I use the usual trick of partsing + index switch.

So:

$$\delta[\ln(\det M)] = - \frac{1}{g} C_{abc} \int dx dy \Theta_d(x) \overrightarrow{D}_{dc}^\mu(x) \left[\delta(x-y) \left(\partial_\mu^{(x)} M_{ba}^{-1}(y,x) \right) \right]$$

B. Calculation of the variation of the Field Jacobian for constrained gauge transformation.
Covariant gauge only. A-fields only.

1. My earlier page on this subject is still correct. There I showed this general result valid for any gauge:

$$\delta J = - \frac{1}{g} \int dx dy \left[D_{ca}^\mu(x) \delta(x-y) \right] \left(\frac{\nabla \Theta_a(y)}{\nabla A_c^\mu(x)} \right)$$

Here lets get $\partial \Theta / \partial A$:

$$\int M_{ab}(x,y) \Theta_b(y) dy = h_a(x)$$

$$\Rightarrow \int \underbrace{\left(\frac{\nabla M_{ab}(x,y)}{\nabla A_c^\mu(z)} \right)}_{C_{abc} \partial_\nu^{(x)} [\delta(x-z) \delta(x-y)]} \Theta_b(y) dy + \int M_{ab}(x,y) \frac{\nabla \Theta_b(y)}{\nabla A_c^\mu(z)} dy = \frac{\nabla h_a(x)}{\nabla A_c^\mu(z)}$$

First term is:

$$\begin{aligned} & C_{abc} \int dy \Theta_b(y) \left\{ \delta(x-z) (\partial_\nu^x \delta(x-y)) + \delta(x-y) (\partial_\nu^x \delta(x-z)) \right\} \\ &= C_{abc} \left[\int dy (\partial_\nu^{(y)} \Theta_b(y)) \delta(x-z) \delta(x-y) + \int dy \Theta_b(y) \delta(x-y) (\partial_\nu^x \delta(x-z)) \right] \\ &= C_{abc} \delta(x-z) [\partial_\nu^{(x)} \Theta_b(x)] + [\partial_\nu^{(x)} \delta(x-z)] \Theta_b(x) \\ &= C_{abc} \partial_\nu^{(x)} [\delta(x-z) \Theta_b(x)] \\ &= C_{abc} \partial_\nu^{(x)} [\Theta_b(x) \delta(x-z)] \\ &= C_{abc} [\underline{\partial_\nu \Theta_b}]_{xz} \end{aligned}$$

So we get $\underbrace{C_{abc} [\underline{\partial_\nu \Theta_b}]} + [\underline{M_{ab}}] \left[\frac{\nabla \Theta_b}{\nabla A_c^\mu} \right] = \left[\frac{\nabla h_a}{\nabla A_c^\mu} \right]$

$$\begin{aligned} & - \cancel{C_{bac}} + \epsilon_{\phi\phi} \\ & - C_{bac} [\underline{\partial_\nu \Theta_b}] = - (C^b)_{ac} [\underline{\partial_\nu \Theta_b}] \end{aligned}$$

Thus we can write

$$\underline{\underline{M_{ab}}} \left[\underline{\underline{\frac{\nabla \theta_b}{\nabla A_\nu^c}}} \right] = \left[\underline{\underline{\frac{\nabla h_a}{\nabla A_\nu^c}}} \right] + (C^b)_{ac} [\underline{\underline{\partial_\nu \circ \theta_b}}]$$

Write as space-coln:

$$\left(\left[\underline{\underline{M}} \right] \left[\underline{\underline{\frac{\nabla \theta}{\nabla A_\nu}}} \right] = \left[\underline{\underline{\frac{\nabla h}{\nabla A_\nu}}} \right] + \left[\underline{\underline{C}} \right]^b [\underline{\underline{\partial_\nu \circ \theta_b}}] \right) \quad \left(\left[\underline{\underline{C}} \right]^b_{xy} = C_{by} \right)$$

Now solve:

$$\left[\underline{\underline{\frac{\nabla \theta}{\nabla A_\nu}}} \right] = \left[\underline{\underline{M}} \right]^{-1} \left[\underline{\underline{\frac{\nabla h}{\nabla A_\nu}}} \right] + \left[\underline{\underline{M}} \right]^{-1} \left[\underline{\underline{C}} \right]^b [\underline{\underline{\partial_\nu \circ \theta_b}}]$$

← agrees with earlier result after repair.

Now write out

$$\frac{\nabla \theta_a(y)}{\nabla A_c^M(x)} = \int dz \, \underline{\underline{M}}_{ad}^{-1}(y,z) \left[\frac{\nabla h_d(z)}{\nabla A_\nu^c(x)} + [C^b]_{dc} \partial_\nu^{(z)} [\theta_b(z) \delta(z-x)] \right]$$

Go ahead, parts over:

$$\begin{aligned} \frac{\nabla \theta_a(y)}{\nabla A_c^M(x)} &= \int dz \, \underline{\underline{M}}_{ad}^{-1}(y,z) \frac{\nabla h_d(z)}{\nabla A_\mu^c(x)} \\ &\quad - \int dz [\partial_\mu^{(z)} \underline{\underline{M}}_{ad}^{-1}(y,z)] [C^b]_{dc} \theta_b(z) \delta(z-x) \end{aligned}$$

For the moment, let's forget about the h-term, Pretend $h = \text{constant}$, as in Lee Zinn.
Then we have only the second term which is:

$$\frac{\nabla \Theta_a(y)}{\nabla A_c^M(x)} = - C_{bdc} \Theta_b(x) \left[\partial_\mu^{(x)} M_{ad}^{-1}(y, x) \right]$$

Therefore,

$$\delta J = + \frac{1}{g} C_{bdc} \int dx dy \Theta_b(x) \left(\vec{D}_{cb}^{(x)}(x) \delta(x-y) \right) \left(\partial_\mu^{(x)} M_{ba}^{-1}(y, x) \right)$$

$\Rightarrow$

$$\delta J = - \frac{1}{g} C_{adc} \int dx dy \Theta_d(x) \left(\vec{D}_{cb}^{(x)}(x) \delta(x-y) \right) \left(\partial_\mu^{(x)} M_{ba}^{-1}(y, x) \right)$$

(C)

Proof of product invariance for the Lee case, covariant gauge only.

1. So far we have computed the variation in the separate pieces, now we need merely add them together. Here is what I get:

$$\left\{ \begin{aligned} \delta[\ln(\det M)] &= -\frac{1}{g} C_{abc} \int dx dy \Theta_d(x) \vec{D}_{dc}^\mu(x) \left[\delta(x-y) (\partial_\mu^{(x)} \tilde{m}_{ba}^{-1}(y,x)) \right] \end{aligned} \right. \quad (1)$$

$$\left\{ \begin{aligned} \delta J &= -\frac{1}{g} C_{adc} \int dx dy \Theta_d(x) \left(\vec{D}_{cb}^\mu(x) \delta(x-y) \right) (\partial_\mu^{(x)} \tilde{m}_{ba}^{-1}(y,x)) \end{aligned} \right. \quad (2)$$

I do have a particular Θ in mind, but it turns out you can show that these two things sum to zero for arbitrary parameters Θ . [The fact of constraint has, of course, already been used in the derivation of the dJ expression. Constraint was not used to get the other one]. In other words, it will be possible to show this:

$$\begin{aligned} & C_{abc} \int dy \vec{D}_{dc}^\mu(x) \left[\delta(x-y) (\partial_\mu^{(x)} \tilde{m}_{ba}^{-1}(y,x)) \right] \\ & + C_{adc} \int dy \left[\vec{D}_{cb}^\mu(x) \delta(x-y) \right] (\partial_\mu^{(x)} \tilde{m}_{ba}^{-1}(y,x)) = 0 \quad (3) \end{aligned}$$

If equation (3) can be shown, then for sure $(1)+(2)=0$ for any function Θ since, in the x integration, Θ multiplies an identical zero function!!!

First, examining the A -independent terms in (3), ie, the terms in D 's which just involve derivative d . You see that there are essentially three terms. They are:

$$\begin{aligned} & \underline{C_{abd}} \int dy [\partial_x^\mu \delta(x-y)] (\partial_\mu \tilde{m}_{ba}^{-1}) \\ & + C_{abd} \int dy [\delta(x-y)] \partial_\mu^\mu \partial_x^\mu \tilde{m}_{ba}^{-1}(y,x) \\ & + \underline{C_{adb}} \int dy [\partial_x^\mu \delta(x-y)] (\partial_\mu \tilde{m}_{ba}^{-1}). \end{aligned}$$

The first and third terms exactly cancel, leaving only the second term which is:

$$+ C_{abd} \int dy \delta(x-y) [\Box_x^2 \tilde{m}_{ba}^{-1}(y,x)] \quad (4)$$

Now consider the other remaining terms, the ones involving A 's. From (3) we get at once:

$$-g[C_{abc}C_{dce} + C_{adc}C_{bce}]A_e^\mu(x) \int dy \delta(x-y) [\partial_\mu^x \tilde{M}_{ba}^{-1}(y,x)]$$

Luckily these C's are in the right form to combine via Jacobi thus:

$$C_{abc}C_{edc} + C_{adc}C_{bec}$$

$$= h_{ab,ed} + h_{ad,be} = -h_{ea,bd} = -C_{eac}C_{bdc}$$

$$\Rightarrow +gC_{bdc}C_{eac}A_e^\mu(x) \int dy \delta(x-y) [\partial_\mu^x \tilde{M}_{ba}^{-1}(y,x)]$$

$$= -gC_{abd}C_{ace}A_e^\mu(x) \int dy \delta(x-y) [\partial_\mu^x \tilde{M}_{bc}^{-1}(y,x)] \quad (5)$$

It will now be shown that the sum of Qe's (4) + (5) equals zero and that therefore equation (3) is true. Adding (4)+(5) we get:

$$C_{abd} \int dy \delta(x-y) \left[\left(\square_x^2 \tilde{M}_{ba}^{-1}(y,x) \right) - gC_{ace}A_e^\mu [\partial_\mu^{(x)} \tilde{M}_{bc}^{-1}(y,x)] \right] \quad (6)$$

However, we can compute the box object as follows. Recall from Lee-paper1 how the transpose business worked:

$$\widehat{M}_{ab}^{(x)} \tilde{M}_{bc}^{-1}(x,y) = \left[-\frac{1}{g} (\partial_\mu^x D_{ab}^\mu(x)) \right] \tilde{M}_{bc}^{-1}(x,y) = \delta_{ac} \delta(x-y)$$

$$\Rightarrow \left[-\frac{1}{g} (D_{ab}^\mu(x) \partial_\mu^x) \right] \tilde{M}_{bc}^{-1T}(x,y) = \delta_{ac} \delta(x-y)$$

The reason this is true is best seen in spacematrix notation; also, recall how LEEZ used this a lot. Note that T here means transpose in both space and color, so we have:

$$\left(-\frac{1}{g} D_{ab}^\mu(x) \partial_\mu^{(x)} \right) \tilde{M}_{cb}^{-1}(y,x) = \delta_{ac} \delta(x-y) \quad (7)$$

$$\left(-\frac{1}{g} \partial_\mu^{(x)} D_{ab}^\mu(x) \right) \tilde{M}_{bc}^{-1}(x,y) = \delta_{ac} \delta(x-y) \quad (8)$$

Thus, we have rules for applying derivatives to either the first or second argument of the M^{-1} Greens function.

In particular, write out (7) in more detail:

$$-\frac{1}{g} [\Box_{(x)}^2 \mathcal{M}_{ba}^{-1}(y, x)] + C_{abe} A_e^\mu [\partial_\mu^{(x)} \mathcal{M}_{bc}^{-1}(y, x)] = \delta_{ab} \delta(x-y)$$

$$\Rightarrow \boxed{\Box_{(x)}^2 \mathcal{M}_{ba}^{-1}(y, x) = -g \delta_{ab} \delta(x-y) + g C_{ace} A_e^\mu(x) [\partial_\mu^{(x)} \mathcal{M}_{bc}^{-1}(y, x)]} \quad (9)$$

If we now stuff this result (9) back into equation (6) we finally get our null result:

$$(6) = C_{abd} \int dy \delta(x-y) [-g \delta_{ab} \delta(x-y)]$$

$$= -g C_{aad} \delta(0)$$

$$= 0 \text{ by color !}$$

Thus we have finally shown that the product $[dA] \det M[A]$ is invariant in first order as you move away from the gauge surface, provided you constrain the way Lee Zinn do Q!!! This then yields their nice derivation of the alpha-Landau gauge.

D.

Modification for use in Tyein thing. ?

1. My results are find for proof of the Lee result (covariant gauge), but they don't quite apply to the Tyutin thing because their D is my M^{-1} transpose, so I have to make a change in my computation of the variation in J. The other piece stays the same.

2. So after some fiddling here is what I find:

$$\Theta_a(x) = \int dy \tilde{M}_{ab}^{-1}(x,y) h_b(y) = \int dy h_b(y) \tilde{M}_{ba}^{-1}(y,x)$$

$$\text{or } \Theta = \frac{1}{k} \cdot [\tilde{M}^{-1}] \Rightarrow \int dy \Theta_b(y) M_{ba}(y,x) = h_a(x)$$

Then get:

$$\int dy \frac{\delta \Theta_b(y)}{\delta A_c^{\mu}(z)} M_{ba}(y,x) + \int dy \Theta_b(y) \frac{\delta M_{ba}(y,x)}{\delta A_c^{\mu}(z)} = \frac{\delta h_a(x)}{\delta A_c^{\mu}(z)}$$

$$(\text{after much fiddle}) \rightarrow C_{abc} (\partial_{\nu}^{(x)} \Theta_b(x)) \delta(x-z)$$

$$\Rightarrow \left[\frac{\delta \Theta_a(y)}{\delta A_c^{\mu}(x)} = + C_{bdc} (\partial_{\mu} \Theta_b(x)) \tilde{M}_{da}^{-1}(x,y) + \int dz \left(\frac{\delta h_d(z)}{\delta A_c^{\mu}(x)} \right) \tilde{M}_{da}^{-1}(z,y) \right] \leftarrow \text{Tyutin mode!}$$

If you compare, you can see that this is very close to the previous Lee-mode result. You can almost connect them by replacing M with its complete transpose everywhere. But, as you can see in the first term above, there is also a parts change which I had know way to know about without doing this calculation. If you take this result and do an x-parts with the first term (without justification since no x-integral here), and if you replace M by its complete transpose, you duplicate the previous result exactly.

$$\text{They use } h_d(z) = k \cdot (\partial_{\nu} A_d^{\nu}(z))$$

$$\Rightarrow \text{second term in box} = -k [\partial_{\mu}^{(x)} \tilde{M}_{ca}^{-1}(x,y)]$$

I then get:

$$\delta J = -\frac{1}{g} \int dx dy \left[\vec{D}_{ca}^{\mu}(x) \delta(x-y) \right] \left\{ C_{bdc} (\partial_{\mu}^{(x)} \Theta_b(x)) \tilde{M}_{da}^{-1}(x,y) - k [\partial_{\mu}^{(x)} \tilde{M}_{ca}^{-1}(x,y)] \right\}$$

I suspect that if you played with this you would get the same dJ as in the mther problem, and they would then cancel in the same way. But here there is still a lot of work to do. You have to remove all derivatives from Θ 's. So you have to parts out the first term. That is no real problem. But the second term above does not contain Θ explicitly, so you have to go "find it" using something like this:

$$\int dx A_c^{\mu}(x) [-k \partial_{\mu}^{(x)} \tilde{M}_{ca}^{-1}(x,y)] = \Theta_a(y).$$

I don't quite see how its going to work, but the thing is there in full detail in Tyutin paper. I will accept their result. My goal was really to show the thing for Lee only, where it is basic to what they are trying to do.

The general gauge case, Lee mode.

1. Since we now have a new and more general object M , we have to write it, then compute dM/dA to get a new matrix equation for object dg/da . This is needed on the way to computing the delta-J variation. So let's do that for a while:

2. The dJ-variation. First, here is what we know:

$$[\underline{M}_{ab}] = -\frac{1}{g} \left[\frac{\nabla f_a}{\nabla A_d^\mu} \right] [\underline{D}_{db}^\mu]$$

OR

$$M_{ab}(x, y) = -\frac{1}{g} \int dt \frac{\nabla f_a(x)}{\nabla A_d^\mu(t)} \left[\vec{D}_{db}^\mu(t) \delta(t-y) \right]$$

$$\Rightarrow \frac{\nabla M_{ab}(x, y)}{\nabla A_c^\nu(z)} = -\frac{1}{g} \int dt \left[\frac{\nabla^2 f_a(x)}{\nabla A_c^\nu(z) \nabla A_d^\mu(t)} \right] \left[\vec{D}_{db}^\mu(t) \delta(t-y) \right] \quad \#1$$

$$-\frac{1}{g} \int dt \left[\frac{\nabla f_a(x)}{\nabla A_d^\mu(t)} \right] \left[\underbrace{\frac{\nabla D_{db}^\mu(t)}{\nabla A_c^\nu(z)} \delta(t-y)}_{-g C_{dbc} \delta_{\mu\nu} \delta(t-z)} \right] \quad \#2$$

$$\#2 = C_{dbc} \left[\frac{\nabla f_a(x)}{\nabla A_d^\nu(z)} \right] \delta(y-z)$$

It might be better to do parts right at the start in definition of M as follows:

$$M_{ab}(x, y) = +\frac{1}{g} \left[\vec{D}_{bd}^\mu(y) \frac{\nabla f_a(x)}{\nabla A_d^\mu(y)} \right]$$

$$\frac{\nabla M_{ab}(x, y)}{\nabla A_c^\nu(z)} = +\frac{1}{g} \left[\underbrace{\left(\frac{\nabla D_{bd}^\mu(y)}{\nabla A_c^\nu(z)} \right) \frac{\nabla f_a(x)}{\nabla A_d^\mu(y)}}_{+g C_{dbc} \delta_{\mu\nu} \delta(y-z)} + \vec{D}_{bd}^\mu(y) \frac{\nabla^2 f_a(x)}{\nabla A_d^\mu(y) \nabla A_c^\nu(z)} \right]$$

$$= C_{dbc} \frac{\nabla f_a(x)}{\nabla A_d^\nu(y)} \delta(y-z) + \frac{1}{g} \vec{D}_{bd}^\mu(y) \frac{\nabla}{\nabla A_d^\mu(y)} \cdot \frac{\nabla f_a(x)}{\nabla A_c^\nu(z)}$$

So:

$$\begin{aligned} \frac{\nabla M_{ab}(x,y)}{\nabla A_c^v(z)} &= C_{dbc} \delta(y-z) \frac{\nabla f_a(x)}{\nabla A_d^v(z)} + \frac{1}{g} \overrightarrow{D}_{bd}^\mu(y) \frac{\nabla}{\nabla A_c^\mu(y)} \frac{\nabla f_a(x)}{\nabla A_c^v(z)} \\ &= \left\{ C_{dbc} \delta(y-z) + \frac{1}{g} \delta_{cd} \overrightarrow{D}_{be}^\mu(y) \frac{\nabla}{\nabla A_c^\mu(y)} \right\} \frac{\nabla f_a(x)}{\nabla A_d^v(z)} \end{aligned}$$

This is nice notation because now everything acts on the gauge surface function which is way over on the right. Just for fun, here is some fancy notation that I might consider using if it is convenient:

$$\begin{aligned} M_{ab}(x,y) &= \frac{1}{g} \left\{ \overrightarrow{D}_{bd}^\mu(y) \frac{\nabla f_a(x)}{\nabla A_d^\mu(y)} \right\} \\ &= \frac{1}{g} \int dt \underbrace{\left[\overrightarrow{D}_{bd}^\mu(y) \delta(y-t) \right]}_{\overrightarrow{D}_{bd}^\mu(y,t)} \underbrace{\left[\frac{\nabla f_a(x)}{\nabla A_d^\mu(y)} \right]}_{\equiv R_{da}^\mu(y,x)} \end{aligned}$$

$$\Rightarrow M_{ba}^T(y,x) = \frac{1}{g} \int dt \overrightarrow{D}_{bd}^\mu(y,t) R_{da}^\mu(y,x)$$

$$\Rightarrow [\underline{M}_{ba}^T] = \frac{1}{g} [\underline{D}_{bd}^\mu] [\underline{R}_{da}^\mu]$$

$$\Rightarrow \boxed{[\underline{M}_{ba}^T] = \frac{1}{g} [\underline{D}_{bd}^\mu] [\underline{R}_{da}^\mu]}$$

Notice how this object R is defined (it agrees exactly with Fradkins). Both the color and space indices are reversed from the way I usually use this thing. So roughly you could say $R = (df/dA)^T$ in complete sense. I'm not yet convinced this notation helps.

Now lets go on to compute the dg/dA object:

$$\int dy \mathcal{M}_{ab}(x, y) \Theta_b(y) = \mathcal{I}_a = \text{constant.}$$

$$\Rightarrow \int dy \frac{\nabla \mathcal{M}_{ab}(x, y)}{\nabla A_c^\nu(z)} \Theta_b(y) + \int dy \mathcal{M}_{ab}(x, y) \frac{\nabla \Theta_b(y)}{\nabla A_c^\nu(z)} = 0.$$

$$\begin{aligned} \Rightarrow \int dy \mathcal{M}_{ab}(x, y) \frac{\nabla \Theta_b(y)}{\nabla A_c^\nu(z)} &= - \int dy \Theta_b(y) \left\{ C_{dbc} \delta(y-z) + \frac{1}{g} \delta_{cd} \vec{D}_{be}^\mu(y) \frac{\vec{\nabla}}{\nabla A_e^\mu(y)} \right\} \frac{\nabla f_a(x)}{\nabla A_d^\nu(z)} \\ &= - C_{dbc} \int dy \Theta_b(y) \delta(y-z) R_{da}^\nu(z, x) \\ &\quad - \frac{1}{g} \delta_{cd} \int dy \Theta_b(y) \vec{D}_{be}^\mu(y) \frac{\vec{\nabla}}{\nabla A_e^\mu(y)} R_{da}^\nu(z, x) \end{aligned}$$

$$\Rightarrow \left([\underline{\mathcal{M}_{ab}}] \left[\frac{\nabla \Theta_b}{\nabla A_c^\nu} \right] \right)^T = - C_{dbc} [\underline{\Theta_b}] [\underline{R_{da}^\nu}] - \frac{1}{g} \delta_{cd} \left(\int dy \Theta_b(y) \vec{D}_{be}^\mu(y) \frac{\vec{\nabla}}{\nabla A_e^\mu(y)} \right) [\underline{R_{da}^\nu}] \left. \vphantom{\left(\int dy \Theta_b(y) \vec{D}_{be}^\mu(y) \frac{\vec{\nabla}}{\nabla A_e^\mu(y)} \right)} \right\} \text{ignore}$$

Now let's define an object like R for dg/da derivative. Do this:

$$\boxed{S_{cb}^\nu(z, y) \equiv \frac{\nabla \Theta_b(y)}{\nabla A_c^\nu(z)}}$$

Then:

$$\int dy \mathcal{M}_{ab}(x, y) \frac{\nabla \Theta_b(y)}{\nabla A_c^\nu(z)} = \int dy S_{cb}^\nu(z, y) \mathcal{M}_{ba}^\top(y, x)$$

Then,

$$\begin{aligned} [\underline{S_{cb}^\nu}] [\underline{\mathcal{M}_{ba}^\top}] &= - C_{cd}^b [\underline{\Theta_b}] [\underline{R_{da}^\nu}] \\ &\quad - \frac{1}{g} \delta_{cd} \left\{ \left(\int dy \Theta_b(y) \vec{D}_{be}^\mu(y) \frac{\vec{\nabla}}{\nabla A_e^\mu(y)} \right) [\underline{R_{da}^\nu}] \right\} \left. \vphantom{\left(\int dy \Theta_b(y) \vec{D}_{be}^\mu(y) \frac{\vec{\nabla}}{\nabla A_e^\mu(y)} \right)} \right\} \text{OK} \end{aligned}$$

OR

$$[\underline{\underline{S}}^{\nu}] [\underline{\underline{m}}^{\tau}] = - [\underline{\underline{O}}_b] [\underline{\underline{C}}^b] [\underline{\underline{R}}^{\nu}] - \frac{1}{g} \left\{ \left(\int dy \, O_b(y) \widehat{D}_{be}^{\mu}(y) \frac{\vec{\nabla}}{\nabla A_e^{\mu}(y)} \right) [\underline{\underline{R}}^{\nu}] \right\}$$

$$\Rightarrow [\underline{\underline{S}}^{\nu}] = - [\underline{\underline{O}}_b] [\underline{\underline{C}}^b] [\underline{\underline{R}}^{\nu}] [\underline{\underline{m}}^{\tau}]^{-1} - \frac{1}{g} \left\{ \left(\int dy \, O_b(y) \widehat{D}_{be}^{\mu}(y) \frac{\vec{\nabla}}{\nabla A_e^{\mu}(y)} \right) [\underline{\underline{R}}^{\nu}] \right\} [\underline{\underline{m}}^{\tau}]^{-1}$$

Define:

$$-\frac{1}{g} \left(\int dy \, O_b(y) \widehat{D}_{be}^{\mu}(y) \frac{\vec{\nabla}}{\nabla A_e^{\mu}(y)} \right) \equiv \vec{F} = \text{an operator}$$

Then

$$[\underline{\underline{S}}^{\nu}] = - [\underline{\underline{O}}_b] [\underline{\underline{C}}^b] [\underline{\underline{R}}^{\nu}] [\underline{\underline{m}}^{\tau}]^{-1} + (\vec{F} [\underline{\underline{R}}^{\nu}]) [\underline{\underline{m}}^{\tau}]^{-1}$$

This, then, is our solution for dg/DA. Notice that F is some kind of scalar operator. I guess I could represent it as a color/space matrix, but I'll leave it as is because I hope eventually it will cancel against a similar object. For covariant gauge this second term vanishes. Let's pause to check that the first term agrees with what I got before in the covariant case (a consistency check). It certainly looks the same:

check: $f_a(x) = \partial_{\mu}^{(x)} A_a^{\mu}(x)$

$$\Rightarrow \frac{\nabla f_a(x)}{\nabla A_d^{\nu}(y)} = \delta_{ad} [\partial_{\nu}^{(x)} \delta(x-y)] = R_{da}^{\nu}(y, x)$$

$$\Rightarrow R_{da}^{\nu}(y, x) = - \delta_{da} [\partial_{\nu}^{(y)} \delta(y-x)]$$

$$\rightarrow [\underline{\underline{R}}^{\nu}] = - [\underline{\underline{1}}] [\underline{\underline{\partial}}_{\nu}] \quad / \text{covariant gauge.}$$

$$\Rightarrow [\underline{\underline{S}}^{\nu}] = + [\underline{\underline{O}}_b] [\underline{\underline{C}}^b] [\underline{\underline{\partial}}_{\nu}] [\underline{\underline{m}}^{\tau}]^{-1}$$

$$\begin{aligned}
 \Rightarrow S_{ca}^{\vee}(x, y) &= C_{cd}^b \left(\underline{\underline{\Theta_b \circ \partial_{\nu}}} \right) \left[\underline{\underline{M_{da}^{-1}}} \right]_{xy} \\
 &= C_{bcd} \int dt \Theta_b(x) [\partial_{\nu}^{(x)} \delta(x-t)] M_{da}^{-1}(t, y) \\
 &= C_{bcd} \int dt \Theta_b(x) \delta(x-t) \partial_{\nu}^{(x)} M_{da}^{-1}(t, y) \\
 &= C_{bcd} \Theta_b(x) \partial_{\nu}^{(x)} M_{da}^{-1}(x, y) \\
 &= C_{bcd} \Theta_b(x) [\partial_{\nu}^{(x)} M_{ad}^{-1}(y, x)] \\
 &= - C_{bdc} \Theta_b(x) [\partial_{\nu}^{(x)} M_{ad}^{-1}(y, x)] \\
 &= \frac{\nabla \Theta_a(y)}{\nabla A_c^{\vee}(x)} \quad \checkmark
 \end{aligned}$$

Or, from the start,

$$\begin{aligned}
 S_{ca}^{\vee}(x, y) &= C_{bcd} \left(\underline{\underline{\Theta_b \circ \partial_{\nu} \circ M_{da}^{-1}}} \right)_{xy} \\
 &= C_{bcd} \Theta_b(x) \partial_{\nu}^{(x)} M_{da}^{-1}(x, y) \quad \checkmark
 \end{aligned}$$

So, in the covariant case this does duplicate our former result, I am glad. to say.

Now we have a general formula for dJ as follows:

$$\begin{aligned}
 \delta J &= -\frac{1}{g} \int dx dy \ D_{ca}^{\vee}(x, y) \frac{\nabla \Theta_a(y)}{\nabla A_c^{\vee}(x)} \\
 &= -\frac{1}{g} \int dx dy \ D_{ca}^{\vee}(x, y) S_{ca}^{\vee}(x, y) \\
 &= -\frac{1}{g} \int dx dy \ D_{ca}^{\vee}(x, y) S_{ac}^{\vee}(y, x) \\
 &= -\frac{1}{g} \text{tr} \left[\left[\underline{\underline{D_{ca}^{\vee}}} \right] \left[\underline{\underline{S_{ac}^{\vee}}} \right] \right] = -\frac{1}{g} \text{Tr} \left(\left[\underline{\underline{D_{ca}^{\vee}}} \right] \left[\underline{\underline{S_{ac}^{\vee}}} \right] \right) \\
 &= -\frac{1}{g} \text{Tr} \left(\left[\underline{\underline{D_{ca}^{\vee}}} \right]^T \left[\underline{\underline{S_{ac}^{\vee}}} \right] \right) \quad \text{since } \text{tr}(AB^T) = \text{tr}(A^TB).
 \end{aligned}$$

But,

$$[\underline{\underline{D}}^\nu]^T = -[\underline{\underline{D}}^\nu]$$

proof:

$$D_{ab}^\nu(x, y) = \overrightarrow{D_{ab}^\nu(x)} \delta(x-y)$$

$$= -\overrightarrow{D_{ba}^\nu(y)} \delta(y-x)$$

$\partial/\partial x \rightarrow -\partial/\partial y$ etc.

$$-D_{ab}^\nu(x, y) = -\overrightarrow{D_{ba}^\nu(y, x)}$$

$$= -\overrightarrow{D_{ba}^\nu(y)} \delta(y-x)$$

Therefore:

$$\delta J = + \frac{1}{g} \text{Tr} \left([\underline{\underline{D}}^\nu] [\underline{\underline{S}}^\nu] \right)$$

So, using our compact color/space matrix expression for S appearing on page 4, we have a complete result for δJ in any gauge.

$$\delta J = -\frac{1}{g} \text{Tr} \left([\underline{\underline{D}}^\nu] [\underline{\underline{\Theta}}_b] [\underline{\underline{C}}^b] [\underline{\underline{R}}^\nu] [\underline{\underline{M}}^{-T}] \right) + \frac{1}{g} \text{Tr} \left([\underline{\underline{D}}^\nu] (\vec{F} [\underline{\underline{R}}^\nu]) (\underline{\underline{M}}^{-T}) \right)$$

(1)

3. Now compute the variation in the FP determinant object. A comment is in order here.

Again we will need the object $dM(x,y)/dA(z)$. In the dJ business just discussed, this entity had to be integrated against a function of variable y , and for that reason we wrote the object out as a y - z operator acting on an xz matrix (see page 2 above). The reason was that then the xz matrix just sits there, impervious to the y -integrations.

Now however, to compute dM we are going to take the same object and integrate it against a function of z , namely $dA(z)$. This suggests to me that I try to write this thing as a y - z operator times an xy matrix. Here is what I get:

$$\frac{\nabla M_{ab}(x,y)}{\nabla A_c^\mu(z)} = \left\{ C_{dbc} \delta(y-z) \delta_{\mu\nu} + \frac{1}{g} \vec{D}_{bd}(y) \frac{\vec{\nabla}}{\nabla A_c^\mu(z)} \right\} \underbrace{\frac{\nabla f_a(x)}{\nabla A_d^\nu(y)}}_{R_{da}^\nu(y,x)}$$

Then,

$$\delta M_{ab}(x,y) = \int dz \underbrace{\delta A_c^\mu(z)}_{\left[-\frac{1}{g} \vec{D}_{cf}^\mu(z) \Theta_f(z) \right]} \left[\frac{\nabla M_{ab}(x,y)}{\nabla A_c^\mu(z)} \right]$$

I will parts this D^μ over right away to get:

$$\begin{aligned} \delta M_{ab}(x,y) &= + \frac{1}{g} \int dz \Theta_f(z) \vec{D}_{fc}^\mu(z) \left[\frac{\nabla M_{ab}(x,y)}{\nabla A_c^\mu(z)} \right] \\ &= (- [\vec{F}] M_{ab}(x,y)) \end{aligned}$$

Here I suddenly realize that I have that same invariant operator F I had before. To write this last line I don't need the thing at the top of this page (yet). So lets go ahead and compute the variation in $\ln \det M$:

$$\delta [\ln(\det M)] = -\text{Tr} \left(\{ \vec{F}[\underline{m}] \} [\underline{m}^{-1}] \right)$$

But this does not seem to help very much, so I'll go back and use top-of-page result:

$$\delta M_{ab}(x,y) = \frac{1}{g} C_{dbc} \int dz \Theta_F(z) \left[\overrightarrow{D_{fc}^{\wedge}}(z) \delta(z-y) \right] R_{da}^{\wedge}(y,x) \\ - \frac{1}{g} \overrightarrow{D_{bd}^{\vee}}(y) \left[(\overrightarrow{F}) R_{da}^{\vee}(y,x) \right]$$

Lets deal first with the second term which I will now express as:

$$[\underline{\underline{\delta m^T}}] = - \frac{1}{g} [\underline{\underline{D^{\vee}}}] \left(\overrightarrow{F} [\underline{\underline{R^{\vee}}}] \right) + \text{omitted first term.}$$

Then the contribution to $\delta [\ln(\det M)]$ is

$$\delta [\ln(\det M)] = \text{Tr} \left[[\underline{\underline{\delta m}}] [\underline{\underline{m^{-1}}}] \right] = \text{Tr} \left[[\underline{\underline{\delta m^T}}] [\underline{\underline{m^{-1T}}}] \right]$$

since $\text{tr}(AB) = \text{tr}(B^T A^T) = \text{tr}(A^T B^T)$. Thus.

$$\delta [\ln(\det M)] = - \frac{1}{g} \text{Tr} \left\{ [\underline{\underline{D^{\vee}}}] \left(\overrightarrow{F} [\underline{\underline{R^{\vee}}}] \right) [\underline{\underline{m^{-1T}}}] \right\} \quad (2)$$

This exactly cancels the second term in Eq. (1), as we had hoped!!! Thus, we need only consider the first term above. Notice that this same cancellation occurred in Lee appendix. Now I can forget about the functional differential operator F . No need to worry about exactly what it does, since terms cancelled. So:

$$[\underline{\underline{\delta m_{ba}^T}}] = - \frac{1}{g} C_{bc}^d [\underline{\underline{D_{cf}^{\wedge}}} \Theta_F] [\underline{\underline{R_{da}^{\wedge}}}] = \text{first term.}$$

$$\Rightarrow \delta [\ln(\det M)] = - \frac{1}{g} C_{bc}^d \text{tr} \left\{ [\underline{\underline{D_{cf}^{\wedge}}} \Theta_F] [\underline{\underline{R_{da}^{\wedge}}}] [\underline{\underline{m_{ab}^{-1T}}}] \right\}$$

To get the above "first term" I had to parts the D over onto the Θ (which I think is where it was in the first place). This result here is similar to but different from the first term in (1). I know we somehow will need the Jacobis.....

Now, let's momentarily consider just the d^u contained in the D^u in our result:

$$\text{term 1} = -\frac{1}{g} C_{ebc} \text{tr} \left\{ \delta_{ed} [\underline{\partial}_\mu \underline{\Theta}_c] [\underline{R}_{da}^\mu] [\underline{M}_{ab}^{-1}] \right\}$$

Notice that the f color sum died, and we put in a new delta δ_{ed} by hand. The reason for this is so I can use my theorem derived elsewhere, namely:

$$\delta_{ed} [\underline{\partial}_\mu \underline{\Theta}_c] = \underbrace{[\underline{D}_{ed}^\mu] [\underline{\Theta}_c]}_{\rightarrow \text{term 1A}} - \underbrace{[\underline{\Theta}_c] [\underline{D}_{ed}^\mu]}_{\rightarrow \text{term 1B}}$$

I will now show that term (1B) in fact vanishes. Watch:

$$\begin{aligned} \text{term (1B)} &= +\frac{1}{g} C_{ebc} \text{tr} \left\{ [\underline{\Theta}_c] [\underline{D}_{ed}^\mu] [\underline{R}_{da}^\mu] [\underline{M}_{ab}^{-1}] \right\} \\ &= [\underline{M}_{ea}^T] \cdot g \end{aligned}$$

$$= C_{ebc} \text{tr} \left\{ [\underline{\Theta}_c] \delta_{eb} [\underline{1}] \right\} = C_{bbc} \text{tr} \{ [\underline{\Theta}_c] \} = 0 \text{ by color.}$$

Thus we are left with term (1A) plus the term in the result on past page that we have not yet included. Adding these two terms here, we are left with:

$$\begin{aligned} &-\frac{1}{g} C_{ebc} \text{tr} \left\{ [\underline{D}_{ed}^\mu] [\underline{\Theta}_f] [\underline{R}_{da}^\mu] [\underline{M}_{ab}^{-1}] \right\} \\ &-\frac{1}{g} C_{dbc} \text{tr} \left\{ -g C_{cfh} [\underline{A}_h^\mu] [\underline{\Theta}_f] [\underline{R}_{da}^\mu] [\underline{M}_{ab}^{-1}] \right\} \end{aligned}$$

Notice that this first term is very close to our page 6 equation (1) result, Eq. (1), but not quite. For example, in the above thing, a color index on R does not tie to the C , whereas it does on page 6. So let's expand the first term so we then have three terms:

$$= -\frac{1}{g} \text{tr} \left\{ \left[C_{ebf} \delta_{ed} [\underline{\partial}^\mu \underline{\Theta}_f] - g (C_{ebf} C_{edh} + C_{dbc} C_{cfh}) [\underline{A}_h^\mu \underline{\Theta}_f] \right] : [\underline{R}_{da}^\mu] [\underline{M}_{ab}^{-1}] \right\}$$

Now comes the famous Jacobi combo:

$$C_{bf} C_{dh} + C_{db} C_{fh} = - C_{fd} C_{bh}$$

/see "group"

Then the square bracket becomes

$$\begin{aligned} & \left[C_{dbf} \left[\underline{\underline{\partial}}^\mu \right]_{\uparrow} + g C_{fd} C_{bh} \left[\underline{\underline{A}}^\mu_h \right] \right] \left[\underline{\underline{\theta}}_f \right] \\ & \quad \quad \quad \delta_{bx} \\ & = C_{fd} \left[\left[\underline{\underline{\partial}}^\mu \right] \delta_{bx} - g C_{bh} \left[\underline{\underline{A}}^\mu_h \right] \right] \cdot \left[\underline{\underline{\theta}}_f \right] \\ & = C_{fd} \left[\underline{\underline{D}}^\mu_{bx} \right] \left[\underline{\underline{\theta}}_f \right] = - \left[\underline{\underline{D}}^\mu_{bx} \right] \left[\underline{\underline{\theta}}_f \right] C_{fd} \end{aligned}$$

Thus we get

$$\begin{aligned} \delta[\ln(\det M)] &= + \frac{1}{g} \text{Tr} \left\{ \left[\underline{\underline{D}}^\mu_{bx} \right] \left[\underline{\underline{\theta}}_f \right] C_{fd} \left[\underline{\underline{R}}^\mu_{da} \right] \left[\underline{\underline{M}}^{\tilde{-1}}_{ab} \right] \right\} \\ &= + \frac{1}{g} \text{Tr} \left\{ \left[\underline{\underline{D}}^\nu \right] \left[\underline{\underline{\theta}}_b \right] \left[\underline{\underline{C}}^b \right] \left[\underline{\underline{R}}^\nu \right] \left[\underline{\underline{M}}^{\tilde{-1}} \right] \right\} \quad (3) \end{aligned}$$

You see finally that this cancels the first term of (1), so we are now done!!!!!!!!!!!!!!

In other words, $(2)+(3)+(1) = 0$ and we have shown the product invariance that we set out to show, in any gauge, Lee mode.

A summary of the general gauge invariance theorem.

1. The notation is extremely complicated, so I will try to pick out the critical equations. First, here are the basic definitions and relations:

a) meaning of transpose T :

$$\boxed{F_{ab}^T(x,y) = F_{ba}(y,x)} \quad / \text{does both} \quad (1.1)$$

b) constraint object

$$\left[\frac{\nabla \theta}{\nabla A^\nu} \right]_{\approx xy}^{ab} = \frac{\nabla \theta_a(x)}{\nabla A_b^\nu(y)} = \left[\underline{\underline{S^\nu}} \right]_{\approx xy}^{ab} = S_{ab}^\nu(x,y) = S_{ba}^\nu(y,x)$$

Thus, we use

$$\boxed{\left[\frac{\nabla \theta}{\nabla A^\nu} \right]_{\approx} = \left[\underline{\underline{S^\nu}} \right]^T} \quad (1.2)$$

c) gauge surface object

$$\left[\frac{\nabla f}{\nabla A^\nu} \right]_{\approx xy}^{ab} = \frac{\nabla f_a(x)}{\nabla A_b^\nu(y)} = R_{ab}^\nu(x,y)^T = R_{ba}^\nu(y,x)$$

Thus, we use

$$\boxed{\left[\frac{\nabla f}{\nabla A^\nu} \right]_{\approx} = \left[\underline{\underline{R^\nu}} \right]^T}$$

$$\boxed{\left[\underline{\underline{R^\mu}} \right] = - \left[\underline{\underline{1}} \right] \left[\underline{\underline{\partial^\mu}} \right] \text{ covariant gauge}}$$

(1.3)

d) One object M

$$\boxed{\begin{aligned} \left[\underline{\underline{M}} \right] &= -\frac{1}{g} \left[\frac{\nabla f}{\nabla A^\mu} \right] \left[\underline{\underline{D^\mu}} \right] \\ \left[\underline{\underline{M^T}} \right] &= +\frac{1}{g} \left[\underline{\underline{D^\mu}} \right] \left[\underline{\underline{R^\mu}} \right] \end{aligned}} \quad (1.4)$$

$$\boxed{\left[\underline{\underline{D^\mu}} \right]^T = - \left[\underline{\underline{D^\mu}} \right]} \quad (1.5)$$

2.

δJ

In computing the field-jacobian variation, we need first to know how to implement the condition of constraint. Here is how it went:

$$\begin{bmatrix} \underline{\underline{M}} \end{bmatrix} \begin{bmatrix} \underline{\underline{\Theta}} \end{bmatrix} = \begin{bmatrix} \underline{\underline{h}} \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} \underline{\underline{\Theta}} \end{bmatrix} \begin{bmatrix} \underline{\underline{M}}^T \end{bmatrix} = \begin{bmatrix} \underline{\underline{h}} \end{bmatrix} \quad (2.1)$$

Differentiate to get

$$\begin{aligned} \begin{bmatrix} \underline{\underline{S}}^V \end{bmatrix} \begin{bmatrix} \underline{\underline{M}}^T \end{bmatrix} + \begin{bmatrix} \underline{\underline{\Theta}}_b \end{bmatrix} \begin{bmatrix} \underline{\underline{C}}^b \end{bmatrix} \begin{bmatrix} \underline{\underline{R}}^V \end{bmatrix} - \vec{\mathcal{F}} \begin{bmatrix} \underline{\underline{R}}^V \end{bmatrix} &= 0 \\ \begin{bmatrix} \underline{\underline{M}} \end{bmatrix} \begin{bmatrix} \underline{\underline{\nabla \Theta}} \end{bmatrix} - \begin{bmatrix} \underline{\underline{\nabla f}} \end{bmatrix} \begin{bmatrix} \underline{\underline{C}}^b \end{bmatrix} \begin{bmatrix} \underline{\underline{\Theta}}_b \end{bmatrix} - \vec{\mathcal{F}} \begin{bmatrix} \underline{\underline{\nabla f}} \end{bmatrix} &= 0 \end{aligned} \quad (2.2)$$

where

$$\begin{aligned} \vec{\mathcal{F}} &\equiv -\frac{1}{g} \text{Tr} \left\{ \begin{bmatrix} \underline{\underline{\Theta}}_b \end{bmatrix} \begin{bmatrix} \underline{\underline{D}}_{be}^A \end{bmatrix} \begin{bmatrix} \underline{\underline{\nabla}} \end{bmatrix} \right\} \\ &= -\frac{1}{g} \left(\int dy \, \Theta_b(y) \vec{D}_{be}^A(y) \frac{\vec{\nabla}}{\nabla A_e^A(y)} \right) \end{aligned} \quad (2.3)$$

Solving the above for $(\nabla \Theta / \nabla A)$ we get.

$$\begin{aligned} \begin{bmatrix} \underline{\underline{S}}^V \end{bmatrix} &= \left\{ \left(\vec{\mathcal{F}} \begin{bmatrix} \underline{\underline{R}}^V \end{bmatrix} \right) - \begin{bmatrix} \underline{\underline{\Theta}}_b \end{bmatrix} \begin{bmatrix} \underline{\underline{C}}^b \end{bmatrix} \begin{bmatrix} \underline{\underline{R}}^V \end{bmatrix} \right\} \begin{bmatrix} \underline{\underline{M}}^T \end{bmatrix}^{-1} \\ \begin{bmatrix} \underline{\underline{\nabla \Theta}} \end{bmatrix} &= \begin{bmatrix} \underline{\underline{M}} \end{bmatrix}^{-1} \left\{ \vec{\mathcal{F}} \begin{bmatrix} \underline{\underline{\nabla f}} \end{bmatrix} + \begin{bmatrix} \underline{\underline{\nabla f}} \end{bmatrix} \begin{bmatrix} \underline{\underline{C}}^b \end{bmatrix} \begin{bmatrix} \underline{\underline{\Theta}}_b \end{bmatrix} \right\} \end{aligned} \quad (2.4)$$

Then we get

$$\begin{aligned} \delta J &= +\frac{1}{g} \text{Tr} \left(\begin{bmatrix} \underline{\underline{D}}^V \end{bmatrix} \left\{ \vec{\mathcal{F}} \begin{bmatrix} \underline{\underline{R}}^V \end{bmatrix} - \begin{bmatrix} \underline{\underline{\Theta}}_b \end{bmatrix} \begin{bmatrix} \underline{\underline{C}}^b \end{bmatrix} \begin{bmatrix} \underline{\underline{R}}^V \end{bmatrix} \right\} \begin{bmatrix} \underline{\underline{M}}^T \end{bmatrix}^{-1} \right) \\ &= -\frac{1}{g} \text{Tr} \left(\begin{bmatrix} \underline{\underline{M}} \end{bmatrix}^{-1} \left\{ \vec{\mathcal{F}} \begin{bmatrix} \underline{\underline{\nabla f}} \end{bmatrix} + \begin{bmatrix} \underline{\underline{\nabla f}} \end{bmatrix} \begin{bmatrix} \underline{\underline{C}}^b \end{bmatrix} \begin{bmatrix} \underline{\underline{\Theta}}_b \end{bmatrix} \right\} \begin{bmatrix} \underline{\underline{D}}^V \end{bmatrix} \right) \end{aligned} \quad (2.5)$$

In this last we have used the general formula for a variation dJ :

$$\delta J = + \frac{1}{g} \text{Tr} \left(\underline{\underline{D}}^\nu \underline{\underline{S}}^\nu \right) - \frac{1}{g} \text{Tr} \left(\underline{\underline{\frac{\nabla \theta}{\nabla A}}} \underline{\underline{D}}^\nu \right) \quad (2.6)$$

③ Now we turn to the second part, the variation in $\det M$. First I found this:

$$[\delta \underline{\underline{M}}^T_{ba}] = - \frac{1}{g} [\underline{\underline{D}}^\nu_{bc}] (\vec{F} [\underline{\underline{R}}^\nu_{ca}]) - \frac{1}{g} C_{bc}^e \{ \delta_{ed} [\underline{\underline{D}}^\mu_{cf} \underline{\underline{\theta}}_f] \} [\underline{\underline{R}}^\mu_{da}] \quad \text{or}$$

I then expanded a piece of the second term:

$$\{ \delta_{ed} [\underline{\underline{D}}^\mu_{cf} \underline{\underline{\theta}}_f] \} = \underbrace{[\underline{\underline{D}}^\mu_{ed}][\underline{\underline{\theta}}_c] - [\underline{\underline{\theta}}_c][\underline{\underline{D}}^\mu_{ed}]}_{\text{exp.}} - \delta_{ed} g C_{cfh} [\underline{\underline{A}}^\mu_h \underline{\underline{\theta}}_f]$$

$$\Rightarrow C_{ebc} \{ \delta_{ed} [\underline{\underline{D}}^\mu_{cf} \underline{\underline{\theta}}_f] \} = C_{dbc} [\underline{\underline{\theta}}^\mu_c][\underline{\underline{\theta}}_e] - C_{ebc} [\underline{\underline{\theta}}_e][\underline{\underline{D}}^\mu_{ed}] \\ - g (C_{ebf} C_{edh} + C_{dbc} C_{cfh}) [\underline{\underline{A}}^\mu_h \underline{\underline{\theta}}_f]$$

$$\Rightarrow C_{dbc} [\underline{\underline{D}}^\mu_{cf} \underline{\underline{\theta}}_f] = - C_{ebc} [\underline{\underline{\theta}}_e][\underline{\underline{D}}^\mu_{ed}] - [\underline{\underline{D}}^\mu_{bc}] C_{cd}^f [\underline{\underline{\theta}}_f] \quad (*)$$

This is the line that really makes it go. It is a combination of my one theorem which pulls apart d_u , and the Jacobi identity. Substituting this back in the above you get:

$$[\delta \underline{\underline{M}}^T_{ba}] = - \frac{1}{g} [\underline{\underline{D}}^\nu_{bc}] (\vec{F} [\underline{\underline{R}}^\nu_{ca}]) + \frac{1}{g} \{ C_{ebc} [\underline{\underline{\theta}}_e][\underline{\underline{D}}^\mu_{ed}] + [\underline{\underline{D}}^\nu_{bc}] C_{cd}^f [\underline{\underline{\theta}}_f] \} [\underline{\underline{R}}^\mu_{da}]$$

But identify DR as M^T so that

$$[\delta \underline{\underline{M}}^T] = - [\underline{\underline{C}}^c][\underline{\underline{\theta}}_c][\underline{\underline{M}}^T] - \frac{1}{g} [\underline{\underline{D}}^\nu] \{ (\vec{F} [\underline{\underline{R}}^\nu]) - [\underline{\underline{C}}^b][\underline{\underline{\theta}}_b][\underline{\underline{R}}^\nu] \}$$

④ This comes from identity (*) which I'll write like so:

$$[\underline{\underline{C}}^c][\underline{\underline{D}}^\mu_{ab}] = [\underline{\underline{D}}^\mu_{ab}][\underline{\underline{C}}^c][\underline{\underline{\theta}}_c] - [\underline{\underline{\theta}}_c][\underline{\underline{C}}^c][\underline{\underline{D}}^\mu_{ab}]$$

I now see how to write all these things in matrix notation. First, by direct computation of the variation in M we get:

$$[\underline{\underline{\delta m^T}}] = -\frac{1}{g} [\underline{\underline{D^v}}] (\tilde{F} [\underline{\underline{R^v}}]) + \frac{1}{g} [\underline{\underline{C^c}}] [\underline{\underline{D^u}}]_c [\underline{\underline{R^u}}] \quad (3.1)$$

Then we insert into these these identities, in order:

$$\begin{aligned} [\underline{\underline{C^c}}] [\underline{\underline{D^u}}]_c &= [\underline{\underline{D^u}}] [\underline{\underline{C^c}}] [\underline{\underline{\Theta_c}}] - [\underline{\underline{\Theta_c}}] [\underline{\underline{C^c}}] [\underline{\underline{D^u}}] \\ [\underline{\underline{D^u}}] [\underline{\underline{R^u}}] &= g [\underline{\underline{m^T}}] \end{aligned} \quad \leftarrow \text{Jacobi} \quad (3.2)$$

Insertion of these then yields,

$$\begin{aligned} [\underline{\underline{\delta m^T}}] &= -[\underline{\underline{\Theta_c}}] [\underline{\underline{C^c}}] [\underline{\underline{m^T}}] \\ &\quad - \frac{1}{g} [\underline{\underline{D^v}}] \left\{ \tilde{F} [\underline{\underline{R^v}}] - [\underline{\underline{\Theta_c}}] [\underline{\underline{C^c}}] [\underline{\underline{R^v}}] \right\} \end{aligned} \quad (3.3)$$

The formula for variation of the detM thing is:

$$\delta [\ln(\det M)] = \text{Tr} \left([\underline{\underline{\delta m^T}}] [\underline{\underline{m^T}}]^{-1} \right) \quad (3.4)$$

Inserting the above we find:

$$\begin{aligned} \delta [\ln(\det M)] &= -\text{Tr} \left([\underline{\underline{\Theta_c}}] [\underline{\underline{C^c}}] \right) \\ &\quad - \frac{1}{g} \text{Tr} \left([\underline{\underline{D^v}}] \left\{ \tilde{F} [\underline{\underline{R^v}}] - [\underline{\underline{\Theta_c}}] [\underline{\underline{C^c}}] [\underline{\underline{R^v}}] \right\} [\underline{\underline{m^T}}]^{-1} \right) \end{aligned} \quad (3.5)$$

The first trace vanishes by color; the second is equal and opposite in sign to the dJ variation. Thus we can add to get:

$$\delta J + \delta [\ln(\det M)] = 0 \quad [dA'] [\det M(A')] = [dA] [\det M(A)] \quad (3.6)$$