

General Field Theory

Weak Interactions

Math

Commins

Old Field Theory Notes

New Field Theory Notes

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MATH

Some Feynman identities

① The basic one is this

$$\frac{1}{A \cdot B} = \int_0^1 dx_1 \int_0^1 dx_2 \delta(x_1 + x_2 - 1) \frac{1}{[A x_1 + B x_2]^2} = \int_0^1 dx_1 \frac{1}{[A x_1 + B(1-x_1)]^2}$$

and generalizes to this:

$$\frac{1}{A_1 A_2 \cdots A_n} = (n-1)! \int_0^1 dx_1 \cdots dx_n \delta(x_1 + \cdots + x_n - 1) \frac{1}{[A_1 x_1 + A_2 x_2 + \cdots + A_n x_n]^n}$$

② Tim Ball uses this generalized version of above:

$$\frac{1}{A^\alpha B^\beta} = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 dz_1 \int_0^1 dz_2 \delta(1-z_1-z_2) \frac{(z_1)^{\alpha-1} (z_2)^{\beta-1}}{[z_1 A + z_2 B]^{\alpha+\beta}}$$

(which I have checked)

↖ for gen, see + Hol-Velt (A.12)

Examples of useful Minkowski integrals (after you do the feynman parametrization):

$$\textcircled{1} \quad \int_M d^4k \frac{(k^2)^m}{[k^2 - A]^n} = i\pi^2 (-1)^{m+n} A^{m+2-n} B(m+2, n-m-2) \quad /GR3.194.3$$

$$\textcircled{2} \text{ useful fact: } \int_0^\infty dk \frac{k^\beta}{(k^2 + M^2)^\alpha} = \frac{1}{2} \frac{\Gamma(\frac{\beta+1}{2}) \Gamma(\alpha - \frac{\beta+1}{2})}{\Gamma(\alpha) (M^2)^{\alpha - \frac{1}{2}(\beta+1)}} \quad (\text{Ar) of t. vet.}$$

On integrating a scalar function in $\mathbb{R}^n$

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10.

① The problem: we are confronted with this integral:

$$I = \int_{\text{all } \mathbb{R}^n} d^n k f(k^2)$$

$$k^2 = k_1^2 + k_2^2 + \dots + k_n^2$$

We want to be able to do the "angular" integral first + then leave ourselves with a one dimensional integral.

② Example:

$$\int d^3 k f(k) = \left[\int_0^\infty k^2 dk f(k) \right] \times 4\pi$$

③ When we integrate over angles, we are left with:

$$\int_{\text{angles}} d^n k = A(k) dk$$

$$d^n k = k^{n-1} dk d\Omega_n$$

where $A(k)$ = area of the n -sphere. From p. 14 of Stakgold we happen to know that:

$$A^{(n)}(k) = k^{n-1} \times \left[\frac{n \pi^{n/2}}{(n/2)!} \right]$$

In other words:

$$\int_{\text{angles}} d\Omega_n = \frac{n \pi^{n/2}}{(n/2)!} \equiv D_n \quad \left(\frac{1}{2}\right)! = \frac{\sqrt{\pi}}{2}$$

Thus:

$$I = \int d^n k f(k) = \frac{n \pi^{n/2}}{(n/2)!} \int_0^\infty k^{n-1} dk f(k)$$

④ Table:

n	D_n	"area"
1	2	2
2	2π	$2\pi R$
3	4π	$4\pi R^2$
→ 4	$2\pi^2$	$2\pi^2 R^3$
5	$8\pi^2/3$	$8\pi^2 R^4/3$
6	π^3	$\pi^3 R^5$

N-dimensional Euclidean $d^N x$.

① Start here: $r_2^2 = x_1^2 + x_2^2$ $\left. \begin{array}{l} r_2 \cos \theta_2 = x_2 \\ r_2 \sin \theta_2 = x_1 \end{array} \right\} \text{ defines } \theta_2 \quad \left(\begin{array}{l} x_2 = r_2 \cos \theta_2 \\ x_1 = r_2 \sin \theta_2 \end{array} \right)$

Then:

$$d^2 x = dx_1 dx_2 = r_2 dr_2 d\theta_2 \quad \theta_2 \text{ goes } 0, 2\pi$$

② Next, $r_3^2 = r_2^2 + x_3^2$ $\left. \begin{array}{l} r_3 \cos \theta_3 = x_3 \\ r_3 \sin \theta_3 = r_2 \end{array} \right\} \text{ defines } \theta_3 \quad \left(\begin{array}{l} x_3 = r_3 \cos \theta_3 \\ r_2 = r_3 \sin \theta_3 \end{array} \right)$

$$\begin{aligned} d^3 x &= dx_1 dx_2 dx_3 = d^2 x dx_3 \\ &= (r_2 dr_2 d\theta_2) dx_3 = r_2 d\theta_2 (dr_2 dx_3) \\ &= r_2 d\theta_2 (r_3 dr_3 d\theta_3) \\ &= dr_3 (r_3 r_2) d\theta_3 d\theta_2 = dr_3 (r_3^2 \sin \theta_3) d\theta_3 d\theta_2 \quad \theta_3 \text{ goes } 0, \pi \end{aligned}$$

③ Next, $r_4^2 = r_3^2 + x_4^2$ $\left. \begin{array}{l} r_4 \cos \theta_4 = x_4 \\ r_4 \sin \theta_4 = r_3 \end{array} \right\} \text{ defines } \theta_4 \Rightarrow dr_3 dx_4 = r_4 dr_4 d\theta_4$

$$d^4 x = d^3 x dx_4 = d\theta_2 d\theta_3 r_2 r_3 (dr_3 dx_4) = d\theta_2 d\theta_3 d\theta_4 r_2 r_3 r_4 dr_4$$

$$r_2 = r_3 \sin \theta_3 = r_4 \sin \theta_4 \sin \theta_3$$

$$r_3 = r_4 \sin \theta_4$$

$$\Rightarrow r_2 r_3 r_4 = r_4^3 \sin^2 \theta_4 \sin \theta_3$$

$$\text{So } d^4 x = r_4^3 dr_4 d\Omega_4 \quad d\Omega_4 = \sin^2 \theta_4 d\theta_4 \sin \theta_3 d\theta_3 d\theta_2$$

④ Review the above:

$$d^2 x = r_2 dr_2 d\Omega_2 \quad d\Omega_2 = d\theta_2$$

$$d^3 x = r_2 dr_2 d\Omega_2 dx_3 = (r_2 d\Omega_2) (r_3 dr_3 d\theta_3)$$

$$= r_3^2 dr_3 d\Omega_3 \quad d\Omega_3 = \sin \theta_3 d\theta_3 d\Omega_2$$

$$d^4 x = (r_3^2 dr_3 d\Omega_3) dx_4 = r_3^2 d\Omega_3 dr_3 r_4 dr_4 d\theta_4$$

$$= r_4^3 dr_4 d\Omega_4 \quad d\Omega_4 = \sin^2 \theta_4 d\theta_4 d\Omega_3$$

The general result is now obvious.

⑤ First, here is the reason all higher angles go to only π .

In general you have:

$$r_n \cos \theta_n = x_n$$

$$r_n \sin \theta_n = r_{n-1}$$

But r_{n-1} is positive, so $\theta_n \in (0, \pi)$ only! Only in lowest case is the 2π correct.

⑥ General result is

$$dx^n = (r_n)^{n-1} dr_n d\Omega_n$$

$$d\Omega_n = \sin^{n-2} \theta_n d\theta_n d\Omega_{n-1}$$

Let's now change notation. Lower all angle indices so:

in new notation.

$$d\Omega_2 \rightarrow d\Omega_1 = d\theta_1$$

$$d\Omega_3 \rightarrow d\Omega_2 = \sin \theta_2 d\theta_2 d\Omega_1$$

$$d\Omega_4 \rightarrow d\Omega_3 = \sin^2 \theta_3 d\theta_3 \sin \theta_2 d\theta_2 d\Omega_1$$

Then:

$$\left\{ \begin{aligned} dx^n &= r^{n-1} dr d\Omega_{n-1} \end{aligned} \right.$$

$$\text{where } d\Omega_{n-1} = (\sin \theta_{n-1})^{n-2} d\theta_{n-1} (\sin \theta_{n-2})^{n-3} d\theta_{n-2} \dots$$

$$\dots \sin^2 \theta_2 d\theta_2 d\Omega_1$$

Especially:

$$dx^n = r^{n-1} dr d\Omega_n$$

$$d\Omega_n = d\theta_1 d\theta_2 \dots d\theta_n \times \sin \theta_2 \times \sin^2 \theta_3 \times \sin^3 \theta_4 \dots \sin^{n-1} \theta_n$$

$$= \prod_{i=1}^n [(\sin \theta_i)^{i-1} d\theta_i]$$

$$\text{where } \theta_1 \in (0, 2\pi), \text{ all others } \in (0, \pi)$$

⑦ What are the angular integrals?

$$\int d\Omega_n = \left[\int_0^{2\pi} d\phi_1 \right] \left[\int_0^\pi \sin \phi_2 d\phi_2 \right] \left[\int_0^\pi \sin^2 \phi_3 d\phi_3 \right] \dots \left[\int_0^\pi \sin^{n-1} \phi_n d\phi_n \right]$$

Let $h_i = \int_0^\pi (\sin x)^i dx$. Then

$$\int d\Omega_n = 2\pi \cdot h_1 h_2 \dots h_{n-1} \quad n=2,3,\dots$$

Now $h_i = 2 \int_0^{\pi/2} dx (\sin^i x) = 2 \cdot 2^{i-1} B\left(\frac{i+1}{2}, \frac{i+1}{2}\right)$

$$\begin{aligned} i &= \mu-1 & \mu-2 &= i-1 \\ \mu &= i+1 \end{aligned}$$

$$= \frac{2^i \Gamma\left(\frac{i+1}{2}\right)}{\Gamma(i+1)} = \sqrt{\pi} \frac{\Gamma\left[\frac{1}{2}(i+1)\right]}{\Gamma\left[\frac{1}{2}(i+2)\right]} \checkmark$$

Then: $\int d\Omega_n = 2\pi (\sqrt{\pi})^{n-1} \underbrace{\frac{\Gamma(\frac{1}{2}(1+1))}{\Gamma(\frac{1}{2}(1+2))}}_{h_1} \cancel{\frac{\Gamma(\frac{1}{2}(2+1))}{\Gamma(\frac{1}{2}(2+2))}} \dots \cancel{\frac{\Gamma(\frac{1}{2}(n-1+1))}{\Gamma(\frac{1}{2}(n-1+2))}}$

$$= \frac{2\pi (\sqrt{\pi})^{n-1}}{\Gamma(\frac{1}{2}(n+1))}$$

$$\Rightarrow \int d\Omega_{n-1} = \frac{2\pi (\sqrt{\pi})^{n-2}}{\Gamma(\frac{n}{2})} = 2(\sqrt{\pi})^n / \Gamma(\frac{n}{2})$$

So $\boxed{\int d\Omega_{n-1} = \frac{2(\sqrt{\pi})^n}{\Gamma(\frac{n}{2})} = \text{area of unit sphere embedded in } n\text{-space} = \text{area of an } (n-1)\text{dim surface.}}$

Then,

$$\boxed{\int d^n x f(r) = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})} \int_0^\infty r^{n-1} f(r) dr}$$

By the way, volume:

$$V_n = \int d^n x = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})} \underbrace{\int_0^1 r^{n-1} dr}_{\frac{1}{n}} = \frac{2(\sqrt{\pi})^n}{n \Gamma(\frac{n}{2})}$$

So

$$\boxed{\int d^n x = \int r^{n-1} dr d\Omega_{n-1} = \frac{2\pi^{\frac{n}{2}}}{n \Gamma(\frac{n}{2})} = \text{volume of unit sphere in } n\text{-space.}}$$

⑧ Now let integral depend on the angle.

$$\int d^n \vec{p} f(p^2, k^2, \hat{p} \cdot \hat{k}) \quad \hat{p} \cdot \hat{k} = \cos \theta_{n-1}$$

$$= \int d^n r f(r^2, k^2, \cos \theta_{n-1})$$

$$= \int r^{n-1} dr \underbrace{d\Omega_{n-1}}_{= (\sin \theta_{n-1})^{n-2} d\theta_{n-1} d\Omega_{n-2}} f(r^2, k^2, \cos \theta_{n-1})$$

$$= \frac{2(\sqrt{\pi})^{n-1}}{\Gamma(\frac{n-1}{2})} \int_0^\infty r^{n-1} dr \int_0^\pi d\theta (\sin \theta)^{n-2} f(r^2, k^2, \cos \theta)$$

Let $z = \cos \theta$ $\sin \theta = \sqrt{1-z^2}$ $dz = -\sin \theta d\theta$.

$$\int_0^\pi d\theta (\sin \theta)^{n-2} = \int_{-1}^1 \left[\frac{-dz}{\sqrt{1-z^2}} \right] (\sqrt{1-z^2})^{n-2} = \int_{-1}^1 dz (\sqrt{1-z^2})^{n-3}$$

So:

$$\int_E d^n k f(k^2, q^2, k \cdot q)$$

$$= \frac{2(\pi)^{\frac{n-1}{2}}}{\Gamma(\frac{n-1}{2})} \int_0^\infty dr r^{n-1} \cdot \int_{-1}^1 dz (\sqrt{1-z^2})^{n-3} f(r^2, q^2, r q z)$$

In general, if you use neg. of BD metric,

then

$$\int_M d^n k f(k) = i \int_E d^n k f(k)$$

assuming Wick rot. is OK.

Related Integrals

$$\textcircled{1} \int d^2 k \exp(-ak^2 - q \cdot k) = \pi^w \bar{a}^w \exp\left[\frac{q^2}{4a}\right]$$

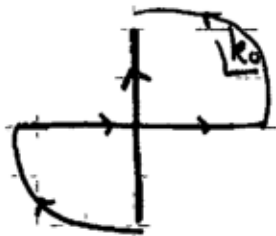
I did this + result agrees with Liebsant.

Standard Wick Rotation

9. 21.

① We seek $\int d^4k f(k)$ Minkowski, where $k^2 = k_0^2 - \underline{k}^2$.

② If $f(k)$ is analytic in quadrants I, III and has vanishing integral on great circle in k_0 -plane [ie, goes as $1/k^2$ or better], then you can Wick rotate:



$$\int_{-\infty}^{+\infty} dk_0 = \int_{-i\infty}^{+i\infty} dk_0 = i \int_{-\infty}^{\infty} dk_4$$

③ Let $k_4 = -ik_0$ $k_4^2 = -k_0^2$ $dk_4 = -i dk_0$

$$k_0 = -i\infty \Rightarrow k_4 = -\infty$$

$$k_0 = +i\infty \Rightarrow k_4 = +\infty$$

④ Now we have:

$$\begin{aligned} \int_M d^4k f(k^2) &= i \int_E d^4k f(-k^2) = i 2\pi^2 \int_0^\infty k^3 dk f(-k^2) \\ &= i \pi^2 \int_0^\infty k^2 dk^2 f(-k^2) \\ &= i \pi^2 \int_0^\infty x f(-x) dx \end{aligned}$$

Other metric $k_4^2 = k^2 - k_0^2 = k^2 + k_4^2$

Then $\int_M d^4k f(k^2) = (i) \int_E d^4k f(k^2)$



$$\frac{1}{2} \pi r^2$$

$$\frac{1}{2} \pi r^2$$

Euclidean 4-space or $SO(4)$ integrals.

① Let $(x, y, z, w) = 4$ cartesian coordinates. Let $R^2 = x^2 + y^2 + z^2 + w^2 =$ the big R . Define angles in just the way you do it for $O(3)$:

$$\begin{aligned} w &= R \cos \alpha & / \text{the effective "z"} \\ r &= R \sin \alpha & / r = \text{usual 3-D } r. \end{aligned}$$

$$z = r \cos \theta = R \sin \alpha \cos \theta$$

$$y = r \sin \theta = R \sin \alpha \sin \theta$$

$$x = y \cos \phi = R \sin \alpha \sin \theta \cos \phi$$

$$y = y \sin \phi = R \sin \alpha \sin \theta \sin \phi$$

Thus, we have these relations

$$\begin{aligned} w &= R \cos \alpha \\ z &= R \sin \alpha \cos \theta \\ x &= R \sin \alpha \sin \theta \cos \phi \\ y &= R \sin \alpha \sin \theta \sin \phi \end{aligned}$$

② I have computed the 4×4 jacobian matrix on scratch paper + I found:

$$dx dy dz dw = d^4x = R^3 dR \cdot \sin^2 \alpha d\alpha \cdot \sin \theta d\theta \cdot d\phi$$

$$\text{Or, } d^4x = R^3 dR d\Omega_3$$

$$d\Omega_3 = \sin^2 \alpha d\alpha \cdot \sin \theta d\theta \cdot d\phi$$

③ Compute total solid angle

$$\int d\Omega_3 = \underbrace{\int_0^\pi d\alpha \sin^2 \alpha}_{\frac{\pi}{2}} \cdot \underbrace{\int_0^\pi d\theta \sin \theta}_2 \cdot \underbrace{\int_0^{2\pi} d\phi}_{2\pi}$$

So $\boxed{\int d\Omega_3 = 2\pi^2}$

Thus, $A_4 = 2\pi^2 R^3$ = area of 4 sphere radius R .
/ agrees with Stat. page 14

Thus, $V_4 = \int_0^R R'^3 dR' \cdot \int d\Omega$
 $= \frac{1}{4} R^4 \cdot 2\pi^2 = \frac{1}{2} \pi^2 R^4 = V_4$

/ also agrees with Stat.

④ Integrals of special form:

$$\boxed{\int d^4x f(R) = 2\pi^2 \int_0^\infty dR R^3 f(R)}$$

$$\boxed{\int d^4x f(R, \vec{R} \cdot \vec{A}) = 4\pi \int_0^\infty R^3 dR \int_{-1}^1 dz \sqrt{1-z^2} f(R, RAz)}$$

Here we choose axes $\Rightarrow \vec{A} = A\hat{n}$, then $\vec{R} \cdot \vec{A} = \cos \alpha$.

Practice Integral d^4k . General Conversion Rule.

$$I \equiv \int d^4k_M \frac{1}{(k^2)^{1+\alpha}} \frac{1}{[(q-k)^2]^{1+\beta}}$$

This is of the form: $\int d^4k_M f(k^2, q^2, k \cdot q)$ all Minkowski vectors.

$$\begin{aligned} \text{So far, } k^2 &= k_0^2 - \vec{k} \cdot \vec{k} \\ q^2 &= q_0^2 - \vec{q} \cdot \vec{q} \\ k \cdot q &= q_0 k_0 - \vec{q} \cdot \vec{k} \end{aligned}$$

$$\begin{aligned} \text{Now, define } k_4 &\equiv -ik_0 \quad \text{then:} & k^2 &= -k_4^2 - \vec{k} \cdot \vec{k} \equiv -k_E^2 \\ q_4 &\equiv -iq_0 & q^2 &= -q_4^2 - \vec{q} \cdot \vec{q} \equiv -q_E^2 \\ & & k \cdot q &= -q_4 k_4 - \vec{q} \cdot \vec{k} \equiv -q_E \cdot k_E \end{aligned}$$

We then have, assuming you can do the Wick:

$$I = +i \int d^4k_E f(-k_E^2, -q_E^2, -k_E \cdot q_E)$$

So this is our first theorem. In general

$$\text{Wick} \Rightarrow \int_M d^4k f(k^2, q_1^2, q_2^2, \dots) = +i \int_E d^4k f(-k_E^2, -q_1^2, -q_2^2, \dots)$$

in the BD metric, assuming rotation is OK

Now we can specialize.

$$\begin{aligned} \int_M d^4k f(k^2, q^2, k \cdot q) &= +i 4\pi \int_0^\infty R^3 dR \int_{-1}^1 dz \sqrt{1-z^2} f(-R^2, -q^2, -Rqz) \\ \int_M d^4k f(k^2) &= +i 2\pi^2 \int_0^\infty R^3 dR f(-R^2) \end{aligned}$$

Now let's apply this to integral I:

$$I = \int_M d^4k \cdot \int_0^\infty dx_1 \int_0^\infty dx_2 \delta[x_1 + x_2 - 1] \frac{1}{\text{stop.}}$$

A technique for proving the Finkman Identity.

1. Here is what we want to prove for any positive integer n :

$$\frac{1}{a_1 a_2 \dots a_n} = (n-1)! \int_0^\infty (dz)_n (S_n)^{-n}$$

where we have defined:

$$(dz)_n \equiv \prod_{i=1}^n \left(\int_0^\infty dz_i \right) \delta\left(\sum_{i=1}^n z_i - 1\right) \quad S_n \equiv \sum_{i=1}^n a_i z_i$$

2. I proposed a proof by induction. Assume it is true for $n-1$, then ~~verify~~ verify that it is true for n . We start in this way:

$$(n-1)! \int_0^\infty (dz)_n (S_n)^{-n} = (n-1)! \prod_{i=1}^{n-2} \left(\int_0^\infty dz_i \right) \int_0^\infty dx \int_0^\infty dy \delta\left(\sum_{i=1}^{n-2} z_i + x + y - 1\right) \\ \times \left[\sum_{i=1}^{n-2} a_i z_i + a_{n-1} x + a_n y \right]^{-n}$$

So far I have done nothing more or less than rename the last two integration variables. I wish to examine the $dx dy$ integrations, holding all the other z 's fixed. I want to define new variables to replace x and y like so:

$$\begin{aligned} t &\equiv x+y & dx dy &= \frac{1}{2} dt ds & x &= \frac{1}{2}(t+s) \\ s &\equiv x-y & & & y &= \frac{1}{2}(t-s) \end{aligned}$$

$$\theta(x) \theta(y) = \theta(x) \theta(y) \theta(x+y) = \theta(t+s) \theta(t-s) \theta(t)$$

$$\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy f(x,y) = \frac{1}{2} \int_{-\infty}^{\infty} ds \int_{-\infty}^{\infty} dt f\left(\frac{t+s}{2}, \frac{t-s}{2}\right).$$

Now rename ... no not yet: we get

$$(n-1)! \prod_{i=1}^{n-2} \left(\int_0^\infty dz_i \right) \frac{1}{2} \int_{-\infty}^{\infty} ds \int_{-\infty}^{\infty} dt \cdot \theta(t+s) \theta(t-s) \theta(t) \delta\left(\sum_{i=1}^{n-2} z_i + t - 1\right) \\ \times \left[\sum_{i=1}^{n-2} a_i z_i + a_{n-1} \frac{1}{2}(t+s) + a_n \frac{1}{2}(t-s) \right]^{-n}$$

Back the dt off to the left, and put two thetas into endpoints:

$$= (n-1)! \prod_{i=1}^{n-2} \left(\int_0^\infty dz_i \right) \int_0^t dt \delta\left(\sum_{i=1}^{n-2} z_i + t - 1\right) \cdot \underbrace{\frac{1}{2} \int_{-t}^t ds \left[\sum_{i=1}^{n-2} a_i z_i + a_{n-1} \frac{1}{2}(t+s) + a_n \frac{1}{2}(t-s) \right]^{-n}}_{\equiv I}$$

Next, explicitly execute the ds integration to get:

$$\frac{1}{z} \cdot \frac{1}{(-n+1)} \cdot \frac{1}{z(q_{n-1}-q_n)} \left[\sum_{i=1}^{n-2} a_i z_i + \frac{1}{z} q_{n-1} (t+s) + \frac{1}{z} q_n (t-s) \right]^{-n+1} \Big|_{-t}^t$$

$$= \frac{1}{(n-1)} \cdot \frac{1}{(q_n - q_{n-1})} \left\{ \left[\sum_{i=1}^{n-2} a_i z_i + \frac{1}{z} q_{n-1} t \right]^{-n+1} - \left[\sum_{i=1}^{n-2} a_i z_i + q_n t \right]^{-n+1} \right\}$$

Now take the variable t and rename it z_{n-1} so answer becomes:

$$\frac{1}{a_1 a_2 \dots a_n} = ? (n-2)! \frac{1}{(q_n - q_{n-1})} \int_0^\infty (dz)_{n-1} \left\{ \left[\sum_{i=1}^{n-2} a_i z_i \right]^{-n+1} - \left[\sum_{i=1}^{n-2} a_i z_i + q_n z_{n-1} \right]^{-n+1} \right\}$$

2. Now what should I do next? So far I like it because I have caused the once-lowered $(dz)_{n-1}$ integration to appear. I have yet to prove that this is equal to the thing on the left. But now lets use the fact that this thing is supposed to be true for $n-1$. Then:

$$(n-2)! \int_0^\infty (dz)_{n-1} \left[\sum_{i=1}^{n-2} a_i z_i \right]^{-n+1} = \frac{1}{a_1 a_2 \dots a_{n-2} q_{n-1}}$$

$$(n-2)! \int_0^\infty (dz)_{n-1} \left[\sum_{i=1}^{n-2} a_i z_i + q_n z_{n-1} \right]^{-n+1} = \frac{1}{a_1 a_2 \dots a_{n-2} q_n} \quad / \text{ie, same as abt with } q_{n-1} \rightarrow q_n.$$

$$\text{So } \frac{1}{a_1 a_2 \dots a_n} \stackrel{?}{=} \frac{1}{(q_n - q_{n-1})} \left[\frac{1}{a_1 a_2 \dots a_{n-2} q_{n-1}} - \frac{1}{a_1 a_2 \dots a_{n-2} q_n} \right]$$

$$= \frac{1}{a_1 a_2 \dots a_{n-2}} \left[\frac{q_n - q_{n-1}}{q_n q_{n-1}} \right] \frac{1}{(q_n - q_{n-1})} = QED.$$

Thus it is proved. If true for $n=1$, then must be true for $n=2$, and so on. But is this proof valid for low n ? I showed that $(n-1)$ implies n . Must show true for $n=2$ explicitly:

$$\frac{1}{a_1 a_2} = ? = \int_0^\infty dz_1 \int_0^\infty dz_2 \delta(z_1 + z_2 - 1) [a_1 z_1 + a_2 z_2]^{-2}$$

$$\frac{1}{ab} = \int_0^\infty dx \int_0^\infty dy \delta(x+y-1) [ax+by]^{-2} \quad ?$$

$$= \frac{1}{2} \int_0^1 dt \int_{-t}^t ds \cdot \delta(t-1) \cdot \left[a \frac{1}{2} (t+s) + b \frac{1}{2} (t-s) \right]^{-2}$$

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$$= \frac{1}{2} \int_{-1}^1 ds \left[a^{\frac{1}{2}(1+s)} + b^{\frac{1}{2}(1-s)} \right]^{-2} = \frac{1}{2} \cdot \frac{1}{(-2+1)} \frac{1}{\frac{1}{2}(a-b)} \left[a^{\frac{1}{2}(1+s)} + b^{\frac{1}{2}(1-s)} \right]^{-1} \Big|_{-1}^1$$

$$= \frac{1}{(b-a)} \cdot \{ [a]^{-1} - [b]^{-1} \} = \frac{1}{b-a} \left(\frac{1}{a} - \frac{1}{b} \right) = \frac{1}{ab} \checkmark.$$

Well the case $n=2$ is same as any higher n , so I already proved $n=2$ when I did the general induction proof. Now lets show for $n=1$:

$$\frac{1}{a_1} = \int_0^\infty dz_1 \delta(z_1 - 1) [a_1, z_1]^{-1} = [a_1]^{-1} \checkmark.$$

And so completes this induction proof of the finkman identity. There was no need to sum over permutations or anything fancy like that.

Evaluation of a Feynman Integral #1.

1. In the study of mass renormalization in scalar-boson QED, you soon encounter this integral by considering the one photon loop on the electron propagator:

$$I = \int d^4k \frac{1}{k^2 - \lambda^2} \cdot \frac{1}{(p-k)^2 - m_0^2} \cdot \left(\frac{\lambda^2}{k^2 + \lambda^2} \right) = I(p^2, \lambda^2, \Lambda^2, m_0^2)$$

Here you see a photon prop with a mass inserted, the electron prop at m_0^2 , and a factor put in to remove the logarithmic UV divergence. For small k , it does not appear to me that you get an IR divergence. But lets just treat this as an integral to practice on.

2. First, use Feynman parameters to get:

$$I = 2\Lambda^2 \int_0^1 d\alpha_1 \int_0^1 d\alpha_2 \int_0^1 d\alpha_3 \delta(1-\alpha_1-\alpha_2-\alpha_3) \cdot \int d^4k \cdot \left[(k^2 - \lambda^2)\alpha_1 + (p^2 + k^2 - m_0^2 - 2p \cdot k)\alpha_2 + (k^2 + \lambda^2)\alpha_3 \right]^{-3}$$

$$= 2\Lambda^2 \int_0^1 d\alpha_1 \int_0^{1-\alpha_1} d\alpha_2 \int d^4k \frac{1}{[k^2 - 2\alpha_2 p \cdot k - \{ \alpha_1(\lambda^2 + \lambda^2) + \alpha_2(\lambda^2 + m_0^2 - p^2) - \lambda^2 \}]^3}$$

Now complete the square, then shift k to new k' :

$$= 2\Lambda^2 \int_0^1 d\alpha_2 \int_0^{1-\alpha_2} d\alpha_1 \cdot \int d^4k' \frac{1}{[(k')^2 - \underbrace{\{ \alpha_2^2 p^2 + \alpha_2(\lambda^2 + m_0^2 - p^2) + \alpha_1(\lambda^2 + \lambda^2) - \lambda^2 \}}_{\equiv A}]^3}$$

Now I can look up this integral in my table:

$$\int d^4k \frac{1}{[k^2 - A]^3} = \frac{-i\pi^2}{2A}$$

$$\Rightarrow I = -i\pi^2 \Lambda^2 \int_0^1 d\alpha_2 \int_0^{1-\alpha_2} d\alpha_1 \frac{1}{(\lambda^2 + \lambda^2)\alpha_1 + \underbrace{[\alpha_2^2 p^2 + \alpha_2(\lambda^2 + m_0^2 - p^2) - \lambda^2]}_{\equiv B}}$$

We now do the trivial log integral:

$$\int_0^{1-\alpha_2} d\alpha_1 \frac{1}{(\lambda^2 + \Lambda^2)\alpha_1 + B} = \frac{1}{(\lambda^2 + \Lambda^2)} \ln \left[\frac{(\lambda^2 + \Lambda^2)(1-\alpha_2) + B}{B} \right] \checkmark$$

where $B = \alpha_2^2 p^2 + \alpha_2(\Lambda^2 + m_0^2 - p^2) - \Lambda^2$

$$\text{NUM} = \alpha_2^2 p^2 + \alpha_2(m_0^2 - \lambda^2 - p^2) + \lambda^2$$

$$\Rightarrow I = +i\pi^2 \frac{\Lambda^2}{(\lambda^2 + \Lambda^2)} \int_0^1 dx \ln \left[\frac{p^2 x^2 + (\Lambda^2 + m_0^2 - p^2)x - \Lambda^2}{p^2 x^2 + (m_0^2 - \lambda^2 - p^2)x + \lambda^2} \right]$$

3. Now define a sub-integral like so: But first redefine the meaning of all variables by dividing them by p. So:

$$I_1 \equiv \int_0^1 dx \ln(x^2 + (\Lambda^2 + m_0^2 - 1)x - \Lambda^2) \equiv I_1(\Lambda^2)$$

Then:

$$I = +i\pi^2 \left(\frac{\Lambda^2}{\lambda^2 + \Lambda^2} \right) [I_1(\Lambda^2) - I_1(+\lambda^2)]$$

4. It remains then only to compute the integral I_1 :

$$\begin{aligned} I_1 &= \int_0^1 dx [\ln(x+\alpha) + \ln(x+\beta)] \quad \text{where} \quad \alpha = \left(\frac{\Lambda^2 + m_0^2 - 1}{2} \right) \pm \\ &= \sum_{i=1}^2 \int_0^1 dx \ln(x + a_i) \quad \frac{1}{2} \sqrt{(\Lambda^2 + m_0^2 - 1) + 4\Lambda^2} \end{aligned}$$

where

$$a_{1,2} = \frac{1}{2}(\Lambda^2 + m_0^2 - 1) \pm \frac{1}{2} \sqrt{\Delta(1, m_0^2, -\Lambda^2)}$$

$$\text{But } \int_0^1 dx \ln(x+a_i) = \left[(x+a_i) \ln(x+a_i) - (x+a_i) \right] \Big|_0^1 \\ = (1+a_i) \ln(1+a_i) - a_i \ln a_i - 1$$

$$\text{So } I_1(-\Lambda^2) = \sum_{i=1}^2 \left[(1+a_i) \ln(1+a_i) - a_i \ln a_i - 1 \right]$$

$$\text{where } a_{1,2} = \frac{1}{2} (m_0^2 - 1 - (-\Lambda^2)) \pm \frac{1}{2} \sqrt{\Delta(1, m_0^2, -\Lambda^2)}$$

5. Therefore, here is a complete and exact result:

$$I = +i\pi^2 \left(\frac{\Lambda^2}{\Lambda^2 + \Lambda^2} \right) \sum_{i=1}^4 \left[(1+a_i) \ln(1+a_i) - a_i \ln a_i - 1 \right]$$

$$\text{where } a_{1,2} = \frac{1}{2} (m_0^2 - 1 + \Lambda^2) \pm \frac{1}{2} \sqrt{\Delta(1, m_0^2, -\Lambda^2)} \\ a_{3,4} = \frac{1}{2} (m_0^2 - 1 - \Lambda^2) \pm \frac{1}{2} \sqrt{\Delta(1, m_0^2, \Lambda^2)}$$

6. Now lets look at some interesting limits: first, take LAMBDA cap to infinity, to get the UV behavior:

$$a_1 = -1 + \frac{m_0^2}{\Lambda^2} \quad a_2 = \Lambda^2 + m_0^2 - \frac{m_0^2}{\Lambda^2} \approx \Lambda^2$$

$$\Rightarrow I = +i\pi^2 \left\{ \frac{m_0^2}{\Lambda^2} \ln\left(\frac{m_0^2}{\Lambda^2}\right) - (-1) \ln(-1) - 1 \right\} / a_1$$

$$+ \left(1 + \frac{\Lambda^2}{m_0^2} \right) \ln\left(1 + \frac{\Lambda^2}{m_0^2} \right) - \left(\Lambda^2 \ln \Lambda^2 - 1 \right) \right\} / a_2$$

$$a_2\text{-term} = \ln(1 + m_0^2 + \Lambda^2) + \left(\frac{m_0^2}{\Lambda^2} \right) \ln \left[\frac{1 + m_0^2 + \Lambda^2}{m_0^2 + \Lambda^2} \right] - 1$$

$$a_1\text{-term} = -\frac{m_0^2}{\Lambda^2} \ln\left(\frac{\Lambda^2}{m_0^2}\right) \rightarrow 0 \quad \ln\left(1 + \frac{1}{m_0^2 + \Lambda^2}\right) \approx \frac{1}{m_0^2 + \Lambda^2}$$

$$I \approx +i\pi^2 \ln\left(\frac{\Lambda^2}{p^2}\right)$$

This is the controlling term for large Λ . The next lower term is a constant which I could find if I wanted. I have replaced the p^2 scaling factor. You see the log divergence as expected at the start, and you see that nothing very special happens as you go to the mass shell.

Finally, let's hold Λ fixed and go to the IR limit where $\lambda \rightarrow 0$. Then I find that:

$$\begin{aligned}\sqrt{\Delta(1, m_0^2, \lambda^2)} &= \left[(1 - m_0^2)^2 - 2\lambda^2(1 + m_0^2) + \mathcal{O}(\lambda^4) \right]^{\frac{1}{2}} \\ &= (1 - m_0^2)^{2 \cdot \frac{1}{2}} + \frac{1}{2} (1 - m_0^2)^{2 \cdot (-\frac{1}{2})} [-2\lambda^2(1 + m_0^2)] + \mathcal{O}(\lambda^4) \\ &= (1 - m_0^2) - \lambda^2 \frac{(1 + m_0^2)}{(1 - m_0^2)} + \mathcal{O}(\lambda^4)\end{aligned}$$

Then:

$$\begin{aligned}a_3 &= \frac{1}{2} \left[m_0^2 - 1 - \lambda^2 + 1 - m_0^2 - \lambda^2 \left(\frac{1 + m_0^2}{1 - m_0^2} \right) \right] = -\frac{1}{2} \lambda^2 \left[1 + \frac{1 + m_0^2}{1 - m_0^2} \right] \\ a_4 &= \frac{1}{2} \left[m_0^2 - 1 - \lambda^2 - 1 + m_0^2 + \lambda^2 \left(\frac{1 + m_0^2}{1 - m_0^2} \right) \right] = m_0^2 - 1 + \frac{1}{2} \lambda^2 \left[\frac{1 + m_0^2}{1 - m_0^2} - 1 \right]\end{aligned}$$

So $a_3 = -\lambda^2 \left(\frac{1}{1 - m_0^2} \right) \approx 0$

$$a_4 = m_0^2 - 1 + \lambda^2 \left(\frac{m_0^2}{1 - m_0^2} \right)$$

$$\begin{aligned}\Rightarrow I &= +i\pi^2 \left[1 \ln 1 - \left(\frac{\lambda^2}{m_0^2 - 1} \right) \ln \left(\frac{\lambda^2}{m_0^2 - 1} \right) \right] \quad \therefore \quad / \text{ for } a_3 \\ &\quad + \left(m_0^2 + \lambda^2 \left(\frac{m_0^2}{1 - m_0^2} \right) \right) \ln \left(m_0^2 + \lambda^2 \left(\frac{m_0^2}{1 - m_0^2} \right) \right) \\ &\quad - (m_0^2 - 1) \ln(m_0^2 - 1)\end{aligned}$$

$$I = +i\pi^2 \left(\frac{\lambda^2}{m_0^2 - p^2} \right) \ln \left(\frac{\lambda^2}{m_0^2 - p^2} \right)$$

~~This says that for fixed Λ^2 and $p^2 \neq m_0^2$, this integral is also singular at this IR end. How do I understand that???~~

$$I = \text{nonsingular at } \lambda \rightarrow 0$$

7. Suppose we take our general result first to the mass shell, holding Λ and λ finite. Note that mass shell means $m_0^2 = 1$. Then we get:

$$a_{1,2} = \frac{1}{2} \frac{\Lambda^2}{m_0^2} \pm \frac{1}{2} \sqrt{\frac{\Lambda^4}{m_0^4} + 4 \frac{\Lambda^2}{m_0^2}} = \frac{\Lambda^2}{m_0^2} - 1 + \mathcal{O}\left(\frac{\Lambda^2}{m_0^2}\right)$$

$$a_{3,4} = -\frac{1}{2} \frac{\lambda^2}{m_0^2} \pm \frac{1}{2} \sqrt{\frac{\lambda^4}{m_0^4} - 4 \frac{\lambda^2}{m_0^2}} = -\frac{\lambda^2}{m_0^2},$$

Recompute these with care please:

$$a_1 = \frac{1}{2} \Lambda^2 + \frac{1}{2} (\Lambda^4 + 4\Lambda^2)^{\frac{1}{2}}, \Lambda \gg 1 \Rightarrow \boxed{a_1 \approx \Lambda^2}$$

$$a_2 = \frac{1}{2} [\Lambda^2 - (\Lambda^4 + 4\Lambda^2)^{\frac{1}{2}}] = \frac{1}{2} \Lambda^2 [1 - (1 + \frac{4}{\Lambda^2})^{\frac{1}{2}}]$$

$$= \frac{1}{2} \Lambda^2 [1 - \{1 + \frac{1}{2} (\frac{4}{\Lambda^2}) + \frac{1}{2} (-\frac{1}{2}) \frac{1}{2} (\frac{4}{\Lambda^2})^2 + \dots\}]$$

$$= \frac{1}{2} \Lambda^2 [-\frac{2}{\Lambda^2} + \frac{2}{\Lambda^4} + \dots] = -1 + \frac{1}{\Lambda^2} \Rightarrow \boxed{a_2 = -1 + \frac{1}{\Lambda^2}}$$

$$a_3 = +\frac{1}{2} [-\lambda^2 + \sqrt{\lambda^4 - 4\lambda^2}] = \frac{1}{2} [-\lambda^2 + i\sqrt{4\lambda^2 - \lambda^4}]$$

$$\text{But } \sqrt{4\lambda^2 - \lambda^4} \approx 2\lambda \sqrt{1 - \frac{\lambda^2}{4}} = 2\lambda (1 - \frac{\lambda^2}{4})^{\frac{1}{2}} \approx 2\lambda$$

$$\text{So } \begin{array}{|c|} \hline a_3 \approx i\lambda \\ \hline a_4 \approx -i\lambda \\ \hline \end{array} \Rightarrow \left. \begin{array}{l} | \ln | - (i\lambda) \ln(i\lambda) \\ + | \ln | - (-i\lambda) \ln(-i\lambda) \end{array} \right\} \text{no sing do I see!}$$

So even if you go to the mass shell first, this integral is still not IR singular. Somehow it differs from the true spinor-electron integral.

#2

Practice Integral: (see me)

$$I = \int_M d^4 k \frac{1}{[k^2]^{1+\alpha} [(q-k)^2]^{1+\beta}}$$

① First, use gen FI to combine denominators:

$$\frac{1}{[k^2]^{1+\alpha} [(q-k)^2]^{1+\beta}} = \frac{\Gamma(\alpha+\beta+2)}{\Gamma(\alpha+1)\Gamma(\beta+1)} \int_0^1 dz \frac{z^\alpha (1-z)^\beta}{[zk^2 + (1-z)(q-k)^2]^{\alpha+\beta+2}}$$

② Complete the square:

$$zk^2 + (1-z)(q^2 - 2q \cdot k + k^2) = k^2 - 2(1-z)k \cdot q + q^2(1-z)$$

$$= [k - (1-z)q]^2 - (1-z)^2 q^2 + q^2(1-z) = [k']^2 + q^2 z(1-z)$$

③ Shift $d^4 k$ to $d^4 k'$ to get

$$I = \frac{1}{\Gamma(\alpha+1)\Gamma(\beta+1)} \int_0^1 dz \cdot z^\alpha (1-z)^\beta \cdot \underbrace{\int_M d^4 k' \frac{1}{[k'^2 + q^2 z(1-z)]^{\alpha+\beta+2}}}_{= I_1}$$

④ Use contour rule to get [q^2 is as it was]

$$I_1 = 2\pi^2 i \int_0^\infty k^3 dk \frac{1}{[-k^2 + q^2 z(1-z)]^{\alpha+\beta+2}}$$

Let $k^2 = q^2 r$, assume $q^2 > 0$ so $r > 0$. Then $2k dk = q^2 dr$ $k^3 dk = q^3 r \cdot \frac{1}{2} q^2 dr$

$$[-k^2 + q^2 z(1-z)] = [-q^2 r + q^2 z(1-z)] = q^2 [-r + z(1-z)], \text{ so}$$

$$I_1 = 2\pi^2 i \cdot \frac{1}{2} q^4 \cdot (q^2)^{-\alpha-\beta-2} \cdot \int_0^\infty r dr \cdot [-r + z(1-z)]^{-\alpha-\beta-2}$$

$$\text{Let } -r = -s z(1-z) \text{ so } [-r + z(1-z)] = [-s z(1-z) + z(1-z)] = z(1-z) [-s+1]$$

Then

$$I_1 = 2\pi^2 i \cdot \frac{1}{2} (q^2)^2 \cdot (q^2)^{-\alpha-\beta-2} \cdot [z(1-z)]^2 \cdot [z(1-z)]^{-\alpha-\beta-2} \cdot \int_0^\infty ds \cdot \frac{s}{(1-s)^{\alpha+\beta+2}}$$

See GR p.285:

$$I_1 = 2\pi^2 i \cdot \frac{1}{2} \cdot (q^2)^{-\alpha-\beta} [z(1-z)]^{-\alpha-\beta} \cdot (-1)^{-2} B(2, \alpha+\beta+2-2) = i\pi^2 [q^2 z(1-z)]^{-\alpha-\beta} B(2, \alpha+\beta)$$

This leaves us with

$$\begin{aligned}
 I &= \frac{1}{B(\alpha+1, \beta+1)} \int_0^1 dz \, z^\alpha (1-z)^\beta \cdot i\pi^2 (q^2)^{-\alpha-\beta} (z)^{-\alpha-\beta} (1-z)^{-\alpha-\beta} B(z, \alpha+\beta) \\
 &= i\pi^2 (q^2)^{-\alpha-\beta} \cdot \frac{\Gamma(z) \Gamma(\alpha+\beta)}{\Gamma(\alpha+\beta+z)} \cdot \frac{\Gamma(\alpha+\beta+z)}{\Gamma(\alpha+1) \Gamma(\beta+1)} \int_0^1 dz \, z^{-\beta} (1-z)^{-\alpha} \\
 &= i\pi^2 \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha+1) \Gamma(\beta+1)} \left(\frac{1}{q^2}\right)^{\alpha+\beta} \underbrace{\int_0^1 dz \, \frac{1}{z^\beta (1-z)^\alpha}}_{\int_0^1 dy \, \frac{1}{(1-y)^\beta y^\alpha} = I_3} \quad y=1-z
 \end{aligned}$$

This verifies (3.9) of Ball-Lack 1978 paper. But

$$I_3 = \int_0^1 dx \, x^{-\beta} (1-x)^{-\alpha} = B[1-\beta, 1-\alpha], \text{ so}$$

$$I = \frac{i\pi^2 \Gamma(\alpha+\beta)}{\Gamma(\alpha+1) \Gamma(\beta+1)} \frac{\Gamma(1-\alpha) \Gamma(1-\beta)}{\Gamma(2-\alpha-\beta)} \left(\frac{1}{q^2}\right)^{\alpha+\beta}$$

Summary:

$$\begin{aligned}
 I &\equiv \int_M d^4k \, \frac{1}{(k^2)^{1+\alpha} (k^2)^{1+\beta}} \quad \text{where } k' = q-k \\
 &= i\pi^2 \cdot \frac{\Gamma(1-\alpha)}{\Gamma(1+\alpha)} \cdot \frac{\Gamma(1-\beta)}{\Gamma(1+\beta)} \cdot \frac{\Gamma(\alpha+\beta)}{\Gamma(2-\alpha-\beta)} \cdot \left(\frac{1}{q^2}\right)^{\alpha+\beta}
 \end{aligned}$$

Feynman Integral #3

$$I = \int \frac{d^4 k}{m} \frac{1}{(k^2 - m^2)} \frac{1}{(k'^2 - m^2)}$$

where $k' = q - k$

① This is dk/k log divergent so we must regulate. Let's regulate like this:

$$I(\Lambda) = \int \frac{d^4 k}{(k^2 - m^2)} \left[\frac{1}{k^2 - m^2} - \frac{1}{k^2 - \Lambda^2} \right]$$

$\underbrace{\hspace{10em}}_{(m^2 - \Lambda^2)/D}$
↑
method chosen.

$$\Rightarrow I(\Lambda) = (m^2 - \Lambda^2) \int \frac{d^4 k}{(k^2 - m^2)} \frac{1}{(k'^2 - m^2)} \frac{1}{(k^2 - \Lambda^2)}$$

② Put in usual F parameters:

$$= 2(m^2 - \Lambda^2) \int_0^1 dx \int_0^{1-x} dy [x(k^2 - m^2) + y(k^2 - m^2) + (1-x-y)(k^2 - \Lambda^2)]^{-3}$$

③ Simplify factor + complete square + shift origin to get

$$= 2(m^2 - \Lambda^2) \int_0^1 dx \int_0^{1-x} dy [k^2 + y(\Lambda^2 - m^2) + x(\Lambda^2 - m^2 + q^2) - x^2 q^2 - \Lambda^2]^{-3}$$

Let $A \equiv \Lambda^2 + x^2 q^2 - x(\Lambda^2 - m^2 + q^2) - y(\Lambda^2 - m^2)$ / tends to be > 0 !

Then use $\int \frac{d^4 k}{(k^2 - A)^3} = -i\pi^2/2A$

$$\Rightarrow I(\Lambda) = 2(m^2 - \Lambda^2) \frac{(-i\pi^2)}{(-2)} \int_0^1 dx \int_0^{1-x} dy [y(\Lambda^2 - m^2) + x(\Lambda^2 - m^2 + q^2) - x^2 q^2 - \Lambda^2]^{-1}$$

$$= \frac{2(\Lambda^2 - m^2)}{2(\Lambda^2 - m^2)} (-i\pi^2) \int_0^1 dx \ln \left[\frac{(1-x)(\Lambda^2 - m^2) + x(\Lambda^2 - m^2 + q^2) - x^2 q^2 - \Lambda^2}{x(\Lambda^2 - m^2 + q^2) - x^2 q^2 - \Lambda^2} \right]$$

$$\Rightarrow I(\Lambda) = (-i\pi^2) \int_0^1 dx \ln \left[\frac{x^2 - x + (m^2/q^2)}{x^2 - x \left[\left(\frac{\Lambda^2 - m^2}{q^2} \right) + 1 \right] + (\Lambda^2/q^2)} \right], \text{ exact.}$$

Write this as

$$I(\Lambda) = (-i\pi^2) \int_0^1 dx \ln \left[\frac{(x-a_+)(x-a_-)}{(x-b_+)(x-b_-)} \right]$$

where

$$\begin{aligned} a_{\pm} &= +\frac{1}{2} \pm \frac{1}{2} \left(1 - 4(m^2/q^2) \right)^{\frac{1}{2}} \\ b_{\pm} &= +\frac{1}{2} \left(\frac{\Lambda^2}{q^2} - \frac{m^2}{q^2} + 1 \right) \pm \frac{1}{2} \lambda \left(\frac{\Lambda^2}{q^2}, \frac{m^2}{q^2}, 1 \right) \quad \lambda = \sqrt{\Delta} \end{aligned}$$

④ Now,

$$\begin{aligned} \int_0^1 dx \ln(x-a) &= [(x-a)\ln(x-a) - (x-a)]_0^1 \\ &= [(1-a)\ln(1-a) - (1-a)] - [(-a)\ln(-a) - (-a)] \\ &= \ln(1-a) + a \ln\left(\frac{a}{a-1}\right) - 1, \text{ exact.} \end{aligned}$$

Thus, the complete + exact result is

$$\begin{aligned} \frac{I(\Lambda)}{(-i\pi^2)} &= \left[\ln(1-a_+) + a_+ \ln\left(\frac{a_+}{a_+-1}\right) - 1 \right. \\ &\quad + \ln(1-a_-) + a_- \ln\left(\frac{a_-}{a_--1}\right) - 1 \\ &\quad - \ln(1-b_+) - b_+ \ln\left(\frac{b_+}{b_+-1}\right) + 1 \\ &\quad \left. - \ln(1-b_-) - b_- \ln\left(\frac{b_-}{b_--1}\right) + 1 \right] \end{aligned}$$

⑤ The terms involving a_+ and a_- are "constant" in Λ . Assume $\Lambda =$ larger than all other variables. Then look at b -term.

$$\lambda \left(\frac{\Lambda^2}{q^2}, \frac{m^2}{q^2}, 1 \right) \approx \left[\frac{\Lambda^2}{q^2} - \frac{m^2}{q^2} - 1 - \frac{2 \cdot \frac{m^2}{q^2} \cdot 1}{\Lambda^2/q^2} \right]$$

$$\Rightarrow b_+ = \left(\frac{\Lambda^2}{q^2} - \frac{m^2}{q^2} \right) + \mathcal{O}\left(\frac{1}{\Lambda^2}\right)$$

$$b_- = 1 + \frac{m^2}{\Lambda^2} + \mathcal{O}\left(\frac{1}{\Lambda^4}\right)$$

⑥ Look at b_- terms:

$$-\ln\left(-\frac{m^2}{\Lambda^2}\right) - \left(1 + \frac{m^2}{\Lambda^2}\right) \ln\left(\frac{1 + \frac{m^2}{\Lambda^2}}{\frac{m^2}{\Lambda^2}}\right)$$

$$= -\ln\left(-\frac{m^2}{\Lambda^2}\right) - \ln\left(\frac{\Lambda^2 + m^2}{m^2}\right) - \frac{m^2}{\Lambda^2} \ln\left(\frac{\Lambda^2 + m^2}{m^2}\right)$$

$$b_- = -\ln\left[-\frac{m^2}{\Lambda^2} \left(\frac{\Lambda^2 + m^2}{m^2}\right)\right] - \frac{m^2}{\Lambda^2} \ln\left(\frac{\Lambda^2 + m^2}{m^2}\right) \quad \leftarrow \text{log has no effect on power.}$$

$$= -\ln\left[-1 - \frac{m^2}{\Lambda^2}\right] - \mathcal{O}\left(\frac{1}{\Lambda^2}\right) = -\ln(-1) - \ln\left(1 + \frac{m^2}{\Lambda^2}\right) + \mathcal{O}\left(\frac{1}{\Lambda^2}\right)$$

$$= -\ln(-1) + \mathcal{O}\left(\frac{1}{\Lambda^2}\right) =$$

$$= [-i\pi + \mathcal{O}\left(\frac{1}{\Lambda^2}\right)] = b_- \text{ contribution}$$

$-1 = e^{-i\pi}$
 $\ln(-1) = i\pi$ choice made off the m .

⑦ Now look at b_+ : $b_+ = \frac{\Lambda^2 - m^2}{q^2} + \mathcal{O}\left(\frac{1}{\Lambda^2}\right) \equiv a + \mathcal{O}\left(\frac{1}{\Lambda^2}\right)$

So $-\ln(1-a) - a \ln\left(\frac{a}{a-1}\right)$

$$\approx -\ln(-a) + a \ln\left(1 - \frac{1}{a}\right) = -\ln(-a) + a \left[-\frac{1}{a}\right] = -\ln(-a) - 1$$

Now $-\ln(-a) = -\ln(-1) - \ln\left(\frac{\Lambda^2 - m^2}{q^2}\right)$ so

$$[-1 - i\pi - \ln\left(\frac{\Lambda^2 - m^2}{q^2}\right)] = b_+ \text{ contribution}$$

⑧ Overall result:

$$I(\Lambda) = +i\pi^2 \left(\ln\left[\frac{\Lambda^2 - m^2}{q^2}\right] \right) + (\text{constant in } \Lambda) + \mathcal{O}\left(\frac{1}{\Lambda^2}\right)$$

where $\text{const} = (-i\pi^2) \left[-1 - \underline{2i\pi} + \ln(1-a_+) + a_+ \ln\left(\frac{a_+}{a_+-1}\right) + \ln(1-a_-) + a_- \ln\left(\frac{a_-}{a_--1}\right) \right]$

where $a_{\pm} = a_{\pm}(m^2/q^2)$

⑨ Now, we have shown that

$$I(q^2, \Lambda^2, m^2) = (-i\pi^2) \int_0^1 dx \ln \left[\frac{(x-a_+)(x-a_-)}{(x-b_+)(x-b_-)} \right]$$

where

$$2a_{\pm} = 1 \pm \sqrt{1 - 4\left(\frac{m^2}{q^2}\right)}$$

$$2q^2 b_{\pm} = (\Lambda^2 - m^2 + q^2) \pm \lambda(\Lambda^2, m^2, q^2)$$

We now wish to assess dI/dq^2 near $q^2 = m^2$. First note:

$$\lambda(\Lambda^2, m^2, m^2) = \Lambda \sqrt{\Lambda^2 - 4m^2} \approx \Lambda^2 - 2m^2 + \mathcal{O}(1/\Lambda^2)$$

$$\Rightarrow \begin{cases} 2m^2 b_+ \approx 2\Lambda^2 - 2m^2 + \mathcal{O}(1/\Lambda^2) & 2a_+ = 1 + \sqrt{-3} \\ 2m^2 b_- \approx 2m^2 + \mathcal{O}(1/\Lambda^2) & 2a_- = 1 - \sqrt{-3} \end{cases}$$

Next: $\left. \frac{d\lambda}{dq^2} \right|_{q^2=m^2} = -\frac{\Lambda^2}{\lambda} = -1 - 2m^2/\Lambda^2 + \mathcal{O}(1/\Lambda^4)$

Therefore: $\frac{d(2q^2 b_+)}{dq^2} = -2m^2/\Lambda^2$ $\frac{d(2q^2 b_-)}{dq^2} = 2 + 2m^2/\Lambda^2$

Thus:
$$\begin{cases} \left[\frac{db_+}{dq^2} \right] = -\frac{\Lambda^2}{m^4} + \frac{1}{m^2} + \mathcal{O}\left(\frac{1}{\Lambda^2}\right) \\ \left[\frac{db_-}{dq^2} \right] = \mathcal{O}\left(\frac{1}{\Lambda^2}\right) \\ \left[\frac{da_+}{dq^2} \right] = \frac{1}{m^2 \sqrt{-3}} \\ \left[\frac{da_-}{dq^2} \right] = \frac{-1}{m^2 \sqrt{-3}} \end{cases}$$

Now

$$\begin{aligned} \left(\frac{dI}{dq^2} \right)' &= (-i\pi^2) \frac{d}{dq^2} \int_0^1 dx \ln(x-a_+) = (-i\pi^2) \int_0^1 dx \frac{1}{x-a_+} \left[-\frac{da_+}{dq^2} \right] \\ &= i\pi^2 \left[\frac{da_+}{dq^2} \right] \ln\left(\frac{1-a_+}{-a_+}\right) \end{aligned}$$

Now for a_+

$$\left(\frac{1-a_+}{-a_+}\right) = \frac{\frac{1}{2} - \frac{1}{2}\sqrt{-3}}{-\frac{1}{2} - \frac{1}{2}\sqrt{-3}} = \frac{1-\sqrt{-3}}{-1-\sqrt{-3}}$$

$$\ln(\text{same}) = \ln(-1) + \ln\left[\frac{1-\sqrt{-3}}{1+\sqrt{-3}}\right]. \quad \text{Let } (-1) = e^{\pm i\pi} = (+i)^2 \text{ so}$$

$$= i\pi + \ln\left(\frac{1-i\sqrt{3}}{1+i\sqrt{3}}\right) = i\pi - \ln\left(\frac{1+i\sqrt{3}}{1-i\sqrt{3}}\right) = i\pi - \ln(e^{i2\theta})$$

$$= i\pi - i2\theta \quad \text{where } \theta = \tan^{-1}(\sqrt{3}/1) = 60^\circ = \frac{\pi}{3}$$

$$= i\left[\pi - \frac{2\pi}{3}\right] = i\pi/3. \quad \text{ie } \left[\ln\left(\frac{1-a_+}{-a_+}\right) = i\frac{\pi}{3}\right]$$

Similarly,

$$\ln\left(\frac{1-a_-}{-a_-}\right) = \ln(-1) + \ln\left[\frac{1+i\sqrt{3}}{1-i\sqrt{3}}\right] = i\pi + i\frac{2\pi}{3} = i\frac{5\pi}{3} = \ln\left(\frac{1-a_-}{-a_-}\right)$$

My confidence is low on these since I did not investigate cuts.

δI due to the a_+ is then:

$$\text{let } \sqrt{-3} = +i\sqrt{3}$$

$$\frac{dI}{dq^2} = (i\pi^2) \frac{1}{m^2\sqrt{-3}} \left[\left(\frac{i\pi}{3}\right) - \left(\frac{i5\pi}{3}\right) \right] = \left[(-i\pi^2) \left[\frac{2\pi}{3\sqrt{3}m^2} \right] = \frac{dI}{dq^2} \right]_{a_+, a_-}$$

Next:

$$\left(\frac{dI}{dq^2}\right)_{b_+, b_-} = (-i\pi^2) \frac{db_+}{dq^2} \ln\left(\frac{1-b_+}{-b_+}\right) + \text{same.}$$

Now:

$$\ln\left(\frac{1-b_+}{-b_+}\right) = \ln\left[\frac{2m^2 - (2m^2 b_+)}{-(2m^2 b_+)}\right] = \ln\left[\frac{2m^2 - 2\Lambda^2 + 2m^2 + \mathcal{O}(1/\Lambda^2)}{-2\Lambda^2 + 2m^2 + \mathcal{O}(1/\Lambda^4)}\right]$$

$$= \ln\left(1 - \frac{m^2}{\Lambda^2} + \mathcal{O}(1/\Lambda^4)\right) \approx -m^2/\Lambda^2 + \mathcal{O}(1/\Lambda^4)$$

$$\ln\left(\frac{1-b_-}{-b_-}\right) = \mathcal{O}(\ln\Lambda^2)$$

$$\text{So } \left(\frac{dI}{dq^2}\right)_{b_+} = (-i\pi^2) \left[\frac{-\Lambda^2}{m^4} \right] \left[\frac{-m^2}{\Lambda^2} \right] + (-i\pi^2) \left[\mathcal{O}\left(\frac{1}{\Lambda^2}\right) \right] \left[\mathcal{O}(\ln\Lambda^2) \right] = (-i\pi^2) \frac{1}{m^2}$$

Here is the result

$$\frac{1}{(-i\pi^2)} \frac{dI}{dq^2} \Big|_{q^2=m^2} = (+) \frac{2\pi}{3\sqrt{3}m^2} + \frac{1}{m^2} = \left[\frac{(\text{number})}{m^2} \right]$$

Notice that, although my number may be wrong, it does not depend on Λ . Thus, there seems to be, in ϕ^3 theory, only a finite Z to this order. In QED this thing is not 1.

Commins

STABLE PARTICLES

/ pseudoscalar octet
/ baryon octet

<u>particle</u>	<u>lifetime</u>	<u>decay products</u>	
γ	∞	—	
p	∞	—	
ν	∞	—	
e	∞	—	
n	918 sec	$p e \nu$	weak
μ	2.2 μ s	$e \nu \bar{\nu}$	weak
K_L	51 ns	$3\pi, \pi l \nu^{(66)}$	weak
$\pi^\pm$	26 ns	$\mu^\pm \nu$	"
$K^\pm$	12 ns	$\mu^\pm \nu^{(60)}, \pi\pi^{(30)}$	"
Λ	260 ps	$p\pi^-, n\pi^0$	"
Ξ^-	170 ps	$\Lambda\pi^-$	"
Σ^-	150 ps	$n\pi^-$	"
K_S	89 ps	2π	"
Σ^+	80 ps	$p\pi^0$	"
π^0	80 μ ps	$\gamma\gamma$	EM
η	5 μ ps	$\gamma\gamma^{(30)}, 3\pi^{(54)}$	EM
Σ^0		$\Lambda\gamma$	EM.

— processes
 — parameters
 — ideas + phrases

② $\mu \rightarrow e \bar{\nu} \nu$ ch 2 G

③ $\begin{cases} \nu \bar{e} \rightarrow \nu \mu \\ \nu \bar{e} \rightarrow \nu e \\ \bar{\nu} e \rightarrow \bar{\nu} e \\ e \bar{e} \rightarrow \nu \bar{\nu} \\ \nu \mu \rightarrow \nu \mu \end{cases}$ ch 3 G W-boson

④ $\begin{array}{l} \pi^\pm \rightarrow \mu^\pm \nu \\ \pi^\pm \rightarrow e^\pm \nu \\ K^\pm \rightarrow \mu^\pm \nu \\ K^\pm \rightarrow e^\pm \nu \end{array}$ $\begin{array}{l} f_\pi \\ f_K \end{array}$ "P₁₂" CVC Goldberger-Treiman $n \rightarrow p e \nu$
 $\pi^+ \rightarrow \pi^0 e^+ \nu$ f_π only CVC $\Rightarrow f_3 = 0$
 $B' \rightarrow B(\ell \nu)$ $f_1, f_2, f_3, g_1, g_2, g_3$ CVC, second-class currents g_2

⑤ $n \rightarrow p e \nu$ C_V, C_A "allowed" approx Fermi, Gamow-Teller weak-magnetism, $f_2 \sim$ magnetic moments of n, p second-class g_2

⑥ $\bar{\mu} p \rightarrow n \nu_\mu$ capture

⑦ Rules:

1. $|ds| \leq 1$
2. when $|ds| = 0, |dt| = 1$
3. when $|ds| = 1, |dt| = 0$
4. when $|ds| = 1, |dt| = \frac{1}{2}$

SU(3) review

Cabibbo current-octet hypothesis + rotation

Gell-Mann's $SU_2 \times SU_2$

hadronic current charges $SU_3 \times SU_3$ current algebra

⑧ Adler-Weisberger relation (C_A in terms of $\sigma_{\pi^+ p}$)
 $B' \rightarrow B(\ell \nu)$ decays (gen. beta decay) D, F

⑨ $B' \rightarrow B \pi$ A, B s, p waves α, β, γ
 isospin spurions from $\Delta I = \frac{1}{2}$ rule
 λ_6 -dominance and Lee-Sugawara

- ⑩ K_L and K_S
regeneration
 $K^+ \rightarrow \pi^0 (l\nu)$
 $K_L^0 \rightarrow \pi^\pm (l\nu)$
Adams-Gatto Theorem: $f_+(0) = 1 + \mathcal{O}(\lambda^2)$
Soft Pion Theorem
 $K \rightarrow \pi\pi (l\nu)$ " K_L " a_1, a_2, a_3, a_4
neutral currents and
 $K_L^0 \rightarrow \mu^+ \mu^-$ $\swarrow$ SU_3 breaking

- ⑪ $K \rightarrow \pi\pi$ K_L, K_S ϵ CP-violation
regeneration $\eta^{+-}, \phi^{+-}, R(\epsilon),$ mass ($K_L - K_S$)
 $K \rightarrow \pi\pi\pi$ Dalitz diagram $\sigma =$ slope parameter

- ⑫ parity violating potential add to nuclear potential
forbidden events
circular pol.
asymmetry

- ⑬ $\nu p \rightarrow \mu X$ etc. structure functions scaling

- ⑭ Cl^{37} exper!

1. Introduction. Brief history: in 1932 the beta decay spectrum was seen to be continuous; this seemed to violate energy conservation since reaction was supposed to be $n \rightarrow p e^-$. Pauli suggested a neutrino, then Fermi made a theory using this idea in 1934. Fermi's lagrangian was just $J^\mu j_\mu$ where J^μ is simple γ^A current, for nucleon and j_μ is the simple γ^A current for electron-neutrino. Of course this does not show V-A form, but it worked for beta decay.

There are many unstable particles. The ones which are stable against both strong and EM decay live 10^{-10} seconds which is a long time; ie, a width of 10^{-6} eV. If there is some EM decay, like $\pi^0 \rightarrow \gamma\gamma$ then expect shorter life 10^{-16} sec, or width of 1 eV. Strong decay particles are 10^{-23} sec or width 100 MeV.

In 1958 Feynman and GellMann came up with our present form of the lagrangian; recall by the way that it was 1953 when GellMann Nishijima posulated the strangeness quantum number to explain the low production rate of single strange particles. Thus the Feynman GellMann theory had strangeness changing currents etc etc.

In fact, the weak current is sum of four currents: electronic and muonic currents, and $dS=0$ and $dS=1$ hadronic currents. Each of these hadronic currents has a V and A piece, whereas the leptonic currents are each V-A. Separate electron and muon conservation is noted, and the helicity story is noted. Figure 1.4 on $\pi_{\mu 2}$ decay makes the complete P and C violation of weak interactions very clear. Neutrinos are of helicity -1 only; electrons are also, to the extent you can neglect their mass. Ie, left-handed neutrinos. By the way, semileptonic decay means both hadrons and leptons ~~in the final state~~ ^{involved}. Also by the way, the θK paradox was that the K decays both into 2 and 3 pions which seems to break parity conservation. Thus Lee and Yang. This was 1956. Feyn Gell was 1958.

Commins has huge table showing all process types. Electron neutrino elastic scattering has now been done I think. Muon decay is the only all-leptonic decay process. Semi-leptonic includes beta decay, neutrino absorption (inverse beta decay), pion decay, the pp process. Then you go to semi-leptonic stuff with strange particles: kaon, sigma and lambda decays here. Also, the direct interaction of the hadronic weak currents should effect np elastic reactions.

The $dS = dQ$ rule applies to reactions which are strangeness-violating and which are semileptonic. The changes apply to the hadron involved. This rule is built into the Cabibbo form.

The $|dI| = \frac{1}{2}$ rule. This rule says that when you have a reaction with a hadron changing $|dS| = 1$, its isospin must change by $\pm \frac{1}{2}$ unit. Again the Cabibbo form will incorporate this rule.

Since K(long) goes into $\pi^0\pi^0$, we have CP violation, hence T violation. This still a mystery.

Ch2: Muon Decay. The amplitude M is written in (2.6). After squaring, you have the produce of two traces $T1$ and $T2$ all of which boils down to (2.26). Notice the neutrino projection operator (2.22) is just $\frac{1}{2}(1 - \gamma_5)$ since no helicity option, whereas for lepton like electron you have mass and spin. Next thing you want to do is integrate over the di-neutrino phase space weighted by some momenta as in (2.28). By fooling around you deduce the tensor I_{ab} which is this phase space and you are left with (2.37). Finally you go to muon rest frame with the muon polarized along the plus z , decay rate is (2.40).

In (2.40) quantity ϵ is the fraction of max of electron energy. You see that the degree of front-back assymetry (coefficient of $\cos\theta$) is energy dependent as plotted on next page. Since this formula neglects electron mass, you see that rate is zero into electrons with positive helicity.

Next section shows how L is non C invariant. Then comes the piece about trying to construct a very general amplitude, more general than $V-A$. You now have many constants C as in (2.44) which is same as constants a and b of (2.45). The final decay rate in general scheme is (2.53). The 4 parameters called ρ, η, ξ and δ are theoretically given from the constants as shown. The functions $g(x)$ and $h(x)$, explicitly stated in next section, are the radiative corrections.

What are these Michel parameters? Clearly ξ controls the scale of the fore-aft assymetry term, and is -1 in the $V-A$ theory. Parameter ρ controls the shape of the high end of the electron energy spectrum in unpolarized decay (where the $\cos$ term averages away and $x = 1$ roughly.) Then η does the low energy end, and finally δ is the assymetry shape parameter. Experiment shows that these 4 parameters are all consistent with the $V-A$ theory, see page 49.

Next subject is the magnetic moment anomaly $\frac{1}{2}(g-2)$ of the muon. The first order theoretical prediction is just $g/2$ the same as for the electron, but the order α^2 and α^3 values are different for electron and muon.

The Garwin-Lederman method is this: you stop muons and make them rotate at their Larmor frequency like a compass needle rotating. Using one fixed detector and noting the muon arrival time, the time distribution of counts in the detector tells you a) the gyromagnetic g , ie, the anomaly a , from the observed frequency of the decay distribution; b) the magnitude of the assymetry. Obviously if there were no front-back assymetry this would not work. This experiment has been used to measure the assymetry shape factor δ mentioned above. A G-L experiment can also be used to get μ_μ , the magnetic moment of the muon. But this type exper is not good enough to get the anomaly because you need accurate muon mass beyond ability to measure. What happens in fact is that the anomaly is measured in another experiment very accurately, then the just-mentioned μ measurement gives an accurate mass for the muon. The experiment which gets a_μ is the famous one that uses muons rotating in a ring.

Finally, Gene notes that you can make long-lived muonium atoms μe . As with hydrogen, there is some hyperfine splitting which you can calculate in QED. Unlike hydrogen, there is nothing unknown here (like proton shape and size effects). Of course you get the theoretical result very accurately, and this is used as a measurement of α . Another measurement of α comes from Josephson experiment. Now α is measured two ways and they agree to 1 ppm. By the way, Josephson is this: you put a voltage V on a J junction, and you should see an ac signal at frequency $\omega = 2eV/\hbar$.

Elastic Lepton Weak Processes

Ch. 3: Recall that the muon decay $/M^2 = 64G^2 (p.q)(p.q)$. By just changing names around a little you get the various scattering and annihilation processes, ie, anything you can do with two ^{lepton} currents.

However, there is a slight difference in the cross sections. For $l\nu$ process, you find that the DCS is only energy dependent, no angular. See (3.19). However, for the $l\bar{l}$ process there is some angle dependence in the DCS. Reason is that here you are going through a vector resonance and this picks out $J=0$ and $J=1$, whereas in the $l\nu$ process you are exchanging the vector and you get the contact effect so only $J=0$.

Obviously, with either of these elastic cross sections you are in trouble with unitarity because only 1 or 2 partial waves involved and energy is upstairs. Of course this is just first order perturbation theory, you might hope that higher orders would fix things up. But higher 4-point interactions just make things worse. Even when you put in the W boson, you still violate unitarity in say the zeroth partial wave, although the total cross section goes to a constant instead of blowing up. The problem with the W is that the $k^\mu k^\nu$ term in the propagator ruins your ability to renormalize. So the W improves unitarity a little but does not do it completely. By the way, The W theory gives a β which differs a little from .75 of the V-A Fermi theory.

Chapter 4: Hadronic Weak Currents.

Previous chapters were: introduction, muon decay, other purely leptonic weak processes. For first time we now look into hadronic decays.

First subject is the famous $\pi_{\mu 2}$ pion decay. Your pion goes into vacuum through a hadronic current: this current must be A_0 type since axial and no strangeness change. Why axial? Because only 4-vector around is q^μ which is polar. Thus matrix element of current must be polar, but pion is pseudoscalar, so current operator must be axial vector. Pion decay constant f_π is defined. You can determine this constant from the decay rate of pions through this mode (which, by the way, is essentially 100% of charged pion decays).

Next, consider the π_{e2} decay model. Calculate in same way with same constant! Result differs from the $\pi_{\mu 2}$ only by mass factors, hence you know the ratio as in (4.9); branch fraction is a mere 10^{-4} . Although electron has more phase space, this mode is suppressed because you are trying to force electron into wrong helicity state (pion has no spin).

Next comes the decays $K_{\mu 2}$ and K_{e2} . Compute in exact same way, except constant now called f_K . The $K_{\mu 2}$ model is now only 63%, and K_{e2} is way down for same reason as before, even more so. So from rate into the $K_{\mu 2}$ mode you can compute f_K . The ratio of f_K/f_π is $\tan\theta = .27$, Cabibbo. later we will learn that you can always associate $\sin\theta$ with non-strange and $\cos\theta$ with strange, changing or strangeness violating reactions.

Now move on to other decays. Consider those like $\pi^+ \rightarrow \pi^0 e^+ \nu_e$ which is called pion beta decay. Now you need a different matrix element between two pion states. There are now two 4-vectors available, both polar, so current chosen will be V_0 . For each 4-vector you will have a form factor, usually called f_+ and f_- and usually defined with a $\sin\theta$ showing.

Digression: if weak hadronic vector current is raising/lowering part of isotopic current triplet whose third component is the I_3 piece of the EM current, (isovector triplet hypothesis), then what can you say about pion beta decay? First of all, since EM current is conserved, the second EM form factor vanishes and as we well know there is only pion EM form factor called F_1 . In iso-notation you can write EM current matrix element as $\phi^+ I_3 \phi$ with F_1 . Thus to get the vector, nonstrange element V_0 you replace with I_+ say.

Beware: it is the I_+ components weighted by $\cos\theta$ that are assumed to be the isospin partners of the EM current. This is CVC. Thus, the constants f_+ and f_- mentioned above for pion beta decay are: $f_+ = \sqrt{2} F_1$ and $f_- = 0$. Thus, it is simply the EM pion form factor that enters in the weak pion beta decay, the linkage being CVC.

By the way, pion beta decay is 10^{-8} fraction of charged pion decays. All charged pion decay modes are weak and involve neutrinos. In contrast, all neutral pion decays are basically electromagnetic. The reason of course is that for a π^0 decay you have to dump the charge on some lighter particle, and the only candidates are leptons which implies a weak decay.

In the same mode as above you can have $K^+ \rightarrow \pi^0 \mu^+ \nu_\mu$. Your current is still V_1 strangeness changing, so $\cos$ is replaced with a $\sin$. However, the difference now is that I don't think CVC applies to a V_1 current, only to a V_0 current, so I would guess that you still have two form factors to deal with since no known conserved current. CVC of course does not apply to either A_0 or A_1 .

To finish off pion beta decay, Gene goes on to compute the rate. Since the q^u transferred off the pion is so small, you can use $F_1 = 0 \times 1.0$, ie, the EM pion form factor normalization. Then your pion beta decay rate involves only $G^2 \cos^2 \theta$ as shown in (4.63). Now, we measure G from muon decay, and we measure θ from ratio of kaon/pion decay constants, so this result is a pure prediction. Agrees with experiment to 10% exper error.

So far we have done Meson to Meson + Pair. What about Meson to Meson + Meson + Pair? Now you have 3 polar vectors since 3 4-momenta, and you suddenly also have an axial vector of the form $\epsilon_{\dots} k \dot{p}$. Thus, if you are talking $K^+ \rightarrow \pi^+ \bar{\nu} e^+$ your A_1 current has 3 terms, and your V_1 current has 1 term, so now four form factors to play with, at the start. Gene does not pursue this decay very much. Since $dS=1$ there is no CVC, so will be a mess. Gene omits things that are a mess. Fraction is of course way down on this.

Now we move to baryon stuff. Regular beta decay is B to $B + \text{pair}$. Now baryons have spin so have to deal with that. The B/B matrix element must always be of the form $\bar{u}(p) \dots u(p')$; this is just an alternative to the M-function formalism. By playing with things, you see that with your Dirac γ_μ and your two available momenta, you can make 3 polar vectors and 3 axial as in (4.29), with f_i and g_i as the 6 form factors.

Now comes CVC again. The EM current would have the 3 f-type form factors since current is of the V_0 type. But EM current conservation reduces this to the usual two form factors. As written in (4.39), the first form factor gives charge plus usual moment, and second form factor gives the moment anomaly. If you proceed to write this EM matrix element in iso-notation, you get (4.45). Whereas the corresponding pion matrix element for EM had only an I_3 piece, the nucleon EM matrix element has an I_3 piece and an $I=0$ piece. This $I=0$ piece arises from the hypercharge y of the baryon via the GM Nish formula, whereas pion had no y .

So examine (4.45). There are 2 form factors in each term. Only the second term will be assumed by CVC to be the I_3 piece of a triplet. So what happens? Well, for the vector piece of the current matrix element (which is $dS=0$ since nuclear beta decay) you can set $f_3 = 0$ by current conservation, and then $F_V = f_1$ and $F_V = f_2$ as in (4.52). Ie, the two vector couplers get connected with EM form factors, the third is set to zero. You are in great shape, except you don't know the axial form factors.

What are arguments for CVC? The γ^μ weak form factor f_1 is experimentally found to normalize out to unity ($C_V = 1$), just like the EM form factor. I think this is the main motivation.

Now, what are second-class currents? One presumes that ~~the~~, of the 6 form factors appearing in nuclear beta decay, 4 are completely caused by strong interactions (all but f_1 and g_1). Thus, you might expect the four other terms to be G-parity invariant. This assumption combined with T invariance (which forces all form factors to be real) yields result $g_2=0$. The g_2 term, if it is experimentally present, is called a second class current.

Assuming no g_2 , the beta decay current now has the two EM form factors and the still unknown g_1 and g_3 axial form factors. By the way, g_3 is called "the induced pseudoscalar form factor" of beta decay.

Finally, we have Goldberger-Trieman. Consider nucleon beta decay. We have just argued that in the axial piece you have only the g_1 and the g_3 terms. The one-pion exchange model is a model for the g_3 term. This model of course involves the pion decay constant f_π (of weak interactions, appearing in $\Pi_{\mu\nu}$) and the hadronic pion-nucleon coupler called g_0 here. Thus, the model here says precisely (4.74), where you put in a little adjustment to keep in mind that you should have used $g(q^2)$ as a running PI-N constant. So we now have statement (4.76) with g_3 now replaced by $(f_\pi g_0)$.

Next, look back at regular $\Pi_{\mu\nu}$ decay and observe that the A_0 current (which was involved there) is conserved to the extent that $m_\pi = 0$. Now since we are dealing with the same current operator A_0 (but between nucleon states) we will assume conservation of this current, PCAC. This of course makes a relation between g_1 and $(f_\pi g_0)$. But $g_1 = -C_A = 1.23$ as measured. You can measure each of these three constants separately and the result works!

What does Goldberger-Trieman prove? It is evidence for $(PCAC) \wedge (g_2=0) \wedge$ (OPE model). Another way to say GTR is that g_3 is related to g_1 .

Obviously, after all this preparation on the form of the nuclear beta decay matrix element, we should do beta decay. Next chapter.

Chapter 5: Nuclear Beta Decay.

This is a long and complicated subject, but an excellent exercise in applying what has been learned in the earlier chapters. Here are the basic facts: you are dealing with a hadronic weak current. In principle that means, since nuclear implies baryons, there could be the 6 form factors $f_1, f_2, f_3, g_1, g_2, g_3$. By the way, the form factors f_2 and g_2 are called the "induced tensor coupling" and, I guess, the "induced pseudotensor coupling" because these multiply the $\sigma_{\mu\nu}$ and $\sigma_{\mu\nu}\gamma_5$ terms in the baryon weak current. Of course the g_2 term is also called the second-class current term and is supposed to vanish by G-parity argument. The f_3 and g_3 are called induced scalar and induced pseudoscalar. The word induced means they are caused by presence of strong interactions.

assuming V, A theory.

In the pure lepton world you have only f_1 and g_1 non-induced couplings. Any other couplings are "induced". Of course in CVC theory you set $f_3 = 0$ and you identify f_1 and f_2 with your EM form factors. At $q^2 = 0$ f_1 is called C_V and g_1 is called C_A for obvious reasons.

In a plane-wave theory, the beta decay amplitude is given by (5.1). There you see the usual leptonic V-A current hitting onto the 5-term hadronic current (recall $f_3=0$). But in this decay you have to treat the nucleons as having definite wave-functions, they are not plane waves. Plane waves are only useful when something is non-localized. A nucleus is localized and has some definite wavefunction for neutron distribution in it. So we can rewrite the amplitude putting in wavefunctions and replacing the momenta (which appear in both induced tensor terms and the induced pseudoscalar term, ie, in g_2, f_2, g_3 terms) with position gradients. The symbol $j_\mu(x)$ is shorthand for the weak leptonic current.

Now the key simplification in beta decay is this: the Q-values (energy released) are on the order of 1 MeV for all nuclear beta decays. Since nucleuses are much heavier than this, they don't recoil much. Thus, you can make them non-rel and use the large-small spinor reduction. Also, obviously the momentum transfer t for the nucleus (the q^2 which appears in the various form factors) is nearly 0. Thus, all form factors reduce to constants like $f_1(0) = C_V$ and $g_1(0) = -C_A$.

Moreover, once you write out the various terms in the reduced-Pauli language as in (5.8), you realize that each gradient $\vec{\nabla}$ appearing is, in momentum space, a $\vec{q}$ but this is small. So in the "allowed approximation" you just drop all gradient terms. But, the only non-gradient terms are the C_V and C_A terms; the f_2, g_1, g_2, g_3 terms then all go away (note that $(E+w)$ terms also go away since this is q^0). Therefore, once you go into the "allowed approx", you can learn nothing at all about these other form factors. You can learn only about C_V , C_A , and G_F = the weak coupling appropriate for beta decay.

So, everything reduces to (5.14) which shows the amplitude (called U) as two terms. Basically, $U = (G \cos \theta / \sqrt{2}) (C_V \langle i | j_0(0) | i \rangle - C_A \langle i | \vec{\sigma} \cdot \vec{j}(0) | i \rangle)$, where $\langle i |$ is the nuclear wavefunction value of operator $\vec{\tau}^+$ (and is thus always some number like 1.00 or $\sqrt{2}$), and $\langle i | \vec{\sigma} \cdot \vec{j}(0) | i \rangle$ is the same thing but with a Pauli $\vec{\sigma}$ operator in there, ie, this is sort of the nuclear spin. For an nucleus, you sum over all nucleons and you have little $\vec{\tau}^+$ operators which allow the possibility of each nucleon converting. Ie, you get a coherent sum of amplitudes as the total amplitude. The isospin stuff enters just as notation, ie, wavefunction χ is an isotopic vector, doublet for neutron decay eg.

So the big thing we learn is that there are basically two terms, the CV and the CA terms. These terms are called Fermi and Gamow-Teller as we shall see.

Since we assumed that lepton wavefunctions are plane waves over nucleus, Gene argues that this means $L=0$ for leptons. Well that's not right. Really the variation of the plane waves of the leptons is so small over the nucleus that you approximate $j(x)$ by $j(0)$. I.e., the actual decay occurs in a small localized place; roughly speaking, the leptons originate at the same spatial point and therefore can have no orbital L . They can have $S=0$ or $S=1$, and hence leptons have $J=0$ or $J=1$. If leptons carry off $J=1$, then nuclear spin must have changed by $J=1,0$. What I mean is this: $J_i \subset J_f \oplus 1$.

This requires a little discussion. Consider the matrix element $\langle f | \hat{T} | i \rangle$. The Pauli current operator transforms as a spin-1 object. The states i and f also have rotational transformation properties, i.e., nuclear spin J_i and J_f . This matrix element will vanish unless $J_i \subset J_f \oplus 1$. In general, if you have a transition $J_i = 1$ to $J_f = 1$, this matrix element will have some value, albeit unknown. The other element will also have a value, i.e., $\langle M | i \rangle$ where I have omitted the $\hat{T}$ in both cases. Thus, this is a "mixed F and GT transition". If you have $J=0$ goes to $J=0$, that must be pure Fermi, because then the $\hat{T}$ vanishes from rotation group theory. Or, if you have $J=0$ to $J=1$, then pure GT, since the F vanishes from same group theory.

So, depending on how the nuclear spin changes, you can have F, GT or mixed transitions. ~~That's why it's that~~ In (5.14) these facts are obvious without any talk about the lepton L . It's just that the $L=0$ discussion explains why there are no $dJ=2$ beta decays, eg.

Another detail: electron wave function, since it is charged, is "distorted" by the nuclear electrostatic field, so you need $F(Z,E)$ to correct for this. This correction is a function of nuclear charge Ze , and electron energy E . So with this correction we now have amp^2 as in (5.17) with (5.18).

Fact: since leptons always have the $(1+\gamma_5)$ factor, the longitudinal polarization of any electron emitted in any weak interaction, such as beta decay, is $-v/c$. How did I not know this? In the neutrino limit you of course get -1 . This fact has been tested experimentally in various beta decays.

OK, now put in your amplitude, square it, do the traces etc and get the rate shown in (5.30) which applies only for $J=\frac{1}{2}$ to $J=\frac{1}{2}$. Discuss this rate: You see the obvious phase space and Coulomb correction $F(Z,E)$ already mentioned. The overall rate normalization shows G^2 as well as $\xi = (5.31)$. In the brackets, you see a 1, then all kinds of angular terms. The theoretical coefficients of these angular terms are given as a, A, B, D ; the idea is to measure these coefficients

experimentally then compare.

But first, if you integrate over everything to get the total decay rate, none of these coefficients matters. Suppose you integrate over everything except the electron energy. The (5.40) gives you the theoretical "electron energy spectrum". I.e, you look at 1,000,000 beta decays and record the electron energy in each case, this gives you a spectrum. Experimentally, what you learn from this spectrum is that it has the right shape. This fact then tells you that the emitted neutrino $\bar{\nu}_e$ must have been lighter than 60 eV, or else spectrum would have been wrong. A similar beta decay experiment which emits a positron only gives $\nu_e < 4.1$ KeV.

So much for the electron energy spectrum. Now integrate this to get your total decay rate. The relevant dE integration is called the Fermi Integral and is called f_0 . Thus, the lifetime τ is given by (5.42) and involves only G_F, Δ, ξ . This implies of course a certain "comparative half-life" which is the half-life ~~which~~ in some sense a half life; this thing is called "ft" and $ft = (5.44)$. Thus, you can determine the product $G_F^2 \xi$ by measuring lifetimes. So if you somehow knew ξ you could deduce G_F . So, go look at some $J=0$ to $J=0$ transitions which are of course pure Fermi, then you know $\langle 1 \rangle$; but you do not know C_A yet, nor do you know $\langle 1 \rangle$, the Wigner-Eckart reduced $\langle \hat{r} \rangle$. But, if you go off and measure the front-back electron asymmetry in another experiment, you know A of (5.33) and therefore ξ of (5.47) and therefore ξ . Thus, you can measure G for various decays, result shown in Fig. 5.2. Notice that all G 's are smaller than G from muon decay, evidence for $\cos\theta$ presence. Once you know G_F , you go back to regular neutron decay where $\langle 1 \rangle = \sqrt{3}$ (since no messy wavefunction anymore) and you conclude that $C_A = 1.23$, the famous result. By the way, neutron lifetime is 11 minutes half-life.

Now let's go back and look again at those angular terms in (5.30). Note that in general J_i, J_f case you have to add in terms on bottom of page 110. Look at the terms. The a term is a scalar under parity, as is the D term. But the A and B terms are pseudoscalars. Thus, if all these terms are present, you have a parity violating interaction. Eg, parity invariance would tell you that $A = -A$. Similarly, if D is there, you have T violation. Confirm this by looking at (5.35).

So, first angular thing is the "electron-neutrino angular correlation" as seen in (5.52). In the theory, $a =$ as shown in terms of ξ . Thus, for each decay a and ξ (or x) must have a certain relation, i.e, they must lie on the main straight line of Fig. 5.3. Experimentally you measure a ~~and~~ by measuring electron p and recoil ion p and deducing p_ν . As noted above, you separately

measure g from the front-back electron asymmetry in polarized decays, the coefficient A . Since points do come out on straight line, you like.

Next section deals with J^A (A, B, D terms). Here of course you need to use polarized nuclei in your experiment. Obviously, the fore-aft coefficient A involving only the electron is easiest to get. Since this is pseudoscalar term, the fact that $A \neq 0$ in Co^{60} decay started the parity revolution. (1957) If you measure A and B , you get a separate determination of $C_A = -1.26$. D always comes out near 0, indicating that T and hence PC are ~~OK~~ in beta decay. Of course small ~~xxxxxxxxxxxxxxxx~~ inconsistencies can always be attributed to neglected terms in the allowed transition.

Now, when you elevate back up from the Dirac-Pauli level to the Gamma matrix level, you realize that, after dropping the various terms, your true amplitude for the hadronic current is the same as for the leptonic, except the $(1+\gamma_5)$ is replaced with $(1+C_A \gamma_5)$. But don't forget that this is only in the "allowed approximation". So it is a V, A theory again. In general you might have expected P, S, T terms (note: we assumed vector-vector at the start, way back with those form factors). The experiments combined with general forms of course support but do not prove the V, A weak hadronic current.

What the hell is "weak magnetism"? This is an effect which involves the f_2 form factor "induced tensor" term which is completely neglected in the allowed approx. If you include this term and look at certain beta decays which tend to isolate it, it causes a little correction to the electron energy spectrum. This correction involves $f_2(0)$, and therefore (recall CVC) the nucleon magnetic moments. The prediction is (5.70). Measurements of this shape correction agree well. More support for CVC. In EM current, you would call this f_2 tensor term the "magnetic" form factor since it gives the magnetic anomaly. Thus, this f_2 term is called the "weak magnetism" effect. So much for f_2 . Another corrective term is the second-class g_2 . Evidence for this term is inconclusive to date. No comments on the g_3 term.

{ Add: for Fermi transition, $\Delta I = 0$ } selection rules in addition to those for J_i, J_f .
for F or GT , $P_f = P_i$

Chapter 6: Muon Capture. Basic reaction here is $\bar{\mu} + p \rightarrow n + \nu_\mu$. If you did this at FNAL, you would get and compute a cross section. But, this chapter concerns instead this process occurring as follows: a muon is captured by an atom and replaces a 1S electron. Since muon mass is larger than electron, the Bohr radius is very small; ie, this muon spends time in the nucleus, so the capture rate is significant. My main difficulty in understanding this is that you get a rate, not a cross section. Ie, somehow this rate is like a cross section

multiplied by a flux; the flux is caused by the presence of the muon right there in the atom. I am so used to plane waves and cross sections, but I know it is possible to fold in the various particle wavefunctions (if they are not plane waves) to get the total transition probability, which in this case is a capture rate. In fact we had to do some of this wavefunction folding back in nuclear beta decay also. Notice that this folding gives you a very particular rate: from some initial muon 1S wavefunction and some nucleon wavefunction, into some particular final nuclear wavefunction; the final muon I guess can escape in a plane wave; it is the only plane wave here among the 4 particle states.

So, formula (6.5) really gives you a rate into that final nuclear state; you need only to integrate over neutrino phase space to get that rate.

Lets take time out to work on the matrix element for a while. You see in (6.6) that the amplitude is the spatial integral of the ~~xxxx~~ overlap of the four wavefunctions at each point x ; at each point, however, the amplitude is just the leptonic current dotted into the 4-term hadronic current (here we see only f_1, f_2, g_1, g_3 ; the g_2 second-class term is dropped.) The kinematics is such that you are looking at this 2-to-2 process at threshold in the initial channel. This fact means that variables are fixed. The transfer $t = q^2$ off the nucleon is fixed as shown in (6.13); the neutrino energy is fixed, see (6.12). You are in the cms frame of course. Thus, all form factors are evaluated not at $q^2=0$ but at some spacelike fixed point. Note that $q_\mu = q$ is the transfer, not the muon 4-momentum.

So, from your EM theory and CVC you deduce what f_1 and f_2 are at the relevant points. $-g_1 = C_A$ to .999; the only other constant is that g_3 . You define $g_p = mg_3$ where m = muon mass; then use Goldberger Trieman to get this $g_p = 7C_A$ approx. Now you have a good idea of all the constants involved, assuming no g_2 term.

The muon wavefunction is given by (6.18), has only upper component because so low energy. For the neutrino, p is not negligible so you have the usual (6.21). Note that this σ is a matrix in the lepton Pauli space. Meanwhile

WAIT; maybe the reason the muon has only an upper component is that experimentally you are forced to average over muon helicities. So $\sigma \cdot \hat{p}$ never appears anywhere. In contrast, the nucleon before and after spin does appear, as does the final neutrino spin; ie, you could prepare your nucleus in polarized state before you shoot in your muons.

When the dust settles, the amplitude you get is shown in (6.28). You have that folding of the wavefunctions and a spinor matrix element of an operator given in (6.29). The constants G_V and G_A and G_P are all known as discussed above. You must remember that σ_A operates in the nucleon 2x2 space, whereas σ in the lepton.

OK, so the amplitude is now clear. To get rate, use up the eldta function and exhibit neutrino direction integral as in (6.35). By the way, notice in (6.33) for the general nucleus that Gene has gone back to the isonotation, so you now have three spaces to worry about. The notation of (6.36) is not quite clear; the nuclear wave functions $f(x)$ now appear as states, so a spatial integration is implicit; you even see the x inside the muon wavefunction. Now the τ operator does its usual lowering function. So you get a nuclear $\langle 1 \rangle$ and $\langle \nu \rangle$ type matrix element, as in beta decay. As noted, these would be exactly the beta decay objects if muon were a plane wave; the exiting electron in beta decay is a plane wave, but here the absorbed muon is not.

Now comes a trick; you want to sum over available final nuclear states. If you fudge a little by pulling bracketed factor out of Z^4 , then this sum goes directly onto matrix elements squared as shown in (6.40). But then you can use unitarity to ~~xxx~~ absorb this sum; your squared matrix elements then contain double-sums over all nucleons in nucleus, each has a double spatial integral, so non-local. However, if you execute the neutrino direction integral before doing the double coordinate space integrals, you reduce the latter to a double-radial integral as in (6.45). The final rate result is in (6.45), and the double nuclear sum is still in there of course. If you can ignore the non-diagonal terms in this sum, you simplify to (6.46). For light small nuclei your muon-nuclear overlap is small so get (6.47). Simplify this some more to get approximate end result (6.48), which shows that Z^4 rate increase!

Comments: for larger Z , your $1S$ wavefunction is heavier at the origin, as shown in (6.20); this gives Z^3 . The 4th Z arrives from the diagonal sum of (6.47). The crude result (6.48) tells you that for a $p\mu$ atom (muonic hydrogen gas) the capture rate is 1000 times less than the muon decay rate. But for large Z atoms, even $Z=10$, it seems that the muon capture rate will exceed the muon decay rate.

Notice one bad aspect of this process; in essence there is only one final particle, the neutrino, so you are forced to do the complete phase space; thus, the only data you can measure is the total rate; it is unlike beta decay where you get asymmetries, angular correlations etc. Not a very rich experiment.

What does experiment say? First, you can do the gaseous $p\mu$ atoms. But the two fermions can have spin $S=0$ or 1 , and these two states have different matrix elements. Gen goes back and undoes some of the rate, showing ν once again. Well, the conclusions are not very exciting. You can do this muonic hydrogen, or capture in He^3 or in He^4 . Section closes with "radiative muon capture" and some hard-to-do experiment with ~~xxxx~~ nucleus recoil angular distribution. Does not prove or disprove anything; is compatible with the theory.

Chapter 7: the Cabibbo Trick. Multiplets reviewed. Empirical selection rules are noted: for $dS=0$ decays, $|dI| = 1$; ie, both V_0 and A_0 behave as isovectors. for $dS=1$ decays, $dS=dQ$; other rule: no ($dS=2$ -or-more) decays. for $dS=1$ decays, $|dI|=\frac{1}{2}$; ie, both $V_{\frac{1}{2}}$ and $A_{\frac{1}{2}}$ behave as isospinors.

These are rules I should keep in mind. Examples: $\pi^+ \rightarrow \pi^0 \pi^+$ is blocked by $dS=dQ$ rule; $K^+ \rightarrow \pi^0 \pi^+$ also blocked by same. Notice that the $|dI|=\frac{1}{2}$ rule contains the $dS=dQ$ rule due to GMN formula. However, there is some evidence of $|dI|=\frac{1}{2}$ rule violation.

The example given for valid $|dI|=\frac{1}{2}$ rule is the decay $K_S^0 \rightarrow \pi^+ \pi^-$. Since final state must have $L=J=0$, isospin state can be only $I=0$ or 2 . But if current operator is a $I=\frac{1}{2}$ object, only the $I=0$ amplitude is non-vanishing. Therefore, using trivial clebsches, you can deduce that K -short should go into charged pair twice as often as neutral pair, ie, ratio of branching fractions should be 2 . My tables show this fraction is in fact $69/31 = 2.2$.

Consider now $K^+ \rightarrow \pi^+ \pi^0$. Again, as above, you require $I=0$ by the $|dI|=\frac{1}{2}$ rule; but $I_2=1$ for final state, so rule says this is blocked! However rate shows up as 30% of the $K_{\pi\pi}$ rate. Is this really a weak decay? It's rate is in the weak time scale ball park. I am used to considering this as a Zweig forbidden hadronic decay. Maybe Zweig prohibition is exact and it only leaks through weakly; then cylinder correction should be zero.

Anyway, in the $K^+ \rightarrow \pi^+ \pi^0$ you must conclude that it is leaking through in the $I=2$ channel. Doing isospin comparisons, you conclude that $A_2/A_0 = 8\%$, a measure of the degree of $|dI|=\frac{1}{2}$ rule violation.

Next Gene goes into $SU(2)$ and $SU(3)$ discussion. He writes down the octet 3×3 matrices in usual way. I have reviewed this but have not set my own conventions yet.

Now, how do you associate the currents $V_0, V_{\frac{1}{2}}, A_0, A_{\frac{1}{2}}$ with octet currents? Well, in CVC you have already said that $V_0^* = j_1 + i j_2$ and $J_{em} = j_3$. Ie, you have an isovector, so far. Now, recall that the key $SU(3)$ assumption is the identification of F_8 with hypercharge $y = B+S$. Consider the weak decay $K^+ \rightarrow \pi^0 \pi^+$. (strange version of pion beta decay*) This involves only the current $V_{\frac{1}{2}}$ (see p 84). Clearly this current is acting like V -spin raise/lower, so you conclude that $V_{\frac{1}{2}}^* = j_4 + i j_5$. By the way, J_{em} = neutral current so is like U -spin direction. Perhaps J_{em} = mix of j_3 and j_8 ; the CVC only says that V_0 currents are partners of the isovector portion of the EM current; as usual, thing GMN formula.

What about the U -spin raise/lower operators which would relate to currents j_6 and j_7 which no one ever mentions? These would be involved in a decay say of $K^0 \rightarrow \pi^0 \pi^0$; the catch is that this would violate charge conservation; so there

seems to be no way to "access" those members of the current octet. That makes me uncomfortable. At any rate, I now understand how the various identifications are made. Similar to the association of the current with an iso-vector in beta decay i.e., recall presence of $\tau^{\pm}$ in there. Seems solid.

The A_0 and A_1 axial vector currents are connected with their own axial octet of currents. Gene calls these $g_1 + ig_2 = A_0^{\dagger}$ etc. Earlier I asked: does axial $dS=0$ current look like a vector in isospin; answer is yes.

Now comes the Cabibbo stuff. Go back first to the leptons. Just as you did for n and p in hadronic beta decay current, you can put e and ν_e in a doublet and imagine a group "weak isospin" SU_2 . Since ν_e is definitely left handed, you might put it only with e_L component of e . In this viewpoint, the regular V and A portions of the weak leptonic current look like weak-isospin raising operators, so the charges connected with these currents ~~A_{lep}~~ A_{lep} and V_{lep} are called F_+ and F_+^5 where the 5 reminds you of axial. You can of course imagine diagonal currents and charges (these would be neutral currents) which have charge L_3 and L_3^5 . Clearly you have yourself an algebra $SU(2) \times SU(2)$ where one $SU(2)$ is for the vector charges and the other for the axial charges. This is the chiral group. That the charges respect chiral algebra is not questionable, you show it directly. Gene is not saying yet what you should do with this group relevant to the weak leptonic currents. He mentions this because he next says: presumably the hadronic weak currents (i.e., the related charges) form $SU(3) \times SU(3)$, where again, one $SU(3)$ is for the vector octet and the other for the axial octet of currents; these are the j_i and the g_i currents.

Obviously this extension of CVC to the "octet hypothesis of the weak hadronic currents" is not exact nor proveable, so the $SU_3 \times SU_3$ things is not as good as the $SU_2 \times SU_2$ chiral algebra.

The Cabibbo business now arrives. You know experimentally that $dS=1$ decays are suppressed relative to $dS=0$, and there are no $dS=2$. The Cabibbo Hypothesis is really the octet hypotheses; obviously if weak hadronic ~~dS~~ currents transform as U, I, V spin raise and lower operators, you can get only $dS=0$ and $dS=1$. Suppose there were no $dS=1$. Then you would say that $J_{had} = j_1 + ij_2$ period, and by "universality" you would give this current exactly the same strength as the leptonic current $J_{lep} = j_{1,lep} + ij_{2,lep}$. I.e., weak-isospin raise and lower operators, same G . But then you imagine that somehow the presence of strong interactions causes a rotation of this $J(had)$ vector in the 8-dimensional space; the length of the vector does not change however. You choose just the right rotation so that you pick up a component along the $j_4 + ij_5$ axis in this space. You choose the rotation angle so that $dS=0$ and $dS=1$ parts of the current have correct experimental strength.

Moreover, you assume that the axial current is rotated exactly as much as the vector hadronic current. This leads to the concise result (7.79) which is the real thing. Obviously tons of predictions will follow.

What about charm. Gene's book is 1973. I would guess offhand: put weak currents into a 15 rep of SU(4). Do the same Cabibbo rotation to get (7.79), then rotate again to bring in some $dC=1$ weak currents. I.e., charm-changing weak currents. These will then be involved in decays of D mesons. Not clear, but seems you will need at least two Cabibbo angles. Maybe more. You might use the ~~Bxxx~~ f_D/f_K decay constants ratio to get $\tan\phi$, where ϕ is second Cabibbo angle.

The Cabibbo business is finally very clear and straightforward.

Chapter 8: Baryon Semileptonic Decays. This chapter opens with a review of the Adler-Weisberger relation. Lets see if I can't penetrate this and see how it really goes. You start with a commutator of axial + and - charges put between two protons. Put in some intermediate states and of course you get matrix elements of the A_0^0 0-th component of the weak, axial, hadronic current. For neutron intermediate state, this is precisely the axial part of beta decay and you relate to C_A . The RHS of this ~~strong~~^{hadronic} charges commutator gives a constant. Thus, I think you get the left hand side of the AW relation.

Now the problem is what to do about higher intermediate states? Instead of writing the charge I_+^5 as a spatial integral of A_0^0 , this time you can introduce a $\partial/\partial t$ at the expense of energy denominator as in (8.12), then you can replace d/dt with $\partial_\mu A_0^\mu$. Finally, this last is replaced with pion field via the Goldberger-Trieman.

Pause to recall how GTR works. You make a OPE model to explain g_3 in terms of f_π and $g_{\pi NN}$. Then you use PCAC to relate g_1 to g_3 , since second-class $g_2=0$. But $g_1 = C_A$. Thus, you have simple relation between these guys: $f_\pi, g_{\pi NN}, C_A$. Now, if you like you can use (8.10) version which says that $\partial_\mu A_0^\mu =$ pion field with constant as shown. From all this, you get for your "higher intermediate state" sum the form (8.15). But since pion field is there, the only possible contributing intermediate states are PI-N states. Somehow (Gene skips this) you can then relate this term of the Adler-Weisberger relation to integral over PI-N cross section, which certainly sounds reasonable. The net result is (8.16). Amazingly, when you numerically integrate the PI-N cross section difference, you do get the right answer for C_A .

What does this verify? First, we used a portion of the SU3xSU3 algebra, but only the non-strange portion. This whole thing does not involve strangeness at all.

Second, we used Goldberger Trieman, but we already know that works. We only uses PCAC inasmuch as it gives the GT relation. Really, the main thing this tests is the algebra of charges, only in the non-strange sector. Ie, we are testing the concept that weak hadronic currents have isospin structure. By the way, strong interaction currents must all be isoscalars! Just because weak interactions violate isospin does not rule out a role for isospin. The statement that the $dS=0$ currents are isovectors simply bursts with predictive power.

Now we go to the baryon decays. Obviously the simplest weak baryon decays are those emitting a $\bar{\nu}\nu$ pair. There are 10 such decays. The matrix elements are as shown in (8.19) for $dS=0$ and (8.20) for $dS=1$ decays. You preselect your baryons as b and b' . The currents in there must have various terms involving $\bar{B}B$ since you have to destroy one baryon and create another.

Question: how do you combine the baryon fields in this bilinear way so that what you have is an octet? (Recall that Cabibbo says currents are supposed to transform as octet.) Obviously there are two ways, the 8_s and the 8_a in the product of 8×8 . Question: what are the entries in the 3×3 bilinear $\bar{B}B$ matrices that you connect with the 8_s and 8_a ? Typical entries are given in (8.26).

Now a subtle point. Consider your $dS=0$ currents, both vector and axial. The vector are like $j_1 + i j_2$ and axial are like $g_1 + i g_2$. Thus, the $dS=0$, vector weak hadronic current looks like isospin-raising operator, and we know that such an operator can never change total isospin I . Recall that a pure Fermi beta decay involves only the C_V term, ie, only the $dS=0$ vector term, and for such decays $dI=0$. Same idea.

In contrast, your axial $dS=0$ current looks like an axial-isospin raiser, and thus cannot change axial-isospin. But axial-isospin is not the same as regular isospin. Thus, the $dS=0$ axial current can change total isospin.

So what? The point comes now: when you make your 8_s and 8_a entries of $\bar{B}B$ terms, you find that all your 8_s entries contain at least one term which changes total isospin of the baryon. Thus, for the vector current you must rule out all 8_s type $\bar{B}B$ terms.

But, according to Cabibbo, once you have removed the Symmetric combination from the $dS=0$ vector current, you should also remove it from the $dS=1$ current because this thing is just an induction from the $dS=0$ via the Cabibbo rotation. On the other hand, a direct $dS=1$ argument would say: well, these currents are not I_+ style currents, so you could not a priori say $dI=0$.

So, imagine putting in the $\bar{B}B$ linear combinations and letting the fields hit onto the states to make $u(b)$ spinors. You then get (8.30) and (8.31). The

symbols a and s in there are just the coefficients like $2/\sqrt{6}$ appearing in (8.26). You see that the $dS=0$ and $dS=1$ matrix elements each have only one vector term which is to say, the 8_a term. Note also that these terms each have overall coefficient 1, aside from the $\cos$ and $\sin$. This 1 is $f_1(0)$ of regular nuclear beta decay. Recall that this was the experimental fact that suggested CVC, ie, that $f_1=1$, ie, that the beta decay G is same as muon decay G aside from cosine.

However, for axial currents, the isospin argument given above fails and you are stuck with both 8_a and 8_s type terms, the usual F and D terms. Again, looking at regular n decay, you see that $F+D = C_A$.

So, after a lot of yak, I now see what (8.30) and (8.31) are saying. All 10 allowed baryon simple (semileptonic) decays can be fitted with just two constants, D and F .

Table 8.1 in fact lists the effective C_V and C_A for all the decays. By the way, notice that all this business of combining $8 \otimes 8$ into 8_a and 8_s implicitly means we are assuming exact $SU(3)$ invariance. In essence we are applying $SU(3)$ Wigner-Eckhart theorem to the matrix elements here. However, because there are two octets in the 8×8 , you cannot predict everything from $SU(3)$, even exact $SU(3)$.

Notice that some times $C_V=0$ for some processes. This is simply because this process is $SU(3)$ -absent from the 8_a term. Since axial has more terms than vector, ie axial has $8_a + 8_s$, it is of course more likely that an effective C_V or f_1 should vanish than for the g_1 to vanish.

By the way, it should also be noted that we are treating all these baryon decays in the same approximation as we treated the beta decay, ie, we assume small momentum transfer, evaluate all form factors at $q^2=0$, drop all small terms such as the f_2 weak-magnetism. Ie, in this limit the decay theory is just a f_1, g_1 theory.

Now, lets do a fit. Choose F and D as shown in (8.58), use the usual Cabibbo angle. This then enables us to compute all these decays. Ie, we can get all the decay rates. Most total rates fit well.

But are there other things you can measure experimentally to test this Theory of Baryon Decays? Sure, everything you measured in beta decay you can also measure here. Eg, the ρ correlation, which is a function of the angle θ between $\hat{p}_e$ and $\hat{p}_\nu$ (this was the coefficient a in beta decay) is some definite function of your f_1 and g_1 . By measuring correlation, you can check to see if $f_1=0$ in those decays where it is supposed to be. You can also look at the lepton and recoil baryon energy spectra; the fore-aft lepton asymmetry; the recoil polarization; (these latter two in polarized initial baryon decays). These peripheral experiments often yield information on the C_A/C_V ratio $= g_1/f_1$.

Apparently nothing drastically wrong has yet been found.

Now a few more topics are in this chapter. Why does SU3 invariance work so well here if masses are so severely broken? Some people set out to find out, leading to the Ademollo-Gatto Theorem. This theorem is derived later in this book but says this: the SU3-breaking term in Hamiltonian must be $F_8 = Y$ = hypercharge (octet dominance). This term has relative coefficient .1 if superstrong H has weight 1.0. Theorem claims this: the F_8 correction has no effect on our analysis to first order in parameter .1, so expect corrections to be a mere 1%.

Another subject concerns generalized GT relations. For regular beta decay we related f_1, g_1, C_A . Starting with any other baryon decay you can make a similar theorem. Eg, in a strangeness violating decay, the meson-pole term which gives g_3 will be a K so you will involve $f_K, g_{\Lambda p K}$ and $g_1 = C_A$ for this decay. You can get some formulas like (8.35) which relate ratio of C_A for two different processes to the strong coupling constant ratio.

Now suddenly Gene writes down the two strong coupling lagrangian terms as in (8.37). The two relative couplings are here called f and d. By just examining the terms, you can get the strong-SU3 prediction for strong coupling constant ratios. Interestingly when you are all done, you find that d and f of the hadronic lagrangian are the same as D and F of the baryon weak decay lagrangian. Not too surprising since it is all su3 8x8 business.

The final punch line then is this: you fit all your baryon decays via certain value of F and D. These values agree with hadronic process measurements of f and d.

So this chapter has been: generalized beta decay. Roughly, the generalization is SU2 to SU3 for behavior of weak currents (flavor groups), with Cabibbo relative weights.

Chapter 9: Nonleptonic baryon decays. Consider now the possibility of heavy stable baryons weak decaying into lighter ones with no leptonic emission. Since initial baryons are "stable", you know you are talking weak or EM decays. Also don't care about decays with photons. So, any purely nonleptonic weak decay of a baryon into a baryon must be at most $\Delta S=1$ since weak. But the only mass that will fit in there is a pion mass. So all decays in this chapter are hyperon into baryon + pion. All are $\Delta S=1$.

This is the first time in this book we have looked at weak decays with no leptons at all. Go back to a decay like $\Lambda \rightarrow p(\pi^-)$. There you were able to

essentially "factorize" the final state, and group the leptons off by themselves in a leptonic current, and group the two baryons together in a hadronic current. In the reaction $\Lambda \rightarrow p\pi$ with no leptons what are you going to do? Gene suggests you write $\langle p\pi | J_0 | \Lambda \rangle$. I.e, now both currents are hadronic and get in the hadronic space, so I guess the point is you cannot factorize this thing into, say, $\langle p | J_1 | \Lambda \rangle \langle 0 | J_0 | \pi \rangle$. The difference must be that the p and π in the final state interact strongly, unlike the interaction of the p and ν of a leptonic decay. This must be what precludes this factorization. Related to perturbation theory question.

So think about one of these decays, say $\Lambda \rightarrow p\pi^-$. In partial wave terms, initial state has $J=\frac{1}{2}$, final state has $S=\frac{1}{2}$, so you must have $L=0$ or 1 . I.e, for all these pion decays of baryons into baryons, there is only an s-wave and a p-wave amplitude.

Now, independently, you may go on to show that the amplitude has a most general form (9.3) with amplitudes A and B. Not immediately obvious that A is the s-wave and B is the p-wave. Where are the 4 other terms you usually start with? Answer: because only three momenta here, you can always write pion p in terms of the fermion p's, and these then kill extra terms as illustrated with example on page 199.

So great, for each decay there are just two complex amplitudes to find. Now lets go through the Pauli reduction. Your initial and final baryon are polarized, say, along directions $\hat{n}_i$ and $\hat{n}_f$ which you are free to choose. The amplitudes A and B are converted to s and p. Then from (9.5) you see very clearly why s and p are the s-wave and p-wave amplitudes.

We are not even going to try to predict total rates; there is no way to connect s and p to the constant G. So, we only care about the relative rate which is given by (9.7). Eg, for T invariance, you expect $\beta = 0$. You should be able to measure the various coefficients to get ratio s/p type information.

Notice a problem, however. After the p and π are created "at a point", they are certainly going to do some $PI-N$ scattering, why not, for a while they are within a fermi. This effect is called "final-state interaction" and I have never learned much about it. But I am not at all surprised to learn that we will get involved with only the s and p wave ^{πN} phase shifts, since those are the only waves there are.

In the decay of unpolarized Λ , you expect the proton to have no average transverse polarization because the initial state has no polarization. This is a trivial electrostatic symmetry argument. By measuring the longitudinal polarization, you can deduce $\alpha = .65$. Somehow you can measure the 3 parameters α, β, γ which is in effect 3 of the 4 real numbers present in s and p. The 4th

number is the total rate.

Now, there must be some way to relate these various decays. Since all are $dS=1$, you know that the weak current or effective hamiltonian here should transform as $I=\frac{1}{2}$. If it did, you would "see" the $dI=\frac{1}{2}$ rule being repeated in these interactions. So for the moment let's just assume this is so. Then you install the spinor behavior of the hamiltonian by adding a "spurion" to the initial state; trivial.

Once the spurion is in there, isospin is now conserved, so you use standard isospin techniques to get ratios of things. In the end, these isospin ratios give you 6 conditions on the $2 \times 7 = 14$ amplitudes involved in the 7 decays. In particular there are two branching fractions that should come out being 2, but which are experimentally only close (to 10%.)

So is there anything else you can say beyond this $dI=\frac{1}{2}$ rule implications? I.e., if you assume a certain $SU(2)$ flavor behavior of the effective hamiltonian, you get these $dI=\frac{1}{2}$ -rule predictions. Why not make some $SU(3)$ -behavior assumption for the hamiltonian? And by the way, since the effective $\mathcal{H}$ is supposed to be $J_0 J_1$, ~~and~~ you would expect to see some $dI=3/2$ and $dI=\frac{1}{2}$; but you don't see the $dI=3/2$. Why is this? Well we already assumed in the above paragraph that no $dI=3/2$ when we put $\mathcal{H}$ spurion.

Perhaps you are supposed to take a certain combination of the J_0 and J_1 currents for H which combination goes as an octet member? Obviously you want to choose an octet member that has $I=\frac{1}{2}$. By insisting that H have properties $dQ=0$, $dS=1$ ~~and~~, you conclude that only members 6 and 7 are possible. Later it will be shown that can be only one or the other; both would yield CP violation.

So try H as a 7-member or 6-member of an octet. Write out all the $SU(3)$ scalar combinations of $B, \bar{B}, M$, and h = spurion looking like a 7-matrix. Use CP to reduce number of amplitudes to 3 instead of 9 (among the s-waves). Then examine expanded H to get ratios on page 209. Most are duplicates of the $dI=\frac{1}{2}$ rule spurion method above, but clearly you should get something more. The one extra relation for the s-waves (for the p-waves if you chose member-7) is (9.47) found by Lee-Sugawara.

Go back: we have 7 processes of interest. The $dI=\frac{1}{2}$ rule gave 3 relations among the 7 s-wave amplitudes. Lee-Sugawara is a 4th relation. Are these relations true or not? Figure 9.1 plots for 5 of the 7 processes ($\Lambda^0 \rightarrow n \pi^0$ and $\Xi^0 \rightarrow \Lambda^0 \pi^0$ are not there because these are the two with no charged particles, experiment nearly impossible). So the figure shows 5 arrows, each one representing the s and p amplitudes, assumed real. One arrow is repeated twice because of some experimental ambiguity. All relations seem to hold within experimental error.

Notice that the simple ΔI rules are not on this picture because they involve the non-charged decays. This picture is a test of the rule 9.21 from $\Delta I = \frac{1}{2}$, and the Lee-Sugawara 9.47 rule, for both s and p, in fact. The simple ΔI rules are born out by comparing branching fractions and they look OK.

Chapter 10: Kaon semileptonic decays. We start off with the old story of K_L and K_S . From the viewpoint of strong interactions and em, the states K_0 and $\bar{K}_0$ are distinct states; ie, strong and em do not at all mix such states (eg, you could not couple K_0 to $\bar{K}_0$ because this would be $\Delta S = 2$). But of course the weak interaction can mix these two states via an intermediate state of $S=0$, say $\pi\pi$. Note that this mixing must be second order in G , so is very small.

Now when you speak of a state having a definite lifetime, what do you mean? Lifetime means lifetime with respect to decay. Since kaons decay only by weak interactions, only these interactions are involved in the question. A "state" should be diagonal in all quantum numbers which commute with the relevant hamiltonian. Thus, you want neutral kaon states that are diagonal in CP; you don't care about diagonal in S because $(S, H_W) \neq 0$. In fact, CP is all you have in the way of quantum numbers conserved in weak decays. This is why you make the K_L and K_S combinations.

So which one is long and which is short? The state with $CP=+1$ can decay into $\pi\pi$, so this will be K_S with lifetime 100 picosec. ~~This state~~ This K_S state virtually 100% into charged and neutral $\pi\pi$, with 10^{-3} rate to $\pi\pi\pi$. Obviously any decay can add bremsstrahlung and be down order α .

The $CP=-1$ state is blocked into $\pi\pi$ and as a result lives 600 times longer, ie, 60,000 psec = 60 nanosec. The obvious decays here are $K_L^0 \rightarrow \pi(2\pi)$ which together give 66% of the decays. The remaining 34% goes into $\pi\pi\pi$ which is $CP=-1$. There are some decays into $\pi\pi\pi$ indicating small CP violation. Also decay into $\pi\pi\pi$. Notice that the K_L allows more vision into various decay products because experiment has more time, not so swamped.

So, once you decide that K_L and K_S are relevant, the next topic is what is meant by regeneration. Very simple. You produce say K_0 in πp scattering. Note by the way that you cannot produce $\bar{K}_0$ because there are no $S=+1$ baryons. So you make yourself a nice K_0 beam. The "wavefunction" for this beam is $K_S + K_L$ as a coherent amplitude sum. But of course your K_S decay quickly leaving relatively more K_L down the beam. After a while you have only K_L . But now run this beam of K_L into some crud and of course the $\bar{K}_0$ component can react with protons whereas the K_0 cannot. So, in effect, the crud removes more of the $\bar{K}_0$ component of the beam, thus unbalancing the ~~mixture~~ equal mixture of $K_0 + \bar{K}_0 \equiv K_L$. So you in effect cause some K_S 's to reappear, they are "regenerated".

Clearly the KL-KS mass difference can be roughly explained by the CP selection of intermediate states, and result should be quadratic in G. Thus, expect mass difference to be same order of magnitude as any weak decay rate. In fact, $dM = .44\% \sim 10^{23} \text{ sec}^{-1}$. Of course this is very small compared to the absolute kaon mass which is roughly 10^{23} sec^{-1} , ie, typical hadronic size.

What can we deduce about decays of the form $K \rightarrow \pi \ell \nu$? The first fact is that, for all such decays, there are only 3 amplitudes involved, called f, g, and A^+ on page 219. Of these, g violates the $dS=dQ$ rule. Is there any g present? Comment: when you write the hadronic current matrix element, you want to be using isospin and dS etc, so you put in K_0 , not K_S , as in (10.19). Now consider $K^0 \rightarrow \pi^0 \ell^+ \nu$. This decay is really your only source of decay electrons. You can compute the rate of e^+ to e^- production as a function of distance down the beam, as in (10.23). Then do experiment to get exper measure of g/f . You find .04, so seems to be 4% violation of the $dS=dQ$ rule. But errors are large.

So, presumably $g=0$ and we are left with amplitudes f and A^+ . These are then trivially related by the $dI=\frac{1}{2}$ rule, ie, by isospin, so $A^+ = f/\sqrt{2}$. Thus, you can equate some rates and make a few predictions. It works.

We have now boiled things down to the study on just one amplitude, namely $A^+ = \langle \pi^0 | V_0 | K^0 \rangle$. We now write this in terms of the two form factors f_+ and f_- . If this were a V_0 current we would replace $f_+ = \sqrt{2} F_1$ (the EM form factor) using CVC, and we would also say things about values at $q^2=0$. Ie, we could use for V_0 current all the power of CVC. Now, can we apply CVC to V_1 current? This is not axial, so we are not talking PCAC. In perfect SU(3), since V_0 and V_1 are in same octet according to Cabibbo Hypothesis, answer is yes. The concept of perfect SU(3) justification was that order λ^2 Ademollo-Gatto theorem.

So lets analyze $K^+ \rightarrow \pi^0 \ell^+ \nu$. We start with the two form factors f^+ and f^- but because e is so light, the f_- term goes away (note $\xi \equiv f_-/f_+$). The amplitude then involves only $G \sin \theta f_+(q^2)$. Now, small but finite q^2 is involved because there is large Q value, but set $f_+(0) = 1$ by CVC applied to V_1 as discussed above, and include a small empirical q^2 linear term in $f_+(q^2)$. Compute total rate and compare to exper to get still another measure of Cabibbo angle. Or use old C angle and conclude that theory and exper agree for $K^+ \rightarrow \pi^0 \ell^+ \nu$ process. Once again, basically only one constant enters here and you set it to 1 by CVC, this allowing total rate computation. By looking only at the pion energy spectrum, you can evaluate that small linear term in $f_+(q^2)$. Then look at electron energy spectrum at fixed pion energy. This gives confirmation of vector type theory and conflicts violently with either S or T theory as seen in figure p 231.

What about the $K^+ \rightarrow \pi^+ \ell^+ \nu$ decay? Well, it has that extra f_- or ξ term, so obviously what you want to do is measure ξ . This is done by computing the

muon polarization in terms of ξ and then measuring it. Results are shown in table 10.2. Roughly $\xi = f^-/f^+ = -1$, dimensionless, although you have to account for q^2 dependence somehow. Then, using this ξ in formul (10.79) for the μ/e ratio in K^+ decays (or K_L decays, since ratio) you compute .53 for K^+ and .50 for K_L (keep in mind that such ratios always involve mass differences and sometimes radiative corrections). My data book says 32/48 for K^+ which is .7, and 27/39 for K_L which is again .7. Strangly results are both the same but are both off. No reason given yet for this discrepancy! (neutral currents?)

So where are we? Gene discussed the simple K_{12} decays back in chapter 4 in terms of the simple constant f_K . In this chapter he has so far done the K_{13} decays which involve one pion. The trick here was a version of CVC to say $f_+ = 1$. The last semileptonic decay is the K_{14} involving two pions. As noted back in chapter 4, there will be 4 form factors here, and both V and A currents operate. I think the useful fact here is this: your final state of $\pi\pi$ has 200 MeV of kinetic energy, but my intuition says that momenta will be equipartitioned, so pions may only typically have 10 MeV of energy, ie, they are relatively soft. In general, all momenta p are relatively small for both pions? Anyway, I am trying to justify two facts: (1) the throwing away of the a_4 term in (10.98) since bilinear in pion momenta; (2) the near constancy of the other form factors in their arguments.

Now, we have three form factors with no knowledge of them yet. By taking the two different soft-pion limits and using the soft-pion theorem, you conclude that $a_3 = 0$ and $a_1 = a_2$, at least when all are evaluated at that soft pion point. From the other limit we conclude that $a_1 + a_2 = 4f_+/f_\pi$. Good grief, where did these old form factors come from? How did they get into soft pion theorem? Well the SPT has f_π sitting in it intrinsically because the SPT contains Goldberger Trieman/PCAC stuff in its derivation. On the other hand, the SPT removes a pion from final state, and this brings in f_+ . So result no surprise.

So of the four constants a_1 through a_4 , you set $a_3 = a_4 = 0$ in effect, and you set $a_1 = a_2 = 2f_+/f_\pi$, numbers which you know, and then you can compute the total rate! compare with exper: it works, a great victory for the Soft Pion Theorem!

So lets take a look at this Soft Pion Theorem. Stated on page 234. You pull your soft pion out of the final state and you get $i\sqrt{2}/f_\pi \langle \pi | [F_5^0, \alpha] | \pi \rangle$. How does it work? Obviously by pulling the final state pion, you generate a pion field, which can be put into a commutator for free, see (10.85). Then you replace the pion field with $f_\pi \cdot \partial_\mu g^\mu$, ie, the divergence of the axial current; this is just Goldberger trieman and you see f_+^2 appear. But in the

limit of $q \rightarrow 0$ (soft pion) you can effectively replace $\partial_\mu g_i^\mu$ with g_i integrated over space as in (10.88), which yields the charge F_1^5 . Thus QED. Only problem is that I don't think the cosine should be in there. Later when he applies the rule, Gene does not show the cosine.

What about Ademollo-Gatto which I skipped. This is another one of those things which start with a current algebra statement with charges F_i . Very similar to Adler-Weisberger: you sandwich in between pion states, not proton states, put in intermediate state set. When the intermediate state is a K, you get term involving f_+ and f_- of course. If there were no other terms at all, you could conclude that $f_+(0) = 1$ (this step involves going to infinite momentum frame). Now consider those extra terms. The only available states are single particle states out of the 8×8 O^- octet, and many particle states (recall in KW it is the $PI-P$ states that are important.) Look at (10.44). For large energy difference, it is clear that the "other terms" will all be order λ^2 where $\lambda = .1$ is the SU3 breaking Hamiltonian scale. Thus we have proved that $f_+(0) = 1 + \text{order}(\lambda^2)$. Earlier I conjectured this by arguing that CVC should apply to the A_1 currents, based on SU(3). Here we see that even if there is SU(3) breaking, you still have $f_+ = 1$ to 1%.

This is the general theorem I think which says couplings obey SU(3) much better than mass differences. I.e., mass differences are order $\lambda = 10\%$ whereas couplings are SU(3) invariant to order $\lambda^2 = 1\%$. Good argument to know. Chew used it a lot, but never used phrase Ademollo-Gatto.

Final chapter comments are on neutral currents. Processes of interest here are $K_L^0 \rightarrow \mu^+ \mu^-$ or $e^+ e^-$. Clearly these decays violate strangeness and must therefore be weak, and you can see that the current involved must be neutral. Experimentally, Gene says less than 10^{-9} . But my 1976 data book says these events do occur at rate 10^{-8} . I.e., di-muon events in K_L decays is good evidence that we now have neutral currents. Maybe this now agrees with the $K_L \rightarrow \mu^+ \mu^-$ rate. Some experimenters must have made a mistake in this business.

Chapter 11: K_{L2} and K_{L3} decays. Fact 1: two pions can only be in state $I=0$ or 2 for the K_{L2} decays. Exper seems to indicate a 5% $I=2$ amplitude, which is in violation of the $dI=\frac{1}{2}$ rule. Eg, the ratio of K_{short} into $\pi^+ \pi^- / \pi^0 \pi^0$ should be .2.0 but seems to be .2.2

How should you analyze things? For K_{L2} you cannot do a current=current thing with G because this is nonleptonic. Similar to the hyperon decays where you have to just make up some paratments and see how far you get. Obviously for the K_{L2} you have $L=0$ so only s-wave amplitude. But as noted you can have $I=0,2$, so there are two amplitudes called A_0 and A_2 .

Now obviously you are going to want to make comparisons between all the K decays, half of which involve the neutral kaons. Thus, you immediately have the additional complication of the $K^0-\bar{K}^0$ system. You have to understand that "system" before you can make phenomenological analysis.

Much of this chapter is about that system. We start off with a mass matrix which would be diagonalized by the states K_0 and $\bar{K}_0$ were there no H_W . The weak interaction, as it violated strangeness conservation, connects these two states causing off diagonal entries in the mass matrix in the $K_0, \bar{K}_0$ basis, see (11.29). So obviously, you re-diagonalize and call your new states K_L and K_S . How are these expressed in terms of K_0 and $\bar{K}_0$? Try general form (11.37) and CPT then leads you to the most general form (11.46), showing the one parameter ϵ . Clearly, if $\epsilon \neq 0$ you have some CP-violation.

From (11.50) you can see that $\langle K_S | K_L \rangle \neq 0$ when $\epsilon \neq 0$. In other words, these two states can couple in presence of CP violating interaction; if there were no CP violation, each would be a CP eigenstate. Actually, I think they are CP eigenstates anyway, CP-violation just allows them to couple.

If there were no coupling, no CP violation, then you know that K_S and K_L would have to have same mass to 2 parts in 10^{15} . Assuming CPT, they have to have exactly the same mass. Thus, the fact that there even is a K-long/K-short mass difference is evidence of CP-violation.

Gene throws in a little analysis of regeneration, though not used later. I.e., what happens when you run a neutral kaon beam down through some material? In vacuum you know that mass eigenstates will be K_S and K_L . But, in some material the new eigenstates will be different, namely K_S' and K_L' . In the end, these new eigenstates are given as in (11.67) and (11.68). There you see the regeneration parameter r . This arises because the scattering on protons or nuclei of K_0 differs from $\bar{K}_0$ for reasons that are clear. I.e., you re-diagonalize your mass matrix once again and get these new eigenstates.

Besides the regeneration complication which you cannot avoid in an experiment, there are also strong interaction s-wave phase shifts in the $I=0$ and $I=2$ channels. In effect, these phase shifts change the phase of the 2-pion final state by $e^{i\delta_I}$, so turns out to be very simple correction to account for. I don't know the theory of this, however. So, when you include these phase shifts and the usual isospin Clebsch's, your final possible pion states are as in (11.70) and (11.71). Combine this with your earlier construction of the K_L and K_S states and you end up with main results (11.77) through (11.80). As noted, everything is now described by A_0, A_2 and ϵ .

Recall that A_2 is measure of violation of the $dI=\frac{1}{2}$ rule, and ϵ measures the CP violation.

What can be experimentally measured? Consider K_L/K_S ratios. Obviously such ratios are small because in exact CP conservation, K_L cannot go to $\pi\pi$. Two ratios are called η^{+-} and η^{00} . See (11.81). If $dI=\frac{1}{2}$ is OK, then no A_2 and then these ratios should both equal ϵ .

Experiments say $|\eta^{+-}| = 2 \times 10^{-3}$, although some recent controversy in 1976 particle data book. The mass difference is measured by the very clever gap-regenerator method to accuracy of 1%. Notice from (11.21), by the way, that in order to be able to compute ΔM , you need to compute M_{21} , that off diagonal strength. That in turn requires you to compute quadratic terms like that shown in (11.21) (first term vanishes since $dS=2$). Since, well, the result will be order G^2 and probably proportional to ϵ , but you cannot really compute it because unknown intermediate states, ie, a long calc is needed to try to compute ΔM . No trivial relation between ΔM and ϵ .

The phase of η^{+-} is called ϕ_{+-} and is measured by simply looking at the $\pi\pi$ production rate versus time in a vacuum regenerator. You get 45° with error of 4° . You can also try to get $|\eta^{00}|$ and phase ϕ_{00} ; also people can measure $\text{Re}(\epsilon)$ to 10%.

The results show clearly that there is CP violation, but not much else can be said. Obviously there is also some T violation going on. What is the cause of this CP violation? Is it just that the weak interaction violates CP? Or is it maybe that strong or em are violating CP and it shows up here? Or is there some superweak interaction which is doing the CP violation? Strong and EM are observed to respect CPT and P to high degree, so only possibility is some small C/T violation there? No one really knows. In Wolfenstein's simple "superweak" theory, you just assume there is some extra $dS=2$ interaction which lets $K_L \rightarrow K_S$ which then goes to $\pi\pi$. This little theory predicts $\eta^{+-} = \eta^{00}$ which looks as though it may be true. But $dI=\frac{1}{2}$ also predicts this.

So, to summarize the $K_{\pi\pi}$ story, there are three parameters in the phenomenological "theory" called A_0, A_2 and ϵ . What you can measure are $\eta^{+-}, \phi_{+-}, \eta^{00}, \phi_{00}, \text{Re}(\epsilon), \Delta M$. There is no "test" of any theory here, you just define some parameters and measure them. You observe ~~xxx~~ both from mass difference and fact that $K_L^0 \rightarrow \pi\pi$ that there is CP violation. So CP violation is really the main point of interest in the $K_{\pi\pi}$ Story.

What about $K_{3\pi}$? Obviously even less will be doable. There are several such decays of kaons, obviously try to make some predictions using isospin etc.

Since $K_{3\pi}$ are three-body decays with (near) equal masses, you can plot events in a very simple Dalitz plot, see figure 11.8. As shown by (11.163), the density of points at some place in the diagram (say T_1, T_2 or r, θ) tells you the square of the amplitude. Notice that there is no Dalitz diagram for a 2-body decay since cms kinematics fixes T_1 and T_2 by masses. I.e., for $K_{2\pi}$ decay, $T_1=T_2$ = known number.

So what can you do with amplitudes $A_i(r, \theta)$ where i = process label? First there is symmetry θ to $-\theta$ because two pions are usually the same. Second is this: pions have at most the entire $Q = 75$ MeV, usually much less, typically 25 MeV which means pions are slow. I.e., not much phase space available for these decays, so you might say this: the amplitude at some point in the Dalitz diagram is the amplitude at the center $r=0$ plus a small correction. Hence (11.166) which incorporates this statement with the θ symmetry. The correction turns out to be a function of $y=r\cos\theta$, the size of the correction is measured by parameter ρ called "slope parameter".

Isospin and centrifugal barrier also have something to say here. In fact, considering pion plus dipion, you can correlate CP, I and L of final state; the net result is shown in table 11.4 which also shows the $dI=\frac{1}{2}$ rule. Result is that $K^{\pm}$ and K_L can both go healthily into $I=1$ tripion state, whereas K_S can't go in any I-spin final state. Of course there will be some bleedthru because $dI=\frac{1}{2}$ is violated, CP is violated, and barrier is violated.

As usual, the $dI=\frac{1}{2}$ rule by itself makes various ratio predictions, see Table 11.5 page 283. Obvious small violations of the rule, as usual.

Finally, soft-pion theorem has some stuff to say here, because it allows you to extract pions from final state to relate $K_{3\pi}$ to $K_{2\pi}$ decays. And pions are definitely soft. Results seem to work OK.

To summarize, the $K_{3\pi}$ amplitudes are parametrized by A at center of diagram, plus a slope parameter. There is no theory to compute either parameter (for each decay). Relations between decays in the $K_{3\pi}$ class arise from isospin, $dI=\frac{1}{2}$ rule, barrier effect, 2-pion symmetry, softness of pions. Obviously a painful subject. You have a theory which works nicely for semi leptonic and pure leptonic sectors; but it says little about nonleptonic decays.

Chapter 12: Parity Violation in Nuclear Forces. The point here is very simple: There should be a weak interaction of the form $\langle n p | \vec{J}_0 \vec{J}_0 + \vec{J}_1 \vec{J}_1 | n p \rangle$, i.e. on top of the huge parity conserving strong interaction we should expect a parity violating weak interaction. Recall that EM Hamiltonian also conserves parity. How are you going to find this effect? Certainly not looking at the np cross

section. Clearly, what you do is say that the effective nuclear potential now has a small parity violating term. This then admixes states of opposite parity, which were formerly unmixed. Thus, this admixing causes photo-decays which you would normally think not possible.

So one trick is to look for some strictly forbidden transition under strong + EM, and see if in fact it leaks through a little. By the way, the wavefunction admixing effect is about $F = 10^{-7}$, so your leakthru transition will be down by huge factor of 10^{14} from a regular allowed transition. Iel, this requires good experimntal technique to see. As example, Gene uses a certain forbidden decay $O^{16} \rightarrow C^{12} + \alpha$ where the O^{16} is prepared in a certain excited state. If you see alphas of this characteristic energy, then you are seeing the forbidden transition. This has been seen and measured, and result seems about what you would expect.

Besides looking for forbidden events, you can look at allowed transitions which may have mixtures of forbidden events in them. Ie, it is probably hard to find things that are reall forbidden, so you have to accept a mixture. But that is enough.

There are two classic experiments. In one, you look at a simple photon emitted in a nuclear transition. Ie, a gamma ray. If the states involved are getting parity mixed by presence of parity violating term in hamiltonian, then you expect to see circular polarization of emitted photon (ie, emitted from unpolarized nucleus, and final nuclear polarization is not measured). Just draw picutre and you can see that there should be no net longitudinal polarization of a phton in reaction $n \rightarrow p + \gamma$. Circular pol = longitudinal polarization. So set up your polarized iron target and look for this gamma ray polarization. Typical net circular polarizations are 10^{-7} effect, so you need many many events to get statistics on this. Often your process is started by thermal neutron hitting some nucleus, getting "captured" into an excited state which then dumps a gamma ray.

The other obvious experiment is to look at fore-aft gamma asymmetry when initial nucleaus is polarized (captures a polarized neutron). This effect is sometimes relatively large, 10^{-4} , and has been seem. Chapter ends with quick discussion of the "theory" involved, not very exciting at present.

By the way, what about parity violation in atomic transitions? Now you are asking about $\langle \bar{e} p | H_w | e p \rangle$. Since lepton factors off, you see that this is a neutral current effect. So parity violation in atomic transitions has to do with neutral weak currents, and is really a different subject altogether (not in our theory so far); of course experiments are very similar. I think this is what Commins is now up to with Steve Chu.

Chapter 13: neutrino scattering. Obviously you don't want to look for weak scattering in a process that is dominated by other forces, like ep. That is why the real thing to do is $\nu p \rightarrow e X$. (Standard charged-current deal). The kinematics here is the same as deep inelastic ep scattering, more or less. I.e., interesting variables are q^2 and ν .

First consider elastic $\nu_\mu n \rightarrow \mu^- p$. Obviously you put in your usual B to B weak hadronic current with f_1 through g_3 form factors. Now however you will need the large q^2 behavior of these form factors. Analogously to the form usually taken and/or measured for the EM formfactors (vector dominance etc) you might take forms as shown in (13.19). You might assume that $f_3 = g_2 = 0$ (though not obvious that CVC still works at "high energy"). If precision experiments could be done, you could test theory. At the time of writing, only a poor statistics CERN 1967 experiment had been done with neutrinos.

What about inelastic stuff, like nucleon throws off a pion. Adler has some current algebra relation about composition of products in deep inelastic. In the deep inelastic, in general you do not know the tensor that gets multiplied by the (weak or EM) lepton trace; hence the usual structure functions W_1 and W_2 for EM; and also W_3 for neutrinos, which arises from presence of axial piece not present in EM. This is the same old hadronic story: when you have too many hadrons, you cannot make up form factors, so you set up some phenomenological parameters, A and B in hyperon decay, structure functions here. All structure functions are expected to show Bjorken scaling.

Chapter 14: Neutrino Astrophysics. You can compute solar neutrino flux; Davis et al measured it in cave in South Dakota, number was not in agreement with solar theory. (the famous cleaning fluid experiment). Gene goes on to do the neutrinos out of hot stars, and cosmological topics. I am not reading this chapter now.

Excellent book! However, most recent references are about 1971; it is now 1978. Much has happened in between. Neutral currents found, so the 1958 Feynman-Gellman theory now obsolete; must use something with weak currents, like Weinberg-Salam. Of course any new theory must duplicate all the stable-particle decay results described in this book! Charm also found, so the whole theory has to be generalized to include both strangeness and charm changing currents. Then there is QCD for strong interactions, quarks looking like leptons, and so on and so forth. A review of last 7 years would be helpful at this point, rather than detailed reading of special papers.

Watanabe.⁷ It should be noted, however, that some variants of their theory may be found which cannot be disproved by one negative experiment alone. Therefore, it is desirable that both of these experiments be performed. It seems to be rather difficult to invent a theory of the Tanikawa type which does not give a resonance in any of the processes $\nu + p$, $\nu + n$, $e + p$, and $e + n$, where ν and e denote either particles or antiparticles.

We conclude that it is probably worthwhile to search for possible anomalies in electron and neutrino reactions over the energy range which may be covered with the present accelerators before looking into weak processes at higher energies.

I should like to thank C. R. Schumacher for checking some of the calculations.

* Supported in part by the joint program of the Office of Naval Research and the U. S. Atomic Energy Commission.

¹R. P. Feynman and M. Gell-Mann, Phys. Rev. **109**, 193 (1958); E. C. G. Sudarshan and R. E. Marshak, Phys. Rev. **109**, 1860 (1958).

²M. Schwartz, Phys. Rev. Letters **4**, 360 (1960); T. D. Lee and C. N. Yang, Phys. Rev. Letters **4**, 307 (1960). Similar considerations have recently been made by Y. Yamaguchi (to be published), and N. Cabibbo and R. Gatto (to be published).

³It is usually assumed that neutrinos emitted in the $\pi - \mu$ decay and β decay are of the same kind. There is no positive proof for this, however. Obviously, it is desirable that this be examined experimentally.

⁴The possibility that weak interactions are mediated by a boson has been considered by many authors: H. Yukawa, Proc. Phys.-Math. Soc. Japan **17**, 48 (1935); J. Schwinger, Ann. Phys. **2**, 407 (1957); T. D. Lee and C. N. Yang, Phys. Rev. **108**, 1611 (1957); R. P. Feynman and M. Gell-Mann, Phys. Rev. **109**, 193 (1958); Y. Tanikawa, Progr. Theoret. Phys. (Kyoto) **3**, 338 (1948); Y. Tanikawa and S. Watanabe, Phys. Rev. **113**, 1344 (1959).

⁵S. L. Glashow, Phys. Rev. (to be published).

⁶Lee and Yang (reference 2) showed that the production cross section of such a boson in the neutrino-nucleon collision is about 10^{-35} cm² for an incident neutrino energy much greater than 2 Bev. This is more practical than the resonance scattering method as a direct test of the intermediate boson hypothesis. It should be noted that the absence of $\mu \rightarrow e + \gamma$ is not conclusive evidence against the intermediate-boson hypothesis. See G. Feinberg, Phys. Rev. **110**, 1482 (1958).

⁷Y. Tanikawa and S. Watanabe, Phys. Rev. **113**, 1344 (1959).

⁸The theory of Tanikawa and Watanabe leads to the V-A theory only after spinors are rearranged by the Fierz transformation. In this theory, therefore, it is not easy (if not impossible) to find a natural explanation for the conservation of the vector current part of the weak interaction, if it is in fact conserved (R. P. Feynman and M. Gell-Mann, reference 1). If the presence of the weak magnetism [M. Gell-Mann, Phys. Rev. **111**, 362 (1958)] is established experimentally, it may again be hard to explain by Tanikawa's theory. The experimental result is not conclusive yet.

⁹In this case, formulas (8) and (10) must be modified slightly to take account of an additional decay mode of the B particle.

¹⁰R. Hofstadter, F. Bumiller, and M. R. Yearian, Revs. Modern Phys. **30**, 482 (1958).

AXIAL VECTOR CURRENT CONSERVATION IN WEAK INTERACTIONS*

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In analogy to the conserved vector current interaction in the beta decay suggested by Feynman and Gell-Mann, some speculations have been made about a possible conserved axial vector current.¹⁻³ One can formally construct an axial vector nucleon current, which satisfies a continuity equation, (ie, is conserved)

$$\partial_\mu A_\mu = \Gamma_\mu^A(p', p) = i\gamma_5 \gamma_\mu - 2M\gamma_5 q_\mu / q^2, \quad q = p' - p, \quad (1)$$

where p and p' are the initial and final nucleon

momenta. Such an attempt has some appeal in view of the apparently modest renormalization effect on the axial vector beta decay constant ($g_A/g_V = 1.25$), although the second appealing point,¹ namely, the possible forbidding of $\pi \rightarrow e + \nu$, has now lost its relevance.

The expression (1), unfortunately, can be easily ruled out experimentally, as was pointed out by Goldberger and Treiman,² since it introduces a large admixture of pseudoscalar interaction.

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On the other hand, Eq. (1) arouses theoretical curiosity as to the origin of the second term if it really exists; according to our conventional field theory, we would have to interpret the denominator q^2 as implying a massless, pseudoscalar, and charged quantum bridging the nucleon and lepton currents.

We would like to suggest that there may not be a strict pseudovector current conservation, but that we may have an approximate conservation which becomes rigorous in the limit $q^2 \gg m_\pi^2$, m_π being the pion mass. Specifically, we propose that the axial vector part of the nucleon beta decay vertex has the following form and properties:

$$g_A \Gamma_\mu^A(p', p) = g_V \left[i\gamma_5 \gamma_\mu F_1(q^2) - \frac{2M\gamma_5 q_\mu}{q^2 + m_\pi^2} F_2(q^2) \right],$$

$$F_1(0) = g_A/g_V \approx F_2(0),$$

$$F_1(q^2) \sim F_2(q^2) \quad \text{for } q^2 \gg m_\pi^2. \quad (2)$$

The pion is then the analog of the massless quantum mentioned above. This is consistent with the dispersion relations expected for Γ_μ^A . Namely, F_1 and F_2 should have in general the form

$$F_i(q^2) = F_i(-m_\pi^2) - (q^2 + m_\pi^2) \int_{m_0^2}^{\infty} \frac{\rho_i(m^2) dm^2}{(q^2 + m^2)(m^2 - m_\pi^2)}$$

$$(i = 1, 2), \quad (3)$$

where $m_0 = 3m_\pi$ unless there are new particles of low mass. Thus the F 's will be slowly varying for $|q^2| \ll m_0^2$. The conditions in Eq. (2) imply that $F_1/F_2 \approx 1$ for all q^2 . If $m_\pi = 0$ and $F_1/F_2 \approx 1$, then we restore exact current conservation,² and we also expect $F_1(0) = g_A/g_V = 1$.

If we adopt Eq. (2), the second term of Γ_μ^A immediately gives a relation between g_A , the pion decay (pseudovector) constant g_π , and the pion-nucleon (pseudoscalar) coupling G_π :

$$2Mg_A \approx 2Mg_V F_2(-m_\pi^2) = \sqrt{2} G_\pi g_\pi. \quad (4)$$

With $g_A = 1.25$, $g_V = 1.75 \times 10^{-49}$ erg cm³,⁴ $G_\pi^2/4\pi = 13.5$, this gives a π - μ decay life of 2.7×10^{-8}

sec as compared with the observed value 2.56×10^{-8} sec.

Goldberger and Treiman⁵ have arrived at the same relation Eq. (4) (in the limit of their self-energy integral $J \rightarrow \infty$) from an entirely different approach. In our opinion, this is not a coincidence, as will be explained elsewhere.

We are tempted to extend this approximate conservation of the axial vector (and naturally also the vector current) to the strangeness non-conserving beta decays. We take, for example, the ΛN axial vector in the form

$$\Gamma_\mu^A(p_N', p_\Lambda) \approx i\gamma_5 \gamma_\mu - \frac{(M_\Lambda + M_N)\gamma_5 q_\mu}{q^2 + m_K^2}, \quad (5)$$

and attribute the second term to the pseudoscalar K meson.⁶ The degree of accuracy of the relation (5) will be poorer than in the previous case in view of the Λ - N mass difference (which destroys vector conservation) and the large K -meson mass. At any rate, we obtain an analog of Eq. (4):

$$(M_\Lambda + M_N)g_A' \approx G_K g_K, \quad (6)$$

which relates the Λ beta decay axial vector coupling g_A' , the ΛNK coupling G_K , and the K_μ decay coupling g_K .

With the observed $K_{\mu 2}$ lifetime 2.1×10^{-8} sec and a tentative value $G_K^2/4\pi = \frac{1}{4}G_\pi^2/4\pi$, we get

$$g_A'/g_A \approx 1/10. \quad (7)$$

This is not inconsistent with the observed beta decay of Λ which seems an order of magnitude less than predicted from a universal coupling scheme $g_V' = g_A' = g_V$.⁷

We can still go further, though the argument becomes more arbitrary. Let us assume that a fundamental weak coupling ($\bar{N}N\bar{N}\Lambda$) gives rise to an effective V - A interaction (or at least part of it) of the form

$$g''(\bar{N}_\mu^V - \Gamma_\mu^A) \bar{N} N (\bar{N}_\mu^V - \Gamma_\mu^A) \bar{N} \Lambda. \quad (8)$$

Here $\Gamma_\mu^V = i\gamma_\mu$ which is approximately conserved by itself, and Γ_μ^A stands for Eq. (2) or (5). We see easily that Eq. (8) contains information about the $\Lambda \rightarrow N + \pi$ decay matrix element:

$$(2M_N g''/\sqrt{2} G_\pi) q_\mu (\bar{N}_\mu^V - \Gamma_\mu^A) \bar{N} \Lambda. \quad (9)$$

Combined with the assumption of $\Delta T = \frac{1}{2}$ selection rule, this gives a lifetime of 2.5×10^{-10} sec

for $g'' = g_V$ as compared with the observed value 2.8×10^{-10} sec.

It is possible to apply this kind of consideration to other hyperons. Moreover, if the Feynman-Gell-Mann coupling scheme such as $(\pi\pi\nu)$ is formally extended to $(K\pi\nu)$, etc. as has been tried by some people, all the observed decay processes may be covered. Here we would like to point out that if all baryons should satisfy Eqs. (4) and (6), the ratios g_A/G_π and g_A'/G_K must be approximately common constants.

Our final remark concerns the theoretical basis for the assumptions made here. If the baryons are derived from some fundamental field ψ which possesses an invariance under a transformation of the type $\psi \rightarrow \exp(i\vec{\alpha} \cdot \vec{\tau} \gamma_5) \psi$,⁸ then there will be a conservation of the pseudovector charge-current. A finite observed mass can be compatible with the conservation if the particle is coupled with a boson as was noted in Eq. (1).

This situation may be understood by making an analogy to the theory of superconductivity originated by Bardeen, Cooper, and Schrieffer,⁹ and refined by Bogoliubov.¹⁰ There gauge invariance, the energy gap, and the collective excitations are logically related to each other as was shown by the author.¹¹ In the present case we have only to replace them by γ_5 invariance, baryon mass, and the mesons. In fact, the mathematical method used in superconductivity may be taken over to study the self-energy problem of elementary particles. It is interesting that pseudoscalar mesons automatically emerge in this theory as bound states of baryon pairs. The nonzero meson masses and baryon mass splitting would indicate that the γ_5 invariance of the bare baryon

field is not rigorous, possibly because of a small bare mass of the order of the pion mass.

The above-mentioned model of elementary particles will be studied in a separate paper.

* This work was supported by the U. S. Atomic Energy Commission.

¹J. C. Taylor, Phys. Rev. **110**, 1216 (1958);

²J. C. Polkinghorne, Nuovo cimento **8**, 179 and 781 (1958).

³M. L. Goldberger and S. B. Treiman, Phys. Rev. **110**, 1478 (1958).

⁴A. I. Alikhanov, Ninth Annual International Conference on High-Energy Physics, Kiev, 1959 (unpublished).

⁵M. L. Goldberger and S. B. Treiman, Phys. Rev. **110**, 1178 (1958); M. L. Goldberger, Revs. Modern Phys. **31**, 797 (1959).

⁶It is also possible to associate a scalar K meson with the ΛN vector current conservation, while leaving the axial vector unaccounted for.

⁷Again Eq. (5) and the subsequent conclusions are essentially the same as those of C. H. Albright, Phys. Rev. **114**, 1648 (1959) and B. Sakita, Phys. Rev. **114**, 1650 (1959), which are based on the Goldberger-Treiman method. For the Λ -decay case below, see L. Tenaglia, Nuovo cimento **14**, 499 (1959).

⁸F. Gürsey (private communication) has recently obtained similar results on the π decay based on this γ_5 invariance. We do not here specify the interaction of the ψ field, which may be of the nonlinear Heisenberg type, or due to an intermediate boson (different from π or K).

⁹J. Bardeen, L. N. Cooper, and J. R. Schrieffer, Phys. Rev. **106**, 162 (1957).

¹⁰N. N. Bogoliubov, V. V. Tolmachev, and D. V. Shirkov, A New Method in the Theory of Superconductivity (Academy of Sciences of USSR, Moscow, 1958).

¹¹Y. Nambu, Phys. Rev. **117**, 648 (1960).

no reason given as to why the axial-isospin gauge invariance is broken.
obvious $\pi \rightarrow \mu$

ERRATUM

HELICITY OF NEGATIVE MUONS FROM PION DECAY. W. A. Love, S. Marder, I. Nadelhaft, and R. T. Siegel [Phys. Rev. Letters **2**, 107 (1959)].

In this Letter we presented preliminary results of an experiment designed to measure the forward-backward asymmetry of β rays emitted from B^{12} , which has been produced by absorption of polarized muons in carbon (in C_5H_{12}). Continuing

measurements have yielded an amended result for the asymmetry parameter a , as defined in this Letter, of $a = (0.64 \pm 0.58)\%$. It thus appears that in pentane the B^{12} is depolarized by one of the various interactions (quadrupole coupling, multiple electron capture and loss, etc.) which might cause spin reorientation. Therefore, a definite conclusion about the helicity of the negative muon cannot be drawn from our results to date.

Old Field Theory Notes

Index of Old Field Theory Notes

- ① Pages of notes on discrete symmetries P, C, $\mathcal{T}$
- ② Quick review of field theory method.
- ③ The T -matrix
- ④ Field theory session
- ⑤ ee scatter: a review of the "two methods"
- ⑥ Feynman Rules for QED
- ⑦ LSZ Reduction Formulas
- ⑧ Standard Functions
- ⑨ Norms
- ⑩ Commutators
- ⑪ Propagators
- ⑫ Photon spin sums
- ⑬ Transverse photon prop. story
- ⑭ The Π - N page
- ⑮ Disc current
- ⑯ Dimensions of Fields
- ⑰ Unitarity
- ⑱ Poincaré vs Lorentz groups
- ~~⑲ Noether for coord transformations~~
- ~~⑳ Noether reminders~~

Contents of this section:

1. Various page involving discrete symmetries like P,C,T, G:
 - a) how the field operators transform
 - b) how the creation operators transform
 - c) what to do about squares of operators like P^2 , T^2 etc.
 - d) time reversal
 - e) G-parity
 - f) parity and effect on lagrangian and current; pairs like $B\bar{B}$ and $F\bar{F}$
 - g) charge conjugation and pairs
2. Reviews of field theory method
 - a) the two methods of field theory
 - b) the U-matrix and the S-matrix
 - c) another review of the two methods, including Wick
 - d) ee scattering as an example
3. Feynman rules for QED.
4. The Reduction Formulas
5. Propagator functions
6. Field theory Norms (like C_5 etc.)
7. More commutators and Propagators
8. The photon spin sum business
9. The transverse photon propagator
10. The old pion nucleon page.
11. Some detail on the Dirac current.
12. Dimensions in field theory.
13. Unitarity in field theory.
14. Poincare versus Lorentz groups.
15. Crossing in field theory.
16. Noether Theorem for internal and external symmetries (various versions)
17. Feynman graph combinatorics, loops, internal lines etc .
18. Feynman rule for vertex in spinor QED and scalar QED.
19. Proof by induction of Feynman Identity.

Field Operator Transformations

1. Parity:

$$P \phi(\vec{x}, t) P^{-1} = \pm \phi(-\vec{x}, t)$$

$$P \psi(\vec{x}, t) P^{-1} = \gamma_0 \psi(-\vec{x}, t)$$

$$P \vec{A}(\vec{x}, t) P^{-1} = -\vec{A}(-\vec{x}, t)$$

2. Charge Conjugation:

$$C \phi(x) C^{-1} = \phi^\dagger(x)$$

$$C \psi(x) C^{-1} = C \gamma^0 \psi^\dagger(x) = i \gamma^2 \psi^\dagger(x) = C \bar{\psi}(x)$$

$$C A(x) C^{-1} = -A(x)$$

3. Time Reversal:

$$T \phi(x, t) T^{-1} = \pm \phi(x, -t)$$

$$T \psi(x, t) T^{-1} = T \psi(x, -t) = i \gamma^1 \gamma^3 \psi(x, -t)$$

$$T A(x, t) T^{-1} = -A(x, -t)$$

4. TCP:

$$(PC) \phi(x) (PC)^{-1} = \pm \phi^*(-x)$$

$$(PC) \psi(x) (PC)^{-1} = +i \gamma^5 \psi^\dagger(-x)$$

$$(PC) A(x) (PC)^{-1} = -A(-x)$$

5. Poincare:

$$U(\Lambda, a) \phi(x) U^{-1}(\Lambda, a) = \phi(\Lambda x + a)$$

$$U(\Lambda, a) \psi(x) U^{-1}(\Lambda, a) = S^{-1}(\Lambda) \psi(\Lambda x + a)$$

$$U(\Lambda, a) A^i(x) U^{-1}(\Lambda, a) = (\Lambda^{-1})^i_j A^j(\Lambda x + a)$$

Additional information:

$$C \gamma_\mu C^{-1} = -\gamma_\mu^T$$

$$T \gamma_\mu T^{-1} = +\gamma_\mu^T = \gamma_\mu^{*\dagger}$$

$$x'_\mu = \Lambda_\mu^\nu x_\nu$$

$$\gamma^5 = +i \gamma^0 \gamma^1 \gamma^2 \gamma^3$$

$$S^{-1}(\Lambda) \gamma^\nu S(\Lambda) = \Lambda^\nu_\mu \gamma^\mu$$

$$S = e^{\frac{-i}{4} \omega^{\mu\nu} \gamma_{\mu\nu}}$$

$$-C = C^\dagger = C^{-1} = C^T = -i \gamma^2 \gamma^0$$

$$+T = T^\dagger = T^{-1} = i \gamma^1 \gamma^3 = -T^{*\dagger} = -\tilde{T}$$

$$\Lambda^\mu_\nu = g^\mu_\nu + 4 \omega^\mu_\nu$$

P, C, T on creation etc operators.

$$P a(k) P^{-1} = \pm a(-k)$$

$$P a^\dagger(k) P^{-1} = \pm a^\dagger(-k)$$

$$P b(p,s) P^{-1} = b(-p,s)$$

$$P d^\dagger(p,s) P^{-1} = -d^\dagger(-p,s)$$

$$G a_+(k) G^{-1} = a_+(k)$$

$$G a_-(k) G^{-1} = a_-(k)$$

$$G b(p,s) G^{-1} = d(p,s) \cdot \epsilon_s$$

$$G d^\dagger(p,s) G^{-1} = b^\dagger(p,s) \cdot \epsilon_s$$

$$J = UK$$

$$U b(p,s) U^{-1} = -b(-p,s)$$

$$U d^\dagger(p,s) U^{-1} = -d^\dagger(-p,s)$$

My field theory says what?

① $P^2 \psi P^{-1} = (\gamma_0)^2 \psi = \psi$ so $\boxed{P^2 = 1}$ ✓

② $G^2 \psi G^{-1} =$ unclear because G is anticommuting w/ ψ .

③ $S^2 \psi S^{-1} = T^2 \psi = \psi$ so $\boxed{S^2 = 1}$ ✓

④ $(P3) \psi (P3)^{-1} = i \gamma^0 \gamma^3 \gamma_0 \psi(-x) = i \gamma_0 \gamma^3 \psi(-x) = (3P) \psi (3P)^{-1}$

$\Rightarrow \boxed{P3 = 3P}$ ✓

⑤ BUT, BE CAREFUL! because T is an anti-unitary operator, it complex conjugates any number it has to move through. SO reconsider the above in this light:

$S \psi S^{-1} = T \psi(x, t) \Rightarrow S^2 \psi S^{-1} = T^* T \psi = -\psi$ so $\boxed{S^2 = -1}$

$(P3) \psi (P3)^{-1} = P [T \psi] P^{-1} = T \gamma_0 \psi(-x)$

$(3P) \psi (3P)^{-1} = 3 [\gamma^0 \psi] 3^{-1} = \gamma_0 T \psi(-x)$ } $\Rightarrow \boxed{P3 = 3P}$

So, in my field theory I have:

$\left. \begin{array}{l} P^2 = +1 \\ S^2 = -1 \\ [P, S] = 0 \end{array} \right\} \text{ for spin field!}$

⑥ Looking at QFT, I think "t" means T everything in sight:

$G \psi G^{-1} = C \bar{\psi} = C \psi^\dagger \gamma_0 = C_{ij} \psi_k^\dagger(\gamma_0)_{kj} = C_{ij}(\gamma_0)_{ik} \psi_k^\dagger$

$= C \gamma_0 \psi^\dagger$ (since $\gamma_0 = \gamma_0$) $= +i \gamma^0 \gamma^0 \psi^\dagger = +i \gamma^2 \psi^\dagger$

Then: $G \psi^\dagger G^{-1}$ (note: $G^{-1} = C^\dagger = G$) $= \gamma_0 \psi^\dagger C = -i \gamma^2 \psi^\dagger$

So, $C \psi^\dagger C^{-1} = (i \gamma^2 \psi^\dagger)^\dagger = \psi \gamma^2 i^\dagger = \psi \gamma^2$

$\Rightarrow G \psi^\dagger G^{-1} = \psi \gamma^2 \gamma_0 = \psi (i \gamma^2 \gamma^0) = \psi C = -i \psi^\dagger C$ ✓

So: $G \psi G^{-1} = i \gamma^2 \psi^\dagger = \psi^\dagger (i \gamma^2)$ since $\gamma^2 \gamma^2 = 1$ $\gamma^2 = \gamma^2$
 $C \psi^\dagger C^{-1} = \psi (i \gamma^2) = (i \gamma^2) \psi$ $(i \gamma^2)^\dagger = (i \gamma^2)$

$\Rightarrow G^2 \psi G^{-1} = (i \gamma^2) \psi = \psi$

$\Rightarrow \boxed{G^2 = 1}$

① Keep going:

$$(PC)^\dagger (PC)^{-1} = (i\gamma^2)^\dagger P^\dagger \gamma^2 P = (i\gamma^2)^\dagger \gamma^0 \gamma^2 P^\dagger$$

$$(GP)^\dagger (GP)^{-1} = \gamma^0 i\gamma^2 \gamma^0 \gamma^2 P^\dagger = -i\gamma^2 \gamma^0 P^\dagger$$

$$\Rightarrow \boxed{PC = -GP}$$

$$(GZ)^\dagger (GZ)^{-1} = G^\dagger [T^\dagger \gamma^0] G^{-1} = \gamma^0 i\gamma^2 \gamma^0 P^\dagger$$

$$(PC)^\dagger (PC)^{-1} = (i\gamma^2)^\dagger P^\dagger \gamma^2 P = (i\gamma^2)^\dagger \gamma^0 \gamma^2 P^\dagger$$

$$= (i\gamma^2)^\dagger \gamma^0 \gamma^2 P^\dagger$$

$$= (i\gamma^2)^\dagger \gamma^0 \gamma^2 P^\dagger = (i\gamma^2)^\dagger \gamma^0 \gamma^2 P^\dagger$$

$$\text{Now: } \gamma^2 \gamma^0 = \gamma^0 \gamma^2 = -\gamma^0 \gamma^2 = -\gamma^0 \gamma^2$$

$$\boxed{GZ = -ZG}$$

② What about PC?

$$(PC)C = -P$$

$$C(PC) = -P$$

$$(PG)Z = -PC$$

$$Z(PC) = -P$$

$$(PC)P = -G$$

$$P(PC) = +G$$

$$\boxed{(PG)Z = -Z(PG)}$$

$$\boxed{(PC)P = -P(PC)}$$

$$\boxed{(PC)G = +G(PC)}$$

③ Squared?

$$(PC)^2 = PCPC = 1$$

$$(PZ)^2 = PZPZ = -1$$

$$(GZ)^2 = GZGZ = +1$$

$$(PC)^2 = PCPC = 1$$

$$(PC)PC = +1$$

$$GZGZ = -1$$

$$(PC)^2 = 1$$

Summary of BD for K field:

$$\underline{P}^2 = +1 \quad (\underline{PC})^2 = -1 \quad (\underline{PC3})^2 \equiv \underline{T}^2 = -1 \quad \checkmark$$

$$\underline{Z}^2 = -1 \quad (\underline{P3})^2 = -1 \quad (\underline{T} \equiv \underline{PC3} \text{ here.})$$

$$\underline{G}^2 = +1 \quad (\underline{C3})^2 = +1$$

$$\underline{P3} = +\underline{3P}) \times \quad \underline{T} \cdot \underline{3} = -\underline{3T} \quad \checkmark$$

$$\underline{C3} = -\underline{3C} \quad \underline{T} \cdot \underline{P} = -\underline{P} \cdot \underline{T} \quad \checkmark$$

$$\underline{PC} = -\underline{CP} \quad \underline{T} \cdot \underline{C} = +\underline{C} \cdot \underline{T}$$

Some of these equations agree with Lee and Wick, some do not. Eg, (2.18) through (2.21) of Lee Wick agree with the above. I think we can get rid of the disagreement by redefining the T operator so that new $T \equiv CT$, or something like that. OK, but I don't want to do that now. What to keep the above relations strictly for the BD conventions.

-

$$34\bar{3}^1 = 74$$

$$3b\bar{3}^1 = b(-p, -s) \quad \text{etc.}$$

-

- $$J|p,s\rangle = Jb^\dagger(p,s)\tilde{J}^{-1}|0\rangle = b^\dagger(-p,-s)|0\rangle = |-p,-s\rangle$$

4. Now assume QED so write H as $J_\mu A^\mu$ where J is the Dirac current. What can be said?

$$m' \sim \langle \vec{p}, s | \sum_{\mu} \vec{J}_{\mu} | \vec{p}', s' \rangle \sum A^{\mu} \vec{z}'$$

$$\sim \langle p, s | J^\mu | \bar{p}', s' \rangle A_\mu$$

$$m \sim \bar{u}(p', s') \gamma^\mu u(p, s)$$

$$m' \sim \bar{u}(p,s) \gamma_\mu u(p',s')$$

(cross channel exchange)

They are: $\overset{\vee}{J}, \overset{\vee}{P}, \overset{\vee}{C}, \overset{\vee}{G}, \overset{\vee}{I}, \overset{\vee}{B}, \overset{\vee}{Q}, \overset{\vee}{Y}$

Note: usually you assume a T and this gives P .
 Y and Q are obvious, as is B .

Toget C or P for pairs, see next two pages.

Note: since $I_3 = Q - \frac{Y}{2}$, sometimes an $I=0$ can be ruled out. If, eg, $Q=1$ and $Y=2$, you know that $Q=0$ is another allowed state in the 2^N , so $I=0$ impossible.

Note: C only has neutral eigenstates.
G can have charged eigenstates.

$$G = C e^{i\pi T_2}$$

Note: $Q P = \bar{P}$, so Q can change hypercharge.
 $e^{i\pi T_2}$ cannot change hypercharge since it is a rotation within an isospin multiplet.

Thus, G parity only good for $y=0$ systems!
Same goes for B

Case 1: system of systems of definite G . Then $G = G_1 \times G_2 \times \dots$

Case 2: for bos-abos a Ser-afser, as long as $y=0$ and $B=0$
we can say: $G = (-1)^{l+s+t}$. Example is $\bar{p}n$.

PARITY PAGE

Note: spin is pseudovector + is unaffected by space inversion.

Note 2: pseudoscalar mesons ($B=0$) have negative parity relative to the vacuum.

Note 3: The parity (intrinsic) of a baryon cannot be determined relative to the vacuum. Usually baryons are given +. Then the antibaryons must be -.

Field Theory:

$$\begin{aligned} P \mathcal{L}(x, t) P^{-1} &= \mathcal{L}(-x, t) \\ P j_\mu(x, t) P^{-1} &= j^\mu(-x, t) \end{aligned}$$

$$P \phi(x, t) P^{-1} = \pm \phi(-x, t)$$

$$P a(\underline{k}) P^{-1} = \pm a(-\underline{k})$$

$$P \psi(x, t) P^{-1} = \gamma_0 \psi(-x, t)$$

$$P b(\underline{p}, s) P^{-1} = b(-\underline{p}, s)$$

$$P d(\underline{p}, s) P^{-1} = -d(-\underline{p}, s)$$

Parity of two-particle system:

$$P = \eta_1 \eta_2 (-1)^L$$

Note: $\eta(\text{boson}) = \eta(\text{antiboson})$
 $\eta(\text{fermion}) = -\eta(\text{antifermion})$

$$\begin{aligned} P(B\bar{B}) &= -(-1)^L \\ P(M\bar{M}) &= +(-1)^L \end{aligned}$$

Charge Conjugation of Pairs

1. bos-bos.

$$\psi_{fg}(\vec{s}\vec{s}') = \int d^3k d^3k' f(k,k') g(s,s') b^\dagger(k,s) d^\dagger(k',s') |0\rangle$$

facts: $f(k,k') = (-1)^L f(k',k)$
 $g(s,s') = (-1)^S g(s',s)$ (for integer spins only!)*

Thus: $C = (-1)^{L+S}$

2. fer-fer.

$$\psi_{fg}(n\bar{n}) = \int d^3k d^3k' f(k,k') g(s,s') b^\dagger(k,s) d^\dagger(k',s') |0\rangle$$

$g(s,s') = (-1)^{S+1} g(s',s)$ (for half int spin)*
 extra - since fermions

$C = (-1)^{L+S}$

3. self conj bos pair:

$$\psi(p^0 p^0) = \dots a^\dagger(k,s) a^\dagger(k',s') |0\rangle$$

Clearly $C = (+1)$.
 But may write as $(-1)^{L+S}$ since this is overall sym + must be +1 for bosons.

* The spin state is: (function)

$$\begin{aligned} g(s_1, s_2) &= |S, m\rangle = \sum_{m_1, m_2} \langle s_1 m_1 s_2 m_2 | S m \rangle |s_1 m_1\rangle |s_2 m_2\rangle \\ &= (-1)^{S-s_1-s_2} g(s_2, s_1) \end{aligned}$$

see p S26. of PDG.

Quick review of field theory method

- ① First, as outlined in chapter 16 of B+D II, you can write any S-matrix element in terms of "amputators" acting on a V.E.V of a T.O.P. of a bunch of interacting, Hermitian, fields. The method of obtaining such an expression is called "the LSZ reduction formula method". So it all boils down to:

$$M_{is} = \overline{\mathcal{O}}_p \left[\langle 0 | T \{ \phi(x_1) \phi(x_2) \dots \phi(x_n) \} | 0 \rangle \right]$$

where of course all the fields can be different. The object in the $[\]$ is called a "tau function". The fields are most definitely not free fields. Sometimes this "tau function" is called "the complete n-particle Green's function". The operators remove the external leg propagators.

- ② Next, we consider a U-matrix connecting the Heisenberg picture to the Schrödinger picture: $\phi(x,t) = U^\dagger(t) \phi_{in}(x,t) U(t)$. This allows the Green's functions to be written as:

$$G(x_1, x_2) = \langle 0 | T(\phi(x_1) \phi(x_2)) | 0 \rangle = \langle 0 | T(\phi_{in}(x_1) \phi_{in}(x_2) e^{-i \int_{-\infty}^{\infty} \mathcal{H}_I(x) dx}) | 0 \rangle$$

where $\mathcal{H}_I = \mathcal{H}_I(\phi_{in}, \partial_\mu \phi_{in})$, which should be normal ordered, so as to make sure that $\langle 0 | \mathcal{H}_I | 0 \rangle = 0$. This is a series expansion for the Green's function. The zeroth order term in this example is:

$$\langle 0 | T(\phi_{in}(x_1) \phi_{in}(x_2)) | 0 \rangle = \text{wavy line}$$

just a straight propagator. The next term is:

$$G^{(1)}(x_1, x_2) = -i \int d^4x \langle 0 | T(\phi_{in}(x_1) \phi_{in}(x_2) : \mathcal{H}_I(x) :) | 0 \rangle$$

For a ϕ^3 theory, this would be 0. The next term is:

$$G^{(2)}(x_1, x_2) = \frac{(-i)^2}{2!} \int d^4y_1 \int d^4y_2 \langle 0 | T(\phi_{in}(x_1) \phi_{in}(x_2) : \mathcal{H}_I(y_1) : : \mathcal{H}_I(y_2) :) | 0 \rangle$$

A contraction would be



The amputators would cut off the external propagators.

"second method"

- ③ A different version of perturbation theory amputates at the start on all external fields. Then you have:

$$S \sim \langle 0 | a(p_1) T \left(: \phi^3(y_1) : : \phi^3(y_2) : \right) a^\dagger(p_2) | 0 \rangle \frac{(-i)^2}{2!}$$

where here a.F.T. has removed some integrals? Again, you can't do that forthwith. Note that the normal ordering eliminates self-energy tadpoles.

- ④ In the end, all this stuff leads to the "Feynman rules" of the theory.

The U-matrix and field theory

4-27-75

Reference: pp. 181 → 188 Sakurai

① Pictures: We are used to using operators and state vectors in the Schrödinger picture. Eg, $O^{(S)}$ = an operator, $\phi^{(S)}$ = a state vector. We know that a) all time dependence is in the state vector, namely, $i \dot{\phi}^{(S)} = H^{(S)} \phi^{(S)}$; this is the S.E. Decompose $H^{(S)} = H_0^{(S)} + H_I^{(S)}$. Now transform to the interaction picture as follows:

$$\begin{aligned}\phi(t) &= e^{i H_0^{(S)} t} \phi^{(S)} \\ O(t) &= e^{i H_0^{(S)} t} O^{(S)} e^{-i H_0^{(S)} t}\end{aligned}$$

You find that:

$$(1) \quad i \dot{\phi}(t) = H_I \phi(t) \quad \dot{O} = i [H_0, O]$$

Notice that $H_0^{(S)} = H_0$. So, in this new picture, the operators are "driven" by H_0 . If H_I were 0, this would be the Heisenberg picture. Also in the new picture, the state vectors are "driven" only by H_I , so they change "slowly" in time.

Also notice that, at $t=0$, an "interaction picture operator" exactly coincides with its timeless Schrödinger counterpart.

② The U-operator. We are now "in the interaction picture". Define U to be the operator which propagates the interaction picture state vector (slowly):

$$\phi(t) \equiv U(t, t_0) \phi(t_0) \Rightarrow i \frac{\partial U(t, t_0)}{\partial t} = H_I U(t, t_0) \quad \text{--- /from (1)}$$

Usually $H_I^{(S)}$ has no time dependence, but H_I does have such dependence. Now integrate both sides from t_1 to t_2 (variable t)

$$i [U(t_2, t_0) - U(t_1, t_0)] = \int_{t_1}^{t_2} H_I(t) U(t, t_0) dt$$

Now set $t_1 = t_0$ and use $U(t_0, t_0) \equiv 1$. Then, let $t_2 = t$.

$$i [U(t, t_0) - 1] = \int_{t_0}^t H_I(t') U(t', t_0) dt'$$

$$\text{Then: } U(t, t_0) = 1 - i \int_{t_0}^t H_I(t') U(t', t_0) dt'$$

This is the very famous integral equation which is at the heart of field theory calculations. Sometime I should think more about

This equation qua integral equations with a kernel, resolvent, Neumann series, convergence, etc. Eg, is it Fredholm?

③ The S-Matrix. Although Sakurai avoids IN and OUT states, I find them useful Eg,

$$\begin{aligned}\phi_{in} &= \phi(t=-\infty) \\ \phi_{out} &= \phi(t=+\infty)\end{aligned}$$

Then,

$$S_{fi} = \langle \phi_{out} | \phi_{in} \rangle = \langle \phi(+\infty) | \phi(-\infty) \rangle$$

But, use

$$\phi(+\infty) = U(+\infty, -\infty) \phi(-\infty)$$

i.e

$$\phi_{out} = U(+\infty, -\infty) \phi_{in}$$

Therefore:

$$S_{fi} = \phi_{out}^\dagger \phi_{in} = \phi_{in}^\dagger U^\dagger(+\infty, -\infty) \phi_{in}$$

$$= \langle \phi_{in} | U^\dagger(+\infty, -\infty) | \phi_{in} \rangle$$

So, more or less, we have:

$$S = U(+\infty, -\infty) = \text{huge master operator}$$

$$= T \{ \exp(-i \int d^4x \mathcal{H}_I) \}$$

④ Comments: I have been very sloppy but the idea is to generate a rough support for the "second method" field theory method. I have not shown why $\mathcal{H}_I$ should be form of free fields, etc, but I suspect that Schwinger can answer that one.

Motivation: I was trying to read p 135 of Feynman on parton calculations and was unable to recall clearly the trace business. Also, I have been wanting to see how field theory looks in my all-norm notation. Finally, I have been interested in the various pseudofieldtheoretic calculations of the massive vector mesons. All these things justified this long session.

Applied Field Theory. I reviewed the bare bones of field theory, skipped over most derivations and details, otherwise I would never have finished. I found that there are basically two methods of doing calculations:

First Method: in this method (LSZ) you use the reduction formulas to express an S-matrix element in terms of a giant vacuum expectation value of a time ordered product of lots of fields. Prior to perturbation theory, there is a field for each external particle, and these are all "full" fields.

One then goes to a perturbation theory by transforming these full-Heisenberg Picture fields to IN fields via the operator $U(t, t')$. This causes all the external particle fields to become "in" fields (free fields) and you are left with $U(-\infty, +\infty)$ inside the T-product with these external fields. This operator U is then replaced with the usual exponentiated interaction Hamiltonian of free fields, and each occurrence of H is normal ordered to prevent the possibility of tadpole type graphs etc.

You are now ready to do "contractions". Choose an "order" and insert the corresponding number of H 's and d^4x integrals. Then pair off the fields in all possible ways using Wick's theorem. In the case of ee elastic scattering in second order, each contraction is a product of 5 propagators. After doing all contractions, you put the huge mess back into your LSZ reduction whereupon all the external propagators are "amputated". Then you can convert the whole thing to k-space and get out the Feynman rules.

What can be said about norms for this first method? First of all, the "square root factor" or "external wavefunction" or whatever you want to call it must be the same as the factor occurring in (ϕ, a) field/creator commutation relation. This "external wavefunction" is determined in effect by the sizes of single particle states and by the size of the spinors for Diracions. Remember that fields are the same "size" for everyone. Thus, in general "external wavefunction factor" $\neq C_4$. Nevertheless, it is still C_4 that occurs in the definition of T .

Now in most norm systems C_4 = external wavefunction factor, but Gas. as usual is a notable exception. (see Kine sheet daggers).

When all is said and done, we have a matrix element M_{Feyn} given exactly by the Feynman rules with no external leg-factors. The connection between M_F and the all-norm T_{Fi} is:

$$T_{fi} = +i(-C_1) \cdot \frac{(2\pi)^4}{C_3} \cdot \prod_{\text{all}} \left(\frac{\sqrt{\frac{C_2 C_6}{(2\pi)^3}}}{C_4} \right) \cdot M_{\text{Feyn}}.$$

In any of the standard (dagger) norm systems this reduces to:

$$T_{fi} = +i(-C_1) M_{\text{Feyn}} \leftarrow$$

For Gasiorowicz the result is instead,

$$T_{fi} = +i \prod_{\text{all}} \sqrt{\frac{1}{(2\pi)^3}} \cdot M_{\text{Feyn}}.$$

For Gasiorowics it is as if he had an extra feynman rule which reads: "For each external line include a factor of $(-i)$."

For Kallen the result is:

$$T_{fi} = -i \left(\prod_{\text{all}} C_6 \right) \cdot M_{\text{Sym}} \quad C_6 = 1 \text{ or } 2E$$

We should note in passing that T_{fi} is not always C_6 independent. This is because the phase space "knows" about C_6 in certain norm systems, like Bjorken Drell and Gasiorowics.

In this first method (LSZ) each time you "pull" a particle out of the state and into a field, you get a factor of i . This is simply the i that occurs in the inversion formula (12.9) page 27 resulting from a time derivative on a free field. The renormalization factor of Z I always set equal to one, and I use renormalized (physical) masses and couplings in any calculation.

Second Method. This technique is much more efficient for doing a calculation. Here, you use say IN creation and annihilation operators for your external particles, and you replace S directly with the exponentiated interaction Hamiltonian. Then at once you do the contractions. Each time you contract a field with a creation operator, you pick up one of those "external wavefunction factors". The number of spacetime integrations equals the order of the calculation. It is this method that allows a fast derivation of the feynman rules for any theory. This is the method used by Sakurai, see pages 138 etc.

The factorial factor. In either perturbation method, there is always an $n!$ in the denominator for an n th order calculation, but somehow this never appears in the result. The reason is that there are $n!$ "contractions" which belong to the same topologically distinct feynman graph, so the factorial is always cancelled.

Wicks Theorem. This is one of those combinatoric things that is best proven only by examples, because the general proof is too hard to understand for more than one day. First, you prove Wicks theorem for two fields. To do this, you choose a particular time sequence so you can remove the T -product. Then you observe that the time ordered thing differs from the normal ordered thing only by some field commutators, and these are always c -numbers, so the T order and N order can only differ by a c -number. Then you take a VEV to show that that c -number is just the VEV of the TOP.

Next, you move on to three fields. Again, pick a definite time order. The game is to get "from" this particular order, "to" the normal order. This involves shifting all plus fields to the LEFT where they can do no harm. Each such "shift" causes a term with a propagator for the two interchanged fields, and all the other fields just sit there. Eventually, all fields that do not appear in propagators end up normal ordered. I should do a better proof some day on this.

Note that any VEV of a TOP of an odd number of fields must vanish, due directly to Wicks theorem.

see E. Moller process notes.

Traces. Whenever there is a Dirac particle in a feynman amplitude, traces will occur when the amplitude is squared. These arise from the projection operators and the Inversion theorem.

$$u\bar{u} = \left(\frac{\not{p} + m}{E} \right) \left(\frac{1 + \gamma_5 \not{S}}{2} \right) \quad \quad \quad v\bar{v} = \left(\frac{\not{p} - m}{E} \right) \left(\frac{1 + \gamma_5 \not{S}}{2} \right)$$

$$[\bar{u}(2) \not{X} u(1)]^* = \bar{u}(1) \not{\tilde{X}} u(2) \quad \quad \quad \not{\tilde{X}} = \gamma^0 \not{X}^\dagger \gamma^0$$

Rather than first write the amplitudes, then square them, then convert to traces, etc, it is easiest to use the "trace diagrams" as illustrated on the back of page 12, there for electron electron scattering in second order QED. The idea is to form all the "unitarity-like" diagrams with appropriate statistics minus signs. Then each "line" may be closed onto itself and stands for a trace. Each time you hit an external particle, insert a projector. Each time you hit a vertex, insert the feynman rule vertex factor (eg, $-ie\gamma^\mu$). The trace closes when you return to start. Sometimes there is only one trace, sometimes two or more. Don't forget the propagators apropos. When you do all this, you have at once M^2 and you are ready to go to a cross section or whatever.

Compton provides another example of the technique. (See page 15). Again you use vertex factors in the various traces, except here it happens that polarization 4-vectors are tied to each vertex. Also, in this example there are real fermion propagators and these two must be inside the traces.

In general, any quantity from the "top" unitarity graph should be complex conjugated, eg, propagators with ie 's, polarization vectors, coupling constants. Usually we do not worry about such things.

In actually executing the traces, there are very important tricks to be observed. See BD on this. I have not gone through all these things here.

Spin Sums: For Dirac electrons, spin summing is trivial. A spin sum merely clears the projector of its spin portion. If you are averaging over initial spins, do not forget to put in a $\frac{1}{2}$ for each initial fermion. For final fermions, summing is done by simply ignoring the spin part of the projector.

For photons and massive vector mesons the spin sum is not so trivial. For external photons, there are only two transverse polarization states although you could combine them to get circulars. It is best to do photon polarization sums as the very end of a calculation (as is not the case for electrons). Basically you have to do the sum explicitly. I have done an example on a separate sheet (ie, the Compton case). Note that the three and four dot products of a polarization vector with its momentum vanish (the three-dot vanishing is not relativistic statement... we adjust our frame by arranging for this to be true, this is the radiation gauge)

For massive vector mesons, there are three polarizations to worry about. The four-dot condition still holds. The parallel polarization vector gets stretched, see back of page 20. Again, you have to do spin sums explicitly.

Propagators. How do you compute field commutators at unequal time? You write your lagrangian and compute your canonical momenta. Then impose the equal time comrels. Make momentum expansions and inversions, and then compute the creator comrels by looking at equal time. Then, using these comrels, the unequal time field commutators follow. See page 20.

omitted
The RHO decay. Now we are getting into the realm of phenomenological field theory calculations, as in weak interactions. For the rho interaction with two pions, we postulate a lagrangian,

$$\mathcal{L} = -ig \vec{\phi} \times \vec{\phi} \cdot \vec{g}$$

As shown on page 24, this contains two terms involving the neutral rho. Then, using the "second method" we can get the feynman rules and in particular the decay matrix element. In a dagger standard norm we find:

$$|T_{fi}| = 2g p \cdot \epsilon$$

Here, p is the 4-momentum of either of the final pions. The factor of 2 arises because there are two contributing terms in the lagrangian.

Using this amplitude, we can quickly compute the width of the RHO ϕ into these two pions:

$$\Gamma_{\phi \rightarrow \pi\pi} = \frac{1}{12\pi} \cdot g^2 \cdot m_\phi^2 \cdot \left(1 - \frac{4m_\pi^2}{m_\phi^2}\right)^{3/2} \quad g = f_{\rho\pi\pi}$$

From the lagrangian you can show that g is dimensionless. Here, I have ~~xxxxxx~~ trivially averaged over initial polarisations of the RHO. The result is independent of polarization anyway.

Using the first method, we can show that:

$$\langle \pi^+(p) | j^+(0) | \phi^0(p, \lambda) \rangle = G_1(\pi^+) \cdot G_1(\phi^0) \cdot 2ig p \cdot \epsilon(p, \lambda)$$

Except for normalization, this is the result quoted in Gas page 455 top. We could have arrived at this form from Lorentz invariance without ever using a lagrangian and perturbation theory. Thus, we can justify that phenom. lagrangian in this way.

Rho production. I tried doing the little perturbation calc that Gas shows but I did not get the right thing exactly. At that point I started this summary.

Formalism.

A Review of ee scattering in QED etc. + (1st + 2nd methods) + (s. decay) + (hace stuff) + (massive V_μ rules) + ...

① This is a warm up exercise for poster reading. I would like to go through the entire thing, start to finish. This stuff is all very vague in my head right now. It all starts with the statement of a Lagrangian.

② Lagrangian. One usually uses:

$$\mathcal{L} = \bar{\Psi}(i\partial - eA)\Psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

There are millions of technical details to think about, like normal ordering. After stating $\mathcal{L}$, we have to define asymptotic states, then define an S-matrix operator. This leads quickly to the LSZ technique, where we learn how to "pull" particles from the "in" state & transfer their life to a full field. This quickly leads to the general reduction formulae. Then we move on to chapter 17 on perturbation theory.

Here, $U(t)$ is supposed to be a transformation relating the "interacting fields" to the "in fields". We are able to construct a differential equation for $U(t)$ whose driving term (input) is essentially the "interaction Hamiltonian". $U(t, t')$ is then invented. The integral equation for $U(t, t')$ admits of the usual Neuman like solution, (17.11) p. 177, which can be time ordered and cast into the form (17.13), to wit:

$$U(t, t') = T \left(\exp \left[-i \int_{t'}^t dx \mathcal{H}_I(\phi_{in}(x)) \right] \right)$$

This is the tool which allows us to do pert. theory. We have

$$\langle 0 | T(\phi(x_1) \phi(x_2) \dots \phi(x_n)) | 0 \rangle \leftarrow \text{a "Wightman Function", 1956.}$$

$$= \langle 0 | T(\bar{U}(t_1) \phi_{in}(x_1) U(t_1, t_2) \phi_{in}(x_2) \dots \phi_{in}(x_n) U(t_n) | 0 \rangle$$

This was trivial step. Next, you are allowed to reorder the guts of a T product however you like, so first we do:

$$= \langle 0 | T(\bar{U}(t_1) \phi_{in}(x_1) \phi_{in}(x_2) \dots \phi_{in}(x_n) \cdot U(t_1, t_n) U(t_n) | 0 \rangle$$

$$\text{Now } \bar{U}(t_1) U(t_1, t_n) U(t_n) = \bar{U}(t_1) U(t_1, t) U(t, t_n) U(t_n, t) U(t) \\ = \bar{U}(t_1) U(t, -t) U(-t).$$

$$\Rightarrow = \langle 0 | (\bar{U}(t_1) \cdot \phi_{in}(x_1) \phi_{in}(x_2) \dots \phi_{in}(x_n) \cdot U(t, -t) \cdot U(-t) | 0 \rangle$$

propagator.

Since these VEV things don't really do much to the vac, we disp. them for now.

$$\left[\langle 0 | T(\phi(x_1) \phi(x_2) \dots \phi(x_n)) | 0 \rangle \right]^{U(-\infty, \infty)} = \langle 0 | T(\phi_{in}(x_1) \phi_{in}(x_2) \dots \phi_{in}(x_n) \exp[-i \int_{-\infty}^{\infty} dt H_I(t)]) | 0 \rangle / 1$$

- ③ Now all the real work is done. Keep in mind that H is normal ordered so that ~ well wait to see why this matters. The next important thing is

Dyson-Wick expansion

- ④ Wick's Theorem Example:

$$\begin{aligned} T(\phi_{in}(x_1) \phi_{in}(x_2) \phi_{in}(x_3) \phi_{in}(x_4)) \\ = \langle 0 | T(\phi_{in}(x_1) \phi_{in}(x_2)) | 0 \rangle : \phi_{in}(x_3) \phi_{in}(x_4) : + \text{perms.} \\ + \langle 0 | T(\phi_{in}(x_1) \phi_{in}(x_3)) | 0 \rangle \langle 0 | T(\phi_{in}(x_2) \phi_{in}(x_4)) | 0 \rangle + \text{perms.} \\ + : \phi \phi \phi \phi : \end{aligned}$$

- ⑤ Of course these pairs are the various propagators:

$$\langle 0 | T(\phi(x_1) \phi(x_2)) | 0 \rangle = i \Delta_F(x_1 - x_2)$$

$$= \int \frac{d^4 k}{(2\pi)^4} e^{-ik \cdot (x_1 - x_2)} \left(\frac{+i}{k^2 - m^2 + i\epsilon} \right)$$

$$\langle 0 | T(\psi_\beta(x_1) \psi_\alpha(x_2)) | 0 \rangle = i S_F(x_1, x_2)$$

$$\begin{aligned} &= \int \frac{d^4 k}{(2\pi)^4} e^{-ik \cdot (x_1 - x_2)} \left(\frac{+i (\not{k} + m)_{\beta\alpha}}{k^2 - m^2 + i\epsilon} \right) \\ &= \left(\frac{+i}{\not{k} - m} \right) \end{aligned}$$

$$\langle 0 | T(A_\mu(x_1) A_\nu(x_2)) | 0 \rangle = i D_F^{\mu\nu}(x_1, x_2)$$

$\rightarrow -g_{\mu\nu}$

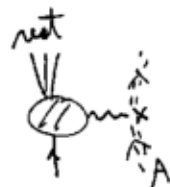
$$= \int \frac{d^4 k}{(2\pi)^4} e^{-ik \cdot (x_1 - x_2)} \left(\frac{+i}{k^2 + i\epsilon} \right) \times \left[\sum_{\lambda=1}^2 \epsilon_\mu(k) \epsilon_\nu(k) \right] + \text{gauge.}$$

Feynman Rules (Q.E.D.)

$$S-1 = R = -i(2\pi)^4 \delta^4(\text{ext}) \underbrace{(\prod \int \text{legs})}_{(\prod C_i)} \cdot \mathcal{M}$$

Rules for $\mathcal{M}$:

- ✓ 1) $(-ie)\gamma^\mu$ each vertex
- ✓ 2) ϵ_μ each external photon. $\bar{u}, u$ etc. for external fermions.
- ✓ 3) $\frac{i}{\not{p} - m + i\epsilon} = iS_F(p)$ each internal fermion line (p along arrow)
- ✓ 4) $\frac{-iq_{\mu\nu}}{q^2 + i\epsilon} = iD_F(q)_{\mu\nu}$ each internal photon line
- ✓ 5) $\int \frac{d^4q}{(2\pi)^4}$ each internal loop
- ✓ 6) -1 for internal fermion loop
- ✓ 7) -1 relative for exchange initial + final ferm/antiferm.
- 8) $+i\delta m$ counterterm.



well, $dQ_F = \left[\frac{(2\pi)^3 \delta A}{2MA} \right] \cdot 2\pi \delta(\text{energy}) \cdot \prod_{\text{rest}} \left(\frac{\delta_f d^3 p_F}{2E_f} \right)$
 good for any $C_3 = (2\pi)^4$ norm.

Coulomb's "external potential".



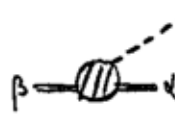
$$S-1 = 2\pi \delta(E) \underbrace{(\prod \int \text{legs})}_{(\prod C_i)} \cdot \mathcal{M}$$

- ✓ 1) $-ie\gamma^\mu$ at such a vertex
- ✓ 2) $-\frac{Ze}{q^2} \delta_{\mu 0} \frac{1}{|q|^2}$ for the γ line. ($e < 0$; $q_0 = -Ze$)



Reduction Formulas

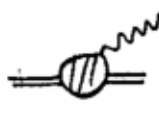
(for removing one particle from the S-matrix)



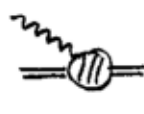
$$(+)\left(\frac{i}{\sqrt{2}}\right) \int d^4x \left[e^{-ip \cdot x} \frac{1}{\sqrt{2}} \vec{K}_x \right] \langle \beta | \phi | \alpha \rangle \left[e \right]$$



$$(+)\left(\frac{i}{\sqrt{2}}\right) \int d^4x \left[e \right] \langle \beta | \phi^\dagger | \alpha \rangle \left[\vec{K} \frac{1}{\sqrt{2}} e^{+ip \cdot x} \right]$$



$$(-)\left(\frac{i}{\sqrt{2}}\right) \int d^4x \left[e^{-ik \cdot x} \frac{1}{\sqrt{2}} \epsilon^\mu(k, \lambda) \vec{Q}_x \right] \langle \beta | A_\mu | \alpha \rangle \left[e \right]$$



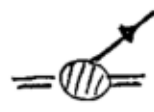
$$(-)\left(\frac{i}{\sqrt{2}}\right) \int d^4x \left[e \right] \langle \beta | A_\mu | \alpha \rangle \left[\vec{Q}_x \epsilon^\mu(k, \lambda) \frac{1}{\sqrt{2}} e^{ik \cdot x} \right]$$



$$(-)\left(\frac{i}{\sqrt{2}}\right) \int d^4x \left[e \right] \langle \beta | \bar{\psi} | \alpha \rangle \left[\vec{D}^* \frac{1}{\sqrt{2}} u(p, s) e^{-ip \cdot x} \right]$$



$$(-)\left(\frac{i}{\sqrt{2}}\right) \int d^4x \left[e^{+ip \cdot x} \frac{1}{\sqrt{2}} \bar{u}(p, s) \vec{D} \right] \langle \beta | \psi | \alpha \rangle \left[e \right]$$



$$(+)\left(\frac{i}{\sqrt{2}}\right) \int d^4x \left[e^{-ip \cdot x} \frac{1}{\sqrt{2}} \bar{v}(p, s) \vec{D} \right] \langle \beta | \bar{\psi} | \alpha \rangle \left[e \right]$$



$$(+)\left(\frac{i}{\sqrt{2}}\right) \int d^4x \left[e \right] \langle \beta | \bar{\psi} | \alpha \rangle \left[\vec{D}^* \frac{1}{\sqrt{2}} v(p, s) e^{+ip \cdot x} \right]$$

Remarks: Time increases to the left. All momenta are defined as being to the left. Incoming particles always have a minus in the exponential. The spinor indices do not close right onto the field operator shown, but thru the D operator. Thus, the Dirac ordering cannot be changed. State bracket is always (out/in). Alpha and Beta represent all the remaining particles.

$$T(A(x) B(y)) = R(A(x) B(y)) + B(y) A(x)$$

$$\hookrightarrow = \Theta(x_0 - y_0) [A(x), B(y)]_?$$



Scalar and Spinor Invariant Propagator & Commutator Functions

(exact Bjorken-Drell notation)

$S^{(\pm)}(z) \equiv \int du (\pm \not{p} + m) e^{\mp i p \cdot z}$ $i S_F = \theta(z_0) S^{(+)} + \theta(-z_0) S^{(-)}$ $= \langle 0 T(\psi \bar{\psi}) 0 \rangle$ $i(-S) = S^{(+)} - S^{(-)}$ $= \langle 0 \{ \psi, \bar{\psi} \} 0 \rangle$ $(i \not{\partial} - m) \begin{bmatrix} S^{(+)} \\ S^{(-)} \\ -S \end{bmatrix} = 0$ $S^{(+)}(x-x') = \langle 0 \psi(x) \bar{\psi}(x') 0 \rangle$	$\Delta^{(\pm)}(z) \equiv \int du e^{\mp i p \cdot z}$ $i \Delta_F = \theta(z_0) \Delta^{(+)} + \theta(-z_0) \Delta^{(-)}$ $= \langle 0 T(\phi \phi^*) 0 \rangle$ $i \Delta = \Delta^{(+)} - \Delta^{(-)}$ $= \langle 0 (\phi, \phi^*) 0 \rangle$ $(\square + m^2) \begin{bmatrix} \Delta^{(+)} \\ \Delta^{(-)} \\ \Delta \end{bmatrix} = 0$ $\Delta^{(+)}(x-x') = \langle 0 \phi(x) \phi^*(x') 0 \rangle$
$S^{(-)}(x-x') = - \langle 0 \bar{\psi}(x') \psi(x) 0 \rangle$ $(i \not{\partial} - m) S_F = + \delta^4(z)$	$\Delta^{(-)}(x-x') = + \langle 0 \phi^*(x') \phi(x) 0 \rangle$ $(\square + m^2) \Delta_F = - \delta^4(z)$
$\begin{bmatrix} S^{(+)} \\ S^{(-)} \\ S_F \\ (-S) \end{bmatrix} = (i \not{\partial} + m) \begin{bmatrix} \Delta^{(+)} \\ \Delta^{(-)} \\ \Delta_F \\ \Delta \end{bmatrix}$ $du \equiv \frac{d^3 \vec{p}}{(2\pi)^3 2E}$	1. Variables always in same order unless otherwise specified. 2. Note that S and S₁ are defined with a minus compared to Δ and Δ₁.
$(-S_1) = S^{(+)} + S^{(-)}$	$\Delta_1 = \Delta^{(+)} + \Delta^{(-)}$

Field Theory Normalization

- ① Always choose $C_3 = (2\pi)^4$ in field theory this always comes out in a natural way.
- ② Assume $C_6 = 1$ for non-Dirac particles to facilitate certain formulae.
- ③ Use the Bjorken metric such that $A \cdot B = A_0 B_0 - \vec{A} \cdot \vec{B}$. This helps in writing those exponentials.
- ④ Choose a norm such that $C_4^2 = C_2 C_6 / (2\pi)^3$. This means there will be no extraneous external leg factors other than the usual C_4 pulled in the definition of T_{fi} . Any of these norms is OK on this point: (those with $\dagger$ in kinematics sheet). Then:

$$C_5 \equiv \left[\frac{C_6}{(2\pi)^3 C_2} \right]^{1/2} = C_4 / C_2$$

- ⑤ The field expansions in momentum space are:

scalar neutral field: $\phi(x) = \int \frac{d^3k}{2E} \cdot C_5 \left[a(k) e^{-ik \cdot x} + a^\dagger(k) e^{+ik \cdot x} \right]$

charged scalar field: $\phi(x) = \int \frac{d^3k}{2E} \cdot C_5 \left[a_+(k) e^{-ik \cdot x} + a_-^\dagger(k) e^{+ik \cdot x} \right]$

Dirac field: $\psi(x) = \sum_s \int \frac{d^3k}{2E} \cdot C_5 \left[b(k, s) u_s(k) e^{-ik \cdot x} + d^\dagger(k, s) v_s(k) e^{+ik \cdot x} \right]$

photon field: $A^\mu(x) = \sum_{\lambda=1}^2 \int \frac{d^3k}{2E} \cdot C_5 \left[a(k, \lambda) \epsilon^\mu(k, \lambda) e^{-ik \cdot x} + a^\dagger(k, \lambda) \epsilon^{\mu*}(k, \lambda) e^{+ik \cdot x} \right]$

massive vector field: $\phi^\mu(x) = \sum_{\lambda=1}^3 \int \frac{d^3k}{2E} \cdot C_5 \left[a(k, \lambda) \epsilon^\mu(k, \lambda) e^{-ik \cdot x} + a^\dagger(k, \lambda) \epsilon^{\mu*}(k, \lambda) e^{+ik \cdot x} \right]$

- ⑥ The relevant commutators are:

$$[\phi(x), a^\dagger(p)] = C_4 \cdot e^{-ip \cdot x}$$

$$[\phi(x), a_+^\dagger(k)] = C_4 \cdot e^{-ik \cdot x}$$

$$[a_+(k), \phi^\dagger(x)] = C_4 \cdot e^{+ik \cdot x}$$

$$[\psi(x), b^\dagger(p, s)] = C_4 \cdot u_s(p, s) e^{-ip \cdot x}$$

$$[b(p, s), \bar{\psi}(x)] = C_4 \cdot \bar{u}_s(p, s) e^{+ip \cdot x}$$

$$[A^\mu(x), a^\dagger(k, \lambda)] = C_4 \cdot \epsilon^\mu(k, \lambda) e^{-ik \cdot x}$$

Field Commutators

Scalar field: $[\phi(x), \phi^\dagger(y)] = i \Delta(x-y)$

Dirac field:

$$\{\psi_\alpha(x), \bar{\psi}_\beta(y)\} = i (i \not{\partial} + m)_{\alpha\beta} \Delta(x-y) \equiv -i \bar{S}_{\alpha\beta}(x-y)$$

Photon field:

$$[A^\mu(x), A^\nu(y)] = -i g^{\mu\nu} \Delta(x-y)$$

Massive vector field:

$$[\phi^\mu(x), \phi^\nu(y)] = -i (g^{\mu\nu} + \frac{1}{m^2} \partial^\mu \partial^\nu) \Delta(x-y)$$

① Note that $\Delta(x-y) = \Delta(x-y; m)$ where m is the appropriate mass.

PROPAGATORS

Scalar field:

$$i \Delta_F(x-y) = \langle 0 | T(\phi(x) \phi^\dagger(y)) | 0 \rangle$$

$$= \int \frac{d^4 k}{(2\pi)^4} e^{-ik \cdot (x-y)} \underbrace{\left[\frac{+i}{k^2 - m^2 + i\epsilon} \right]}_{= i \Delta_F(k)}$$

Dirac field:

$$i S_F(x-y)_{\alpha\beta} = \langle 0 | T(\psi_\alpha(x) \bar{\psi}_\beta(y)) | 0 \rangle$$

$$= \int \frac{d^4 k}{(2\pi)^4} e^{-ik \cdot (x-y)} \underbrace{\left[\frac{+i}{k \not{=} - m + i\epsilon} \right]_{\alpha\beta}}_{= i S_F(k)_{\alpha\beta}}$$

Photon field:

$$i D_F(x-y)_{\mu\nu} = \langle 0 | T(A_\mu(x) A_\nu(y)) | 0 \rangle$$

$$= \int \frac{d^4 k}{(2\pi)^4} e^{-ik \cdot (x-y)} \underbrace{\left[\frac{-i g_{\mu\nu}}{k^2 + i\epsilon} \right]}_{= i D_F(k)_{\mu\nu}}$$

Massive vector meson field:

$$i D_F(x-y)_{\mu\nu} = \langle 0 | T(\phi_\mu(x) \phi_\nu(y)) | 0 \rangle$$

$$= \int \frac{d^4 k}{(2\pi)^4} e^{-ik \cdot (x-y)} \underbrace{\left[\frac{-i g_{\mu\nu} + i k_\mu k_\nu / m^2}{k^2 - m^2 + i\epsilon} \right]}_{= i D_F(k)_{\mu\nu}}$$

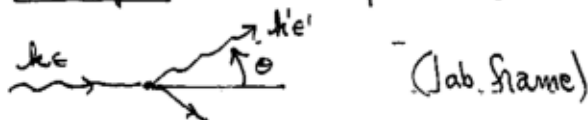
Photon Spin Sum

- ① In a calculation, choose a frame (BD choose lab) and keep the ϵ 's all through the calc, keeping in mind that:

$$\epsilon^0 = 0 \quad \vec{\epsilon} \cdot \vec{k} = 0 \quad \epsilon_\mu k^\mu = 0 = \epsilon \cdot k$$

In the end, you will have objects like $\epsilon \cdot a$ appearing in your cross section. Then, do a specific transverse pole sum.

- ② Example: in BD p.131 we consider:



One finds the Klein-Nishina formula which says:

$$\frac{d\sigma}{d\Omega} = -\frac{\alpha^2}{4m_e^2} \left(\frac{k'}{k}\right)^2 \left[\left(\frac{k'}{k} + \frac{k}{k'} - 2\right) + 4(\epsilon \cdot \epsilon')^2 \right]$$

Here, $d\sigma$ means electron spin summed. So we now seek:

$$\frac{1}{2} \sum_{\epsilon, \epsilon'} [A + B(\epsilon \cdot \epsilon')^2] = 2A + \frac{1}{2} B \sum_{\epsilon, \epsilon'} (\epsilon \cdot \epsilon')^2$$

We set up so that

$$\uparrow \epsilon_1 \sim \odot \epsilon_2 \rightarrow \quad \text{and}$$



$$\text{Then: } \vec{\epsilon}_1 \cdot \vec{\epsilon}_1 = \cos \theta, \quad \vec{\epsilon}_1 \cdot \vec{\epsilon}_2 = 0, \quad \vec{\epsilon}_2 \cdot \vec{\epsilon}_1 = 0, \quad \vec{\epsilon}_2 \cdot \vec{\epsilon}_2 = 1.$$

$$\text{Thus: } \sum_{\epsilon, \epsilon'} (\epsilon \cdot \epsilon')^2 = [1 + 0 + 0 + \cos \theta] = (1 + \cos \theta)$$

$$\begin{aligned} \Rightarrow 2A + \frac{1}{2} B(1 + \cos \theta) &= 2 \left[A + \frac{1}{4} B(1 + \cos \theta) \right] \\ &= 2 \left[\frac{k'}{k} + \frac{k}{k'} - 2 + 1 \cdot (1 + \cos \theta) \right] = 2 \left[\frac{k'}{k} + \frac{k}{k'} - \sin^2 \theta \right] \end{aligned}$$

Then the fully averaged Compton result is: $(\alpha = \frac{e^2}{4\pi})$

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{LAB}} = \frac{e^2}{2m_e^2} \left(\frac{k'}{k}\right)^2 \left(\frac{k'}{k} + \frac{k}{k'} - \sin^2 \theta\right)$$

The Story on the Transverse Photon Propagator

① First, let's write the photon field expansion:

$$\vec{A}(x) = \sum_{\lambda=1,2} \frac{d^3k}{2E} \cdot \vec{e}(k, \lambda) \left[a(k, \lambda) e^{-ik \cdot x} + a^\dagger(k, \lambda) e^{ik \cdot x} \right]$$

We are in the "radiation gauge" where $A_0 = \phi = 0$ and $\vec{\nabla} \cdot \vec{A} = 0$, so the photons are transverse. A_0 is not quantized. In this gauge we define:

$$\langle 0 | T(A_\nu(x') A_\mu(x)) | 0 \rangle = i D_F^{\mu\nu}(x', x)$$

This is transverse because these A_μ are only summed over transverse photons in their momentum expansion. We get at once,

$$i D_F^{\mu\nu}(x' - x)_{\mu\nu} = \int \frac{d^4k}{(2\pi)^4} e^{-ik \cdot (x' - x)} \cdot \left(\frac{-i}{k^2 + i\epsilon} \right) \cdot \sum_{\lambda=1,2} \vec{e}_\nu(k, \lambda) \vec{e}_\mu(k, \lambda)$$

where $\epsilon_0 = 0$. But, easy to show that:

$$\sum_{\lambda=1,2} \vec{e}_\nu \vec{e}_\mu = -g_{\mu\nu} + \frac{k_\mu k_\nu}{k^2}$$

In practice, the $\sim k^0$ & $\sim k^\mu$ terms give nothing, because of "gauge invariance". For example:

$$\begin{aligned} \left(\int_1^2 \right) &= \bar{u}(2) \gamma^\mu u(1) \cdot k_\mu = \gamma^\mu \cdot k_\mu \\ &= \bar{u}(2) \not{k} u(1) = \bar{u}(2) (\not{p}_2 - \not{p}_1) u(1) = \bar{u}(2) (\not{m} - \not{m}) u(1) = 0. \end{aligned}$$

I.e., γ^μ is conserved current. Recall that

$$j^\mu \equiv \bar{\Psi} \gamma^\mu \Psi$$

$$\partial_\mu j^\mu = 0 \sim \bar{\Psi}_1 \gamma^\mu \Psi_2 \cdot (p_2 - p_1)_\mu$$

I.e., ∂_μ always gets separated by k_μ in momentum space. Why is this called "gauge invariance"? Because if you do a gauge transformation on $\int d^4x \mathcal{L}(x)$, you have

$$\bar{\Psi} \gamma^\mu \Psi A_\mu \rightarrow \bar{\Psi} \gamma^\mu \Psi (A_\mu + \partial_\mu \Lambda) = \bar{\Psi} \gamma^\mu \Psi A_\mu + \underbrace{\partial_\mu (\bar{\Psi} \gamma^\mu \Psi) \Lambda}_{=0}$$

It is entirely due to the observed fact that we have gauge invariance in the first place. Recall that $Q = \text{charge}$ is the single generator of this:

(X) gauge transformation

So, throwing out the $\sim k^4$ and $\sim k^2$ terms, we get

$$\sum_{\lambda=1,2} \epsilon_{\mu} \epsilon_{\nu} = -g_{\mu\nu} - \frac{k^2}{(k^2)^2} \eta_{\mu} \eta_{\nu} \quad \text{or} \quad \eta_{\mu} = (1, 0, 0, 0)$$

So:

$$\sum_{\lambda=1,2} \epsilon_{\mu} \epsilon_{\nu} + \frac{k^2}{(k^2)^2} \eta_{\mu} \eta_{\nu} = -g_{\mu\nu}$$

transverse only

Coulomb

So, if we included the Coulomb effect, then we'd be including everything and we'd have

$$iD_F(x-x') = \int \frac{d^4 k}{(2\pi)^4} e^{ik(x-x')} \frac{(-ig_{\mu\nu})}{k^2 + i\epsilon} \equiv iD_F(k)_{\mu\nu}$$

BtD define:

$$D_F(x-x') \equiv \int \frac{d^4 k}{(2\pi)^4} \frac{e^{ik(x-x')}}{k^2 + i\epsilon}$$

no Lorentz indices

So I will define:

$$\begin{cases} iD_F(k)_{\mu\nu} = \frac{-ig_{\mu\nu}}{k^2 + i\epsilon} \\ iD_F(x-x')_{\mu\nu} = \int \frac{d^4 k}{(2\pi)^4} e^{ik(x-x')} iD_F(k)_{\mu\nu} \end{cases}$$

$$c_{\pm} = \frac{1}{2}(c_1 \pm i c_2)$$

$$\phi_{\pm} = \frac{1}{\sqrt{2}}(\phi_1 \mp i \phi_2)$$

PION NUCLEON PAGE

$$\phi_1 = \frac{1}{\sqrt{2}}(\phi_+ + \phi_-)$$

$$c_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

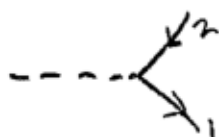
$$\phi_2 = \frac{i}{\sqrt{2}}(\phi_+ - \phi_-)$$

$$c_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$\mathcal{L} = -ig_0 \bar{\psi} \gamma_5 \psi \phi = -g_0 \bar{p} i \gamma_5 p \pi_0 + g_0 \bar{n} i \gamma_5 n \pi_0 - \sqrt{2} g_0 \bar{p} i \gamma_5 n \pi_+ - \sqrt{2} g_0 \bar{n} i \gamma_5 p \pi_-$$

To compute any amplitude, use:

$$\mathcal{M} = \chi^\dagger(1) \hat{\phi}^{(1)} \chi(2) \quad \bar{u}(p_1) \gamma_5 \left[\frac{u}{v} \right] (p_2)$$



Charged pions always introduce a $\sqrt{2}$:

$$\hat{\phi}^{(+)} = \begin{bmatrix} 0 & \sqrt{2} \\ 0 & 0 \end{bmatrix}$$

$$\hat{\phi}^{(\pm)} = \frac{1}{\sqrt{2}}(1, \pm i, 0)$$

$$\hat{\phi}^{(0)} = \begin{bmatrix} 0 & 0 \\ \sqrt{2} & 0 \end{bmatrix}$$

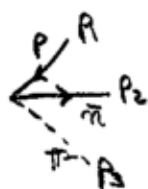
$$c \cdot \pi = c_3 \pi_0 + \sqrt{2} c_+ \phi_+ + \sqrt{2} c_- \phi_-$$

$$\hat{\phi}^{(0)} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Rules: 1) $\chi = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ for p or $\bar{p}$; $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ for n or $\bar{n}$.

2) use u, u for particles; $\bar{v}, v$ for anti particles.

Example:



$$\mathcal{M} \sim \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 & \sqrt{2} \\ \sqrt{2} & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{\sqrt{2}} g \bar{v}(p_2) \gamma_5 u(p_1)$$

$$i\mathcal{L} = (-i)\mathcal{H} = g_0 \bar{p} \gamma_5 p \pi_0 - g_0 \bar{n} \gamma_5 n \pi_0 + \sqrt{2} g_0 [\bar{p} \gamma_5 n \pi_+ + \bar{n} \gamma_5 p \pi_-]$$

Hermiticity, adjointing, normal ordering, and the dirac current.

1. Adjointing operators.

$$(AB)^\dagger = B^\dagger A^\dagger \quad \text{since}$$

When you adjoint a product of two operators, the order reverses and there are no minus signs even if these are anticommuting dirac fields. Also, this reversal of the order has nothing whatsoever to do with matrices. When you adjoint an

$$\left\{ \begin{array}{l} AB|p\rangle = |q\rangle \\ \Downarrow \\ \langle p|(AB)^\dagger = \langle q| \\ \langle p|B^\dagger A^\dagger = \langle q| \end{array} \right.$$

operator, the adjoint operation has nothing to do with transposing and complex conjugating anything.

2. Adjointing matrix representations of operators.

Square matrices have the above property which is one reason that matrices can be used to represent operators in the first place. With matrices, the conventional adjoint operation applies: twiddle and *.

3. Playing with the dirac current operator.

In either the first or second quantized theories, the dirac current j_μ is hermitian. The only possible point of confusion is that in the second quantized theory you do not throw in a minus sign because at no point are the fields anticommutated. See 1 above.

$$\begin{aligned} j_\mu &= \bar{\psi} \gamma_\mu \psi & j_\mu^\dagger &= [\psi^\dagger (\gamma_0 \gamma_\mu)_{\alpha\beta} \psi]^\dagger = (\gamma_0 \gamma_\mu)^*_{\alpha\beta} [\psi^\dagger_\alpha \psi_\beta]^\dagger \\ & & &= (\gamma_\mu^\dagger \gamma_0)_{\beta\alpha} [\psi^\dagger_\beta \psi_\alpha] = j_\mu. \end{aligned}$$

Notice that the adjoint of a number is just its *. As shown in lecture 22 of 230A, the hermiticity of j is unaffected by normal ordering.

4. When does that anticommutation minus sign come into play? When you charge conjugate the dirac current!

$$\begin{aligned} G j_\mu \bar{G}^\dagger &= G \psi^\dagger \bar{G}^\dagger \gamma_0 \gamma_\mu G \psi \bar{G}^\dagger = \psi^\dagger \gamma_0 C^\dagger \gamma_0 \gamma_\mu C \psi^\dagger \\ &= \psi^\dagger (\gamma_0 \gamma_\mu)^T \psi^\dagger = -\psi^\dagger \gamma_0 \gamma_\mu \psi = -j_\mu \end{aligned}$$

By the way, the first quantized theory does not give the above minus sign. e must be changed to $-e$ by hand.

$$1: \psi_c = C \gamma^0 \psi^\dagger$$

$$2: \psi_c = G \psi \bar{G}^\dagger = C \gamma^0 \psi^\dagger$$

Dimensions in Field Theory

$$E = \frac{1}{L}$$

$$L = T$$

Scalar field: $\phi \sim E$

$$a, a^\dagger \sim E^{-3/2}$$

Photon field: $A_\mu \sim E$

$$a, a^\dagger \sim E^{-3/2}$$

Dirac field: $\psi, \psi^\dagger \sim E^{3/2}$

$$b, d^\dagger \sim E^{-3/2}$$
$$b^\dagger, d$$

$u, v = \text{dimensionless}$

$e, g = \text{dimensionless} \quad \left(\frac{e^2}{4\pi} = \alpha \right)$

Consider: (1) $\bar{\psi}(i\not{\partial} - m)\psi = E^3 \cdot E = E/L^3 \checkmark$

$$(2) -\frac{1}{2}m^2\phi^2 = E^4 = E/L^3 \checkmark$$

$$(3) p_\mu \rightarrow p_\mu - eA \quad E = E \checkmark$$

$$(4) -e \bar{\psi} \gamma_\mu \psi A^\mu = E^3 E = E/L^3 \checkmark$$

$$(5) g_0 \bar{\psi} \gamma_5 \underline{\underline{\pi}} \psi = E^3 \cdot E = E/L^3 \checkmark$$

Weaker: $j_\mu = \bar{\psi} \dots \psi \sim E^3$

$$j_\mu j^\mu \sim E^6$$

$$\mathcal{L} \propto j_\mu j^\mu G = E^4 \Rightarrow G = E^{-2}$$

Unitarity in Field Theory

1. Because you are dealing with a hermitian lagrangian, unitarity in a field theory is automatically repeated. So is Löröntz invariance and conserved quantum numbers. We always make such a big deal about how hard it is to satisfy unitarity, but here in field theory you don't even give it a thought. Formally, unitarity follows from $S = e^{iL}$. But you can show unitarity order by order in, say, QED. Here are some graphs:

$$\text{Im } T_{ij} = \int d\phi_k T_{ik}^* T_{kj}$$

$$\text{Im } \text{[diagram]} = 0 \quad \text{order } \alpha$$

$$\text{Im } \text{[diagram]} = \int d\phi_k (\text{[diagram]}) \times (\text{[diagram]}) \quad \text{order } \alpha^2$$

2. In fact, unitarity is trivial in QED. It holds for each graph. All you do is put the intermediate lines on mass shell Cutkosky-like.
3. Of course the other symmetries follow from invariances of the Lagrangian. Such as: lorentz invariance, isospin invariance, and so on.
4. Well, what ever happened to pion-nucleon field theory? I gather that the large dimensionless coupling ruins perturbation theory, but what about the axiomatics? And then there is the problem of fundamental entities which appear in the lagrangian.

Comments on the Poincare group versus Lorentz group problem.

The problem here is this: why is the RHO sometimes a vector $(\frac{1}{2}, \frac{1}{2})$ of the LG, but other times it is a $(0,1)$ of the same group, and other times a mass= m , spin=1 rep of the Poincare group. How are these ideas all related?

I don't have a full answer yet. Here are some ideas though.

1. I know that I can construct spinor states of the form $|p^{\mu}\rangle$ which transform as the vector rep of LG. In such a case, neither of these indices is a helicity. I can then make a vector wavefunction by changing basis, so that we have $|p^{\mu}\rangle$ and $\phi^{\mu}(p)$ as our momentum-space wavefunction. This transforms in obvious way

$$[U(L) \phi]^{\mu}(p) = L^{\mu}_{\nu} \phi^{\nu}(Lp)$$

or something like that. This thing is just a function. Obviously, you also know how the state transforms:

$$U(L) |p^{\mu}\rangle = L^{\mu}_{\nu} |lp^{\nu}\rangle$$

Next, you should be able to find out how the creation operator transforms. Then finally you find out how the field operator goes.

The conclusion is roughly this: the field operator $\phi(x)$ of field theory transforms just like the wavefunction (spinor) $\phi(p)$, except perhaps under translations. Since these two things are roughly related by simple fourier transform, that is just what you expect. So, both wavefunctions and fields should belong to representations of the Lorentz group rather than the Poincare group. At least we can see the Lorentz group in both p and x space.

2. On the other hand, if you take any massive particle and put it at rest, it must have some spin s due to the Poincare group. I.e., it must belong to some rep of the Poincare. The RHO has $s=1$ for sure. The spinor state I would normally make for the RHO would transform with $(0,s) = (0,1)$, not $(\frac{1}{2}, \frac{1}{2})$. This latter has the wrong number of states (4). Nevertheless, in theories of the RHO, the three states are often embedded into a vector field. I think this is why you need gauge invariance, to compensate for this erroneous embedding of three objects into a 4-plet.

In S-matrix theory, I don't think there is any reason at all to think of the RHO as a 4-vector.

New Field Theory Notes

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5/7/79

1. Consider the main reduction formula (16.81) of Bjorken Drell: the fields inside the T operator have a definite order. Now let's go at once to perturbation theory where this T-function is expanded as in (17.22). The IN fields in (17.22) are ordered in the same way as the full fields in (16.81).

Now, the interaction hamiltonian must contain an even number of fermion fields (as, eg, in QED) to conserve fermion superselection. Thus, you can move a field through the symbolic exponential in (17.22) top line. The fact that H_I is normal ordered makes no difference to this argument. The point is that it has even number of fermion fields.

2. Now we have to say something about how we shall write states. Notice that:

$$a_1^\dagger a_2^\dagger |0\rangle \equiv |1,2\rangle$$

$$\Rightarrow \langle 1,2| \equiv (|1,2\rangle)^\dagger = \langle 0| a_2 a_1$$

This is important. We have defined a state $|1,2\rangle$ and never deal with a state $|2,1\rangle$, at least not here. When you conjugate the state, the ordering of the labels in the final state is the reverse of the order of the annihilation operators! Often people do not order things in this way, but it seems right to me.

3. Therefore, it would be my inclination to reverse the label order for the final state in (16.81). Then this equation combined with the other equation would give:

$$\begin{aligned} & \langle p_n, \dots, p_2, p_1 | S | q_1, \dots, q_m \rangle \\ &= \left(\frac{i}{\sqrt{2\pi}} \right)^{m+n} \prod_{i=1}^m \int d^4 x_i \prod_{j=1}^n \int d^4 y_j \cdot f_{q_i}(x_i) (\overrightarrow{D}_x + m) \cdot (\overleftarrow{D}_y + m) f_{p_j}^*(y_j) \\ &= \langle 0 | T (\phi(y_1) \phi(y_2) \dots \phi(y_n) \phi(x_1) \dots \phi(x_m)) | 0 \rangle \\ &= \langle 0 | T (\phi_{in}(y_1) \phi_{in}(y_2) \dots \phi_{in}(y_n) \phi_{in}(x_1) \dots \phi_{in}(x_m) e^{-i \int dt H}) | 0 \rangle \end{aligned}$$

Now you can see what crossing will do. Without changing any orders, you can cross particle $y_n = p_n$ from the final state to a position to the left of $q_1 = x_1$ in the initial state. The total 4-momentum gets negated because f goes to f^* . Any spin index.. well, look at self-reduction page. A $u(p,s)$ in f becomes $u(-p,s)$ which is $v(p,s)$ or something, so essentially spin stays put. This crossing is in accord with Taylor's crossing rule, except they have no phase "over the top".

A Paradox + Its Resolution["Do not use Euler equation while doing Noether"]

● Paradox: consider any variation $\delta\phi$ of your field. You know that

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\phi}\delta\phi + \pi_\mu \delta(\partial_\mu\phi)$$

But from Euler you have that $\partial\mathcal{L}/\partial\phi = \partial_\mu\pi^\mu$, so

$$\begin{aligned}\delta\mathcal{L} &= (\partial_\mu\pi^\mu)\delta\phi + \pi_\mu \partial_\mu(\delta\phi) \\ &= \partial_\mu [\pi^\mu \delta\phi]\end{aligned}$$

Since the variation in $\mathcal{L}$ is a total derivative, we have shown that $\delta S = 0$ and therefore all variations $\delta\phi$ are symmetries of your theory.

● Resolution: The Lagrangian $\mathcal{L}(\phi, \partial_\mu\phi)$ is defined for $\phi \in \mathcal{A}$, the set of arbitrary functions; whereas the Euler equation is valid not for any $\phi \in \mathcal{A}$, but for $\phi \in \mathcal{S}$, the set of solutions. If $\phi \in \mathcal{S}$, then sure, any variation of ϕ around a solution field will yield $\delta S = 0$, that's where you got Euler in the first place.

When we speak of a "symmetry of $\mathcal{L}$ " we mean a $\delta\phi$ such that $\delta\mathcal{L} = 0$, or at least such that $\int \delta\mathcal{L} = 0$, for any field ϕ . Thus, you are not allowed to use the Euler equation when you are discussing this matter.

Noether's Theorem

Comment: I want a clear statement of this thing that works both for coordinate transformations and for internal transformations like isospin. Should think later also of local gauge xforms.

Derivation: (note: the extension to many fields is trivial, do it later, now assume only one field). We start by postulating a certain form for the change in our field:

$$\delta\phi(x) = \underset{\substack{\uparrow \\ \text{the "parameter" }}}}{a(x)} \cdot \underset{\substack{\uparrow \\ \text{some operator}}}{\vec{\sigma}}\phi$$

Since $\partial_\mu[\delta\phi] = (\partial_\mu a) \vec{\sigma}\phi + a(\partial_\mu \vec{\sigma}\phi)$, we can write out $\delta\mathcal{L}$ in the obvious way, + we can then define some new derivatives:

$$\delta\mathcal{L} = \left(\frac{\partial\mathcal{L}}{\partial\phi}\right) [a \cdot \vec{\sigma}\phi] + \pi^\mu \left[(\partial_\mu a) \vec{\sigma}\phi + a(\partial_\mu \vec{\sigma}\phi)\right]$$

where $\pi^\mu \equiv \partial\mathcal{L}/\partial[\partial_\mu\phi]$. So now define, or rather, observe that:

$$\frac{\partial[\delta\mathcal{L}]}{\partial a} = \left(\frac{\partial\mathcal{L}}{\partial\phi}\right) \vec{\sigma}\phi + \pi^\mu (\partial_\mu \vec{\sigma}\phi)$$

$$\frac{\partial[\delta\mathcal{L}]}{\partial[\partial_\mu a]} = \pi^\mu \vec{\sigma}\phi$$

Then we can say:

$$\begin{aligned} \int d^4x \delta\mathcal{L} &= \int d^4x \left[\frac{\partial[\delta\mathcal{L}]}{\partial a} a + \frac{\partial[\delta\mathcal{L}]}{\partial[\partial_\mu a]} (\partial_\mu a) \right] \\ &= \int d^4x \left[\frac{\partial[\delta\mathcal{L}]}{\partial a} - \partial_\mu \left(\frac{\partial[\delta\mathcal{L}]}{\partial[\partial_\mu a]} \right) \right] a(x) \end{aligned}$$

Now, because $\delta\phi$ is supposedly a symmetry transformation, we should find that

$$\oint d^4x (\delta\mathcal{L}) = 0.$$

In particular, [same local gauge for later] in order to satisfy this condition for arbitrary parameter function $a(x)$, we conclude that

$$\frac{\partial[\delta\mathcal{L}]}{\partial a} = \partial_\mu \left(\frac{\partial[\delta\mathcal{L}]}{\partial[\partial_\mu a]} \right) \quad \text{"Symmetry Equation"}$$

Pause: obviously this proof is close to the derivation of the Euler equation itself. Here we say,

$$\delta S = 0 = \int d^4x \delta\mathcal{L} = \int d^4x \left[\left(\frac{\partial\mathcal{L}}{\partial\phi} \right) - \partial_\mu \left(\frac{\partial\mathcal{L}}{\partial[\partial_\mu\phi]} \right) \right] \delta\phi(x)$$

The Euler + Symmetry equations are similar, but do not mix them up. In particular, we do not make use of the Euler in deriving the Sym. Eq.

Globality: usually our symmetry is global, ie, we have the symmetry when $a(x) = \text{independent of } x$. In this case we may say

$$\delta\mathcal{L}|^{\text{GLO}} = \frac{\partial[\delta\mathcal{L}]}{\partial a} \cdot a$$

and then
$$\int d^4x \delta\mathcal{L}|^{\text{GLO}} = \int d^4x \left(\frac{\partial[\delta\mathcal{L}]}{\partial a} \right) a = 0$$

Thus, it must in this case be true that the quantity $\frac{\partial[\delta\mathcal{L}]}{\partial a}$ at worst a 4-divergence [or it may vanish].

ie, it must be possible to write:
$$\frac{\partial[\delta\mathcal{L}]}{\partial a} = \partial_\mu \Lambda^\mu$$

Therefore, using our earlier expression for $\partial[\delta\mathcal{L}]/\partial[\partial_\mu\phi]$, we find that the Sym. Equ. states:

$$\partial_\mu \Lambda^\mu = \partial_\mu [\pi^\mu \vec{\sigma}\phi]$$

Thus, we arrive at our conserved current:

$$J_\mu = \pi_\mu \vec{\sigma}\phi - \Lambda_\mu$$

$$\partial^\mu J_\mu = 0$$

where $\delta\phi = a \cdot \vec{\sigma}\phi$ defines operator $\vec{\sigma}$

$\frac{\partial[\delta\mathcal{L}]}{\partial a} = \partial_\mu \Lambda^\mu$ defines object Λ^μ

Comment: by the way, I have been using symbol ∂_μ as a total derivative, perhaps should call it D_μ or d_μ . I.e., ($\partial_\mu \mathcal{L} = 0$ for a partial, but $D_\mu \mathcal{L} \neq 0$ for total)

Comment: since it is merely a notation, you might abbreviate

$$\frac{\partial[\delta\mathcal{L}]}{\partial a} = \frac{\partial\mathcal{L}}{\partial a} \text{ etc}$$

where this would have to mean $\frac{\partial\mathcal{L}(\phi, \partial_\mu\phi)}{\partial a} = \overbrace{\frac{\partial\mathcal{L}(\phi, \partial_\mu\phi)}{\partial a}}^{=0} + \frac{\partial[\delta\mathcal{L}]}{\partial a}$.
I find my notation less confusing.

It is in fact the variation $\delta\mathcal{L}$ that is a function of parameter a .

Translation Application

$$\delta\phi_i = \phi'_i(x) - \phi_i(x) = g_\mu \partial^\mu \phi_i(x) \quad \text{so} \quad \vec{\sigma} = \partial^\mu.$$

$$\begin{aligned} \frac{\partial [\delta\mathcal{L}]}{\partial a^\mu} &= \frac{\partial [\mathcal{L}'(x) - \mathcal{L}(x)]}{\partial a^\mu} = \left[\frac{\partial \mathcal{L}}{\partial x_\nu} \delta x^\nu \right] = \frac{\partial}{\partial a^\mu} \left[\left(\frac{\partial \mathcal{L}}{\partial x_\nu} \right) a^\nu \right] = \frac{\partial \mathcal{L}}{\partial x_\mu} \\ &= \partial_\mu \mathcal{L}. \end{aligned}$$

$$\text{So, } \frac{\partial [\delta\mathcal{L}]}{\partial a^\nu} = \partial_\nu \mathcal{L} = \partial_\mu [g_{\mu\nu} \mathcal{L}] = \partial^\mu [g_{\mu\nu} \mathcal{L}]$$

$$\text{Thus, } \Lambda_\mu = g_{\mu\nu} \mathcal{L}.$$

Current is then:

$$J_\mu^{(\nu)} = \pi_\mu (\partial^\nu \phi) - g_{\mu}^{\nu} \mathcal{L}$$

Usually this is written

$$T_{\mu\nu} = \pi_\mu (\partial_\nu \phi) - g_{\mu\nu} \mathcal{L} = \text{canonical stress tensor.}$$

↑
not symmetric.

$$\partial^\mu T_{\mu\nu} = 0$$

As usual, the conserved "charges" are

$$\int d^3x T_{0\nu} = P_\nu = \text{the translation generators.}$$

$$\left(\frac{dP_\nu}{dt} = 0 \right)$$

Footnote: $\delta\phi_i(x) = \phi'_i(x) - \phi_i(x) = e^{-iq_\mu \tilde{P}^\mu} \phi_i(x) - \phi_i(x) = e^{+q_\mu \partial^\mu} \phi_i(x) - \phi_i(x)$
 $= \phi_i(x+a) - \phi_i(x) = g_\mu \partial^\mu \phi_i(x)$
 $\delta\mathcal{L}(x) = g_\mu \partial^\mu \mathcal{L}(x)$ in same way. ✓

The $M^{\mu\nu}$ Case

5

$$\delta\phi = \phi'_i(x) - \phi_i(x) = [e^{\ominus \tilde{M}^{\mu\nu}}]_{ij} \phi_j(x) - \phi_j(x) = \ominus [\tilde{M}^{\mu\nu}]_{ij} \phi_j(x)$$

Note that this is a Type ② transformation. We identify:

$$\vec{O} = [\tilde{M}^{\mu\nu}]_{ij} = \delta_{ij} \tilde{M}^{\mu\nu} + [M^{\mu\nu}]_{ij} = \delta_{ij} (x^\mu \partial^\nu - x^\nu \partial^\mu) + \sum_{ij}^{\mu\nu}$$

$$a = \ominus$$

Now to find Λ^λ we do as follows

$$\mathcal{L}'(x) - \mathcal{L}(x) = \delta\mathcal{L} = [e^{\ominus \tilde{M}^{\mu\nu}}] \mathcal{L}(x) - \mathcal{L}(x) = \ominus \tilde{M}^{\mu\nu} \mathcal{L}$$

$$\begin{aligned} \text{Thus } \frac{\partial[\delta\mathcal{L}]}{\partial a} &= \tilde{M}^{\mu\nu} \mathcal{L} = (x^\mu \partial^\nu - x^\nu \partial^\mu) \mathcal{L} \\ &= \partial_\lambda \underbrace{[(x^\mu g^{\nu\lambda} - x^\nu g^{\mu\lambda}) \mathcal{L}]}_{\equiv \Lambda^\lambda} \end{aligned}$$

Note extra term is $(g_\lambda^\mu g^{\nu\lambda} - g_\lambda^\nu g^{\mu\lambda}) \mathcal{L} = 0$

So, conserved current is

$$\begin{aligned} J^\lambda &= \pi^\lambda \vec{O} \phi - \Lambda^\lambda = \pi_i^\lambda [\tilde{M}^{\mu\nu}]_{ij} \phi_j - (x^\mu g^{\nu\lambda} - x^\nu g^{\mu\lambda}) \mathcal{L} \\ &= \pi_i^\lambda \sum_{ij}^{\mu\nu} \phi_j + \underbrace{\pi_i^\lambda (x^\mu \partial^\nu - x^\nu \partial^\mu) \phi_i - (x^\mu g^{\nu\lambda} - x^\nu g^{\mu\lambda}) \mathcal{L}}_{\equiv T^{\lambda\nu}} \\ &= \end{aligned}$$

So

$$\begin{aligned} J_{(\mu\nu)}^\lambda &= \pi_i^\lambda \sum_{ij}^{\mu\nu} \phi_j + (x^\mu T^{\lambda\nu} - x^\nu T^{\lambda\mu}) \equiv M^{\lambda\mu\nu} \\ \partial_\lambda M^{\lambda\mu\nu} &= 0 \end{aligned}$$

Conserved charges are

$$\int d^3x M^{0\mu\nu} = M^{\mu\nu}$$

$$\frac{dM^{\mu\nu}}{dt} = 0$$

... The Dilation Case

$$\delta\phi = \phi'(x) - \phi(x) = e^{da} \phi(e^a x) - \phi(x) \approx (1+da)\phi(x+ax) - \phi(x)$$

$$\approx da \phi(x) + a x_\mu \partial^\mu \phi \approx a(d + x_\mu \partial^\mu) \phi(x)$$

$$\text{Thus, } \vec{J} = [d + x_\mu \partial^\mu]$$

$$\theta = a$$

Now to find Λ^1 . [Wait. Are we sure this is a symmetry?

$$\mathcal{L}(\phi, \partial_\mu \phi) = m^2 \phi^2, \text{ say } = \mathcal{L}(\phi)$$

$$\mathcal{L}'(\phi) = \mathcal{L}(\phi') = m^2 [e^{da} \phi(e^a x)]^2$$

$$\text{Then } S' = \int d^4x [m^2 e^{2da} \phi^2(e^a x)]. \text{ Let } y^\mu = e^a x^\mu \Rightarrow d^4y = e^{4a} d^4x.$$

Then $S' = \int e^{-4a} d^4y m^2 e^{2da} \phi^2(y)$. With $d=1$, this would not be a scale invariant action! But clearly $\lambda\phi^4$ is. What about

$$\int d^4x \partial_\mu \phi \partial^\mu \phi = \int d^4x \partial_\mu [e^a \phi(e^a x)] \partial^\mu [e^a \phi(e^a x)]$$

$$\text{Let } y^\mu = e^a x^\mu \Rightarrow \partial_\mu = \frac{\partial}{\partial x^\mu} = e^a \frac{\partial}{\partial y^\mu} = e^a \partial_\mu^{(y)}$$

$$\Rightarrow S' = \int (e^{-4a} d^4y) \cdot e^{2a} \cdot e^{2a} \partial_\mu^{(y)} \phi(y) \partial^\mu \phi(y) = S \checkmark.$$

So, a lagrangian like $\mathcal{L}(\phi, \partial_\mu \phi) = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \lambda \phi^4$ is scale invariant.]

Now back to Λ^1 :

$$\delta\mathcal{L} = \mathcal{L}'(x) - \mathcal{L}(x) = a[d + x \cdot \nabla] \mathcal{L} \Rightarrow \frac{\partial[\delta\mathcal{L}]}{\partial a} = [4d + x_\mu \partial^\mu] \mathcal{L}$$

Probably you choose d such that this $\nearrow$ is a 4. So for

ϕ we choose $d=1$. In general,

Then $\Lambda^1 = x^\lambda \mathcal{L}$ and

$$\frac{\partial[\delta\mathcal{L}]}{\partial a} = [4 + x_\mu \partial^\mu] \mathcal{L} = 2_\lambda [x^\lambda \mathcal{L}]$$

$\nwarrow$ dimension of $\mathcal{L}$

$$J^\lambda = \pi^\lambda [d + x \cdot \nabla] \phi - x^\lambda \mathcal{L} = \text{conserved dilation current.}$$

For several fields you would get

$$J^\lambda = \pi_i^\lambda (d_i + x_\mu \partial^\mu) \phi_i - x^\lambda \mathcal{L} = \pi_i^\lambda d_i \phi_i + x_\mu T^{\lambda\mu} = J_{\text{dilatation}}^\lambda$$

$$D = \int d^3x J_0 = \text{the "charge" = scale generator.}$$

Feynman Graph Combinatorics

① Parameters of a Feynman graph in a ϕ^n theory are:

$n = \# \text{ fields at vertex}$

$I = \# \text{ internal lines}$

$E = \# \text{ external lines}$

$L = \# \text{ loops}$

$V = \# \text{ vertices.}$

How are these quantities related?

Theorem 1: for a "connected" Feynman graph [in sense of  ruled out], this is true: $(I - V) + 1 = L$

Proof: start with V vertices. . If connected, there must be at least $(V-1)$ internal lines. If $I = V-1$, then you have a tree graph, no loops. Each added internal line forces an extra loop.

Corollary: for tree graphs, $I = V-1$.

Theorem 2: for any Feynman graph in ϕ^n , $nV = 2I + E$.

Proof: conservation of lines!

Theorem 3: in a theory with one interaction Lagrangian that involves several fields, we have:

$$L = (\sum I_i) - V + 1$$

Proof: same as above, applies only to connected graphs.

Methods of Putting in an UV cutoff to get a finite integral.

1. Consider an integral of the following form where there is some parameter inside:

$$I(a) = \int_0^{\infty} \frac{dx}{x+a}$$

It is obvious from inspection that this thing is logarithmically divergent. Here are three ways of putting in a cutoff:

a) just stop the integral at a finite value:

$$I(a) \rightarrow I(a, \Lambda) = \int_0^{\Lambda} \frac{dx}{x+a} = \ln(x+a) \Big|_0^{\Lambda} = \ln\left(\frac{\Lambda+a}{a}\right)$$

b) do a subtraction:

$$\begin{aligned} I(a) \rightarrow I(a) - I(\Lambda) &= \int_0^{\infty} dx \left[\frac{1}{x+a} - \frac{1}{x+\Lambda} \right] \\ &= \ln\left(\frac{\infty+a}{a}\right) - \ln\left(\frac{\infty+\Lambda}{\Lambda}\right) = \ln\left(\frac{\Lambda}{a}\right). \end{aligned}$$

By subtracting the integrand at some point (the subtraction point) you get a new integral which is convergent, although here I evaluated the thing by shortcut.

c) add a multiplying factor:

$$\begin{aligned} I(a) \rightarrow I(a, \Lambda) &= \int_0^{\infty} \frac{dx}{x+a} \cdot \left(\frac{\Lambda}{x+\Lambda}\right) = \underbrace{\left(\frac{\Lambda}{\Lambda-a}\right)}_{\rightarrow 1} \underbrace{\int_0^{\infty} dx \left(\frac{1}{x+a} - \frac{1}{x+\Lambda}\right)}_{\ln\left(\frac{\Lambda}{a}\right)} \\ \frac{1}{x+a} \cdot \frac{1}{x+\Lambda} &= \left(\frac{1}{x+a} - \frac{1}{x+\Lambda}\right) \frac{1}{\Lambda-a} \end{aligned}$$

2. Comment: for finite LAMBDA each of these methods gives a slightly different result. But for large LAMBDA all results are the same, basically $\ln(\Lambda/a)$, which shows up the logarithmic divergence. You can use any method you want:

d) change the number of dimensions of spacetime, I don't know how to do this now.

The Minimal Substitution and Feynman rules for various kinds of QED.

1. First, calibrate things by doing regular QED a la BD. Here is the lagrangina and the minimal substitution:

$$\mathcal{L}_0 = + \bar{\Psi} (i \not{\partial} - m) \Psi \quad \Rightarrow \quad \begin{aligned} i \partial_\mu &\rightarrow i \partial_\mu - e_0 A_\mu \\ p_\mu &\rightarrow p_\mu - e_0 A_\mu \quad \text{if } e^{-i p \cdot x} \end{aligned}$$

Thus, $\mathcal{L}_I = - e_0 \bar{\Psi} \not{A} \Psi$ / as in (15.5).

Now get the feynman rule for a QED vertex byb considering a simple S-matrix element (even though kinematically blocked):

$$\begin{aligned} \text{Diagram: } \text{incoming } p, \text{ outgoing } p', \text{ vertex } &= \langle p', k | S - 1 | p \rangle \approx i \langle p', k | \underbrace{-e_0 \bar{\Psi} \not{A} \Psi}_{\bar{u}(p')} | p \rangle \\ &= \bar{u}(p') \underbrace{[-i e_0 \gamma_\mu]}_{\epsilon^\mu} u(p) \end{aligned}$$

↳ this is the Feynman Rule for the QED vertex.

2. Now redo this for a charged scalar field. First, notice that althought there is a $\frac{1}{2}$ in the lagrangian for a neutral scalar field as in (12.2), there is none in the charged case because you added two Klein Gordon fields, see (12.54). Also, you must keep the daggar also on d_u to get the right result:

$$\mathcal{L}_0 = (\partial_\mu \phi)^\dagger (\partial^\mu \phi) - m^2 \phi^\dagger \phi \quad \partial_\mu \rightarrow \partial_\mu + i e_0 A_\mu$$

$$\mathcal{L}_I = (i e_0 A_\mu \phi)^\dagger \partial^\mu \phi + (\partial_\mu \phi)^\dagger i e_0 A^\mu \phi = i e_0 A^\mu [(\partial_\mu \phi)^\dagger \phi - \phi^\dagger (\partial_\mu \phi)]$$

$$\begin{aligned} \text{Diagram: } \text{incoming } p, \text{ outgoing } p', \text{ vertex } &= i \langle p', k | i e_0 A^\mu \{ \underbrace{(\partial_\mu \phi)^\dagger}_{(-i p'_\mu)^*} \phi - \phi^\dagger \underbrace{(\partial_\mu \phi)}_{(-i p_\mu)} \} | p \rangle \\ &= i (i e_0) \epsilon^\mu (i p'_\mu + i p_\mu) = [-i e_0 (p_\mu + p'_\mu)] \epsilon^\mu \end{aligned}$$

↳ Feynman Rule for scalar QED vertex.

About the symmetric Belinfante stress tensor.

① First, let us recall these basic Noether results:

$$T^{\mu\nu} \equiv \Pi^\mu \partial^\nu \phi - g^{\mu\nu} \mathcal{L} \quad (1-1) \quad \text{"canonical stress tensor"} \quad \partial_\mu T^{\mu\nu} = 0 \quad (1-2)$$

$$M^{\lambda\mu\nu} \equiv (x^\mu T^{\lambda\nu} - x^\nu T^{\lambda\mu}) + \Pi^\lambda \Sigma^{\mu\nu} \phi \quad (1-3) \quad \partial_\lambda M^{\lambda\mu\nu} = 0 \quad (1-4)$$

"angular momentum current"

$$P^\nu = \int d^3x T^{0\nu}$$

$$M^{\mu\nu} = \int d^3x M^{0\mu\nu}$$

② Now define: $F^{\lambda\mu\nu} \equiv (\Pi^\lambda \Sigma^{\mu\nu} \phi - \Pi^\mu \Sigma^{\lambda\nu} \phi - \Pi^\nu \Sigma^{\lambda\mu} \phi) \frac{1}{2} \quad (2-1)$

By the way, $\Pi^\lambda \Sigma^{\mu\nu} \phi$ means $\Pi^\lambda_i \Sigma^{\mu\nu}_{ij} \phi_j$ where $\Pi^\lambda_i = \partial \mathcal{L} / \partial [\partial_\lambda \phi_i]$.

③ Here are some other facts we shall need:

$$\partial_\lambda \Pi^\lambda = \partial \mathcal{L} / \partial \phi, \quad \text{the Euler equation}$$

④ $(\frac{\partial \mathcal{L}}{\partial \phi}) \Sigma^{\mu\nu} \phi + \Pi_\alpha \Sigma^{\mu\nu} (\partial^\alpha \phi) = \Pi^\nu (\partial^\mu \phi) - \Pi^\mu (\partial^\nu \phi), \quad \text{Lorentz invariance}$

See CT notes for derivation of this last. We can combine them to get

$$\partial_\lambda [\Pi^\lambda \Sigma^{\mu\nu} \phi] = \Pi^\nu (\partial^\mu \phi) - \Pi^\mu (\partial^\nu \phi) \quad (3-1)$$

⑤ Now define a new stress tensor by:

$$\Theta^{\mu\nu} \equiv T^{\mu\nu} + \partial_\lambda F^{\lambda\mu\nu} \quad (4-1)$$

⑥ Fact: $F^{\lambda\mu\nu} = -F^{\mu\lambda\nu}$, i.e., $F^{\lambda\mu\nu} = F^{\hat{\lambda}\hat{\mu}\hat{\nu}}$. Note, however, that F is not a totally antisym tensor!

⑦ Theorem: $\partial_\mu \Theta^{\mu\nu} = 0$.

Proof: true because $\partial_\mu T^{\mu\nu} = 0$ and $\partial_\mu F^{\lambda\mu\nu} = 0$ by symmetry.

⑧ Theorem: $\int d^3x \Theta^{0\nu} = \int d^3x T^{0\nu} = P^\nu$.

Proof: $\int d^3x \partial_\lambda F^{\lambda 0\nu} = \int d^3x \partial_0 F^{00\nu} = \int d^3x \partial_0 (0) = 0$

⑧ Theorem: $\Theta^{\mu\nu}$ is symmetric.

Proof: construct it!

$$\Theta^{\mu\nu} = T^{\mu\nu} + \partial_\lambda F^{\lambda\mu\nu} = \pi^\mu \partial^\nu \phi - g^{\mu\nu} \mathcal{L} + 2\lambda [\pi^\lambda \Sigma^{\mu\nu} \phi] \frac{1}{2} \\ + 2\lambda [\pi^\mu \Sigma^{\lambda\nu} \phi + \pi^\nu \Sigma^{\lambda\mu} \phi] \left(-\frac{1}{2}\right)$$

Use equation (3-1) to get

$$\Theta^{\mu\nu} = \frac{1}{2} [\pi^\mu (\partial^\nu \phi) - \partial_\lambda (\pi^\mu \Sigma^{\lambda\nu} \phi) - g^{\mu\nu} \mathcal{L}] + (\mu \leftrightarrow \nu)$$

Regroup to get

$$\Theta^{\mu\nu} = \frac{1}{2} [T^{\mu\nu} - \partial_\lambda (\pi^\mu \Sigma^{\lambda\nu} \phi)] + (\mu \leftrightarrow \nu) \quad (8-1)$$

Thus $\Theta^{\mu\nu}$ is symmetric, and of course then $\partial_\lambda \Theta^{\mu\nu} = 0$ too.

⑨ Define: $\tilde{M}^{\lambda\mu\nu} = x^\mu \Theta^{\lambda\nu} - x^\nu \Theta^{\lambda\mu} \quad (9-1)$

We can use Eq. (8-1) and definition (1-3) to show

$$\tilde{M}^{\lambda\mu\nu} = M^{\lambda\mu\nu} + [(x^\mu \partial_\alpha [F^{\alpha\nu}] - x^\nu \partial_\alpha [F^{\alpha\mu}]) - \pi^\lambda \Sigma^{\mu\nu} \phi] \quad (9-2)$$

⑩ Theorem: $\partial_\lambda \tilde{M}^{\lambda\mu\nu} = 0$.

Proof: obvious from def (9-1): $= g_{\lambda}^{\mu} \Theta^{\lambda\nu} - g_{\lambda}^{\nu} \Theta^{\lambda\mu} = \Theta^{\mu\nu} - \Theta^{\nu\mu} = 0$.
 It follows from symmetry and conservation of $\Theta^{\mu\nu}$.

⑪ Theorem: $\int d^3x \tilde{M}^{0\mu\nu} = \int d^3x M^{0\mu\nu} = M^{\mu\nu}$.

Proof: look at (9-2).

$$\int d^3x (x^\mu \partial_\alpha F^{\alpha 0\nu} - x^\nu \partial_\alpha F^{\alpha 0\mu}) = - \int d^3x (g_{\alpha}^{\mu} F^{\alpha 0\nu} - g_{\alpha}^{\nu} F^{\alpha 0\mu}) \\ = - \int d^3x (F^{\mu 0\nu} - F^{\nu 0\mu})$$

Here I could do parts with d^3x because $F^{\Delta 00\nu} = 0$. Now look at def (2-1) to get

$$F^{\mu 0\nu} - F^{\nu 0\mu} = \text{twice the antisym part of } F^{\mu\nu} = -\pi^0 \Sigma^{\mu\nu} \phi.$$

Thus,

$$-\int d^3x (F^{\mu\nu} - F^{\nu\mu}) = \int d^3x [\pi^0 \Sigma^{\mu\nu} \phi]$$

and this cancels the last term of $\tilde{M}^{\mu\nu}$ shown in (9-2). QED.

(12) References: { Coleman Jackiw 1971
CCJ
Belinfante }.

(13) Summary:

The linear momentum current $T^{\mu\nu}$ (nonsymmetric) and the angular momentum current $M^{\lambda\mu\nu}$,

$$\begin{aligned} T^{\mu\nu} &= \pi^\mu \partial^\nu \phi - g^{\mu\nu} \mathcal{L} \\ M^{\lambda\mu\nu} &= (x^\mu T^{\lambda\nu} - x^\nu T^{\lambda\mu}) + \pi^\lambda \Sigma^{\mu\nu} \phi \end{aligned} \quad \begin{aligned} \partial_\mu T^{\mu\nu} &= 0 \\ \partial_\lambda M^{\lambda\mu\nu} &= 0, \end{aligned}$$

may be replaced by new objects which are "equivalent" in that they yield the same conserved charges P^ν and $M^{\mu\nu}$. New guys are

$$\left\{ \begin{aligned} \Theta^{\mu\nu} &= \frac{1}{2}(T^{\mu\nu} + T^{\nu\mu}) - \frac{1}{2} \partial_\lambda (\pi^\lambda \Sigma^{\lambda\mu\nu} \phi + \pi^\nu \Sigma^{\lambda\mu\lambda} \phi) \\ \Theta^{\mu\nu} &= \Theta^{\nu\mu} \quad \partial_\mu \Theta^{\mu\nu} = 0 \quad \int d^3x \Theta^{0\nu} = \int d^3x T^{0\nu} = P^\nu \\ \Theta^{\mu\nu} &= T^{\mu\nu} + \partial_\lambda F^{\lambda\mu\nu} \quad \text{where } F^{\lambda\mu\nu} = \frac{1}{2} (\pi^\lambda \Sigma^{\mu\nu} \phi - \pi^\mu \Sigma^{\lambda\nu} \phi - \pi^\nu \Sigma^{\lambda\mu} \phi) \\ \tilde{M}^{\lambda\mu\nu} &= x^\mu \Theta^{\lambda\nu} - x^\nu \Theta^{\lambda\mu} \quad \partial_\lambda \tilde{M}^{\lambda\mu\nu} = 0 \\ \int d^3x \tilde{M}^{0\mu\nu} &= \int d^3x M^{0\mu\nu} = M^{\mu\nu}. \end{aligned} \right.$$

The new objects are of course also conserved in the same way. They are nice in that $\Theta^{\mu\nu}$ is symmetric.

How to define the Lorentz and Translation Generators.

1. First, let's get straight the notion of using a coordinate transformation R to define a transformation in the Q.M. Hilbert Space. You define $\mathbb{R}$ in terms of R in this way:

$$\langle x | \mathbb{R} | \phi \rangle \equiv \langle R^{-1} x | \phi \rangle$$

You could, if you wanted, make an alternative definition by using R instead of R^{-1} . BUT, if R in addition to being a transformation is also representing some group, then you must use R^{-1} so that the operators $\mathbb{R}$ also represent the group, ie, you want $R_1 R_2 = R_3$ to be reflected by $\mathbb{R}_1 \mathbb{R}_2 = \mathbb{R}_3$ for the operators. Notice, by the way, that the above setup is just the usual "regular representation":

$$[\mathbb{R} \phi](x) = \phi(R^{-1} x)$$

2. So here is how you do angular momentum. Define $\mathbb{M}^{\mu\nu}$ like so:

$$\langle x | e^{-i \Theta \mathbb{M}^{\mu\nu}} | \phi \rangle \equiv \langle (e^{+i \Theta \mathbb{M}^{\mu\nu}})^\alpha{}_\beta x^\beta | \phi \rangle$$

Here both M and $\mathbb{M}$ will satisfy the same Lie Algebra. (If you were to use opposite sign that would not be true). Go to infinitesimal to learn that: (note: $M = M_{BD}$)

$$\langle x | \mathbb{M}^{\mu\nu} | \phi \rangle = [M^{\mu\nu}]^\alpha{}_\beta x^\alpha \partial^\beta \langle x | \phi \rangle$$

$$\text{But } = i [\delta^\mu{}_\alpha \delta^\nu{}_\beta - \delta^\mu{}_\beta \delta^\nu{}_\alpha] x^\alpha \partial^\beta \phi(x) = i (x^\mu \partial^\nu - x^\nu \partial^\mu) \phi(x) \equiv \tilde{M}^{\mu\nu} \phi(x)$$

3. Since the translation group is abelian and $t_1 t_2 = t_2 t_1$, we don't have to worry about the distinction made above. This means that the sign of operator P is just ~~an~~ a convention, so I will do it the way I have always done it and in a way that duplicates classical definition of the relation between angular and linear momentum.

So:

$$\langle x | e^{-i a_\mu P^\mu} | \phi \rangle \equiv \langle x + a | \phi \rangle \quad \text{or} \quad \langle x | \mathbb{P} | \phi \rangle = \langle T x | \phi \rangle$$

$$\Rightarrow \langle x | P^\mu | \phi \rangle = i \partial^\mu \phi(x) \equiv \tilde{P}^\mu \phi(x).$$

$$\text{Then: } \tilde{M}^{\mu\nu} = x^\mu \tilde{P}^\nu - x^\nu \tilde{P}^\mu, \text{ eg, } \tilde{M}^{12} = x^1 \tilde{P}^2 - x^2 \tilde{P}^1 = "L_3"$$

4. Summary of results:

$$\langle x | P^\mu | \phi \rangle = \tilde{P}^\mu \langle x | \phi \rangle \quad \text{where} \quad \tilde{P}^\mu \equiv i \partial^\mu$$

$$\langle x | \mathbb{M}^{\mu\nu} | \phi \rangle = \tilde{M}^{\mu\nu} \langle x | \phi \rangle \quad \text{where} \quad \tilde{M}^{\mu\nu} \equiv i (x^\mu \partial^\nu - x^\nu \partial^\mu)$$

Generalization to include spin.

(M80)

- ① Has write $|\phi\rangle = |\phi_c\rangle \otimes |\phi_s\rangle$ where $c = \text{configuration space}$
 $s = \text{spin space}.$

ie, $\mathcal{H}^{\text{SM}} = \mathcal{H}^c \otimes \mathcal{H}^s$

$\{|x\rangle\} = \text{basis for } \mathcal{H}^c$

$\{|e\rangle\} = \text{basis for } \mathcal{H}^s$

- ② A wavefunction with spin transforms like so:

$$\phi'_i(x) = [R\phi](x) = R_{ij} \phi_j(\tilde{R}^{-1}x)$$

Note: $[R_1 R_2 \phi](x) = R_{ij}^{(1)} \phi_j [R_2 \phi]_j(R_1^{-1}x) = R_{ij}^{(1)} \{ R_{jk}^{(2)} \phi_k(\tilde{R}_2^{-1} \tilde{R}_1^{-1}x) \} = (R^{(12)})_{ij} \phi_j(\tilde{R}_{12}^{-1}x).$

We can write this as

$$\langle x, e_i | \phi'_c, \phi'_s \rangle = R_{ij} \langle \tilde{R}^{-1}x, e_j | \phi_c, \phi_s \rangle \Rightarrow \langle x | \phi'_c \rangle \langle e_i | \phi'_s \rangle = \langle \tilde{R}^{-1}x | \phi_c \rangle R_{ij} \langle e_j | \phi_s \rangle$$

ie. $\langle x | \phi'_c \rangle = \langle \tilde{R}^{-1}x | \phi_c \rangle$ and $\langle e_i | \phi'_s \rangle = R_{ij} \langle e_j | \phi_s \rangle.$

So, R_{ij} is some FDR of some group, ie, of the Lorentz group (need not be irreducible, eg, might be the 4x4 Dirac spinor rep), and R_{ij} acts only in the spin space. Note that the use of R_{ij} and not R_{ij}^{-1} is correct in the consistency sense mentioned on earlier page. Ie, now we have three different representations of the Lorentz group acting in different spaces.

3. Now we can define $M^{\mu\nu}$ when spin is present:

$$\langle x, e_i | \bar{e}^{+i\theta M^{\mu\nu}} | \phi_c, \phi_s \rangle = [\bar{e}^{+i\theta M^{\mu\nu}}]_{ij} \langle (e^{+i\theta M^{\mu\nu}})^{\mu}_{\nu} x^{\mu}, e_j | \phi_c, \phi_s \rangle$$

Can use the star notation to say

$$\begin{aligned} \mathbb{L} &= \mathbb{L}^c \otimes \mathbb{L}^s = e^{-i\theta M_c^{\mu\nu}} \otimes e^{-i\theta M_s^{\mu\nu}} \\ &= e^{-i\theta [M_c^{\mu\nu} \oplus M_s^{\mu\nu}]} = e^{-i\theta M^{\mu\nu}} \text{ where } A \oplus B = A \otimes 1 + 1 \otimes B \end{aligned}$$

Then

$$\begin{aligned} \langle x, e_i | \mathbb{L}^c \otimes \mathbb{L}^s | \phi_c, \phi_s \rangle &= \langle x | \mathbb{L}^c | \phi_c \rangle \langle e_i | \mathbb{L}^s | \phi_s \rangle \\ &= \langle e^{+i\theta M^{\mu\nu}} x | \phi_c \rangle \cdot [e^{-i\theta M^{\mu\nu}}]_{ij} \langle e_j | \phi_s \rangle \end{aligned}$$

Thus we have:

$$\begin{aligned} \langle x | e^{-i\theta M_c^{\mu\nu}} | \phi_c \rangle &= \langle (e^{+i\theta M^{\mu\nu}})^\alpha_\beta x^\beta | \phi_c \rangle \Rightarrow \langle x | M_c^{\mu\nu} | \phi_c \rangle = \tilde{M}^{\mu\nu} \langle x | \phi_c \rangle \\ \langle e_i | e^{-i\theta M_s^{\mu\nu}} | \phi_s \rangle &= [e^{-i\theta M^{\mu\nu}}]_{ij} \langle e_j | \phi_s \rangle \Rightarrow \langle e_i | M_s^{\mu\nu} | \phi_s \rangle = [M^{\mu\nu}]_{ij} \langle e_j | \phi_s \rangle \end{aligned}$$

Corollaries:

$$\langle x | M_c^{\mu\nu} | x' \rangle = \tilde{M}^{\mu\nu}_{(x)} \delta^4(x-x')$$

$$\langle e_i | M_s^{\mu\nu} | e_j \rangle = [M^{\mu\nu}]_{ij}$$

Finally:

$$\begin{aligned} \langle x, e_i | M^{\mu\nu} | \phi_c, \phi_s \rangle &= \langle x | e_i | M_c^{\mu\nu} \otimes 1 + 1 \otimes M_s^{\mu\nu} | \phi_c, \phi_s \rangle \\ &= \tilde{M}^{\mu\nu} \langle x | \phi_c \rangle \langle e_i | \phi_s \rangle + \langle x | \phi_c \rangle M^{\mu\nu}_{ij} \langle e_i | \phi_s \rangle \\ &= \left\{ \tilde{M}^{\mu\nu} \delta_{ij} + [M^{\mu\nu}]_{ij} \right\} \langle x | \phi_c \rangle \langle e_j | \phi_s \rangle \\ &= [\tilde{M}^{\mu\nu}]_{ij} \langle x, e_j | \phi_c, \phi_s \rangle \end{aligned}$$

Thus:

$$\langle x, e_i | M^{\mu\nu} | \phi \rangle = [\tilde{M}^{\mu\nu}]_{ij} \phi_j(x)$$

$$\text{where } [\tilde{M}^{\mu\nu}]_{ij} \equiv \tilde{M}^{\mu\nu} \delta_{ij} + [M^{\mu\nu}]_{ij}$$

Wavefunction Variations:

$$\delta\phi_i(x) \equiv \phi'_i(x) - \phi(x)$$

For $M^{\mu\nu}$ we get

$$\delta\phi_i(x) = [e^{-i\theta M^{\mu\nu}} \phi]_i(x) - \phi(x) \approx -i\theta \langle x, i | M^{\mu\nu} | \phi \rangle = -i\theta [\tilde{M}^{\mu\nu}]_{ij} \phi_j(x)$$

For P^μ we get

$$\delta\phi_i(x) \approx -i a_\mu \langle x, i | P^\mu | \phi \rangle = (-i a_\mu) (i \partial^\mu) \phi_i(x) = + a_\mu \partial^\mu \phi_i(x)$$

A Lagrangian would go like a spinless ϕ under $M^{\mu\nu}$.

$$\delta\mathcal{L}(x) = -i\theta \tilde{M}^{\mu\nu} \mathcal{L}(x)$$

Introduction of Fields. Global, comrels, variations, and $\mathcal{Z}^{\mu\nu}$.

1. We have already introduced the object R_{ij} as a spin matrix, Rx as a transformation on x , and R as an operator in the QM Hilbert Space. We now define a "field with spin" as follows:

$$\phi'_i(x) = \tilde{R}^{-1} \phi_i(x) R = R_{ij} \phi_j(\tilde{R}x)$$

You have to do it just this way. If, eg, you were to switch the R and R^{-1} which surround the field, you would find a group multiplication inconsistency. Notice that if you take the VEV of this field, thus getting the classical field, you would find a classical field that transforms consistently with our previous wavefunction transformations.

2. For linear and angular momentum, then, we have shown that:

$$\begin{cases} e^{+i\Theta M^{\mu\nu}} \phi_i(x) e^{-i\Theta M^{\mu\nu}} = [e^{-i\Theta M^{\mu\nu}}]_{ij} \phi_j((e^{+i\Theta M^{\mu\nu}})^{\alpha}_{\beta} x^{\beta}) \\ e^{+i q_{\mu} P^{\mu}} \phi_i(x) e^{-i q_{\mu} P^{\mu}} = \phi_i(x+a) \end{cases}$$

3. Now put in small parameters to get the commutation relations:

$$\begin{cases} i\Theta [M^{\mu\nu}, \phi_i(x)] = \delta \phi_i(x) = -i\Theta [\tilde{M}^{\mu\nu}]_{ij} \phi_j(x) \\ i q_{\mu} [P^{\mu}, \phi_i(x)] = \delta \phi_i(x) = -i q_{\mu} \tilde{P}^{\mu} \phi_i(x) \end{cases}$$

$$\Rightarrow \begin{cases} [\phi_i(x), M^{\mu\nu}] = [\tilde{M}^{\mu\nu}]_{ij} \phi_j(x) = \{[x^{\mu} \partial^{\nu} - x^{\nu} \partial^{\mu}] \delta_{ij} + M^{\mu\nu}_{ij}\} \phi_j(x) \\ [\phi_i(x), P^{\mu}] = \tilde{P}^{\mu} \phi_i(x) = i \partial^{\mu} \phi_i(x). \end{cases}$$

4. Examine BD (11.73) to learn:

$$\boxed{\Sigma^{\mu\nu}_{ij} = -i [M^{\mu\nu}]_{ij}}$$

$$\text{Then: } ([\phi_i(x), M^{\mu\nu}] = i ([x^{\mu} \partial^{\nu} - x^{\nu} \partial^{\mu}] \delta_{ij} + \Sigma^{\mu\nu}_{ij}) \phi_j(x)$$

Comment: for a free field we can make this expansion:

$$\phi(x) = \sum_{p,n} \left\{ \langle x|p\rangle \langle e|n\rangle \phi(p,n) + \langle p|x\rangle \langle n|e\rangle \phi^\dagger(p,n) \right\}$$

where you sum over all momenta and spin states n . I am allowing here for a change of spin basis. Then we have:

$$\phi_i(x) = \sum_{p,n} \left[\phi_i^{(p,n)}(x) \phi(p,n) + \overline{\phi_i^{(p,n)}} \phi^\dagger(p,n) \right]$$

$$\Rightarrow \langle 0 | \phi_i(x) \phi^\dagger(p,n) | 0 \rangle = \langle 0 | \phi_i(x) | p,n \rangle = \phi_i^{(p,n)}(x).$$

Thus, by using a single particle state on one side we can get a connection between the regular wavefunctions and the fields; we could have used this to show the transformation properties of the fields if we liked.

Dilatation Generator.

- ① Define a wavefunction scale transformation as follows:

$$[\exp(-iaD)\phi](x) = e^{da} \phi(e^a x)$$

As usual, let a be small to show that

$$\langle x | D | \phi \rangle = i(d + x^\mu \partial_\mu) \phi(x) \equiv \tilde{D} \langle x | \phi \rangle \quad \tilde{D} = i(d + x^\mu \partial_\mu)$$

- ② Finite transformation then has the form on a field:

$$e^{+iaD} \phi(x) e^{-iaD} = e^{da} \phi(e^a x)$$

- ③ Go then to ∞ -tesimals to get

$$+ia[D, \phi(x)] = \delta \phi(x) = -ia \tilde{D} \phi(x)$$

$$\Rightarrow [\phi(x), D] = \tilde{D} \phi(x)$$

④ Algebra: $[\tilde{D}, \tilde{M}^{\mu\nu}] = [i(d + x^\lambda \partial_\lambda), i(x^\mu \partial^\nu - x^\nu \partial^\mu)] = 0$

$$[\tilde{D}, \tilde{P}^\mu] = [i(d + x^\lambda \partial_\lambda), i\partial^\mu] = -[x^\lambda \partial_\lambda, \partial^\mu] = \partial^\mu \phi = -i \tilde{P}^\mu \phi$$

$$[\tilde{D}, \tilde{D}] = -[x^\mu \partial_\mu, x^\nu \partial_\nu] = 0$$

Dilatation and Green's Functions.

① Assume that scale invariance is a true invariance for your theory, you have

$$\langle 0 | T(\phi_1(x_1) \dots \phi_n(x_n)) | 0 \rangle = e^{\alpha(d_1 + \dots + d_n)} \langle 0 | T(\phi_1(e^\alpha x_1) \dots \phi_n(e^\alpha x_n)) | 0 \rangle.$$

Let $\lambda = e^\alpha$. Then

$$[\text{scale invar}] \Rightarrow G^{(n)}(x_1, x_2, \dots, x_n) = (\lambda)^{d_1 + d_2 + \dots + d_n} G^{(n)}(\lambda x_1, \lambda x_2, \dots, \lambda x_n)$$

② Now look at Fourier:

$$\tilde{\phi}^{(n)}(p_1, \dots, p_n) \equiv \int_{\pi's} d^4x_1 \dots d^4x_n e^{ip_1 x_1} \dots e^{ip_n x_n} G^{(n)}(x_1, \dots, x_n)$$

Now let $x'_i = \lambda x_i$ so $d^4x'_i = \lambda^4 d^4x_i$. Then

$$= C_n \cdot (\lambda^4)^{-n} \int d^4x'_1 \dots d^4x'_n e^{i p_1 (x'_1/\lambda)} \dots (\lambda)^{d_1 + \dots + d_n} G(x'_1, \dots, x'_n)$$

$$= (\lambda)^{d_1 + \dots + d_n - 4n} \tilde{\phi}^{(n)}(p_1/\lambda, p_2/\lambda, \dots, p_n/\lambda).$$

③ Conclusions:

$$G^{(n)}(x_1, \dots, x_n) = [\lambda]^{\sum d_i} G^{(n)}(\lambda x_1, \dots, \lambda x_n)$$

$$\tilde{\phi}^{(n)}(p_1, \dots, p_n) = [\lambda]^{\sum d_i - 4n} \tilde{\phi}^{(n)}(p_1/\lambda, \dots, p_n/\lambda)$$

these are true if theory has scale invariance.

④ Remove deltas: $G^{(n)}(p_1, \dots, p_{n-1},) (2\pi)^4 \delta^4(p_1 + \dots + p_n)$

$$= (\lambda)^{[\sum d_i - 4n]} G^{(n)}\left(\frac{p_1}{\lambda}, \dots, \frac{p_{n-1}}{\lambda}\right) (2\pi)^4 \delta^4\left(\frac{p_1}{\lambda} + \frac{p_2}{\lambda} + \dots + \frac{p_n}{\lambda}\right)$$

$$= \lambda^4 \delta^4(p_1 + \dots + p_n)$$

$$\Rightarrow G^{(n)}(p_1, \dots, p_{n-1},) = (\lambda)^{[\sum d_i - 4n + 4]} G^{(n)}\left(\frac{p_1}{\lambda}, \dots, \frac{p_{n-1}}{\lambda}\right).$$

⑤ Now go to proper vertex fctn.

$$G^{(n)}(p_1, \dots, p_{n-1}) = G_1^{(2)}(p_1) G_2^{(2)}(p_2) \dots G_n^{(n)}(p_n) \Gamma^{(n)}(p_1, p_2, \dots, p_{n-1}) + \text{other terms.}$$

$$\begin{aligned} &= \downarrow \qquad \qquad \qquad \downarrow \qquad \qquad \qquad \downarrow \qquad \qquad \qquad \downarrow \qquad \qquad \qquad = \downarrow \\ &(\lambda)^{\sum d_i - 4n + 4} G^{(n)}(p_1/\lambda, \dots) \quad (\lambda)^{2d_1 - 4} G_1^{(2)}(p_1/\lambda) \quad [?] \end{aligned}$$

So we get

$$[?] = (\lambda)^{[d_1 + d_2 + \dots + d_n - 4n + 4 + 2d_1 - 2d_2 - \dots - 2d_n + 4n]} \frac{G^{(n)}(p_1/\lambda, \dots, p_{n-1}/\lambda)}{\underbrace{G_1^{(2)}(p_1/\lambda) \dots G_n^{(n)}(p_n/\lambda)}} = \Gamma^{(n)}(p_1/\lambda, \dots, p_{n-1}/\lambda).$$

Thus:

$$\Gamma^{(n)}(p_1, p_2, \dots, p_{n-1}) = (\lambda)^{(4 - \sum d_i)} \Gamma^{(n)}(p_1/\lambda, \dots, p_{n-1}/\lambda).$$

⑥ Summary:

$$G^{(n)}(\lambda x_1, \lambda x_2, \dots, \lambda x_n) = (\lambda)^{-\sum d_i} G^{(n)}(x_1, x_2, \dots, x_n)$$

$$G^{(n)}(\lambda p_1, \lambda p_2, \dots, \lambda p_{n-1}) = (\lambda)^{+\sum d_i - 4n + 4} G^{(n)}(p_1, p_2, \dots, p_{n-1})$$

$$\Gamma^{(n)}(\lambda p_1, \lambda p_2, \dots, \lambda p_{n-1}) = (\lambda)^{4 - \sum d_i} \Gamma^{(n)}(p_1, p_2, \dots, p_{n-1})$$

⑦ Easy to compute dimensions (Energy & Mass)

$$\dim[G^{(n)}(x_1, \dots, x_n)] = \sum d_i$$

$$\dim[G^{(n)}(p_1, \dots, p_{n-1})] = \sum d_i - 4n + 4$$

$$\dim[\Gamma^{(n)}(p_1, \dots, p_{n-1})] = 4 - \sum d_i$$

Obviously same as above exponents.

Summary for P^μ , $M^{\mu\nu}$, and D .

① Lie Algebra: ($M = M_{BD} = M(\text{McKamel})$)

$$[M^{\mu\nu}, M^{\alpha\beta}] = i \left\{ g^{\nu\alpha} M^{\mu\beta} - g^{\mu\alpha} M^{\nu\beta} + g^{\nu\beta} M^{\alpha\mu} - g^{\mu\beta} M^{\alpha\nu} \right\}$$

$$[P^\alpha, M^{\mu\nu}] = i \left\{ g^{\alpha\mu} P^\nu - g^{\alpha\nu} P^\mu \right\}$$

$$[P^\alpha, D] = i P^\alpha$$

$$[P^\mu, P^\nu] = [D, D] = [D, M^{\mu\nu}] = 0$$

② Finite transformations of fields:

$$e^{+i\theta M^{\mu\nu}} \phi_i(x) e^{-i\theta M^{\mu\nu}} = [e^{-i\theta M^{\mu\nu}}]_{ij} \phi_j([e^{+i\theta M^{\mu\nu}}]_{\alpha\beta} x^\beta)$$

$$e^{+i\theta_\mu P^\mu} \phi_i(x) e^{-i\theta_\mu P^\mu} = \phi_i(x+a)$$

$$e^{+iaD} \phi_i(x) e^{-iaD} = e^{da} \phi_i(e^a x)$$

③ Comrels:

$$[\phi_i(x), M^{\mu\nu}] = [\tilde{M}^{\mu\nu}]_{ij} \phi_j(x)$$

$$[\tilde{M}^{\mu\nu}]_{ij} = \tilde{M}^{\mu\nu} \delta_{ij} + [M^{\mu\nu}]_{ij}$$

$$[\phi_i(x), P^\mu] = \tilde{P}^\mu \phi_i(x)$$

$$\tilde{P}^\mu = i \partial^\mu$$

$$[\phi_i(x), D] = \tilde{D} \phi_i(x)$$

$$\tilde{D} = i(d + x^\mu \partial_\mu)$$

Also:

$$\tilde{M}^{\mu\nu} = i(x^\mu \partial^\nu - x^\nu \partial^\mu)$$

$$[M^{\mu\nu}]_{ij} = i \Sigma_{ij}^{\mu\nu}$$

$$\Rightarrow [\tilde{M}^{\mu\nu}]_{ij} = i \left([x^\mu \partial^\nu - x^\nu \partial^\mu] \delta_{ij} + \Sigma_{ij}^{\mu\nu} \right)$$

Fourier Transform of Translation-Invariant Functions

① Start with $f(x_1, x_2, \dots, x_n)$. Normally we might say

$$F(p_1, p_2, \dots, p_n) \equiv \int dx_1 \dots dx_n e^{ip_1 x_1} \dots e^{ip_n x_n} f(x_1, x_2, \dots, x_n)$$

② Now suppose $f(x_1+a, x_2+a, \dots, x_n+a) = f(x_1, x_2, \dots, x_n)$. Then we can if we like choose $a = -x_n$ so that

$$f(x_1, x_2, \dots, x_n) = f(\underbrace{x_1 - x_n}_{\equiv \xi_1}, \underbrace{x_2 - x_n}_{\equiv \xi_2}, \dots, \underbrace{x_{n-1} - x_n}_{\equiv \xi_{n-1}}, 0) = f(\xi_1, \xi_2, \dots, \xi_{n-1}, 0)$$

Then

$$F(p_1, p_2, \dots, p_n) = \int d\xi_1 d\xi_2 \dots d\xi_{n-1} dx_n e^{ip_1(\xi_1+x_n)} e^{ip_2(\xi_2+x_n)} \dots e^{ip_{n-1}(\xi_{n-1}+x_n)} \\ \times e^{ip_n x_n} f(\xi_1, \dots, \xi_{n-1}, 0)$$

Now go ahead and do the dx_n integration to get

$$= (2\pi)^4 \delta^4(p_1 + p_2 + \dots + p_n) \cdot \underbrace{\int d\xi_1 \dots d\xi_{n-1} e^{ip_1 \xi_1} \dots e^{ip_{n-1} \xi_{n-1}} f(\xi_1, \dots, \xi_{n-1}, 0)}_{\equiv \tilde{f}(p_1, p_2, \dots, p_{n-1})}$$

③ Now how do you invert?

$$f(x_1, x_2, \dots, x_n) = (2\pi)^{4n} \int dp_1 dp_2 \dots dp_n e^{-ip_1 x_1} \dots e^{-ip_n x_n} F(p_1, p_2, \dots, p_n) \\ = (2\pi)^{4(n-1)} \int dp_1 dp_2 \dots dp_{n-1} e^{-ip_1 \xi_1} \dots e^{-ip_{n-1} \xi_{n-1}} \tilde{f}(p_1, p_2, \dots, p_{n-1})$$

④ So it is perhaps best to do it both ways. The second way being:

$$f(\xi_1, \xi_2, \dots, \xi_{n-1}, 0) = \left(\frac{1}{(2\pi)^4}\right)^{n-1} \int dp_1 \dots dp_{n-1} e^{-ip_1 \xi_1} \dots e^{-ip_{n-1} \xi_{n-1}} \tilde{f}(p_1, p_2, \dots, p_{n-1})$$

The key fact is that $f(x_1, \dots, x_n)$ is really only a function of the differences!

Greene Functions and Fourier Transform.

- ① Let $G^{(n)}(x_1, \dots, x_n) = \langle 0 | T(\phi_1(x_1) \dots \phi_n(x_n)) | 0 \rangle =$ full greene.
Translation invariance tells you: $G(x_1, x_2, \dots, x_n) = G(x_1+a, x_2+a, \dots, x_n+a)$.
Reason: $\exp(iaP) \phi(x) \exp(-iaP) = \phi(x+a)$.

② Define:

$$\mathcal{G}^{(n)}(p_1, p_2, \dots, p_n) \equiv \int dx_1 \dots dx_n e^{ip_1 x_1} \dots e^{ip_n x_n} G^{(n)}(x_1, \dots, x_n)$$

But now translation tells you

$$\begin{aligned} &= (2\pi)^4 \delta^4(p_1 + p_2 + \dots + p_n) \underbrace{\int dx_1 \dots dx_{n-1} e^{ip_1 x_1} \dots e^{ip_{n-1} x_{n-1}} G^{(n)}(x_1, \dots, x_{n-1}, 0)}_{\equiv \mathcal{G}^{(n)}(p_1, p_2, \dots, p_{n-1})} \\ &\equiv \mathcal{G}^{(n)}(p_1, p_2, \dots, p_{n-1}) \end{aligned}$$

Note: p_n is a dependent variable in $G^{(n)}$; some choose to include it, others not.

- ③ Inversion formulae: ($b = 1/(2\pi)^4$)

$$G^{(n)}(x_1, x_2, \dots, x_n) = b^n \int dp_1 \dots dp_n e^{-ip_1 x_1} \dots e^{-ip_n x_n} \mathcal{G}^{(n)}(p_1, p_2, \dots, p_n)$$

Insert then to get

$$\begin{aligned} G^{(n)}(x_1, x_2, \dots, x_n) &= b^{n-1} \int dp_1 \dots dp_{n-1} \cdot e^{-ip_1(x_1-x_n)} \dots e^{-ip_{n-1}(x_{n-1}-x_n)} \\ &\quad \times \mathcal{G}^{(n)}(p_1, p_2, \dots, p_{n-1}) \end{aligned}$$

④ Propagators as $n=2$:

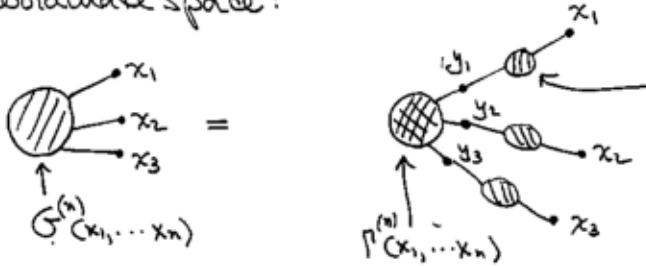
$$\begin{aligned} \underbrace{G^{(2)}(p)}_{\equiv P(p)} &= \int dx_1 e^{ipx_1} \underbrace{G^{(2)}(x_1, 0)}_{\equiv P(x_1)} \end{aligned}$$

$$\underbrace{G^{(2)}(x_1, x_2)}_{P(x_1-x_2)} = \frac{1}{(2\pi)^4} \int dp_1 e^{-ip_1(x_1-x_2)} \underbrace{G^{(2)}(p_1)}_{P(p_1)}$$

This agrees with BD p 387, eg $G P(x) = S_F(x)$, $P(p) = S_F(p) = \frac{1}{\not{p} - m_0}$

The Proper Vertex Functions.

① In coordinate space:



$$G_i^{(2)}(x_1, y_1) = P_i(x_1, y_1) = [i D(x_1, y_1)]$$

Thus:

$$G^{(n)}(x_1, x_2, \dots, x_n) = \int dy_1 \dots dy_n P_1(x_1, y_1) \dots P_n(x_n, y_n) \underbrace{\Gamma^{(n)}(y_1, y_2, \dots, y_n)}_{1PI} + \text{other terms}$$

② Now do Fourier to get:

$$\begin{aligned} G^{(n)}(p_1, p_2, \dots, p_n) &= \int dy_1 \dots dy_n \int dx_1 \dots dx_n \\ &\times e^{ip_1 x_1} \dots e^{ip_n x_n} P_1(x_1, y_1) \dots P_n(x_n, y_n) \cdot \Gamma^{(n)}(y_1, y_2, \dots, y_n) \\ &= \int dy_1 \dots dy_n \cdot \int d\xi_1 \dots d\xi_n \cdot e^{ip_1 y_1} \dots e^{ip_n y_n} \\ &\times e^{ip_1 \xi_1} \dots e^{ip_n \xi_n} P_1(\xi_1) \dots P_n(\xi_n) \Gamma^{(n)}(y_1, \dots, y_n) \\ &= P_1(p_1) P_2(p_2) \dots P_n(p_n) \cdot \tilde{\Gamma}^{(n)}(p_1, p_2, \dots, p_n). \end{aligned}$$

③ Now extract $(2\pi)^4 \delta^4(\sum p_i)$ to get

$$G^{(n)}(p_1, p_2, \dots, p_{n-1}) = P_1(p_1) \dots P_n(p_n) \Gamma^{(n)}(p_1, p_2, \dots, p_{n-1}) + \text{others}$$

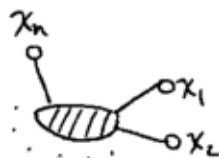
where

$$P_i(p_i) = G_i^{(2)}(p_i,)$$

Graphical Language for Green's Functions etc.

① Define these basic objects

$$i^{n-1} G^{(n)}(x_1, x_2, \dots, x_n) =$$



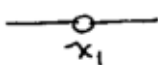
= complete Greens.

$$\Gamma^{(n)}(x_1, \dots, x_n) =$$



= proper 1PI vertex.

$$\int dx_1 \dots$$



$$\delta^4(x_1 - x_2)$$



② Review basics of $\Gamma[\phi]$ and $\mathcal{L}[\psi]$: (let δ mean ∇).

a) $\phi(x) = \frac{\delta \mathcal{L}}{\delta \mathcal{J}(x)}$ and $\Gamma[\phi] \equiv \mathcal{L}[\psi] - \int dx \phi(x) \mathcal{J}(x) \Rightarrow \frac{\delta \Gamma[\phi]}{\delta \phi(x)} = -\mathcal{J}(x).$

b) $\frac{\delta^2 \mathcal{L}}{\delta \mathcal{J}(x_1) \delta \mathcal{J}(x_2)} = \frac{\delta \phi(x_2)}{\delta \mathcal{J}(x_1)} = i G^{(2)}(x_1, x_2) =$ $(= "X^{-1}" \text{ Abusive.})$

c) $\frac{\delta^2 \Gamma}{\delta \phi(x_1) \delta \phi(x_2)} = -\frac{\delta \mathcal{J}(x_2)}{\delta \phi(x_1)} = \Gamma^{(2)}(x_1, x_2) =$ $(= "X" \text{ Abusive.})$

③ Theorem: $= - \left(\text{Diagram of a single propagator between x1 and x3} \right)$

Proof:

$$\text{LHS} = \int dx_2 \left[\frac{\delta \phi(x_2)}{\delta \mathcal{J}(x_1)} \right] \left[-\frac{\delta \mathcal{J}(x_3)}{\delta \phi(x_2)} \right] = -\frac{\delta \mathcal{J}(x_3)}{\delta \mathcal{J}(x_1)} = -\delta(x_3 - x_1).$$

[Also works with to right of]

④ Theorem:

$$x_1 \text{---} \text{[shaded circle]} \text{---} y_1 \text{---} \text{[cross-hatched circle]} \text{---} y_2 \text{---} \text{[shaded circle]} \text{---} x_2 = - \left(x_1 \text{---} \text{[shaded circle]} \text{---} x_2 \right)$$

Proof: obvious from last theorem. Write out longhand to get

$$\int dy_1 dy_2 iG^{(2)}(x_1, y_1) \Gamma^{(2)}(y_1, y_2) iG^{(2)}(y_2, x_2) = - iG^{(2)}(x_1, x_2)$$

OR

$$\int dy_1 dy_2 \left[\frac{\delta^2 \mathcal{L}}{\delta \mathcal{J}(x_1) \delta \mathcal{J}(y_1)} \right] \left[\frac{\delta^2 \Gamma}{\delta \phi(y_1) \delta \phi(y_2)} \right] \left[\frac{\delta^2 \mathcal{L}}{\delta \mathcal{J}(y_2) \delta \mathcal{J}(x_2)} \right] = - \left[\frac{\delta^2 \mathcal{L}}{\delta \mathcal{J}(x_1) \delta \mathcal{J}(x_2)} \right]$$

⑤ Theorem:

$$\text{[shaded circle]} \leftarrow x_0 = \text{[shaded circle]} \text{---} y_0 \text{---} \text{[shaded circle]} \text{---} x_0$$

I.e., $[\leftarrow x_0] \Rightarrow \delta / \delta \mathcal{J}(x_0)$

This theorem states that

$$\frac{\delta}{\delta \mathcal{J}(x_0)} \Gamma^{(n)}(x_1, \dots, x_n) = \int dy_0 \Gamma^{(n+1)}(x_1, \dots, x_n, y_0) \cdot iG^{(2)}(y_0, x_0)$$

Proof:

$$\hookrightarrow = \frac{\delta}{\delta \mathcal{J}(x_0)} \cdot \frac{\delta^n \Gamma}{\delta \phi(x_1) \dots \delta \phi(x_n)} = \int dy_0 \underbrace{\frac{\delta \phi(y_0)}{\delta \mathcal{J}(x_0)}}_{iG^{(2)}} \frac{\delta^{n+1} \Gamma}{\delta \phi(y_0) \delta \phi_1 \dots \delta \phi_n}$$

⑥ Theorem:

$$\text{[shaded circle]} \leftarrow x_0 = \text{[shaded circle]} \text{---} x_0$$

Proof: obvious

$$\delta / \delta \mathcal{J} \left(\frac{\delta^n \mathcal{L}}{\delta \mathcal{J}_1 \dots \delta \mathcal{J}_n} \right) = \frac{\delta^{n+1} \mathcal{L}}{\delta \mathcal{J}_0 \mathcal{J}_1 \dots \delta \mathcal{J}_n}$$

① Theorem:

$$x_1 \circ \text{blob} \circ x_2 = x_1 \circ \text{blob} \circ y_1 \circ \text{blob} \circ y_2 \circ \text{blob} \circ x_2$$

Proof: $x_1 \circ \text{blob} \circ x_2 = \left[x_1 \circ \text{blob} \circ x_2 \right] \leftarrow x_3$

$$= - \left[x_1 \circ \text{blob} \circ y_1 \circ \text{blob} \circ y_2 \circ \text{blob} \circ x_2 \right] \leftarrow x_3$$

$$= - \left[x_1 \circ \text{blob} \circ \text{blob} \circ \text{blob} \circ x_2 \right] - \left[x_1 \circ \text{blob} \circ \text{blob} \circ \text{blob} \circ x_2 \right]$$

$$- \left[x_1 \circ \text{blob} \circ \text{blob} \circ \text{blob} \circ x_2 \right]$$

$$= + x_1 \circ \text{blob} \circ x_2 + x_1 \circ \text{blob} \circ x_2 - \left[x_1 \circ \text{blob} \circ \text{blob} \circ \text{blob} \circ x_2 \right]$$

based on earlier theorem. Thus. QED.

② Theorem: operation $\leftarrow$ is distributive over all blobs.

Proof: blobs are always multiplied and $\leftarrow$ indicates a single derivative, QED.

③ Theorem:

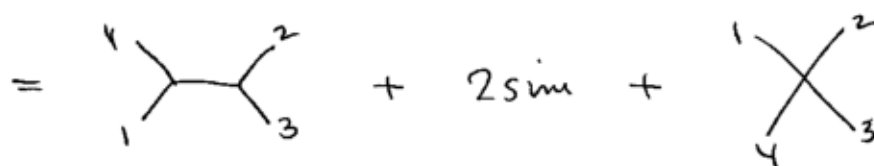
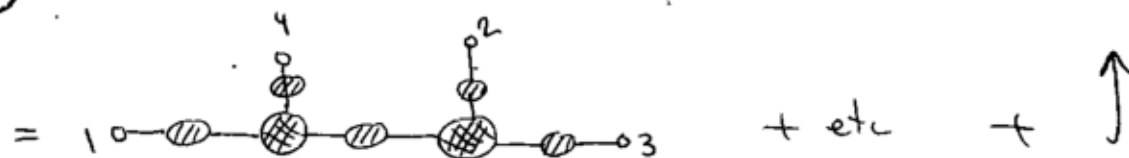
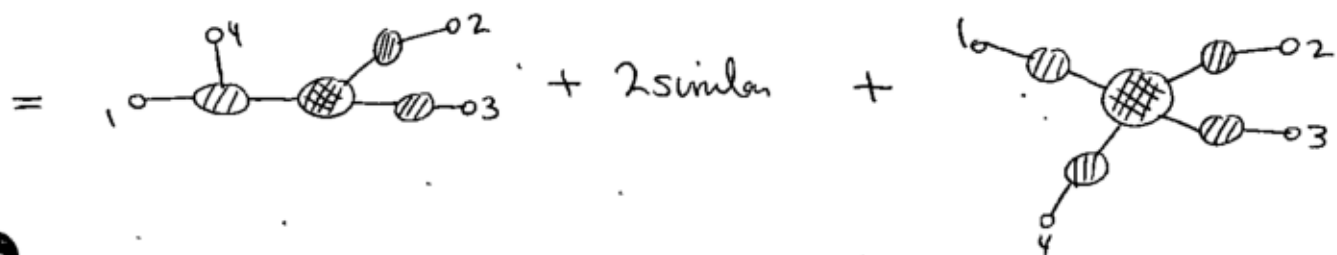
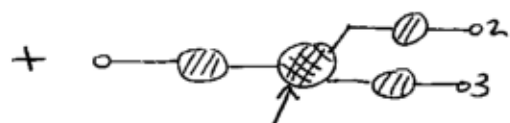
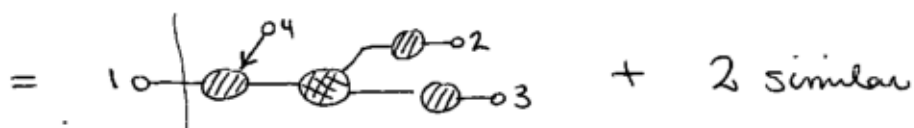
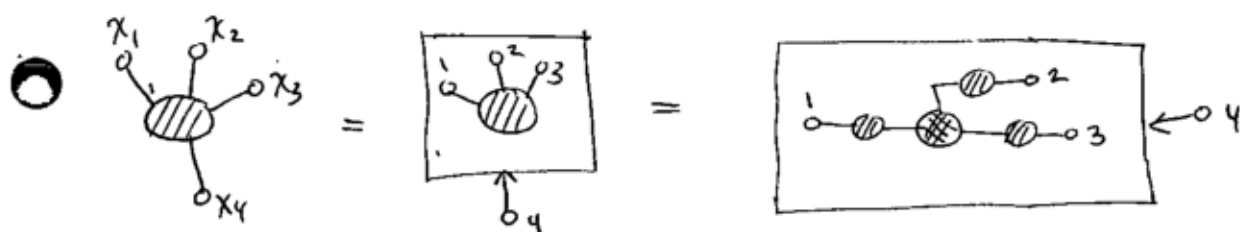
$$x_1 \circ \text{blob} \circ x_2 = x_1 \circ \text{blob} \circ x_2 + \text{cyclic } 123$$

$$+ \left\{ x_1 \circ \text{blob} \circ x_2 + \text{cyclic } 123 \right\}$$

Proof: just do it using above theorem.

↑ This exhausts skeletons that are not 1PI.

Elaborate this last result:



Rescaling fields + effect on 1PI proper vertices

① First consider the coordinate-space Green Functions:

$$G^{(n)}(x_1, x_2, \dots, x_n) = \langle 0 | T (\phi(x_1) \phi(x_2) \dots \phi(x_n)) | 0 \rangle$$

Let each field scale like so:
then you will find that

$$\phi_i = Z_i^{\frac{1}{2}} \phi$$

$$G^{(n)}(x_1, \dots, x_n) = \prod_{i=1}^n (Z_i)^{\frac{1}{2}} G^{(n)}(x_1, \dots, x_n)$$

② Now recall that:

$$G^{(n)}(p_1, p_2, \dots, p_{n-1}) = \text{thing without the delta}$$

$$= \int dx_1 \dots dx_{n-1} e^{ip_1 x_1} \dots e^{ip_{n-1} x_{n-1}} G^{(n)}(x_1, \dots, x_{n-1}, 0)$$

Clearly you find that:

$$G^{(n)}(p_1, p_2, \dots, p_{n-1}) = \prod_{i=1}^n \sqrt{Z_i} G^{(n)}(p_1, p_2, \dots, p_{n-1})$$

③ Next, consider that

$$G^{(n)}(p_1, p_2, \dots, p_{n-1}) = P_1(p_1) \dots P_n(p_n) \Gamma^{(n)}(p_1, \dots, p_{n-1}) + \text{other}$$

① But,

$$P_i(p_i) = G^{(2)}(p_i) = Z_i G^{(2)}(p_i) = Z_i P_i(p_i)$$

Thus we get

$$\prod_{i=1}^n \sqrt{Z_i} G^{(n)}(p_1, \dots, p_{n-1}) = \prod_{i=1}^n (Z_i) P_i(p_i) \cdot \left[\prod_{i=1}^n \left(\frac{1}{\sqrt{Z_i}} \right) \right] \Gamma^{(n)}(p_1, \dots, p_{n-1})$$

Thus, since we want

$$G^{(n)}(p_1, \dots, p_{n-1}) = P_1(p_1) \dots P_n(p_n) \Gamma^{(n)}(p_1, \dots, p_{n-1})$$

we must do this:

$$\Gamma^{(n)}(p_1, p_2, \dots, p_{n-1}) = \prod_{i=1}^n \left(\frac{1}{\sqrt{Z_i}} \right) \Gamma^{(n)}(p_1, \dots, p_{n-1})$$

Truncated Greens Functions.

1. Recall BD discussion of IN fields, and OUT fields. These are supposed to be free, non-interacting asymptotic states. When you examine an S-matrix element, you do so in terms of these free, asymptotic states, i.e., $S_{fi} = (f, OUT/i, IN) = (f, IN/S/i, IN)$. I think the idea is that the external lines of a diagram representing an S-matrix element are supposed to represent these free states. Such "states" do not self-interact, and thus do not "build up" the factor Z_3 due to their self-interaction. The external lines are supposed to be propagating as free particles without even self-interactions.

So how would you relate ϕ_{in} to the unrenormalized field ϕ^0 ? Well, the propagator you would get by fourier transforming $(\phi_{in}\phi_{in})$ would be just $1/(p^2 - m^2)$, whereas the unrenormed (but "interacting") fields would yield $(\phi_0\phi_0) = Z_3/(p^2 - m^2)$. Thus, you can imagine how these two kinds of fields might be related by a square root of Z_3 , as suggested in the "asymptotic condition" (16.20) of BB.

Dont be confused. Here we are discussing the relation between a free-field (which makes just the bare propagator) and an interacting field. In the past I have often compared two kinds of interacting fields: the unrenormalized which I call ϕ_0 , and the renormalized which I call ϕ . These two interacting fields are simply related by a multiplicative factor.

2. I claim that the LSZ formula really says this: compute your momentum-space Greens function, then knock off the contributions which are due to self-energy insertions on external lines. The result is the "truncated or amputated" Greens function and is precisely the S-matrix element you seek. You knock off or amputated the external Greens Propagators because you are supposed to be computing S-matrix element between asymptotic states. In other words, an S-matrix element is a multi-pole residue of a Greens function. Then the object you get is similar to the 1PI function (which also has its external self-energy insertions knocked out) but the amputated guy is not 1PI.

Re-insert deltas on both sides to get:

$$i T^{(n)}(p_1, p_2, \dots, p_n) = (1/i)^n (p_1^2 - m^2) \dots (p_n^2 - m^2) \mathcal{Q}^{(n)}(p_1, p_2, \dots, p_n)$$

Now elevate $\mathcal{Q}$ into its fourier rep

$$= (i)^n \int d x_1 \dots d x_n e^{\pm i p_1 x_1} \dots e^{\pm i p_n x_n} \\ \times (\Box_{x_1}^2 + m^2) \dots (\Box_{x_n}^2 + m^2) \underbrace{G^{(n)}(x_1, x_2, \dots, x_n)}_{= \langle 0 | T(\phi(x_1) \dots \phi(x_n)) | 0 \rangle}$$

The $\pm$ choices depend whether particles is ingo or outgo and depends on labelling convention, but the sign has no effect on the sign of Box² operator. So here you see the "amputation" going on in the x-space itself. Thus, we arrive at precisely BD's LSZ formula (16.81) except I dont have any Z's. The reason is simply that I am using renormalized fields, whereas they are using unrenormalized fields. I could get their result if I make the usual replacement that $\phi = (Z_3)^{-\frac{1}{2}} \phi_0$. Then you should imagine BD's fields as ϕ_0 's.

4. By the way, according to the way I did it, I just pulled off external propagator insertions. Thus, the first term in my S-matrix element here called iT will be the 1PI proper vertex. Ie, this is the connected-part of the T-matrix.

5. These Z's sitting outside the formula in BD are sometimes called "external wave-function renormalization", which terminology relates to paragraph 1 above. For me, these Z's are just the Z's that get you to a renormalized Greens function, so I dont have to call them anything.

3. Implement the above comments in scalar field theory, say ϕ^4 or something. Then let $G(p_1, \dots, p_n)$ be the renormalized and thus finite n-point Greens function. It contains the overall delta of course. Now, remember that LSZ gives you a matrix element of $S-1$ rather than S , so you are really getting the T-matrix. Lets say that $S-1=iT$ which is a standard form. Then here is what we can say:

$$\mathcal{G}^{(n)}(p_1, p_2, \dots, p_n) = \int \frac{dq_1}{(2\pi)^4} \dots \frac{dq_n}{(2\pi)^4} \mathcal{G}^{(2)}(p_1, q_1) \dots \mathcal{G}^{(2)}(p_n, q_n) i \mathcal{T}^{(n)}(q_1, q_2, \dots, q_n)$$

↑ script-T

Here I am using all-script functions in momentum space because these are the complete ones which include the $(2\pi)^4 \delta(\dots)$ overall. Now lets expose all those delta functions and the result is much simpler, and we go to non-script functions:

$$G^{(n)}(p_1, p_2, \dots, p_{n-1}, p_n) = G^{(2)}(p_1) G^{(2)}(p_2) \dots G^{(2)}(p_n) i T^{(n)}(p_1, p_2, \dots, p_{n-1}, p_n)$$

Here I put the last momentum argument in parentheses because really p_n should be regarded as a function of the other $n-1$ momenta, since we are now conserving 4-momenta. So invert to get:

$$iT^{(n)}(p_1, p_2, \dots, p_{n-1}) = [G^{(2)}(p_1)]^{-1} \dots [G^{(2)}(p_n)]^{-1} G^{(n)}(p_1, p_2, \dots, p_{n-1})$$

By the way,
$$\begin{cases} FT \langle \phi \phi \rangle = (2\pi)^4 \delta(\dots) G^{(2)}(p) \\ \langle \phi \phi \rangle = G^{(2)}(x_1, x_2) \end{cases}$$

But $G^{(2)}(p) = i D(p)$ and $D(p) = \frac{1}{p^2 - m^2}$

Thus, $D^{-1} = (p^2 - m^2)$ and:

$$iT^{(n)}(p_1, p_2, \dots, p_{n-1}) = \left(\frac{1}{i}\right)^n (p_1^2 - m^2)(p_2^2 - m^2) \dots (p_n^2 - m^2) G^{(n)}(p_1, p_2, \dots, p_{n-1})$$

Of course the implication is that you are taking a limit here as p_i^2 go on shell. This is how you are taking a multi-pole residue to get the S-matrix element. If we were not to put things back into x-space, we would get $p^2 = -d^2$ so each factor makes a (-1) which combines with each factors's $1/i$ to make a plus i . Thus:

The Identification of a Propagator.

① From Coleman's matrix theorem we know this: (no color).

$$\int [d\phi] e^{-\int dx \left[\frac{1}{2} \phi(x) \vec{M} \phi(x) + \phi(x) J(x) \right]} = \frac{1}{\sqrt{\det \vec{M}}} e^{+\int dx \left[\frac{1}{2} J(x) \vec{M}^{-1} J(x) \right]}$$

② In Dirac notation:

$$\int [d\phi] e^{-\left[\frac{1}{2} \langle \phi | M | \phi \rangle + \langle \phi | J \rangle \right]} = \frac{1}{\sqrt{\det M}} e^{+\frac{1}{2} \langle J | M^{-1} | J \rangle}$$

where $\langle \phi | M | \phi \rangle = \langle \phi | x \rangle \langle x | M | \phi \rangle = \langle \phi | x \rangle \vec{M} \langle x | \phi \rangle$

$$= \langle \phi | x \rangle \underbrace{\langle x | M | y \rangle}_{\equiv M(x, y)} \langle y | \phi \rangle$$

$$\langle J | M^{-1} | J \rangle = \langle J | x \rangle \underbrace{\langle x | M^{-1} | y \rangle}_{\equiv M^{-1}(x, y)} \langle y | J \rangle$$

So we can say

$$\int [d\phi] e^{-\int dx dy \left[\frac{1}{2} \phi(x) M(x, y) \phi(y) + \phi(x) \delta(x-y) J(y) \right]} = \left[\det M(x, y) \right]^{-\frac{1}{2}} e^{+\int dx dy \left[\frac{1}{2} J(x) M^{-1}(x, y) J(y) \right]}$$

③ If this is $W[J]$, then you can see that

$$\frac{1}{W[J]} \frac{\delta^2 W[J]}{\delta J(x) \delta J(y)} \bigg|_{J=0} = M^{-1}(x, y)$$

① Now stuff in (-i) to get

$$\int [d\phi] e^{+i \int dx \left[\frac{1}{2} \phi(x) \vec{m} \phi(x) + \phi(x) \mathcal{J}(x) \right]}$$

$$= \frac{1}{\sqrt{\det M}} e^{-i \int dx dy \left[\frac{1}{2} \mathcal{J}(x) \vec{M}^{-1}(x, y) \mathcal{J}(y) \right]}$$

If this were $W[\mathcal{J}]$, then we would say

$$\left. \frac{1}{W} \frac{\delta^2 W}{\delta \mathcal{J}(x) \delta \mathcal{J}(y)} \right|_{\mathcal{J}=0} = \frac{\int [d\phi] (i\phi(x))(i\phi(y)) e^{iS}}{\int [d\phi] e^{iS}} = (i)^2 \langle 0 | T(\phi(x)\phi(y)) | 0 \rangle$$

$$\Rightarrow G_c^{(2)}(x, y) = (-i)^2 \left. \frac{1}{W} \frac{\delta^2 W[\mathcal{J}]}{\delta \mathcal{J}(x) \delta \mathcal{J}(y)} \right|_{\mathcal{J}=0}$$

$$= (-i)^2 \cdot [-i \vec{M}^{-1}(x, y)] = +i \underbrace{\vec{M}^{-1}(x, y)}_{D(x, y)}.$$

So we identify $\vec{M}^{-1}(x, y) = D(x, y)$, the propagator.

$$W[\mathcal{J}] = \int [d\phi] e^{+i \int dx \left[\frac{1}{2} \phi(x) \vec{m} \phi(x) + \phi(x) \mathcal{J}(x) \right]} = \frac{e^{-i \int dx dy \frac{1}{2} \mathcal{J}(x) \vec{M}^{-1}(x, y) \mathcal{J}(y)}}{\sqrt{\det M}}$$

$$\Rightarrow G_c^{(n)}(x_1 \dots x_n) = (-i)^n \left. \frac{\delta^n W}{\delta \mathcal{J}(x_1) \dots \delta \mathcal{J}(x_n)} \right|_{\mathcal{J}=0}$$

$$\Rightarrow G_c^{(2)}(x, y) = +i \vec{M}^{-1}(x, y) = i D(x, y) = \langle 0 | T(\phi(x)\phi(y)) | 0 \rangle$$