

Infrared Divergence

Yennie, Frautschi & Suura (1961)

Grammar & Yennie (1973)

Levy & Sucher (1969) Eikonal Approximation

Phil Lucht notes (1978)

Yennie, Frautschi & Suura

1961

Phil Lucht notes 1978

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The Infrared Divergence Phenomena and High-Energy Processes*

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A general treatment of the infrared divergence problem in quantum electrodynamics is given. The main feature of this treatment is the separation of the infrared divergences as multiplicative factors, which are treated to all orders of perturbation theory, and the conversion of the residual perturbation expansion into one which has no infrared divergence, and hence no need for an infrared cutoff. In the infrared factors, which are exponential in form, the infrared divergences arising from the real and virtual photons cancel out in the usual way. These factors can then be expressed solely in terms of the momenta of the initial and final charged particles and an integral over the region of phase space available to the undetected photons; they do not depend upon the specific details of the interaction. Electron scattering from a static potential is treated in considerable detail, and several other examples are discussed briefly. As an important byproduct of the general treatment, it is found that when the infrared contributions are separated in a particular way, they dominate the radiative corrections at high energies and together with certain "magnetic terms" and vacuum polarization corrections seem to give all the contributions proportional to $\ln(E/m)$. All of these corrections can be easily estimated (in most cases) simply from a knowledge of the external momenta of the charged particles; this then provides a very powerful and accurate way of estimating radiative corrections to high-energy processes.

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Introduction. As I hoped, this introduction makes much more sense now that I have read the paper and thought about things. I will try to just list off the things they have to say.

1. Their prototype example is electron scattering off a potential. First, by drawing the Purcell pictures, you can convince yourself that photons should tend to come off in the direction of the incident and scattered electron. This fact is then verified by the mathematical form of the factor shown in (1.1). This appears in Jackson classical as well as BD RQFT calculation. Two facts: the higher the energy E and the larger the scattering angle, the more radiation you are going to get. Both rather obvious just by thinking about the "proper fields" i.e. the Purcell diagrams.

Now, the energy in the radiation is constant at small k which implies a dk/k photon distribution, the infrared blowup situation.

2. When you compute the effects of both real and virtual photons, you find that the dk/k singularity is cancelled and you get a residual $\log(dE)/E$ factor in your radiative corrections. Angular integration over the divergent ~~xxxx~~ angular peaks gives another log to radiative corrections of the form $\log(E/m) \approx A$. Thus, after cancellation of the IR divergence, your radiative corrections are typically of this "double log" form, to first order in ALPHA. I.e., it is really an exponential but you can consider the first term which is then that double log.

The electron case gives the typical result: the cross section with some definite resolution dE is given by a correction factor times the cross section you would compute for the process ignoring soft photons all together, and the correction factor is maybe 1.2 or so, i.e., not very large. But finite! The explicit answer for electron scattering is given facing page 400.

3. History of this subject. Bloch and Nordsieck in 1937 first noted the problem of soft photons unavoidably accompanying your process; they first showed how the real and virtual photon divergences cancel, ^{to lowest order only} and how you get a finite radiative correction that is not very large. Then in 1950's Lomon and ~~xxxx~~ friends did a semiclassical analysis which required an arbitrary hard/soft photon cutoff. Then in 1954, Jauch and Rorlich showed the virtual/real cancellation to all orders of ALPHA (for electron scattering I guess). This paper claims to be a refinement of the JR argument.

4. A crucial point in the argument is that you only have to worry about photons on external lines. Intuitively this is because soft photons are large and don't much get into the high energy small space guts of a Feynman graph. This is of course just an intuitive statement. You really have to show it, and this involves considering "overlapping divergences" where one end of a photon or both land somewhere inside a messy diagram. In Appendix A they deal with this problem but not completely and rigorously, but they at least consider it which is an improvement over Jauch-Reich.

This external lines thing is of course what lets you extract the IR factors and get a calculable result.

A second improvement over JR is that they give a series expansion for the non-IR radiative correction terms.

Section 5. I could only skim this. Relates to "Kallen's argument about the inconsistency of QED" which I am not familiar with. Basically they say that Kallen's factors must be modified by IR exponentials such as in (5.4) and maybe this makes some difference. Another thing I don't know about is "spurious charge renormalization".

Some interesting remarks, though on page 431: the vertex and electron propagator are only IR singular when you are on the mass shell.

Section 6: Summary. There are several other types of "corrections" besides the IR ones, but they tend to be single log rather than double log, so that the infrared corrections tend to be the main corrections. These corrections include: vacuum polarization (ie, electron corrections to photon lines), "magnetic terms" whatever that means.

Section 2: Radiative Corrections to Electron Scattering.

The general scenario here is that of a single electron coming along and scattering as many times as you like off a potential. I.e., we don't consider the heavy nucleus in itself here. The blob shown in figure 1 is supposed to be a sum of all possible Feynman graphs! The scattering center picks up no energy, so $\epsilon = E - E'$ is the energy total of any emitted physical photons.

(a) Assume for the moment that there are no physical photons. Of course there always will be because a deflected particle must radiate. But here we want to examine only the internal photons. Arrange your infinite set of Feynman graphs by their number n of internal or virtual photons. Let M_n be the sum of all graphs having precisely n virtual photons, no radiated photons. What question do we want to ask?!? First, by recursion you can derive (2.8) which shows the amplitude M_n where all IR terms have been factored out. Virtual photons only cause IR singularity when both ends are on external lines, as was shown somehow in Appendix A. EG, the first term in the expansion (2.8) for M_n 's integrand $\mathfrak{M}_n$ shows $S(k_1) \dots S(k_n)$ and is thus infrared singular in all the k 's. But since these k 's here are integration variables (which is not the case for external photons), you are really asking about singular portions of the multi-integration. This singularity isolated into the object called B . Look ahead to (2.23) where you get the IR singular part of B . Recall that $d^4k = k^3 dk$, whereas the object $S(k)$ (which is the virtual coupling of that photon to external lines) is singular in the sense that it behaves as $1/k^2$. So now you have $k dk$ which is not yet singular. But when you add the photon propagator, you get another k^{-2} and so you get $k^{-1} dk$ which is log singular.

So when you speak of "IR singular" you speak of the fact that B has singularities in the variable λ . You do not mean so much that $S(k)$ is singular. But since $S(k)$ is $1/k^2$ and contributes to the singularity, you might refer to this $S(k)$ thing as being an "infrared" term.

So the "infrared" piece of M_n is all in the object $B(\lambda)$. Notice that M_n involves all orders of field theory except there are only n virtual photons. The complete M is the full sum of everything, and again the only singularity in λ appears in B which is now in an exponent.

So what makes the "virtual photons singularity"? It is a combination of the photon propagators with the electron lines which are created by adding those photons. Hard to say it is just the photons themselves.

So, you know the IR singularity due to integral photons to all orders in terms of an object $S(k)$ which is essentially something of first order! The "all orders"ness merely exponentiates the thing, something I am used to from Regge theory.

(b) So far we know the singularity in the photon mass λ of the complete electron scattering amplitude due to the effect of all internal photons. Next we ask what is the effect of external radiated photons which up to now we have ignored.

Consider an amplitude which is radiating n photons. Since this amplitude contains all possible internal photons, it contains the $\exp(\alpha B)$ factor just discussed. Let us now compute directly the DCS for radiating n photons of energy total $= \epsilon = E' - E$. This is given in (2.13), showing the phase space and the amplitude squared. Again you do an iteration to express the amp^2 in terms of IR factors called $\tilde{S}(k_i)$ which are little amp^2 factors for ~~putting~~ adding an external photon of momentum k_i . This "add-a-photon" term appears and gives the entire λ -singular part. Look at 2.24 to see this. Notice that $d^3k = k^2 dk$, so the combination of k^{-2} from $\tilde{S}$ and the k^{-1} from the energy denominator causes $\tilde{B}$ to be singular in the variable λ at $\lambda = 0$.

The $\tilde{B}$ factor is essentially determined by a first-order Feynman graph, actually the two order α graphs shown in Fig 3 (a) and (b). It gets exponentiated in the usual way, so the big net result is this:

$$\text{DCS to radiate photons of energy } \epsilon \text{ in electron potential scattering} \\ = \exp(\alpha [B + \tilde{B}]) \text{ times something non-singular in } \lambda.$$

Thus, the IR part has been completely isolated and factorized off. And we have done things to all orders in the field theory! All you have to show is that $B + \tilde{B} \doteq 0$, and this is something you show in first-order, so to speak. Actually, you want to show that the singularity in B at $\lambda = 0$ cancels the singularity in $\tilde{B}$ at $\lambda = 0$. Then there is no IR singularity left.

Comment": Experimentally, what are we saying here? If you compute up your total Feynman amplitude for radiating no photons at all, just scatter the electron, you expect a finite answer but you get an infinite answer due to an infrared singularity. Luckily, it is impossible to do this experiment! So you don't care that it is singular. In a real experiment, there will always be some photons coming off, no matter how small you make your resolution. What you measure is the DCS for electron scattering plus unknown number of photons with energy less than something. So you have to add the elastic rate to the rate with photons. In this (incoherent) sum, the IR singularities cancel and no singularity appears in something that you can do with an experiment.

(c) Here we have explicit formulas for B and $\tilde{B}$. These are derived in the appendix A, but you can see where they come from. Why are there no unknown "blobs" in these formulas? Because $S(k)$ is a multiplicative factor, not the whole thing. You do not include the blob in a multiplying factor.

How do you show that the $\lambda = 0$ singularities of B and $\tilde{B}$ cancel? Well, you just do it by brute force. In the limit that p and p' are very large (high energy) and $\epsilon = \text{small}$ (forward scattering), you can write B and $\tilde{B}$ in closed form as (2.27) and (2.28). There you see explicitly the $\lambda = 0$ singularity. It is logarithmic in both cases and cancels between the two terms.

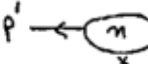
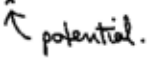
Sometimes the theory is reformulated in terms of a low-end momentum cutoff $k_{\min}$ instead of a photon mass λ . Same result, same cancellation of course. Its just that you want to use the same cutoff method in both cases!

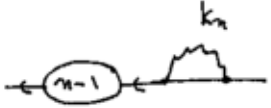
(d) the nonIR terms: If you know B , the entire amplitude for electron scattering is given by the various m_1 . See 2.3. The "non-infrared" terms are the ones that are left after the cancellation of the infrared singular terms. It is these non-IR terms that tell you "the answer", ie, the DCS for electron scattering. These terms are the "radiative corrections" that you often compute around a blob. Not surprising that the things we computed above are same things people computed for radiative corrections to certain processes involving one electron line. Actually, I think the term "radiative corrections" means only "virtual photon corrections". The corrections due to soft external photons are considered in the next section.

(e) the Non IR real photon terms. Now we want to include the effect of soft Brem and compute explicitly the electron scattering DCS. Earlier we only ~~showed~~ that the IR singular parts cancel. Here we want to know some real numbers for what is left over. The answer is given I think in 2.57.

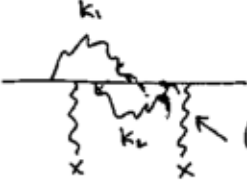
(f) Now suppose we do the experiment where we force $E' - E$ to be less than some small number ΔE . Ie, we try our best to look at elastic events with no photons. Then 2.58 shows you what you are measuring.

notes on section (2.2).


 $p' \leftarrow \text{circle}(n) \leftarrow p$ = all graphs with n internal photons

 potential.


 $+ \text{etc.}$

$g_n(k_1 \dots k_n) =$ amplitude less photon propagators.


 I need not count as virtuals

Now consider:

$$\underbrace{g_n(k_1 \dots k_{n-1}, k_n)}_{\text{circle}(n)} = \underbrace{S(k_n)}_{\text{circle}(n-1)} \underbrace{g_{n-1}(k_1 \dots k_{n-1})}_{\text{circle}(n-1)} + \underbrace{\beta_n^{(1)}(k_1 \dots k_{n-1}; k_n)}_{\text{mixed.}}$$

only Fig (2), a, b, c.

These are the terms which are " k_n -infrared"

But $g_{n-1} = S(k_{n-1})g_{n-2} + \beta_{n-1}^{(1)}$

$\Rightarrow g_n = S(k_n)S(k_{n-1})g_{n-2} + S(k_n)\beta_{n-1}^{(1)}(k_1 \dots k_{n-2}; k_{n-1}) + \beta_n^{(1)}$

But better to symmetrize this as

$$\begin{aligned}
 g_n(k_1 \dots k_n) = & S(k_n)S(k_{n-1})g_{n-2}(k_1 \dots k_{n-2}) \leftarrow \text{sym in } k_n, k_{n-1} \\
 & + S(k_n)\beta_{n-1}^{(1)}(k_1 \dots k_{n-2}; k_{n-1}) \\
 & + S(k_{n-1})\beta_{n-1}^{(1)}(k_1 \dots k_{n-2}; k_n) \leftarrow \text{same} \\
 & + \left\{ -S(k_{n-1})\beta_{n-1}^{(1)}(k_1 \dots k_{n-2}; k_n) + \beta_n^{(1)}(k_1 \dots k_{n-1}; k_n) \right\} \leftarrow \text{also same.}
 \end{aligned}$$

Thus $\{ \}$ is non-IR in both, call it $\beta_n^{(2)}(k_1 \dots k_{n-2}; k_{n-1}, k_n)$

Thus we find:

$$g_n(k_1 \dots k_n) = \left. \begin{aligned} & S(k_n) S(k_{n-1}) g_{n-2}(k_1 \dots k_{n-2}) \\ & + S(k_n) \beta_{n-1}^{(1)}(k_1 \dots k_{n-2}; k_{n-1}) \\ & + S(k_{n-1}) \beta_{n-1}^{(1)}(k_1 \dots k_{n-2}; k_n) \\ & + \beta_n^{(2)}(k_1 \dots k_{n-2}; k_{n-1}, k_n) \end{aligned} \right\} A$$

Now keep iterating:

$$g_n(k_1 \dots k_n) = A + S(k_n) S(k_{n-1}) \left[S(k_{n-2}) S(k_{n-3}) g_{n-4} \right.$$

$$\begin{aligned} g_n(k_1 \dots k_n) = & S(k_n) S(k_{n-1}) \left[S(k_{n-2}) g_{n-3}(k_1 \dots k_{n-3}) + \beta_{n-2}^{(1)}(k_1 \dots k_{n-3}; k_{n-2}) \right] \\ & + S(k_n) \beta_{n-1}^{(1)}(\dots k_{n-2}; k_{n-1}) \\ & + S(k_{n-1}) \beta_{n-1}^{(1)}(\dots k_{n-2}; k_n) \\ & + S(k_{n-2}) \beta_{n-1}^{(1)}(\dots k_{n-2}; k_{n-2}) + \text{etc.} \end{aligned}$$

OK, a little combi problem. Answer is

$$\begin{aligned} g_n(k_1 \dots k_n) = & S(k_1) \dots S(k_n) \beta_0 \\ & + \sum_{i=1}^n S(k_1) S(k_2) \dots \underbrace{S(k_i)}_{\text{missing}} \dots S(k_n) \beta_1(k_i) \\ & + \sum_{i,j=1}^n S(k_1) S(k_2) \dots \underbrace{S(k_i)}_{\text{missing}} \dots \underbrace{S(k_j)}_{\text{missing}} \dots S(k_n) \beta_2(k_i, k_j) \\ & + \dots \sum_{i=1}^n S(k_i) \beta_n(k_1 \dots [k_i] \dots k_n) + \beta_n \quad (2.8) \end{aligned}$$

Examine (2.9):

$$\sum_{r=0}^{\infty} \frac{1}{r!(n-r)!} \prod_{i=1}^r S(k_i) \beta_{n-r}(k_{r+1} \dots k_n)$$

$$= \beta_n + \underbrace{\frac{1}{(n-1)!} S(k_1) \beta_{n-1}(k_2 \dots k_n)}_{\sum_{\text{perms.}} \quad r=1 \text{ yields second last term of (2.8)}}$$

$$+ \sum_{\text{perm}} \frac{1}{2!(n-2)!} S(k_1) S(k_2) \beta_{n-2}(k_3 \dots k_n)$$

$\sum_{\text{perm}}$ means all perms of $(1, 2, 3 \dots n)$. There are $n!$ such terms. Of these, $(n-1)!$ have 1 in first position. These are all the same, hence $\div (n-1)!$

here $(n-2)!$ have $S(k_1) S(k_2)$, and another $(n-2)!$ have $S(k_2) S(k_1)$. all are same, so $\div 2!(n-2)!$

last term in series is $r=n$:

$$\sum_{\text{perm}} \frac{1}{n!} S(k_1) \dots S(k_n) \beta_0 \quad \text{all } n! \text{ are the same.}$$

Examine (2.10)

$$M_n = \frac{1}{n!} \left(\prod_{i=1}^n \int \frac{d^4 k_i}{k_i^2 - \lambda} \right) \underbrace{S_n(k_1 \dots k_n)}_{\text{part of symm.}}$$

symmetrized! $1e, = \tilde{S}(1, 2, \dots, n) + \tilde{S}(2, 1, \dots, n) + \dots = n!$ terms.

$$= \sum_P \sum_{r=0}^n \frac{1}{r!(n-r)!} \prod_{i=1}^r S(k_i) \beta_{n-r}(k_{r+1} \dots k_n)$$

Repeat:

$$M_n = \frac{1}{n!} \int \dots \int \frac{d^4 k_1 \dots d^4 k_n}{(k_1^2 - \lambda)(k_2^2 - \lambda) \dots (k_n^2 - \lambda)} \cdot \sum_P \sum_{r=0}^{\infty} \frac{1}{r!(n-r)!} \\ \times \prod_{i=1}^r S(k_i) \beta_{n-r}(k_{r+1} \dots k_n) \\ = \sum_{r=0}^{\infty} \frac{1}{r!(n-r)!} \cdot \sum_P \frac{1}{n!} \prod_{j=1}^n \left(\frac{\int d^4 k_j}{k_j^2 - \lambda} \right) \left[\left(\prod_{i=1}^r S(k_i) \right) \beta_{n-r}(k_{r+1} \dots k_n) \right]$$

Look at term "r": variables $k_1 \dots k_r$ appear only in here.

Write $\prod_{j=1}^n \left(\frac{\int d^4 k_j}{k_j^2 - \lambda} \right) = \prod_{j=1}^r \left(\dots \right) \prod_{j=r+1}^n \left(\dots \right).$

Then ~~slide~~ put $\prod_{i=1}^r S(k_i)$ with $\prod_{j=1}^r \left(\dots \right)$, make as one product:

$$= \sum_{r=0}^{\infty} \frac{1}{r!(n-r)!} \cdot \sum_P \frac{1}{n!} \left\{ \prod_{i=1}^r \left(\frac{\int d^4 k_i S(k_i)}{k_i^2 - \lambda} \right) \right\} = \left(\dots \right)^r,$$

$$\times \left[\prod_{j=r+1}^n \left(\int \dots \right) \beta_{n-r}(k_{r+1} \dots k_n) \right] \\ = \prod_{k=1}^{n-r} \int \frac{d^4 k_k}{k_k^2 - \lambda} \cdot \beta_{n-r}(k_1 \dots k_{n-r}).$$

I guess under full integration over all $d^4 k_i$, $\sum_P \rightarrow n!$.

So (2.10) looks OK.

Now $S(k_i)$ is order $\epsilon^2 = \alpha$ + will depend on $\vec{p}, \vec{p}'$;
and $\vec{p}' = fch(\epsilon)$, easy to derive.

$$M_n = \sum_{r=0}^n \frac{1}{r!} [\alpha B]^r \cdot m_{n-r}$$

$$M_0 = m_0$$

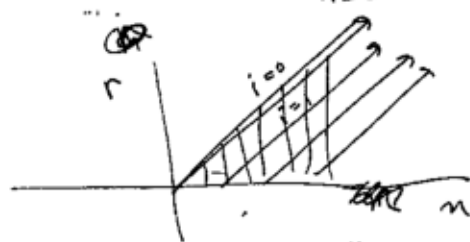
$$M_1 = m_1 + [\alpha B] m_0$$

$$M_2 = m_2 + [\alpha B] m_1 + \frac{[\alpha B]^2}{2!} m_0 \text{ etc.}$$

where:

$$B = \int \frac{d^4 k S(k)}{k^2 - \lambda^2} \quad \text{is a simple integral.}$$

$$M = \sum_{n=0}^{\infty} M_n = \sum_{n=0}^{\infty} \left[\sum_{r=0}^n m_{n-r} \frac{(\alpha B)^r}{r!} \right]$$



Let

$$i = n-r, \quad j = n+r.$$

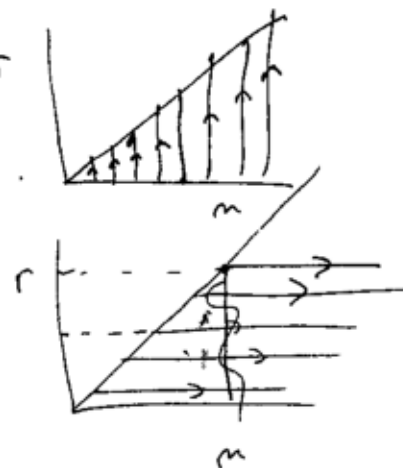
$$j-i = 2r$$

$$\sum_{n=0}^{\infty} \sum_{r=0}^n f_{n,r} = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} f_{i,j} = \sum_{i=0}^{\infty} m_i \frac{(\alpha B)^{j-i}}{(j-i)!}$$

Redo:

$$\sum_{n=0}^{\infty} \sum_{r=0}^n f_{n,r} =$$

$$\sum_{r=0}^{\infty} \sum_{n=r}^{\infty}$$

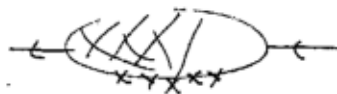


So get

$$\sum_{r=0}^{\infty} \sum_{n=r}^{\infty} m_{n-r} \frac{(\alpha B)^r}{r!} = e^{\alpha B} \sum_{i=0}^{\infty} m_i$$

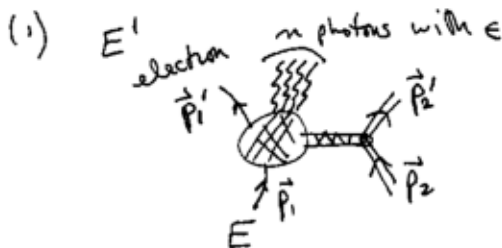
$$S_0 \quad M = e \sum_{n=0}^{\infty} M_n.$$

This M is complete amplitude.



Those regions of the phase space integrations where M might diverge have been isolated into $B!!!$

notes on section (2b) We now compute this:



$$d\sigma = |M|^2 \frac{d^3 p_i'}{2E_i'} \cdot \frac{d^3 p_2'}{2E_2'} \left(\prod_{i=1}^n \frac{d^3 k_i}{2E_i} \right) \delta^4(p_i + p_2 - p_i' - p_2' - \sum k_i)$$

$$S \quad d\sigma = |M|^2 \cdot \frac{(p_i')^2 dp_i' d\Omega_i'}{4E_i' E_2'} \cdot \delta(E + E_2 - E_i' - E_2' - \sum_{i=1}^n E_i) \cdot \prod \left(\frac{d^3 k_i}{2E_i} \right)$$

normal.

But $(p_i')^2 dp_i' = p_i' E_i' dE_i'$ so

$$d\sigma = \frac{|M|^2 p_i'}{4m_2} dE_i' d\Omega_i' \text{ etc.}$$

Let $\epsilon = E_i' - E$ so $d\epsilon = dE_i'$. Then

$$\frac{d\sigma}{d\epsilon} = \frac{|M|^2 p_i'(\epsilon)}{4m_2} \int d\Omega_i' \delta(\epsilon - \sum E_i) \cdot \prod \left(\frac{d^3 k_i}{2E_i} \right)$$

$$a \quad \frac{d\sigma}{d\epsilon} = \left(\prod_{i=1}^n \int \frac{d^3 k_i}{2E_i} \right) \cdot \delta(\epsilon - \sum E_i) \cdot \underbrace{\int d\Omega_i' \frac{p_i'(\epsilon) |M|^2}{4m_2}}$$

But, $M =$ has same virtual-infrared as

$\Rightarrow \Rightarrow \Rightarrow$

- So show these $M = e^{iB} \tilde{M}$. Then end up with (2.13).
 Note: we have done a DCS where E_i held fixed, hence $\epsilon = \sum k_i$ held fixed.

(2) Now $\frac{1}{n!}$ due to identical photons, a sym, a whatever. Do recursion to locate IR part of $\tilde{g}_n$, similar to (2.8) $\rightarrow$ (2.10).

(3) Last time we looked only at M . Now we really construct $d\epsilon$. Hence now we have $\delta(\epsilon - \sum k_i)$, and phase space in (2.17). I think the new (2.10) is:

$$\frac{d\sigma_n}{d\epsilon} = \sum_{r=0}^n \frac{1}{r!(n-r)!} \left[\int \frac{d^3k}{2E_k} \tilde{S}(k) \right]^r \prod_{i=1}^{n-r} \frac{d^3k_i}{2E_i} \tilde{\beta}_{n-r} \\ \times \int dy e^{iy(\epsilon - \sum k_i)} \quad \text{energy correlator!}$$

Go back to (2.9).

$$\tilde{g}_n = \sum_{\mathbf{L}} \sum_{r=0}^{\infty} \frac{1}{r!(n-r)!} \prod_{i=1}^r \tilde{S}(k_i) \tilde{\beta}_{n-r}(k_{r+1} \dots k_n)$$

$$\frac{d\sigma_n}{d\epsilon} = e^{2\alpha B} \sum_r \frac{1}{n!} \frac{1}{r!(n-r)!} \prod_{i=1}^r \tilde{S}(k_i) \tilde{\beta}_{n-r}(k_{r+1} \dots k_n) \cdot \prod_{i=1}^n \frac{d^3k_i}{2E_i} \\ \times \int dy e^{iy(\epsilon - \sum k_i)}$$

Now group: $\left[\prod_{i=1}^r \tilde{S}(k_i) \frac{d^3k_i}{2E_i} e^{-iyk_i} \right] \cdot \left(\prod_{i=r+1}^n \tilde{\beta}_{n-r} \cdot \prod_{i=r+1}^n \frac{d^3k_i}{2E_i} \int dy e^{-iy(\sum k_i)} \right)$
 $= \left[\int \frac{d^3k_i}{2E_i} e^{iyk_i} \tilde{S}(k_i) \right]^r \times_{n-r}$

So we get (2.22) where all $k \rightarrow \infty$ singular stuff is exposed.

$$\frac{d\sigma}{d\epsilon} = \mathcal{O} \left(\frac{2d(B+\tilde{B})}{d\epsilon} \right)$$

Added Comments on Section 2 on Electron Scattering.

I am getting more of an overview now as to what they are doing. In subsection (a) they computed all virtual photon IR stuff and they concluded that the complete electron scattering amplitude M can be written as (2.3). The thing B contains all the potential IR singularities, and the "rest" or non-IR part of the complete blob electron scattering amplitude is given in terms of the m_i objects. In turn, as shown in (2.12), these m_i are given in terms of the $\tilde{p}_i$ which are certain remainders which were obtained in the iteration.

Now, we go to section (b) where they compute the IR effect of real emitted photons. Now the various objects $\tilde{p}_i$ which make up $\tilde{q}_n$ are really functions of the old m_i . I.e., in order to compute accurately the $\tilde{p}_i$, you need to know the m_i , which is to say, you need to know the non-IR terms due to the virtual photon effects.

So let's say you have worked out the virtual photon corrections and you know all the m_i . Then you know the $\tilde{p}_i$. Thus, from formula (2.17) you know exactly in full details what the DCS with energy loss ϵ is. The sum in (2.17) is over the number of real emitted photons. Of course you know $\tilde{B}$ from a first order computation

Everything is then summarized in (2.21) and (2.22). Notice that the effect of virtual photons is contained both in $\tilde{B}$ and in the various $\tilde{p}_i$.

So obviously, what you want to do is compute what the DCS should be for electron scattering when there is ϵ energy loss. Then go compare it to experiment. Of course this is just a QED problem, but you are summing over all orders and you have to deal with the IR "problem".

In particular, at high energy the infrared exponent $N \times B + \tilde{B}$ is explicitly given in (2.31). So it remains only to find the "non-IR" part of the cross section, which is called $d\hat{\sigma}/d\epsilon$.

Section (c) computes the exponent as I just showed, it shows how the IR singularities cancel, and it tells what the finite piece is!

Section (d) then tries to compute some of the m_i , it only gives m_1 . Again, the point here is to try to compute the m_i so that you can know the $\tilde{p}_i$.

Section (e) then concentrates on the first few $d\hat{\sigma}/d\epsilon$ terms involving zero and then one non-IR photon, which just means keeping first two terms in the series of formula (2.21). The total result keeping only these terms is stated then in (2.57). Again, you see the IR exponent which was already computed, and you see a function F known as in (2.45) as a gamma function, and you see the $\tilde{p}_0$ and $\tilde{p}_1$ functions of course, though the section one is recoded as G_1 .

Finally in section (f) you get an answer for the cross section when you have unobserved photons of energy up to some small ΔE . $\tilde{p}_0$ is just the usual elastic cross section with no photons, something you could compute to any order. In principle, this is the complete answer if you could compute the m_i hence the $\tilde{p}_i$ hence G_1 . (over)

Now try to summarize: you want to compute the observed electron DCS, but you know there are going to be soft photons which you will accidentally also measure so you have to include these also. You know that you have to separate out the internal photon effects as well as the external photon effects in order to get the IR sing to cancel. You do things at once to all orders; you find that to all orders the IR sing cancellation arises from something which you can compute in lowest order.

But you want more than just showing that there are no new IR sing left over. You want to know exactly to all orders (in a certain sense) what the observed DCS will be if there is some resolution soft photon loss. You can compute the elastic DCS so any order you want, but you want to include soft photon effects to all orders, sort of a hybrid situation.

You do this and the result is an explicit formula for the DCS. It involves certain functions which can be computed by considering ~~the~~ the effects to all orders of both internal and external soft photons. So it all boils down to some brute force calculations.

What is odd here is that usually I think of just adding as many Feynman graphs as you want to get an accurate answer. But here you really have to add an infinite number of graphs to get the right answer.

Section 3: Further Examples.

- (a) Consider electron scattering from potential accompanied by one hard Brem photon. You still have to think about the unknown number of soft photons emitted, so this is just an application of the preceeding stuff. The hot photon has k , the soft photons have $\text{EPSILON}-k$ energy. Skimmed only,.
- (b) Inelastic (deep) electron scattering. Radiative corrections make radiative tail. You add correction photons to the electron/photon blob.
- (c) Multiphoton production. Ie, just multiplie Bremsstrahlung, work done by Gupta and repeated here using their new methods.
- (d) Recoil in electron scattering. Here you do radiative corrections on both sides of the carrier photon. Ie, there is an electron blob to correct and a photon blob to correct also.
- (e) Brem off nucleons in nucleon-nucleus scattering.

Comments: The topic here is really that of radiative corrections to scattering processes in field theory. No doubt good references here if I ever want to delve into that subject; right now I want to concentrate on QCD Dyson equation and Jim Ball's papers, so put this on the back burner.

Added notes for Section 3.

Last notes were no good because I didnt understnad the functions of Sections 2.

(a) electron scattering with large energy loss. In the general formula, the integral over β_i which involves i emitted non-IR photons is now called G_i and the sum of all the G_i is called $\hat{G}$. Since usually $\alpha A \ll 1$, the electron DCS is given by my equation (3.5A) page 402.

Now when you have a large energy loss, I think the emission of more than one phton in the non-IR is supporessed, so you use only the G_1 term. You can then go ahead and compute this in the Born approx [now we are back to adding a few Feynman graphs] and the result is (3.9) plotted in the figure. This is something you could compute without going through all this paper. The point here is that we can compute the corrections to this result.

(b) Inelastic electron scattering. Assume now that the "target" can absorb some energy, namely ϵ' , and there are soft photons with energy $\epsilon \rightarrow \epsilon'$... Formula (3.14) gives the DCS in both these energies: you see the usual IR exponent, and all non-IR photons are now being ignored so we only deal with $\tilde{\beta}_0$ which here is $\tilde{\sigma}(\epsilon')$, the inelastic excitation cross section. We want to know simply $d\sigma/d\epsilon$ integrating over the target, because this is the thing you usually observe. This is given in (3.15); if there were no photon emission, the observed DSC would tell you the excitation function. But the presence of the photons causes a convolution of this excitation thing with the photon emission, see (3.15).

This is precisely the theory of the radiative tail. To get the actual excitation spectrum you have to somehow de-convolute.

(c) many photon emission. I have no new comments here: they duplicate Guptas results with their methods.

Section 4: Infrared Analysis in the general Case.

So far we have considered only electron scattering off a potential. Now consider a general Feynman graph with many incoming and exiting charged lines. We now want to consider the IR effect of any number of virtual photons on these lines, and any number of external soft emitted photons. All we want to do is generalize the earlier case where there was only one pair of external charged lines.

No doubt it will turn out that you only have to consider photons on external charged lines, so things will be controllable. The general reduction for the amplitude is given in (4.2) where the $\tilde{S}$ is the multiplying IR factor for photons on external lines in all ways, and where $\tilde{B}_i$ is sort of the Feynman graph with i radiating photons, but including virtual photon corrections. When things are regrouped you get the same kind of exponential factor out ~~from~~ front with a B and a $\tilde{B}$. These factors are given in subsection (b) and are given basically by summing over line-pairs, hence the B_{ij} objects. It turns out that now B can be complex so you only use $\text{Re}(B)$.

The IR singularity cancels between the B and the $\tilde{B}$ as before. However, due to kinematic complications involving the way the soft photons come off, it is too hard to write a general formula giving the various radiative corrections for an arbitrary process. But they try to give some examples and general guidelines.

I have just skimmed this section. The main idea is that you can ignore virtual and emitted photons from inside the blob and therefore the IR factor can be isolated as a multiplicative factor which ends up in the exponent. The virtual and radiated photon IR factors have singularities which cancel. And there are left over nonsingular parts which are the radiative corrections.

Appendix A: extraction of the infrared factors...

Case a: Consider a general Feynman diagram. Pick out one electron line and trace it all the way through the diagram. We want to know: suppose we take an external photon with momentum k and sum over all the stick-on places for this line. As we let k go to zero, can we separate out the piece of the result that is infrared singular?

If there is some singularity, what causes it? Look at (A.3). When you stick this photon onto an external line, you create one extra internal electron propagator. I.e., you make a $1/(\not{p} - \not{k} - m)$. When this operator is put up against $u(p)$, you see in (A.3) that a $1/k$ term is generated. That extra electron prop yields a $1/k$ singularity.

When you sum over all insertions for an external photon in this way, you can classify the insertions into three kinds: insertion on the left end, insertion on the right end, and insertion inside the mess of the diagram. It turns out (this is somehow a result of Ward like thing) that the inside insertions as in figure 3(c) do not yield this infrared singularity. So basically, instead of having to add an infinite number of things, you just add graphs of (a) and (b). The infrared-singular piece of the blob with this addition is given in (A.11). Of course for a true external photon you will dot this into a polarization e_μ and you can then drop the k_μ terms by gauge invariance. Also, set $k^2 = 0$.

So we have now figured out what is the infrared-in- k piece of the amplitude when one extra photon is inserted everywhere on one line.

Case b: More complicated. Now you want to sum over all ways of adding a virtual photon to one line so both ends of the photon are on the line. Figure 2 shows the 6 ways this might be done. It turns out when one or both ends of the photon land inside the blob, there is no IR singular contribution. Thus you need to consider only the parts (a),(b),(c) of the figure. These yield the infrared singular amplitude shown in (A.19) where you see three terms after you square. It happens that this virtual photon thing is tied up with mass and charge renormalization in the UV end of things, but I see no reason to worry about that here. We just want the IR piece.

Comments: there must be a more elegant way to do this, something less brute force. Maybe in their later paper they will explain a new method. For now this is OK. After all, this is 1960, a fairly pioneering paper in IR behavior. Prior to the idea of containment this was not such a hot subject.

Grammer + Yennie

1973

Improved Treatment for the Infrared-Divergence Problem in Quantum Electrodynamics*

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Although the conceptual difficulties associated with infrared divergences in quantum electrodynamics have long since been resolved, a convenient technique for identifying the appropriate infrared-finite part of Feynman graphs has been lacking. Such a technique is presented here. The polarization sums for both real and virtual photons are rearranged into two parts. One of these (called the K -polarization sum) resembles a gauge transformation whose structure permits a simple demonstration of infrared factorization and exponentiation. It is easy to see by inspection that the residual factors (with G -polarization sums) are infrared free. This is done for each separate graph, in contrast to earlier treatments where contributions from sets of graphs had to be considered together. The technique is illustrated by a detailed treatment of the radiative corrections to lowest-order potential scattering, and generalizations to other processes are indicated.

I. INTRODUCTION

The infrared-divergence problem in quantum electrodynamics has a long history which we shall not review here.¹⁻³ It is known that the infrared-divergent contributions to an observable cross section cancel in any given order of perturbation theory, provided all appropriate real- and virtual-photon corrections are taken into account together. Aside from the leading divergence in each order, the complete proof of this cancellation is rather complicated. In practical calculations, one often uses a photon-mass or other infrared cutoff at intermediate stages of the work and then does the infrared cancellation by hand at the very end. Clearly it would be desirable to have a procedure whereby only infrared-free expressions would have to be evaluated.

Recently several authors have given reformulations of quantum electrodynamics using the concept of coherent photon states.⁴ While these reformulations may improve the logical foundations of the subject, they so far seem to do very little for actual practical problems. In effect, they seem to justify the use of a minimum photon momentum in actual calculations. The conceptual problems treated by these authors arise because one would like to be able to define asymptotic in and out fields. In the real world, any experiment is carried out during a finite time interval so the emission of very soft photons is necessarily inhibited. But no physical result is sensitive to the way this happens (precisely because there is an infrared cancellation), and it is not interesting to make a detailed analysis of the dependence on the time duration of an experiment, just as the dependence of scattering results on the details of the wave packets has not been a very practical

subject even if it is conceptually important. Therefore, in the present paper, we shall avoid these conceptual issues and simply assume that there is some convenient infrared cutoff.

Our aim is to present a more transparent demonstration of the infrared separation into divergent factors and infrared-free expressions. The technique will be to rewrite the polarization sums for both real and virtual photons as a sum of two modified polarization sums. One of these polarization sums (K type) will include the infrared-divergent terms which very conveniently factor out and exponentiate when summed over all numbers of photons. The other (G type) will lead to infrared-free expressions which no longer require a cutoff. The infrared factors depend only on the momenta of external charged particles and are already well known.² When real and virtual corrections are suitably combined, they lead to infrared-finite results, as is also well known.

The difficulty of giving such a demonstration is to show that it works to all orders, including the less-divergent terms which are contained within some high-order infrared divergence. In our approach, the infrared factorization becomes a simple problem in algebra and combinatorics; it is separated from the actual analysis of integrals. The demonstration that the residual factor is infrared-free is also rather simple as it involves only looking at, but not carrying out, the integrations. This is to be contrasted with the complicated proof given in YFS (Ref. 2) in which sets of integrals had to be combined in order to obtain the infrared factorization. In the present demonstration, sets of integrals are again combined, but in a simpler manner because the K -type polarization sum has a resemblance to a gauge trans-

1. Introduction. This paper is written about 12 years after the earlier one. They have found a much better way to do things, as described below. Although paper was received in August 1973, paper is restricted to QED and QCD is not mentioned.

Between the two papers the theory of "coherent states" appeared, but they don't think this has much use in practice, so they skip it.

2. Virtual photons. As in the YFS paper, the canonical example is electron scattering, here where there is only one interaction with the potential. Here is how the method works: you want to sum over all virtual photons on the scattered electron line, skip electron loops for now. The trick is to break each virtual photon propagator into a K and G term, then expand the whole thing. Using fancy combinatorics and the Feynman identity (which makes the Ward identity), you can show that the sum of all insertions of a K photon onto a graph with a set of G photons causes a simple multiplicative factor aB where $a = 1/137$. When you then go ahead and sum all graphs including all mixtures of both kinds of photons, you get the YFS result: the K photons sum to make the exponential, and then the m_l residual IR-finite functions are the Feynman graphs with only G photons included. Much nicer than the earlier proof.

So the Feynman identity is now what tells you to ignore inside insertions, somehow. I have not really followed the details but could if I had to. For closed electron loops, you always get zero in the limit that an attached photon is soft (I don't yet know why this is so) so you can forget them. There is a detail about photons which are self-energy corrections on the external lines, but not a big deal.

In last section of this part of the paper they show exactly how it is that when you compute a simple Feynman graph with only G photons, there is never any divergences. Nor do you seem to need any kinds of a cutoff.

3. Real photons. Now consider the situation where you have n unobserved soft photons and maybe also some observed ones. The amplitude is as in (3.1) where you see the virtual exponential already exposed and you see all the polarizations of the unobserved ones. In computing the DCS you have to sum over all these polarizations. These sums make propagator like numerators $g_{\mu\nu}$. As in the virtual case, you break these down into a G and K type term. Somehow the K type terms cause the other exponential and you get the result shown in (3.21). This is the same as the old result, but it was derived in a much better way.

4. Generalizations. Comptons scattering is easy since we already did the general case with observed photons. You can divide the p -leg from the p' -leg at either hard photon and get equivalent results.

Next the closed dfermion loop detail is mentioned and gives nothing.

Finally, electron-electron scattering is more complicated but can be done, as can multiplie Coulomb scattering.

length of paper: 26 standard pages.

LÉVY & SUCHER

1969

Eikonal Approximation in Quantum Field Theory*

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The eikonal approximation for high-energy collisions, long familiar in the theory of potential scattering, is considered from the viewpoint of relativistic quantum field theory. We study, in particular, the Feynman amplitude $M(s,t)$ describing the scattering of two spin-0 particles, a and b , interacting by the exchange of spin-0 mesons. We show that if $M_n(s,t)$, the contribution to $M(s,t)$ arising from all n th-order Feynman diagrams in which exactly n mesons are exchanged between a and b , is written in an appropriately symmetrized way, and if the terms in any a or b particle propagator which are quadratic in the internal momenta are then dropped, the resulting expression, $M_n^{\text{eik}}(s,t)$, may be evaluated in closed form, and the sum over n , which defines $M^{\text{eik}}(s,t)$, may be carried out. The representation of $M^{\text{eik}}(s,t)$ found in this way involves the exponential of a function χ of a relative space-time variable $x=(x^0, \mathbf{x})$ and the external momenta; χ is a relativistic generalization of the eikonal χ_{pot} familiar from the theory of high-energy potential scattering. $M^{\text{eik}}(s,t)$ is both crossing-symmetric and time-reversal-invariant. In the static limit ($m_b \rightarrow \infty$), χ tends to χ_{pot} for the appropriate Yukawa potential and $f^{\text{eik}} = -M^{\text{eik}}/8\pi\sqrt{s}$ has a limiting form $f_{\text{pot}}^{\text{eik}}$, which we also derive directly from the theory of potential scattering; for small scattering angles, $f_{\text{pot}}^{\text{eik}}$ coincides with the standard result. The amplitude for particle-antiparticle scattering is studied in the same model. It is shown that the eikonal $\chi_2(x)$ associated with the contribution of all annihilation-type diagrams has a logarithmic singularity at $x=0$ whose coefficient is proportional to $\alpha(t)+1$, where $\alpha(t)$ is the Regge-trajectory function obtained from the asymptotic behavior of the ladder-type diagrams alone. Another connection with Regge behavior is made by showing that the summation of a certain infinite class of radiative corrections to the lowest-order γ - e Compton amplitude gives rise, in our eikonal approximation, to an eikonal $\chi(x)$ which has a similar logarithmic singularity with strength $1+\beta(t)$; here $\beta(t)$ is the trajectory function, introduced less directly in earlier work, which reproduces the major part of the spectrum of positronium on setting $\beta(t)=l=n-1$. A generalization of a simple algebraic identity used in the derivation of the above results, in the form of an integral representation, permits their extension to the case where one or more particles are off the mass shell. This is illustrated by a computation of an eikonal-type approximation to the Green's function for a relativistic particle moving in an external scalar field and by the summation of an infinite class of contributions to the vertex function in the model referred to above. The possibility of applying an off-shell eikonal approximation to the analysis of production processes is emphasized.

I. INTRODUCTION

IN recent years, there has been great interest in high-energy approximations to scattering amplitudes which exhibit an exponential dependence on some of the kinematical variables, especially in connection with the revival of Regge theory. Simple approximations of this type have been known for a long time in the theory of nonrelativistic potential scattering: These are the so-called eikonal type of approximations.¹⁻³ For example, for a spinless particle of mass m scattered by an external potential $V(x)$ the scattering amplitude $f(\mathbf{p}', \mathbf{p})$ may be approximated, for large $|\mathbf{p}|$ and small scattering angle and under suitable restrictions on V ,⁴ as

$$f(\mathbf{p}', \mathbf{p}) = \frac{|\mathbf{p}|}{2\pi i} \int d^2b e^{i(\mathbf{p}' - \mathbf{p}) \cdot \mathbf{b}} (e^{i\chi(\mathbf{b})} - 1), \quad (1.1)$$

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† On leave of absence from the Faculté des Sciences of the University of Paris, Paris, France.

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¹ G. Molitère, Z. Naturforsch. 2, 133 (1947).

² R. J. Glauber, in *Lectures in Theoretical Physics*, edited by W. E. Britten and L. G. Dunham (Wiley-Interscience, Inc., New York, 1959), Vol. I, p. 315.

³ L. I. Schiff, Phys. Rev. 103, 443 (1956); see also D. S. Saxon and L. I. Schiff, Nuovo Cimento 6, 614 (1957).

⁴ See, e.g., Ref. 2, pp. 342-344.

where $\mathbf{b}$ is a 2-component vector orthogonal to $\hat{\mathbf{p}} = \mathbf{p}/|\mathbf{p}|$ and the "eikonal" χ is defined by

$$\chi(\mathbf{b}) = \frac{-m}{|\mathbf{p}|} \int_{-\infty}^{\infty} V(\mathbf{b} + \hat{\mathbf{p}}\xi) d\xi. \quad (1.2)$$

It seems worthwhile to ask to what extent analogous approximations for a two-body scattering amplitude may be obtained in quantum field theory. In the present paper we show that there is indeed a natural relativistic generalization of the eikonal approximation. The techniques used are rather simple and may be useful for deeper investigations of the asymptotic behavior of scattering amplitudes.

The usual derivations of equations such as (1.1) are based on calculations which start with expressions for the scattering amplitude and the Schrödinger wave function in position space. In Sec. II, we reconsider the problem of nonrelativistic potential scattering in momentum space, starting with an exact expression for the n th-order term f_n in the Born expansion of f . We show that if in the energy denominators appearing in this expression, terms of the form $\mathbf{K}^2$ are dropped relative to terms of the form $\mathbf{p} \cdot \mathbf{K}$, where $\mathbf{K}$ is a partial sum of internal momenta, the resulting approximation to f_n may be evaluated in closed form, with the help of an identity used in earlier, closely related work in quantum

Ball, Horn & Zachariassen
(1977)

NEAR MASS SHELL SINGULARITIES IN QUANTUM ELECTRODYNAMICS *

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We discuss the near mass shell infrared behavior of QED by performing an explicit sum over all Feynman diagrams in the eikonal approximation. We review the infrared singularities of exclusive amplitudes in particular limits ((a) small photon mass or dimension $\neq 4$, (b) equal off shell p_i^2 , (c) large momentum transfers) as special cases of a general parametric formula. In the parametric representation the infrared singularities always exponentiate. This allows us to derive simple differential equations for Laplace transforms of the scattering amplitudes. Similar differential equations have been conjectured to hold in QCD and we summarize the present evidence regarding this assumption.

1. Introduction

It is frequently suggested that confinement of color in non-Abelian gauge theories is a consequence of their highly singular infrared (IR) behavior (1). Consequently it is of considerable interest to understand just what this behavior is.

Non-Abelian gauge theories are an extension of Abelian ones, and for the Abelian case the IR behavior is completely understood (2). It is therefore natural to try to build the study of the non-Abelian IR regime on this base, in order to emphasize in exactly what ways the Abelian situation is modified. This motivates our discussion and review of IR singularities in QED in a language perhaps more suitable to the problem at issue in the non-Abelian QCD case. Then we outline the simplest possible conjecture as to how QCD modifies what happens in QED. We do not claim to exhi-

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Questions:

- 1) why is S called unrenormalized on top of page 514 ? NA
- 2) what happened to the i's in (2.10) ? NA
- 3) how are 2.16 and 2.17 computed ?
- 4) why can external closed loops be ignored as in figure 1? NA
- 5) point out that Levy Sucher is misquoted. NA

Ball, Horn and Zach: "near-mass-shell sings in QED " (rec'd Sept. 1977, the month I arrived in Utah)

1. Interuduction. The IR theory of QED is under control, references to the Yennie papers which I have now studied. They want to re-examine QED in hopes of seeing how to go about attacking QCD.

Two methods of doing regulation: first, you can put in a photon mass, or you can vary the number of dimensions; second, you can merely compute things which are not IR singular and make no change to your theory. In particular, off-shell amplitudes tend to not have the IR singularities.

The basic question you are asking is different in the two theories. In QED you want to show that the observed inclusive cross sections are finite. You can never measure an exclusive cross section, so the fact that these come out infinite in QED is no problem.

In QCD however you are never supposed to get any gluons out, soft or otherwise. Thus, here you want to show that the color thresholds are absent. Recall that gluon carries color, so you want to see that there are no multi-gluon branch points; if there are, then you can produce gluons through unitarity.

Thus there are no "inclusive cross sections" with unobserved soft gluon emission. The goal is this: consider off-shell amplitudes (since you thereby avoid the IR singularities) and see if gluon thresholds are inhibited somehow. That is the problem.

2. Infrared behavior of the IR propagator.

We want to know what the full propagator looks like near $p^2 = m^2$ after you dress on all the photons. Apparaently you only have to add all graphs of the form of figure 4 to ~~the~~ get the full blob as far as the IR singular piece is concerned.

They explicitly add all the graphs and get the answer given in (2.10). The characteristic exponentiation is obtained. Near $x=0$, the behavior of the integral in (2.10) is controlled by the behavior of $H(s)$ at large s , hence they figure out ~~(2.11)~~ (2.12). Using a mass counterterm, this all boils down to (2.13). We have of course an explicit expression for $G(s)$ as shown in (2.12). By the way we have scalar electrons here so far.

Now lets go for the answer: what does this renormalized or dressed propagator look like near $p^2 = m^2$? He claims that you can just examine $G(s)$ for large s and get the result (2.18): you see the pole and a cut due to the multi-photons states. On the other hand, you can first put in a photon cutoff mass and take s to infinity to get the form 2.16.

I think their point is that things are clearer if you stay away from the exact point $p^2 = m^2$ as ask: what is the nature of the singularity there? Then you see that cut. Not clear what you get in the photon-mass case.

derive (2.15)

$$S_\lambda = \int_0^\infty ds e^{is(p^2 - m^2) + G_\lambda(s)}. \quad \text{Why does this have a pole?}$$

Assume $p^2 - m^2 = \epsilon$ is small, $S_\lambda \approx \int_0^\infty ds e^{is\epsilon + G_\lambda(s)}.$

Well, consider:

$$\int_0^\infty ds e^{-sa}.$$

This has a pole at $a=0$ because that is where integral diverges. I.e., caused by upper end. So you care about integral at large s .

Near $p^2 = m^2$, integral is dominated by high end:

$$S_\lambda \Big|_{p^2 \approx m^2} \approx e^{G_\lambda(\infty)} \cdot \frac{i}{p^2 - m^2}$$

So $\lim (p^2 - m^2) S_\lambda = i e^{G_\lambda(\infty)}. \quad \checkmark \quad \underline{\underline{(2.15)}}$

Meanwhile

$$G_\lambda(\infty) = -i \int \frac{d^4 k}{(2\pi)^4} \frac{4\pi e^2}{k^2 - \lambda^2} \frac{e^{is(k^2 - 2p \cdot k)} - 1}{(k^2 - 2p \cdot k)^2} \Big|_{p^2 = m^2} \left(p^2 - s \frac{(p \cdot k)^2}{k^2} \right)$$

assume $\rightarrow 0$ as $s \rightarrow \infty$, hence (2.6a).

So you must assume a direction in the S -plane.

Compute (2.16b)

$$4e^2 i \int \frac{d^3 k}{(2\pi)^4} \frac{1}{(k^2 - \lambda^2)} \frac{1}{[k^2 - 2p \cdot k]^2} \left(m^2 - 5 \frac{(p \cdot k)^2}{k^2} \right) \Big|_{p^2 = m^2}.$$

Only care about near $k=0$. $p \cdot k = Ek - pk \cos \alpha$
 $= k(E - p \cos \alpha).$

so

$$\frac{4e^2 i}{(2\pi)^4} \int \frac{d^4 k}{(k^2 - \lambda^2)^{1/2} \underbrace{(k^2 - 2[E - p \cos \alpha])^2}_{\uparrow \text{drop}}} (m^2 - 5 [E - p \cos \alpha]^2)$$

$$\propto \int \frac{d^4 k}{k^2 (k^2 - \lambda^2)}$$

Start again: $p \cdot k = pkz$.

$$G_\lambda(\omega) = \frac{i 4e^2}{(2\pi)^4} 4\pi \int_0^1 dz \sqrt{1-z^2} \cdot \int_0^\infty k^3 dk \cdot \frac{1}{k^2 - \lambda^2} \cdot \frac{1}{k^2} \cdot \frac{1}{(k - 2pz)^2} \cdot p^2 (1 - z^2)$$

well, for small k^2 region

$$\int_0^\infty k^2 \frac{dk^2}{2} \frac{1}{k^2 - \lambda} \frac{1}{k^2} \cdot \text{const} \propto \int_0^\infty \frac{dx}{x - \lambda} \approx \underline{\underline{\ln \lambda}}.$$

Check (3.21). Page 925 of GR says:

$$\int_0^x \left(\frac{e^{-t} - 1}{t} \right) dt = \underbrace{Ei(-x)}_{-e^{-x} \psi(1,1;x)} - C - \ln x.$$

$$\underbrace{-e^{-x} \psi(1,1;x)}_{x^{-1} \text{ as } x \rightarrow \infty}.$$

$$\text{So } \lim_{x \rightarrow \infty} \int_0^x \left(\frac{e^{-t} - 1}{t} \right) dt = -C - \ln x.$$

Hence:

$$\int_0^a \frac{dz}{z} (e^{iAz} - 1)$$

$$\text{Let } t = -iAz$$

$$= \int_0^{-iAa} \frac{dt}{t} (e^{-t} - 1) \approx -\ln(-iAa) \sim -\underbrace{\ln A}_{\text{dominates}} - \ln(-in)$$

Appendix: Generalized Levy Sucher Identity.

There is a certain amount of history behind this thing. First, suppose you want to make a sum over all virtual photon insertions on a scalar electron line. Assume there are no external photons. If you are interested only in the IR behavior of the result, you can make the approximation called Rytov or super-eikonal of neglecting denominator terms which are quadratic in the photon momenta. You now want to sum, in this approximation, all graphs of the form shown in figure 4, ie virtual photons no matter how they land, nexted or whatever.

This sum is a special case of the Generalized IS Idneity derived in this index, ie, there are M internal photons and no external ones. Actually, we are summing over all possible ways to insert M internal photons, we are not yet summing M . The result is given in (A.12) for this special case, and this then appears in (2.7) of the text. There they have summed over n photon loops. I gather that no-one ever did this simple calculation before?????

The original Levy-Sucher identity answered the question: how do you sum over all ways of inserting N external photons onto an electron line. That requires (in the Rytoc) simply a permutation sum of terms like (A.2) and the sum is (A.3). This original Levy-Sucher I have now derived and I see how it works.

The generalized question is this: how do you sum over all ways of inserting N external and M internal photons?? The answer is given in (A.7) as a single integral of a product of two products. This thing is derived by a recursion relation rather than by some brute force proof.

A little differential operator may be used to convert a photon from external to internal. Thus, in (A.15) they show how the case of M -internal photons can be written as a derivative of the M -external, original Levy-Sucher result.

Laplace-Sudler Identity.

$$\sum_{\mathbf{L}} \frac{1}{x} \frac{1}{(x+a_1)} \frac{1}{(x+a_1+a_2)} \dots \frac{1}{(x+a_1+a_2+\dots+a_n)} = A.$$

$$= \sum_{\mathbf{L}} \int_0^\infty dx_1 \int_0^\infty dx_2 \dots \int_0^\infty dx_n e^{-x_1} e^{-(x+a_1)x_1} e^{-(x+a_1+a_2)x_2} \dots e^{-(x+a_1+\dots+a_n)x_n}.$$

$$= \sum_{\mathbf{L}} \left(\prod_{i=1}^n \int_0^\infty dx_i \right) \int_0^\infty dx_0 e^{-x(x_0+a_1+a_2+\dots+a_n)} \times e^{-a_1(a_1+x_2+\dots+x_n)} \times e^{-a_2(a_2+\dots+x_n)} \times \dots \times e^{-a_n x_n}.$$

$$\frac{1}{x} \prod_{i=1}^n \left(\frac{1}{x + \sum_{j=i}^n a_j} \right) = \int_0^\infty dx e^{-xs} \prod_{i=1}^n \left(\frac{1-e^{-a_i s}}{a_i} \right)$$

Now let:

$$\begin{aligned} x_0 + x_1 + \dots + x_n &= \beta_0 \\ x_1 + \dots + x_n &= \beta_1 \\ &\vdots \\ x_n &= \beta_n. \end{aligned}$$

$$\begin{aligned} x_0 &= \beta_0 - \beta_1 \\ x_1 &= \beta_1 - \beta_2 \\ &\vdots \\ x_n &= \beta_n - 0 \end{aligned}$$

What is the jacobian?

$$J = \begin{vmatrix} \partial \beta_0 / \partial x_0 & \partial \beta_0 / \partial x_1 & \partial \beta_0 / \partial x_2 & \dots & \partial \beta_0 / \partial x_n \\ \partial \beta_1 / \partial x_0 & & & & \\ & \ddots & & & \\ & & \partial \beta_n / \partial x_n & & \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & \dots & 1 \\ 0 & 1 & 1 & \dots & 1 \\ 0 & 0 & 1 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{vmatrix}$$

So what is det? Det = 1!

What is domain in β -space? For each set of $\{a_i\}$, $\{\beta_i\}$ is uniquely set.

How about

$$\prod_{i=0}^n \left(\int_0^\infty dx_i \theta(x_i) \right) = \prod_{i=0}^n \left(\int_{-\infty}^\infty d\beta_i \theta(\beta_i - \beta_{i-1}) \right)$$

$$\text{So } \theta(\beta_0 - \beta_1) \theta(\beta_1 - \beta_2) \dots \theta(\beta_n) \Rightarrow \beta_0 \geq \beta_1 \geq \beta_2 \dots \geq \beta_n \geq 0.$$

So we now have:

$$A = \sum_{\beta} \prod_{i=0}^n \left(\int_0^{\infty} d\beta_i \right) e^{-x\beta_0 - a_1\beta_1 - a_2\beta_2 - a_3\beta_3 \dots - a_n\beta_n}$$

$\beta_0 > \beta_1 > \beta_2 \dots > \beta_n$

This thing is integrated over some sector of β -space. How many sectors are there?

$$\left. \begin{array}{l} \beta_0 > \beta_1 > \beta_2 \\ \beta_0 > \beta_2 > \beta_1 \\ \beta \end{array} \right\} \text{if I enumerate all possibilities, I get all of } \beta\text{-space}$$

$$A = \int_0^{\infty} d\beta_0 e^{-x\beta_0} \underbrace{\sum_{\beta_1 > \beta_2 > \dots > \beta_n} \prod_{i=1}^n \left(\int_0^{\infty} d\beta_i \right) e^{-a_1\beta_1 - a_2\beta_2 \dots - a_n\beta_n}}_{= \prod_{i=1}^n \left(\int_0^{\infty} d\beta_i e^{-a_i\beta_i} \right)}$$

Look at domain union

$$(\beta_0 > \beta_1 > \beta_2 > \beta_3) \cup (\beta_0 > \beta_1 > \beta_3 > \beta_2) \cup \text{etc} = (\beta_0 > \beta_i > 0)$$

$$A = \sum_{\beta_0 > \beta_i > 0} \prod_{i=0}^n \left(\int_0^{\infty} d\beta_i \right) e^{-x\beta_0 - a_1\beta_1 \dots - a_n\beta_n}$$

$$= \int_0^{\infty} d\beta_0 e^{-x\beta_0} \left[\int_0^{\beta_0} d\beta_1 \int_0^{\beta_1} d\beta_2 \dots \int_0^{\beta_{n-1}} d\beta_n \right] e^{-a_1\beta_1 - a_2\beta_2 \dots}$$

$$= \int_0^{\infty} d\beta_0 e^{-x\beta_0} \prod_{i=1}^n \left[\int_0^{\beta_0} ds e^{-a_i s} \right] = \int_0^{\infty} ds e^{-xs} \prod_{i=1}^n \left(\frac{1 - e^{-a_i s}}{a_i} \right)$$

$= \frac{1}{a_i} (e^{-a_i \beta_0} - 1)$ QED

Misc.

IR behavior of NAGT, II"

Phys. Rev. D15, 2937 (1977)

The Introduction

① These guys have an ongoing program of IR research.

CT, PRL 35, 338 (1975)

CT, PR D13, 3370 (1976)

CT, proceed. of 3rd Orbis Scientiae, Univ. Mainz 1976, Perlmutter ed.These studies are in momentum space with gluon mass μ as regulator.

② Recall how in looking for HE behavior of field theory, you add up ladders using the "leading log" in each order of g . Each ladder gives $(\ln s)^n$ but when you add up all the ladders you get S^{γ} , regge poles. There seems to be a similar analysis to get the LE IR behavior, and this also leads to some kind of exponential in leading-log.

③ In this paper, authors convert to coordinate space view rather than p-space. The

p-space gluon propagator is

We are interested in $\bar{g}^2(q^2)$ as

$$g^2 D_{\mu\nu}(q) \rightarrow g_{\mu\nu} \bar{g}^2(q^2)/q^2$$

 $q^2 \rightarrow 0$. Maybe $\bar{g}^2(q^2) \approx q^{-2}$,which is a harmonic osc. binding potential ($\bar{q}^1 \rightarrow$ linear potential).

④ "In QED, the leading-log's contain all IR sing effects" [see Yennie et al.]:

OK. Recall that they would write for each Feynman graph [each order],

Graph = B x rest where B contained IR-sing part. Then they compute

B + keep only the leading-log. $\ln(m^2/\delta)$ or whatever. [δ = photon mass]Then you add up all orders + get $\exp(B)$ out front. Then $B + \bar{B} = 0$

+ you are all set. So non-leading log's in B apparently make no difference in this analysis.

⑤ This paper will be in effect an internal analysis. In the HE case I know that internal $\Rightarrow$ ladders $\Rightarrow$ leading-log approx to get S^{γ} . Similarly here, except low energy analysis. They hope [based on QED] that nonleading logs will not affect IR behavior in NAGT.

⑥ A mass shell analysis is meaningless if particle has no mass (i.e., propagator has no pole since confined); this makes p-space analysis clumsy, leaves $\bar{x}$ -space OK.

Reference here data:

[t'Hooft NPB75, 461 (1974)
" 2-Dim QCD in lightcone gauge
used $N \rightarrow \infty$ where $SU(N_{col})$; large N allows you to ignore certain graphs
quark poles washed out!

[Callan, Coles, Gross PRD13, 1649 (76)
continuation of above t'Hooft idea

Einhorn [PRD14, 3451 (76)] showed that the $N \rightarrow \infty$ trick is not needed; those ignored graphs have no IR singe anyway.

⑩ If you ignore quark loops, QCD becomes a classical theory (string).
2nd quantized $\rightarrow$ 1st quantized.

Bars [] starts with string + ends up with QCD,
ie, goes the other way.

⑪ Outline of this paper:

- II. IR singe in $\bar{x}$ -space.
- III. Eikonal construction
- IV. NAGT
- V. Eikonalized Fermion Propagators.
- VI. Summary.
- App: two-dim QCD.

Polyakov [PhysLett 59B, 82 (1975)] made up a "pseudoparticle" in 3D QED.

⑫ This Intro is loaded with refs which I could trace if I had time. But goal now is Tim's engineering calculation.