

$\lambda\phi^3$ in $d=6$

ϕ^4 in $d=4$

QED

The first thing I learned was how you renormalize a field theory and get a finite result. Basically, you show finiteness by examining integral equations. The trick is to use counterterms (ie, rescalings) which contain the cutoff. It took me a long time to understand how you do all this, but I think I've now got it. Can take QED, eg, and compute something and get a clear finite answer. I understand these things without having understood Hepp. The overlap analysis was crucial to showing how a certain Z_5 factor was removed, similar to BD deal.

All the above work was done without thinking about "renormalization point" or anything fancy. I choose to renormalize at the physical mass m^2 : ie, I set up my "renormalization conditions" at $p^2 = m^2$. Basically, this had to do with setting the scale of the vertex and propagator. I used mass counterterm method in the above, never had a bare mass.

In the next section of notes I considered the idea of using some other point to define the renormalization conditions. I think that each "renormalization part" in a field theory needs renorm conditions, maybe more than one if D.G.T.O. Obviously when you have a divergent n-point function you are going to have to effect subtraction of the divergence with counterterms, or rescaling of basic objects like the charge. If you have to subtract some function, you lose control over its finite constant part. Thus, you must specify the value of the function somewhere. This is what a renormalization condition does. The value of p^2 where you do the subtraction or specification or whatever is called the renorm point. So I did ϕ^3 again with an arbitrary renorm point u_1 . The Z's you get coming out of the original bare theory then depend on u_1 , so therefore does the charge. My mass m_1 happened not to depend on u_1 , but Abarb's did so I kept this possible dependence.

Then I asked: suppose you choose some other renorm point u_2 . Then you get a different but finite theory, your Z's are a little different. These theories 1 and 2 are then connected by a finite GR multiplication rule. Its not very complicated: if you change your renorm point u_1 to u_2 , you want to impose the same renorm condition on each renorm part, but you do so at different value of p^2 . Same as letting keep same renorm point and allowing different scale sets, eg, different "residue" etc. These scale sets are not observables; you can always scale up your propagator and scale down your vertex and have same result. Only the Z's have changed, etc. Physical finite theory is the same.

This leads to the idea of the invariant vertex function or charge d. You find a combination which is the same despite rescalings: this is the object of physical interest. You ask how it behaves at large and small q^2 and get asymptotic freedom perhaps, or just a finite infrared charge in the case of QED. This "running coupler" is the object of especial interest: you want to know all about it. Some information can be obtained from the BETA function.

$$\phi^3 \text{ in } d=6$$

Contents of this Section.

1. Summary of ϕ^3 in $d=6$, ie, summary of notes in later section of this binder,. This work was done using usual mass renorm point m . Main idea was to show the finiteness of the renormed theory due to integral equations.
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17. A FEW residual QED notes, to be killed soon.

Summary of ϕ^3 Theory $d=6$

1. Propagator

(a) unrenormalized:

$${}^0D = \frac{1}{p^2 - m^2 - \Sigma^2(p^2)} = \frac{1}{(p^2 - m^2)[1 - {}^0\Sigma^2(p^2)]}$$

$${}^0\Sigma^2(p^2) = {}^0\Sigma^2(m^2) + (p^2 - m^2) {}^0\tilde{\Sigma}^2(p^2) \quad {}^0\Sigma^2(m^2) = 0 \text{ see below.}$$

$${}^0D(p^2 \approx m^2) = \frac{Z_3}{p^2 - m^2} \Rightarrow Z_3^{-1} = 1 - {}^0\tilde{\Sigma}^2(m^2) \text{ defines } Z_3$$

(b) renormalized:

$$D = \frac{1}{p^2 - m^2 - \Sigma^2(p^2)} = \frac{1}{(p^2 - m^2)[1 - \Sigma^2(p^2)]} = Z_3^{-1} {}^0D$$

$$\Rightarrow \Sigma^2(p^2) = (p^2 - m^2)(1 - Z_3) + Z_3 {}^0\Sigma^2(p^2)$$

$$\Rightarrow \Sigma^2(m^2) = Z_3 {}^0\Sigma^2(m^2)$$

$$\tilde{\Sigma}^2(p^2) = (1 - Z_3) + Z_3 {}^0\tilde{\Sigma}^2(p^2) = Z_3 \cdot [{}^0\tilde{\Sigma}^2(p^2) - {}^0\tilde{\Sigma}^2(m^2)]$$

(c) Renorm. Conditions:

$$1. \quad \Sigma^2(m^2) = 0 \text{ (adjust } \delta m^2 \text{ to make true in L.O.)}$$

$$\Rightarrow \text{pole is at mass } m.$$

$$2. \quad \tilde{\Sigma}^2(m^2) = 0 \text{ is automatic, given } D = Z_3^{-1} {}^0D$$

$$\Rightarrow \text{pole residue is unity}$$

(adjust $Z_3(\lambda)$ to make this true in L.O.)

(d) Lowest Order Unrenorm Self energy:

$${}^0\Sigma^2(p^2)^{(2)} = \delta m^2^{(2)} + \frac{\lambda_0^2}{2} \left[(A\Lambda^2 + m^2 B \ln \Lambda) + (p^2 - m^2) \{ B \ln \Lambda + C(p^2/m^2) \} \right]$$

$$= \text{---} \times \text{---} + \frac{1}{2} \text{---} \bigcirc \text{---}$$

$$\begin{aligned} {}^0D &= Z_3 D & \Gamma &= \lambda \tilde{\Gamma} \\ {}^0\Gamma &= Z_3^{-3/2} \Gamma \\ {}^0K &= Z_3^{-2} K \\ \lambda_0 &= Z_5 Z_3^{-3/2} \lambda \\ {}^0\phi &= Z_3^{1/2} \phi \end{aligned}$$

2. Vertex



(a) unrenormalized:

$$^0\Gamma(p_1^2, p_2^2, p_3^2) = 1PI = \text{symmetric}$$

$$^0\Gamma(m^2, m^2, m^2) = ^0\Gamma(\text{sym}) = \sum_5^{-1} \lambda_0$$

$$\text{sym} \Rightarrow p_i \cdot p_j = m^2 (2\delta_{ij} - 1)$$

$$^0\tilde{\Gamma} \equiv ^0\Gamma / \lambda_0$$

defines $\sum_5$

$$p_1 \cdot p_2 = -m^2$$

(b) renormed:

$$\Gamma \equiv \sum_3^{3/2} ^0\Gamma$$

$$\tilde{\Gamma} \equiv \Gamma / \lambda$$

$$\Rightarrow \Gamma(\text{sym}) = \sum_3^{3/2} \sum_5^{-1} \lambda_0$$

$$\tilde{\Gamma} = \sum_5 ^0\tilde{\Gamma}$$

$$\tilde{\Gamma}(\text{sym}) = \sum_3^{3/2} \sum_5^{-1} \frac{\lambda_0}{\lambda}$$

(c) renorm condition:

$$3. \quad \begin{aligned} \Gamma(\text{sym}) &= \lambda \\ \tilde{\Gamma}(\text{sym}) &= 1 \end{aligned} \quad \Rightarrow \quad \lambda = \sum_3^{3/2} \sum_5^{-1} \lambda_0$$

(adjust $\sum_5$ in L.O. to make true).



(d) lowest order vertex 1renorm.

$$^0\Gamma(p_i^2) = \lambda_0 + \lambda_0^3 [Flu + \tilde{f}^+(p_1^2, p_2^2, p_3^2)]$$

$$= \text{diagram 1} + \text{diagram 2}$$

3. Kernel + general $\mathcal{Z}^5$

$$(a) \quad ^0K(p_1^2, p_2^2, p_3^2, p_4^2; p_1 \cdot p_2, p_1 \cdot p_3) = 4\text{-pt } 1PI \text{ minus } \text{diagram} \text{ graphs.}$$

$$^0K = \text{diagram 1} + \text{diagram 2} + \text{diagram 3} + \text{diagram 4} + \text{etc.}$$

$$K \equiv (\sum_3^{\frac{1}{2}})^4 ^0K$$



$$\mathcal{Z}_N = (\sum_3^{\frac{1}{2}})^N \mathcal{Z}_N \quad \text{where } \mathcal{Z}_N = \text{proper } N\text{-point.}$$

Note: these $\sum_3^{\frac{1}{2}}$ factors are like those in LSZ = "wavef renorm"

(b) K and Σ_N can be written as skeleton expansions,

$$K = \sum_s f_s(\tilde{\Gamma}, D, \lambda)$$

To any finite order in λ , K will be divergence-free, since $\tilde{\Gamma}$ and D are divergence free and $D = -2$ on overall integrals, and $\exists$ no bad subgraphs among skeleton graphs.

4. The Self-Energy Dyson Equation

(a) The unrenormalized version:

$$\begin{aligned} {}^0\Sigma^2(p^2) &= \frac{1}{2} \lambda_0 \int D^0 D^0 \tilde{\Gamma} + \delta m^2 = \frac{1}{2} \text{diagram} + \text{cross} \\ &= \lambda^2 \Sigma_3^{-1} \left\{ \Sigma_5 \cdot \frac{1}{2} \int D D \tilde{\Gamma} \right\} + \delta m^2 \\ &\quad \xrightarrow{\text{logs cancel}} \equiv E(p^2) \end{aligned}$$

$$\begin{aligned} E(p^2) &\equiv \frac{1}{2} \Sigma_5 \int D D \tilde{\Gamma} = E(m^2) + (p^2 - m^2) \tilde{E}(p^2) = \frac{1}{2} \Sigma_5 \text{diagram} \\ &\quad \text{has no divergences} \\ &= \frac{1}{2} \int \tilde{\Gamma} D D \tilde{\Gamma} - \frac{1}{4} \int \tilde{\Gamma} D D K D D \tilde{\Gamma} \\ &= \frac{1}{2} \text{diagram} - \frac{1}{4} \text{diagram} \quad \text{as will later be shown.} \end{aligned}$$

Thus, we see that:

$$\begin{cases} {}^0\Sigma^2(m^2) = \lambda^2 \Sigma_3^{-1} E(m^2) + \delta m^2 \leftarrow = 0 \text{ sets } \delta m^2. \\ {}^0\Sigma^2(p^2) = \lambda^2 \Sigma_3^{-1} \tilde{E}(p^2) \end{cases}$$

(b) renormalized version:

$$\begin{aligned} \tilde{\Sigma}^2(p^2) &= \lambda^2 [\tilde{E}(p^2) - \tilde{E}(m^2)] = \lambda^2 \Delta \tilde{E}(p^2) \\ \tilde{\Sigma}^2(m^2) &= 0 \text{ since } \delta m^2 = -\lambda^2 \Sigma_3^{-1} E(m^2). \end{aligned}$$

$$\Sigma_3 = 1 + \lambda^2 \tilde{E}(m^2)$$

$$D(p^2) = \frac{1}{(p^2 - m^2) [1 - \lambda^2 \Delta \tilde{E}(p^2)]}$$

5. The Vertex Dyson Equation.

(a) unrenormed version:

$${}^0\Gamma(p_i) = \lambda_0 + \frac{1}{2} \int {}^0\tilde{\Gamma} {}^0D {}^0D {}^0K \quad \text{or} \quad {}^0\tilde{\Gamma}(p_i) = 1 + \frac{1}{2} \int {}^0\tilde{\Gamma} {}^0D {}^0D {}^0K$$

(b) renormed version:

$$\tilde{\Gamma}(p_i) = Z_5 + \frac{1}{2} \int \tilde{\Gamma} D D K = Z_5 + \frac{1}{2} \text{diagram}$$

eliminate Z_5 from this to get E deal.

Let: $F(p_1^2, p_2^2, p_3^2) \equiv \frac{1}{2} \int \tilde{\Gamma} D D K = [F_{\text{lu}} + f^+(p_1^2, p_2^2, p_3^2)] \lambda^2$

Then: $Z_5 = 1 - F(\text{sym}) = 1 - [F_{\text{lu}} + f^+(\text{sym})] \lambda^2$

$\Rightarrow \tilde{\Gamma}(p_i) = 1 + \Delta f^+(p_i^2) \lambda^2 = \text{obviously finite.}$

6. Second-Order Calculations

(a) $E^{(0)}(p^2) = \frac{1}{2} \text{diagram} = \underbrace{\frac{1}{2} (A\Lambda^2 + m^2 B_{\text{lu}})}_{E(m^2)} + (p^2 - m^2) \underbrace{\frac{1}{2} [B_{\text{lu}} + C(p^2)]}_{\tilde{E}(p^2)}$

$\Rightarrow Z_3^{(2)} = 1 + \lambda^2 \tilde{E}^{(0)}(m^2) = 1 + \lambda^2 \frac{1}{2} [B_{\text{lu}} + C(m^2)]$

$\tilde{\Sigma}^{(2)}(p^2) = \lambda^2 \cdot \frac{1}{2} \cdot \Delta C(p^2)$

$D(p^2)^{(2)} = \frac{1}{(p^2 - m^2) [1 - \frac{1}{2} \lambda^2 \Delta C(p^2)]}$

$\delta m^2^{(2)} = -\frac{1}{2} \lambda^2 [A\Lambda^2 + m^2 B_{\text{lu}}]$

$Z_5^{(2)} = 1 - \lambda^2 [F_{\text{lu}} + f^+(\text{sym})]$

$\tilde{\Gamma}(p_i)^{(2)} = 1 + \lambda^2 \cdot \Delta f(p_i^2)$

$\delta \lambda^{(3)} \equiv (\lambda_0 - \lambda) = \lambda \left(\sum_3^{-3/2} Z_5 - 1 \right) = \lambda \left[-\frac{3}{2} Z_3^{(2)} + Z_5^{(2)} \right]$

$= -\frac{1}{4} \lambda^3 \left[(3B + 4F)_{\text{lu}} + (3C(m^2) + 4f^+(\text{sym})) \right]$

7. Renormalized Lagrangian and graphical interpretation of the renormalized theory.

It is clear that the program outlines above, basically that described in BD, is equivalent to simultaneously rescaling the field from an unrenormed field ϕ_0 to a renormed field ϕ , and at the same time rescaling the charge. I.e, we take the bare lagrangian,

$$\mathcal{L}(\phi_0, \lambda_0) = \frac{1}{2} (\partial_\mu \phi_0 \partial^\mu \phi_0 - m^2 \phi_0^2) + \frac{\lambda_0}{3!} \phi_0^3 + \frac{1}{2} \delta m^2 \phi_0^2$$

and make these substitutions,

$$\phi_0 = \sqrt{Z_3} \phi \quad \lambda_0 = Z_5 Z_3^{-3/2} \lambda$$

to get this new "renormalized lagrangian":

$$\mathcal{L}(\phi, \lambda) = \frac{1}{2} Z_3 (\partial_\mu \phi \partial^\mu \phi - m^2 \phi^2) + \frac{(Z_5 \lambda)}{3!} \phi^3 + \frac{1}{2} (\delta m^2 Z_3) \phi^2$$

where Z_3 and Z_5 are power series in λ^2 . One can then use the "Feynman rules" implied by this lagrangian,

$$\text{---} = \frac{Z_3^{-1}}{p^2 - m^2} \quad \text{---} \text{---} = (Z_5 \lambda) \quad \text{---} \times \text{---} = (Z_3 \delta m^2)$$

and one finds that the result of any calculation is finite, if you compute to a finite order in λ^2 . In fact, this is a way to set the various Z's and δm^2 . Usually, one computes things using the BD framework however, though this in principle works as well.

The idea of having unrenormed and renormed fields is simply this:

$${}^0D = \langle 0 | T(\phi_0 \phi_0) | 0 \rangle \quad D = \langle 0 | T(\phi \phi) | 0 \rangle = Z_3^{-1} {}^0D.$$

8. Counterterm philosophy. One can obtain the above "renormed lagrangian" in another way.

Start with the original L but in terms of ϕ , then add counterterms to get new lagrangian. In this view, there is no field ϕ_0 . The original L has divergences, but by adding counterterms you get rid of those pieces. Since the counterterms arise from rescaling, it is obvious that the counterterms are of the form of terms already present. Once we choose a particular m and λ , the Z's are determined, so the counterterms are fixed. Changing m and λ to new values changes the Z's and changes the counterterms. The freedom to add arbitrary counterterms (of said form) reflects the freedom to have m and λ be whatever values you like: they cannot be computed in a field theory!

1. Propagator:

$$\begin{aligned} \Sigma^2(m^2) = 0 & \Rightarrow \text{pole at } m \\ \tilde{\Sigma}^2(\mu^2) = 0 & \Rightarrow D(\mu^2) = (\mu^2 - m^2)^{-1}. \end{aligned}$$

$$\Rightarrow 0 = (1 - \Sigma_3) + \Sigma_3 \circ \tilde{\Sigma}^2(\mu^2) \Rightarrow \Sigma_3^{-1} = 1 - \circ \tilde{\Sigma}^2(\mu^2)$$

$$\circ \tilde{\Sigma}^2(p^2)^{(2)} = \delta m^2^{(1)} + \frac{1}{2} \lambda_0^2 \left[(A \tilde{N} \cancel{m^2} B \ln \frac{\Lambda}{m}) + (p^2 - m^2) \left\{ B \ln \frac{\Lambda}{m} + C(p^2/m^2) \right\} \right]$$

↑ does not know about μ !

$$\text{Thus, } \Sigma_3^{-1(2)} = 1 - \frac{1}{2} \lambda_0^2 \left[B \ln \frac{\Lambda}{m} + C(\mu^2/m_0^2) \right].$$

But, $m = m(\mu; \lambda_0, m_0, \Lambda)$ if m were defined differently. $m = \mu \cdot D(\lambda_0, \frac{m_0}{\Lambda}, \frac{\mu}{\Lambda})$

$$\Sigma_3^{-1(2)}(\mu; \lambda_0, m_0, \Lambda) = 1 - \frac{1}{2} \lambda_0^2 \left[B \cdot \ln\left(\frac{\Lambda}{\mu}\right) + C\left(\frac{m_0}{\mu}\right) \right].$$

$$\text{Generally speaking, } \Sigma_3(\mu; \lambda_0, m_0, \Lambda) = \sum_{n=1}^N (\lambda_0)^{2n} \left[\ln\left(\frac{\Lambda}{\mu}\right) \right]^n + C\left(\frac{m_0}{\mu}, \lambda_0\right)$$

2. Vertex:

$$\Sigma_5^{-1} = \frac{P^0(\mu_1^2, \mu_2^2, \mu^2; m_0, \lambda_0, \Lambda)}{\lambda_0} = \Sigma_5^{-1}(\mu; \lambda_0, m_0, \Lambda).$$

$$\text{Thus, } \Sigma_5 = 1 - \lambda^2 [F \ln \Lambda + f^+(\mu^2)].$$

$$\Rightarrow \tilde{\Gamma} = 1 + \lambda^2 [f^+(p_i^2) - f^+(\mu^2)].$$

3. New subtraction rule for self-energy:

$$\tilde{\Sigma}^2(p^2) = \Sigma_3 [\circ \tilde{\Sigma}^2(p^2) - \circ \tilde{\Sigma}^2(\mu^2)] \Rightarrow \tilde{\Sigma}^2(p^2)^{(2)} = \frac{1}{2} \lambda^2 \left[C\left(\frac{p^2}{\Lambda_0}\right) - C\left(\frac{\mu^2}{\Lambda_0}\right) \right].$$

Dimensionalities in ϕ^3 in $d=6$

Look at the lagrangian. Will be integrated over d^6x so must be E^6 . If mass is mass, then $\dim(\phi) = E^2$. Thus, coupling constant is dimensionless! Ie, $L \phi^3$ has the entire 6 dimensions in the ϕ^3 . Then, eg, the propagator in x-space has $\dim(D(x)) = 4$. But then you fourier transform to get $D(p) = d^6x D(x) = -6+4 = -2$ which is still the pole.

How to do ϕ^3 using a renorm point different from the mass m .

1. Start off by looking at "propagator" section of notes and see which equations change:

$$^0D = \frac{1}{p^2 - m^2 - ^0\Sigma^2(p^2)} \quad \checkmark$$

$$^0\Sigma^2(p^2) = ^0\Sigma^2(m^2) + (p^2 - m^2) i \tilde{\Sigma}^2(p^2) \quad \checkmark$$

$^0\Sigma^2(m^2) = 0$ $\checkmark$ still so pole is at m^2 , ie, so D will have a pole there.

$$^0D(p^2 = \mu^2) = \frac{Z_3}{(\mu^2 - m^2)} \quad \text{now define } Z_3, \text{ so } \underline{\underline{Z_3^{-1} = 1 - ^0\tilde{\Sigma}^2(\mu^2)}}.$$

Now go to renormed prop:

$$D = \frac{1}{p^2 - m^2 - \Sigma^2(p^2)} \quad \checkmark = \frac{1}{(p^2 - m^2) [1 - \tilde{\Sigma}^2(p^2)]} \quad \checkmark$$

$$\left\{ \begin{array}{l} \Sigma^2(m^2) = Z_3 \cdot \Sigma^2(m^2) \quad \checkmark \\ \tilde{\Sigma}^2(p^2) = (1 - Z_3) + Z_3 \cdot \tilde{\Sigma}^2(p^2) \end{array} \right\} \text{ but } Z_3 \text{ now}$$

$$= Z_3 \left[^0\tilde{\Sigma}^2(p^2) - ^0\tilde{\Sigma}^2(\mu^2) \right] \quad \Leftarrow \text{so new subtr. point!}$$

$$\text{Thus, } \underline{\underline{^0\tilde{\Sigma}^2(\mu^2) = 0}} \quad \text{and} \quad D(p^2 = \mu^2) = \frac{1}{\mu^2 - m^2}.$$

2. I now wish to argue that our requirements that $\Sigma^2(m^2) = 0$ and that $\tilde{\Sigma}^2(\mu^2) = 0$ cause both these pieces of the renormalized self-energy to depend implicitly on μ . If this is true, then of course the renormed propagator depends on μ . Lets see how this goes:

Logic: a) $\tilde{\Sigma}^2(\mu^2) = 0 \Rightarrow Z_3^{-1} = 1 - \tilde{\Sigma}^2(\mu^2; m^2, \lambda^2, \Lambda^2) \Rightarrow Z_3 = Z_3(\mu^2, m^2, \lambda^2, \Lambda^2).$

b) $^0\Gamma(\mu^2, \mu^2, \mu^2; m^2, \lambda^2, \Lambda^2) = Z_5^{-1} \lambda_0 \Rightarrow Z_5 = Z_5(\mu^2, m^2, \lambda^2, \Lambda^2).$

c) $\lambda = Z_3^{3/4} Z_5^{-1} \lambda_0 = f(\mu^2, m^2, \lambda^2, \Lambda^2) \lambda_0$

$\Rightarrow$ solve for $\lambda_0 \Rightarrow \lambda_0 = \lambda_0(\mu^2, m^2, \lambda^2, \Lambda^2)$

d) Thus you can say $Z_3 = Z_3'(\mu^2, m^2, \lambda^2, \Lambda^2)$ $\leftarrow$ now λ appears.

c) Consider:
$$\begin{aligned}\Sigma^2(m^2) &= Z_3 \cdot {}^0\Sigma^2(m^2; \lambda_0^2, \Lambda^2) \\ &= Z_3(\mu^2, m^2, \lambda^2, \Lambda^2) \cdot {}^0\Sigma^2(m^2; \lambda_0^2(\mu^2, m^2, \lambda^2, \Lambda^2), \Lambda^2) \\ &= f(\mu^2, m^2, \lambda^2, \Lambda^2).\end{aligned}$$

Since $\Sigma^2(m^2)$ depends on μ^2 . So pole-at- m^2 says:

$$\boxed{\Sigma^2(p^2=m^2; \mu^2, \lambda^2, \Lambda^2) = 0}$$

I.e., we choose a certain mass counterterm $\delta m^2(m^2, \mu^2, \lambda^2, \Lambda^2)$ to make this true.

f) So, if you keep this same form + vary μ^2 holding λ^2 and Λ^2 fixed, you will change m^2 !

g) Similarly, $\tilde{\Sigma}^2(p^2) = \tilde{\Sigma}^2(p^2; \mu^2, m^2, \lambda^2, \Lambda^2)$.

h) Now look at vertex:

$$\Gamma = Z_3^{3/2} \Gamma^0(p; \lambda_0^2, m^2, \Lambda^2) \Rightarrow \Gamma = \Gamma(p; \mu^2, m^2, \lambda^2, \Lambda^2).$$

Then $\tilde{\Gamma}(p; \mu^2, m^2, \lambda^2, \Lambda^2)$.

And $\boxed{\tilde{\Gamma}(\mu^2; \mu^2, m^2, \lambda^2, \Lambda^2) = 1.}$

but no dep on this.

First, the point of the above is to show that really all renormalized objects are going to be functions of μ because of two sources: the Z's, and converting λ_0 to λ . It would be a truly stunning coincidence if somehow a renormed object (like self-energy, vertex, propagator, kernel, etc) did not depend on the renorm point μ .

So, this being the case, consider the two boxed equations above: these are the conditions that put the pole at mass m , and the charge at value λ . Suppose you start off with some particular choice of μ and Λ . You also choose m and λ . You then construct the above functions of all those arguments. I.e., the "theory" determines these functions. Now, suppose you vary μ a little, but you insist that the above conditions still be true. The claim is that both your mass and charge will move a little. Of course you could make the mass and charge not move by changing your counterterm functions like δm^2 or the Z's. But the idea is to keep them the same functions that they were.

Thus we get the important point: a change of renorm point changes your mass and charge, if you hold the form of your counterterm functions fixed.

This is also what the differential renormalization group equation will tell us: that a change in the renormalization point will be accompanied by a change in the mass and the charge!

3. About finite renormalizations in ϕ^3 theory.

$$D' \equiv z_3 D \quad \tilde{\Gamma}' \equiv z_5^{-1} \tilde{\Gamma} \quad \lambda' \equiv z_5 z_3^{-3/2} \lambda$$

The Bare Mass Viewpoint.

1. Motivation: I think this viewpoint is more useful when you are talking about renorm group transformations. I don't yet know how, but that is my motivation.

2. Consider the unrenormalized, bare propagator. It has a pole at m_0 . Question: where is the pole of the unrenormalized but fully dressed propagator? Answer: it is at a position which you cannot fix but which is fixed by the "theory". I.e:

$$^0D(p^2; \lambda_0, m_0, \Lambda) = \frac{1}{p^2 - m_0^2 - \Sigma^2(p^2; \lambda_0, m_0, \Lambda)}$$

I.e., $\Sigma^2(p^2; \lambda_0, m_0, \Lambda) = \text{---} \text{---} + \text{etc} = \text{determined}.$

Call this pole position M_0 . Then $M_0^2 - m_0^2 - \Sigma^2(p^2 = M_0^2; \lambda_0, m_0, \Lambda) = 0$
determines M_0 .

So $M_0 = M_0(\lambda_0, m_0, \Lambda)$

3. Now go to the renormalized, dressed propagator. It is supposed to be related to the unrenorm by a Z_3 factor and it has the form:

$$D = \frac{1}{p^2 - m^2 - \Sigma^2(p^2)} \quad \text{pole at } m^2 \Rightarrow \Sigma^2(p^2 = m^2) = 0.$$

$$D = Z_3^{-1} {}^0D \Rightarrow \tilde{D} = Z_3 ({}^0D) \Rightarrow p^2 - m^2 - \Sigma^2(p^2) = Z_3 [p^2 - m_0^2 - \Sigma_0^2(p^2; \lambda_0, m_0)]$$

Renorm condition therefore is $0 = m^2 - m_0^2 - \Sigma^2(p^2 = m^2; \lambda_0, m_0, \Lambda)$.

Or: $\boxed{\Sigma^2(p^2 = m^2; \lambda_0, m_0, \Lambda) = m^2 - m_0^2}$ i.e., $f(m, \lambda_0, m_0, \Lambda) = 0$.

Thus, if you had chosen definite values of m_0, λ_0 , and Λ , this would force your mass m to go at a particular value. Usually, however, you think of m and Λ as being given, then this renorm condition defines a curve through the space m_0, λ_0 .

3. Now let's worry about Z_3 . Here is how you want to expand the self-energies:

$$^0\tilde{\Sigma}^2(p^2; m_0, \lambda_0, \Lambda) = ^0\tilde{\Sigma}^2(m^2; m_0, \lambda_0, \Lambda) + (p^2 - m^2) ^0\tilde{\Sigma}^2(p^2; m_0, \lambda_0, \Lambda)$$

I.e., imagine that once we have computed this mass m in terms of m_0, λ_0, Λ as discussed above, we then proceed to expand the self energy around that value and we thereby effect a well-defined definition for the twiddled nought function. Similarly:

$$\tilde{\Sigma}^2(p^2) = \tilde{\Sigma}^2(m^2) + (p^2 - m^2) \tilde{\Sigma}^2(p^2)$$

$$\begin{aligned} \text{Thus: } p^2 - m^2 - \left[\underbrace{\tilde{\Sigma}^2(m^2)}_{=0} + (p^2 - m^2) \tilde{\Sigma}^2(p^2) \right] &= \\ Z_3 \left\{ p^2 - m_0^2 - \left[\underbrace{^0\tilde{\Sigma}^2(m^2)}_{=m^2 - m_0^2} + (p^2 - m^2) ^0\tilde{\Sigma}^2(p^2) \right] \right\} &= \\ = Z_3 \left\{ (p^2 - m^2) [1 - ^0\tilde{\Sigma}^2(p^2)] \right\} &= \\ = (p^2 - m^2) [1 - \tilde{\Sigma}^2(p^2)] & \\ \Rightarrow Z_3 [1 - \tilde{\Sigma}^2(p^2)] = [1 - \tilde{\Sigma}^2(p^2)] & \\ \Rightarrow Z_3 [1 - ^0\tilde{\Sigma}^2(m^2)] = 1 \quad \text{since } \tilde{\Sigma}^2(m^2) = 0 \quad (\text{unit residue}). & \\ \Rightarrow \left\{ \begin{array}{l} Z_3^{-1} = 1 - ^0\tilde{\Sigma}^2(p^2 = m^2; \lambda_0, m_0, \Lambda) \\ = Z_3^{-1}(\lambda_0, m_0, \Lambda). \end{array} \right\} \Rightarrow Z_3 = Z_3(\lambda_0, m_0, \Lambda) & \text{since } m = m(\lambda_0, m_0, \Lambda) \end{aligned}$$

So, if Z_3 is given this numerical value, then the pole residue will be unity.

4. Next, what about Z_5 ? Compute the unrenormed vertex, then go to sym point and you got it:

$$\begin{aligned} \Sigma_5^{-1} &= ^0\Gamma(p_i^2 = m^2; \lambda_0, m_0, \Lambda) / \lambda_0 \\ &= \Sigma_5^{-1}(\lambda_0, m_0, \Lambda). \end{aligned} \quad (5)$$

5. Now what about the renormalized charge? Both Z 's are determined, so it also is determined, you cannot set it. In fact we have, from the ~~renorm~~ definition of renormed charge, that:

$$\begin{aligned} \lambda &= \hat{\Gamma}(\text{sym}) = Z_3^{3/2} ^0\Gamma(\text{sym}) = Z_3^{3/2} Z_5^{-1} \lambda_0 \\ \text{so } \lambda &= Z_3^{3/2} Z_5^{-1} \lambda_0 = \lambda(\lambda_0, m_0, \Lambda) \end{aligned} \quad (4)$$

6. We could have introduced a Z to describe the mass renormalization, had we chosen to:

$$m^2 = Z_1 m_0^2 \Rightarrow \Sigma^2(p^2 = Z_1 m_0^2; \lambda_0, m_0, \Lambda) = m_0^2 (Z_1 - 1)$$

solve to get $Z_1 = Z_1(\lambda_0, m_0, \Lambda)$ Replaces $m = m(\lambda_0, m_0, \lambda)$.

7. So lets summarize: you start off with an unrenorm lagrangian with m_0 and λ_0 . You calculate that the dressed, unrenorm prop has a pole at some qierd place M_0 . By examining a certain equation involving the function $\tilde{Z}_0$, you deduce that it is possible to construct a renormed propagator which has a polea at m , residue 1. The value of m is predetermined, and so is the Z_3 that makes the residue 1. Finally, you go on to compute Z_5 and λ .

The following items are completly determined by m_0 and λ_0 : Z_3, Z_5, m, λ .

~~xxxx~~

8. Lets now write out the mass-set equation and the charge-set equations:

$$(1) \Rightarrow m = m(\lambda_0, m_0, \Lambda) \quad (2) \Rightarrow Z_3 = Z_3(\lambda_0, m_0, \Lambda)$$

$$(4) \Rightarrow \lambda = \lambda(\lambda_0, m_0, \Lambda) \quad (3) \Rightarrow Z_5 = Z_5(\lambda_0, m_0, \Lambda)$$

Presumably we can invert equations (1) and (4) to get values of the bare mass and charge which will produce any desired renormed mass and charge. Then we get:

$$\begin{aligned} m_0 &= m_0(\lambda, m, \Lambda) & Z_3 &= Z_3(\lambda, m, \Lambda) \\ \lambda_0 &= \lambda_0(\lambda, m, \Lambda) & Z_5 &= Z_5(\lambda, m, \Lambda) \end{aligned}$$

Now we simply pick any mass and charge we want, m and λ , then we can compute m_0 , λ_0 , and Z_3 and Z_5 that make the theory work: ie, m will be the mass, λ will be the charge; residue will be unity. Again: if someone hands you m and λ , you will have to adjust m_0 and λ_0 to specific values to make the theory work.

Renorm with (a) arbitrary renorm point μ , and (b) bare mass viewpoint.

1. As in the first "bare mass" sheet, we compute the pole M_0 of the unren prop. No one really cares what this value is. Then we define the renormed prop by some Z_3 and we want it to have pole at m , not at renorm point μ . Thus, we get:

$$D = \frac{1}{p^2 - m^2 - \tilde{\Sigma}(p^2)} \Rightarrow \tilde{\Sigma}^2(p^2 = m^2) = 0 \Rightarrow \text{pole at } m.$$

$$\Rightarrow \text{same Eq (1) as before} \Rightarrow \boxed{m = m(\lambda_0, m_0, \Lambda)}$$

Although we have chosen μ as our "renorm point", we have not used it yet. The idea of "mass" requires us to look at $p^2 = m^2$ and the renorm point is not yet invoked.

2. Now we invoke the renorm point to require that:

$$D(p^2 = \mu^2) = \frac{1}{\mu^2 - m^2 - \tilde{\Sigma}(\mu^2)} = \frac{1}{[\mu^2 - m^2][1 - \tilde{\Sigma}(\mu^2)]} = \frac{1}{\mu^2 - m^2} = \frac{1}{p^2 - m^2}.$$

In other words, we are requiring that, at the renorm point, the propagator be equal to some specific number (which number we know) (because we know m by the above). There are probably other ways to impose this renorm condition. As μ approaches m , this condition duplicates the usual "mass shell" renorm condition; for general μ notice that we are not at a pole. It is probably a good idea to not impose this condition at a pole!

What Z_3 will cause the above to be true? Look at page (2) of last section and duplicate all definitions there regarding self-energies. Then you will find that to satisfy

$$\begin{aligned} \tilde{\Sigma}^2(\mu^2) = 0 &\Rightarrow Z_3^{-1} = 1 - \tilde{\Sigma}^2(p^2 = \mu^2; \lambda_0, m_0, \Lambda) \\ \Rightarrow Z_3 &= Z_3(\mu, \lambda_0, m_0, \Lambda) \end{aligned}$$

Now you see clearly that, besides being dependent on the bare mass and charge and cutoff, Z_3 depends for sure on the renorm point because μ appears in the definition of Z_3 !!!!

3. Now do Z_5 :

$$\begin{aligned} Z_5^{-1} &\equiv \Gamma^0(p_i^2 = \mu^2; \lambda_0, m_0, \Lambda) / \lambda_0 \\ \Rightarrow Z_5 &= Z_5(\mu, \lambda_0, m_0, \Lambda). \end{aligned}$$

So to summarize: Z_3 is determined by a condition on the renormed self-energy (2-point fn) at the renorm point; Z_5 is determined by a condition on the unrenormed vertex (3-pt) at the renorm point.

4. Finally, we can compute the charge, just as before. We get:

$$\lambda = \mathcal{Z}_3(\mu, \lambda_0, m_0, \Lambda)^{3/2} \mathcal{Z}_5(\mu, \lambda_0, m_0, \Lambda)^{-1} \lambda_0 \Rightarrow \boxed{\lambda = \lambda(\mu, \lambda_0, m_0, \Lambda)}$$

5. Summary:

$$\begin{aligned} \text{potential } m : & \quad \begin{cases} m = m(\lambda_0, m_0, \Lambda) \\ \lambda = \lambda(\mu, \lambda_0, m_0, \Lambda) \end{cases} \\ \text{charge } \lambda : & \\ \text{by definition: } & \begin{cases} \mathcal{Z}_5 = \mathcal{Z}_5(\mu, \lambda_0, m_0, \Lambda) \\ \mathcal{Z}_3 = \mathcal{Z}_3(\mu, \lambda_0, m_0, \Lambda) \end{cases} \end{aligned}$$

6. Invert to get:

$$\boxed{\begin{aligned} m_0 &= m_0(\mu, \lambda, m, \Lambda) \\ \lambda_0 &= \lambda_0(\mu, \lambda, m, \Lambda) \\ \mathcal{Z}_5 &= \mathcal{Z}_5(\mu, \lambda, m, \Lambda) \\ \mathcal{Z}_3 &= \mathcal{Z}_3(\mu, \lambda, m, \Lambda) \end{aligned}}$$

Connection with the notes of Abarbanel:

Abarb Eq. (17) reads, when converted to my notation:

$$iP^2 = p^2 - m^2 - \Sigma^2(p^2)$$

$$\frac{\partial}{\partial p^2} iP^2 = 1 - \frac{\partial \Sigma^2(p^2)}{\partial p^2} = 1 \Rightarrow \left. \frac{\partial \Sigma^2(p^2)}{\partial p^2} \right|_{p^2=\mu^2} = 0$$

$$\text{But } \Sigma^2(p^2) = \Sigma^2(\mu^2) + (p^2 - \mu^2) \tilde{\Sigma}^2(p^2)$$

$$a + b \cdot (p^2 - \mu^2) + c(p^2 - \mu^2)^2 + \dots$$

$$= A + Bp^2 + Cp^4 + \dots$$

So his deriv is:

$$\left. \frac{\partial \Sigma^2(p^2)}{\partial p^2} \right|_{p^2=\mu^2} = \tilde{\Sigma}^2(\mu^2) + (\mu^2 - \mu^2) \frac{\partial \tilde{\Sigma}^2}{\partial p^2} \Big|_{\mu^2} \neq \tilde{\Sigma}^2(\mu^2) !$$

$$\text{If we defined } \Sigma^2(p^2) = \Sigma^2(\mu^2) + (p^2 - \mu^2) \tilde{\Sigma}^2(p^2)$$

$$\text{Then he is saying } \tilde{\Sigma}^2(\mu^2) = 0.$$

Thus, Abarbanel's "residue condition" is not the same as mine. He has a certain equation ~~that his self-energy~~ that my self-energy function must satisfy. But his equation is not the same as mine. Thus, my Z_3 will be different from his Z . His Z will depend on the same parameters as my Z_3 , but these Z 's will be different functions of their parameters.

I think the lesson here is simple: any "residue type condition" will do the trick, as long as it is a specific condition on the self-energy and thus it determines the function Z .

Similarly, look at Abarb equation (20). In my notation it reads:

$$0 - m^2 - \Sigma^2(0) = -m^2 \Rightarrow \Sigma^2(0) = 0 \Rightarrow D(p^2=0) = \frac{1}{-m^2}.$$

Obviously the symbol m so introduced by Abarbanel has nothing to do with the location of the pole of the renormalized propagator! Abarb even states this on page 404. Good.

2. So how should I compare my "conditions" to his "conditions". I think the reason for having "conditions" is this: the conditions determined the Z 's, and the Z 's then determined the counterterms. So having conditions just tells you what counterterms to add to the theory to get a finite theory. Our conditions are different, so our counterterms are different, but both our theories will be finite. My m has mass pole significance, his m has no significance except as noted above, .

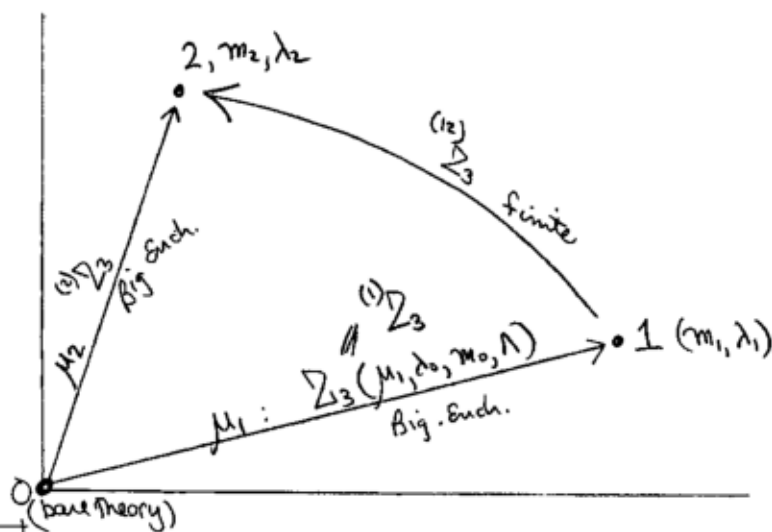
(Abarb con'd)

So, either set of conditions (his or mine) lead to specific functions which are the Z's. One interesting difference is this: my condition leads to an equation $m = m(m_0, \lambda_0, \mu)$ and simply does not involve the renorm point μ . This is really an accident because Z_3 should have appeared in the m equation but it did not. Thus, the derivative he calls η on page 407 always vanishes for me: if I change my renorm point μ while holding m_0 and λ_0 fixed, my m does not move!!!

How to "emanate" from the bare theory in two different ways:

1. Start off with the bare theory with m_0 and λ_0 . Choose a renorm point call it μ_1 . Compute up objects m_1 and λ_1 , each will be a function of $\mu_1, m_0, \lambda_0, \Lambda$. (Except, if m is really the mass, it won't be a function of μ_1 , but let's keep general anyway and pretend it is a function of μ_1 .) The "transformation" which takes us from the bare theory to the 1-theory is a Big Enchilada and is characterized by some Z 's. I.e., we do multiplicative renorm by the Z 's that are relevant.

2. Alternatively, you could choose same m_0 and λ_0 but now renorm point is μ_2 . Go for another big enchilada and arrive at the 2-theory. Claim: then the finite 1-theory and the finite 2-theory will be connected by a finite multiplicative renorm of objects!!!



$$\left\{ \begin{array}{l} \downarrow \\ \sum_3^{-1} D \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} \downarrow \\ \sum_3^{-1} D \end{array} \right\}$$

$$^{(12)} Z_3 = \sum_3^{-1} \sum_3$$

$$^{(12)} Z_5 = \left[\sum_5 \right]^{-1} \cdot \left[\sum_5 \right]$$

$$\lambda_2 = \sum_3^{3/2} \sum_5^{-1} \lambda_1$$

$(m_2 \equiv m_1 \text{ for any } m.)$

The claim is that $^{(12)} z_3$, for example, is finite, has no logs of cutoff in it. Let's check this in perturbation theory to make sure:

$$\left\{ \begin{array}{l} \sum_3^{[2]} = 1 + \frac{1}{2} \lambda_1^2 [B \ln \Lambda + C(m_1^2)] \\ \sum_3^{[2]} = 1 + \frac{1}{2} \lambda_2^2 [B \ln \Lambda + C(m_2^2)] \end{array} \right.$$

$\hookrightarrow \lambda_1^2 + O(\lambda_1^4)$

Sure enough, it's finite!

$$\begin{aligned} \leftarrow \sum_3^{(12)} &\cong (1 - \frac{1}{2} \lambda_1^2 [B \ln \Lambda + C(m_1^2)]) (1 + \frac{1}{2} \lambda_1^2 [B \ln \Lambda + C(m_1^2)]) \\ &\approx 1 + \frac{1}{2} \lambda_1^2 [C(m_2^2) - C(m_1^2)] + O(\lambda_1^4) = \text{finite} \checkmark \end{aligned}$$

Say this in words: the finite R - group transformation is induced simply by changing the renormalization point, holding m_0 and λ_0 and Λ fixed. This transformation is simply a multiplicative rescaling of objects by some finite Z's, just as in QED.

3. Construction of a possible "invariant charge" object:

$$\left(\frac{\lambda_1^2 \cdot [{}^1D(p^2)]^3}{[{}^1\tilde{\Gamma}(p_1^2, p_2^2, p_3^2)]^2} \right) = \left(\frac{\lambda_2^2 \cdot [{}^2D(p^2)]^3}{[{}^2\tilde{\Gamma}(p_1^2, p_2^2, p_3^2)]^2} \right)$$

$$\begin{aligned} \dim \lambda &= 0 \checkmark \\ \dim D &= -2 \checkmark \\ \dim \tilde{\Gamma} &= 0 \checkmark \end{aligned}$$

Thus, also true if all momenta set equal! Define:

$$(p)^6 \tilde{d}\left(\frac{\lambda_1^2}{\mu_1^2}, \frac{m_1^2}{\mu_1^2}, \frac{p^2}{\mu_1^2}\right) \equiv \left(\frac{\lambda_1^2 [{}^1D(p^2)]^3}{[{}^1\tilde{\Gamma}(p^2, p^2, p^2)]^2} \right) \quad \dim = -6$$

Then we have constructed an invariant object

$$\tilde{d}\left(\frac{\lambda_1^2}{\mu_1^2}, \frac{m_1^2}{\mu_1^2}, \frac{p^2}{\mu_1^2}\right) = \tilde{d}\left(\frac{\lambda_2^2}{\mu_2^2}, \frac{m_2^2}{\mu_2^2}, \frac{p^2}{\mu_2^2}\right)$$

Then:

$$\tilde{d}\left(\frac{\lambda_1^2}{\mu_1^2}, \frac{m_1^2}{\mu_1^2}, 1\right) = \frac{\lambda_1^2 \mu_1^6}{(\mu_1^2 - m_1^2)^3} = \left(\frac{\lambda_1^2}{\mu_1^2}\right) \cdot \left[\frac{\mu_1^6}{(\mu_1^2 - m_1^2)^3}\right] = \lambda_1^2 \left[\frac{\mu_1^2}{\mu_1^2 - m_1^2}\right]^3$$

Now redefine:

$$d_1 = \frac{(\mu_1^2 - m_1^2)^3}{\mu_1^6} \tilde{d}_1 \Rightarrow d(-, -, 1) = \frac{\lambda_1^2}{\mu_1^2}$$

Then let $e_1^2 \equiv \lambda_1^2 / \mu_1^2 =$ dimensionless charge!

Summary:

$$d(e_1^2; \frac{m_1^2}{\mu_1^2}, \frac{p^2}{\mu_1^2}) \equiv \frac{(\mu_1^2 - m_1^2)^3}{\mu_1^6} \cdot (p)^4 \cdot (e_1^2 \mu_1^2) \frac{[{}^1D(p^2)]^3}{[{}^1\tilde{\Gamma}(p^2, p^2, p^2)]^2}$$

Again:

$$d(e_1^2; \frac{m_1^2}{\mu_1^2}, \frac{p^2}{\mu_1^2}) \equiv e_1^2 \cdot (\mu_1^2 - m_1^2)^3 \cdot \left(\frac{p^2}{\mu_1^2}\right)^2 \cdot \frac{[{}^1D(p^2)]^3}{[{}^1\tilde{\Gamma}(p^2, p^2, p^2)]^2}$$

No define this new, dimensionless charge:

$$e_1^2 \equiv \frac{\lambda_1^2}{\cancel{\mu_1^6}} \left(\frac{\mu_1^6}{(\mu_1^2 - m_1^2)^3} \right) = \frac{\lambda_1^2}{\cancel{\mu_1^6}} \left[\frac{1}{(1 - \frac{m_1^2}{\mu_1^2})^3} \right]$$

Then let

$$d(e_1^2; \frac{m_1^2}{\mu_1^2}, \frac{p^2}{\mu_1^2}) \equiv \tilde{d}(\frac{\lambda_1^2}{\mu_1^2}; \frac{m_1^2}{\mu_1^2}, \frac{p^2}{\mu_1^2})$$

Ie, this thing is a function of three variables, now I am changing the variables from one set to another set. Now notice that:

$$d(e_1^2, 1, \frac{m_1^2}{\mu_1^2}) = \tilde{d}(-, -, 1) = e_1^2 !$$

Now restate:

$$e_1^2 \equiv \frac{\lambda_1^2 \mu_1^6}{(\mu_1^2 - m_1^2)^3}$$

$$d(e_1^2, \frac{p^2}{\mu_1^2}, \frac{m_1^2}{\mu_1^2}) \equiv (p)^{\frac{6}{2}} \lambda_1^2 [D(p^2)]^3 / [\tilde{\Gamma}(p^2, p^2, p^2)]^2$$

Then we know that

$$d(e_1^2, \frac{p^2}{\mu_1^2}, \frac{m_1^2}{\mu_1^2}) = d(e_2^2, \frac{p^2}{\mu_2^2}, \frac{m_2^2}{\mu_2^2})$$

$$d(e_1^2, 1, \frac{m_1^2}{\mu_1^2}) = e_1^2$$

$$d(e_2^2, 1, \frac{m_2^2}{\mu_2^2}) = e_2^2$$

Thus, I have been able to come up with a definition of a dimensionless charge e and an invariant function d which obeys the same rules as I found before in the QEDlike case.

(cribbled)

2. Properties of the invariant charge function.

a) mass scale m present.

One can repeat all the above choosing μ_2^2 instead of μ_1^2 . One will then end up with a charge e_2 instead of e_1 . The propagators D_1 and D_2 are then connected by some finite renormalization which we will call Z and which is unrelated to the Z_3 used in the last section. Thus:

$$e_2^2 = \sum e_1^2 \quad \left[D_2 = \sum^{-1} D_1 \right] \quad \text{not true for } \phi^3 \quad (14)$$

But, $e_1^2 D_1 = e_2^2 D_2$. Thus

$$d(e_1^2, \frac{q^2}{\mu_1^2}, \frac{m_1^2}{\mu_1^2}) = d(e_2^2, \frac{q^2}{\mu_2^2}, \frac{m_2^2}{\mu_2^2}) \quad \forall q^2! \quad \checkmark \quad (15)$$

Special cases of (15) are:

$$\left. \begin{aligned} q^2 = \mu_1^2 &\Rightarrow e_1^2 = d(e_2^2, \frac{\mu_1^2}{\mu_2^2}, \frac{m_2^2}{\mu_2^2}) \\ q^2 = \mu_2^2 &\Rightarrow e_2^2 = d(e_1^2, \frac{\mu_2^2}{\mu_1^2}, \frac{m_1^2}{\mu_1^2}) \end{aligned} \right\} \quad \checkmark \quad (16)$$

where we recall from (9) and (11) that

$$\left. \begin{aligned} d(e_1^2, 1, \frac{m_1^2}{\mu_1^2}) &= e_1^2 \\ d(e_2^2, 1, \frac{m_2^2}{\mu_2^2}) &= e_2^2 \end{aligned} \right\} \quad \checkmark \quad (17)$$

The Z which links (μ_1, e_1) and (μ_2, e_2) is given by

$$Z = \frac{d(e_1^2, \frac{\mu_2^2}{\mu_1^2}, \frac{m_1^2}{\mu_1^2})}{e_1^2} = \frac{e_2^2}{d(e_2^2, \frac{\mu_1^2}{\mu_2^2}, \frac{m_2^2}{\mu_2^2})} \quad \checkmark \quad (18)$$

Replace e_2^2 in (15) by (16b) to get this system:

$$\left\{ \begin{aligned} d(e_1^2, \frac{q^2}{\mu_1^2}, \frac{m_1^2}{\mu_1^2}) &= d(d(e_2^2, \frac{\mu_2^2}{\mu_1^2}, \frac{m_2^2}{\mu_2^2}), \frac{q^2}{\mu_2^2}, \frac{m_2^2}{\mu_2^2}) \quad \checkmark \quad \leftarrow \text{Functional Equation with variable } q^2 \\ d(e_1^2, 1, \frac{m_1^2}{\mu_1^2}) &= e_1^2 \quad \checkmark \quad \leftarrow \text{Boundary Condition on } d(x, y, z) \end{aligned} \right. \quad (19)$$

b) no mass scale present

Here, thinking of pure YMT, we set $m_1 = m_2 = 0$ and drop the third argument of $d(x, y, z)$ and we rewrite all the previous equations as follows:

$$d(e_1^2, \frac{q^2}{\mu_1^2}) = d(e_2^2, \frac{q^2}{\mu_2^2}) \quad (20)$$

$$e_1^2 = d(e_2^2, \frac{\mu_1^2}{\mu_2^2}) \quad \text{and } (1 \leftrightarrow 2) \quad (21)$$

$$d(e_1^2, 1) = e_1^2 \quad \text{and } (1 \leftrightarrow 2) \quad (22)$$

$$\Sigma = \frac{e_2^2}{e_1^2} = \frac{d(e_1^2, \frac{\mu_2^2}{\mu_1^2})}{e_1^2} = \frac{e_2^2}{d(e_2^2, \frac{\mu_1^2}{\mu_2^2})} \quad (23)$$

$$\begin{cases} d(e_1^2, \frac{q^2}{\mu_1^2}) = d\left(d(e_1^2, \frac{\mu_2^2}{\mu_1^2}), \frac{q^2}{\mu_2^2}\right) \\ d(e_1^2, 1) = e_1^2 \end{cases} \quad \forall q \quad (24)$$

These equations apply either to pure YMT or to QED in an appropriate high energy limit according to Weinberg's Theorem which allows you to neglect the electron mass as a scale. One would like to solve the functional equation for $d(x, y)$ with the boundary condition shown. This will be done below after other stuff.

3. How to make functions $f(e_2)$ which look like they are $f(e_2, u_2/u_1)$

Set $q = \lambda \mu_2$ in (15) to get

$$d(e_2^2, \lambda^2, \frac{m_2^2}{\mu_2^2}) = d(e_2^2, \lambda^2 \frac{\mu_2^2}{\mu_1^2}, \frac{m_1^2}{\mu_1^2}) \quad (25)$$

Now do $\partial/\partial \lambda$ on both sides then set $\lambda=1$ to get

$$d_2(e_2^2, 1, \frac{m_2^2}{\mu_2^2}) = \frac{\mu_2^2}{\mu_1^2} d_2(e_2^2, \frac{\mu_2^2}{\mu_1^2}, \frac{m_1^2}{\mu_1^2}) \quad (26)$$

Replace e_1^2 in both the above equations using (16) to get ($\lambda=1$)

$$H_1(e_2^2, \frac{\mu_1^2}{\mu_2^2}, \frac{m_2^2}{\mu_2^2}) \equiv d(d(e_2^2, \frac{\mu_1^2}{\mu_2^2}, \frac{m_2^2}{\mu_2^2}), \frac{\mu_2^2}{\mu_1^2}, \frac{m_1^2}{\mu_1^2}) = d(e_2^2, 1, \frac{m_2^2}{\mu_2^2}) \quad (27)$$

$$H_2(e_2^2, \frac{\mu_1^2}{\mu_2^2}, \frac{m_2^2}{\mu_2^2}) \equiv \frac{\mu_2^2}{\mu_1^2} d_2(d(e_2^2, \frac{\mu_1^2}{\mu_2^2}, \frac{m_2^2}{\mu_2^2}), \frac{\mu_2^2}{\mu_1^2}, \frac{m_1^2}{\mu_1^2}) = d_2(e_2^2, 1, \frac{m_2^2}{\mu_2^2}) \quad (28)$$

These functions H_1 and H_2 are examples of functions which appear to depend on three arguments. Due to the invariance of d , both H_1 and H_2 are in fact independent of the second argument, as one sees by looking on the extreme right. If we now let $m=0$ so no mass scale in theory, these equations become:

$$H_1(e_2^2, r^2) \equiv d(d(e_2^2, r^2), \tilde{r}^2) = d(e_2^2, 1) \quad (29)$$

$$H_2(e_2^2, r^2) \equiv \tilde{r}^2 d_2(d(e_2^2, r^2), \tilde{r}^2) = d_2(e_2^2, 1) \quad (30)$$

In this case, the functions H appear to depend on the variable r , but they really don't depend on the variable r . Obviously these are the kind of statements that will allow us to show why the BETA function is a function only of one argument.



4. The (differential) Renormalization Group Equations for d and f.

a) RGE for function d.

$$\mu_2 \left(\frac{d}{d\mu_2} \left[d(e_1^2, \frac{q^2}{\mu_1^2}, \frac{m_1^2}{\mu_1^2}) \right] \right)_{e_1, \mu_1, m_1} = 0 \quad \text{by obviousity} \quad \checkmark \quad (q^2, m_1^2 \text{ also fixed}) \quad (31)$$

Therefore,

$$0 = \mu_2 \left(\frac{d(e_1^2, q^2/\mu_1^2, m_1^2/\mu_1^2)}{d\mu_2} \right)_{e_1, \mu_1, m_1} \quad (32)$$

$$\Rightarrow 0 = \left[\mu_2 \frac{\partial}{\partial \mu_2} + \underbrace{\left(\mu_2 \frac{\partial e_2}{\partial \mu_2} \right)_{\mu_1, e_1, m_1}}_{\equiv \beta} \frac{\partial}{\partial e_2} + \underbrace{\left(\frac{\mu_2}{m_2} \frac{\partial m_2}{\partial \mu_2} \right)_{\mu_1, e_1, m_1} \frac{\partial}{\partial m_2}}_{\equiv \eta} \right] d(e_1^2, \frac{q^2}{\mu_1^2}, \frac{m_1^2}{\mu_1^2}) \quad (33)$$

In equation (33) the partial wrt μ_2 is supposed to pick out the explicit μ_2 dependence which appears in the second and third arguments of $d(x, y, z)$. The implicit dependence through the first argument is taken by d/de_2 with, of course, the second two arguments held fixed as implied by the partial deriv notation. Below we will discuss further the object " β ", and the possibility of d/dm_2 .

b) RGE for function f.

↓ define f.

drop

Recall $d(e_1^2, \frac{q^2}{\mu_1^2}, \frac{m_1^2}{\mu_1^2}) = e_2^2 f(e_1^2, \frac{q^2}{\mu_1^2}, \frac{m_1^2}{\mu_1^2})$

Plug this into (33) to find:

$$\left[\mu_2 \frac{\partial}{\partial \mu_2} + \beta \cdot \frac{\partial}{\partial e_2} + \eta \cdot \frac{\partial}{\partial m_2} + 2 \cdot \gamma \right] f(e_1^2, \frac{q^2}{\mu_1^2}, \frac{m_1^2}{\mu_1^2}) = 0$$

where

$$\beta \equiv \mu_2 \left(\frac{\partial e_2}{\partial \mu_2} \right)_{\mu_1, e_1, m_1}$$

$$\gamma \equiv \frac{1}{2} \mu_2 \left(\frac{\partial [\ln e_2^2]}{\partial \mu_2} \right)_{\mu_1, e_1, m_1} = \frac{1}{e_2} \beta$$

$$\eta \equiv \mu_2 \left(\frac{\partial m_2}{\partial \mu_2} \right)_{\mu_1, e_1, m_1}$$

Show that $\beta = \beta(e)$ in massless theory.

① Look back at "emanation" sketch. We can "solve" the 1-theory to get

$$\lambda_0 = \lambda_0(m_1, \lambda_1, \Lambda, \mu_1) \quad / \text{as shown in red box, other page}$$

$$m_0 = m_0(m_1, \lambda_1, \Lambda, \mu_1) \quad 0 \leftrightarrow 1$$

Now look at the 2-theory:

$$\lambda_2 = \lambda_2(m_2, \lambda_2, \Lambda, \mu_2) \quad 0 \leftrightarrow 2$$

$$m_2 = m_2(m_2, \lambda_2, \Lambda, \mu_2)$$

Then replace m_0, λ_0 via above to get

$$\lambda_2 = \lambda_2(m_1, \lambda_1; \mu_1, \mu_2; \Lambda)$$

$$m_2 = m_2(m_1, \lambda_1; \mu_1, \mu_2; \Lambda)$$

But, m_2, λ_2 are finite, as are $m_1, \lambda_1, \mu_1, \mu_2$. There cannot be Λ^2 dep.
So

$$\boxed{\begin{matrix} \lambda_2 = \lambda_2(m_1, \lambda_1; \mu_1, \mu_2) \\ m_2 = m_2(m_1, \lambda_1; \mu_1, \mu_2) \end{matrix}} \leftarrow \text{you can "move" from th-1} \rightarrow \text{th-2 in this way.}$$

② Now go to the e-charges + conclude

$$\begin{cases} e_2 = e_2(m_1, e_1; \mu_1, \mu_2) \\ m_2 = m_2(m_1, e_1; \mu_1, \mu_2) \end{cases}$$

③ Now define:

$$\beta_2 = \mu_2 \left[\frac{\partial}{\partial \mu_2} e_2(m_1, e_1; \mu_1, \mu_2) \right] \Big|_{m_1, e_1, \mu_1}$$

$$= \frac{1}{2e_2} \mu_2 \cdot \left[\frac{\partial}{\partial \mu_2} e_2^2(m_1, e_1; \mu_1, \mu_2) \right] \Big|_{m_1, e_1, \mu_1}$$

But $e_2^2 = d(e_1^2, \frac{\mu_2^2}{\mu_1^2}, \frac{m_1^2}{\mu_1^2})$ deriv wrt sec arg.

Thus, $\left(\frac{\partial e_2^2}{\partial \mu_2} \right) \Big|_{m_1, e_1, \mu_1} = \frac{2\mu_2}{\mu_1^2} d_2(e_1^2, \frac{\mu_2^2}{\mu_1^2}, \frac{m_1^2}{\mu_1^2})$ by Eq. (26) p-6

$$\Rightarrow \beta_2 = \frac{1}{e_2} \left(\frac{\mu_2^2}{\mu_1^2} \right) d_2(e_1^2, \frac{\mu_2^2}{\mu_1^2}, \frac{m_1^2}{\mu_1^2}) = \frac{1}{e_2} d_2(e_2^2, 1, \frac{m_2^2}{\mu_2^2})$$

Thus, in general we can say that

$$\beta_2 = \beta_2(e_2, \frac{m_2^2}{\mu_2^2})$$

Then in a massless theory, $m_1 = m_2 = 0$ get $\{\beta_2 = \beta_2(e_2)\}$.

④ In principle, β_2 could have depended on: $\mu_2, m_1, e_1, \mu_1 = m_2, e_2, \mu_1, \mu_2$

Then dimensionless $\rightarrow$

$$\beta = \beta(e_2, \frac{m_2^2}{\mu_2^2}, \frac{\mu_1}{\mu_2})$$

$$\begin{matrix} e_1(m_2, e_1, \mu_1, \mu_2) \\ \mu_1(m_2, e_1, \mu_1, \mu_2) \end{matrix}$$

but due to fixed equation, our argument is not present!

⑤ Now consider:

$$\tilde{\beta}_2 \equiv \left[\mu_2 \frac{\partial \lambda_2(m_1, \lambda_1, \mu_1, \mu_2)}{\partial \mu_2} \right]_{m_1, \lambda_1, \mu_1}$$

$$= \left(\frac{\partial \lambda_2}{\partial e_2} \right) \left(\frac{\partial e_2}{\partial \mu_2} \right) = \left(1 - \frac{m_2^2}{\mu_2^2} \right)^{3/2} \underbrace{\mu_2 \frac{\partial e_2}{\partial \mu_2}}_{\beta_2} \checkmark$$

$$\begin{aligned} \Rightarrow \tilde{\beta}_2 &\equiv \left(\mu_2 \frac{\partial \lambda_2}{\partial \mu_2} \right)_{m_1, \lambda_1, \mu_1} = \left(1 - \frac{m_2^2}{\mu_2^2} \right)^{3/2} \cdot \beta_2(e_2, \frac{m_2^2}{\mu_2^2}) \\ &= \tilde{\beta}_2(e_2, \frac{m_2^2}{\mu_2^2}) = \text{dimless.} \end{aligned}$$

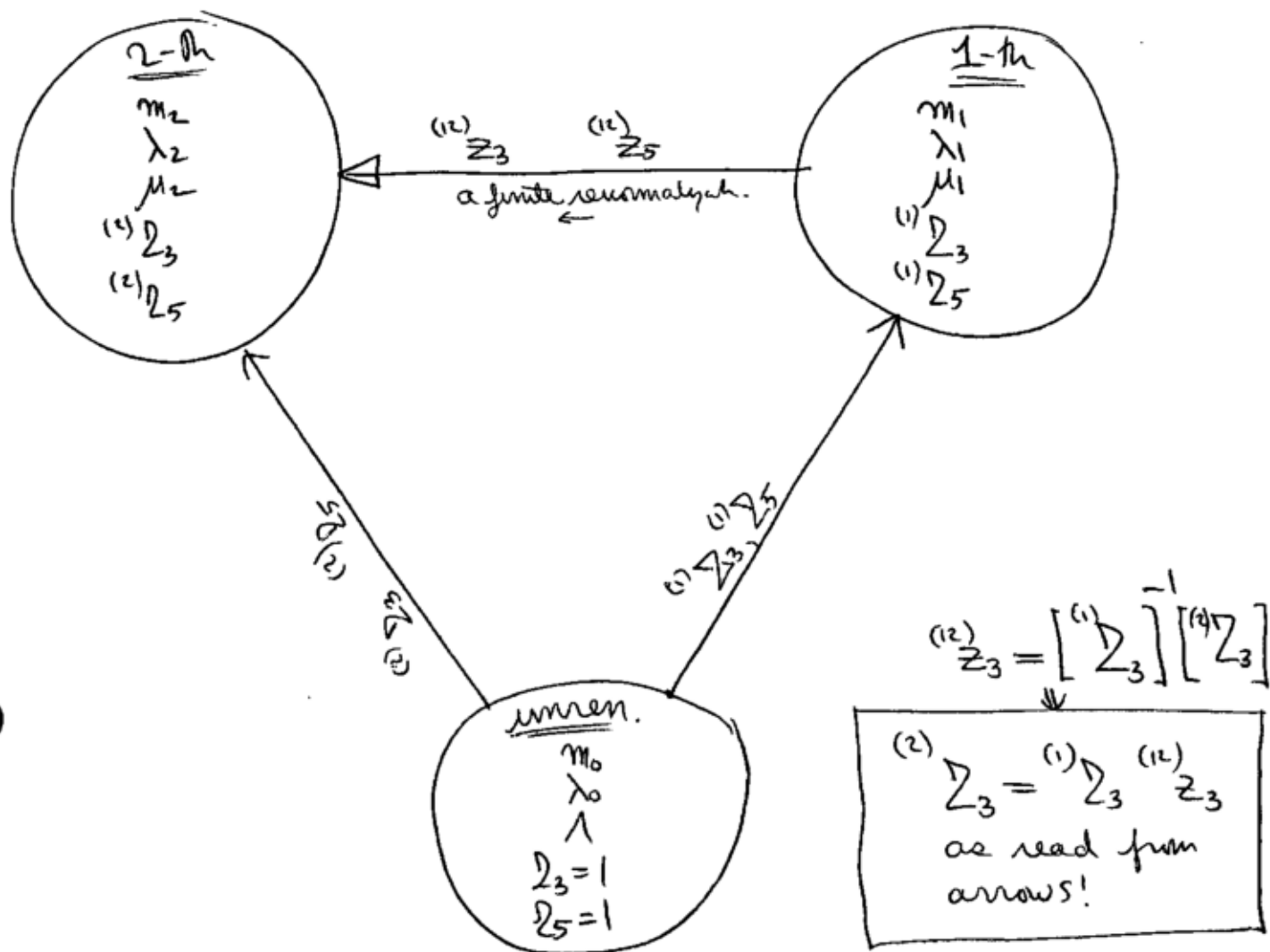
Thus, the BETA defined in this way has the same property as the other BETA: it is a function only of the arguments shown, and is dimensionless.

6. Now go to massless theory and observe these facts:

$$m_1 = 0 \Rightarrow e_1^2 = \left(\frac{\lambda_1^2}{\mu_1^2} \right) = \text{obvious dim-less change.} \checkmark$$

$$\tilde{\beta}_2 \equiv \left(\mu_2 \frac{\partial \lambda_2}{\partial \mu_2} \right)_{\lambda_1, \mu_1} = \tilde{\beta}_2 \left(\frac{\lambda_2}{\mu_2^2} \right) = \tilde{\beta}_2(e_2) \text{ only} \checkmark$$

Location of Theories in Renorm- Λ space



If you hold m_0, λ_0, Λ fixed + look at 2-theory, you say:

$$\begin{aligned} {}^{(2)}\Sigma_3 &= \Sigma_3(\mu_2; m_0, \lambda_0, \Lambda) \\ m_2 &= m_2(\mu_2; m_0, \lambda_0, \Lambda) \\ \lambda_2 &= \lambda_2(\mu_2; m_0, \lambda_0, \Lambda). \end{aligned}$$

But, if instead you hold μ_1, m_1, λ_1 fixed, you get

$$\begin{aligned} {}^{(2)}\Sigma_3 &= \tilde{\Sigma}_3(\mu_2; m_1, \lambda_1, \mu_1) \\ m_2 &= \tilde{m}_2(\mu_2; m_1, \lambda_1, \mu_1) \\ \lambda_2 &= \tilde{\lambda}_2(\mu_2; m_1, \lambda_1, \mu_1) \end{aligned}$$

Rephrase the RGE for propagator.

$$\begin{array}{c} m_2 \lambda_2 \mu_2 \leftarrow m_1, \lambda_1, \mu_1 \\ \text{mod } \Lambda \end{array}$$

- ① Imagine that m_1, λ_1, μ_1 are held fixed and you vary μ_2 . The bare theory can be left out of the discussion.

Then:

$$\mu_2 \frac{\partial}{\partial \mu_2} [{}^1D(p^2; m_1, \lambda_1, \mu_1)] = 0$$

because the 1-theory is held fixed — it never heard of μ_2 .

② But,

$${}^1D(p^2; m_1, \lambda_1, \mu_1) = \left[{}^{(12)}Z_3(\mu_2; m_1, \lambda_1, \mu_1) \right] \cdot {}^2D(p^2; m_2, \lambda_2, \mu_2)$$

$\uparrow$ old theory $\uparrow$ new theory.

Thus we get:

$$\left\{ \underbrace{\mu_2 \left[\frac{\partial}{\partial \mu_2} {}^{(12)}Z_3(\mu_2; m_1, \lambda_1, \mu_1) \right]}_{Z_3 \gamma_2} + Z_3 \mu_2 \frac{d}{d\mu_2} \right\} {}^2D(p^2; m_2, \lambda_2, \mu_2) = 0$$

But,

$$\mu_2 \frac{d}{d\mu_2} {}^2D = \mu_2 \frac{\partial}{\partial \mu_2} {}^2D + \underbrace{\left(\mu_2 \frac{\partial m_2}{\partial \mu_2} \right) \frac{\partial}{\partial m_2}}_{\equiv m_2 \eta_2} {}^2D + \underbrace{\left(\mu_2 \frac{\partial \lambda_2}{\partial \mu_2} \right) \frac{\partial}{\partial \lambda_2}}_{\equiv \beta_2} {}^2D$$

Then: $\gamma_2 \equiv \mu_2 \frac{\partial}{\partial \mu_2} (\ln Z_3) = \frac{1}{Z_3} \cdot \mu_2 \left(\frac{\partial Z_3}{\partial \mu_2} \right)$. So

$$\left\{ \gamma_2 + \mu_2 \frac{\partial}{\partial \mu_2} + m_2 \eta_2 \frac{\partial}{\partial m_2} + \beta_2 \frac{\partial}{\partial \lambda_2} \right\} {}^2D(p^2; m_2, \lambda_2, \mu_2) = 0$$

where:

$$\beta_2 \equiv \mu_2 \left(\frac{\partial \lambda_2(\mu_2; \mu_1, m_1, \lambda_1)}{\partial \mu_2} \right) \Big|_{\mu_1, m_1, \lambda_1}$$

$$\eta_2 \equiv \frac{\mu_2}{m_2} \cdot \left(\frac{\partial m_2(\mu_2; \mu_1, m_1, \lambda_1)}{\partial \mu_2} \right) \Big|_{\mu_1, m_1, \lambda_1}$$

$$\gamma_2 \equiv \mu_2 \left(\frac{\partial}{\partial \mu_2} [\ln Z_3(\mu_2; \mu_1, m_1, \lambda_1)] \right) \Big|_{\mu_1, m_1, \lambda_1}$$

The RGE for general 1PI greens.

$$\textcircled{1} \quad {}^0G^{(n)}(p_1 \dots p_n) = {}^0G^{(2)}(p_1) \dots {}^0G^{(2)}(p_n) {}^0\Gamma^{(n)}(p_1 \dots p_n) + \text{etc.}$$

$\uparrow$
 0D

As usual, let ${}^0D = Z_3 D$. Then. $G = \langle d-d \rangle = Z_3^{-\frac{n}{2}} {}^0G$

${}^0\phi = \sqrt{Z_3} \phi$

So:

$$Z_3^{\frac{n}{2}} G^{(n)}(p_1 \dots p_n) = Z_3^n D^{(2)}(p_1) \dots D^{(2)}(p_n) \cdot [\Gamma^{(n)} \quad Z^a]$$

$$\Rightarrow a = -\frac{n}{2}$$

$\Rightarrow$

$$\underbrace{{}^0\Gamma^{(n)}(p_1 \dots p_n)}_{\text{old}} = Z_3^{-\frac{n}{2}} \Gamma^{(n)}(p_1 \dots p_n)$$

$\textcircled{2}$ Suppress p_i ; hold them 1 fixed, then:

$$\underbrace{{}^1\Gamma^{(n)}(m_1, \lambda_1, \mu_1)}_{\text{old}} = ({}^{(n)}Z_3)^{-\frac{n}{2}} {}^2\Gamma^{(n)}(m_2, \lambda_2, \mu_2)$$

Only change is in 1st term:

$$\mu_2 \frac{\partial}{\partial \mu_2} (Z_3^{-\frac{n}{2}}) = -\frac{n}{2} (Z_3)^{-\frac{n}{2}} \cdot \frac{1}{Z_3} \cdot \underbrace{\mu_2 \frac{\partial Z_3}{\partial \mu_2}}_{Z_3 \gamma_2} = -\frac{n}{2} (Z_3)^{-\frac{n}{2}} \gamma_2$$

Result will then be:

$$\left\{ -\frac{n}{2} \gamma_2 + \mu_2 \frac{\partial}{\partial \mu_2} + \eta_2 m_2 \frac{\partial}{\partial m_2} + \beta_2 \frac{\partial}{\partial \lambda_2} \right\} {}^2\Gamma^{(n)}(p_i; m_2, \lambda_2, \mu_2) = 0$$

(agree w-42 above).

Renorm Dimension

$$\begin{aligned} \textcircled{1} \quad {}^0G &= (\sqrt{Z_3})^n G \quad \Rightarrow \quad \text{Renorm dim} = n. \\ {}^0\Gamma &= (\sqrt{Z_3})^n \Gamma \quad \Rightarrow \quad \text{Renorm dim} = -n \\ {}^0D &= (\sqrt{Z_3})^2 \Gamma \quad \Rightarrow \quad \text{Renorm dim} = +2. \\ {}^0X &= (\sqrt{Z_3})^{RD} X \quad \Rightarrow \quad \text{Renorm dim} = RD. \end{aligned}$$

② General RGE:

$$\left\{ \frac{1}{2} RD, \gamma_2 + \mu_2 \frac{\partial}{\partial \mu_2} + \eta_2 m_2 \frac{\partial}{\partial m_2} + \beta_2 \frac{\partial}{\partial \lambda_2} \right\}^2 X(m_2, \mu_2) = 0.$$

Calculation of Beta in ϕ^3 .

① Let $\tilde{\beta}_1 \equiv \mu_1 \frac{\partial \lambda_1}{\partial \mu_1}(\mu_i; \lambda_0, m_0 \Lambda) = \left(1 - \frac{m_1^2}{\mu_1^2}\right)^{3/2} \beta_1(e_1; \frac{m_1^2}{\mu_1^2})$

direct:

$$\lambda_1 = \underbrace{\Sigma_3^{-1} \Sigma_5^{-1}}_{\equiv \Sigma_4} \lambda_0 \Rightarrow \tilde{\beta}_1 = \lambda_0 \mu_1 \left(\frac{\partial \Sigma_4}{\partial \mu_1} \right).$$

② Now, $\Sigma_3^{-1}(\mu_i; \lambda_0, m_0 \Lambda) = 1 - \tilde{\Sigma}^2(p^2 = \mu_i^2; \lambda_0, m_0 \Lambda)$

$$\begin{aligned} \text{But } \tilde{\Sigma}^2(p^2) &= \lambda^2 \Sigma_3^{-1} E(p^2) + \delta m^2 \\ &= \lambda^2 \Sigma_3^{-1} E(m^2) + \delta m^2 + (p^2 - m^2) \tilde{E}(p^2) \lambda^2 \Sigma_3^{-1} \end{aligned}$$

③ $\Rightarrow \tilde{\Sigma}^2(p^2) = \lambda^2 \Sigma_3^{-1} \tilde{E}(p^2)$

$\Rightarrow \tilde{\Sigma}^2(p^2)^{(2)} \approx \lambda_0^2 \tilde{E}(\mu_i^2) \text{ etc}$

$\Rightarrow \Sigma_3^{(2)} = 1 + \lambda_0^2 [B \ln \Lambda + C(\mu_i^2)].$

④ $\Sigma_5^{-1} = P^0(\mu_i^2)/\lambda_0 = 1 + \lambda_0^2 [F \ln \Lambda + f^+(\mu_i^2)].$

$$\begin{aligned} \Rightarrow \Sigma_4 &\equiv \Sigma_3^{-1} \Sigma_5^{-1} = [1 + \lambda_0^2 (B \ln \Lambda + C(\mu_i^2))]^{3/2} [1 + \lambda_0^2 (F \ln \Lambda + f^+(\mu_i^2))] \\ &\approx [1 + \frac{3}{2} \lambda_0^2 (B \ln \Lambda + C(\mu_i^2))] [1 + \lambda_0^2 (F \ln \Lambda + f^+(\mu_i^2))] \\ &\approx 1 + \lambda_0^2 \left[\underbrace{\frac{3}{2} B + F}_{\substack{\uparrow \\ \text{put in scale, allow for } m=0 \text{ log.}}} \right] + \left[\frac{3}{2} C(\mu_i^2) + f^+(\mu_i^2) \right] + O(\lambda_0^4). \end{aligned}$$

put in scale, allow for $m=0$ log.

Since, $\mu_1 \left(\frac{\partial \lambda_1}{\partial \mu_1} \right) = -\lambda_0^2 \left(\frac{3}{2} B + F \right) + \mu_1 \frac{\partial}{\partial \mu_1} \left(\frac{3}{2} C + f^+ \right) \lambda_0^2$

$$\Rightarrow \tilde{\beta}_1 \equiv \mu_1 \frac{\partial \lambda_1}{\partial \mu_1} (\mu_1; \lambda_0, m_0, 1)$$

$$= \lambda^3 \left[-\frac{3}{2} B - F + \frac{3}{2} \mu_1 \frac{\partial}{\partial \mu_1} C \left(\frac{m_0^2}{\mu_1^2} \right) + \mu_1 \frac{\partial}{\partial \mu_1} f^+ \left(\frac{m_0^2}{\mu_1^2} \right) \right]$$

Since $\lambda_0^3 = \lambda^3 + \text{H.O.}$

However, I don't think the C and f terms are really present. These are dimensionless functions and therefore should have been dimensioned to the renorm point.

No. In a massive theory ($m_0 \neq 0$), the functions C and F^+ can exist and these terms will thus contribute to BETA. In a massless theory, by dimensional argument C and f cannot depend on u_1 , so those derivatives must vanish, in which case the entire answer is as shown.

$$\tilde{\beta}_1 = -\lambda^3 \left(\frac{3}{2} B + F \right). \quad m_0 = m = 0.$$

On the connectin between our invariant function d, and the so-called "running coupler".

1. Assume massless theory so no mass scale, no mass present, all masses = 0. Then we know the following equations to be true:

$$(1) \quad \left[\mu \frac{\partial}{\partial \mu} + \beta \frac{\partial}{\partial e_1} \right] d(e_1^2, \frac{q^2}{\mu_1^2}) = 0 \quad \beta = \beta(e) \quad \text{set } q = \mu_2$$

$$(2) \quad d(e_1^2, 1) = e_1^2$$

$$(3) \quad \beta(e) = \mu_2 \left(\frac{\partial e_1}{\partial \mu_2} \right)^{e_1, \mu_1} = \frac{1}{e_2} \left(\frac{\mu_2^2}{\mu_1^2} \right) \frac{\partial}{\partial (\mu_2^2 / \mu_1^2)} d(e_1^2, \frac{\mu_2^2}{\mu_1^2})$$

$$(4) \quad \beta(e_1) = \frac{1}{e_1} \left[\frac{\partial}{\partial x} (d[e_1^2, x]) \right]^{x=1}$$

$$(5) \quad e_2^2 = d(e_1^2, \mu_2^2 / \mu_1^2)$$

② Now invent new symbols:

$$e_1 \rightarrow g \quad \mu_1 \rightarrow \mu \quad \mu_2 / \mu_1 \rightarrow e^{2t} = \mu / \mu_1$$

$$e_2 \rightarrow \bar{g} \quad \mu_2 \rightarrow \mu_2$$

The above equations now read:

$$(1) \quad \left[-\frac{\partial}{\partial t} + \beta(g) \frac{\partial}{\partial g} \right] d(g^2, e^{2t}) = 0$$

$$(2) \quad d(g^2, 1) = g^2$$

$$(3) \quad \beta(\bar{g}) = \frac{1}{\bar{g}} \cdot \frac{1}{2} \frac{\partial}{\partial t} d(g^2, e^{2t})$$

$$(4) \quad \beta(g) = \frac{1}{g} \left[\frac{\partial}{\partial x} d(g^2, x) \right]^{x=1}$$

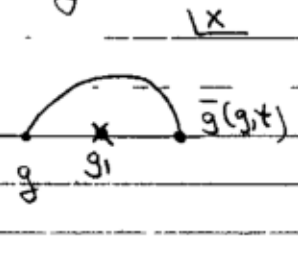
$$(5) \quad \bar{g}^2 = d(g^2, e^{2t})$$

⑥ Zero of Beta?

Suppose $\beta(g_1) = 0$. Then $1/\beta(x)$ has a pole at g_1 .

Consider

$$t = \int_g^{\bar{g}(g,t)} \frac{dx}{\beta(x)}$$



As $\bar{g} \rightarrow g_1$, you cannot deform contour + integral blows up. Thus,

$$+\infty = \int_g^{\bar{g}(g,+\infty)} \frac{dx}{\beta(x)} \Rightarrow \bar{g}(g,+\infty) = g_1 \quad / = \sqrt{\bar{F}'(\infty)}.$$

Here, g_1 is called a fixed point. As $g^2 \rightarrow \infty$, $t \rightarrow \infty$, you approach g_1 .

If $g_1 = 0$, you have "asymptotic freedom". So fixed pts are zeros of $\beta(g)$; [slope conditions also...].

Thyrid calc of beta

$$b. \beta = \frac{1}{2e} \mu \left(\frac{\partial e^2}{\partial \mu} \right)_{\Lambda, e_0}$$

$$\Sigma = \frac{e^2}{e_0^2} \quad (12)$$

$$= \frac{1}{2\sqrt{Z}e_0} \cdot e_0^2 \mu \left(\frac{\partial \Sigma}{\partial \mu} \right)_{\Lambda, e_0}$$

$$= \frac{e_0}{2\sqrt{Z}} \cdot \mu \frac{\partial}{\partial \mu} \left[1 + \Pi_S(e_0^2, \mu^2, \Lambda^2, m_0^2) \right]^{-1}$$

$$= \frac{e_0}{2\sqrt{Z}} \cdot (-1) \Sigma^2 \cdot \mu \frac{\partial}{\partial \mu} \left[\underbrace{\Pi_S(e_0^2, \frac{\mu^2}{\Lambda^2})}_{A e_0^2 \ln(\frac{\mu}{\Lambda})} \right] \quad m=0.$$

$$\underbrace{A e_0^2 \ln(\frac{\mu}{\Lambda})}_{A e_0^2} \quad A = \text{some \#}.$$

$$= - e_0^3 \Sigma^{3/2} A = - A \cdot (e_0 \sqrt{Z})^3$$

$$= - A e^3 = \text{fcn of } e \text{ only!}$$

Sail.

The connection between global and differential renormalization group equations, why $\beta = \beta(g)$ only, and related facts.

1. Renormalization of the Gauge Propagator, definition of functions F, f, d .

$$\begin{aligned} \text{wavy line with a blob} &= i\pi(e_0^2, q^2, \Lambda^2, m_1^2) [-g_{\mu\nu} q^2 + q_\mu q_\nu] + \text{don't care} \\ \text{wavy line} &= iD^{\text{BARE}} = i \left[\frac{-1}{q^2} \left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \right) \right], \text{ transverse gauge, } d=0. \\ \text{wavy line with a blob} &= iD_{\mu\nu}(e_0^2, q^2, \Lambda^2, m_1^2) \quad / \text{ suppress } \mu\nu \text{ from now on.} \end{aligned}$$

← Here m_1 is the physical electron mass, e_0 is the bare electron charge, Λ = a cutoff. If Λ implemented in gauge invariant way, there are no "don't care" terms in the gauge self-energy. One should imagine the objects π and D as being computed to some definite order in e_0^2 , say to order n , ie, to order $(e_0^2)^n$. The lowest order term in D is zeroth order, the lowest order term in π is first-order [for this reason, BD use $e_0^2 \pi^{\text{BD}} = \pi$]. The function π to any finite order is $\ln \Lambda$ divergent. From the self-energy Dyson equation we know that:

$$D(e_0^2, q^2, \Lambda^2, m_1^2) = \left[\frac{1}{1 + \pi(e_0^2, q^2, \Lambda^2, m_1^2)} \right] D^{\text{BARE}} \quad (1)$$

Let us now choose an arbitrary point $q^2 = \mu_1^2$ [one should imagine this to be a negative real number so you stay off the normal thresholds etc of various functions] and simply define a Z_3 in this way:

$$Z_3^{-1} \equiv 1 + \pi(e_0^2, \mu_1^2, \Lambda^2, m_1^2) \sim \ln(\Lambda) \text{ divergent} \quad (2)$$

Now using this number Z_3 we make a multiplicative renormalization of the charge and gauge propagator [the electron propagator goes with Z_2 , the vertex with $Z_1=Z_2$ by Ward, but we ignore these functions here] as follows:

$$D_1 \equiv Z_3^{-1} D \quad e_1^2 \equiv Z_3 e_0^2 \quad (3)$$

Maintaining accuracy to order n (ie, making an error in the next order $n+1$) we can incorporate this multiplicative renormalization as a subtraction in the denominator:

$$D_1 = \left[\frac{1}{1 + \pi(e_0^2, q^2, \Lambda^2, m_1^2) - \pi(e_0^2, \mu_1^2, \Lambda^2, m_1^2)} \right] D^{\text{BARE}} \quad (4)$$

see S
renorm
section.

Now define a subtracted, scalar self-energy function like so:

$$\Pi_S(e_0^2, q^2, \mu_1^2, m_1^2) \equiv \lim_{\Lambda \rightarrow \infty} \left[\Pi(e_0^2, q^2, \Lambda^2, m_1^2) - \Pi(e_0^2, \mu_1^2, \Lambda^2, m_1^2) \right] \quad (5)$$

Presumably the $\ln \Lambda$ divergence is removed in making this subtraction, so the limit $\Lambda \rightarrow \infty$ can be taken and the function so defined is independent of Λ . We then replace the variable e_0 with e_1 as defined in (3) on both sides of the equation (4) to get the following propagator which is "renormalized" in that it is independent of the cutoff, and which is "normalized" to be equal to the bare propagator at the momentum point $q^2 = \mu_1^2$:

$$D_1(e_1^2, q^2, \mu_1^2, m_1^2) = \left[\frac{1}{1 + \Pi_S(e_1^2, q^2, \mu_1^2, m_1^2)} \right] D^{\text{BARE}} \quad (6)$$

$$\text{where } \Pi_S(e_1^2, \mu_1^2, \mu_1^2, m_1^2) = 0. \quad (7)$$

We might now define a function F so that:

$$\left. \begin{aligned} F(e_1^2, q^2, \mu_1^2, m_1^2) &\equiv [1 + \Pi_S(e_1^2, q^2, \mu_1^2, m_1^2)]^{-1} \\ F(e_1^2, \mu_1^2, \mu_1^2, m_1^2) &= 1 \\ D_1(e_1^2, q^2, \mu_1^2, m_1^2) &= F(e_1^2, q^2, \mu_1^2, m_1^2) D^{\text{BARE}} \end{aligned} \right\} \quad (8)$$

Thus, by doing this multiplicative renormalization by the logarithmically divergent Z_3 [a procedure one implements by adding counterterms to the lagrangian which are of the same form as terms already present], we end up with a finite propagator and a finite charge e_1 , a finite electron mass m from the start, and a finite renormalization point μ_1^2 . Parameters e_0 and Λ are gone. We shall refer below to this type of infinite renormalization as The Big Enchilada. $m_1 = m_e$, the physical mass.

Were one to choose $\mu_1^2 = 0$ in QED one would find that $e_1 = e$, the measured physical electron charge. For $\mu_1^2 = -1.0 \text{ GeV}^2$ one gets some charge $e_1 \neq e$, $e_1 = \text{finite}$. One could compute e_1 in terms of e given a knowledge of the function $d(x, y, z)$ to be defined below. [also can compute m_1 ?]

Since the function F is dimensionless we replace it with function f :

$$\left. \begin{aligned} f(e_1^2, \frac{q^2}{\mu_1^2}, \frac{m_1^2}{\mu_1^2}) &\equiv F(e_1^2, q^2, \mu_1^2, m_1^2) \\ f(e_1^2, 1, \frac{m_1^2}{\mu_1^2}) &= 1 \end{aligned} \right\} \quad (9)$$

So,

$$D_1(e_1^2, q^2, \mu_1^2, m_1^2) = f(e_1^2, \frac{q^2}{\mu_1^2}, \frac{m_1^2}{\mu_1^2}) D^{\text{Bare}}. \quad (10)$$

Instead of dealing with f , it is more convenient to replace it with a function d in this way:

$$d(e_1^2, \frac{q^2}{\mu_1^2}, \frac{m_1^2}{\mu_1^2}) \equiv e_1^2 f(e_1^2, \frac{q^2}{\mu_1^2}, \frac{m_1^2}{\mu_1^2}) \quad (11)$$

$$[e_1^2 D_1(e_1^2, q^2, \mu_1^2, m_1^2)] = d(e_1^2, \frac{q^2}{\mu_1^2}, \frac{m_1^2}{\mu_1^2}) D^{\text{Bare}} \quad (12)$$

The obvious convenience of the function d is that it is invariant under further multiplicative renormalizations of the charge and the propagator, since it is equal to their product, modulo $D(\text{bare})$. For this reason d is sometimes called "the invariant charge" (BS). Below we will see that d is the "running coupling constant" of YMT, with its arguments slightly redefined.

note 1. Electron mass renormalization is taken care of by mass counterterm dm which one adjusts in any order so that $\Delta(m, dm) = 0$; then Z_2 can be written in terms of Σ' which is $\ln \Lambda$ divergent.

note 2. For the gauge particle there is no mass renormalization unless Π has a pole or singularity at $q^2 = 0$. In QED there is no singularity since the lowest singularity is at $q^2 = 4m^2$. In QCD this threshold and all others fall into the origin and one does not know the result. Also, Z_3 is given in terms of Π at some point, not Π' .

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Notes on ϕ^3 -Theory in $d=6$ dimensions.

Comments: this was just a testing ground for me to study the idea of renormalization. Yes, this theory is not useful because it is unstable due to the shape of the potential, and so on. But it is the simplest coupling you can imagine, there is no spinology to get confused about, so I feel it is the right place to begin ones study of renormalization.

Index to Notes:

1. Prelim attempt to compute 4th order self-energy.

This was where I began. I computed unrenormalized objects in terms of λ_0 . To get the 4th order self-energy I used the asymmetric Dyson equation. My notation such as Σ_1 and Σ_2 etc was set in another set of notes stored in Renorm Binder. The result of this calculation was at first that the renormed 4th order self energy seemed to still have a log in it, but it almost cancelled. My mistake here was that I was forgetting to expose both divergent subintegrations in the 4th order loop graph. In effect, this modification causes a 2 to appear (red) which, when carried through, causes a finite self-energy in fourth order! Most other results of this section are correct.

2. The overlapping divergence in ϕ graph.

Here I exposed for the first time both divergent subintegration logs. This yielded finally an accurate expression for the integral of $D^0 D^0 f$ where f was the finite part of the vertex in third order. Rather than being just quadratic plus log as I first thought, I found that this expression really had $(\log)^2$ terms due to the overlap log interacting with the overall log.

3. New calculation of 4th order self-energy accounting for the overlap.

This time I wrote a renormed version of the asymmetric Dyson equation. I inserted the correct form for the graph discussed above and after doing the various subtractions etc, I came up with a well-defined and finite result for the 4th order self-energy. This was satisfying because up to that point I had the feeling that this object really was not finite except on shell. This section ends with a summary page showing all results to this point.

4. Reinterpret results using a charge counterterm.

This turned out to be no big deal. I wanted to have λ and λ_0 appear in the Lagrangian, but their sum can still be called λ_0 . The results of working in this way are so that you see only the symbol λ and not λ_0 . But this is the same as just dealing with renormalized objects and equations as I started to do later.

5. Simple self-energy algorithm.

Here I came up with a very short formula for the renormed self energy using the renormed asymmetric Dyson. The result was Z_5 times delta-h. At this point it was a real mystery how the missed overlap singularities of the asymmetric integral were going to exactly cancel the logs present in Z_5 ! I was really stuck.

6. Vertex Integral Equation.

After an abortive attempt, I came up with an integral equation for the vertex which had no overlaps. By excluding two-particle states from the horizontal channel of K you remove the overlap problem, at least naively. In fact Bjorken and Drell analyse this in more detail and do indeed conclude that there is no overlap in the QED analog. So in a red box I was able to correctly state the "vertex equation" in terms of renormed objects and for the first time I was really aware that a single subtraction removes all problems; and I had a way to compute Z_5 . It is this vertex equation that provides iteration used in later analyses.

7. Proof of renorm of ϕ^3 .

Here I wrote down the basic iterative idea, using the three main ingredients: skeleton expansions, vertex Dyson, and self-energy Dyson. The program at this stage was complete except I didn't know how to handle the overlap in the general self-energy case.

8. How you develop powers of logs in computing Γ^0 .

When I first considered the overlap problem, my approach was to work with bare objects and to try to "expose" all the overlap singularities etc. Later I found a better way to do this, but it motivated an investigation of what Z_5 and Γ^0 really look like in terms of their log structure. Basically this is how it works: you know that in terms of λ , Z_5 can have only one power of log arising from the overall divergence. But this is multiplied by a power series in λ and when you shift to λ_0 you find that Z_5 is a polynomial in $(\ln \lambda)$. This polynomial is easy to interpret in terms of simple Feynman graphs. I realized that the Feynman rules for doing renormalized calculations include this rule: vertices should be subtracted. Actually the rule is to use renormed vertices which are finite. I.e., you should work through the integral equations. Generally I put solid boxes around a vertex that is subtracted.

9. K has no divergent subintegrations.

Here I show using the skeleton expansion exactly why it is that the renormed kernel K can have no divergent subintegrations inside itself, even though K^0 does.

10. Theorem: if you ignore overlap, self-energy will not be finite.

This was a warmup to motivate my search for the self-energy problem's solution. I just showed that, if you ignore overlap hidden divergences, you don't stand a chance of getting a finite self energy!

At this point I went and studied Bjorken and Drell. Before doing that, I somehow figured out that you had to do a multi-kernel expansion of the vertex in order to examine the overlap problem; then I was going to push n kernels to the left and m kernels to the right or something. I noticed that BD used a similar expansion and finally I read that famous section 19.11 on why renorm is finite.

The major deal here was the use of a symmetric Dyson equation rather than the asymmetric one. In this symmetric Dyson, the ~~maximax~~ subdivergences always cancel between the two terms. My notes on this reading are filed in BD with other earlier notes on that last chapter.

11. Discussion of finite 4th order using BD technique.

Here I found out exactly how the dangerous subintegrations cancel in this simple case. When you put together all graphs which appear in the symmetric self-energy dyson, you find that you have precisely pulled out both subdivergences so the result has only overall divergence. This led to my precise formulation of how you compute self-energies: construct object $E(p^2)$, subtract and divide by pole, then subtract again. The first subtraction kills overall constant, the second kills the log multiplying $(p^2 - m^2)$ and the result is then finite. Divergences from subintegration always cancel if you include all terms. The self-energy problem was finally solved!

12. 6th order self-energy.

Here I wrote down and classified some of the graphs as they appear in $E(p^2)$. I realized that you really want to classify terms by their kernel number, as in BD. I showed that for $n=0,1$ there are no divergent subintegrations, i.e., no overlap problem. For $n=2$ I had to sit down and ponder.

13. The Subgraph Theorem:

This just says that the divergence of a subintegration cannot be removed by an overall subtraction unless all external momenta flow through your subgraph.

14. The Finiteness proof.

Here I develop the kernel-number expansion for the vertex as in BD. Then I consider explicitly the 2-kernel sector and I examine the various subintegrations to show that they are finite. This always involves a tricky combination of terms. Thus, you can see that you are getting cancellation of bad subintegrations between various of your 2-kernel terms. The general idea is that divergent subintegrations cancel in each order of kernel count. BD do it for n-kernel piece, I did it for 2-kernel piece. I don't really have it down rigorously, but I get the main ideas.

15. Summary of the ϕ^3 renorm program.

Here I redo what I did in 7 above. There are really only three ingredients called A,B,C. There is then a general algorithm for computing everything in the theory to any order you want; renormed and bare objects, Z's, dm^2 , everything.

16. Here I applied the algorithm in lowest order and was able to duplicate my earlier results. I think it's in the bag now.

17. Discussion of renormed graph expansion and counterterm philosophy.

Preliminary Attempt to Get a Finite 4th order self-energy in ϕ^3 theory with $d=6$.

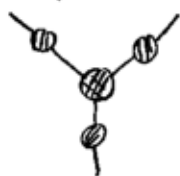
Comment: In the following discussion, all quantities will be un-renormalized, with the exception of the 4th order self-energy itself which I will use in its renormed form. There will be three different constants Z ; there is no Ward identity to reduce this number of constants.

1. Definition of the three constants:

$$\bar{Z}_3^{-1} = 1 - \tilde{\Sigma}^2(m^2) \quad \text{so that} \quad {}^0D(p^2 = m^2) = \frac{Z_3}{p^2 - m^2}$$

$${}^0\Gamma(\text{sym}) = \bar{Z}_5^{-1} \lambda_0 \quad \lambda_0^2 = \bar{Z}_4^{-1} \cdot \lambda^2$$

In order for the theory to renormalize correctly, I anticipate this relation between the various Z 's:



$$\lambda_0 (\sqrt{Z_3})^3 \bar{Z}_5^{-1} = \lambda \Rightarrow \lambda_0^2 = Z_3^3 \bar{Z}_5^{-2} \lambda^2 \Rightarrow \boxed{\bar{Z}_4^{-1} = Z_3^3 \bar{Z}_5^{-2}}$$

2. Compute vertex through third-order, get Z_5 .

$$\begin{aligned} {}^0\Gamma(p_1, p_2, p_3)^{[3]} &= \lambda_0 + \lambda_0^3 [F \ln \Lambda + f(p_1, p_2, p_3)] = \text{diagram} + \text{diagram} \\ &= \lambda_0 [1 + \lambda_0^2 (F \ln \Lambda + f(p_i))] \end{aligned}$$

$$\Rightarrow \bar{Z}_5^{-1 [2]} = 1 + \lambda_0^2 (F \ln \Lambda + f(\text{sym}))$$

3. Compute second-order self-energy, get Z_3 .

$$\text{diagram} = \text{diagram} + \frac{1}{2} \text{diagram} = \delta m^2^{[2]} + \frac{1}{2} \cdot \text{diagram}$$

$$\Rightarrow \tilde{\Sigma}^2(p^2)^{[2]} = \frac{1}{2} \lambda_0^2 [B \ln \Lambda + C(p^2)]$$

$$A \Lambda^2 + (p^2 - m^2) [B \ln \Lambda + C(p^2)]$$

$$\Rightarrow \bar{Z}_3^{-1 [2]} = 1 - \frac{1}{2} \lambda_0^2 [B \ln \Lambda + C(m^2)]$$

Here, second-order mass counterterm was adjusted to kill the quadratic divergence A .

4. Compute renormalized second-order self energy

$$\tilde{\Sigma}^2(p^2) = \frac{\tilde{\Sigma}^2(p^2) - \tilde{\Sigma}^2(m^2)}{1 - \tilde{\Sigma}^2(m^2)} = \frac{1}{2} \tilde{\lambda}_0^2 \Delta C(p^2) + O(\lambda_0^4) \quad \Delta C(p^2) \equiv C(p^2) - C(m^2)$$

$$\sum_{4}^{[1]} = 1 \Rightarrow \tilde{\lambda}_0 = \tilde{\lambda} + O(\lambda_0^4) \Rightarrow \tilde{\Sigma}^2(p^2) \stackrel{[2]}{=} \frac{1}{2} \tilde{\lambda}^2 \Delta C(p^2)$$

This shows that the second-order renormalized self-energy is finite when expressed in terms of the renormalized coupling constant, because function $C(p^2)$ is finite.

5. Compute unrenormalized fourth-order self energy.

Basically there are three terms here: you can feed back the second-order correction into either propagator, and you can feed-back the vertex correction. The propagator in second-order is given by:

$$\begin{aligned} {}^0D^{[2]}(p^2) &= \frac{1}{(p^2 - m^2)[1 - \tilde{\Sigma}^2(p^2)]} = \frac{1}{(p^2 - m^2)(1 - \frac{1}{2} \tilde{\lambda}_0^2 [B \ln \Lambda + C(p^2)])} \\ &= {}^0D^{(0)}(p^2) + \frac{1}{2} \tilde{\lambda}_0^2 [B \ln \Lambda + C(p^2)] {}^0D^{(1)}(p^2). \end{aligned}$$

Corrections are:

$$\begin{aligned} {}^0\tilde{\Sigma}^{(4)}(p^2) &= \frac{1}{2} \lambda_0 \int d^4k {}^0D^{(0)}(k^2) \left\{ \frac{1}{2} \tilde{\lambda}_0^2 [B \ln \Lambda + C(k^2)] \right\} {}^0D^{(0)}(p^2) + (\text{similar}) \\ &\quad + \frac{1}{2} \lambda_0 \int d^4k {}^0D^{(0)}(k^2) {}^0D^{(0)}(p^2) \left\{ \lambda_0^3 [F \ln \Lambda + \tilde{f}(k, k, p)] \right\} \end{aligned}$$

$$\begin{aligned} &= \frac{1}{2} \lambda_0^4 (B \ln \Lambda + 2F \ln \Lambda) \left\{ A \Lambda^2 + (p^2 - m^2) [B \ln \Lambda + C(p^2)] \right\} \\ &\quad + \frac{1}{2} \lambda_0^4 \left\{ A_1 \Lambda^2 + (p^2 - m^2) [B_1 \ln \Lambda + C_1(p^2)] \right\} \\ &\quad + \frac{1}{2} \lambda_0^4 \left\{ A_2 \Lambda^2 + (p^2 - m^2) [B_2 \ln \Lambda + C_2(p^2)] \right\} + \delta m^{(4)} \end{aligned}$$

These last two integrals are what you get including first $C(k^2)$ and second $f(.,.,.)$. I am assuming that the functions C and f don't change the UV behavior of these integrals so they look roughly the same, in UV, as the raw integral.

We should comment on the various divergences. The second-order propagator correction which we are feeding back already includes the effect of the second-order mass counterterm. That is why the propagator correction has pole also at $p^2 = m^2$. Then the overall loop integration in the first term generates new quadratic and log divergences. In particular, we get a $\log^2$ term arising from the feedback log and the overall log.

Now we adjust the 4th order mass counterterm to kill all divergent constants.

Thus:

$$\tilde{\Sigma}^2(p^2) \stackrel{(4)}{=} \frac{1}{2} \lambda_0^4 \left\{ (B + 2F) \ln \Lambda \cdot [B \ln \Lambda + C(p^2)] + (B_1 + B_2) \ln \Lambda + C_1(p^2) + C_2(p^2) \right\}$$

6. Compute renormalized 4th-order self-energy. We do this using the general formula given back in 4. Now however we have to include the integration of the second-order denom with the second-order numerator which gives extra 4th order:

$$\tilde{\Sigma}^{[4]}(p^2) = \frac{\frac{1}{2} \lambda_0^4 \{ (B+2F) \ln \Lambda \cdot \Delta C(p^2) + \Delta C_1(p^2) + \Delta C_2(p^2) \} + \frac{1}{2} \lambda_0^2 \{ \Delta C(p^2) \}}{1 - \frac{1}{2} \lambda_0^2 \{ B \ln \Lambda + C(m^2) \}}$$

$$\Rightarrow \tilde{\Sigma}^{[4]}(p^2) = \frac{1}{4} \lambda_0^4 \{ (2B+2F) \ln \Lambda \cdot \Delta C(p^2) + 2\Delta C_1 + 2\Delta C_2 + (B \ln \Lambda + C(m^2)) \Delta C(p^2) \}$$

$$= \frac{1}{4} \lambda_0^4 \{ (3B+2F) \ln \Lambda \cdot \Delta C(p^2) + 2\Delta C_1(p^2) + 2\Delta C_2(p^2) + C(m^2) \Delta C(p^2) \}$$

$$\Rightarrow \tilde{\Sigma}^{[4]}(p^2) = \frac{1}{2} \lambda_0^2 \Delta C(p^2) + \frac{1}{4} \lambda_0^4 (3B+2F) \ln \Lambda \cdot \Delta C(p^2) + \lambda_0^4 \times (\text{finite})$$

Now do your charge renormalization to get rid of divergences:

$$\lambda_0^2 = \tilde{\Sigma}_4^{-1} \lambda^2 = \lambda^2 + \underbrace{\left(\tilde{\Sigma}_4^{-1[2,3,\dots]} \right)}_{\substack{D\lambda_0^2 + E\lambda_0^4 + \dots \\ D\lambda^2 + D'\lambda^4 + \dots}} \lambda^2 \quad \lambda_0^4 = \tilde{\Sigma}_4^{-2} \lambda^4 = \lambda^4 + \mathcal{O}(\lambda^6)$$

$$\Rightarrow \tilde{\Sigma}^{[4]}(p^2) = \frac{1}{2} \tilde{\Sigma}_4^{-1} \lambda^2 \Delta C(p^2) + \frac{1}{4} \lambda^4 (3B+2F) \ln \Lambda \Delta C(p^2) + \lambda^4 (\text{finite}) + \mathcal{O}(\lambda^6)$$

$$= \frac{1}{2} \lambda^2 \Delta C(p^2) + \underbrace{\left[\frac{1}{2} \tilde{\Sigma}_4^{-1(2)} \lambda^2 \Delta C(p^2) + \frac{1}{4} \lambda^4 (3B+2F) \ln \Lambda \cdot \Delta C(p^2) \right]}_{\text{finite}} + \text{finite}$$

$$\Rightarrow \tilde{\Sigma}_4^{-1(2)} = -\frac{\lambda_0^2}{2} (3B+2F) \ln \Lambda + \lambda_0^2 \cdot \text{finite}$$

$$\Rightarrow \tilde{\Sigma}_4^{-1[2]} = 1 - \lambda_0^2 \left(\frac{3}{2} B + \frac{1}{2} F \right) \ln \Lambda + \lambda_0^2 \cdot (\text{finite})$$

So, if in fact $\tilde{\Sigma}_4$ has a divergent part as shown here, you will achieve your goal of obtaining a finite 4-th order self energy!!!

7. But what is $\tilde{\Sigma}_4$ to second order based on relation to other $\tilde{\Sigma}$'s ??

$$\tilde{\Sigma}_4^{-1} = \tilde{\Sigma}_3^{-1} \tilde{\Sigma}_5^{-2} \Rightarrow \tilde{\Sigma}_4^{-1(2)} = (1 - 3\tilde{\Sigma}_3^{(2)}) \cdot (1 + 2\tilde{\Sigma}_5^{(2)}) = -3\tilde{\Sigma}_3^{(2)} + 2\tilde{\Sigma}_5^{(2)}$$

$$= -3 \left\{ +\frac{1}{2} \lambda_0^2 [B \ln \Lambda + C(m^2)] \right\} + 2 \left\{ -\lambda_0^2 [F \ln \Lambda + f(\text{sym})] \right\}$$

$$\Rightarrow \Sigma_4^{-1 [2]} = 1 - \lambda_0^2 \left(\frac{3}{2} B + 2 F \right) \ln \Lambda - \lambda_0^2 \left(\frac{3}{2} c(m^2) + 2 f(\text{sym}) \right)$$

It agrees, except for the factor-of-2 discrepancy in the F term as indicated by the arrows.

8. Possible source of discrepancy. I am not sure of this section, but have an idea. I think it has to do with the overlapping singularity.

(a) the term F in the third correction term for the 4th order self energy represents the divergent subintegration on the right side. Somehow I have neglected to display the divergent subintegration on the left side. This is somehow still contained in the f integration.

In other words, I am guessing now that my form for the f integral is wrong, and in fact looks like this:

$$\int d^4k {}^0D^{(0)}(k) {}^0D^{(0)}(k') f(k, k', p)$$

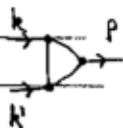
$$= F \ln \Lambda [A \Lambda^2 + (p^2 - m^2) \{ B \ln \Lambda + C(p^2) \}] + A_2 \Lambda^2 + (p^2 - m^2) [B_2 \ln \Lambda + C_2(p^2)]$$

Ie, f has some pinch or something in it which generates this extra term which in effect doubles the weight of F in the self energy. Then if you follow the red ink 2's I have put in, things work out consistently and I have finally been able to produce a finite 4th order self-energy.

The overlapping divergence question in $d=6$ case.

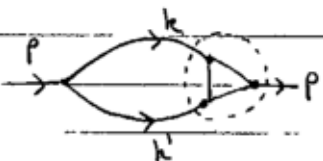


① I have claimed that:



$$= \lambda_0^3 [F \ln \Lambda + f(k, k', p)]$$

② Now consider:



$$= \lambda_0 \int d^6 k \cdot D^0(k) D^0(k') \cdot \{ F \ln \Lambda + f(k, k', p) \} \lambda_0^3$$

This is meant to be an exact expression for this Feynman graph

ie



$$= \int d^6 k d^6 k' \lambda_0^4 \cdot D^0(k) D^0(k') D^0(k+k') D^0(k_1) D^0(k'_1)$$

has three kinds of divergences:

This double integral really



a) hold k fixed, integrate over k_1 : $\lambda_0 \int d^6 k D^0(k) D^0(k') \left[\int d^6 k_1 D^0(k+k_1) D^0(k_1) D^0(k'_1) \right]$

$F \ln \Lambda + f(k, k', p)$

b) hold k_1 fixed, integrate over k :

$$\lambda_0^4 \int d^6 k_1 D^0(k_1) D^0(k'_1) \left[\int d^6 k D^0(k+k_1) D^0(k_1) D^0(k'_1) \right]$$

$F \ln \Lambda + f(k, k', p)$

These are the divergent sub-integrations.

Note they are each only $6/6 = \log$.

c) overall divergence $k, k_1 \rightarrow \infty$ together $= \frac{12}{10} = \Lambda^2 + \log$.



③ Therefore, if we wish to show all divergences in this $\text{---}\bigcirc\text{---}$ graph, we can say:



$$\text{---}\bigcirc\text{---} = 2 F \ln \Lambda \cdot \lambda_0^4 \cdot \int d^6 k D^0(k) D^0(k')$$

$$+ [A_3 \Lambda^2 + (p^2 - m^2) \left[\ln \Lambda + C_3(p^2) \right]]$$

$\uparrow$
 B_3

$\uparrow$
finite

But as usual, $\int dk D^0(k) D^0(k') = \Lambda^2 + (p^2 - m^2) [B \ln \Lambda + C(p^2)]$

Therefore:

$$\begin{aligned} \text{Diagram} &= \lambda_0^4 \int dk D^0(k) D^0(k') [F \ln \Lambda + f(k, k', p)] \\ &= \lambda_0^4 \left\{ (2F \ln \Lambda) (\Lambda^2 + (p^2 - m^2) (B \ln \Lambda + C(p^2))) \right. \\ &\quad \left. + A_3 \Lambda^2 + (p^2 - m^2) (B_3 \ln \Lambda + C_3(p^2)) \right\} \end{aligned}$$

$$= \lambda_0^4 \{ 2AF \Lambda^2 \ln \Lambda + A_3 \Lambda^2 \}$$

$$+ \lambda_0^4 (p^2 - m^2) \{ 2BF (\ln \Lambda)^2 + 2FC(p^2) \ln \Lambda + B_3 \ln \Lambda + C_3(p^2) \}$$

Therefore, arising from this graph only we would say,

$$\sum_1^{(4)} \Sigma_1^{(2)}(p^2) = \lambda_0^4 \{ \Delta C(p^2) \cdot 2F \ln \Lambda + \Delta C_3(p^2) \}$$

④ If we now put in the Dyson $\frac{1}{2}$ and include the propagator corrections, we shall get:

$$\sum_1^{(4)} \Sigma_1^{(2)}(p^2) = \frac{1}{2} \lambda_0^4 \{ \Delta C(p^2) \cdot [2F \ln \Lambda + B \ln \Lambda] + \Delta C_1(p^2) + \Delta C_3(p^2) \}$$

This is the 2 we needed to make all work.

⑤ Therefore,

$$\begin{aligned} \int dk D^0(k) D^0(k') f(k, k', p) &= \Lambda^2 (AF \ln \Lambda + A_3) \\ &\quad + (p^2 - m^2) \{ BF (\ln \Lambda)^2 + [F \cdot C(p^2) + B_3] (\ln \Lambda) + C_3(p^2) \} \end{aligned}$$

Get 4th order self-energy in ϕ^3 with $d=6$ using renormed objects.

① Renormalized vertex and propagator:

$$^{\circ}D(p^2) = Z_3 \cdot D(p^2)$$

$$^{\circ}\Gamma(p_i) = Z_3^{-3/2} \Gamma(p_i)$$

② What are the D and Γ objects to lowest order?

$$\Sigma_1^{(2)}(p^2) = \frac{1}{2} \lambda^2 \Delta C(p^2) \Rightarrow D(p^2)^{(2)} = \frac{1}{(p^2 - m^2) [1 - \frac{1}{2} \lambda^2 \Delta C(p^2)]}$$

$$\Rightarrow D(p^2)^{(2)} = \frac{1}{2} \lambda^2 \frac{\Delta C(p^2)}{p^2 - m^2}$$

$$D(p^2)^{(0)} = \frac{1}{p^2 - m^2}$$

Notice that this renormed object is finite! Next

$$\Gamma(p_i)^{(3)} = Z_3^{3/2} \left\{ \lambda_0 (1 + \lambda_0^2 [F_{\text{ren}} + f(p_i)]) \right\}$$

$$= \lambda \cdot Z_5 \cdot [1 + \lambda^2 \{ F_{\text{ren}} + f(p_i) \}]$$

$$= \lambda [1 - \lambda^2 (F_{\text{ren}} + f(\text{sym}))] \cdot [1 + \lambda^2 (F_{\text{ren}} + f(p_i))]$$

$$= \lambda [1 + \lambda^2 \Delta f(p_i)] \quad \Delta f(p_i) \equiv f(p_i) - f(\text{sym})$$

$$\Rightarrow \Gamma(p_i)^{(3)} = \lambda^3 \cdot \Delta f(p_i) = \text{finite!}$$

③ Renormed integral equation reads:

$$^{\circ}\Sigma_1^{(2)}(p^2) = \delta m^2 + \frac{1}{2} \int d^d k \cdot \lambda_0 \cdot ^{\circ}D(k^2) ^{\circ}D(k'^2) ^{\circ}\Gamma(k, k', p)$$

$$= \delta m^2 + \frac{1}{2} \underbrace{(\lambda_0 Z_3)}_{(Z_5 Z_3 \lambda)} \cdot \int d^d k \cdot D(k) D(k') \Gamma(k, k', p)$$

$$\Rightarrow ^{\circ}\Sigma_1^{(2)}(p^2) = \delta m^2 + \frac{1}{2} Z_5 Z_3 \cdot \lambda \cdot \int d^d k \cdot D(k) D(k') \Gamma(k, k', p) \quad \checkmark$$

④ Now let's list off the contribution to $^0\Sigma^2(p^2)^{(4)}$:

$$\begin{aligned} ^0\Sigma^2(p^2)^{(4)} &= \delta m^2^{(4)} + \frac{1}{2} \cdot [\Sigma_5^{(2)} + \Sigma_3^{-1(2)}] \cdot \lambda \cdot \int dk \cdot D^{(0)} D^{(0)} \cdot \lambda \quad \checkmark \\ &+ \frac{1}{2} \lambda \cdot \int dk \cdot D^{(0)} D^{(0)} \cdot \lambda^3 \Delta f(p_i) \quad \checkmark \\ &+ \lambda \cdot \int dk \cdot [\frac{1}{2} \lambda^2 \Delta C(k^i)] D^{(0)} D^{(0)} \cdot \lambda \quad \checkmark \end{aligned}$$

Meanwhile,

$$\begin{aligned} \Sigma_5^{(2)} + \Sigma_3^{-1(2)} &= -\lambda^2 (F \ln \Lambda + f(\text{sym})) - \frac{1}{2} \lambda^2 (B \ln \Lambda + C(m^2)) \quad \checkmark \\ &= -\frac{1}{2} \lambda^2 \{ (B+2F) \ln \Lambda + C(m^2) + 2f(\text{sym}) \} \rightarrow \text{next page.} \end{aligned}$$

Now,

$$\int dk \cdot D^{(0)} \cdot D^{(0)} = A \Lambda^2 + (p^2 - m^2) [B \ln \Lambda + C(p^2)]$$

$$\int dk \cdot D^{(0)} \cdot D^{(0)} \cdot \Delta f(p_i) = A_3 \Lambda^2 + (p^2 - m^2) [B_3 \ln \Lambda + C_3(p^2)]$$

$$\int dk \cdot D^{(0)} \cdot D^{(0)} \cdot \Delta C(k^i) = A_4 \Lambda^2 + (p^2 - m^2) [B_4 \ln \Lambda + C_4(p^2)]$$

Therefore,

$$\begin{aligned} ^0\Sigma^2(p^2)^{(4)} &= \delta m^2^{(4)} + \lambda^4 \left\{ -\frac{1}{4} (B+2F) \ln \Lambda + \frac{C(m^2)}{4} - \frac{f(\text{sym})}{4} \right\} \{ A \Lambda^2 + (p^2 - m^2) [B \ln \Lambda + C(p^2)] \} \\ &+ \lambda^4 \left\{ \frac{1}{2} A_3 \Lambda^2 + \frac{1}{2} (p^2 - m^2) [B_3 \ln \Lambda + C_3(p^2)] \right\} \\ &+ \lambda^4 \left\{ \frac{1}{2} A_4 \Lambda^2 + \frac{1}{2} (p^2 - m^2) [B_4 \ln \Lambda + C_4(p^2)] \right\}. \end{aligned}$$

Now select:

$$\begin{aligned} \delta m^2^{(4)} &= -\lambda^4 \left\{ -\frac{1}{4} (B+2F) \ln \Lambda + \frac{C(m^2)}{4} - \frac{f(\text{sym})}{4} \right\} \{ A \Lambda^2 \} \\ &- \lambda^4 \left\{ \frac{1}{2} A_3 \Lambda^2 + \frac{1}{2} A_4 \Lambda^2 \right\}. \end{aligned}$$

Then:

$$\begin{aligned} ^0\Sigma^2(p^2)^{(4)} &= \lambda^4 \left\{ -\frac{1}{4} (B+2F) \ln \Lambda + \frac{C(m^2)}{4} - \frac{f(\text{sym})}{4} \right\} \{ B \ln \Lambda + C(p^2) \} \\ &+ \lambda^4 \left\{ \frac{1}{2} B_3 \ln \Lambda + \frac{1}{2} C_3(p^2) + \frac{1}{2} B_4 \ln \Lambda + \frac{1}{2} C_4(p^2) \right\} \end{aligned}$$

Also,

$$a) \int dk D^{(0)} D^{(0)} = A \Lambda^2 + (p^2 - m^2) [B \ln \Lambda + C(p^2)]$$

$$\begin{aligned} b) \int dk D^{(0)} D^{(1)} \Delta f &= -f(\text{sym}) \left[\int dk D^{(0)} D^{(1)} \right] + \int dk D^{(0)} D^{(0)} \cdot f \\ &= -f(\text{sym}) \left\{ A \Lambda^2 + (p^2 - m^2) [B \ln \Lambda + C(p^2)] \right\} \\ &\quad + \Lambda^2 (A F \ln \Lambda + A_3) + (p^2 - m^2) \left\{ B F (\ln \Lambda)^2 + [F \cdot C(p^2) + B_3] \ln \Lambda + C_3(p^2) \right\} \\ &= \Lambda^2 [A F \ln \Lambda + A_3 - A f(\text{sym})] \\ &\quad + (p^2 - m^2) \left[B F (\ln \Lambda)^2 + \{ F \cdot C(p^2) + B_3 - B \cdot f(\text{sym}) \} \ln \Lambda \right. \\ &\quad \left. + \{ C_3(p^2) - f(\text{sym}) C(p^2) \} \right] \end{aligned}$$

$$c) \int dk D^{(0)} D^{(1)} \Delta C(k^2) = A_4 \Lambda^2 + (p^2 - m^2) [B_4 \ln \Lambda + C_4(p^2)].$$

Therefore,

$$\begin{aligned} {}^0 \Sigma^2(p^2)^{(4)} &= \bar{m}^2{}^{(4)} - \frac{1}{2} \lambda^2 \left(\frac{1}{2} \lambda^2 \right) \left\{ (B+2F) \ln \Lambda + C(m^2) + \frac{1}{2} f(\text{sym}) \right\} \\ &\quad \times \left\{ A \Lambda^2 + (p^2 - m^2) [B \ln \Lambda + C(p^2)] \right\} \\ &\quad + \frac{1}{2} \lambda^4 \Lambda^2 (A F \ln \Lambda + A_3) + \frac{1}{2} \lambda^4 (p^2 - m^2) \left[B F (\ln \Lambda)^2 + \{ F \cdot C(p^2) + B_3 - B \cdot f(\text{sym}) \} \ln \Lambda \right. \\ &\quad \left. + \{ C_3(p^2) - f(\text{sym}) C(p^2) \} \right] \\ &\quad + \frac{1}{2} \lambda^4 [A_4 \Lambda^2 + (p^2 - m^2) [B_4 \ln \Lambda + C_4(p^2)]] \end{aligned}$$

Re-write entire mass:

$$\begin{aligned} {}^0 \Sigma^2(p^2)^{(4)} &= \bar{m}^2{}^{(4)} + \frac{1}{4} \lambda^4 \cdot \Lambda^2 \left\{ (-A) \left\{ (B+2F) \ln \Lambda + C(m^2) + \frac{1}{2} f(\text{sym}) \right\} \right. \right. \\ &\quad \left. \left. + 2 A F \ln \Lambda + 2 A_3 + 2 A_4 \right\} \right. \\ &\quad + \frac{1}{4} \lambda^4 (p^2 - m^2) \left\{ \left[(B+2F) \ln \Lambda + C(m^2) + \frac{1}{2} f(s) \right] \cdot [B \ln \Lambda + C(p^2)] \right. \\ &\quad \left. + 2 B F (\ln \Lambda)^2 + 2 \ln \Lambda \{ F \cdot C(p^2) + B_3 - B f(s) \} \right. \\ &\quad \left. + 2 C_3(p^2) - 2 f(s) C(p^2) + 2 B_4 \ln \Lambda + 2 C_4(p^2) \right\} \end{aligned}$$

Thus, you should choose:

$$-\delta m^2^{(4)} = \frac{1}{4} \lambda^4 \Lambda^2 \left\{ -AB \ln \Lambda + C(m^2) \frac{4}{3} f(s) + 2(A_3 + A_4) \right\}$$

Then you get

$$\begin{aligned} \sum_2^{(4)}(p^2) &= \frac{1}{4} \lambda^4 \left\{ -(B \ln \Lambda)^2 - C(p^2)(B \ln \Lambda) + 2 \ln \Lambda (B_3 + B_4 - B \cdot f(s)) \right. \\ &\quad \left. - C(p^2) [C(m^2) + \frac{4}{3} f(s)] + 2 [\rho_3(p^2) + \rho_4(p^2)] \right\} \end{aligned}$$

$$\Rightarrow \left[\sum_2^{(4)}(p^2) - \sum_2^{(4)}(m^2) \right] = \frac{1}{4} \lambda^4 \left\{ -\Delta C(p^2) [B \ln \Lambda + C(m^2) + \frac{4}{3} f(s)] \right. \\ \left. + 2 [\Delta \rho_3(p^2) + \Delta \rho_4(p^2)] \right\}$$

Now,

$$\sum_2^{(4)}(p^2) = {}^0 \sum_2^{(4)}(p^2)_S \cdot \Delta_3 \quad / \text{exact}$$

$$\begin{aligned} \Rightarrow \sum_2^{(4)}(p^2) &= {}^0 \sum_2^{(4)}(p^2)_S + {}^0 \sum_2^{(4)}(p^2)_S \cdot \Delta_3 \\ &= \frac{1}{4} \lambda^4 \left\{ -\Delta C(p^2) \cdot (B \ln \Lambda + C(m^2) + \frac{4}{3} f(s)) + 2 \Delta \rho_3(p^2) + 2 \Delta \rho_4(p^2) \right\} \\ &\quad + \frac{1}{2} \lambda^2 \cdot \Delta C(p^2) \cdot \frac{1}{2} \lambda^2 [B \ln \Lambda + C(m^2)] \\ &= \frac{1}{4} \lambda^4 \left[-\frac{4}{3} f(s) \cdot \Delta C(p^2) + 2 \cdot \Delta \rho_3(p^2) + 2 \cdot \Delta \rho_4(p^2) \right] \end{aligned}$$

$$\Rightarrow \boxed{\sum_2^{(4)}(p^2) = \frac{1}{2} \lambda^4 [\Delta \rho_3(p^2) + \Delta \rho_4(p^2) - 2 \cdot f(s) \cdot \Delta C(p^2)]}$$

Summary of Results

$$D(p^2)^{(2)} = \frac{1}{2} \lambda^2 \Delta C(p^2) (p^2 - m^2)^{-1}$$

$$D(p^2)^{(0)} = (p^2 - m^2)^{-1}$$

$$\Gamma(k, k', p)^{(0)} = \lambda$$

$$\Gamma(k, k', p)^{(3)} = \lambda^3 \Delta f(k, k', p)$$

$$\Delta f = f(k, k', p) - f(s)$$

$$\int dk D^{(0)} D^{(0)} = A \Lambda^2 + (p^2 - m^2) [B \ln \Lambda + C(p^2)]$$

$$\Sigma_3^{(2)} = \frac{1}{2} \lambda^2 [B \ln \Lambda + C(m^2)] \checkmark$$

$$\Sigma_5^{(2)} = -\lambda^2 (F \ln \Lambda + f(s)) \checkmark$$

$$\overset{\circ}{\Sigma}_1^{(2)}(p^2)^{(2)} = \frac{1}{2} \lambda^2 [B \ln \Lambda + C(p^2)]$$

$$\delta m^2^{(2)} = -\frac{1}{2} A \Lambda^2 \cdot \lambda^2 \checkmark$$

$$\overset{\circ}{\Sigma}_1^{(2)}(p^2)^{(2)} = \frac{1}{2} \lambda^2 \cdot \Delta C(p^2) \checkmark$$

$$\overset{\circ}{\Sigma}_1^{(2)}(p^2) = \delta m^2 + \frac{1}{2} \Sigma_5 \Sigma_3^{-1} \lambda \int dk \cdot D(k) D(k') \Gamma(k, k', p)$$

$$\delta m^2^{(4)} = \frac{1}{4} \lambda^4 \Lambda^2 [A \cdot B \cdot \ln \Lambda - C(m^2) - 2f(s) - 2A_3 - 2A_4]$$

$$\overset{\circ}{\Sigma}_1^{(2)}(p^2)^{(4)} = -\frac{1}{4} \lambda^4 \left\{ (B \ln \Lambda)^2 + C(p^2) (B \ln \Lambda) + 2 \ln \Lambda (B \cdot f(s) - B_3 - B_4) \right. \\ \left. + C(p^2) [C(m^2) + 3f(s)] - 2 \cdot C_3(p^2) - 2 \cdot C_4(p^2) \right\}$$

$$\overset{\circ}{\Sigma}_1^{(2)}(p^2)_s^{(4)} = \frac{1}{4} \lambda^4 \left\{ -\Delta C(p^2) \cdot [B \ln \Lambda + C(m^2) + 3f(s)] + 2 \cdot \Delta C_3(p^2) + 2 \cdot \Delta C_4(p^2) \right\}$$

$$\overset{\circ}{\Sigma}_1^{(2)}(p^2)^{(4)} = \frac{1}{2} \lambda^4 \left[\Delta C_3(p^2) + \Delta C_4(p^2) - \frac{2}{\Lambda} f(s) \cdot \Delta C(p^2) \right]$$

$$\Sigma_3^{-1(4)} = -\overset{\circ}{\Sigma}_1^{(2)}(m^2)^{(4)} = \text{see } -3$$

$$D(p^2)^{(4)} = \left[\overset{\circ}{\Sigma}_1^{(2)}(p^2)^{(4)} + \left\{ \overset{\circ}{\Sigma}_1^{(2)}(p^2)^{(2)} \right\}^2 \right] \cdot D(p^2)^{(0)}$$

Re-interpret results using a charge counter-term.

① In, $\mathcal{L} = \frac{1}{2} (\partial_\mu \phi \partial^\mu \phi - \underbrace{m^2 \phi^2}_{\text{phys. mass}} + \underbrace{\delta m^2 \phi^2 + \delta \lambda \phi^3}_{\text{counterterms}} + \underbrace{\lambda \phi^3}_{\text{phys. charge}}$

Let $\lambda_0 \equiv \lambda + \delta \lambda$. Then integral equation true in terms of λ_0 !

Notice then that

$$\delta \lambda = \lambda_0 - \lambda = \lambda_0 (1 - \sqrt{Z_4}) = \lambda_0 (1 - Z_3^{-3/2} Z_5^{-1})$$

② Second order self energy is same calculation! No change at all

$$\Sigma_1^{(2)}(p^2) = \frac{1}{2} \lambda^2 [B \ln \Lambda + C(m^2)] \quad \delta m^2^{(2)} = -\frac{1}{2} \lambda^2 A \Lambda^2$$

$$\Sigma_2^{(2)}(p^2) = \frac{1}{2} \lambda^2 \Delta C(p^2)$$

$$\Sigma_3^{(2)} = -\frac{1}{2} \lambda^2 [B \ln \Lambda + C(m^2)]$$

③ Charge counterterm calc is nice

$$\lambda_0 + \lambda_0^3 \int \frac{d^4 k}{D \cdot D \cdot D} = \lambda + \delta \lambda + \lambda^3 [F \ln \Lambda + f(p^2)] = \Gamma(p_i)$$

But we require that $\Gamma(\text{sym}) = \lambda = Z_3^{-3/2} \Gamma(\text{sym})$

$$\Rightarrow \lambda = (1 + \frac{3}{4} \lambda^2 [B \ln \Lambda + C(m^2)]) (\lambda + \delta \lambda + \lambda^3 [F \ln \Lambda + f(5)])$$

$$\Rightarrow \lambda = \lambda + \delta \lambda + \lambda^3 \left\{ \frac{3}{4} (B \ln \Lambda + C(m^2)) + (F \ln \Lambda + f(5)) \right\}$$

$$\Rightarrow \delta \lambda^{(3)} = -\frac{\lambda^3}{4} \{ (3B + 4F) \ln \Lambda + 3C(m^2) + 4f(5) \}$$

④ But, $\delta \lambda = \lambda_0 - \lambda = (Z_4^{-1/2} - 1) \lambda$

Thus we have found that: $\Sigma_4^{(-1/2)(2)} = -\frac{\lambda^2}{4} \{ (3B + 4F) \ln \Lambda + 3C(m^2) + 4f(5) \}$

$$\Rightarrow \Sigma_4^{(2)} = +\frac{1}{2} \lambda^2 \{ (3B + 4F) \ln \Lambda + 3C(m^2) + 4f(5) \}$$

← This agrees with our orig Σ_4 calc!

⑤ Next, $\Gamma^{(3)}(p_i) = Z_3^{-3/2} \Gamma = (1 + \frac{3}{4} \lambda^2 (B \ln \Lambda + C(m^2))) (\lambda + \delta \lambda + \lambda^3 (F \ln \Lambda + f(p_i)))$

$$\Rightarrow \Gamma^{(3)}(p_i) = \lambda + \lambda^3 \Delta f(p_i) \quad \checkmark$$

⑥ Now renorm the integral equation:

$$\Sigma^2(p^2) = \Sigma_m^2 + \frac{1}{2} \underbrace{(\lambda + \delta\lambda)}_{\lambda_0} Z_3^{-\frac{1}{2}} \int dk D(k) D(k') P(k, k', p).$$

But $2\alpha Z_3^{-\frac{1}{2}} = \lambda Z_4^{-\frac{1}{2}} Z_3^{-\frac{1}{2}} = \lambda Z_3^{-1} Z_5 =$ same factor as other method.

We can go on to compute the 4th order self-energy. There are 4 kinds of terms, but we are just repeating our calculation of before!

So, this "charge counterterm method" works fine, finally! In this method, everything is expressed in powers of λ rather than λ_0 right from the beginning. That was the advantage I was originally looking for. Still it is very convenient to make use of the symbol λ_0 defined as $\lambda + \delta\lambda$. This calculation simply duplicates the earlier calculation.

What was my earlier mistake that kept wrecking this calculation? I kept getting a renormalized 4th order self-energy that depended on the cutoff. My main mistake was with the overlap subtlety; I was failing to fully expose a divergence, so I kept getting an uncancelled $F \log \Lambda$ term. Recall how the $B \log \Lambda$ term did always cancel. Another mistake I made was leaving out the Dyson $1/2$, but I don't think that was much of an error, could have been absorbed into the dk symbol, but now I am always showing the Dyson $1/2$ factor explicitly.

Simpler self-energy algorithm.

①

We know that, to any order,

$$\Sigma^h(p^2) = \Sigma m^2 + \frac{1}{2} \Sigma_5 \Sigma_3^{-1} \lambda \int dk D(k) D(k') P(k, k', p)$$

$$\text{But LHS} = \Sigma^h(m^2) + (p^2 - m^2) \Sigma^h(p^2).$$

Also:

$$\int dk D(k) D(k') P(k, k', p) = h(p^2, \Lambda, \lambda) \cdot (p^2 - m^2) + \text{constant} \\ \therefore k(\Lambda, \lambda)$$

Therefore,

$$\Sigma^h(p^2) = h(p^2, \Lambda, \lambda) \frac{1}{2} \Sigma_5 \Sigma_3^{-1} \lambda$$

$$\Sigma m^2 = -\frac{1}{2} \Sigma_5 \Sigma_3^{-1} \lambda \cdot k(\Lambda, \lambda)$$

Then

$$\Sigma^h(p^2) = \frac{1}{2} \lambda \Sigma_5 \Delta h(p^2, \Lambda, \lambda)$$

$$\text{where } \Delta h(p^2, \Lambda, \lambda) = h(p^2, \Lambda, \lambda) - h(m^2, \Lambda, \lambda)$$

② Compute lowest term:

$$\left\{ \int dk D^0 \cdot D^0 \cdot \lambda \right\} = \left\{ \int dk D \cdot D \cdot P \right\}^{(1)} = h^{(1)}(p^2, \Lambda, \lambda) (p^2 - m^2) + k^{(1)}(\Lambda, \lambda) \\ = \lambda [A \Lambda^2 + (p^2 - m^2) [B \ln \Lambda + C(p^2)]]$$

$$\Rightarrow \begin{cases} h^{(1)}(p^2, \Lambda, \lambda) = \lambda [B \ln \Lambda + C(p^2)] \\ \Delta h^{(1)}(p^2, \Lambda, \lambda) = \lambda \cdot \Delta C(p^2) \\ k^{(1)}(\Lambda, \lambda) = A \cdot \lambda \cdot \Lambda^2 \end{cases}$$

③ Now look at third-order:

$$\left\{ \int dk D \cdot D \cdot P \right\}^{(3)} = \int dk D^0 \cdot D^0 \cdot P^{(3)} + 2 \int dk D^0 \cdot D^0 \cdot \lambda \\ = \lambda^3 \int dk D^0 \cdot D^0 \cdot [\Delta f(k, k', p) + \Delta C(k^2)]$$

I look up these integrals + find

$$h^{(3)}(p, \Lambda, \lambda) = \lambda^3 \left\{ BF(\ln \Lambda)^2 + [F \cdot C(p^2) + B_3 + B_4 - Bf(s)] \ln \Lambda + G_3(p^2) + G_4(p^2) - f(s)C(p^2) \right\}$$

$$\Delta h(p, \Lambda, \lambda)^{(3)} = \lambda^3 \left\{ \Delta C(p^2) \cdot F \ln \Lambda + \Delta G_3(p^2) + \Delta G_4(p^2) - f(s) \Delta C(p^2) \right\}$$

$$h(p, \Lambda, \lambda)^{(3)} = \lambda^3 \Lambda^2 [AF \ln \Lambda + A_3 + A_4 - A \cdot f(s)]$$

④ Therefore:

$$\Sigma^2(p^2)^{(4)} = \frac{1}{2} \lambda \cdot 1 \cdot \Delta h^{(3)} + \frac{1}{2} \lambda \cdot \Sigma_5^{(2)} \cdot \Delta h^{(1)}$$

$$= \frac{1}{2} \lambda^4 \left\{ \Delta C(p^2) \cdot F \ln \Lambda + \Delta G_3(p^2) + \Delta G_4(p^2) - f(s) \Delta C(p^2) \right\}$$

$$+ \frac{1}{2} \lambda^4 \left\{ (-F \ln \Lambda - f(s)) \cdot (\Delta C(p^2)) \right\}$$

$$= \frac{1}{2} \lambda^4 \left\{ \Delta G_3(p^2) + \Delta G_4(p^2) - 2f(s) \cdot \Delta C(p^2) \right\}$$

This method seems more direct, gives correct answer.

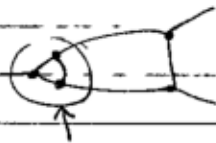
A new vertex integral equation.

① What's wrong with the original one I brought up? This: consider:

$$\lambda_0 \int dk \cdot {}^0D^{(0)}(k) {}^0D^{(0)}(k') {}^0K^{(4)}(k, k', p, p') \quad \text{contribution to } \Gamma^{(5)}$$

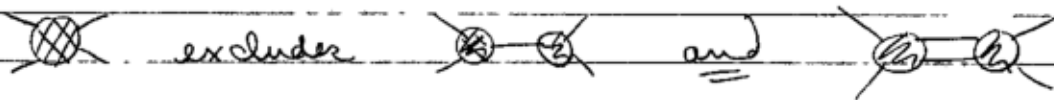
Well, ${}^0K^{(4)}(k, k', p, p') = \text{II} = \lambda_0^4 \int \frac{d^6 k}{k^8} = \text{finite}.$

Thus you say that the above object has only overall divergences. But this is wrong because you have thereby omitted this divergent subintegration:



Thus, you have buried divergences + these have to be exposed.

② So now lets redefine K to be like that in BD:



Then integral equation reads:

$$\text{crossed circle} = \text{crossed circle} + \text{crossed circle with cross} \times \frac{1}{2} \quad \text{as in 293 BD.}$$

Now all left-vertex corrections are included here; K cannot produce a mass of them!

$$K = \text{crossed circle} + o(\lambda')$$

③ Therefore:

$$\Gamma(q, p, p') = \lambda_0 + \frac{1}{2} \int dk \cdot \Gamma(q, k, k') {}^0D(k) {}^0D(k') {}^0K(k, k', p, p')$$

④ Renormalize:

$$\Gamma(q, p, p') = \lambda_0 + Z_3^{-1/2} \int dk \cdot \Gamma(q, k, k') D(k) D(k') K(k, k', p, p')$$

Thus we obtain

$$\Gamma(q, p, p') = Z_5 \lambda + \frac{1}{2} \int dk \cdot \Gamma(q, k, k') D(k) D(k') K(k, k', p, p').$$

④ Now define a new vertex which includes charge:

$$\frac{\Gamma(q, p, p')}{\lambda} = \tilde{\Gamma}(q, p, p') \quad \text{so} \quad \tilde{\Gamma}(\text{sym}) = 1.$$

Then:

$$\tilde{\Gamma}(q, p, p') = Z_5 + \frac{1}{2} \int dk \cdot \tilde{\Gamma}(q, k, k') D(k) D(k') K(k, k', p, p').$$

Then:

$$1 - Z_5 = \frac{1}{2} \int dk \cdot \tilde{\Gamma}(\tilde{q}, k, k') D(k) D(k') K(k, k', \tilde{p}, \tilde{p}')$$

$$\text{Then:} \quad Z_5 = 1 - \frac{1}{2} \int dk \tilde{\Gamma}(\tilde{q}, k, k') D(k) D(k') K(k, k', \tilde{p}, \tilde{p}')$$

Summarize:

$$\tilde{\Gamma}(q, p, p') = Z_5 + \frac{1}{2} \int dk \cdot \tilde{\Gamma}(q, k, k') D(k) D(k') K(k, k', p, p')$$

$$\tilde{\Gamma}(q, p, p') = 1 - \frac{1}{2} \int dk \tilde{\Gamma}(\tilde{q}, k, k') D(k) D(k') K(k, k', \tilde{p}, \tilde{p}') \\ + \frac{1}{2} \int dk \tilde{\Gamma}(q, k, k') D(k) D(k') K(k, k', p, p')$$

$$Z_5 = 1 - \frac{1}{2} \int dk \tilde{\Gamma}(\tilde{q}, k, k') D(k) D(k') K(k, k', \tilde{p}, \tilde{p}')$$

⑤ Now we say:

$$\dim \left(\frac{1}{2} \int d^4k \tilde{\Gamma}(q, k, k') D(k) D(k') K(k, k', p, p') \right) = 6 + 0 - 2 - 2 - 2 = 0$$

$$\Rightarrow = [F \ln \Lambda + f(q, p, p')] \lambda^2$$

where now $F = F^{(0)} + \lambda^2 F^{(2)} + \dots$

⑥ Therefore we get

$$\tilde{\Gamma}(q, p, p') = 1 + \frac{1}{2} \lambda^2 [F \ln \Lambda + f(p_i)] - \lambda^2 [F \ln \Lambda + f(s)]$$

$$= 1 + \frac{1}{2} \lambda^2 \Delta f(p_i) \Rightarrow \boxed{\tilde{\Gamma}^{[n+2]}(q, p, p') = 1 + \lambda^2 \Delta f^{[n]}(q, p, p')}$$

$$\Rightarrow \tilde{\Gamma}(q, p, p') = 1 + \frac{1}{2} \lambda^3 \Delta f(p_i)$$

Now that we have eliminated the hidden singularities which were present in that other integral equations, the singularities trivially cancel. The only singularities are the "overall" ones due to the overall loop integration. This is so because K is finite and there is no overlap problem.

① Iteration: $K^{(2)} = \left(\text{diagram: a circle with a vertical line through the center, labeled } \lambda \text{ at the top} \right) = \lambda^2 D(e) + \text{diagram: a circle with a horizontal line through the center, labeled } K$

$$\{K^{(2)}, D^{(0)}, \tilde{\Gamma}^{(0)}\} \Rightarrow \tilde{\Gamma}^{(2)}$$

$$\{K^{[n]}, D^{[n]}, \tilde{\Gamma}^{[n]}\} \Rightarrow \tilde{\Gamma}^{[n+2]}$$

In general $\{K^{[n+2]}, D^{[n]}, \tilde{\Gamma}^{[n]}\} \Rightarrow \tilde{\Gamma}^{[n+2]}$

Repeat: $\boxed{\{K^{[n+2]} + D^{[n]} + \tilde{\Gamma}^{[n]}\} \Rightarrow \tilde{\Gamma}^{[n+2]} \text{ and } \Sigma_5^{[n+2]}}$

with no divergence!

This is an important step in the chain. From the vertex integral equation we can start with K thru order $n+2$, D through order n , and $\tilde{\Gamma}$ thru order n and we can compute $\tilde{\Gamma}$ to the next higher order, namely, $n+2$. If the three starting objects have no singularities, then the object you produce also has no singularities!

Proof of Renormalization of ϕ^3 theory, $d=6$.

Here are the basic ingredients:

a) skeleton expansion for K does this: $D^{[n]} + \tilde{\Gamma}^{[n]} \rightarrow K^{[n+2]} \checkmark$

Note: $D^{[n]}$ and $\tilde{\Gamma}^{[n]}$ have no singe, there are no melope, and the overall int. is finite $\Rightarrow K^{[n+2]}$ has no singe. $\checkmark$

lowest: $D^{(0)} + \tilde{\Gamma}^{(0)} \rightarrow K^{(2)} = I = \lambda^2 D^{(2)}$

b) combined with the above, vertex integral equation does this

$$D^{[n]} + \tilde{\Gamma}^{[n]} \rightarrow \tilde{\Gamma}^{[n+2]} + \Sigma_5^{[n+2]} \checkmark$$

Note: the $\tilde{\Gamma}^{[n+2]}$ has no singe $\checkmark$ because they are exactly cancelled by Σ_5 as already noted.

lowest: $\tilde{\Gamma}^{(0)} = 1$ $D^{(0)} + \tilde{\Gamma}^{(0)} \rightarrow \tilde{\Gamma}^{(2)} + \Sigma_5^{(2)}$
 $\Sigma_5^{(0)} = 1$ $\checkmark$ $\checkmark$

I already know these.

c) The propagator Dyson tells you this:

$$D^{[n]} + \tilde{\Gamma}^{[n]} \rightarrow \tilde{h}^{[n]} \text{ and } \tilde{k}^{[n]} \text{ by definition.}$$

Then, $\Sigma^2(p^2) = \delta m^2 + \frac{1}{2} \Sigma_5^{-1} \lambda^2 \int dk \cdot \underbrace{D \cdot D \cdot \tilde{\Gamma}}_{\tilde{k}}$

$$\Rightarrow 0 = \delta m^2 + \frac{1}{2} \Sigma_5^{-1} \lambda^2 \tilde{k} = \tilde{k} + \tilde{h}(p^2) \cdot (p^2 - m^2)$$

$$\Rightarrow \Sigma^2(p^2) = \frac{1}{2} \Sigma_5^{-1} \lambda^2 \tilde{h}(p^2)$$

$$\Rightarrow \left(\Sigma^2(p^2) = \frac{1}{2} \Sigma_5 \lambda^2 \Delta \tilde{h}(p^2) \right)$$

$$\Rightarrow 1 - \Sigma_3^{-1} = \frac{1}{2} \Sigma_5^{-1} \lambda^2 \tilde{h}(m^2)$$

$$\Rightarrow \Sigma_3 = 1 + \frac{1}{2} \Sigma_5 \lambda^2 \tilde{h}(m^2)$$

Summarize these last:

$$\begin{cases} \int dk D(k) D(k') \tilde{\Gamma}(k, k', p) = \tilde{k} + \tilde{h}(p^2) \cdot (p^2 - m^2) \\ \Sigma m^2 = -\frac{1}{2} \Sigma_5 \Sigma_3^{-1} \lambda^2 \tilde{k} \\ \Sigma_3 = 1 + \frac{1}{2} \lambda^2 \Sigma_5 \tilde{h}(m^2) \\ \Sigma_1^2(p^2) = \frac{1}{2} \lambda^2 \Sigma_5 \tilde{\Delta h}(p^2) \end{cases} \quad D(p^2) = \frac{1}{(p^2 - m^2)(1 - \tilde{\Sigma}^2(p^2))}$$

d) Thus you see that

$$\Sigma_5^{[n]} + \Sigma_3^{[n]} + \tilde{k}^{[n]} \rightarrow \Sigma m^2^{[n+2]}$$

Reverse: $\Sigma_5^{[n]} + \Sigma_3^{[n]} + D^{[n]} + \tilde{\Gamma}^{[n]} \Rightarrow \Sigma m^2^{[n+2]}$

start: $1 + 1 + D^{(0)} + 1 \Rightarrow \Sigma m^2^{(2)}$

e) also: $\Sigma_5^{[n]} + D^{[n]} + \tilde{\Gamma}^{[n]} \Rightarrow \Sigma_3^{[n+2]}$ and $\Sigma_1^2(p^2)^{[n+2]} = D(p^2)^{[n+2]}$

start: $1 + D^{(0)} + 1 \Rightarrow \Sigma_3^{(2)} + D^{(2)}$

Here is the sequence:

$$\begin{cases} 1) D^{(0)} + \tilde{\Gamma}^{(0)} \rightarrow \tilde{\Gamma}^{(2)} + \Sigma_5^{(2)} & /a+b & (\text{Keld + vertex int}) \\ 2) D^{(0)} + \tilde{\Gamma}^{(0)} + \Sigma_5^{(0)} \rightarrow \Sigma_3^{(2)} + D^{(2)} & /e & (\text{self-energy int}) \\ 3) D^{(0)} + \tilde{\Gamma}^{(0)} + \Sigma_5^{(0)} + \Sigma_3^{(0)} \rightarrow \Sigma m^2^{(2)} & /d & (\text{self-energy int}) \end{cases}$$

$$\begin{cases} 1)' \tilde{D}^{(2)} + \tilde{\Gamma}^{(2)} \rightarrow \tilde{\Gamma}^{(4)} + \Sigma_5^{(4)} \\ 2)' \tilde{D}^{(2)} + \tilde{\Gamma}^{(2)} + \tilde{\Sigma}_5^{(2)} \rightarrow \Sigma_3^{(4)} + D^{(4)} \\ 3)' \tilde{D}^{(2)} + \tilde{\Gamma}^{(2)} + \tilde{\Sigma}_5^{(2)} + \tilde{\Sigma}_3^{(2)} \rightarrow \Sigma m^2^{(4)} \end{cases}$$

and so on to any order you like !!

Comments: as far as I can tell, the only missing ingredient in this proof of the renormalizability of ϕ^3 theory in $d=6$, is the fact that I have not yet proven that the self-energy when iterated does not contain divergences. Recall that you are supposed to find no divergences in each order. I have explicitly shown this in 4th order, and that demonstration involved the overlap problem crucially. In other words, it remains for me to show that the logs in Z_5 magically cancel against the logs in δh , as they did in the 4th order case.

Lets suppose I can prove this. Then am I done? Conveniently, it is relatively easy to see how the divergences go away when you iterate up your vertex function. There was no no overlap in that equation, that's why it was easier.

Look at the overall procedure. Yes, the only missing piece is to show that the self-energy is finite to any order.

How you get powers of log's in the vertex $\Gamma(q, p, p'; \lambda_0)$.

1. First, we must be careful about the notation used for renormalizing objects.

I propose this:

$$\Gamma(q, p, p'; \lambda_0) = Z_3^{-3/2} \cdot \Gamma(q, p, p'; \lambda)$$

where $\lambda = Z_3^{3/2} Z_5^{-1} \lambda_0$. In other words, the functional form of all renormalized objects is defined to automatically be a function of the new charge. As for a Z, you can write them in either of two ways:

$$Z_3 = Z_3(\lambda, \ln \Lambda) = \bar{Z}_3(\lambda_0, \ln \Lambda)$$

I.e., when I am thinking of a Z as being expanded in the bare charge, I will put a bar over it because it will be a different functional form.

2. Now let's review our analysis of the vertex integral equation. We found:

$$\tilde{\Gamma} = Z_5 + \frac{1}{2} \int dk \tilde{\Gamma}(\lambda) D(\lambda) D(\lambda) K(\lambda, p_i)$$

$$Z_5 = 1 - \frac{1}{2} \int dk \tilde{\Gamma}(\lambda) D(\lambda) D(\lambda) K(\tilde{p}_i, \lambda)$$

You assume you know finite renormalized objects in the integrand as series in λ , so the only singularity arising from the integral is the overall log. Thus write:

$$\begin{cases} Z_5 = 1 - \lambda^2 [F(\lambda) \ln \Lambda + f(s; \lambda)] = 1 - \lambda^2 \sum_{i=0}^{\infty} [F_i \ln \Lambda + f_i(s)] (\lambda^2)^i \\ \tilde{\Gamma}(p_i; \lambda) = 1 + \lambda^2 \Delta f(p_i; \lambda^2) = 1 + \lambda^2 \sum_{i=0}^{\infty} \Delta f_i(p_i) \cdot (\lambda^2)^i \end{cases}$$

We have already noted how this relation lets you iterate to compute the renormed vertex to one higher order in λ^2 and no divergences are introduced.

3. However, in order to talk about the overlap problem and other things, I want to restate the above equations in terms of the bare charge. Let's study Z_5 first:

$$Z_5(\lambda_0, \ln \Lambda) = 1 - \sum_{i=0}^{\infty} [F_i \ln \Lambda + f_i(s)] (\bar{Z}_3^3 \bar{Z}_5^{-2} \lambda_0^2)^{i+2}$$

Notice this fact: when Z_5 is expressed in terms of λ , the only power of $(\ln \Lambda)$ you ever get is first power, no matter to what order you compute. But when dealing with $\bar{Z}_5(\lambda_0)$

you will get a polynomial in $(\ln \Lambda)$ if you compute to a finite order. Let's do it:

4. Lets compute Z_5 to fourth order both ways.

$$\Sigma_5(\lambda)^{(2)} = 1 - \lambda^2 (F_0 \ln \Lambda + f_0(s))$$

$$\bar{\Sigma}_5(\lambda_0)^{(2)} = 1 - \lambda_0^2 (F_0 \ln \Lambda + f_0(s))$$

$$\bar{\Sigma}_3(\lambda_0)^{(2)} = 1 + \frac{1}{2} \lambda_0^2 (B_0 \ln \Lambda + G_0(s))$$

$$\Sigma_5(\lambda)^{(4)} = -\lambda^4 (F_1 \ln \Lambda + f_1(s))$$

$$\bar{\Sigma}_5(\lambda_0)^{(4)} = -\lambda_0^4 (F_1 \ln \Lambda + f_1(s)) \{1\} - \lambda_0^2 (F_0 \ln \Lambda + f_0(s)) \{ \bar{\Sigma}_3^3 \bar{\Sigma}_5^{-2} \}^{(2)}$$

But $\{ \bar{\Sigma}_3^3 \bar{\Sigma}_5^{-2} \}^{(2)} = 3 \cdot \bar{\Sigma}_3^{(2)} - 2 \cdot \bar{\Sigma}_5^{(2)} = 3 \left[\frac{1}{2} \lambda_0^2 (B_0 \ln \Lambda + G_0(s)) \right] - 2 \left[-\lambda_0^2 (F_0 \ln \Lambda + f_0(s)) \right]$

$$= \lambda_0^2 \ln \Lambda \left[\frac{3}{2} B_0 + 2F_0 \right] + \left[\frac{3}{2} G_0(s) + 2f_0(s) \right]$$

So:

$$\bar{\Sigma}_5(\lambda_0)^{(4)} = \lambda_0^4 \left\{ F_1 \ln \Lambda + f_1(s) - (F_0 \ln \Lambda + f_0(s)) \left[\left(\frac{3}{2} B_0 + 2F_0 \right) \ln \Lambda + \left(\frac{3}{2} G_0(s) + 2f_0(s) \right) \right] \right\}$$

$$= \lambda_0^4 \left\{ -F_0 \left(\frac{3}{2} B_0 + 2F_0 \right) (\ln \Lambda)^2 + (F_1 - F_0 \left[\frac{3}{2} G_0(s) + 2f_0(s) \right] + f_1(s) \left(\frac{3}{2} B_0 + 2F_0 \right)) (\ln \Lambda) + (f_1(s) - f_0(s) \left[\frac{3}{2} G_0(s) + 2f_0(s) \right]) (\ln \Lambda^0) \right\}$$

So:

$$\Sigma_5(\lambda)^{(4)} = -\lambda^4 [F_1 \ln \Lambda + f_1(s)]$$

$$\bar{\Sigma}_5(\lambda_0)^{(4)} = \lambda_0^4 \left[-F_0 \left(2F_0 + \frac{3}{2} B_0 \right) (\ln \Lambda)^2 + \left(F_1 - \frac{3}{2} F_0 G_0(s) + f_0(s) \frac{3}{2} B_0 \right) (\ln \Lambda) + (f_1(s) - f_0(s) \left[\frac{3}{2} G_0(s) + 2f_0(s) \right]) \right]$$

If you go look now at 6th order Z_5 and so on, you will find in general that:

$$\begin{cases} \Sigma_5(\lambda, \ln \Lambda)^{(2n)} = -\lambda^{2n} [F_{n+1} \ln \Lambda + f_{n+1}(s)] \\ \bar{\Sigma}_5(\lambda_0, \ln \Lambda)^{(2n)} = \lambda_0^{2n} [A_n (\ln \Lambda)^n + A_{n-1} (\ln \Lambda)^{n-1} + \dots + A_0] \end{cases}$$

I now have a way to interpret or understand where these polynomials in $(\ln \lambda)$ are coming from. Recall the definition of Z_5 :

$${}^0P(\text{sym}) = \bar{Z}_5(\lambda_0, \ln \lambda)^{-1} \cdot \lambda_0$$

For example, ${}^0P(5, \lambda_0)^{[7]} = \bar{Z}_5^{-1[6]} \cdot \lambda_0$

$$\Rightarrow {}^0P(5, \lambda_0)^{(7)} = (\bar{Z}_5^{-1})^{(6)} \cdot \lambda_0$$

Pause:

$$\bar{Z}_5^{(2n)} = \lambda_0^{2n} P_n(\ln \lambda)$$

↑ degree of poly in $\ln \lambda$.

$$\Rightarrow \bar{Z}_5^{[2n]} = 1 + \lambda_0^2 P_1(\ln \lambda) + \lambda_0^4 P_2(\ln \lambda) + \dots + \lambda_0^{2n} P_n(\ln \lambda)$$

$$= 1 + \sum_{i=1}^n (\lambda_0^2)^i P_i(\ln \lambda)$$

Then:

$$\bar{Z}_5(\lambda_0) = 1 + \sum_{i=1}^{\infty} (\lambda_0^2)^i P_i(\ln \lambda)$$

$$\Rightarrow \bar{Z}_5(\lambda_0)^{-1} = \sum_{j=0}^{\infty} \left[- \sum_{i=1}^{\infty} (\lambda_0^2)^i P_i(\ln \lambda) \right]^j$$

Now analyze this

$$\bar{Z}_5(\lambda_0)^{-1 (2n)} = \because j=1 \Rightarrow i=n \Rightarrow - \lambda_0^{2n} P_n(\ln \lambda)$$

$$\begin{aligned} j=2 \Rightarrow i=n-1 \text{ combines with } i=1 &\Rightarrow 2(\lambda_0^2)^{2n} P_{n-1}(\ln \lambda) P_1(\ln \lambda) \\ i=n-2 \text{ " " } i=2 &\Rightarrow 2(\lambda_0^2)^{2n} P_{n-2}(\ln \lambda) P_2(\ln \lambda) \\ &\text{etc.} \end{aligned}$$

so

$$j=2 \Rightarrow (\lambda_0^2)^{2n} P_n'(\ln \lambda)$$

Thus, in general:

$$\begin{cases} \bar{Z}_5(\lambda_0)^{-1 (2n)} = (\lambda_0^2)^{2n} P_n'(\ln \lambda) \\ \Rightarrow \bar{Z}_5(\lambda_0)^{-1 [2n]} = 1 + \lambda_0^2 P_1'(\ln \lambda) + \dots + \lambda_0^{2n} P_n'(\ln \lambda) \end{cases}$$

The point here is that if you invert Z_5 , it has the same general form as a series, except the polynomials in $\ln \Lambda$ are now different polynomials. Their degree is the same as before. So continue with the discussion started on top of last page:

$$\begin{aligned} {}^0P(\text{sym}, \lambda_0)^{(7)} &= (\Sigma_5(\lambda_0)^{-1})^{(7)} \lambda_0 \\ &= [1 + \lambda_0^2 P_1'(\ln \Lambda) + \lambda_0^4 P_2'(\ln \Lambda) + \lambda_0^6 P_3'(\ln \Lambda)] \cdot \lambda_0 \end{aligned}$$

$$\Rightarrow {}^0P(\text{sym}, \lambda_0)^{(7)} = \lambda_0^7 P_3'(\ln \Lambda).$$

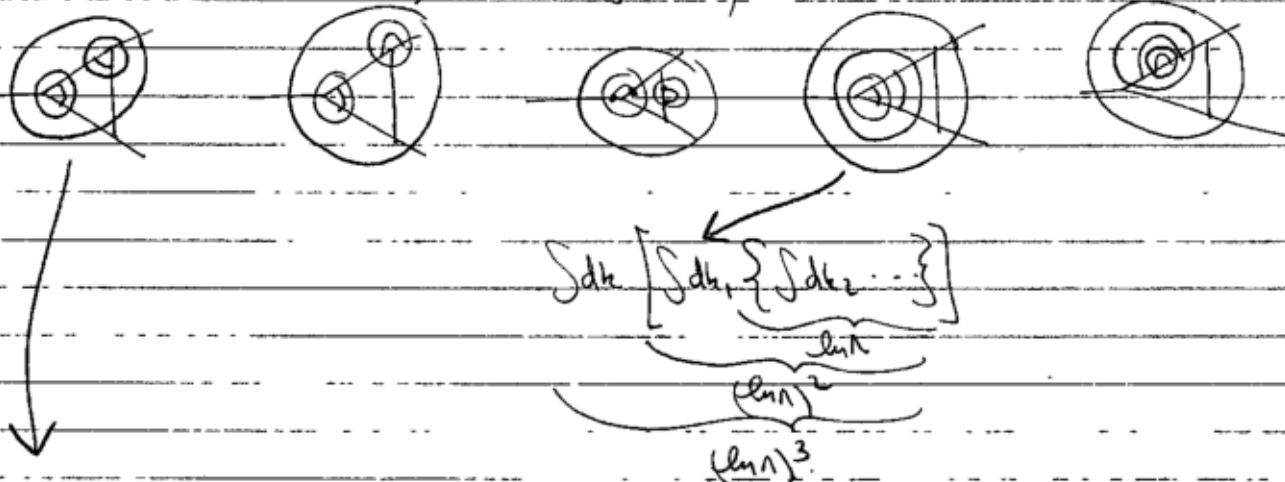
Go back to lower order:

$${}^0P(\lambda_0)^{(3)} = \lambda_0^3 P_1'(\ln \Lambda) = \text{[triangle graph]} \text{ only graph, } (\ln \Lambda) \text{ obvious.}$$

$${}^0P(\lambda_0)^{(5)} = \lambda_0^5 P_2'(\ln \Lambda) = \text{[circle with triangle]} + \text{[circle with triangle]} + \text{[circle with triangle]}$$

Here I am showing how the $\log^2$ terms arise. A subintegration has a log which then multiplies the overall loop with the subintegration shrunk to a point. The overall loop also makes a log, so you get $\log^2$ in each case shown above. Then the finite part of each subintegration yields finite + $\log$ from the overall loop. Thus you see exactly how the polynomial of degree 2 is arising in Z_5 computed to 5th order.

$$\text{Go one more order: } {}^0P(\lambda_0)^{(7)} = \lambda_0^7 P_3'(\ln \Lambda)$$



$$\text{Sdk} \cdot \underbrace{[Sdk_1 \dots]}_{\ln \Lambda} \cdot \underbrace{[Sdk_2 \dots]}_{\ln \Lambda} = (\ln \Lambda)^3$$

All graphs of this type make polynomial in $\ln$ of degree 3. Subintegrations are either nested or disjoint.

there:

$$\triangle = \bigcirc \triangle = P_3(\ln \lambda)$$

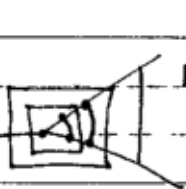
I am finally starting to understand how this renormalization stuff works.

I have not yet worked through the self-energy stuff since it is more complicated, but there is much that I can already say just from the vertex analysis.

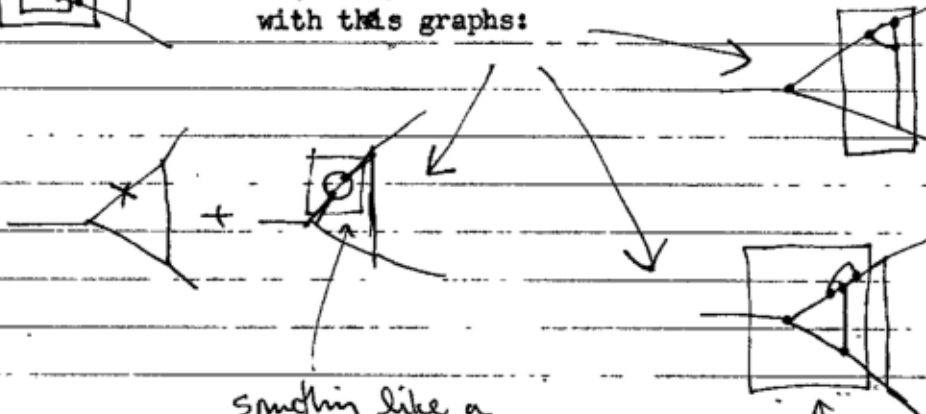
Consider the task of computing Z_5 the 5th order contribution. To do this in terms of the bare coupling constant is very easy. You just write down all the 5th order graphs and compute them. For self-energy insertions, the quadratic divergence is always cancelled by the mass counterterm insertion, so you never get these. You just go ahead and compute things and you get a polynomial of degree 2 in $\log \lambda$. There is no ambiguity in what to do, you just do it.

On the other hand, suppose you want to do the same thing in terms of the renormalized charge, ie, compute Z_5 to fifth order, say. Now the claim is you will only get $\log^1$ from the overall integration. How do you do this? Simple: you just draw all 5th order feynman graphs, now putting λ at each vertex in place of λ_0 . Then you draw lines isolating your vertex insertions, overlapping or disjoint or whatever. According to our integral equation, you are to understand each vertex insertion as being subtracted once at the symmetric point! For example, when you compute $\tilde{\Gamma}^{(3)}$, you look at the integral equation and you see that you are doing a simple and direct subtraction which kills the log and takes f to Δf . In turn, this object $\tilde{\Gamma}^{(3)}$ which has been thus subtracted appears on the right side in the generation of $\tilde{\Gamma}^{(5)}$. The idea then is simple: for each vertex insertion you do one subtraction at the symmetric point. Thus at each nested sublevel the log is removed.

Actually, the discussion above only handles the following graphs:



I have not yet learned how to do the self-energy right, nor have I shown that the renormalized K has no logs. Eg, what happens with this graphs:



something like a
subtr. needed
here.

or at top level, but at a
lower level this is like

~~x333~~ So where am I?

I have not finished yet but I can see that there will emerge a set of "renormalization" rules which must accompany the Feynman rules. One of those rules seems to be: anytime you see a simple vertex insertion, subtract that subintegration, and do so at the symmetric point, this being the point where $\tilde{\eta} = 1$ to all orders. When accompanied with the rest of the renorm rules (which I have yet to derive), you can compute any amplitude to any finite order and get a unique and finite result!!!

For example, suppose you were computing the anomalous magnetic moment of the "electron" in ϕ^3 . You really want to know the vertex computed to third order, ie, the renormalized vertex. You do so and you get a finite result. Note that you certainly cannot ignore the moment graph?! Every graph still contributes something to the finite result!

$$\text{[Diagram: a vertical line with a horizontal line crossing it] + [Diagram: a loop with a vertical line through it] = \lambda + \lambda^3 \Delta f = \text{well defined \& finite result.}$$

How to show that K has no divergences using the skeleton expansion.

(Similarly for the triangle).

1. I have noticed that K does in fact contain subdivergences. However, these are always due to vertex and self-energy insertions and K does not have them as I will now show. Probably this argument could be phrased in terms of an integral equation, but I have been unable to write an integral equation for K . However, I do know the skeleton expansion:

$$\begin{aligned} \text{Diagram of } K &= \text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3} + \text{etc.} \quad (\text{no } \alpha = \text{allowed}) \\ &= F(\Gamma(\lambda_0), D(\lambda_0)) = F\left[\sum_3^{3/2} \Gamma(\lambda), \sum_3 D(\lambda)\right] \end{aligned}$$

Now you just look at the "shape" of any term in the skeleton expansion and factor off the $Z_3^{3/2}$ in the obvious way and you will get:

$$\begin{aligned} &= Z_3^{-2} f(\Gamma(\lambda), D(\lambda)) \\ &= K(\lambda) \end{aligned}$$

Here you see that the renormalized kernel called K is a series in renormalized Γ and D . Since these objects have no divergences, the only divergences in K are those that might arise from overall loop integrations. But power count shows that K has no such divergences. Thus, using the skeleton expansion we can in principle compute K to any order in λ given expressions for Γ and D .

2. This allows us to improve our former argument concerning the vertex. The only missing link is how you iterate to show that self-energy is divergence free. We have already shown how to iterate the vertex Γ and also how to get K and both are finite to any order.

3. This same skeleton argument can be used for the vertex in place of the integral equation argument as follows:

$$\begin{aligned} \text{Diagram of } \Gamma &= \text{Diagram 1} + \text{Diagram 2} + \text{etc.} \Rightarrow \Gamma(\lambda_0) = \lambda_0 + F(\Gamma, D) = Z_3^{-3/2} Z_5 \lambda + F(Z_3 \Gamma, Z_3 D) \\ \Rightarrow \Gamma(\lambda) &= Z_5 \lambda + F(\Gamma, D) \Rightarrow \lambda = Z_5 \lambda + F(\Gamma, D; s) \\ \hookrightarrow \Gamma(s, \lambda) &= \lambda \Rightarrow Z_5 = 1 - F(\Gamma, D; s) / \lambda \\ \Rightarrow \Gamma(\lambda) &= \lambda - F(s) + F = \lambda + \Delta F_{\overline{K}} \quad \text{for cancelled in sub!} \end{aligned}$$

$\xrightarrow{\text{only overall log}}$

Theorem: if you fail to account for overlap, the iterative renorm prog fails for the self-energy.

① Copying from earlier works we have:

$$\Sigma_1^{[n]}(p^2) = \delta m^2 + \frac{1}{2} \lambda^2 \int dk D^{[n]}(k) D^{[n]}(k') \tilde{\Pi}^{[n]}(k, k', p)$$

Assume that D and $\tilde{\Pi}$ have been computed thru n^{th} order and are log-free, i.e., are finite. Then, if you ignore the overlap subdivergences at the left vertex, you would say: The only divergences in the $\int dk$ integral here are the overall ones, so

$$\int dk D^{[n]}(k) D^{[n]}(k') \tilde{\Pi}^{[n]}(k, k', p) = A^{[n]} \Lambda^2 + (p^2 - m^2) [B^{[n]} \ln \Lambda + C^{[n]}(p^2)]$$

where $C^{[n]}(p^2)$ is finite. Thus you are saying:

$$\Sigma_1^{[n]}(m^2) = \delta m^2 + A^{[n]} \lambda^2 \Lambda^2 + \frac{1}{2} \lambda^2 \int dk D^{[n]}(k) D^{[n]}(k') \tilde{\Pi}^{[n]}(k, k', p) = 0, \text{ adjust } \delta m^2$$

and:

$$\Sigma_1^{[n]}(p^2) = \frac{1}{2} \lambda^2 \int dk D^{[n]}(k) D^{[n]}(k') \tilde{\Pi}^{[n]}(k, k', p) = \frac{1}{2} \lambda^2 \Lambda^2 [B^{[n]} \ln \Lambda + C^{[n]}(p^2)]$$

② But if this were true, you would then get

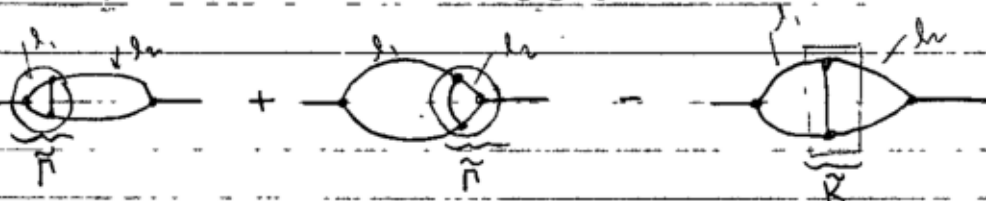
$$\Sigma_1^{[n]}(p^2) = \frac{1}{2} \lambda^2 \Lambda^2 \Delta C^{[n]}(p^2)$$

But this thing obviously contains logs via the Λ^2 + so result is not finite.

Thus I have at least shown the following: if you ignore the overlap subdivergences which occur at the left vertex, then the renormalized self energy must not be finite. What I would now like to show is that if you include the effect of these subdivergences, the renormed self-energy is in fact finite.

Discussion of finiteness of self energy to fourth order using BD technique.

1. First, here are the contributing graphs:



The circled vertices in these graphs are of course renormalized vertices, so not just Feynman graphs. I.e., they have been internally subtracted so only finite part remains. The kernel in the third graph is so simple that subtraction makes no difference. These graphs all come in the one-kernel classification category.

2. Lets look for possible divergent subintegrations which cannot be tamed by overall subtraction because these are the only things that can get us. The combination integral $l_1 l_2$ obviously can be overall subtraction-tamed, so we need only consider l_1 and l_2 separately. But symmetry then says we need only examine the l_1 integration.

3. The first of the above three terms obviously has no danger with regard to the l_1 , so we consider the other two terms. Basically what happens is this: the divergence at the left end cancels between these two terms leaving you OK. Write:

$$\begin{aligned} \int DD \tilde{\Gamma} + \int DD K DD &= \int DD DD + \int DD K DD - \int DD K DD \\ \downarrow & \quad (c) \\ [Z_5 + \int K DD] &= \int DD DD + \int DD K DD - \int DD K DD \\ (2a) \quad (2b) & \quad (2a) \quad (2b) \quad (c) \\ &= Z_5 \left[\text{Born loop} \right] + \text{bar graph} - \text{bar graph} \end{aligned}$$

Here you can see exactly what happens. The subtraction term inside term 2, called here (2a), is no problem because it is log times the Born loop and can thus be cleared by an overall subtraction. It is the other piece of the second term which contains the problem. This other piece is exactly the bar graph which does contain the undesired divergent subintegration. But this graph and thus its dangerous divergent subintegration, is exactly cancelled by the third term!

$$\text{bar graph} = \text{dangerous div subintegration}$$

4. Clarification: the term 2a which is Z_5 times Born loop is not just a constant that can be removed by overall subtraction. The point is that an overall subtraction would remove the log arising in the l_1 integration and that is all you need to show.

Simple discussion of how the BD method gives you a finite 4th order self-energy.

1. The instruction is to add three graphs as discussed on the last sheet:

$$\left[\frac{i}{2} \text{graph 1} + \frac{i}{2} \text{graph 2} - \frac{i}{2} \text{graph 3} \right] = \text{plus } \text{graph 4} \text{ which is no problem.}$$

$$\frac{i}{2} \{ Z_5 [-O] + -O \} + \frac{i}{2} \{ Z_5 [-O] + -O \} - \frac{i}{2} \{ -O \} = 2Z_5 [-O] + \frac{i}{2} \text{graph 4}$$

Because of the way that $\tilde{\Gamma}$ is defined, the terms in this equation are actually second-order in the coupling constant. As will be shown below, the second-order part of Z_5 is negative. The $2Z_5 \times \text{BornLoop}$ sitting there will exactly cancel the two divergent subintegrations on the two sides and you are then left with just the overall divergence which then is removed by your overall subtraction! In fact:

$$Z_5 = 1 - \frac{1}{2} \int dk \tilde{\Gamma} \tilde{D} \tilde{D} \tilde{K} \Rightarrow Z_5^{(2)} = -\frac{1}{2} \left[\text{graph 5} \right] = -\frac{1}{2} \lambda^2 [F \ln \Lambda + f(\text{sym})]$$

$$\frac{i}{2} \text{graph 4} = \frac{1}{2} \lambda^2 [2F \ln \Lambda [-O] + A_3 \tilde{\Lambda}^2 + (p^2 - m^2)(B_3 \ln \Lambda + C_3(p^2))]$$

Thus, $\Pi_{\text{unsubtracted}}^{\text{BD method}} = 2 \left(-\frac{1}{2} \lambda^2 [F \ln \Lambda + f(s)] \right) (-O) + \frac{1}{2} \lambda^2 [2F \ln \Lambda [-O]] + \text{etc.}$

$$\Rightarrow \Pi(p^2) - \Pi(m^2) = -\lambda^2 f(s) (p^2 - m^2) [B \ln \Lambda + C(p^2)] + \frac{1}{2} \lambda^2 (p^2 - m^2) (B_3 \ln \Lambda + C_3(p^2))$$

Well, I see how the divergent subintegrations cancel out in the result for the unsubtracted self-energy. But what is this object in my notation? I should go back and construct the self-energy equation in my own notation:

$$\Sigma^2(p^2) = \delta m^2 + \frac{\lambda^2}{4} Z_5 Z_3^{-1} \int dk \cdot D \cdot D \cdot \tilde{\Gamma} \quad / D, \tilde{\Gamma} \text{ are my renormalized objects}$$

$$\tilde{\Gamma} = Z_5 + \frac{1}{2} \int dk \cdot \tilde{\Gamma} D D K$$

$$\Rightarrow \Sigma^2(p^2) = \delta m^2 + Z_3^{-1} \lambda^2 \left[\underbrace{\frac{1}{2} \int dk \cdot \tilde{\Gamma} \cdot D \cdot D \cdot \tilde{\Gamma}}_{\text{const} + (p^2 - m^2) [\text{stuff}]} - \frac{1}{2} \int dk_1 \int dk_2 \cdot \tilde{\Gamma} D D K D D \tilde{\Gamma} \right]$$

$$\Rightarrow \Sigma^2(p^2) = Z_3^{-1} \cdot [\text{stuff}] = Z_3^{-1} H(p^2) \lambda^2 \quad H(p^2)$$

$$\Rightarrow \tilde{\Sigma}^2(p^2) = \Delta H(p^2) \lambda^2$$

So, $Z_3^0 \Sigma^2(p^2) = Z_3 \cdot \delta m^2 + \lambda^2 \left[\frac{1}{2} \int dk \tilde{\Gamma} D D \tilde{\Gamma} - \frac{1}{2} \int dk_1 \int dk_2 \tilde{\Gamma} D D K D D \tilde{\Gamma} \right]$

$$= \Pi_{\text{unsub.}}^{\text{BD.}}$$

2. So when you subtract this object you don't expect to get something finite. Its only after you factor off the $p^2 - m^2$ factor and then subtract that result should be finite. In BD this factor is the $q^u q^v$ tensor thing, and there is no constant to worry about because gauge invariance blocks mass renormalization. So here is a way I can write my thing:

$$\text{Let } [\pm \int dk \dots] = \mathcal{E}(p^2) = K + (p^2 - m^2) H(p^2).$$

$$\text{Then } \Delta \mathcal{E}(p^2) = (p^2 - m^2) H(p^2) \Rightarrow \frac{\Delta \mathcal{E}(p^2)}{(p^2 - m^2)} = H(p^2)$$

So here is the rule:

$$\Sigma^2(p^2) = [H(p^2) - H(m^2)] \cdot \lambda^2$$

$$\text{where } H(p^2) = \frac{\mathcal{E}(p^2) - \mathcal{E}(m^2)}{p^2 - m^2}$$

$$\text{where } \mathcal{E}(p^2) = \left[\frac{1}{2} \int dk \cdot \tilde{\Gamma} \cdot D D \tilde{\Gamma} - \frac{1}{4} \int dk_L dk_L \tilde{\Gamma} D D K D D \tilde{\Gamma} \right]$$

$$= \frac{1}{2} \text{---} \text{---} \text{---} - \frac{1}{4} \text{---} \text{---} \text{---}$$

$$= Z_5 \int dk D \cdot D \cdot \tilde{\Gamma} = Z_5 \cdot \left[\text{---} \text{---} \text{---} \right]$$

3. This is now very nice. I now understand that the expression $E(p^2) = E$ is devoid of those divergent subintegrations. Thus, you see by writing it in the other way with Z_5 that the divergences arising when the lines on the right of the asymmetric loop are slid to the left, these divergences must be exactly cancelled by the leading Z_5 factor. Thus, the quantity E contains only "overall" divergences. These overall divergences are of the quadratic and log variety. When you do your first subtraction to make $E(p^2) - E(m^2)$, the quadratic divergences go away because they only appear in constants. Then you divide through by the pole, and finally you resubtract. In this second subtraction the log divergences go away! The result is finite!!!!

Calculation of the Sixth Order Self Energy in ϕ^3 theory with $d=6$ via BD method.

1. The first task is to compute what I have called the expression $E(p^2)$. So:

$$K^{(2)} = \left(\text{Diagram 1} + \text{Diagram 2} \right)$$

$$\text{Let } K^{(2)} = \text{Diagram 1} \text{ only.}$$

This is probably then correct:

$$E(p^2) = \frac{1}{2} \left[\int dk \tilde{\Gamma} D D \tilde{\Gamma} - \iint dk_1 dk_2 \tilde{\Gamma} D D K' D D \tilde{\Gamma} \right]$$

Make a list of contributions: (hold $\frac{1}{2}$ out)

a) $2 \cdot \int \tilde{\Gamma}^{(4)} D^{(0)} D^{(0)}$



b) $\int \tilde{\Gamma}^{(2)} D^{(0)} D^{(0)} \tilde{\Gamma}^{(2)}$



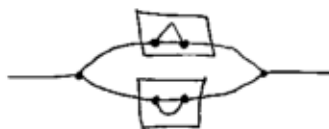
c) $4 \cdot \int \tilde{\Gamma}^{(2)} D^{(2)} D^{(0)} \tilde{\Gamma}^{(0)}$



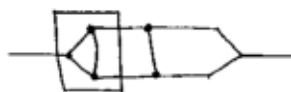
d) $2 \cdot \int D^{(4)} D^{(0)}$



e) $\int D^{(2)} D^{(2)}$



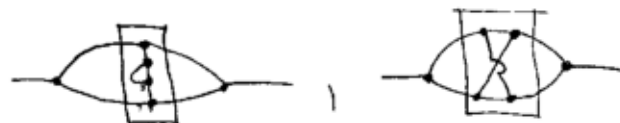
f) $-\iint \tilde{\Gamma}^{(2)} D^{(0)} K^{(2)} D^{(0)} D^{(0)} \times (2)$



g) $-\iint D^{(2)} D^{(0)} K^{(2)} D^{(0)} D^{(0)} \times (4)$



h) $-\iint D^{(0)} D^{(0)} K^{(4)} D^{(0)} D^{(0)}$



typical graph.

no, but all terms not of this form cancel!

2. Once you accept that E has no divergent subintegrations, you see why the calculation gives a finite result: namely, by direct power counting each integral gives a term of the usual form: $\text{const } L^2 + (p^2 - m^2)[B \log L + C(p^2)]$ since $D=2$. The constant goes away in the first subtraction, then the inside log goes away in the second subtraction, and you get a finite result. Remember that each insertion (with box around it in figure) contains no logs because it is a renormalized object. If a vertex insertion, then it is directly subtracted before it goes into the graph. If its a lower order self-energy insertion, the thing inserted is the finite result of the lower order calculation.

3. The fourth order vertex belongs either to the one or two kernel expansion terms. I think it is correct to completely cancel that overall factor of $\frac{1}{2}$ and change K to K' because a twist can always be undone to give a term already present. So, let's now construct the second and fourth order kernels first:

$K^{(2)} = \text{[diagram: two horizontal lines connected by a vertical line]}$

$K^{(4)} = 2 \cdot \text{[diagram: two horizontal lines with a box containing a vertical line and a horizontal line crossing it]} + \text{[diagram: two horizontal lines with a box containing a vertical line and a horizontal line crossing it, with a twist]} + \text{[diagram: two horizontal lines with a box containing a vertical line and a horizontal line crossing it, with a twist and a cross]} \quad \text{always should include } \text{[diagram: twisted box]} + \text{[diagram: crossed box]}$

$\tilde{\Gamma}_{(1)}^{(4)} = \text{[diagram: two horizontal lines with a box containing } K^{(4)} \text{]} = 2 \cdot \text{[diagram: two horizontal lines with a box containing a vertical line and a horizontal line crossing it]} + \text{[diagram: two horizontal lines with a box containing a vertical line and a horizontal line crossing it, with a twist]} + \text{[diagram: two horizontal lines with a box containing a vertical line and a horizontal line crossing it, with a twist and a cross]}$

$\tilde{\Gamma}_{(2)}^{(4)} = \text{[diagram: two horizontal lines with a box containing } K^{(4)} \text{ and } K^{(2)} \text{]} = \text{[diagram: two horizontal lines with a box containing a vertical line and a horizontal line crossing it, with a twist and a cross]}$

$\tilde{\Gamma}_{(1)}^{(2)} = \text{[diagram: two horizontal lines with a box containing } K^{(2)} \text{]} = \text{[diagram: two horizontal lines with a box containing a vertical line and a horizontal line crossing it]} \text{ only.}$

4. Now let's examine the overlapping divergence business and try to see why there are no divergent subintegrations floating around here. First, the basic instruction is to classify all 6th order contributions according to the number of kernels involved. To this order, that number can be 0, 1 or 2 and no more.

First let's deal with the single kernel contributions:

$\text{[diagram: two horizontal lines with a box containing a vertical line and a horizontal line crossing it]} + \text{[diagram: two horizontal lines with a box containing a vertical line and a horizontal line crossing it, with a twist]} = \text{typical single kernel graphs in first term.}$

$\equiv \int \tilde{\Gamma} DD \quad \left\{ \begin{array}{l} \downarrow \\ [2S + \int DD K'] \end{array} \right\} \Rightarrow 2S [-\text{[diagram: two horizontal lines with a box containing a vertical line and a horizontal line crossing it]}] + \int DD K' DD$

$\text{[diagram: two horizontal lines with a box containing a vertical line and a horizontal line crossing it]} + \text{[diagram: two horizontal lines with a box containing a vertical line and a horizontal line crossing it, with a twist]}$

This extra term we just got will then cancel against the single-kernel part of the second term, leaving us with this overall result in the single-kernel sector: Z_5 times a dressed Born Loop. But this object obviously has no overlap problems and becomes finite after the instructions: subtract, divide by pole, subtract again. The overlap problematic part of the first term completely cancelled with the second term, just as happened in the simpler 4th order case.

So much for the single-kernel contributions. The zero kernel ones are at once no problem because they are already in the dressed Born loop form, so to speak.

The real test is to deal with the 2-kernel contributions.

Theorem: The divergence arising from a subintegration cannot be removed by an overall subtraction on your Feynman graph unless the subintegration ties to all the external lines.

The reason for this is very simple and can be illustrated with a few examples. Basically the idea is this: when you speak of an overall subtraction on the Feynman graph, you are referring to subtracting a divergent constant. I.e., in an overall subtraction you can only remove something which is independent of the external momenta. The characteristic of a divergent subintegration which is detached from some of the external momenta is that its contribution to the overall graph is dependent on the momenta which were not touched.

For example, consider this graph:

The diagram shows an equation: a circle with five external lines labeled 1, 2, 3, 4, and 5. A small shaded region on the circle represents a subintegration. This is equal to a divergent constant, written as $(div\ constant)$, multiplied by a bracketed diagram. The bracketed diagram is the same circle with five external lines, but the subintegration region is replaced by a single dot, representing a reduced graph.

Here I have picked out a certain subintegration which we shall presume is divergent. Let's take most optimistic case, i.e., that the ~~case~~ divergence of this subintegration is just a constant (as has often been the case, maybe is always the case). Obviously this divergent constant is going to get multiplied by the "reduced" graph shown in the bracket and clearly this thing is going to depend on those momenta! Thus, such a divergent term could not be removed by an overall subtraction.

Here is another example:

The diagram shows an equation: a circle with five external lines labeled 1, 2, 3, 4, and 5. A small shaded region on the circle represents a subintegration. This is equal to a divergent constant, written as $(div\ const)$, multiplied by a bracketed diagram. The bracketed diagram shows the same circle with five external lines, but the subintegration region is replaced by a dot. A line from external momentum 5 passes through this dot, and the entire bracketed diagram is labeled $f(p_5)$ below it, indicating it is a function of momentum 5.

If your selected divergent subgraph misses even one external momentum, as in this example, the function multiplying the constant will be a function of that momentum and thus cannot be removed by an overall constant.

On the other hand, if your subgraph touches all the external momenta, the "reduced" graph in that case is just a dot with all lines touching, which is to say it is a constant; such a divergence then could be removed by overall subtraction.

Relevance of this theorem:

In the proofs of finiteness of various objects in BD, you often are dealing with objects having $D=0$ which can be reduced to $D=-1$ by an overall subtraction. The proof always requires you to show that there are no subintegrations which diverge which fail to make contact with all the external particles. If there were a $D=0$ subintegration of this type, you would be in trouble because it would be a log that could not be removed by an overall subtraction, and your proof that PI , eg, was finite, would fail.

See footnote on page 329 for confirmation of this theorem.

Proof that the self-energy is finite.

1. First I considered the fourth order self-energy and then started considering the sixth order, but now this does not seem the right way to go. The significant quantity is not the "order" but rather the "kernel number" of the self-energy. The idea is to prove finiteness for each kernel-number sector, regardless of the order. Of course any order, such as the 6th order, can be decomposed into sum of 0, 1 and 2 kernel contributions. I have already verified the finiteness of the 0 and 1st kernel number sectors. Here I will show how it works for 2-kernels. The arguments here could of course be specialized to that portion of the sixth order self-energy, but no need to do so.

2. Lets stick with the notation of BD, except I plan to change a few signs. The notation I am referring to is on page 337 of BD and refers to the finite part of integrals. The net result is an iteration for the finite functions of the vertex by "kernel-number" rather than by specific order. With my choice of signs, here is what you have:

$$\int \Lambda_i DDK = L_{(i+1)} + \Lambda_{(i+1)}(k, k', p)$$

$$\tilde{\Gamma}_{(n)} = 1 + \sum_{i=1}^n \Lambda_i$$

$$\Sigma_5^{(n)} = 1 - \sum_{i=1}^n L_i$$

The first line just says: due to power counting, L_{i+1} will be the constant divergent term in the integral, quadratic plus log; and Λ will be a function that vanishes at the symmetric point, due to norm condition as shown below. Thru this iteration we generate an expression for Σ_5 and for the finite vertex to order n in kernel count. If you are interested in a certain finite order of perturbation theory, then you will only have to go up to a finite number of kernels, so no problem.

3. Lets derive the above equations to make sure we know what we are doing:

$$\tilde{\Gamma} = \Sigma_5 + \int \tilde{\Gamma} DDK \quad \text{/standard vertex Dyson.}$$

$$\Rightarrow \tilde{\Gamma}_{(1)} = \Sigma_{5(1)} + \underbrace{\int \Lambda_1 DDK}_{L_1 + \Lambda_1} = (\Sigma_5^{(1)} + L_1) + \Lambda_1(p) = \tilde{\Gamma}^{(1)}(p)$$

If $\Lambda_1(\text{sym}) \neq 0$, then write $L_1 + \Lambda_1(p) - \Lambda_1(s_y) + \Lambda_1(s_y) = L_1' + \Lambda_1'(p) \checkmark$
So assume L_1, Λ_1 defined $\exists \Lambda_1(\text{sym}) = 0$. Then

$$\tilde{\Gamma}_{(1)}(\text{sym}) = 1 \Rightarrow \Sigma_5^{(1)} = 1 - L_1$$

Now you see exactly when the n-kernel contribution to the vertex (renormed) is:

$$\tilde{\Gamma}^{[3]} = \tilde{\Gamma}^{(1)} + \tilde{\Gamma}^{(2)} + \tilde{\Gamma}^{(3)}$$

$$\tilde{\Gamma}^{(3)} = \text{diagram} = \Lambda_3.$$

You can now see precisely the operational rule for computing this finite object. Start at the innermost nested level, do the innermost integration and throw away the divergent part, keep the finite part called Λ_1 (throw out L_1). Then move up to the next nest and do the next integral. Throw out L_2 but keep the finite part Λ_2 . Finally, go to the outermost level. Throw out L_3 but keep the finite part. Thus, I know exactly how to compute the n-kernel finite vertex function as a certain operation on the kernel. Of course to know the answer, I have to know the kernel, and the propagator, since that is in there also. I.e., I have to have an expression for the finite kernel renormed and finite propagator renormed.

4. Now let's return to our task of examining the "expression E" to see if there are any divergent subintegrations. The claim is that there are not any.

$$\begin{aligned} E(p^2) &= \Lambda_2 \overset{L_3}{DD} + \overset{L_1}{DD} \Lambda_2 + \Lambda_1 \overset{L_2}{DD} \Lambda_1 \\ &\quad - \Lambda_1 \overset{L_2}{DD} K \overset{L_3}{DD} - \overset{L_1}{DD} K \overset{L_2}{DD} \Lambda_1 \end{aligned}$$

These are all the 2-kernel terms.

L_1 integration. Throw out terms with Λ_2 or Λ_1 on left since these are OK.

Then:

$$\text{exp} = \overset{L_1}{DD} \Lambda_2 - \overset{L_1}{DD} K \overset{L_2}{DD} \Lambda_1$$

But $\Lambda_2 = -L_2 + \text{diagram} KDD\Lambda_1$ so

$$\text{exp} = -L_2 \cdot DD + \text{diagram} KDD\Lambda_1 - \text{diagram} KDD\Lambda_1 = -L_2 \cdot DD = -L_2 \left[\text{diagram} \right].$$

Here we are examining the L_1 subintegration to see if there are any dangerous divergences which cannot be tamed by the routine process of subtractions. The discussion of this result in BD appears on page 340 below equation (19.79). Our result here is L_2

times a Born loop deal which contains the l_1 integration. Our question is: is this a worrisome divergent subintegration? The answer is no: it starts out as $D=2$, but lowers to $D=0$ when we factor out the $(p^2 - m^2)$ factor, and finally drops to $D=-1$ when we subtract. The fact that L_2 is a divergent constant does not matter because L_2 arose from integrating over those other variables l_2 and l_3 .

So as far as the l_1 integration is concerned, it does not represent a dangerous divergent subintegration in E . You can see, however, that in those two terms which cancelled, in each of those terms l_1 was a bad subgraph causing a divergence. But they cancelled. This is what usually happens, there is a cancellation between the two major terms of E .

Analysis of the l_3 integration is similar by symmetry so forget it.

l_2 integration. This is different and so must be considered:

$$E(p^e) \approx \Lambda_1 \overset{l_2}{DD} \Lambda_1 - \Lambda_1 \overset{l_2}{DD} KDD - DD KDD \Lambda_1$$

But let $\Lambda_1 = -L_1 + DDK$ to get

$$Exp = \left(-L_1 \overset{l_2}{DD} \Lambda_1 + DD \cancel{KDD} \Lambda_1 \right) - \Lambda_1 DD KDD - DD KDD \Lambda_1$$

$$= -L_1 \overset{l_2}{DD} (-L_1 + KDD) - (-L_1 + DDK) DD KDD$$

$$= +L_1^2 \overset{l_2}{DD} - L_1 \cancel{DD} KDD + L_1 DD \cancel{KDD} - DD KDD KDD$$

$$= L_1^2 \cdot DD - DD KDD KDD = L_1^2 \left[\text{diagram 1} \right] - \left[\text{diagram 2} \right]$$

Consider the first term. The divergent constant L_1^2 comes from the l_1 and l_3 int's and thus does not count. The born loop object then has $D=-1$ and is thus OK. As for the second term, the l_2 loop there has $6-2-2-2-2 = -2$, just as for K itself, and is thus no problem. Thus, there ~~is~~ is no problem associated with the l_2 .

$l_1 l_2$ subintegration. Here we go once again:

$$exp = DD \overset{l_2}{\Lambda_2} + \Lambda_1 DD \Lambda_1 - \Lambda_1 DD KDD - DD KDD \Lambda_1$$

$$= -L_2 \cdot DD + \underbrace{\Lambda_1 DD \Lambda_1 - \Lambda_1 DD KDD}_{\Lambda_1 DD (\Lambda_1 - KDD)} = -L_2 \cdot DD - (L_1 DD) L_1$$

$$= -\overset{l_1, l_2}{DD} L_2 - \overset{l_1, l_2}{\Lambda_1 DD} L_1 \overset{l_3}{\leftarrow} = -\text{diagram 3} - \text{diagram 4}$$

OK. OK

As discussed on the top of page 341, each of these terms is no problem. The second term is really an overall subtraction type term, you can ignore L_1 since it comes from l_3 . The first term is trickier since part of the integration is in the constant.

Obviously, I could reformulate these ideas in a clearer way, because in a subintegration search you are really supposed to hold certain variables fixed. This means you really should go through and undo parts of integrations etc to see what the result is. Such an analysis would then yield the naive counting rules we have been using here.

So I can then consider the $l_1 l_3$ integration and get a similar finite result. Potential problems always cancel between the two main terms!

A summary of the renormalization program of ϕ^3 theory.

1. The claim is that you can compute any of the basic renormalized objects to any finite order and you will get something that is finite, if you follow through with the simple "multiplicative" renorm program. Lets review how this works. First, there are certain important integral equations which describe the theory. They are:

A) vertex equation: $\tilde{\Gamma}^{[n+2]} - 1 = \left[\text{diagram} \right] - \underbrace{\left[\text{sym} \right]}_{1 - \Sigma_5^{[n+2]}}$

effect is: $\{ \tilde{\Gamma}^{[n]}, D^{[n]}, K^{[n+2]} \} \rightarrow \{ \tilde{\Gamma}^{[n+2]}, \Sigma_5^{[n+2]} \}$

B) self energy dyson:

$$\Sigma_2^{[n+2]}(p^2) = \lambda^2 \{ H^{[n]}(p^2) - H^{[n]}(m^2) \}$$

where $H(p^2)^{[n]} = \frac{E(p^2)^{[n]} - E(m^2)^{[n]}}{p^2 - m^2}$

where $E(p^2)^{[n]} = \frac{1}{i} \left[\text{diagram} \right] - \frac{1}{4} \left[\text{diagram} \right]$

Also,

$$\Delta_3^{[n+2]} = 1 + \lambda^2 H(m^2)^{[n]}$$

$$\delta m^2^{[n+2]} = -\lambda^2 E(m^2)^{[n]} (\Delta_3^{-1})^{[n]}$$

} derived below.

effect is: $\{ \tilde{\Gamma}^{[n]}, D^{[n]}, K^{[n]} \} \rightarrow \{ E^{[n]}, H^{[n]} \} \rightarrow \{ D^{[n+2]}, \Delta_3^{[n+2]}, \delta m^2^{[n+2]} \}$

C) skeleton expansion:

$$\text{diagram} = \lambda^2 \left[\text{diagram} \right] + \lambda^4 \left[\text{diagram} \right] + \dots$$

effect: $\{ \tilde{\Gamma}^{[n]}, D^{[n]} \} \rightarrow K^{[n+2]}, G_N^{[n+(N-2)]}$

N-point greens $N = 4, 5, 6, \dots \in \mathbb{N}$

2. Here then is the exact algorithm for computing this theory to any order:

0] start with! $\tilde{\Gamma}^{(0)} = 1$ $K'^{(2)} = \overline{I} = \lambda^2 \frac{1}{(q^2 - m^2)}$
 $D^{(0)} = (p^2 - m^2)^{-1}$
 $K^{(0)} = 0$ $\Pi^{(0)} = \lambda.$

- 1] use A to get $\tilde{\Gamma}^{[2]}, \Sigma_5^{[2]}$
 - 2] use B to get $D^{[2]}, \Sigma_3^{[2]}, \delta m^2^{[2]}$ $\rightarrow$ everything now known to second order.
 - 3] use C to get $K^{[4]}$
 - 4] use A to get $\tilde{\Gamma}^{[4]}, \Sigma_5^{[4]}$
 - 5] use B to get $D^{[4]}, \Sigma_3^{[4]}, \delta m^2^{[4]}$ $\rightarrow$ all now known to 4th order.
 - 6] use C to get $K^{[6]}$
- etc!!

Derivation of the Z_3 and δm^2 iterations.

① First for Z_3 :

a) $Z_3^{-1} = 1 - \sum_0^{\infty} \lambda^2 (m^2)$ / basic def of Z_3 .

b) $Z_3 \circ \sum^{\infty} (p^2) = \lambda^2 H(p^2)$ / in my 4th ad sheet.

Then $Z_3 [1 - Z_3^{-1}] = \lambda^2 H(m^2) = Z_3 - 1$

$$\Rightarrow \boxed{Z_3^{[n+2]} = 1 + \lambda^2 H(m^2)^{[n]}}$$

② Now for mass:

a) $\sum^{\infty} (p^2) = \delta m^2 + Z_3^{-1} \lambda^2 \mathcal{E}(p^2)$

Then set $p^2 = m^2$: $0 = \delta m^2 + Z_3^{-1} \lambda^2 \mathcal{E}(m^2)$

$$\Rightarrow \boxed{\delta m^2^{[n+2]} = -\lambda^2 (Z_3^{-1})^{[n]} \mathcal{E}(m^2)^{[n]}}$$

Execute the Algorithm thru lowest few orders.

1] $n=0$ so $\left[\frac{1}{2} \text{---} \text{---} \text{---} + \frac{1}{2} \text{---} \text{---} \text{---} \right]$

$$= \frac{1}{2} \lambda^2 \left[F_{lnl} + f(q, p, p') + F_{lnl} + f(q, p', p) \right]$$

$$= \lambda^2 [F_{lnl} + f^+(q, p, p')]$$

where $f^+ \equiv \frac{1}{2} [f(p, p') + f(p', p)]$. Then:

$$1 - \Sigma_5^{[2]} = \lambda^2 [F_{lnl} + f^+(\text{sym})] \Rightarrow \boxed{\Sigma_5^{[2]} = 1 - \lambda^2 [F_{lnl} + f^+(\text{sym})]} \quad \checkmark \text{ agree}$$

Then

$$\tilde{\Gamma}^{[2]} - 1 = \lambda^2 [F_{lnl} + f^+] - \text{sym} = \lambda^2 \cdot \Delta f^+$$

$$\Rightarrow \boxed{\tilde{\Gamma}^{[2]} = 1 + \lambda^2 \cdot \Delta f^+(p_i)}$$

2] $E^{(0)}(p^2) = \frac{1}{2} \text{---} \text{---} \text{---} = \frac{1}{2} [A\Lambda^2 + (p^2 - m^2) [B_{lnl} + C(p^2)]]$

$$\Rightarrow H^{(0)}(p^2) = \frac{1}{2} [B_{lnl} + C(p^2)]$$

$$\Rightarrow \boxed{\tilde{\Sigma}_1^{[2]}(p^2) = \lambda^2 \cdot \frac{1}{2} \cdot \Delta C(p^2)} \quad \checkmark$$

Also,

$$\boxed{\Sigma_3^{[2]} = 1 + \lambda^2 \cdot \frac{1}{2} [B_{lnl} + C(m^2)]} \quad \checkmark \text{ agree}$$

Also,

$$\boxed{\Sigma_{m^2}^{[2]} = -\lambda^2 \frac{1}{2} A\Lambda^2 \cdot 1} \quad \checkmark$$

3] $K^{[4]} = \lambda^4 \left(\text{---} \text{---} \text{---} \right) + \lambda^4 \left(\text{---} \text{---} \text{---} \right) + \lambda^4 \left(\text{---} \text{---} \text{---} \right) + \lambda^4 \left(\text{---} \text{---} \text{---} \right)$

$$= \lambda^4 \times [\text{finite}]$$

4] $\tilde{\Gamma}^{[4]} = \left[\frac{1}{2} \text{---} \text{---} \text{---} + \frac{1}{2} \text{---} \text{---} \text{---} + \frac{1}{2} \text{---} \text{---} \text{---} \right] - \text{sym etc}$

Things seem to agree!

Working paper on: a problem on the counterterm interpretation of renormalization.

1. I know that the propagator D can be rendered finite by simply multiplying it by the proper constant. I also know, according to VEVTOP idea, that this same operation can be caused by a simple rescaling of the field. But I also know that a rescaling of the field in any lagrangian (including couplings) can be accomplished by adding the right counterterms to the lagrangian. Therefore, one concludes that the renormalization of the propagator can be "impletmented" by adding counterterms to the lagrangina. Moreover, this argument applies to any n -point Greens Function, and I think I can extend it to any IPI proper function too.

2. My problem is this: to show in low order perturbation theory exactly how this works for the propagator. This means I have to show something about the self-energy. Here is where I am confused: I dont know quite what I want to show. This is where I take up the problem.

3. Comment: consider the relation $D(\lambda) = Z_3^{-1}(\lambda) D^0(\lambda_0)$. Suppose you are interested in the "second-order" part of D , ie, the part which is order λ^2 . There are two contributions to this quantity: first, the 2-vertex graph in D^0 contributes, but also the zero-vertex graph contributes when you include the order λ^2 term in Z_3 . Similarly, the renormed second-order self energy gets a contribution from the obvious 2-vertex graph, but it also gets a contribution via Z from the zero-vertex graph.

Here is another way to say it I think: the residue of the renormalized, bare propagator is not 1 but is Z_3^{-1} which is a polynomial in λ . Thus, the "bare propagator" ie the zero-vertex "graph" actually con tributes in all orders!!!! Of course the residue of the dressed, renormed propagator is 1.

Thus, if you are computing the "second-order self energy", you should include contributions from the 2-vertex graph and the 0-vertex graph. In general, n -th order self-energy will show contributions from all lower graphs because of the Z 's!!!!

Hypothesis: how to do "renormalized feynman graphs"

1. We know that the various Greens Functions can be rendered finite by rescaling the fields and by also redefining the coupling constant. Ie, replace ϕ^0 by $Z_3^{1/2} \phi$ then define your greens functions as VEVTOP's in the new fields: they will be finite when expressed in terms of λ , so you also get rid of λ_0 by replacing it with $Z_5 Z_3^{-3/2} \lambda$. Thus:

$$D = \langle \phi \phi \rangle = D(\lambda) = Z_3^{-1}(\lambda) D_0(\lambda_0 = Z_5 Z_3^{-3/2} \lambda).$$

2. Suppose we make exactly these replacements starting from the bare lagrangian:

$$\begin{aligned} \mathcal{L} &= \frac{1}{2} (\partial_\mu \phi \partial^\mu \phi - m^2 \phi^2) + \frac{\lambda_0}{3!} \phi^3 + \frac{1}{2} \delta m^2 \phi^2 \\ &= \frac{1}{2} Z_3 (\partial_\mu \phi \partial^\mu \phi - m^2 \phi^2) + \frac{(Z_5 \lambda)}{3!} \phi^3 + \frac{1}{2} \delta m^2 Z_3 \phi^2 \end{aligned}$$

Here I have simply rewritten the lagrangian, making the above substitutions. Now lets read off the feynman rules of this new lagrangian:

$$\begin{aligned} \text{---} &= \frac{Z_3^{-1}}{p^2 - m^2} = \text{bare propagator.} \\ \text{Y} &= (Z_5 \lambda) = \text{bare vertex.} \end{aligned}$$

Of course I know that the dressed propagator is finite, as is the dressed vertex. Lets compute these things in lowest few orders and see if you really do get finite results:

zeroth order :

$$\begin{aligned} \text{---} \otimes \text{---} &= \frac{1}{p^2 - m^2} = \text{finite} \\ \text{---} \otimes &= \lambda = \text{finite} \end{aligned} \quad \left. \begin{array}{l} \\ \end{array} \right\} \checkmark \quad \text{---} \otimes \otimes \text{---} = 0.$$

second order :

$$\begin{aligned} \text{---} \otimes \otimes \text{---} &= [\text{---} \otimes \text{---}]^{(2)} + \text{---} \otimes \text{---} + [\text{---} \otimes \text{---}]^{(2)} \\ &= Z_3^{-1} (p^2 - m^2)^{(2)} + (Z_5 \lambda)^2 Z_3^{-2} \frac{1}{2} [A \Lambda^2 + (p^2 - m^2)(B \ln \Lambda + C(p^2)) + \delta m^2] \\ &= -\frac{1}{2} \lambda^2 [B \ln \Lambda + C(p^2)] (p^2 - m^2) + \lambda^2 \frac{1}{2} [A \Lambda^2 + (p^2 - m^2)(B \ln \Lambda + C)] - \frac{1}{2} \lambda^2 \Lambda^2 \\ &= \frac{1}{2} \lambda^2 \Delta C(p^2) \cdot (p^2 - m^2) \\ &= \Sigma^2(p^2)^{(2)} \end{aligned}$$

We get the right answer! Notice that it is crucial to pick up the second-order term from the "bare" propagator graph, since it has residue Z_3^{-1} . The singularities of this graph cancel those arising from the loop integration in the Born Loop graph. Why is this the renormalized self-energy? Because that is supposed to be everything except the term $1/(p^2 - m^2)$. In fact, I think I can argue that this would be a method to calculate Z_3 second order: you put in the log to cancel the other log, and you put in the $C(m^2)$ to get unit residue on the renormed D. Then Z_3 must be as I have it here. If you went to 4th order, you would adjust the fourth order part of Z_3 so that it cancelled all logs created by the higher graphs, and so residue is reset to 1 in that order too. In fact I'm pretty sure this will work exactly. Lets look next at the vertex:

$$\begin{aligned}
 \text{---} \bigcirc \text{---}^{(2)} &= \text{---} \bigcirc \text{---}^{(4)} + \text{---} \bigcirc \text{---}^{(2)} \cdot Z_3^{-3} \cdot [F \ln \Lambda + f^+(p_i)] \\
 &= -\lambda^3 [F \ln \Lambda + f^+(s_m)] + \lambda^3 [F \ln \Lambda + f^+(p_i)] \\
 &= \lambda^3 \cdot \Delta f^+(p_i)
 \end{aligned}$$

Again we get the correct answer: in fact, we could use this to compute Z_5 . The requirements on Z_5 are that you cancel the log and that the third-order term vanishes at the sym point.

So, by playing back and forth between these two equations you should be able to compute the finite vertex or self energy to any order you want, and now it is a direct graphical expansion! I have played with the fourth-order self energy and it seems to work. You then have to adjust $Z_3^{(4)}$ so that $Z^*(m^2) = 0$, ie, unit residue. This time all the fancy double logs cancel, etc.

Certainly this is now the way to prove finiteness etc, but its a reasonable way to interpret things .

3. So, if the above is correct, here is what I have shown:

- (a) if you rescale your fields and charge and think in terms of the new fields, then the theory computed to any order is finite, all logs cancel.
- (b) Another way to say it is this: start with a bare lagrangian but write in in with ϕ . Observe that propagator computed from this lagrangian has logs. Now add counterterms (kinetic, charge and mass) to get the seconline lagrangian on previous page, still ϕ . By thus changing the lagrangian, the new propagator is finite! Thus, by adding counter terms of the form of those already present, you rendered it finite. In this view, you dont speak of any field ϕ_0 . So this is the famous "counter-term implementation".

4. More on counterterms. Notice that we have three counterterms to play with. Basically, we adjust the mass counterterm to get pole to be at chosen m^2 . We choose the Z_3 kinetic counterterm to get the pole to have unit residue. We choose the Z_5 counterterm to get the charge to be the chosen value λ . The freedom to play with the three independent counterterms is the freedom to set mass, residue a and charge to whatever you please.

Of course you should realize that, eg, the 4th order Z_5 will depend on the second-order Z_5 and Z_3 and maybe mass counterterm also. The structure interweaves all lower constnats into the higher order ones, as you would expect.

ϕ^4 in $d=4$

$$\bigcirc \quad D=2$$

$$\bigcirc \quad D=4$$

done after ϕ^3 solved.

4.22.99

Analysis of ϕ^4 theory in $d=4$.

This theory is almost the same as ϕ^6 in $d=6$ which I have already done. But let's go thru each step to make sure nothing is hiding.

1. Propagator: Defined in same way. The only difference between this and ϕ^3 is the appearance of the self energy graph: the functional form is just the same. Thus:

$$\Sigma^2(m^2) = 0 \Rightarrow m = m(\lambda_0, m_0, \Lambda)$$

RC#1 $\rightarrow \boxed{\tilde{\Sigma}^2(\mu^2) = 0} \Rightarrow \Sigma_3^{-1} = 1 - \tilde{\Sigma}^2(p^2 = \mu^2; \lambda_0, m, \Lambda)$

$$\tilde{\Sigma}^2(p^2) = \Sigma_3 \cdot [\tilde{\Sigma}^2(p^2) - \tilde{\Sigma}^2(\mu^2)]$$

$$\begin{aligned} \circ \Sigma^2(p^2) &= \delta m^2 + \underbrace{\frac{\lambda_0}{\Lambda^2}}_{\text{const}} + \frac{1}{3!} \underbrace{\text{loop}}_{\text{const} + (p^2 - m^2) [\ln \Lambda + c(p^2)]} \\ &= \text{const} \end{aligned}$$

Notice, by the way, that whether or not we start with the bare mass m_0 , we always include a mass counterterm. If you are going to allow rescaling D (ie, allow counterterms) then you might as well allow a mass counterterm no matter what. The leaf graph is a constant so can be ignored. The other graph has same form as ϕ^3 loop in $d=6$.

2. Vertex: Z_5 is defined in the same way, but there will be a difference in the charge relationship:

$$\boxed{\Sigma_5^{-1} \equiv \frac{\circ \Gamma(\mu^2)}{\lambda_0}}$$

$$\boxed{\circ \Gamma = \Sigma_3^{-2} \Gamma}$$

RC#2 $\rightarrow \boxed{\tilde{\Gamma}(\mu^2) = 1} \Rightarrow \Gamma_5 / \lambda = 1 \Rightarrow \lambda = \Gamma_5 = \Sigma_3^{-2} \circ \Gamma(\mu^2) = \Sigma_3^{-2} \lambda \Sigma_5^{-1}$

$$\Rightarrow \boxed{\lambda = \Sigma_3^2 \Sigma_5^{-1} \lambda_0}$$

$$\Gamma^p = X + \underbrace{[X + \text{sym}]}_{\lambda_0^2 [F \ln \Lambda + f(p; \dots)]}$$

Same form as earlier.

By the way, odd n -point functions do not exist in this theory, easy to show.

3. Kernel: rule out 1 particle horizontal; two ruled out since cant exist; also rule out three! So:

Comment on How Charge Change is Determined

- ① The grouping business does not really do it:
The group is an identity

$$\phi^3: \quad \Gamma_0 D_0^{3/2} = \Gamma \cdot D^{3/2} \quad / \text{tells you nothing}$$

But, consider $\Sigma_5^{-1} \equiv \Gamma(\mu^1 \dots) / \lambda_0 \leftarrow \text{def } Z_5.$
 $1 \equiv \Gamma(\mu^1 \dots) / \lambda \leftarrow \text{renorm cond.}$

$$\Rightarrow \frac{\Sigma_5^{-1}}{1} = \frac{\Gamma}{\Gamma} \cdot \frac{\lambda}{\lambda_0} \Rightarrow \Sigma_5^{-1} = \Sigma_3^{-3/2} \cdot \frac{\lambda}{\lambda_0}$$

$$\Rightarrow \lambda_0 = \Sigma_3^{-3/2} \Sigma_5 \lambda$$

- ② Now go to ϕ^4 . Here, $\Gamma = \Sigma_3^{-2} \Gamma$, so Σ_5
defined in same way, now sum etc.

$$\Rightarrow \Sigma_5^{-1} = \Sigma_3^{-2} \cdot \frac{\lambda}{\lambda_0} \Rightarrow \boxed{\lambda_0 = \Sigma_3^{-2} \Sigma_5 \lambda} \quad \leftarrow \phi^4.$$

TABLE VII (continued)

Depth = 3200 hg/cm²

Theta, in the center of the bin, ranges from 46.25° to 66.75° in 2½ degree bins.

Multiplicity = 1

Intensity	.231-03	.247-03	.264-03	.266-03	.288-03	.308-03	.320-03	.332-03	.353-03
Error	.106-04	.971-05	.875-05	.677-05	.597-05	.521-05	.491-05	.766-05	.129-04

Multiplicity = 2

Intensity	.307-05	.421-05	.433-05	.480-05	.461-05	.437-05	.422-05	.418-05	.313-05
Error	.895-06	.642-06	.527-06	.571-06	.330-06	.390-06	.234-06	.467-06	.746-06

Multiplicity = 3

Intensity	.216-06	.460-06	.544-06	.632-06	.441-06	.514-06	.423-06	.329-06	.422-06
Error	.142-06	.156-06	.116-06	.126-06	.101-06	.881-07	.753-07	.796-07	.125-06

Multiplicity = 4

Intensity	.133-06	.163-06	.227-06	.242-06	.201-06	.168-06	.235-06	.245-06	.187-06
Error	.730-07	.881-07	.645-07	.753-07	.633-07	.466-07	.417-07	.685-07	.107-06

Multiplicity 5	Average over all angles = $0.5 \times 10^{-7} \pm .1 \times 10^{-7}$
----------------	--

Multiplicity 6	Average over all angles = $0.4 \times 10^{-7} \pm .2 \times 10^{-7}$
----------------	--

Multiplicity 7	Average over all angles = $0.3 \times 10^{-7} \pm .2 \times 10^{-7}$
----------------	--

Average angle = 57.5°

UV behavior of double integrals.

UV

① Consider: $I = \int d^4k d^4k' f(k, k')$ / scalar.

Do angular integrals to get

$$I = \int_0^\infty k^3 dk \int_0^\infty k'^3 dk' F(k, k'). = \int_0^\infty dk \int_0^\infty dk' k^3 k'^3 F(k, k').$$

Now treat like $(k, k') = (x, y)$, so

$$= \int_0^1 r dr \int_0^{\pi/2} d\theta (r \cos \theta)^3 (r \sin \theta)^3 F(r \cos \theta, r \sin \theta)$$

Finally, do angle integral so

$$= \int_0^1 r^4 dr \tilde{F}(r)$$

② Example: $f(k, k') = \frac{1}{k^2 - m^2} \frac{1}{k'^2 - m^2} \frac{1}{(p - k - k')^2 - m^2}$

If we care about UV, expand as
then expand in small p :

$$\frac{1}{k^2} \frac{1}{k'^2} \frac{1}{(p - k - k')^2}$$

$$= \frac{1}{k^2} \frac{1}{k'^2} \frac{1}{(k+k')^2} \left[\frac{1}{(k+k')^2} + \frac{2p \cdot (k+k')}{(k+k')^4} + \mathcal{O}(p^2) \right]$$

also!

Here you see that the Λ term is an odd integral, i.e.,

$$\int d^4k d^4k' \frac{[(k^\mu) + (k')^\mu]}{k^2 k'^2 (k+k')^4}$$

In each subintegral vanishes as $k^\mu \rightarrow -k^\mu$. So $\nexists$ no Λ term!

History only

old.
paradox

Departure Point: Sunday March 11, 1979

I have been trying to understand exactly how you renormalize a theory. I have been working with the ϕ^4 theory because it is the easiest, one field, one coupling, no derivatives, etc.

My problem is as follows: when I compute the renormalized self-energy in this theory, it still seems to depend on the cutoff. The renorm'd self energy is given by the difference of the unrenorm'd objects, divided by something. Even if the log cancels in the numerator, when you divide you are stuck with its presence.

For example, you are supposed to be able to compute the propagator off shell to any finite order in the coupling and you are supposed to get a finite, ie cutoff independent answer. But, suppose I were to compute to say 4th order. The denominator is 1 + second order, and this second order interacts with the second order in the numerator to yield a fourth order term that is linear in the log of cutoff. Thus, you will be stuck with a log cutoff in your propagator.

I don't at the moment see any way around this thing. I want a propagator that is finite to fourth order even away from mass shell. *mu!*

In QED the Ward identity causes my renorm condition to be $Z_3 = 1$ so that you do get just a subtracted photon self energy. But then $PI(0) = 0$ anyway. This is a special case, I would like to understand how this works in ϕ^4 without any Ward identities to add confusion.

In the ϕ^4 theory if you compute the unrenorm'd vertex to some order you will get, say, $Z_1 e$ where e is lambda on this typewriter. The renorm condition will then be that EX:

$$Z^{-2} Z_1 e = e$$

The other condition is that no mass renorm. Thus, two equations in dm^2 and de . Since there is no Ward identity, I see no way to get a relation between Z and Z_1 . Ie, Z_1 is related to Feynman graphs which compose the vertex, whereas Z is related to Feynman graphs which compose the self-energy.

What I have not made use of in ϕ^4 theory are the integral equations. I don't even know what they are!

Footnote : other graphs :

①  $= \int \frac{d^4 k}{k^2 - m^2} = \bar{A} \cdot \Lambda^2 + \text{no finite part?}$

$$I = m^2 \Phi\left(\frac{\Lambda^2}{m^2}\right) = \left[A \cdot \left(\frac{\Lambda^2}{m^2}\right) + B \cdot \ln\left(\frac{\Lambda^2}{m^2}\right) + C\right] \cdot m^2$$

where A, B, C are all constants, i.e., indep of p^2, m^2 .

$$Q = [A \cdot \Lambda^2 + (B \cdot \ln \frac{\Lambda^2}{m^2} + C) m^2]$$

where A, B, C are finite constants, all divergences are shown, result is complete.

(2)

 = $\int d^4k \frac{1}{(k^2 - m^2)(k - p_1 - p_2)^2 - m^2}$

= $A \cdot \ln\left(\frac{\Lambda^2}{m_c^2}\right) + B \left(\frac{(p_1 + p_2)^2}{m_c^2}\right)$

↑ ↑

indep of p's, m. finite & dimensionless
scale fch.

= $\frac{y}{f}$

③ Central part : $x + y + z = 4m^2$

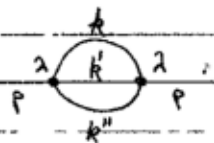
$$a = t = m \Rightarrow a = \frac{4}{3} m^2.$$

$$S_2 \quad (p_1 + p_2)^2 = m^2 + m^2 + 2p_1 \cdot p_2 = \frac{4}{3} m^2$$

$$\Rightarrow 2p_1 \cdot p_2 = \left(\frac{4}{3} - \frac{6}{3}\right)m^2 = -\frac{2}{3}m^2 \Rightarrow p_1 \cdot p_2 = -\frac{1}{3}m^2$$

Footnote: getting graph $\bigcirc$ to standard form.

① Define:
$$I \equiv \int \frac{d^4 k d^4 k'}{(2\pi)^8} \frac{1}{k^2 - m^2} \frac{1}{k'^2 - m^2} \frac{1}{(k - k')^2 - m^2}$$



$$I \equiv \Phi\left(\frac{p^2}{m^2}, \frac{\Lambda^2}{m^2}\right) \cdot m^2$$
 by scaling, since $\delta \Rightarrow D=2$.

② Lead term in Φ is $A \cdot (\Lambda^2/m^2)$, but I don't know constant A yet because I have yet to think about double integrals. But see another footnote. I know that A = dimensionless and indep of p^2 , thus also of m^2 , so $A = \text{a constant}$.

③ Next term could be $B(p^2/m^2) \left(\frac{\Lambda}{m}\right)$. This probably vanishes, but I'll keep it anyway since I'm not sure.

④ Next is $C(p^2/m^2) \ln \Lambda/m$, final is $D(p^2/m^2)$.

Therefore:

$$I = A \cdot \Lambda^2 + B(p^2/m^2) \cdot \Lambda/m + C(p^2/m^2) \ln\left(\frac{\Lambda}{m}\right) m^2 + D(p^2/m^2) \cdot m^2$$

⑤ We want now to expand:

$$I(p^2/m^2) = I(1) + (p^2 - m^2) \tilde{I}(p^2/m^2).$$

$$\text{Thus } \tilde{I} = \frac{I(p^2) - I(m^2)}{p^2 - m^2} = \frac{[A \cdot \Lambda^2 + B(p^2/m^2) \cdot \Lambda/m + C(p^2/m^2) \ln \frac{\Lambda}{m} \cdot m^2 + D(p^2/m^2) \cdot m^2] - [A \cdot \Lambda^2 + B(1) \cdot \Lambda/m + C(1) \ln \frac{\Lambda}{m} \cdot m^2 + D(1) \cdot m^2]}{p^2 - m^2}$$

$$= \frac{\Lambda}{m} \left[\frac{B(p^2/m^2) - B(1)}{p^2/m^2 - 1} \right] + \ln\left(\frac{\Lambda}{m}\right) \left[\frac{C(p^2/m^2) - C(1)}{p^2/m^2 - 1} \right] + \left[\frac{D(p^2/m^2) - D(1)}{p^2/m^2 - 1} \right]$$

If $B = \text{indep of } p^2$, then $\tilde{I}$ has no Λ term.

⑥ On other sheet I have shown $B(p^2/m^2) \equiv 0$ by even/odd. I.e., just expand k'' denom + you see it. Thus:

$$\tilde{I}(p^2/m^2) = \ln\left(\frac{\Lambda}{m}\right) \lim_{p^2 \rightarrow m^2} \left[\frac{C(p^2/m^2) - C(1)}{p^2/m^2 - 1} \right] + \left[\frac{D(p^2/m^2) - D(1)}{p^2/m^2 - 1} \right]$$

$C'(p^2/m^2) \quad D'(p^2/m^2)$

Conclusion: The integral has this form

$$I \equiv \textcircled{P} = m^2 \Phi\left(\frac{p^2}{m^2}, \frac{\Lambda^2}{m^2}\right)$$

$$= \left[A \cdot \left(\frac{\Lambda^2}{m^2}\right) + C(1) \ln\left(\frac{\Lambda^2}{m^2}\right) + D(1) \right] m^2$$

$$+ \left(\frac{p^2}{m^2} - 1\right) m^2 \left[C'\left(\frac{p^2}{m^2}\right) \ln\left(\frac{\Lambda^2}{m^2}\right) + D'\left(\frac{p^2}{m^2}\right) \right]$$

Main points:

- There is no linear divergent term.
- all divergences are exposed
- $C(x)$ and $D(x)$ are finite scalar functions.
- answer is complete, not just $\mathcal{UV}$ result.

① However, $C(p^2/m^2) \equiv 0$ also as follows:

C is coeff of the $\ln \Lambda/m$ term. This is $\mathcal{O}(p^2)$ term in $\mathcal{UV}$. But,

$$= \int \frac{d^4 k d^4 k'}{k^2 k'^2 (k_+ - p)^2} = \int \frac{d^4 k d^4 k'}{k^2 k'^2 k_+^2 \left(1 - \frac{2p \cdot k_+}{k_+^2} + \frac{p^2}{k_+^2}\right)}$$

$$= \int \frac{d^4 k d^4 k'}{k^2 k'^2 k_+^2} \left[\underbrace{1}_{\frac{8}{6} = 1} + \underbrace{\left(\frac{2p \cdot k_+}{k_+^2}\right)}_{\frac{8}{9} = 1} + \underbrace{\left(\frac{4p \cdot k_+ p \cdot k_+}{k_+^4} - \frac{p^2}{k_+^2}\right)}_{\frac{8}{8} = \ln \Lambda} + \dots \right]$$

This $\equiv 0$ under $k_+ \rightarrow -k_+$ etc.

$$I = \int \frac{d^4 k d^4 k'}{k^2 k'^2 k_+^2} = \text{value of } \textcircled{P} \text{ at } p=0$$

Now $\ln \text{ term} = 4p_\mu p_\nu T^{\mu\nu} - p^2 I$

But $T^{\mu\nu} = A \cdot g^{\mu\nu}$

But $T^{\mu\mu} = I = A g^{\mu\mu} = 4A \Rightarrow A = \frac{I}{4} \cdot S_0$

$\ln(\Lambda)$ term $= 4p_\mu p_\nu \left[g^{\mu\nu} \frac{I}{4} \right] - p^2 I \equiv 0$

has neither linear,
nor log!

$\Rightarrow$ $I = \textcircled{P} = A \Lambda^2 + D(1) m^2 + (p^2 - m^2) D'(p^2/m^2)$ ← by this way.

Theorem for ϕ^4 . Theory:

Statement of theorem: the self-energy has $D=2$. The leading term is a constant times Λ^2 . The linear term in Λ vanishes identically. The coefficient of the $\log \Lambda$ divergence is constant times p^2 . And this is all the divergences you have to worry about.

1. First, we can write an integral for the self-energy in terms of the full vertex. This shows the basic form of the self-energy; it is a double integral only, and it has a certain functional form:

$$= \delta m^2 + \int d^4 k d^4 k' f(k^2, k'^2, k_+^2, p^2; k_+, (k_+ + k)^2) + \dots$$

Each propagator depends on its own scalar, and the vertex can depend on the four "masses" and two Mandelstam invariants as chosen in one particular way above. I left out mass m as argument of f . I.e., this is the most general functional dependence you can have for any Feynmann graph in the self-energy.

2. Now, $\dim(f)=-6$ because $D=2$. So we can use scaling argument to rewrite above as:

$$\frac{\int d^4 k d^4 k'}{(k_+^2)^3} F\left(\frac{k^2}{k_+^2}, \frac{k'^2}{k_+^2}, \frac{(k_+ - p)^2}{k_+^2}, \frac{p^2}{k_+^2}; 1, \frac{(k_+ + p)^2}{k_+^2}; \frac{m^2}{k_+^2}\right)$$

like $\frac{x^2}{r^2}$ $\frac{y^2}{r^2}$ damp: damp:

This is still the exact integral. Now if we are interested in UV parts, we can set two arguments to zero and thereby define:

$$F^{UV} \equiv F^{UV}\left(\frac{k^2}{k_+^2}, \frac{k'^2}{k_+^2}, \frac{p^2}{k_+^2}, \frac{k_+ \cdot p}{k_+^2}\right) \equiv F\left(1, 1, 1, 1; 1, 1; 0\right)$$

① ② ③ ④

This thing is dimensionless and independent of mass m . We now expand for a few terms in small p , of course we really mean small p/k_+ since that is what is small. Thus:

$$F^{UV} = \overbrace{F^{UV}\left(\frac{k^2}{k_+^2}, \frac{k'^2}{k_+^2}, 0, 0\right)}^0 + \overbrace{\left(\frac{k_+ \cdot p}{k_+^2}\right) F^{UV}(-, -, 0, 0)}^{-1} + \overbrace{\frac{1}{2} \left(\frac{k_+ \cdot p}{k_+^2}\right)^2 F^{UV}(-, -, 0, 0)}^{-2} + \overbrace{\frac{p^2}{k_+^2} F^{UV}(-, -, 0, 0)}^{-2} + \text{(Remainder)}$$

Ignoring the dimension of p , I have labelled the dimension of each term. You see that the remainder will yield a finite integral, so all UV pieces are now exposed. From this form all results are obvious: (1) the leading term will integrate to constant times Λ^2 where const. is independent of p^2 and m^2 . [upper endpoint evaluation only is implied here]. The second term integrates to zero because odd under negation of all k 's. Since each term is $f(p^2)$, obvious that third and fourth terms will yield p^2 times constant times $\log \Lambda^2$. Then lower endpoints plus remainder will yield finite result.

from  , lowest is 



 $\mathbf{y}_0$

2.

九



Historic only: reminder of the old paradox.

Problem: can the ϕ^4 propagator be made finite (cutoff independent) by multiplicative renormalization, even when it is off shell?

1. On preceding page entitled "theorem" I have shown the , computed to any order, the self-energy has this form:

$$\Sigma^2(p^2) = \Sigma^2(m^2) + (p^2 - m^2) \tilde{\Sigma}^2(p^2)$$

where $\Sigma^2(m^2) = 0 \Rightarrow \delta m^2 = -A \Lambda^2 - B m^2 \log(\frac{\Lambda^2}{m^2}) - m^2 f(1)$ 

and $\tilde{\Sigma}^2(p^2) = [B \log(\frac{\Lambda^2}{m^2}) + C(\frac{p^2}{m^2})]$

where $C(p^2/m^2) = [f(p^2/m^2) - f(1)] m^2 / (p^2 - m^2)$, so $C(1) \neq \text{finite}$.

The function C deviates from zero starting in second-order in the coupling.

2. As an aside, the leaf graph is just a constant and is cancelled directly by δm^2 .
I.e, it has no momentum dependence and thus has no effect on $\tilde{\Sigma}$:

$$\begin{aligned} \text{leaf} &= \int d^4k D(k^2) = \int \frac{d^4k}{k^2} F(\frac{m^2}{k^2}) = \int \frac{d^4k}{k^2} \left[F(0) + \frac{m^2}{k^2} F'(0) + \text{rest} \right] \\ &= a^{(1)} \Lambda^2 + b^{(1)} m^2 \ln \Lambda^2 + m^2 c^{(1)} \end{aligned}$$

Here we have three little numerical constants. Each begins in first order; that is why the mass counterterm has first-order terms in it. All other stuff begins in second order.

3. If we define the renormalized propagator in the usual way, we find:

$$D(p^2) = \frac{1}{p^2 - m^2 - \Sigma^2(p^2)} = \frac{1}{(p^2 - m^2) [1 - \tilde{\Sigma}^2(p^2)]}$$

where $\tilde{\Sigma}^2(p^2) = \frac{\Sigma^2(p^2) - \Sigma^2(m^2)}{1 - \Sigma^2(m^2)} = \frac{C(\frac{p^2}{m^2}) - C(1)}{1 - [B \ln \frac{\Lambda^2}{m^2} + C(1)]}$

Let $\tilde{\Sigma} = 1 - \tilde{\Sigma}_0(m^2) = 1 - [B \ln(\frac{\Lambda^2}{m^2}) + C(1)]$

Then: $\boxed{\tilde{\Sigma}^2(p^2) = \tilde{\Sigma} D^{(1)}(p^2/m^2)}$

This is what I call the renormalized self-energy. To lowest order, it is order λ^2 and is cutoff-independent, as is the resultant propagator to second-order. I.e, we get:

$$D^{(2)}(p^2) = \frac{1}{(p^2 - m^2) [1 - D^{(2)}(p^2/m^2)]}$$

3. So we have made some progress. We have computed in principle the propagator renormalized through second order and have found it is cutoff-independent. The confusion now arises when we try to compute it to higher order:

$$\sum^2 (p^2)^{[2,3,4]} = \sum^{[0,2]} D^{[2,3,4]}$$

If in fact D is ϵ cutoff-independent to any order, you see that the second-order of Z combines with the second-order of D to yield a term of fourth-order in the self-energy which does depend on the cutoff, because that second-order correction to Z has $\log$ cutoff in it.

Thus, it seems that the 4-th order self-energy will be cutoff dependent! How are we going to get around this?

By the way, on shell this function D vanishes to all orders so there is never any problem there. But I am talking about off shell, because propagators appear in Feynman graphs in their off shell sense!

Resolution of a Paradox.

Here was the paradox. Look back at the integral equation for the self energy in ϕ^4 theory. There I argued that any graph which contributes to the main thing must have the result that:

$$\Sigma^2(p^2) = (p^2 - m^2) \left[B \ln \Lambda^2 + C(p^2/m^2) \right] \quad (1)$$

↑
a number

I.e., if you add up all graphs to some order, this must be the form of the result, and the constant B and the function C begin in second order. If this had been true, then you arrive at this result:

$${}^{\text{ren}}\Sigma^2(p^2) = (p^2 - m^2) \left[\frac{C(p^2/m^2) - C(1)}{1 - B \ln \Lambda^2 - C(1)} \right] \quad (2)$$

The claim was then made as follows: since function C is cutoff independent, and since B begins in second order, you are stuck in fourth order with $\ln$ cutoff terms in your renormalized self energy and thus in your renormalized propagator. Result was finite in second order, but then diverges in fourth order even after you have renormalized.

So the problem was that, unless number B happens to be zero in any order, you will sooner or later get uncanceled logs.

Resolution: the form assumed above in equation (1) is only correct in second order. When you iterate the second order correction to the propagator to get the fourth order correction to the self energy, for example, if you were to write the 4th order correction as shown in (1), you would find that the function C in fact depends on the cutoff. The reason is this: it does not depend on the cutoff from the loop integration (superficial thing). That is always the first log. It's just that the second order propagator correction itself contains log, and when that is fed back through, both terms in (1) will be multiplied by this log, so the first term is log squared. ✓

Thus it is at least conceivable that the fourth order term which is log dependent and which you generate in (2) by the interaction of the quadratic term in Z with the quadratic term in C, this 4th order log dependent term could be cancelled by a log dependent fourth order term in C.

Although I have not proved that this must happen, I have at least gotten rid of the paradox which said that the 4th order renormalized self energy had to depend on log cutoff. ✓

The use of charge and mass counterterms in ϕ^4 theory renormalization procedure.

1. Consider the starting lagrangian with bare fields ϕ_0 and m_0 and λ_0 . Suppose you are actually given values of m_0 and λ_0 , and suppose you want the physical mass and charge to come out being m and λ . One has to give up the notation what $dm^2 = m_0 - m$, and that $d\lambda = \lambda_0 - \lambda$. The actual values of these "counterterms" must be set so that the mass and charge come out at values m and λ . One cannot pick a value m and m_0 and expect that dm happens to come out as their difference.

For example, suppose you start out with lagrangian in m_0 and λ_0 . You could then add the above counterterms to shift the thing to m and λ . But you have not really added any counterterms yet, you have just rewritten the lagrangian. Now you add arbitrary counterterms and combine them with the other ones and you call these new, undetermined objects dm and $d\lambda$.

If you want, you can now do the following: specify your m and λ , then compute dm and $d\lambda$ to some order, then define $m_0 = m + dm^2$ and similarly for the charge. Then to any given order, m_0 and λ_0 are computed objects in terms of m and λ . In usual theorems these will always be infinite.

2. In ϕ^4 theory it turns out that λ and λ_0 as defined by $\lambda_0 = \lambda + d\lambda$ always differ by something that is second order. Either can be thought of as the expansion parameter.

3. It is perhaps more convenient to use λ_0 than λ because then you don't have to pepper in counterterm vertex $\delta\lambda$ everytime you have a vertex λ . I.e, these always appear in an additive way. Thus, for example, instead of writing:

$$\lambda + \delta\lambda + \lambda + \delta\lambda$$

We can define λ_0 as noted, and then we have only two graphs:

$$\lambda_0 + \lambda_0$$

4. On the other hand, one may prefer to have an expansion directly in terms of the physical charge λ . It is nice, though, that by going from $\lambda, \delta\lambda$ to λ_0 you replace ~~two~~ two symbols with one.

5. You can do the same thing with mass. But whichever mass you use, you must have the same counterterm. Note that masses don't add quite so simply and in effect reorganize the theory a little, but that is OK.

6. I just feel a little uncomfortable expanding in λ_0 and m_0 since these are really both computed quantities. I'd rather expand directly in the physical objects. The cost is that you have to include counterterms in your feynman diagrams. But you have to worry about them anyway because you have to untangle eventually!

7. For example, suppose you go ahead and use m_0 and λ_0 , thereby avoiding all counter-term feynman graphs except the basic one in the self-energy. Then when you compute to some order you get something like this:

$$1) \quad 0 = \delta m + \lambda_0 f_1(m_0^2) + \lambda_0^2 f_2(m_0^2)$$

$$2) \quad \lambda_0 + \lambda_0^2 h_1(m_0^2) = \lambda \Sigma^2 = \lambda [1 + \lambda_0^2 h_2(m_0^2)]^2$$

$$\Sigma^2 = 1 + \lambda_0^2 h_2(m_0^2)$$

You get very simple expressions for ~~xxxxxx~~ dm and $d\lambda$ as functions of m_0 and λ_0 . But really these are functional equations for dm and $d\lambda$ as functions of the fixed, known objects m and λ . In other words, it is ridiculous to expand in m_0 because you don't even know what m_0 is! And it is not the mass that was in the bare Lagrangian! This is an important point.

What the above equations really say is this:

$$1) \quad 0 = \delta m^2 + (\lambda + \delta\lambda) f_1(m^2 + \delta m^2) + (\lambda + \delta\lambda)^2 f_2(m^2 + \delta m^2)$$

$$2) \quad \lambda + \delta\lambda + (\lambda + \delta\lambda)^2 h_1(m^2 + \delta m^2) = \lambda [1 + (\lambda + \delta\lambda)^2 h_2(m^2 + \delta m^2)]^2$$

Although the charges always appear in a simple way, to actually find dm^2 you have to go in and expand the functions f_1 and f_2 , etc etc, all of which work is no harder than going the other route and explicitly showing the counterterms.

Various Ways to Renormalize ϕ^4 theory

① Start off with the bare Lagrangian: $\mathcal{L}_0 = \frac{1}{2}(\partial_\mu \phi_0 \partial^\mu \phi_0 - m_0^2 \phi_0^2) - \lambda \phi_0^4$

Method A:

② Rewrite it to get mass at m and coupling at λ , but no counterterms yet added, this is just a rewrite:

$$\mathcal{L}_0 = \frac{1}{2}(\partial_\mu \phi_0 \partial^\mu \phi_0 - m_0^2 \phi_0^2) - \lambda \phi_0^4 + \frac{1}{2}(m^2 - m_0^2) \phi_0^2 + (\lambda - \lambda_0) \phi_0^4$$

③ Now add some counterterms which cause these constants to be arbitrary. Then

$$\mathcal{L}_0 = \frac{1}{2}(\partial_\mu \phi_0 \partial^\mu \phi_0 - m^2 \phi_0^2) - \lambda \phi_0^4 + \frac{1}{2} \delta m^2 \phi_0^2 + \delta \lambda \phi_0^4$$

where now δm^2 and $\delta \lambda$ can be anything you want!

④ Expand the scalar self-energy around $p^2 = m^2$ so $\tilde{\Sigma}_0^2(p^2) = \tilde{\Sigma}_0^2(m^2) + (p^2 - m^2) \tilde{\Sigma}'_0(p^2)$.

⑤ Now do the self-energy dyam. Since bare mass has been replaced by m in "bare" propagator, we get

$$D_0 = \frac{1}{p^2 - m^2 - \tilde{\Sigma}_0^2(p^2)} = \frac{1}{(p^2 - m^2) [1 - \tilde{\Sigma}'_0(p^2)]}$$

This last step provided us adjust $\tilde{\Sigma}_0^2(m^2) = 0$. This implies a surface-of-choice in $(\delta m^2, \delta \lambda)$ space.

⑥ Next, lets add more counterterms: $(\tilde{\Sigma}^{-1} - 1) \left\{ \frac{1}{2} \partial_\mu \phi_0 \partial^\mu \phi_0 - m^2 \phi_0^2 + \frac{1}{2} \delta m^2 \phi_0^2 - \lambda \phi_0^4 + \delta \lambda \phi_0^4 \right\} + (\tilde{\Sigma}^{-1} - 1) [-\lambda \phi_0^4 + \delta \lambda \phi_0^4]$

$$\mathcal{L} = \tilde{\Sigma}^{-1} \cdot \frac{1}{2}(\partial_\mu \phi_0 \partial^\mu \phi_0 - m^2 \phi_0^2) - \tilde{\Sigma}^{-1} \phi_0^4 \lambda + \tilde{\Sigma}^{-1} \phi_0^2 \frac{1}{2} \delta m^2 - \tilde{\Sigma}^{-1} \phi_0^4 \delta \lambda$$

Now rescale field so: $\mathcal{L} = \frac{1}{2}(\partial_\mu \phi \partial^\mu \phi - m^2 \phi^2) - \lambda \phi^4 + \frac{1}{2} \delta m^2 \phi^2 - \delta \lambda \phi^4$

⑦ New propagator is then $\phi = \tilde{\Sigma}^{\frac{1}{2}} \phi_0$

$$D = \langle \phi \phi \rangle = \tilde{\Sigma}^{-1} D_0 = \frac{\tilde{\Sigma}^{-1}}{(p^2 - m^2) [1 - \tilde{\Sigma}'_0(p^2)]} = \frac{[1 - \tilde{\Sigma}'_0(m^2)]}{[1 - \tilde{\Sigma}'_0(p^2)]} \frac{1}{(p^2 - m^2)}$$

Select a particular $\tilde{\Sigma}^{-1} = 1 - \tilde{\Sigma}'_0(m^2)$.

① Finally, compute

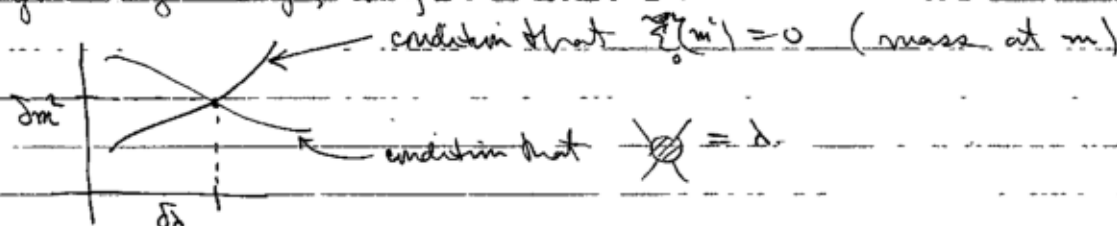
$$\Gamma_0^{(4)}(\dots) = \text{diagram 1} + \text{diagram 2} + \text{diagram 3} + \text{diagram 4} + \dots$$

$$= \langle \phi_0 \phi_0 \phi_0 \phi_0 \rangle.$$

Then define $\Gamma^{(4)}(\dots) = \langle \phi \phi \phi \phi \rangle = \sum_{+2} \Gamma_0^{(4)}(\dots) = f(\delta\lambda, \delta m^2, m^2, \lambda, \dots)$

Then insist that $\Gamma^{(4)}(\text{sym}) = 1$. This puts a second condition on $\delta m^2, \delta\lambda$.

② So you might say, for given m and λ ,



③ So look what this method A is doing. The first thing you do is insist that the bare mass and bare coupling be equal to your final physical values. You move these things right at the beginning by adding some counterterms. Then when you "dress" the theory, your conditions which determined the counterterms values are that there be no further mass renormalization, and no renormalization of the coupling. To any given order in the coupling λ , and for fixed value of the cutoff, you can compute the coordinates of the intersection of the above curves.

[method A, an'd]

⑩ Consider: $D_0 = \frac{1}{[p^2 - m^2 - \Sigma_0^2(p^2)]}$

Define Σ by: $D = \frac{1}{p^2 - m^2 - \Sigma^2(p^2)}$ where $D = \tilde{\Sigma}^{-1} D_0$

How are Σ and Σ_0 related? Do algebra here:

$$[p^2 - m^2 - \Sigma^2(p^2)]^{-1} = \tilde{\Sigma}^{-1} [p^2 - m^2 - \Sigma_0^2(p^2)]^{-1}$$

$$\Rightarrow p^2 - m^2 - \Sigma^2(p^2) = \tilde{\Sigma} \cdot (p^2 - m^2 - \Sigma_0^2(p^2)) \Rightarrow \text{A2E}$$

$$\Rightarrow \Sigma^2(p^2) = (p^2 - m^2)(1 - \tilde{\Sigma}) + \tilde{\Sigma} \Sigma_0^2(p^2)$$

Now expand both objects in same way:

$$\Sigma^2(p^2) = \Sigma^2(m^2) + (p^2 - m^2) \tilde{\Sigma}^2(p^2)$$

$$\Sigma_0^2(p^2) = \Sigma_0^2(m^2) + (p^2 - m^2) \tilde{\Sigma}_0^2(p^2)$$

Then box says: $\Sigma^2(m^2) + (p^2 - m^2) \tilde{\Sigma}^2(p^2) = (p^2 - m^2)(1 - \tilde{\Sigma}) + \tilde{\Sigma} \Sigma_0^2(m^2) + \tilde{\Sigma} (p^2 - m^2) \tilde{\Sigma}_0^2(p^2)$

this is two results

$$a) \Sigma^2(m^2) = \tilde{\Sigma} \Sigma_0^2(m^2)$$

$$b) \tilde{\Sigma}^2(p^2) = \tilde{\Sigma} \tilde{\Sigma}_0^2(p^2) + (1 - \tilde{\Sigma})$$

Now, let $\tilde{D} = 1 - \tilde{\Sigma}_0^2(m^2)$ as was done. Then:

$$b) \Rightarrow \tilde{\Sigma}^2(p^2) = [1 - \tilde{\Sigma}_0^2(m^2)]^{-1} \tilde{\Sigma}_0^2(p^2) + [-\tilde{\Sigma}_0^2(m^2)] [1 - \tilde{\Sigma}_0^2(m^2)]^{-1}$$

$$\Rightarrow \tilde{\Sigma}^2(p^2) = \frac{\tilde{\Sigma}_0^2(p^2) - \tilde{\Sigma}_0^2(m^2)}{1 - \tilde{\Sigma}_0^2(m^2)}$$

⑪ So, not surprised to find that you can rewrite our old result:

$$D = \left[\frac{1}{1 - \tilde{\Sigma}^2(p^2)} \right] \frac{1}{(p^2 - m^2)}$$

where $\tilde{\Sigma}^2(m^2) = 0$, so pole residue = 1.

[mixed A , and d]

(90)

⑫ What are the Renorm parts in ϕ^4 ? $\delta=0$ so $D=4-E_B$.

So

$E_B = 2, 3, 4$ are "renorm parts" of the theory

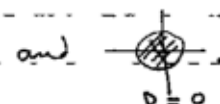
⑬ But recall: $\sum_i n_i b_i^t = E_B^t + 2 I_B^t = n b^t = 4n$.

So $E_B + 2 I_B = 4n \Rightarrow E_B = \text{even!}$ (no graphs exist $E_B=3$)

So only renorm parts of ϕ^4 theory are



$D=2$



$D=0$

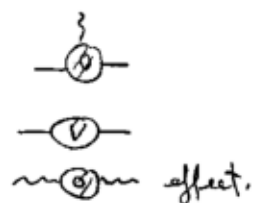
$$\text{So } \Sigma_0(p^2) = \underbrace{\Sigma_0(m^2)}_{D=2} + (p^2 - m^2) \underbrace{\tilde{\Sigma}_0(p^2)}_{D=0}$$

QED

Quantum Electrodynamics

$$D = 4 - \frac{3}{2} E_F - E_B$$

Renormalization of QED



I plan to do this first without putting in the i's, then later go back and do repairs. Comparison to BD notation made somewhere below. All is in my notation.

1. Propagator: Look this up in general renorm notes. Start with bare propagator in the general a-gauge, then add up series. Here is the result:

$$D_{\mu\nu}^{\text{bare}} = \frac{T_{\mu\nu}}{q^4} - \frac{\alpha}{q^4} q_\mu q_\nu \quad \text{where} \quad T_{\mu\nu} \equiv [q_\mu q_\nu - q^2 g_{\mu\nu}]$$

$$D_{\mu\nu}^{\circ} = \frac{1}{[1 + \Pi(q^2)]} \frac{T_{\mu\nu}}{q^4} - \frac{\alpha}{q^4} q_\mu q_\nu \quad \text{where} \quad \Pi_{\mu\nu}(q^2) = T_{\mu\nu} \cdot \Pi(q^2).$$

Here we see the familiar result that the longitudinal part of the propagator remains "unrenormalized", ie, undressed. The reason is that the self-energy is entirely transverse. To show this, one uses the self-energy Dyson equation and the Ward identity: one assumes that the cutoff is implemented in a gauge-invariant manner. See page 302, Eq. 19.22 on this point.

I shall use the standard renormalization point $q^2=0$ rather than an arbitrary point u^2 . Thus, one sees what Z_3 will be:

$$Z_3^{-1} = 1 + \Pi(q^2=0; m_0, e_0, \Lambda)$$

Lets now write the renormalized propagator as follows:

$$D_{\mu\nu} = \frac{1}{1 + \Pi(q^2)} \frac{T_{\mu\nu}}{q^4} - \frac{\alpha}{q^4} q_\mu q_\nu = Z_3^{-1} D_0$$

Thus:	<u>RC#1</u> :	$\Pi(q^2=0) = 0$	/unit residue
	<u>α-cm</u> :	$\alpha = Z_3^{-1} \alpha_0$	/not assume.

According to these definitions, we may relate the renormed and unrenormed self-energy scalar functions in this way:

$$\Pi(q^2) = Z_3^{-1} - 1 + Z_3 \Pi(q^2) = Z_3 [\Pi(q^2) - \Pi(0)]$$

Our assumption of the transversal self-energy yields the above form for the dressed propagator. This simply does not allow any mass renormalization unless PI has a pole, which unitarity says it does not. Thus, gauge invariance in the form of the Ward Identity blocks mass renorm, so in a sense we avoid one problem there. Hence, we never use any kind of mass counterterm for the photon. Not needed. Perturbation theory: clearly the scalar function has $D=0$. Using a gauge invariant cutoff, BD find in vol 1 that:

$${}^0\pi(q^2)^{(2)} = i \frac{e_0^2}{12\pi^2} \left[\ln\left(\frac{\Lambda^2}{m_e^2}\right) - 6 \int_0^1 dz \cdot z \cdot (1-z) \ln\left[1 - \frac{q^2}{m_e^2} z(1-z)\right] \right]$$

some $C(q^2/m_e^2) \rightarrow$ Uehling $2\pi m_e$.

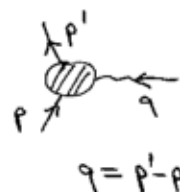
$$\Rightarrow Z_3^{(2)} \approx 1 - \frac{e_0^2}{12\pi^2} \ln\left(\frac{\Lambda^2}{m_e^2}\right).$$

2. The vertex. The momenta are defined like so:

I guess the general kinetic form is: only div. const.

$${}^0\Gamma_\mu(p, p') = A \gamma_\mu + B(p-p)_\mu + C(p'+p)_\mu$$

($\sigma_{\mu\nu}$?)



But if you sandwich between $\bar{u}(p)$ and $u(p)$ and take limit that $q=0$ all components, you find:

$$\bar{u}(p) {}^0\Gamma_\mu(p, p) u(p) = e_0 Z_1^{-1} \cdot (p_\mu/m) = e_0 Z_1^{-1} [\bar{u}(p) \gamma_\mu u(p)]$$

And then Z_1 is some combination of A and C above, evaluated with their arguments on mass shell $p^2=m^2$. Notice that this vertex, unlike the ϕ^3 vertex, is a function of p^2, q^2 and p'^2 and also of p_u and p'_u due to the spinology. Thus, in order to define a renormalization point, you cannot just set thing on mass shell. Just setting to mass shell still leaves two possibly different momenta p^u and p'^u . So here, the thing that corresponds to the symmetric point in ϕ^3 or ϕ^4 is to take $p'_u = p_u$ so that $q_u=0$ as a four vector, and also put p and p' on mass shell.

When this is done, you see that the matrix element of the vertex must be a multiple of p_u since that is the only vector there is, and that can be put in terms of gamma-u. Hence, the above definition of Z_1^{-1} (which I have been calling Z_5^{-1} in other theories). Z_1 is thus a scalar and can depend only on m and cutoff.

So far then both Z_3 and Z_1 have been defined. The scaling rules are:

$$\left. \begin{aligned} {}^0A^\mu &= \sqrt{Z_3} A^\mu \\ {}^0\psi &= \sqrt{Z_2} \psi \end{aligned} \right\} \Rightarrow \begin{cases} {}^0D = Z_3 D \\ {}^0S = Z_2 S \\ {}^0\Gamma = Z_2^{-1} Z_3^{-1} \Gamma \end{cases} \quad \begin{aligned} {}^0\tilde{\Gamma} &= Z_2^{-1} \tilde{\Gamma} \end{aligned}$$

(1PI!)

Let me write schematically:

$$^0\Gamma_\mu(p,p)|^{m^2} \doteq e_0 \Sigma_1^{-1} \gamma_\mu$$

$$^0\tilde{\Gamma}_\mu(p,p)|^{m^2} \doteq \Sigma_1^{-1} \gamma_\mu$$

$$\Gamma_\mu(p,p)|^{m^2} \doteq e \gamma_\mu$$

$$\tilde{\Gamma}_\mu(p,p)|^{m^2} \doteq \gamma_\mu$$

RC #2

Therefore:

$$e_0 \Sigma_1^{-1} \gamma_\mu = \Sigma_2^{-1} \Sigma_3^{-\frac{1}{2}} e \gamma_\mu \Rightarrow e = \Sigma_2 \Sigma_3^{\frac{1}{2}} \Sigma_1^{-1} e_0 = \Sigma_3^{\frac{1}{2}} e_0$$

So, as usual, the definition of Z_1 and the field rescalings lead to a vertex rule which in turn tells how the charge rescales. Ward says $Z_1=Z_2$ but I won't use it yet.

Perturbation theory: the lowest graph for the vertex is obvious, but you have to put in a photon IR cutoff. Same goes for $Z_1=Z_2$, and you have to deal with the soft-photon problem etc. I don't want to get involved in that now, so now make no statement. Obviously, the vertex behaves as UV log of cutoff since $D=0$.

3. Kernel: This is defined to omit single horizontal photon and electron pair. Then it will prove useful in eliminating overlap. Rescaling rule is:

$$^0K = Z_2^{-2} K$$

(Photon)

4. Self-energy Dyson. Again, no mass counterterm so:

$$^0\pi_{\mu\nu}(q^2) = e_0 \text{tr} \int \gamma_\mu ^0S^0 S^0 \gamma_\nu$$

$$= e^2 \Sigma_3^{-1} \cdot \left\{ \Sigma_1 \cdot \int \text{tr} [\gamma_\mu S S \tilde{\Gamma}_\nu] \right\}$$

$$= T_{\mu\nu} ^0 \pi(q^2) \equiv E_{\mu\nu}(q^2) = T_{\mu\nu} E(q^2)$$

Thus, $^0\pi(q^2) = e^2 \Sigma_3^{-1} E(q^2) = e_0^2 E(q^2)$ ↖ [Blen 1 type]

$$\Rightarrow \pi(q^2) = e^2 [E(q^2) - E(0)] = e^2 \Delta E(q^2).$$

Also: $\begin{cases} ^0\pi(0) = e^2 \Sigma_3^{-1} E(0) = \Sigma_3^{-1} - 1 \\ \Rightarrow \Sigma_3 = 1 - e^2 E(0) \end{cases}$

Comments: first, because there is no mass counterterm, the overall subtraction does not appear. The secondary subtraction then yields ΔE for the self-energy renormed, and this is finite apart from possible IR. I.e., the $B \log \Lambda$ cancels here. Also, notice that $E(0)$ is log divergent, certainly does not vanish. Thus, $Z_3 \neq 1$ and $\Pi(0) \neq 0$ for sure.

5. Vertex Dyson. Very easy, same as all previous theories:

$$\begin{aligned} \Gamma_\mu &= e_0 \delta_\mu + \int \Gamma_\mu S S K \\ \tilde{\Gamma}_\mu &= Z_1 \delta_\mu + \int \tilde{\Gamma}_\mu S S K \end{aligned}$$

Then we get:

$$E_{\mu\nu}(p^2) = \int \tilde{\Gamma}_\mu S S \tilde{\Gamma}_\nu - \int \tilde{\Gamma}_\mu S K S \tilde{\Gamma}_\nu = \text{no overlap!}$$

6. Ward Identity and Electron Self-energy Avoidance.

According to the BD, and me, the unrenormalized Ward reads:

$$\begin{aligned} q_\mu \tilde{\Gamma}^\mu(p, p') &= \tilde{S}^{-1}(p') - \tilde{S}^{-1}(p) & q = p' - p \\ q_\mu \tilde{\Gamma}^\mu(p, p') &= \tilde{S}^{-1}(p') - \tilde{S}^{-1}(p) & \text{/renormed.} \end{aligned}$$

$$\Rightarrow \tilde{S}^{-1}(p') = q_\mu \tilde{\Gamma}^\mu(p', p=m) \quad \text{/to be shown below.}$$

Here is the idea: using the vertex Dyson, self energy Dyson for photon, and this connection of S to the vertex via Ward, you can iteratively build up your finite renormalized theory in the same way almost as I did for ϕ^3 . Inductive proof, etc, as shown in BD also. There is an electron self-energy Dyson also, but you can ignore it because of this formula.

Jim's Self Energy Renormalization Question

1. He computes lowest order electron self-energy in QED. He has quadratic divergent term. How should he renormalize it? As I look at my notes, I seems that the renormalize self-energy is given by:

$$\Sigma(p) = \Sigma_1(m^2) \frac{1}{\epsilon} + (\not{p} - m) \tilde{\Sigma}(p^2)$$

where $\Sigma_1(m^2) = 0$ [one renorm cond.]

$$\tilde{\Sigma}(p^2) = e^2 \cdot \Delta \tilde{X}(p^2)$$

$$X(p) = \left[Z_2 \int \gamma_\mu S D \tilde{\Gamma}_\mu \right] = (\not{p} - m) \tilde{X}(p^2) + \text{const.}$$

In lowest order, we want $\tilde{X}$ to zeroth order so

$$\tilde{X}^{(0)}(p) = \int \gamma_\mu S^0 D^0 \tilde{\Gamma}_\mu^0 = \int \gamma_\mu S^0 D^0 \gamma_\mu = \text{---}$$

The problem Jim is running into is of course this: when you compute $\tilde{X}(p^2=m^2)$ in order to get $D\tilde{X}(p^2)$, you hit an IR singularity. What should he do?

- put in a soft photon regulator.
- change his renormalization point.

2. How do you do this second choice? It seems to me that you change one of the two renorm conditions for the electron self-energy: you keep the pole at m^2 , but you redefined Z_2 so that at $p^2 = u^2$ you have "unit residue" so to speak. Then the renormalized self energy is finite.

But what effects result from thus changing the renorm point in this way!!!???

7. The electron self-energy and its Dyson and overlap problems, etc.

As usual we start with the basic definitions:

$$^0S = \frac{1}{\not{p} - m - ^0\Sigma(p)}$$

$$S = \frac{1}{\not{p} - m - \Sigma(p)}$$

$$^0S = \Delta_2 S$$

$$^0\Sigma(p) = \sum_1(p^2) \mathbb{1} + (\not{p} - m) \tilde{\Sigma}(p^2)$$

$$\Sigma(p) = \Sigma_1(m^2) \mathbb{1} + (\not{p} - m) \tilde{\Sigma}(p^2)$$

As usual, want both to have pole at m so

$$^0\Sigma_1(m^2) = \Sigma_1(m^2) = 0.$$

$$S(p^2 = m^2) = \frac{1}{\not{p} - m}$$

Then

$$^0S = \frac{1}{(\not{p} - m)[1 - \tilde{\Sigma}(p^2)]}$$

$$\Rightarrow \Delta_2^{-1} = 1 - ^0\tilde{\Sigma}(m^2)$$

$$S = \frac{1}{(\not{p} - m)[1 - \tilde{\Sigma}(p^2)]}$$

Relation between self-energies is:

$$\tilde{\Sigma}(p^2) = \Delta_2 [\tilde{\Sigma}^0(p^2) - \tilde{\Sigma}^0(m^2)]$$

Dyson:

$$\rightarrow \text{blob} \rightarrow = \rightarrow \text{blob} \rightarrow + \delta m^2 \cdot 1$$

$$^0\Sigma(p^2) = e \int \gamma_\mu ^0S ^0D ^0\Gamma + \delta m^2$$

$$= e^2 \Delta_2^{-1} \left\{ \Delta_2 \int \gamma_\mu S D \tilde{\Gamma}_\mu \right\} + \delta m^2$$

$$X(p^2) = (\not{p} - m) \tilde{X}(p^2) + \text{c.m.t.}$$

Overlap is present:

$$\text{blob} = \text{blob} \leftarrow \text{bad subgraph!}$$

Claim $\Delta X = \text{finite}$. Then.

$$^0\tilde{\Sigma}(p^2) = \tilde{X}(p^2) e^2 \Delta_2^{-1} \Rightarrow$$

$$\boxed{\begin{aligned} \tilde{\Sigma}(p^2) &= e^2 \cdot \Delta \tilde{X}(p^2) \\ &= \text{finite} \end{aligned}}$$

To show $\Delta\tilde{X} = \text{finite}$, solve overlap:

$X(p) =$

But $2\gamma_\mu =$

Thus:

$X(p) =$

$=$

The claim here is that the overlapping singularities cancel.

order 4:

$+$

$-$

$=$

Here you see how it works in 4th order: the third graph has both divergent subintegrations, and these cancel with the first two graphs so the sum has only overall divergences, which are removed when you make ΔX .

Of course all this really doesn't matter in QED since you have the Ward identity telling you that S is finite if the vertex is:

$$\begin{aligned} (p' - p)^\mu \tilde{\Gamma}_\mu(p, p') &= {}^0S^{-1}(p') - {}^0S^{-1}(p) \\ (p' - p)^\mu \tilde{\Gamma}_\mu(p, p') &= S^{-1}(p') - S^{-1}(p) \quad / \text{renormalized} \\ (p' - p)^\mu [\gamma_\mu + \tilde{\Gamma}_\mu(p, p')] &= \not{p}' - \not{p} + \Sigma(p) - \Sigma(p') \\ \Rightarrow (p' - p)^\mu \tilde{\Gamma}_\mu(p, p') &= \Sigma(p) - \Sigma(p') \\ &= \tilde{\Sigma}^2(p^2)(\not{p} - m) - \tilde{\Sigma}^2(p'^2)(\not{p}' - m) \\ \boxed{(p' - p)^\mu \tilde{\Gamma}_\mu(p, p') &= \Sigma^0(p) - \Sigma^0(p')} \end{aligned}$$

The notation $\not{p}' \rightarrow m$ is impossible in literal sense, but is fine if object is known onto $u(p')$. So,

$$q^\mu \tilde{\Lambda}_\mu(p, p') u(p') = \tilde{\Sigma}^2(p^2) (\not{p} - m) u(p')$$

$$\Rightarrow q^\mu \bar{u}(p') \tilde{\Lambda}_\mu(p, p') u(p') = \tilde{\Sigma}^2(p^2) \underbrace{\left[p_\mu \cdot \frac{p^\mu}{m} - m \right]}_{\tilde{\Sigma}^2(p^2) \left[\frac{p^2 + p'^2 - q^2 - 2m^2}{2m} \right]}$$

Probably this really implies $p'^2 = m'^2$ so get a result. Put $q^2 = 0$ but not $q_\mu = 0$. So $q_\mu = \text{lightlike}$. Then

$$\tilde{\Sigma}^2(p^2) = \frac{2m}{(p^2 - m^2)} \cdot \bar{u}(p') \not{(p' - p)}_\mu \tilde{\Lambda}^\mu(p, p') u(p')$$

The spinology is a little vague here, but it seems clear that you can compute the renormalized self-energy completely if you know the renormalized vertex. That is all I want to say for now.

Symbol Translation Dictionary

<u>me</u>		<u>BD</u>
D^{bare}	(bare)	D_F^{bare}
$^{\circ}D$	(dressed)	D_F^1
D	(renormed)	$\tilde{D}_F^1$
S^{bare}	(bare)	S_F
$^{\circ}S$	(dressed)	S_F^1
S	(renormed)	$\tilde{S}_F^1$
Γ_{μ}	(renorm w) charge)	$e \tilde{\Gamma}_{\mu}^{\text{BD}}$
$^{\circ}\tilde{\Gamma}_{\mu}$	(dressed, $\div ch$)	Γ_{μ}^{BD}
$\tilde{\Gamma}_{\mu}$	(renorm, $\div ch$)	$\tilde{\Gamma}_{\mu}^{\text{BD}}$
$^{\circ}\rho_{\mu}$	(dressed, w ch)	$e \rho_{\mu}^{\text{BD}}$
$\tilde{\rho}_{\mu}$		ρ_{μ}^{BD}
$^{\circ}\Sigma(p)$	(dressed)	$\Sigma(p)^{\text{BD}}$
$^{\circ}\Pi(q^2)$	(unren)	$e^2 \Pi^{\text{BD}}(q^2)$
$\Pi(q^2)$	(ren)	$e^2 \Pi_c^{\text{BD}}(q^2)$
$\tilde{\Sigma}(p^2)$	(ren)	$Z_2 \cdot \Sigma_c^{\text{BD}}(p^2)$
		$\Pi_c^{\text{BD}}(q^2) = \Pi^{\text{BD}}(q^2) - \Pi^{\text{BD}}(0)$