

Renormalization Group - papers
(Particle Physics)
Notes

Bogoliubov & Shirkov
Intro to the Theory of Quantized Fields
Chapt 8 The Renormalization Group
(1959)

Callan 1970
Broken Scale Invariance in Scalar Field Theory

Symznik
1969

Symznik
1970

Abarbanel
1974

Gell-Mann & Low
1954

Phil Lucht Notes 1978

Bogoliubov & Shirkov
Intro to the Theory of Quantized Fields
Chapt 8 The Renormalization Group

*AI in 2026: Introduction to the Theory of Quantized Fields (1959) by N.N. Bogoliubov and D.V. Shirkov is considered a classic and foundational text, but it is **outdated for learning modern Quantum Field Theory (QFT)** as a beginner. While its mathematical foundations and historical significance in the development of renormalization group methods remain highly regarded, modern curricula favor newer textbooks for covering contemporary techniques.*

the change of variable

$$[(u\phi) - m]v = y$$

leads to the expression

$$G'(\phi) = \frac{\left[\frac{(u\phi)}{m} - 1 \right]^{\frac{e^2(d_1^0 - 3)}{8\pi^2}}}{(u\phi) - m} I' \quad (41.27)$$

where

$$I' = i \int_0^\infty dy y^{\frac{e^2(3-d_1^0)}{8\pi^2}} e^{iy - ev} = -1 + O'(e^2).$$

By combining expressions (41.26) and (41.27) we obtain the final expression for the electron Green's function in the Bloch-Nordsieck model:

$$G'(\phi) = \frac{1}{m - (u\phi)} \left| 1 - \frac{(u\phi)}{m} \right|^\beta, \quad (41.28)$$

where

$$\beta = -\frac{e^2(3 - d_1^0)}{8\pi^2} = -\frac{\alpha}{2\pi} (3 - d_1^0) \quad (41.29)$$

($\alpha = 1/137$ is the fine structure constant).

On comparing (41.28) with (41.3), we see that the total Green's function differs from the free-field Green's function by the factor

$$\left| 1 - \frac{(u\phi)}{m} \right|^\beta.$$

If the complete function $G'(\phi)$ is expanded in a series in powers of α , then in each approximation we shall obtain logarithmic terms

$$\left| 1 - \frac{(u\phi)}{m} \right|^\beta = e^{\beta \ln \left| 1 - \frac{u\phi}{m} \right|} = 1 - \frac{\alpha}{2\pi} (3 - d_1^0) \ln \left| 1 - \frac{(u\phi)}{m} \right| + \dots,$$

characteristic of the infrared catastrophe.

It will be shown in the next chapter (§ 43.3) that the infrared singularity of the electron Green's function in the usual spinor electrodynamics also has the nature (41.28).

The Renormalization Group

§ 42. The Group of Multiplicative Renormalizations in Spinor Electrodynamics

42.1. Introduction

We shall now turn to the investigation of a method of improving perturbation theory formulas which is based on the renormalization group occurring in field theory; this was already mentioned in Chapter V. The idea of this method is extremely simple, and is based on the fact that the individual terms of perturbation theory expansions for the Green's functions and for the vertex parts are not invariant with respect to the renormalization group, while the quantities themselves are invariant with respect to this group. Therefore, for example, in the ultraviolet region perturbation theory expansions turn out to be given not in terms of powers of the square of the charge e^2 but in terms of powers of the product $e^2 \ln(k^2/m^2)$.

Starting with these considerations, we shall try to transform the usual expansions into ones invariant with respect to the renormalization group. An important technical means of accomplishing this will be the Lie differential equations corresponding to the aforementioned renormalization group.

It should be noted that Stueckelberg and Peterman (118) already remarked on the existence of the renormalization group and on its role in quantum field theory. They also pointed out the possibility of introducing the corresponding infinitesimal operators and in this way indicated the possibility of constructing the Lie differential equations.

Essentially the same renormalization group, but from a different point of view, was utilized by Gell-Mann and Low (48)

Bogoliubov and Shirkov, Chapter 8 on the Renormalization Group

First, the basic and simple idea of the renormalization group (group of multiplicative renormalizations) is set up as in 42.2. It just says that, taking Ward into consideration, if you slide all your Greens functions and your charge at the same time, nothing changes. Your location, so to speak, on this slide is recorded by the renormalization point parameter λ . I have made some notes already on this business, see Renorm in QED stapled section in General Field Theory binder.

As a reminder of their notation, the function d is the photon propagator numerator function, while a and b are the electron propagator numerators. Sometimes these two are treated together as s .

With some straightforward dimensional fiddling (all of which I have verified) you quickly end up, eg, with equation (42.13) which says: the quantities e and d each depend on the renormalization point, but the combination $e^2 d$ is independent of λ and is therefore, reasonably, called the invariant charge. The notation d_0 refers to the usual d , ie, d with the renorm point chosen at $\lambda = 0$. The corresponding charge in this case is the observed charge e_0 .

Now, the basic "global" renormalization group equations are (42.10) for d and (42.15) for $s = a$ or b . These equations are very similar to those discussed by Gell-Mann and Low, but they seem to be more clearly derived and there are not so many different functions. The pair of R-group equations is then rewritten as (42.17 and 18).

It seems to me that, in these equations, (x, y, t, e^2) should be regarded as independent variables, and d and s are the functions, each of 3 variables, that you want to determine. Ie, e^2 is not itself a function here. Now it turns out that you need some "boundary conditions" in order to get a meaningful solution of these "functional" equations. An obvious fact is this: when the parameter e^2 (the variable e^2) is very small, you know d and s as perturbation expansions.

Now, by differentiating the global equations they end up with some differential equations (42.24 and 25). They call these Lie equations, fine. Here is how you are supposed to solve them: first, you somehow figure out what $d(x, y, e^2)$ is for x near unity. This means k^2 near ∞ . If you keep e^2 small, you can use perturbation estimates then for the functions ϕ and γ . Then you integrate the preceding two equations to learn $d(x, y, e^2)$ for general x , perhaps x corresponding to asymptotic momentum.

As an aside, it is noted that the vertex has many invariant functions and each one has an equation similar to the above equation for s or d .

Now let's go right to the UV solution attempt, page 524. Asymptotic means in effect that you can set variable $y=0$, thus simplifying things. The Lie equations

are then (43.3 and 4). First, equation (43.3) is integrated to yield (43.7), a result familiar from Gell-Mann Low or modern BETA functions discussions (later). In 43.7, if you knew the function ϕ , you could solve for d , perhaps. Lets see.

As a first step, they quote a result which gives $d(x, e^2)$ as a perturbation expansion through order e^4 , where $x = p^2/m^2$ is large. This thing is called d_0 because it is computed with the usual $\lambda = 0$. Notice that in effect this thing is not an expansion in e^2 , but in $e^2 \log x$ and x is large!

Now they take some steps I dont quite follow. I presume that (43.9) is the large x result, not really valid for x near 1. But to use this d_0 to compute ϕ , you need $d(x..)$ for x near 1 and they are going to use (43.9). I think they are claiming that this is OK to "logarithmic accuracy". So then ϕ is as given in (43.10) for small z .

But the way, notice in (42.24) that the value of the last argument of ϕ is $e^2 d$, not just e^2 , so really all this stuff (using perturbation series for ϕ) is valid only when $e^2 d$ is small, ie z is small (my ETA in notes). Thus 43.10 gives $\phi(z)$ for small z . Now plug this into (43.7) and we get the result (43.12).

Now i have to ask: why is 43.12 any better than 43.9 ??? It is supposed to be "better" somehow.

The conclusions that BS draw from this section are a little disappointing. True, they have solved for the function $d(x, e^2)$ as in (43.12), but they dont go on to ask: what does e^2 look like at high energy, in some sense.

It seems to me looking at (42.14) that what you really want to know is the behavior of the function $d_0(x, e_0^2)$ since this will tell you what the charge does as a function of renormalization point. I guess that is why Gell-Mann Low have two functions s and s_c , akin to BS's d and d_0 .

So although I am able to derive almost everything here, the conclusions are a let down. However, having derived 43.7, I feel I have derived Gell-Mann Low equation (5.9). In that paper, symbol e_1 is renorm charge and e_0 is bare charge. Recall that if their $\Psi(x)$ has a zero at some finite $x=x_0$, then large k^2 corresponds to a finite value of the bare charge. Bare charge of course means the asymptotic charge.

I am stopping on page 529. Authors go on to treat the use of the R-group to find behavior of electron prop near $p^2=m^2$. The result is (43.27)

Exercise: derive (42.15) of BS.

$$\odot \quad G_2 = z G_1 \Rightarrow a\left(\frac{k^2}{\lambda_2^2}, \frac{m^2}{\lambda_2^2}, e_2^2\right) = z a\left(\frac{k^2}{\lambda_1^2}, \frac{m^2}{\lambda_1^2}, e_1^2\right) \quad (*)$$

Let $k^2 = \lambda_1^2$ so

$$a\left(\frac{\lambda_1^2}{\lambda_2^2}, \frac{m^2}{\lambda_2^2}, e_2^2\right) = z \cdot \underbrace{a\left(1, \frac{m^2}{\lambda_1^2}, e_1^2\right)}_{= a_0 \neq 1}$$

$$\text{So } z = a(\lambda_1^2/\lambda_2^2, m^2/\lambda_2^2, e_2^2) / a(1, m^2/\lambda_1^2, e_1^2)$$

Then we have

← this used to get (42.18)

$$\left[a\left(\frac{k^2}{\lambda_2^2}, \frac{m^2}{\lambda_2^2}, e_2^2\right) a\left(1, \frac{m^2}{\lambda_1^2}, e_1^2\right) = a\left(\frac{\lambda_1^2}{\lambda_2^2}, \frac{m^2}{\lambda_2^2}, e_2^2\right) a\left(\frac{k^2}{\lambda_1^2}, \frac{m^2}{\lambda_1^2}, e_1^2\right) \right]$$

Does not yield their result. Go back to (*) and instead let $k^2 = \lambda_2^2$

$$\odot \quad a\left(1, \frac{m^2}{\lambda_2^2}, e_2^2\right) = z \cdot a\left(\frac{\lambda_2^2}{\lambda_1^2}, \frac{m^2}{\lambda_1^2}, e_1^2\right)$$

Now divide to cancel z.

$$\left[\frac{a\left(1, \frac{m^2}{\lambda_2^2}, e_2^2\right)}{a\left(\frac{k^2}{\lambda_2^2}, \frac{m^2}{\lambda_2^2}, e_2^2\right)} = \frac{a\left(\frac{\lambda_2^2}{\lambda_1^2}, \frac{m^2}{\lambda_1^2}, e_1^2\right)}{a\left(\frac{k^2}{\lambda_1^2}, \frac{m^2}{\lambda_1^2}, e_1^2\right)} \right] \quad \checkmark \quad \text{This is (42.15).}$$

← This is (42.15).

Exercise: derive (42.24).

Start w) (42.17) + do $\partial/\partial x$.

$$\xi = \frac{x}{t}$$

$$= \frac{1}{t}$$

$$\frac{d}{dx} [e^2 d(x, y, e^2)] = e^2 d(t, y, e^2) \frac{\partial \xi}{\partial x} \frac{\partial}{\partial \xi} d(\xi, \frac{y}{t}, e^2 d(t, y, e^2))$$

Question: in (42.17) I gather that we treat x, y, t, e all as independent variables, so

$$\frac{\partial e}{\partial x} = 0$$

Now set $t = x$:

$$\frac{\partial}{\partial x} [e^2 d(x, y, e^2)] = [e^2 d(x, y, e^2)] \cdot \frac{1}{x} \frac{\partial}{\partial \xi} d(\xi, \frac{y}{t}, e^2)$$

$$\times \left[\frac{\partial}{\partial \xi} d(\xi, \frac{y}{x}, e^2 d(x, y, e^2)) \right] \Big|_{\xi=1}$$

$$= \phi\left(\frac{y}{x}, e^2 d(x, y, e^2)\right)$$

Hence (42.24). Notice that $[e^2 d]$ always grouped together.

Now do (42.18).

$$\frac{\partial s(x, y, e^2)}{\partial x} s(1, \frac{y}{t}, e^2 d(t, y, e^2)) = s(t, y, e^2) \frac{\partial \xi}{\partial x} \frac{\partial}{\partial \xi} s(\xi, \frac{y}{t}, e^2 d(\dots))$$

Now set $t = x$ and $\div$:

$$\frac{\partial s(x, y, e^2)/\partial x}{s(x, y, e^2)} = \frac{1}{x} \frac{[\partial s(\xi, y/x, e^2 d)/\partial \xi]_{\xi=1}}{s(1, y/x, e^2 d)}$$

$$\Rightarrow \frac{\partial \ln s(x, y, e^2)}{\partial x} = \frac{1}{x} \left[\frac{\partial \ln s(\xi, y/x, e^2 d)}{\partial \xi} \right]_{\xi=1}$$

$$= \psi(y/x, e^2 d) \checkmark$$

Hence (42.25).

exercise: do (43.2)

$$\frac{d[e^2 d]}{[e^2 d]^2} = \left[\frac{df(x/y)}{dx} \right] dx$$

$$\Rightarrow -[e^2 d]^{-1} = \int dx \frac{d}{dx} f(x/y) + C.$$

$$\Rightarrow d^{-1} = -e^2 \int_1^x dx \frac{d}{dx} f(x/y) + C.$$

Recall that $d(x=1) = 1$, so we can say

$$d^{-1}(x, y, e^2) = -e^2 \int_1^x dx \frac{\partial}{\partial x} f\left(\frac{x}{y}\right) + 1 \quad \text{Hence (43.2)}$$

$f\left(\frac{x}{y}\right) - f\left(\frac{1}{y}\right).$

exercise: do (43.7)

$$\frac{d[e^2 d]}{\phi[e^2 d] [e^2 d]} = \frac{dx}{x} \Rightarrow \int \frac{dz}{z \phi(z)} = \ln x + C.$$

As $x \rightarrow 1$, $\ln x = 0$ and $d=1$, so

$$\int_{e^2}^{e^2 d(x, e^2)} \frac{dz}{z \phi(z)} = \ln x \quad / \text{both sides} = 0 \text{ when } x = 1.$$

Hence (43.7).

And let $e^2 d(z, e^2) = \tau$ in (43.4), so

$$\begin{aligned} \text{RHS} &= \int \left(\frac{dz}{d\tau} \right) \frac{d\tau}{z} \frac{\tau \phi(\tau)}{z} \rightarrow = \int \frac{d\tau}{\tau \phi(\tau)} \frac{\tau \phi(\tau)}{z} \\ \frac{dz}{d\tau} &= \text{LHS of (43.3)} = \frac{[e^2 d]}{x} \frac{d[e^2 d]}{d\tau} = \frac{\tau \phi(\tau)}{x} = \int_{\tau_0}^{\tau} \frac{d\tau' \tau' \phi(\tau')}{\tau \phi(\tau)} \end{aligned}$$

Hence (43.8)

exercise: do (43.10)

I will accept (43.9). Then (42.26) says

$$\phi(\eta^2) = \frac{\partial}{\partial g} d(g, e^2)$$

$$\phi(y, e^2) = \left[\frac{\partial}{\partial g} d(g, y, e^2) \right]^{j=1}$$

But, set $y=0$ and $e^2 \rightarrow e^2 d = \eta$ so

$$\phi(0, e^2 d = \eta) = \left[\frac{\partial}{\partial g} d(g, 0, e^2 d = \eta) \right]^{j=1}$$

$$\underline{a} \quad \phi(\eta) = \frac{\partial}{\partial g} (d(g, \eta))^{j=1}$$

$$\underline{\text{But}}, \quad d(g, \eta) \cong d_0(g, \eta) = 1 + \frac{\eta}{12\pi^2} \ln g + \frac{\eta^2}{16\pi^2} \left[\frac{1}{4\pi^2} \ln g + \frac{1}{9\pi^2} \ln^2 g \right]$$

$$\Rightarrow \phi(\eta) = \frac{\eta}{12\pi^2} + \frac{\eta^2}{16\pi^2} \left[\frac{1}{4\pi^2} \right] + \frac{1}{9\pi^2} \cdot \frac{2\eta \ln g}{g}$$

$$= \left(\frac{\eta}{12\pi^2} + \frac{\eta^2}{64\pi^4} \right) \quad \text{But assume } \eta = e^2 d = z \ll 1.$$

Hence (43.10).

Callan
1970

internal lines. Clearly the contribution of this particular singularity to the s -channel absorptive part is $T_{2 \rightarrow n} \times T_{n \rightarrow 2}$; that is, precisely the n -body intermediate state contribution to the unitarity relation.

Thus it seems that the required singularities are present; what we cannot yet show is that *only* these are present.

Finally, if the situation regarding unitarity can be satisfactorily cleared up, we have defined a theory with all the desired properties of a true bootstrap theory. We have a well-defined set of equations, (3) and (4), for a set of vertex functions and propagators. They incorporate crossing and analyticity, and if they have solu-

tions we have a prescription for calculating any n -legged amplitude. We then have here a set of exact bootstrap equations written in closed form, with which one can study questions of existence and uniqueness of solutions, and which present a basis for systematic approximations.

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Broken Scale Invariance in Scalar Field Theory*

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We use scalar-field perturbation theory as a laboratory to study broken scale invariance. We pay particular attention to scaling laws (Ward identities for the scale current) and find that they have unusual anomalies whose presence might have been guessed from renormalization-group arguments. The scaling laws also appear to provide a relatively simple way of computing the renormalized amplitudes of the theory, which sidesteps the overlapping-divergence problem.

INTRODUCTION

THE perturbation theory of a self-interacting scalar field is about the simplest available model field theory, and a convenient laboratory for testing new ideas in strong-interaction physics. In this paper we shall be concerned with studying the concept of broken scale invariance within such a framework. We shall find that the model calls for some unexpected modifications of our ideas on broken scale invariance. At the same time, the approach suggested by broken scale invariance leads to an interesting, and simple, new approach to renormalization. We hope that this mutual illumination of two interesting questions justifies yet another paper on scalar field theory.

In Sec. I we shall review the general properties of scale invariance as a broken symmetry, leading up to the idea of a scaling law (the analog for scale invariance of PCAC low-energy theorems). In Sec. II we shall see how the general structure of renormalized perturbation theory constrains the allowable form of the scaling law and forces it to differ from naive expectations. In Sec. III we shall show how the existence of the scaling law

leads to a simple prescription for computing the renormalized Green's functions of the theory. Finally, in Sec. IV, we shall demonstrate an interesting connection between the scaling law and the predictions of the renormalization group.

I. BROKEN SCALE INVARIANCE

In simple canonical field theories it is possible to introduce an acceptable energy-momentum tensor^{1,2} $\Theta_{\mu\nu}$ having the following properties: (a) $\Theta = \Theta_{\mu}^{\mu}$ is proportional to those terms in the Lagrangian having dimensional coupling constants (such as mass terms); (b) the charge, $D = \int d^3x S_0$, formed from the current $S_{\mu} = \Theta_{\mu}^{\nu} x_{\nu}$, acts as the generator of scale transformations,

$$[D(x_0), \phi(x)] = -i(d + x \cdot \partial)\phi(x), \quad (1)$$

where d is the dimension of the field; (c) the current S_{μ} satisfies $\partial^{\mu} S_{\mu} = \Theta$ so that it is conserved when there are no dimensional coupling constants in the Lagrangian. With the help of the current S_{μ} and its equal-time commutation relations with fields, given above, one is able

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† Alfred P. Sloan Foundation Fellow.

¹ C. G. Callan, Jr., S. Coleman, and R. Jackiw, *Ann. Phys. (N. Y.)* 59, 42 (1970).

² M. Gell-Mann, University of Hawaii Summer School lectures, 1969 (unpublished).

Summary of the Callan paper.

The so-called scale-Ward identities derived in earlier CCJ papers are stated in the first section. Recall that these state that: a certain differential operator S acting on a Greens function G^n yields F , where F involves the VEV of the fields along with Θ , the divergence of the dilatation current. If you have scale invariance, then this current is conserved, $\Theta=0$, and $F=0$. This formalism was invented to provide a framework for describing the breaking of scale invariance by mass terms in the lagrangian.

In the next section it is shown that something is wrong with these scale-ward identities because they predict something which cannot be true. Namely, you can show that consideration of $S G^n = F^n$ for the "basic n " of the theory (ie, for the renorm parts like $n=2,4$) requires there to be a $d/d\lambda$ term added to the operator S , a term which does not come out of the scale-Ward identities. This analysis requires that you simply pay attention to renormalization theory and how many times each renorm part is subtracted.

The result is this: even if you have naive scale invariance with no mass terms in your lagrangian, the scaling law reads $S G^n=0$ where S must contain that extra term. I think this equation $S G^n=0$ is later called the Callan Symanzik equation. The extra term is sometimes called "implicit scale breaking", nice that it can be incorporated so simply.

A less historically significant part of this paper says that there is an iterative method of computing the G^n which allows you to avoid all overlap problems. But the main result of this paper and the reason it is famous is the statement of that simple scaling law. This law implies that the large momenta behavior of a greens function (ie where masses don't matter) is somehow correlated to dependence of G^n on the coupling constant λ . It is noted also that the scaling law is not rescued merely by allowing the scale dimension of the fields to go anomalous. You really need that extra term in there.

From my present viewpoint, I think the origin of this extra term is clearly explained in the Abarbanel review.

I. Broken Scale Invariance The Ward identities of earlier CC papers are combined into Eq. (2), or Eq. (3) for proper vertex parts. These are "Scale Wards". Another version is (5).

Callan is going to show that these equations are all wrong for some reason. I think (5) will be wrong in two ways: (a) it will turn out that $\delta \neq 1$ as you would suspect; (b) there will appear a new $\partial/\partial\lambda$ term inside the LHS.

II. General Constraints on Scaling Law, ϕ^4 . Expect $\phi = \phi^4 = \mu^2 \phi^2 =$ cause of scale breakage. Now, by computing D for various objects, you learn how much freedom you have, i.e., you learn of your ability to select renorm points. We find:

$$\bar{G}^{(2)} = \text{---} \text{---} \text{---}$$

$$D=2$$

$$\bar{G}^{(2)}(0) = -i\mu^2$$

$$\bar{G}^{(2)}(0) = iZ$$

$$\bar{G}^{(4)} = \text{---} \text{---} \text{---} \text{---}$$

$$D=0$$

$$\bar{G}^{(4)}(0) = -i\lambda$$

$$\bar{F}^{(2)} = \text{---} \text{---} \text{---}$$

$$D=0$$

$$\bar{F}^{(2)}(0) = 2m^2$$

Eg, since $\bar{G}^{(4)}(p_1, \dots, p_4)$ has $D=0$, you know you must do a subtraction to remove divergent part. You can decide to subtract at $p_i \equiv 0$. Thus, the value of $\bar{G}^{(4)}(0,0,0,0)$ can be whatever you want.

Remember that

$$\bar{G}^{(2)}(p) \equiv [G^{(2)}(p)]^{-1}$$

If $p=0$ were a physical point, perhaps μ^2 and λ would be the observed mass and coupling. Anyway, λ is some #.

When we do something, we will subtract, so λ is not going to affect any calculation.

OK, onward. Next it is presumed that any higher $G^{(n)}$ or $F^{(n)}$ can be written as a skeleton expansion where meat is $G^{(2)}$, $G^{(4)}$ etc. Thus, in effect, $G^{(n)}$ is determined by $G^{(2)}$, $G^{(4)}$, and topology.

Now we have an operator S

$$S \bar{G}^{(n)} = -i \bar{F}^{(n)}$$

$$\text{Eq. (5)}$$

$$S = \mu \partial/\partial \mu + n \delta$$

Now here is a Theoremlet: let S be any "distributive" operator (like $\partial/\partial\mu$ is a very derivative). Then

$$S G^{(n)} = -i F^{(n)} \quad \text{for } n=2,3,4 \Rightarrow \text{true for all } n.$$

To prove this, you just pick a skeleton term in $G^{(n)}$, like Fig 1 left. Apply S and you then get a sum of "insertions" as shown. The graphs so made are contained in $F^{(n)}$ of course. The theorem means you just check for exhaustion.

Now $S = \mu \partial/\partial\mu + n\delta$ is distributive.

But so also is $S = \mu \partial/\partial\mu + n\delta + f(\lambda) \partial/\partial\lambda$.

So such a term could be there, but why would you want it? Let's go back to $n \leq 4$ cases.

$$S \bar{G}^{(2)}(0) = -i \bar{F}^{(2)}(0) \quad \text{says} \quad [\mu \partial/\partial\mu + n\delta + f \partial/\partial\lambda] (-i\mu^2) = -i \cdot 2m^2$$

$$\Rightarrow -i 2\mu^2 + n\delta(-i\mu^2) = -i 2m^2 \Rightarrow (1 + \overset{n=2}{\delta}) \mu^2 = m^2$$

Pause: why does $\bar{G}^{(2)}(0)$ involve the same μ as appears in Eq. (5)?

Well, this is the only dimensional parameter. I.e., once you set $\bar{G}^{(2)}(0) = -i\mu^2$, the theory is a form of μ^2 . In fact, everything is form μ, λ — these are the parameters like m, e of QED.

$$\text{Next: apply } \frac{\partial}{\partial\lambda} \Rightarrow S \bar{G}^{(3)}(0) = -i \bar{F}^{(3)}(0) \Rightarrow$$

$$[\mu \frac{\partial}{\partial\mu} + 3\delta + f \frac{\partial}{\partial\lambda}] i 2\lambda = -i \bar{F}^{(3)}(0)'$$

$$\Rightarrow \underbrace{\mu \frac{\partial}{\partial\mu}}_{\text{why } = 0?} + \underbrace{3\delta + f \frac{\partial}{\partial\lambda}}_{\equiv \alpha} = \uparrow \text{ with } i/$$

Guess $D = \text{dimension} \Rightarrow D(\mu, \lambda) = D(\lambda)$ only, so $\partial D/\partial\mu = 0$.

$$\text{Next: } [\mu \frac{\partial}{\partial\mu} + 4\delta + f \frac{\partial}{\partial\lambda}] (-i\lambda) = -i \bar{F}^{(4)}(0).$$

$$\Rightarrow \underbrace{4\delta\lambda + f}_{\equiv \beta} = \bar{F}^{(4)}$$

As noted, $\alpha = \alpha(h, \mu)$ same for β . So:

$$(1+\delta)\mu^2 = m^2$$

$$f \frac{\partial^2}{\partial \lambda^2} + 2\delta = \alpha$$

$$4\delta\lambda + f = \beta$$

What about $\frac{\partial^2}{\partial \lambda^2}$. You might ignore it, not sure why... Then

$$\alpha = 2\delta \Rightarrow f = \beta - 2\alpha\lambda = \mathcal{O}(\lambda^2) \neq 0.$$

I.e., if you go off + compute α, β in pert. theory you get something and you find that $f \neq 0$.

Thus, your theory for $n \leq 4$ is inconsistent with $\mathcal{O}(\lambda^4)$ perturbation theory unless you include the $f \frac{\partial^2}{\partial \lambda^2}$ term in S . I.e., the Ward-Scale identities must be wrong!

Also, notice that $\delta = \frac{1}{2}\alpha =$ some calculation, $\neq 1$. So not only is Ward-scale wrong, but $\delta \neq 1$ either, even with correction $f \frac{\partial^2}{\partial \lambda^2}$ term.

These conclusions are very similar to Coleman-Jackiw who show "anomalies" with their scale-Wards which even $\delta \neq 1$ cannot repair!

III. Computing Basic Green's Fctrs. Lets try to use $SG^{(n)} = F^{(n)}$ with $\frac{\partial}{\partial \lambda}$ in there to compute the full, non-pert theory, $G^{(n)}$. We have:

$$\mu \frac{\partial}{\partial \mu} \bar{G}^{(4)} = - \left[4\delta + f \frac{\partial^2}{\partial \lambda^2} \right] \bar{G}^{(4)} - i \bar{F}^{(4)}$$

$= \Phi(p; \bar{G}^{(2)}, \bar{G}^{(4)}, \bar{F}^{(2)})$ by skeleton idea.

Well, I will summarize: using the little differential equation, Callan can integrate to get an iterative scheme allowing him to compute $G^{(n)}$ to ever-increasing order. The resulting $G^{(n)}$ must be consistent with "the scaling law" (since this "law" is used to construct them. Callan then tries to show through analytic structure that the $G^{(n)}$ so generated really are the true $G^{(n)}$! I am doubtful ~ this.

IV. Interpretation + Connection to the Renorm Group. Suppose you get rid of your mass terms (a go to HE). You would expect to get "naive scaling" albeit with possibly anomalous dimension δ . I.e., $F=0$ and expect $[\mu \frac{\partial}{\partial \mu} + n\delta]G = 0$. But this ain't what happens!

that extra $2/d$ is really there. Somehow the $\lambda\phi^4$ term is itself leaving $d=4$ and causing scale-breaking.

The "scaling law" correlates dependence on complex λ and on momenta p . The seed of AF is present I think. But there is no Gellmann-Low reference here.

Conclusions

- (1) It is nice that the "implicit" scale-breaking due to $\lambda\phi^4$ "drifting" away from $d=4$ (due to interaction) can be stated simply in the scaling law $SG^{(n)} = 0$ with S as shown in (6).
- (2) This law can be used to "do" renormalization, avoiding the overlap difficulties.

Derive Equation (2) using scale invariance

Show that (2) is sensible

$$\bullet \left[\cancel{n+4} - \sum_{i=1}^{(n-1)} p_i \frac{\partial}{\partial p_i} \right] \mathcal{G}^{(n)}(p_1 \dots p_{n-1}) = 0 \quad \text{if scale invar.}$$

But I know that scale invariance also says

$$\mathcal{G}^{(n)}(p_1 \dots p_{n-1}) = (\lambda)^{n(d-4)} \tilde{\mathcal{G}}^{(n)}(p_1/\lambda, \dots, p_{n-1}/\lambda)$$

However, let's now pull the delta:

$$\begin{aligned} (2\pi)^4 \delta^4(p_1 + \dots + p_n) \mathcal{G}_1^{(n)}(p_1 \dots p_{n-1}) \\ = (\lambda)^{n(d-4)} (2\pi)^4 \delta^4\left(\frac{p_1}{\lambda} + \dots + \frac{p_n}{\lambda}\right) \mathcal{G}^{(n)}\left(\frac{p_1}{\lambda} \dots \frac{p_{n-1}}{\lambda}\right) \\ = \lambda^4 \delta^4(p_1 \dots p_n) \end{aligned}$$

Thus,

$$\mathcal{G}^{(n)}(p_1, \dots, p_{n-1}) = [\lambda]^{n(d-4)+4} \mathcal{G}^{(n)}\left(\frac{p_1}{\lambda}, \frac{p_2}{\lambda}, \dots, \frac{p_{n-1}}{\lambda}\right)$$

↑ trivial, given scale invariance
[see dilation notes].

Let $\boxed{D = n(d-4) + 4}$ + rewrite this as $(p_i \rightarrow \lambda p_i)$:

$$\mathcal{G}^{(n)}(p_1, p_2, \dots, p_{n-1}) = \lambda^{-D} \mathcal{G}^{(n)}(\lambda p_1, \lambda p_2, \dots, \lambda p_{n-1})$$

Now apply $\frac{d}{d\lambda}$ to both sides. On left get 0. On right get

$$\underbrace{-D \lambda^{-D-1} \mathcal{G}^{(n)}(\lambda p_1, \dots, \lambda p_{n-1})}_{\lambda^D \mathcal{G}^{(n)}(p_1 \dots p_{n-1})} + \underbrace{(\lambda^{-D}) \frac{d}{d\lambda} \mathcal{G}^{(n)}(\lambda p_1, \dots, \lambda p_{n-1})}_{?}$$

→

Let's compute

$$\frac{d}{d\lambda} G(\lambda p_1, \lambda p_2, \dots, \lambda p_{n-1})$$

$$f_i \equiv \lambda p_i$$

$$= \frac{\partial G(f_1, \dots, f_{n-1})}{\partial f_1} \left(\frac{df_1}{d\lambda} \right) + \dots + \frac{\partial G(f_1, \dots, f_{n-1})}{\partial f_{n-1}} \frac{df_{n-1}}{d\lambda}$$

$$= \sum_{i=1}^{n-1} \frac{\partial G(f_1, \dots, f_{n-1})}{\partial f_i} \Big|_{f_i = \lambda p_i} \cdot p_i$$

But $df_i = \lambda dp_i$ so can say

$$= \sum_{i=1}^{n-1} \frac{1}{\lambda} p_i \frac{\partial}{\partial p_i} G(p_1 \lambda, p_2 \lambda, \dots, p_{n-1} \lambda)$$

Thus we have proven that

$$0 = -D \lambda^{-D-1} \mathcal{G}^{(n)}(p_1 \lambda, p_2 \lambda, \dots, p_{n-1} \lambda) + \lambda^{-D-1} \sum_{i=1}^{n-1} p_i \frac{\partial}{\partial p_i} G(p_1 \lambda, \dots, p_{n-1} \lambda).$$

Ergo;

$$\left[\sum_{i=1}^{n-1} p_i \frac{\partial}{\partial p_i} - D \right] \mathcal{G}^{(n)}(p_1 \lambda, p_2 \lambda, \dots, p_{n-1} \lambda) = 0.$$

Now set $\lambda = 1$ and you have derived Callan #2 when $\exists$ scale invariance.

Derivation of Callan's Eq. (2). [in general]

Go back to the CCJ paper + try to combine (2.13) with (2.11).

First, do $\frac{\partial}{\partial k^\lambda} \Big|_k = 0$ on (2.11) as I did before to get

$$i\tilde{\Gamma}^{(n)}(0; p_1, p_2, \dots, p_{n-1}) = \left[(n-1)g^{\mu\nu} + \sum_{i=1}^{n-1} (p_i)^\mu \left(\frac{\partial}{\partial p_i^\nu} \right)' \right] G(p_1, \dots, p_{n-1})$$

Now take the trace + replace

$$i\Gamma^{(n)}(0; p_1, \dots, p_{n-1}) = i\tilde{\Gamma}^{(n)}(0; p_1, \dots, p_{n-1}) + \underbrace{\sum_{i=1}^n G(p_1, \dots, p_{n-1})}_{= n \cdot G(p_1, \dots, p_{n-1})}$$

Thus we get

$$i\Gamma(0; p_1, \dots, p_{n-1}) + n G(p_1, \dots, p_{n-1}) = \left[4(n-1) + \sum_{i=1}^{n-1} (p_i)^\mu \partial_\mu^{(p_i)} \right] G.$$

Finally,

$$i\Gamma(0; p_1, \dots) = \left[3n-4 + \sum_{i=1}^{n-1} p_i \cdot \frac{\partial}{\partial p_i} \right] G(p_1, \dots, p_{n-1}).$$

Negate:

"S"

$$\boxed{\left[-3n+4 - \sum_{i=1}^{n-1} p_i^\mu \left(\frac{\partial}{\partial p_i^\mu} \right) \right] G^{(n)}(p_1, \dots, p_{n-1}) = -i\Gamma^{(n)}(0; p_1, \dots, p_{n-1})} \quad (2)$$

Finally we have Callan Eq.'s (2)+(3)! Only thing is now he has done it for general field dimension d . I can go check App 2 of CCJ on this, but I have already blown too much time on this Ward stuff, so accept (2) with d in there.

● If you do have scale invariance, then equation says: $SG=0$

Callan (5) (just did One).

$$\mu \frac{d}{d\mu} \left[\mu^d \Phi \left(\frac{p_1}{\mu}, \dots, \frac{p_{n-1}}{\mu} \right) \right] \quad \zeta_i = \frac{p_i}{\mu}$$

$$= d \mu^d \Phi \left(\frac{p_1}{\mu}, \dots \right) + \mu^{d+1} \cdot \left[\frac{\partial \Phi(\zeta_1, \dots, \zeta_{n-1})}{\partial \zeta_i} \cdot \frac{d\zeta_i}{d\mu} \right]$$

but $\frac{d\zeta_i}{d\mu} = \frac{d}{d\mu} (p_i \mu^{-1}) = -p_i \mu^{-2}$

$$\partial \zeta_i = \frac{1}{\mu} \partial p_i \Rightarrow \left[\frac{\partial \Phi}{\frac{1}{\mu} \partial p_i} \cdot \left[\frac{-p_i}{\mu^2} \right] \right]$$

$$\Rightarrow = \left[d \Phi \left(\frac{p_1}{\mu} \right) - \sum p_i \frac{\partial}{\partial p_i} \Phi \left(\frac{p_1}{\mu}, \frac{p_2}{\mu}, \dots \right) \right] \cdot \mu^d.$$

Thus I get:

$$d = 4 - n$$

$$\left\{ \begin{aligned} \mu \frac{d}{d\mu} \bar{G}(p_1, \dots, p_{n-1}) &= \mu^d \left[d - \sum p_i \frac{\partial}{\partial p_i} \right] \underbrace{\Phi \left(\frac{p_1}{\mu}, \frac{p_2}{\mu}, \dots, \frac{p_{n-1}}{\mu} \right)}_{\mu^{-d} \bar{G}(p_1, \dots, p_{n-1})} \\ &= \left[4 - n - \sum p_i \frac{\partial}{\partial p_i} \right] \bar{G}(p_1, \dots, p_{n-1}) \end{aligned} \right.$$

So make a new (4):

$$\left[4 - nd + \mu \frac{\partial}{\partial \mu} - \chi + n \right] \bar{G} = -i \bar{F}$$

$$\Rightarrow \boxed{\left[\mu \frac{\partial}{\partial \mu} + n \underbrace{(1-d)}_{\delta} \right] \bar{G} = -i \bar{F}}$$

Symnzik
1969

Renormalizable Models with Simple Symmetry Breaking

I. Symmetry Breaking by a Source Term

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Abstract. If to a Lagrangian density with invariance under a continuous group of linear transformations of the fields a term linear or bilinear in the fields is added, the symmetry is in general reduced and the currents associated with the original symmetry are only partially conserved. If the theory without the added term is renormalizable, the theory with that term also is, and the needed renormalization conditions are the essential content of the appropriate Ward-Takahashi-Kazes-Rivers identities. The case of symmetry breaking by a term linear in Bose fields (source term) is here analysed completely, in particular with respect to the nonsymmetric limit of vanishing source term, a particular Goldstone mode, and with respect to properties of the ground state energy density as a function of the strength of the source term. Induced and spontaneous breaking of a discrete symmetry are also treated.

Introduction

B. W. Lee [1] has discussed the sigma model [2, 3] from the point of view of renormalized perturbation theory, in order to have available a model that satisfies PCAC¹ and allows to calculate in a formal but consistent way the amplitudes for processes involving nonsoft pions.

We shall show here² that for such models the renormalized perturbation expansions can be very simply obtained if the relations, stemming from PCAC, between vertex functions of different numbers of arguments are exploited. These relations yield all the "renormalization conditions" required in Bogoliubov-Parasiuk-Hepp (BPH)³ renormalization theory in terms of only that many parameters as the unrenormalized Lagrangian has. This technique also covers the Goldstone mode⁴ obtained in the limit of vanishing source but, since it deals with renormalized quantities only,

¹ Ref. [4] discusses the sigma model and related models from the point of view of applications to pion physics.

² A short account was given in Ref. [5].

³ See Ref. [6] and references given there.

⁴ Ref. [7] gives a comprehensive presentation of the relevant material.

allows no direct conclusion concerning e.g. whether, switching off the source term while leaving the rest of the Lagrangian unchanged, the ground state for the latter would be the usual symmetric or the non-symmetric Goldstone one, although this question is meaningful.

The Goldstone situation can in an intuitively appealing way be illuminated in terms of the behaviour of the ground state energy density as a function of the source strength. The discussion hereto, familiar in the classical case, carries over with few changes to quantum field theory, whereby, however, a formal similarity to the theory of condensation of Yang and Lee [8] is noted.

In Section I, the well-known one-particle structure of Green's functions is presented concisely. In Section II, for Green's functions involving current operators that have simple commutators with the fields, Ward-Takahashi-Kazes- and Rivers-type identities are derived. In Section III, the formulas of the first two sections are written for the special case of a symmetric Lagrangian density with added term linear in Bose fields. These relations are used in Section IV to obtain the BPH renormalization conditions in terms of only the number of parameters that appear in the unrenormalized Lagrangian. The same is done, with some necessary precaution to avoid spurious infrared divergences, in Section V for the associated Goldstone mode, i.e. the limit theory with vanishing source and spontaneously broken symmetry, which may also be described directly in terms of a manifestly nonsymmetric Lagrangian. In Section VI, the relation between the theories with and without symmetry breaking source term is discussed and a comparison is made with the breaking of a discrete symmetry. The appendix contains the discussion of the properties of the ground state energy as a function of the source strength, which are obtained using results of Euclidean quantum field theory.

The calculation of the Green's functions involving a current operator will, because of the technique needed hereby, be included in the sequel paper, which deals with symmetry breaking by a term bilinear in fields.

I. One-Particle Structure of Green's Functions

We wish to consider the Poincaré-invariant theory of a multi-component local hermitean field $A(x)$ described by the Lagrangian density

$$L = L(A, \partial A). \quad (I.1)$$

To this end we consider, following Schwinger [9], the related theory described by the Lagrangian density

$$L' = L(A', \partial A') + J A' \quad (I.2)$$

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stability - r=0

Summary of the Symanzki Sigma Model paper.

Probably I should not have read this thing in as much detail as I did, but it was filled with interesting manipulations. My motivation for reading this paper was simply that I was reading the source papers for Callan Symanzik equations, and the Symanzik paper said it was taking its notation from this Symanzik paper.

This paper is mainly interested in the subject of renormalization in the presence of SSB, something I do not want to get into now. The tool Sym. uses in this discussion is the scale Ward identities like those general ones (for any conserved current) that I saw derived in the CCJ paper. These Ward identities are statements about the G^n greens functions and their relation to VEVTOP with divergence of the conserved current inside, ie, the same type of equation Callan used in his paper in the special case of scale symmetry.

So I don't really know how this tool is used, but I was interested to see how you can restate the Ward identities in terms of first the generating functional $G[J]$, and also in terms of the proper vertex generating functional $\Gamma[A]$. These transformed statements of the Ward identities get fancy names like Kazes-Rivers identities.

The connection of this tool to its work is, I think, that the sigma type models break their invariance by a term linear in the field (you can think either of scale breaking somehow, but mainly breaking of CAC to get PCAC). This linear "sigma" term looks just like a source term, and this source term you recall is the basis of the functional approach. So somehow the functional approach gets used to show renormalizability.

I think that as a spinoff this paper led me to a simple graphical language for the functional formalism, in particular, how to relate proper vertices to full vertices and so on, d/dJ is a little arrow strung to a external particle ball, and so on.

Introduction. Recall from the 55 paper that any broken symmetry leads to Ward identities — eg, they were concerned with $\partial_\mu D^\mu = 0$ for dilatations. Obviously $\exists$ a similar Ward identity for PCAC, ie, $\partial_\mu T_{\mu 5} \propto \pi q$. This paper will deal with a σ -model type realization of PCAC, including a possible goldstone mode. The Ward-like identities will be useful for something.

I. Notation. The basic stuff is set up $G[\phi] = \text{my } W[\phi]$, $\ln G[\phi] = \text{my } Z[\phi]$, $G = \text{my } G$, $G_{\text{disc}} = \text{my } G$. $G(x, y, z) \Rightarrow$ one $G^{(2)}$ has been amputated off. For some reason $\phi(x) = \pi(x)$ always appears as $A(x) + G(x)$.

G_{xyz} means $\delta^3 Z / \delta \phi(x) \delta \phi(y) \delta \phi(z)$

Γ_{xy} means $\delta^2 \Gamma / \delta \phi(x) \delta \phi(y)$.

It is noted that there are no poles in $\Gamma^{(n)}$ loops.

OK, I see that $G_i(x) = \text{one } \pi$ of Abus + Lee — want to keep it because this paper will consider SSB!

II. Ward-Takahashi - Kadan - Rivers Identities. I have worked thru this in nearly full detail. Things are set up as in CJ's Ward appendix, here we have Ward (5) for any current j_μ^α . This Ward identity is restated just in terms of $W[\phi]$, then in terms of $\Gamma[\phi]$, though of course not exactly. Results are

(II.10) in terms of $G[\phi]$ objects "Ward-Tak-Kadan"

(II.15) in terms of $\Gamma[A]$ objects "Rivers".

This section was a rather thorough derivation of these identities. We have not specialized yet to PCAC or the like, fully general for any current.

III. The Setup for a σ -model type Lagrangian. In linear σ model, you add a linear term $KA(x)$ to an otherwise scalar Lagrangian, so this term does the $\uparrow$ symmetry breaking. You can in fact show that $\partial_\mu j^\mu \propto K$.

But, $L = L_0 + KA$, K acts very much like the source T , although K is really there + T gets set to zero. What this section does is restates everything in presence of such a linear breaking term. In particular the Wards are restated.

Have not done this section in detail, just a scan.

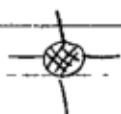
IV. A ϕ -model and renormalization.

Start with a Lag. (4.1) which would be $O(N+1)$ invariant, but the linear term ϕA_0 breaks N generators down to $O(N)$. Thus, you will get $(N/2)$ conserved currents, and N broken (PC) ones as shown.

In Eq. (4.2) we have the usual renormalization equation as in B+S RG chapter.

Now, let's scan for the "renormalization points." I think you should shift fields to get rid of tadpoles, this causes 3-vertices as well as 4's. Thus,

$$D = -N_3 - E_4 + 4 \text{ as in (4.10c).}$$



$$D = 0$$



$$D = 2$$



$$D = 0$$

Somehow Sym. uses the fancy Ward's to "set up" the renormalization points. I.e., in ϕ_0, g_0, c_0 , + you want to see m, g, c in a "renormalized" perturbation expansion.

I guess these "Ward's" play the role of the integral equations in RD's proof that QED is renormalizable.

This section attacks the real bitch of renormalization theory: how to prove renormalizability in the presence of SSB. In 1969 I don't think SSB renorm. was well understood. I will postpone this business to later.

Comment: I suspect Symanzik is one of those ultra-careful dudes. This paper is rather dense with tough formulas; all followable, but the conclusions are not clearly stated.

About $G(x) = v$.

Look at (I.7) which says $G[J] = i \int dx_1 J(x_1) G^{(1)}(x_1) + \text{etc.}$
I do not see any "J-independent terms in here." Now,

$$G^{(1)}(x_1) = \text{diagram: a circle with a dot at } x_1 \text{ and a shaded circle} = \text{tadpole} = \langle 0 | \phi(x_1) | 0 \rangle.$$

When you have SSB, this can be non-zero. So

$$\frac{\delta G[J]}{\delta J(x)} = i G^{(1)}(x) + J\text{-dependent}$$

$$\Rightarrow -i G_x[J] = G^{(1)}(x) + J\text{-dependent.}$$

So this is what $G(x)$ is in this paper.

By translation, this is also v_i of Abus+Lee.

So compare:

$$\begin{array}{ccc} -i G_x[J] - G(x) & \equiv & A(x) & / \text{sym.} \\ \updownarrow & & \updownarrow & \\ \phi(x) - v & \equiv & \bar{\phi}(x) & / \text{abus+lee.} \end{array}$$

② Later in (II.12a,b) you see

$$G_\alpha^\mu(x, J) = G_\alpha^\mu(x) + J\text{-dep} \quad / \text{says (II.7a) definition}$$

$$\langle 0 | j_\alpha^\mu(x) | 0 \rangle = \langle 0 | j_\alpha^\mu(0) | 0 \rangle = V_\alpha^\mu \equiv 0 \text{ because no available } 4 \text{ vec.}$$

Similarly,

$$G_\alpha(x) = \langle 0 | \partial_\mu j_\alpha^\mu | 0 \rangle = \partial_\mu \{ \langle 0 | j_\alpha^\mu(0) | 0 \rangle \} = 0.$$

So these terms appearing in (II.12) can be ignored.

So

$$\boxed{G_\alpha^\mu(x) = G_\alpha(x) = 0}$$

Music. Equations

① Consider (1.9).

$$\mathcal{J}(x) = - \frac{\delta \Gamma[A]}{\delta A(x)} = -i \int dy \tilde{G}(x,y) A(y) + \underbrace{\frac{1}{2} i \iint dy dz G(x,y,z) A(y) A(z) + \dots}_{\text{remainder} = \Gamma'\{A\}}.$$

To show correct, take deriv

$$\left. \frac{\delta \mathcal{J}(x)}{\delta A(y)} \right|_{A=0} = \left. - \frac{\delta^2 \Gamma}{\delta A(x) \delta A(y)} \right|_{A=0} = -\Gamma'(x,y) = -i \tilde{G}(x,y), \text{ OK mod } i's.$$

$$\frac{\delta^2 \mathcal{J}}{\delta A(y) \delta A(z)} = \frac{-\delta^3 \Gamma}{\delta^3 x y z} = -\Gamma'_{xyz} = G(x,y,z) \checkmark$$

Note that $\Gamma'_{xyz} \neq G(x,y,z)$ however because $G(x_i)$ contains pole terms.

Hence (1.9). Hence also (1.16b).

② I.e., (1.9) is just the first derivative of the $\Gamma[A]$ expansion, and big deal.

③ (1.16a) is derivative of the usual $\{G\{\mathcal{J}\}\}$ exp.

$$G\{\mathcal{J}\} = \int dx_1 G(x_1) \mathcal{J}(x_1) + \int dx_1 dx_2 G(x_1, x_2) \mathcal{J}(x_1) \mathcal{J}(x_2) + \dots$$

$$\Rightarrow G_{x_1}\{\mathcal{J}\} = G(x_1) + \int dx_2 G(x_1, x_2) \mathcal{J}(x_2) + \dots \quad \leftarrow \text{span style="border: 1px solid black; padding: 2px;">(1.16a).$$

So $G'_x\{\mathcal{J}\}$ and $\Gamma'_x\{A\}$ involve $n=3$ and higher.

↑
remainder called $G'_x\{\mathcal{J}\}$

Functional Notation for Wards.

① I certainly agree with (II.5) with $\Delta=0 \sim$ a special case of my master theorem in which $B_{ii} = \tau \cdot \phi_i$

② Look at (II.8). If you take $\partial^2 / \partial J(y_1) \dots \partial J(y_n)$ on this + set $J=0$, you clearly obtain the obvious terms of (II.5).

As for the last term:

$$\frac{\delta^n}{\delta J(y_1) \dots \delta J(y_n)} \left[J(x) \underbrace{\frac{\delta \mathcal{L}[J]}{\delta J(x)}}_{\uparrow} \right] = \sum_{i=1}^n \delta(x-y_i) \frac{\delta^n \mathcal{L}}{\delta J(y_1) \dots \delta J(y_{i-1}) \delta J(y_{i+1}) \dots \delta J(y_n)}$$

As usual, all other terms in this huge derivative will vanish when $J \rightarrow 0$.

Thus $\boxed{\text{(II.5)} + \text{(II.8)}}$

Understanding (III.2).

① More graphical notation (see "proper vertex" in gen. F.Th. notes)

$$G_x\{J\} = \Sigma_x[J] = \frac{\delta^2 Z[J]}{\delta J(x)} = \text{diagram: } x \text{ --- } \bigcirc \text{ with lines}$$

The lines without circles are waiting to be acted on with $\delta/\delta J$'s.
For comparison ~ well, maybe this

$$\frac{\delta^2 Z[J]}{\delta J(x)} = x \text{ --- } \bigcirc$$

Then $\delta/\delta J(y) = \text{diagram: } \bigcirc \text{ with line } y \rightarrow$ comes along + hits. To show $G^{(1)}(x)$ I would use same notation, context hopefully will tell about $J=0$ or not.

② Next, $G_x^\mu(x, y) = \text{diagram: } x \text{ --- } \bigcirc \text{ --- } y \text{ or } \text{diagram: } x \text{ --- } \bigcirc \text{ --- } y$

I.e., $\text{---} \bigcirc$ indicates that y_x^μ is replacing ϕ for this point.

Now write $\text{diagram: } x \text{ --- } \bigcirc \text{ --- } y = \text{diagram: } x \text{ --- } \bigcirc \text{ --- } \bigcirc \text{ --- } y = G_x^\mu(x, y)$

Then $G_x^\mu(x, y) = \text{diagram: } x \text{ --- } \bigcirc \text{ --- } y$. We are just removing selfenergies on the line $\text{---} y$ (but not on Π^x).

③ Now: $\int dy G_x^\mu(x, y) G_y\{J\} =$

This represents all graphs that are 1PR in sense shown.

One

$$\text{diagram: } x \text{ --- } \bigcirc = \text{diagram: } x \text{ --- } \bigcirc + \left(\text{1PR w/ } x, \right) \leftarrow \text{insert } G_x^{(i)}$$

How's about (11.13).

① You wonder why no source term. Look at myself:

$$Z[J] = P[\phi] + \int J\phi.$$

$$\text{diagram} = \text{diagram} + \int J\phi.$$

Apply $\delta/\delta\phi(x)$ and make use of:

Σ

$$\text{diagram} = \text{diagram} + \text{diagram} + \text{diagram}$$

These cancel, so $\Rightarrow$

$$P_1(x) = \frac{\delta P[\phi]}{\delta \phi(x)} = -J(x)$$

$$\text{diagram} = - \text{diagram}$$

$$\text{diagram} = \text{diagram}$$

② Σ_0 , the $\int J\phi$ term makes two terms: the one term cancels $\otimes$ and the other gives you what you want.

② But now consider

$$\Sigma^{\mu,i}[J] = P^{\mu}[J]$$

$$\text{diagram} = \text{diagram}$$

Apply deriv:

$$\text{diagram} = \text{diagram} + \text{nothing else wanted!}$$

i.e., this is desired result.

Generalize:

$$\text{diagram} =$$

$$\text{diagram}$$

+ express as usual.

, again what you want.

(A) Look back:

$$0 = \text{diagram} + \text{diagram} \quad \text{since} \quad \text{diagram} + \text{diagram} = 0.$$

Inject $\mathcal{S}/\mathcal{S}$:

$$0 = \text{diagram} + \text{diagram} + \text{diagram} + \text{diagram}$$

↑ cancel

$$\Rightarrow 0 = \text{diagram} + \text{diagram} \quad \checkmark$$

Now inject on this + you are all set as shown earlier.

(B) Suppose the $\mathcal{S}\phi$ were missing. Then

$$\text{diagram} = \text{diagram} \Rightarrow \text{diagram} = \text{diagram}$$

$$\Rightarrow \text{diagram} = \text{diagram} + \text{diagram} = -\text{diagram} + \text{diagram}$$

And this is wrong. But with $\times$ or, it works.

I.e., diagram must be cancelled + replaced by diagram , hence $\mathcal{S}\phi$.

But diagram stays just as it is, so no $\mathcal{S}\phi$.

Pinner's Identities

From (II.12)

$$G_\alpha^\mu(x, \underline{y}) = \int dy G_\alpha^\mu(x, \underline{y}) A(y) + G_\alpha^{\mu i}(x, \underline{y}) \quad (\text{II.12a}) \quad (1)$$

$$G_\alpha(x, \underline{y}) = \int dy G_\alpha(x, \underline{y}) A(y) + G_\alpha^i(x, \underline{y}) \quad (\text{II.12b}) \quad (2)$$

Take ∂_μ^* on first.

$$\partial_\mu^* G_\alpha^\mu(x, \underline{y}) = \int dy \partial_\mu^* G_\alpha^\mu(x, \underline{y}) A(y) + \partial_\mu G_\alpha^{\mu i}(x, \underline{y}) \quad (3)$$

Now rewrite (II.10),

$$\partial_\mu G_\alpha^\mu(x, \underline{y}) = G_\alpha(x, \underline{y}) - \Gamma_x[A] \tau_\alpha i [A(x) + G(x)] \quad (4)$$

Combine (1), (3), (4):

$$\int dy \partial_\mu^* G_\alpha^\mu(x, \underline{y}) A(y) + \overbrace{\partial_\mu^* G_\alpha^{\mu i}(x, \underline{y})}^{\Gamma_\alpha^\mu(x, A)} = \int dy G_\alpha(x, \underline{y}) A(y) + \overbrace{G_\alpha^i(x, \underline{y})}^{\Gamma_\alpha(x, A)} - i [A(x) + G(x)] \tau_\alpha^T \Gamma_x[A]. \quad (5)'$$

Now use (I.10b) to extract the A -linear piece of Γ_x :

$$i \Gamma_x[A] = i \Gamma'_x[A] - i \int dy \tilde{G}(xy) A(y) \quad \checkmark$$

Then:

$$\begin{aligned} & \int dy [\partial_\mu^* G_\alpha^\mu(x, \underline{y}) - G_\alpha(x, \underline{y}) - \tilde{G}(xy)] A(y) \\ &= \Gamma_\alpha(x, A) - \partial_\mu^* \Gamma_\alpha^\mu(x, A) - i A(x) \tau_\alpha^T \Gamma'_x[A] - \end{aligned}$$

Pause. rewrite last term of (5)

$$-i A(x) \tau_\alpha^T \Gamma_x[A] = \underbrace{G(x) \tau_\alpha^T}_{\downarrow G(x) \tau_\alpha^T} i \Gamma'_x[A] + \left[\int dy \tilde{G}(xy) A(y) \right]$$

Now replace

$$\begin{aligned} \int dy [\partial_\mu^* G_\alpha^\mu(x, \underline{y}) - G_\alpha(x, \underline{y})] A(y) &= \int dy \underbrace{A(y)}_{G \tau^T A} \left[i \tau^\alpha G(x) \delta(x-y) A(y) \right] \\ &= i \tau^\alpha G(x) A(x) \end{aligned}$$

Therefore,

$$i\tau G(x)A(x) + \partial_\mu \Gamma^\mu - \Gamma$$

$$= -iA\tau^\top \Gamma_x - G\tau^\top i\Gamma'_x + G(x)\tau^\top \int dy \tilde{G}(x,y)A(y).$$

$$\rightarrow \frac{\partial \Gamma^\mu}{\partial x} = \int -iA\tau^\top \Gamma_x - G\tau^\top i\Gamma'_x + \int dy G(x)\tau^\top \tilde{G}(x,y)A(y) - i\tau G(x)A(x)$$

So

$$\partial_\mu \Gamma_\alpha^\mu(x, A) = \Gamma_\alpha(x, A) - i A(x) \tau_\alpha^T \Gamma_x[A] - i G(x) \tau_\alpha^T \Gamma'_x[A] \quad (\text{II.15})$$

Then if current is conserved and no SSB, you get

$$\partial_\mu \Gamma_\alpha^\mu(x, A) = -i A(x) \tau_\alpha^T \Gamma_x[A] \quad \text{Riviera}$$

Pause, what have we done?

1) we construct a Ward as in ccd for $G^\mu = \langle 0 | T^y (J^\mu \phi \phi \phi \dots) | 0 \rangle$. (II.5)

2) we write same version of $G^\mu(x, J)$ as in (II.10)

3) we identify proper vertices as in (II.12) + (II.13).

4) we end up with Ward in terms of $\Gamma_\alpha^\mu, \Gamma_\alpha, \Gamma_x$

Compare:

$$(\text{II.5}) \quad \partial_\mu G_\alpha^\mu(x, y_1 \dots), \quad G_\alpha(x, y_1 \dots), \quad G_\alpha(y_1 \dots y_n) \delta(x - y_n)$$

$$(\text{II.10}) \quad \partial_\mu G_\alpha^\mu(x; J), \quad G_\alpha(x; J), \quad G_x[J] J(x)$$

$$(\text{II.15}) \quad \partial_\mu \Gamma_\alpha^\mu(x; A), \quad \Gamma_\alpha[x; A], \quad \Gamma_x[A] A(x), \quad G \dots$$

So it is a matter of notation.

now, use (II.14),

$$\partial_\mu G_\alpha^\mu(x, y) = G_\alpha(x, y) + i \tau_\alpha G(x) \delta(x-y)$$

which is $\delta/\delta J(y)$ on (II.10), with $J=0$. Apply $\int dy' \bar{G}(y, y')$ to both sides.

$$\int dy' \bar{G}(y, y') G_\alpha(x, y') = G_\alpha(x, y) \Rightarrow$$

$$\partial_\mu G_\alpha^\mu(x, y) = G_\alpha(x, y) + \frac{\int dy' \bar{G}(y, y') [i \tau_\alpha G(x) \delta(x-y')]}{\bar{G}(y, x) i \tau_\alpha G(x)}$$

Then

$$\partial_\mu G_\alpha^\mu(x, y) = G_\alpha(x, y) + i \bar{G}(y, x) i \tau_\alpha G(x).$$

Then

$$\int dy [\partial_\mu G_\alpha^\mu(x, y) - G_\alpha(x, y)] A(y) = \int dy [i \bar{G}(y, x) i \tau_\alpha G(x) A(y)].$$

$$= i G(x) \tau_\alpha^T \bar{G}(x, y) A(y).$$

$\uparrow$
here

So I get:

$$\int dy i G(x) \tau_\alpha^T \bar{G}(x, y) A(y) + \partial_\mu P^\mu - \Gamma$$

$$= -i [A + G] \tau^T P_x [A]$$

$\Rightarrow$

$$\partial_\mu P^\mu = \Gamma - i [A + G] \tau^T P_x [A] - \frac{i \int dy G(x) \tau_\alpha^T \bar{G}(x, y) A(y)}{\uparrow \text{cancel}}$$

$\uparrow$
This is line #2 of (II.15).

Now, if you like, write

$$-i G(x) \tau^T P_x [A] = -i G(x) \tau^T P'_x [A] + i \int dy \bar{G}(x, y) G(y) \tau^T \bar{G}(x, y) A(y)$$

$$\Rightarrow \partial_\mu P^\mu = \Gamma - i A \tau^T P_x [A] - i G(x) \tau^T P'_x [A]$$

$\uparrow$ This is line #1 of (II.15).

Point: There are no R -linear terms in (II.15).

Symnzik
1970

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Small Distance Behaviour in Field Theory and Power Counting

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Abstract. For infinitesimal changes of vertex functions under infinitesimal variation of all renormalized parameters, linear combinations are found such that the net infinitesimal changes of all vertex functions are negligible relative to those functions themselves at large momenta in all orders of renormalized perturbation theory. The resulting linear first order partial differential equations for the asymptotic forms of the vertex functions are, in quantum electrodynamics, solved in terms of one universal function of one variable and one function of one variable for each vertex function whereby, in contrast to the renormalization group treatment of this problem, the universal function is obtained from nonasymptotic considerations. A relation to the breaking of scale invariance in renormalizable theories is described.

Introduction

The small distance behaviour of Green's and vertex functions in renormalizable quantum field theories has been extensively studied in a formal way via the renormalization group [1, 2] and, with equivalent results, some other approaches [8, 9]. Here we offer an alternative approach to the same problem, which appears to be rather more direct. It leads to formulas that are exact and become the usual asymptotic ones upon a, in principle controllable, neglect.

We study the effect of inserting one extra mass vertex, or a generalized mass vertex in the sense of Wilson [3], into all Feynman diagrams for all vertex functions. (Such vertices are defined as those for which the sum of the mass dimensions of the composing fields, a scalar and the electromagnetic field having dimension one, a spinor field dimension three half, a derivative dimension one, is less than four, e.g. two for a scalar mass vertex and three for a spinor mass vertex.) By such insertion, the superficial divergence D of the corresponding Feynman integral is reduced¹ (e.g. by two and by one, respectively, for a scalar and a spinor mass vertex). Reduction of the superficial divergence, however, results in decrease of the large-momentum behaviour by the corresponding power of an overall scale factor¹. This relation between dimension of

¹ See, e.g., Ref. [2] p. 321; the index $\omega(G)$ we call D . Also *ibid.*, p. 341.

"CS equations" original reference.

Symanzik.

This paper gets results similar to those of GellMann Low and Callan and CJ, namely, that the Greens functions (or proper vertex functions) satisfy certain differential equations like i.17 of this paper; derivatives are with respect to the "parameters" of the theory, mass, coupling constant. Recall that Callen showed the need for the extra coupling constant derivative not normally included in a scaling analysis (ie, a scale invariance analysis, eg, the scale-Ward identities of CCJ). On the other hand Gellmann Low got their asymptotic forms for greens functions from the global renorm group equations like $d = dd$ stuff, same as in BS.

The difference here is that Symanzik derives his equations using the functional methods. In section I he considers ϕ^4 and ends up with the usual Callan -Symznzik equation. The key ingredient is realizing that certain objects have degree D positive and this allows you (requires you) to define various parameters like g and m. These then appear in the CS equations.

The CS equation in turn can be used to get a form for the asymptotic behavior of the greens functions, and you see the deviation from simply naive scaling behavior, ie, the logs. Later papers at this point combine with Wilson to explain scaling results at SLAC.

Next section repeats everything for QED rather than ϕ^4 .]

Appendix I guess shows how the naive scale invariance predictions are wrong.

At the moment this paper is too technical for me. I would rather get back to the Pagels QCD review I started. Once I understand someones derivation of the CS equations, I can go back and see how everyone else derived them in their own way.

Derive the Formula (I.3), (I.5), (I.6a)

① Look back at the way you do perturbation theory: I.e.,

$$\begin{aligned} \bullet G[\mathcal{J}] &= \int [d\phi] e^{-i \int d^4x [\mathcal{L}_0 + \mathcal{L}_I(\phi) + \mathcal{J}\phi]} \\ &= e^{-i \int d^4x \mathcal{L}_I(-i \frac{\delta}{\delta \mathcal{J}})} \underbrace{\int [d\phi] e^{-i \int d^4x [\mathcal{L}_0 + \mathcal{J}\phi]}}_{G_0[\mathcal{J}]} \end{aligned}$$

$$= \exp(i \int d^4x \mathcal{L}_I(-i \frac{\delta}{\delta \mathcal{J}})) G_0[\mathcal{J}].$$

This is precisely what (I.3) says, with $\mathcal{L}_I \rightarrow \Delta \mathcal{L}$, and a $\mathcal{L}$ appears because $\phi = \sqrt{Z} \phi_n$ or something.

② Suppose $\Delta \mathcal{L} = \frac{1}{2} \Delta m^2 \phi^2$, i.e., a mass counterterm. How do you arrive at (I.5)? I think (I.5) says: if you go $G \rightarrow G^{\Delta}$ by adding Δm^2 , probably g will change by some Δg , and Z will renormalize by some ΔZ . I.e., I don't think (I.5) came from (I.3).

③ How's about (I.6a).

$$\begin{aligned} G^{\Delta}(\mathcal{J}) &= \text{const.} \exp\left(i \int d^4x \left(-\frac{1}{2} \Delta m^2 \phi^2 + \mathcal{J} \phi\right)\right) G(\mathcal{J}) \\ &\approx \left(1 + i \int d^4x \left(-\frac{1}{2} \Delta m^2 \frac{\delta^2}{\delta \mathcal{J}^2}\right)\right) G \end{aligned}$$

$$\begin{aligned} \Rightarrow \Delta G(\mathcal{J}) &\approx i \int d^4x \left(-\frac{1}{2} \Delta m^2 \frac{\delta^2}{\delta \mathcal{J}^2}\right) G(\mathcal{J}) \\ &\approx \frac{1}{2} i \Delta m^2 \mathcal{L}_3 \cdot \int d^4x G_{xx}(\mathcal{J}) \end{aligned}$$

• This is roughly (I.6a). I.e., this is ∞ -small

K. Symanzik [DESY, Hamburg]

20 pages

"Small Distance Behavior in F.T. and Power Counting"

Comm. math. phys. 18, 227 (1970)

This paper builds on certain "basic facts" about renormalization. For these basics, we are told to consult two references:

1. Gell-Mann & Low PR 95, 1300 (1954)

2. Bogoliubov + Shirkov book, (1959)

This paper then does two long examples.

I. The ϕ^4 -theory. $\rightarrow$ uses generating functions, Edisc, to get off the ground here we have to go read:

[5] Symanzik, CMP 16, 48 (1970)

II. QED. — same thing.

The gimmick here involves adding some kind of "mass vertex", maybe this means $\rightarrow$

Then follows an appendix on "the breaking of scale invariance"

This method of finding the large momentum behavior of green's functions is supposed to be simpler than other equivalent methods.

HADRON DYNAMICS*

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ABSTRACT

Developments in the dynamics of hadrons and their interactions are presented using the thread of the renormalization group to hold the fabric together. After an introduction to the ideas and equations of the renormalization group we discuss the solution to and usefulness of these equations. Armed with this tool we consider in some depth two seemingly disparate aspects of hadron physics: (1) gauge theories of the strong interactions--in particular, "ultraviolet freedom" and deep inelastic scattering, models of hadron structure, the use and necessity of charm and color, and so on will be discussed. (2) Diffraction scattering and the Pomeron singularity--in particular, the "Reggeon calculus", branch points in the J-plane, decoupling theorems, t-channel discontinuity formula for Reggeons, and so on will be addressed. Some comments will be presented on the relation of these two important subjects.

Throughout the lectures there is an emphasis on pedagogy rather than subtlety. Ideas and elementary examples are stressed and the results of detailed calculations, when presented, are lifted out of the references like magic. This course of lectures should serve as an introduction, then, both to the mainstream of present ideas about hadrons and to the more advanced lectures to be given in the Topical Conference to follow this school.

* Lectures for the SLAC Summer Institute on Particle Physics, 29 July-10 August, 1974.
AF in June 1973.

"We also notice in our audience a bus load of small elderly women from Schnectady, New York. You may stay with us, if you like, but please don't speak out or rustle papers."

A Child's Garden of Grass

J.S. Margolis and R. Clorfene

INTRODUCTION, PHILOSOPHY AND APOLOGY

The subject of these lectures is the most delightful in particle physics: the dynamics and structure of the strong interactions. It is also a field very much in an "open Pandora's box" state: often all one has to go on is hope. On these bare bones there has been a great deal of clever, thoughtful work which attempts to answer some or all of the questions:

1. What is the origin and nature of the J-plane structure, the Pomeron singularity or vacuum singularity, which is responsible for almost constant total cross sections?
2. How can local quantum field theory be consistent with the approximate scaling of deep inelastic structure functions?
3. Are hadrons composites made out of constituents, generically called quarks? How many and what quantum numbers carry these quarks? How is it possible that these presumed constituents have never been seen in the laboratory? Can we make a virtue out of this shyness of quarks?
4. Can we have a Pomeron which gives rise to almost constant cross sections and limiting inclusive cross sections, shows factorization, results in a triple Pomeron coupling, and yet does not decouple from particles?
5. How can one implement unitarity at high energies (direct channel and crossed channel unitarity)? Does this provide the constraints to secure us a solution to the hadron scattering problems?

These are among the major issues of the day. None of them has completely yielded to solution; each of them has been vigorously and imaginatively attacked. The ideas behind this attack are the subject of these lectures. It

6/19/78

Abarbanel: ~~pages~~ Section 1 "The Renorm. Group" / notes 6/19/78, sec. 1.

(pages 401-405). Start with a ϕ^4 Lagrangian which contains ϕ_0, m_0, λ_0 , the bare (unrenormalized) field, mass, coupling constant. In the "counterterm method" you adjust your bare constants m_0, λ_0 in each order so as to get your desired renormalized $m, \lambda \dots$ i.e., pretend that ahead of time you know what values you want for m, λ .

To each order you have cancelled the shift of m away from m_0 . This shift happens to diverge when $D=4$.

The new propagator is $\langle 0 | \phi \phi | 0 \rangle = \frac{\Sigma}{p^2 - m^2}$ to some order. But the old was

$$\langle 0 | \phi_0 \phi_0 | 0 \rangle = \frac{1}{p^2 - m_0^2}. \quad \text{Thus clearly } \sim \text{do I have } \Sigma \text{ backwards?}$$

Abarbanel wants $\langle 0 | \phi \phi | 0 \rangle = \frac{1}{p^2 - m^2}$ so that $\langle 0 | \phi_0 \phi_0 | 0 \rangle = \frac{\Sigma}{p^2 - m^2}$.

Yes, his Σ is backwards from BD.

Anyway, to each order you can compute λ_0, m_0 to achieve your desired λ, m . I.e., to each order you compute the Σ 's.

Desired: $\langle 0 | \phi^2 \phi^2 | 0 \rangle = P_0^{(2)}(p^2) = p^2 - m^2$ / to lowest order

But also $\frac{\partial}{\partial p^2} i P^{(2)}(p^2, m, \lambda, \mu, D) = 1$ when $p^2 = \mu^2$ $m^2 = A\mu^2$.

Desired: $\langle 0 | \phi_0^4 \phi_0^4 | 0 \rangle = P_0^{(4)}(p_1 \dots p_4) = \lambda_0$ / to lowest order

But also $P^{(4)}(p_1 \dots p_4; m, \lambda, \mu, D) = \lambda$ when $p_i \cdot p_j = \text{sym part} \sim \mu^2$ $m^2 = A\mu^2$.

So easy to show that

$$\Sigma = \frac{\partial}{\partial p^2} i P^{(2)}(p^2, m_0, \lambda_0, D) \Big|_{\substack{p^2 = \mu^2 \\ m_0^2 = 2b A \mu^2}}$$

(if you know Σ , then to any given order you can compute $\Sigma(m_0, \lambda_0, \dots)$).

Suppose to order n you have calculated $\Sigma(m_0, \lambda_0, \dots)$. Then use original defs to determine m_0, λ_0 which yield m, λ ; i.e., you can compute $m_0(m, \lambda, \mu)$ etc.

page 406: Let $g = \lambda \mu^{D-4}$ = dimensionless. You know what dimensionless $P(N)$ must have, so

$$\Gamma^{(N)}(p_i, m, g, \mu, D) = [\mu]^{N + \frac{D}{2}(2-N)} \times \left\{ \text{function of dimensionless params.} \right\}$$

$$\downarrow \psi_N\left(\frac{p_i}{\mu}, \frac{m}{\mu}, g, D\right)$$

Thus based purely on dimension, we find this scaling law. This at once leads to

$$\Gamma^{(N)}[\xi p_i, m, g, \mu, D] = (\xi)^{N + \frac{D}{2}(2-N)} \Gamma^{(N)}\left[p_i, \frac{m}{\xi}, g, \frac{\mu}{\xi}, D\right]$$

This is a homogeneous function!

page 407: The fact that $\Gamma^{(N)}$ "never heard of μ " means $\frac{\partial \Gamma^{(N)}}{\partial \mu} = 0$.

Use chain rule to learn then that

$$\left[\mu \frac{\partial}{\partial \mu} + \mu \left(\frac{\partial g}{\partial \mu} \right) \frac{\partial}{\partial g} + \mu \left(\frac{\partial m}{\partial \mu} \right) \frac{\partial}{\partial m} - \frac{N}{2} \mu \left(\frac{\partial \ln \mu}{\partial \mu} \right) \right] \Gamma^{(N)}(p_i, m, g, \mu, D) = 0$$

where derivatives computed somehow. We have

$$m = m(\lambda_0, m_0, \mu, D) \quad \text{so} \quad \frac{\partial m}{\partial \mu} = f(\lambda_0, m_0, \mu, D) \quad \text{but} \quad g = \lambda \mu^{D-4}$$

$$g = \lambda[\lambda_0, m_0, \mu, D] \mu^{D-4} = g(\mu, m_0, \lambda_0, D). \quad \text{So}$$

$$\frac{\partial m}{\partial \mu} = f'(\lambda_0, m_0, g, D) \quad \sim \quad \text{you can replace } \mu \text{ with } g.$$

So (42) - (44) look OK to me. Now combine (42) with homogeneity to get:

$$\left[\frac{\partial}{\partial g} - \beta(g) \frac{\partial}{\partial g} + [1 - \gamma(g)] m \frac{\partial}{\partial m} + \frac{N}{2} \gamma(g) - \left[N + \frac{D}{2}(2-N) \right] \right]$$

$$m \Gamma^{(N)}(\xi p_i, m, g, \mu, D) = 0$$

This is a pde, how do you solve it? This is a bit new to me but result I can quote:

We are particularly interested in how $\Gamma^{(N)}(\epsilon p_i, m, g, \mu, D)$ depends on variable of scale ξ . Let

$$\xi = e^t$$

Then:
$$\Gamma^{(N)}[e^t p_i, \tilde{m}(t), \tilde{g}(t), \mu, D] = \Gamma^{(N)}(p_i, m, g, \mu, D) \times \exp\left[\int_0^t dt' D_N[\tilde{g}(t')]\right]$$

where $D_N[x]$ is (50), and where $\tilde{m}, \tilde{g}$ are auxiliary functions.

$$\tilde{g}(t)' = -\beta[\tilde{g}(t)] \quad \tilde{g}(t=0) = g \rightarrow \tilde{g}(t) \leftarrow$$

$$[\log \tilde{m}(t)]' = 1 - \gamma[\tilde{g}(t)] \quad \tilde{m}(t=0) = m \rightarrow \tilde{m}(t)$$

Clearly you can write $\tilde{g} = \tilde{g}[g, t]$ since g appears in the BC

Then $\tilde{m} = \tilde{m}[m, g, t]$. Note that you can shuffle to get

$$\Gamma^{(N)}(\epsilon p_i, m, g, \mu, D) = \Gamma^{(N)}(p_i, \tilde{m}(-t), \tilde{g}(-t), \mu, D) e^{-\int_{-t}^0 dt' D_N[\tilde{g}(t')]}.$$

↑ This is what you really want.

Main point: If you can find a value of $\xi = e^t$ such that $\tilde{g}(-t) \ll 1$, then you can evaluate $\Gamma^{(N)}$ RHS accurately using perturbation theory! Regardless of how large g is.

(p409) So examine equation for $\tilde{g}[g, t] =$ "the running coupling constant."

$$\frac{d\tilde{g}}{dt} = -\beta[\tilde{g}(t)]$$

Assume that $\beta(x)$ has a zero, so $\frac{d\tilde{g}}{dt} = \beta_1[\tilde{g} - g_1]$ for $\tilde{g}$ near g_1 .

With $\tilde{g}[g, 0] = g$, the solution of this $\uparrow$ is

$$\tilde{g}(t) = g_1 + (g - g_1) e^{\beta_1 t}$$

If $\beta_1 < 0$, then as $t \rightarrow \infty$ [UV limit], $\tilde{g}(-t) \rightarrow g_1$.

How can I phrase this in terms of what I know? Let $t \rightarrow \infty$ in a series of steps [renam group transform]

Then $(R_b)^t g = g^b \rightarrow \infty$.

$$R_b g_1 = g_2 \text{ etc and } R_b g_1 = g_1$$

I.e., as $g \rightarrow \infty$, $g[t] \rightarrow g_1$. I would call g_1 a fixed point. So

"for each zero in $\beta(g)$, at $g = g_i$, such that $\beta_i < 0$, you have an ultraviolet fixed point and $g \rightarrow g_i$. The critical behavior of your field theory is determined by which fixed point!"

Hopefully, the g_1 that counts is small! Then you can use (62) to get the UV behavior of a field theory.

You would like: $\tilde{g}[g, t] |^{t=0} = g = \text{strong coupling}$

$$\tilde{g}[g, t] |^{t=\infty} = g_1 = \text{small.}$$

(page 410) ~ OK, end of section. I rather skipped the last part because in fact I went through all this before.

What is the basic idea described here? First, you have to find a way to define a renormalized field theory which is sort of coordinated at a renormalization point μ .

I.e., this point is used as part of the definition of the renormalized mass and coupling.

Then you are able to show that the renormalized greens functions $\Gamma^{(N)}$ satisfy a certain homogeneity condition, based only on known dimensions of a greens function. The independence

of the unrenormalized greens functions on artificial point μ then leads to a condition

on the renormalized Γ , a condition known as the renormalization group equation for Γ .

When you solve this equation, you learn a certain correlation between scaling all the

momenta, and value of the effectived coupling constant. If a certain function β has a

linear zero with negative slope, then as all p_i in Γ scale off to infinity, you find

that Γ can be computed in terms of another Γ whose effective coupling is g_1 , the place

where β has a zero [this is an UV fixed point]. If $g_1 = 0$, your theory is asymptotically

free. This is thought to be the case for QCD based on parton data in UV region.

Abarbanel Port 2: "Gauge Theories"

[June 13, p. 4]

(pp 413-415) The basics of gauge theory:

- 1) introduction of gauge fields yields local gauge invariance, as in QED.
- 2) in Yang-Mills gauge fields are B_a^μ .
- 3) mass term ruins local gauge invariance [mass for vector B_a^μ].
- 4) mass term also ruins renormalizability since $\delta_i > 0$ for $\bar{\psi} \gamma_\mu \psi B_a^\mu$ in $\mathcal{L}$.
- 5) but SSB allows massive B^μ and maintains local gauge invariant $\mathcal{L}$ and maintains renorm. The Big Discovery! - The shifted Lagrangian then has massive B^μ and hidden symmetry.

(p416) Generalize to arbitrary gauge group.

(p417) Here the specific QCD Lagrangian is stated. It is then claimed that, for this theory, someone has computed

$$\beta(g) = \beta_1 g^3 \quad g \neq 0 \quad \beta_1 = -25/48\pi^2$$

Thus $g=0$ is an UV fixed point and QCD is asymptotically free!Abelian gauge theories with $D=4$ [like QED] are not AF!(p418-420) Since you never take all $p_i \rightarrow \infty$ (they are on mass shell), where is all this AF stuff useful?

- 1) in deep inelastic, data coded in $W_2(\nu, q^2)$
- 2) these are related to $\langle p | [\bar{\psi}_\mu(x), \bar{\psi}_\nu(0)] | p \rangle$.

But using Wilson Operator Product idea, $\hookrightarrow = \sum_n \tilde{C}^{(n)}(x^2) \langle p | \bar{A}_n | p \rangle x, x, \dots, x$.You can show that the c-number $\tilde{C}^{(n)}(x^2)$ satisfy renorm group equations.In Free Field theory, you can show that $FT \tilde{C}^{(n)}(x^2) = \tilde{C}^{(n)}(q^2) \rightarrow C_n$, const. and this implies $\nu W_2(s, q^2) \rightarrow F_2(s)$, BT scaling.So (free theory $\Rightarrow$ BT scaling)In QCD or other AF theory, $\tilde{C}^{(n)}(q^2) \rightarrow C_n \times \left[-\beta_1 g^2 \log\left(\frac{q^2}{\mu^2}\right) \right]^{q_n/2\beta_1}$ as $q^2 \rightarrow \infty$. So AF gives a result similar to a F theory.

(pp 421-424) Model building ~ combining hadron with weak + em.

So, as far as the renormalization group is concerned, this second part of Abarb's paper does not have much to say.

- a) QCD theory, people have computed BETA and they find that theory is indeed asymptotically free in the renorm group sense. There is a fixed point at $g_1=0$ and constant BETA has the correct sign. Moreover,
- b) it is thought that non-abelian gauge theories are the only theories which show this property. Abelian theories do not.
- c) Renorm group is utilized somehow in the Wilson Operator Product idea to show that a theory which is AF should exhibit Bj scaling, just like a free theory does, more or less.

There is no connection made between gauge theory and critical phenomena or my previous knowledge of the renorm group. In fact I don't even know what corresponds to the correlation length. So my two knowledge bins are as yet unconnected.

What is a function of what?

$$\mathcal{L} = \mathcal{L}(\phi_0, m_0, \lambda_0, D) \quad \mathcal{L}_c = \text{counterterm} = \text{same variables!}$$

$$\Gamma_u^{(N)} = \Gamma_v^{(N)}(p_i, m_0, \lambda_0, D) \quad \text{certainly clear.}$$

$$\Gamma^{(N)} = \Gamma^{(N)}(p_i, m, \lambda, \mu, D) \quad , \text{ why depend on } \mu?$$

μ will be a "point" where things are defined.

$$\frac{\partial^2}{\partial p^2} i \Gamma^{(2)}(p^2, m, \lambda, \mu, D) = 1 \quad \text{at the point} \quad p^2 = \mu^2 \quad \text{and} \quad m^2 = A\mu^2$$

Why are we even free to do this? How are the Γ defined in the first place? Everything seems to float! Note: $m \neq$ pole position any more than m_0 .

Note: if you can define Z, Z_λ, Z_m , then you have defined m, λ .

$$(19) \text{ says } \left. \frac{\partial}{\partial p^2} i \Gamma_v^{(2)}(p^2, m_0^2, \lambda_0, D) \right|_{\substack{p^2 = \mu^2 \\ m_0^2 = Z^{-1} Z_m A \mu^2}} = Z^{-1}$$

You can compute the LHS, so this says $f_1(Z, Z_m) = 0$. Better yet,

$$f_1(\mu, \lambda_0, D, A, Z, Z_m) = 0$$

$$(24) \text{ says } f_2(\mu, \lambda_0, D, A, Z, Z_m) = 0.$$

So imagine solving (19) + (21) to get $Z(\mu, \lambda_0, D, A)$
 $Z_m(\mu, \lambda_0, D, A)$

$$\text{Then (23) says: } f_3(\mu, \lambda_0, D, Z, Z_m, Z_\lambda) = 0 \Rightarrow Z_\lambda(\mu, \lambda_0, D, A)$$

Since $m_0^2 = Z^{-1} Z_m A \mu^2$, we can trade A for m_0 :

$$\begin{array}{l} Z(\lambda_0, m_0, D, \mu) \\ \text{Thus we obtain} \end{array} \quad \begin{array}{l} m = m(\lambda_0, m_0, \mu, D) \\ \lambda = \lambda(\lambda_0, m_0, \mu, D) \end{array}$$

$$\text{For fixed } \mu, D, \quad \begin{array}{l} m = m(\lambda_0, m_0) \\ \lambda = \lambda(\lambda_0, m_0) \end{array} \Rightarrow \begin{array}{l} \lambda_0 = \lambda_0(\lambda, m) \\ m_0 = m_0(\lambda, m) \end{array}$$

Your reward is this: having computed Γ to any order, you know

$$\lambda_0(\lambda, m, \mu, D) \quad m_0(\lambda, m, \mu, D)$$

If you change μ , all scales shift, so L^2 change, m and λ change, $\Gamma^{(N)}$ changes. But Γ cannot change, hence (40).

Chain Rule:

$$\mu \frac{\partial}{\partial \mu} \left[\sum \left(m_0, \lambda_0, \mu, D \right) \Gamma^{(N)}(p, m, g, \mu, D) \right] = 0$$

$$\Rightarrow \sum^{-N/2} \cdot \left[\mu \frac{\partial \Gamma^{(N)}}{\partial \mu} + \mu \frac{\partial \Gamma^{(N)}}{\partial m} \frac{\partial m}{\partial \mu} + \mu \frac{\partial \Gamma}{\partial g} \frac{\partial g}{\partial \mu} \right] + \left(-\frac{N}{2} \right) \sum^{-N/2-1} \frac{\partial \sum}{\partial \mu} \mu \Gamma = 0$$

Cancel $\sum^{-N/2}$ so

$$\mu \frac{\partial \Gamma}{\partial \mu} + \left(\mu \frac{dm}{d\mu} \right) \cdot \frac{\partial \Gamma}{\partial m} + \left(\mu \frac{dg}{d\mu} \right) \frac{\partial \Gamma}{\partial g} - \left(\mu \frac{N}{2} \right) \left(\frac{\partial}{\partial \mu} \ln \sum \right) \Gamma = 0$$

$$\left\{ \mu \frac{\partial}{\partial \mu} + \mu \frac{\partial m}{\partial \mu} \frac{\partial}{\partial m} + \mu \frac{\partial g}{\partial \mu} \frac{\partial}{\partial g} - \mu \frac{\partial \ln \sum}{\partial \mu} \frac{N}{2} \right\} \Gamma = 0. \quad (42)$$

Now recall $m = m(\lambda_0, m_0, \mu, D)$ but $g = \lambda \mu^{D-4}$, so replace $\lambda = \lambda(\lambda_0, m_0, \mu, D)$

μ with $g, \lambda \equiv g, \lambda_0, m_0, \mu, D = g, \lambda_0, m_0, D$. $g(\lambda_0, m_0, \mu, D)$

So $m(\lambda_0, m_0, g, D)$ and $\lambda(\lambda_0, m_0, g, D)$. and $\sum(\lambda_0, m_0, g, D)$.

For fixed λ_0, m_0, D these depend only on g . Also

$$\text{So } m \tau \equiv \mu \frac{dm}{d\mu} \Rightarrow \tau = \tau(g) \text{ and } \lambda_0 m_0 D$$

$$\beta \equiv \mu \frac{dg}{d\mu} \Rightarrow \beta = \beta(g).$$

Look back at (37)

$$\begin{aligned} \sum \frac{\partial}{\partial g} \Gamma(p, m, g, \mu, D) &= \sum \left\{ \sum^{N+\frac{D}{2}(2-N)} \cdot \left[\frac{\partial \Gamma}{\partial (\mu \beta)} \frac{\partial}{\partial g} \left(\frac{m}{\beta} \right) + \frac{\partial \Gamma}{\partial (\mu \beta)} \frac{\partial}{\partial g} \left(\frac{\lambda}{\beta} \right) \right] \right. \\ &\quad \left. + \left[N + \frac{D}{2}(2-N) \right] \sum^{N+\frac{D}{2}(2-N)-1} \cdot \Gamma \right\} \end{aligned}$$

$$= \sum^{N+\frac{D}{2}(2-N)} \left\{ -m \frac{\partial}{\partial m} - \mu \frac{\partial}{\partial \mu} + \left[N + \frac{D}{2}(2-N) \right] \right\} \Gamma(p, \frac{m}{\beta}, g, \frac{\lambda}{\beta}, D)$$

$$= \left[N + \frac{D}{2}(2-N) - m \frac{\partial}{\partial m} - \mu \frac{\partial}{\partial \mu} \right] \Gamma(p, m, g, \mu, D) = \sum \frac{\partial}{\partial g} \Gamma(\text{same}). \quad (46)$$

replace μ with g .

Eliminate $(\mu \frac{\partial \Gamma}{\partial \mu})$ between (46), (42) to get:

$$\mu \frac{\partial \Gamma}{\partial \mu} = -\xi \frac{\partial \Gamma}{\partial \xi} + m \frac{\partial \Gamma}{\partial m} + [N + \frac{D}{2}(2-N)].$$

$$\Rightarrow (-\xi \frac{\partial \Gamma}{\partial \xi} + m \frac{\partial \Gamma}{\partial m} + [N + \frac{D}{2}(2-N)] - \frac{N}{2} \xi + \beta \frac{\partial \Gamma}{\partial \beta}) \Gamma = 0 \quad (47)$$

Ch. sign: $\left\{ \xi \frac{\partial \Gamma}{\partial \xi} + m \frac{\partial \Gamma}{\partial m} + \frac{N}{2} \xi - \beta \frac{\partial \Gamma}{\partial \beta} + [N + \frac{D}{2}(2-N)] \right\} \Gamma = 0$

On: [variation in ξ, g, m] of $\Gamma(\xi, g, m, \mu, D) = 0$.

Now $\frac{\partial \Gamma}{\partial t} = \frac{\partial \Gamma}{\partial \log} = \xi \frac{\partial \Gamma}{\partial \xi} = \xi \left[\frac{\partial \log}{\partial \xi} \frac{\partial \Gamma}{\partial \log} \right] \Gamma$

$$\vec{\nabla} \cdot \vec{\nabla} = -\beta \frac{\partial}{\partial g} + (1-\tau) \frac{\partial}{\partial m} \quad \vec{\nabla} = (\partial/\partial g, m \partial/\partial m)$$

Partial in 3-variables!

Let $\begin{cases} \tilde{g}(t) = -\beta[\tilde{g}] \\ \tilde{g}(0) = g \end{cases} \quad \begin{cases} \tilde{m}(t) = 1 - \tau(\tilde{g}) \\ \tilde{m}(0) = m \end{cases}$

Non linear first order system. I.e.

$$-\frac{d\tilde{g}}{\beta[\tilde{g}]} = dt \Rightarrow \int \frac{d\tilde{g}}{\beta[\tilde{g}]} = t + c \quad \frac{d\tilde{m}}{\tilde{m}} = [1 - \tau(\tilde{g}(t))] dt$$

So $\ln \tilde{m}(t) = \int dt [1 - \tau(\tilde{g}(t))] + c \quad \tilde{m}(t) = A e^{\int dt [1 - \tau(\tilde{g}(t))]} \quad A = m$

$$t = c + \int \frac{ds}{\beta(s)} \quad 0 = c - \int \frac{ds}{\beta(s)} \quad \text{so } c = \int$$

So $\left[t = \int \frac{ds}{\beta(s)} \right] \leftarrow \text{my solution for } \tilde{g}(t). \quad \left[\tilde{m}(t) = m e^{\int ds [1 - \tau(\tilde{g}(s))]} \right] \leftarrow \text{my solution for } \tilde{m}(t)$

Check (53):

$\frac{\partial \Gamma}{\partial (e^t p)}$ same first term.

Show that:

$$\frac{d}{dt} \Gamma(e^t p, \tilde{m}(t), \tilde{g}(t), \mu, D)$$

$$\vec{\nabla}(\tilde{g}) \cdot \vec{\nabla} \Gamma(-, m, g) = \vec{\nabla}(\tilde{g}) \cdot \vec{\nabla} \Gamma(\text{same})$$

Well,

$$\begin{aligned} \frac{d}{dt} P[e^t p, \tilde{m}(t), \tilde{g}(t)] &= \frac{\partial P}{\partial (e^t p)} e^t + \frac{\partial \tilde{m}}{\partial t} \frac{\partial P}{\partial \tilde{m}} + \frac{\partial \tilde{g}}{\partial t} \frac{\partial P}{\partial \tilde{g}} \\ &= e^t \frac{\partial P}{\partial (e^t p)} + (1 - \gamma(\tilde{g})) \tilde{m} \frac{\partial P}{\partial \tilde{m}} - \beta(\tilde{g}) \frac{\partial P}{\partial \tilde{g}} \\ &= e^t \cdot \frac{\partial P}{\partial (e^t p)} \cdot \frac{\partial P}{\partial t} = \frac{\partial P}{\partial t} \end{aligned}$$

Thus: $\frac{d}{dt} P[e^t p, \tilde{m}(t), \tilde{g}(t)] = \frac{\partial P}{\partial t} + (1 - \gamma) \tilde{m} \frac{\partial P}{\partial \tilde{m}} - \beta$

$$= \left[\frac{\partial}{\partial t} + (1 - \gamma(\tilde{g})) \tilde{m} \frac{\partial}{\partial \tilde{m}} - \beta(\tilde{g}) \frac{\partial}{\partial \tilde{g}} \right] P[e^t p, \tilde{m}, \tilde{g}]$$

$$= \left[\frac{\partial}{\partial t} + D(\tilde{g}) \cdot \nabla \right] P(\text{all } \sim) = D_V(\tilde{g}) P(\text{same})$$

So (53) now has only one derivative, $\frac{d}{dt}$. Solve (53).

$$\left[\frac{d}{dt} - D(\tilde{g}(t)) \right] f(t) = 0$$

$$\frac{df}{dt} = Df \quad \frac{df}{f} = Df$$

$$\ln f = \int D dt + c$$

$$f(t) = C \exp \left[\int_0^t dt D[\tilde{g}(t)] \right]$$

$$f(0) = C = \checkmark \quad (54)$$

Look back at $\tilde{g}(t)$ solution:

$$dt = - \frac{d\tilde{g}}{\beta(\tilde{g})} \Rightarrow t_2 - t_1 = - \int_{\tilde{g}(t_1)}^{\tilde{g}(t_2)} \frac{ds}{\beta(s)}$$

$$n \quad t - t_1 = - \int_{\tilde{g}(t_1)}^{\tilde{g}(t)} \frac{ds}{\beta(s)}$$

so regard $\tilde{g}(t)$ as $\tilde{g}[t, \tilde{g}(t_1)]$
 $\tilde{g}(\tilde{g}(t_1), t - t_1)$

$$\text{Or } \tilde{g}(t) = \tilde{G}[\tilde{g}(t_1), t - t_1] \text{ for any } t_1$$

$$= \tilde{G}[\tilde{g}, t]$$

$$\tilde{G}[\tilde{g}, 0] = \tilde{G}[\tilde{g}(0), 0 - 0] = \tilde{g}(0) = \tilde{g}$$

$\tilde{g}$ at time t is
a function of its value
at earlier time,
and of time difference.

↑ Obvious from form.

$$\tilde{m}[\tilde{m}(-t), \tilde{g}(-t), t] = ?$$

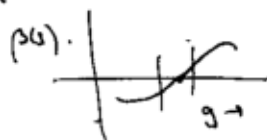
Takes you back:

$$\tilde{m}(t) = \tilde{M}[\tilde{m}(t_1), t - t_1, \tilde{g}(t_1)]$$

$$m = \tilde{m}(0) = \tilde{M}[\tilde{m}(t_1), -t_1, \tilde{g}(t_1)] = \tilde{M}[\tilde{m}(-t), t, \tilde{g}(-t)]$$

Lastly, $\int_0^t D_N [\tilde{g}(g, t')] dt' \rightarrow \int_0^t dt' D_N [\tilde{g}(\tilde{g}(-t), t')]$
 $= \int_0^t dt' D_N [\tilde{g}(\tilde{g}(0), t'-t)] = \int_0^t dt' D_N [\tilde{g}(g, t'-t)] \quad \text{Let } t'' = t' - t$
 $= \int_{-t}^0 dt'' D_N [\tilde{g}(g, t'')] = \int_{-t}^0 dt' D_N [\tilde{g}(t')]. \quad (62) \checkmark$

Near $g \approx g_1$, $\beta(g) \approx \beta_1(g - g_1)$



So $\frac{d\tilde{g}(t)}{dt} = -\beta[\tilde{g}(t)] \approx -\beta_1(\tilde{g}(t) - g_1)$ for $\tilde{g}(t)$ near g_1

So, $\frac{d\tilde{g}(t)}{dt} = -\beta_1 \tilde{g} dt + g_1 \beta_1 dt = dt \beta_1 (g_1 - \tilde{g})$

So $\frac{d\tilde{g}}{g_1 - \tilde{g}} = \beta_1 dt \rightarrow \ln[g_1 - \tilde{g}] \approx \beta_1 t + c$

$g_1 - \tilde{g} = e^{\beta_1 t} c$

$\tilde{g}(t) = g_1 + c e^{-\beta_1 t} \quad \tilde{g}(0) = g = g_1 + c \quad c = g - g_1$

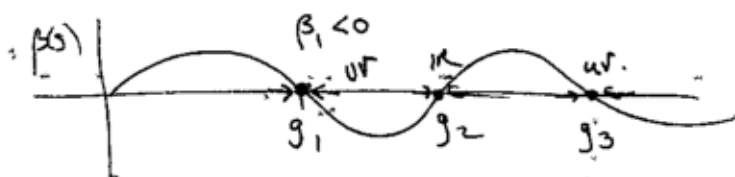
So $\tilde{g}(t) = g_1 + (g - g_1) e^{-\beta_1 t}$ for $\tilde{g}(t)$ near g_1

$\Rightarrow (64)$

If $\beta_1 > 0$, then at some large $t = -T$, (64) is valid because $\tilde{g}(-T) \approx g_1$.

As $t \rightarrow -\infty$, $\tilde{g}(-t) \rightarrow g_1$. What about other zeros? &

For each zero of β , you will get $\tilde{g} \rightarrow g_{\text{zero}}$ in some limit.



Suppose $\Delta = 1 + O(\lambda_0^2)$ in ϕ^3 theory. Then (13) says

$m^2 = (1 + A\lambda_0^2)(1 + B\lambda_0^2)m_0^2$

So what is $\frac{\partial m}{\partial \mu}$? $2m \frac{\partial m}{\partial \mu} = \frac{\partial}{\partial \mu} (1 + (A+B)\lambda_0^2) m_0^2 = \frac{\partial C}{\partial \mu} \lambda_0^2 m_0^2$

So $m \gamma(g) = \mu \cdot \lambda_0^2 m_0^2 \frac{\partial C}{\partial \mu} \approx \mu \Rightarrow (19)$

But $g = \lambda \mu^{D-4} \approx g^2$

Accept (69) - (71), see what it gets you

$$\gamma(g^4) = ag^2$$

$$\gamma(g^4) = cg^2$$

$$\beta(g^4) = bg(g_1^2 - g_2^2)$$

Let $D-4 = d$ $g = \lambda \mu^d$ (72).

$$(75) \Rightarrow \beta(g) = bg(g_1^2 - g_2^2) = \mu \frac{d}{d\mu} [\lambda \mu^d]$$

$$\text{So } b\cancel{g}(g_1^2 - g_2^2) = d\cancel{g} \quad d = bg_1^2 - bg_2^2 = \mu \lambda d \mu^{d-1} = \lambda d \mu^d = dg.$$

On diff of same w/ $g = \lambda \mu^d, \frac{\partial g}{\partial \mu} = \frac{\partial \lambda}{\partial \mu} \mu^d + d \lambda \mu^{d-1}$

$$\text{So } \mu \frac{\partial g}{\partial \mu} = \beta = [\mu \frac{\partial \lambda}{\partial \mu} + d\lambda] \mu^d = [\frac{\mu}{\lambda} \frac{\partial \lambda}{\partial \mu} + d] g.$$

Then, $b\cancel{g}[g_1^2 - g_2^2] = [\frac{\mu}{\lambda} \frac{\partial \lambda}{\partial \mu} + d]\cancel{g}$. What is $\frac{\partial \lambda}{\partial \mu}$? From (2),

$\lambda \approx [1 + O(\frac{1}{\mu^2})] \lambda_0$, so $\frac{\partial \lambda}{\partial \mu} = \text{don't know}$. $= 0$ to order λ_0 . But wait:

$$\lambda = g \mu^{-d} \quad \frac{\partial \lambda}{\partial \mu} = g(-d) \mu^{-d-1} = -gd \mu^{-d-1}$$

Then $\frac{\mu}{\lambda} \frac{\partial \lambda}{\partial \mu} = -\frac{\mu}{\lambda} gd \mu^{-d-1} = -gd \frac{\mu^{-d}}{\lambda} = -gd \frac{1}{g} = -d.$

Ridiculous. $\frac{\partial \lambda}{\partial \mu} = 0!$ So

$$d = b(g_1^2 - g_2^2). \quad (73) \text{ is wrong, proceed.}$$

Try (76):

$$(76) \Rightarrow \ln 2(g) \int_0^g \frac{\gamma(g)}{\beta(g)} dg$$

$$\therefore d(\ln 2(g)) = \frac{\gamma(g)}{\beta(g)} dg \quad - \mu \frac{d(\ln 2(g))}{d\mu} = \frac{\mu \gamma}{\beta} \frac{\partial g}{\partial \mu} = \gamma \quad \checkmark$$

Show $\frac{\mu}{\beta} \frac{\partial g}{\partial \mu} = 1$. $\checkmark$ So (76) is integration of (75) with (77).

But: $\exp \int_0^g dx \frac{cx^2}{bx(g_1^2 - x^2)} = \frac{c}{b} \int_0^g dx \frac{x}{g_1^2 - x^2} =$

$$u = x^2 \\ du = 2x dx$$

$$= \exp \left[\frac{c}{2b} \int_0^{g^2} du \frac{1}{g_1^2 - u} \right] = \exp \left[\frac{c}{2b} \ln \left[\frac{g_1^2 - g^2}{g_1^2} \right] \right]$$

$$= \exp \left[-\frac{c}{2b} \ln \left(1 - \frac{g^2}{g_1^2} \right) \right] = \left(1 - \frac{g^2}{g_1^2} \right)^{-c/2b} \quad (77).$$

But so what? Why is (77) better than (68)? $\square$

Now, $\int_{-\tau}^0 dt' [e^{ct'}] = e^{c \cdot t'} \Big|_{-\tau}^0 = e^{c \cdot 0} - e^{-c \tau} = 1 - e^{-c \tau}$

Use (36):

$$P^{(N)}(s, m, g, \mu, D) = \mu^{[N + \frac{p}{2}(2-N)]} \psi_N\left(\frac{sp}{\mu}, \frac{m}{\mu}, g, D\right) = \text{LHS.}$$

$$P^{(N)}(s, m s^{\frac{2-i}{2}}, g, \mu, D) = \mu^{[N]} \psi_N\left(\frac{s}{\mu}, \frac{m s^{\frac{2-i}{2}}}{\mu}, g, D\right)$$

What does (36) say? $f(L_1, L_2, L_3)$ of dim 3

$= L_1 F\left(\frac{L_2}{L_1}, \frac{L_3}{L_1}\right)$. Yes, obvious. Knowledge of the dimension lets you eliminate one variable. $P \rightarrow \psi$.

Analogy: $f(s_2, z_2) = \psi\left(s\left(\frac{p}{\mu}\right), \left(\frac{m}{\mu}\right)\right)$

$$= \left[s\left(\frac{p}{\mu}\right)\right]^H h\left(\left(\frac{m}{\mu}\right)\left(s\frac{p}{\mu}\right)^B\right) \text{ not quite clear.}$$

Well:

$$\psi_N\left(\frac{p_i}{\mu}, \frac{m}{\mu}, g, D\right) = \left(\frac{p_i}{\mu}\right)^H \cdot \phi_N\left(\left(\frac{m}{\mu}\right)\left(\frac{p_i}{\mu}\right)^B\right)$$

$\uparrow$ $\uparrow$
 y_1 y_2

Except several p variables:

$$f\left(s \frac{p_i}{\mu}, \frac{m}{\mu}\right) = s^H \cdot f\left(\frac{p_i}{\mu}, \frac{m}{\mu} s^B\right) \leftarrow \text{this is true, so what.}$$

OK, seems reasonable. Try again, 3 variables.

$$f(sx, sy, z) = s^H f(x, y, z s^B)$$

end of first lecture Dec 6, 1977.

Abarbanel

(OLD)

"Hadron Dynamics"

SLAC lecture notes 8/5 → 8/7 1974

"Slac summer institute on P. P."

Opens with some questions:

1. origin + structure of the pomeron?
2. how can local GFT → deep inelastic scaling?
3. the constituents problem
4. can pomeron do all those nice things?
5. how to implement unitarity?

He will be dealing with renormalizable field theories. Even tho g is not small, certain products of fields with g may be small — i.e., as small "effective" coupling constant.

Stop. This is excellent stuff — the place to learn about [the renom group]. But, it would cost \$6.⁰⁰ to copy this so maybe I'll wait for it to "come out" — (but will it ever "come out" — being so informal...) Yes, I think it will — ask Joel how to get a hold of it.

Gell-Mann & Low

1954

the results of this experiment $R = \sigma_0 v / \pi b^2 = (1.0 \pm 0.2) \times 10^{16} \text{ sec}^{-1}$ if the limits placed on $\beta_1^0 - \alpha_1^0$ are taken seriously. This assumes the radius of the mesonic Bohr orbit $b = 2.2 \times 10^{-11} \text{ cm}$ and that $v_0 = 8 \times 10^9 \text{ cm/sec}$; $\sigma_0 = (8\pi/9k^2)(v_0/v)(\beta_1^0 - \alpha_1^0)^2$. Panofsky's experiment shows that this capture rate should equal the capture rate for the competing process $p(\pi^-, \gamma)n$.

Bernardini¹¹ has discussed the cross sections for $p(\gamma, \pi^+)n$, $d(\gamma, \pi^+)2n$, and $d(\gamma, \pi^-)2p$ for E_γ between 170 and 190 Mev in the laboratory system. If it is assumed that the ratio of π^- to π^+ production obtained from the second reaction is the same as the photoproduction ratio between the free neutron and free proton then the principle of detailed balance and this ratio can be used to predict the corresponding cross sections for $p(\pi^-, \gamma)n$. If these cross sections are extrapolated to the energy of Panofsky's experiment Bernardini obtains a capture rate that requires the initial slope of $\beta_1^0 - \alpha_1^0$ to be $\pm(9.2^\circ)\eta'$ in contrast to the value of $-(16.5^\circ)\eta'$ obtained from this experiment.

Bethe and Noyes¹⁶ have given an argument to explain this discrepancy in terms of Marshak's¹⁷ suggestion. In brief this argument assumes that the slope of $\beta_1^0 - \alpha_1^0$ obtained from this experiment cannot be extrapolated

¹⁶ H. Bethe and H. P. Noyes, Proceedings of the Fourth Annual Rochester Conference, 1954, University of Rochester.

¹⁷ R. Marshak, Phys. Rev. 88, 1208 (1952).

to the energy of Panofsky's experiment, but that this initial slope is $\pm(9.2^\circ)\eta'$. They then fit this initial slope and the data of this experiment with a smooth curve for $\beta_1^0 - \alpha_1^0$ versus η' . When the values of β_1^0 and α_1^0 from higher energies are extrapolated with this restriction on their difference, it is difficult to fit the data without assuming that α_1^0 varies less rapidly than η' and that β_1^0 varies more rapidly than η' in the energy region between 20 and 42 Mev. For the most probable fit under these assumptions β_1^0 changes sign between 20 and 30 Mev. This energy dependence for β_1^0 suggests a Jastrow¹⁸ potential for this phase shift.

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¹⁸ R. Jastrow, Phys. Rev. 81, 1165 (1951).

Quantum Electrodynamics at Small Distances*

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The renormalized propagation functions D_{FC} and S_{FC} for photons and electrons, respectively, are investigated for momenta much greater than the mass of the electron. It is found that in this region the individual terms of the perturbation series to all orders in the coupling constant take on very simple asymptotic forms. An attempt to sum the entire series is only partially successful. It is found that the series satisfy certain functional equations by virtue of the renormalizability of the theory. If photon self-energy parts are omitted from the series, so that $D_{FC} = D_F$, then S_{FC} has the asymptotic form $A[p^2/m^2]^\alpha [i\gamma \cdot p]^{-1}$, where $A = A(e_1^2)$ and $\alpha = \alpha(e_1^2)$. When all diagrams are included, less specific results are found. One conclusion is that the shape of the charge distribution surrounding a test charge in the vacuum does not, at small distances, depend on the coupling constant except through a scale factor. The behavior of the propagation functions for large momenta is related to the magnitude of the renormalization constants in the theory. Thus it is shown that the unrenormalized coupling constant $e_0^2/4\pi\hbar c$, which appears in perturbation theory as a power series in the renormalized coupling constant $e^2/4\pi\hbar c$ with divergent coefficients, may behave in either of two ways:

- (a) It may really be infinite as perturbation theory indicates;
- (b) It may be a finite number independent of $e^2/4\pi\hbar c$.

1. INTRODUCTION

IT is a well-known fact that according to quantum electrodynamics the electrostatic potential between two classical test charges in the vacuum is not given exactly by Coulomb's law. The deviations are due to

vacuum polarization. They were calculated to first order in the coupling constant α by Serber¹ and Uehling² shortly after the first discussion of vacuum polarization by Dirac³ and Heisenberg.⁴ We may express their results by writing a formula for the potential energy be-

¹ R. Serber, Phys. Rev. 48, 49 (1935).

² A. E. Uehling, Phys. Rev. 48, 55 (1935).

³ P. A. M. Dirac, Proc. Cambridge Phil. Soc. 30, 150 (1934).

⁴ W. Heisenberg, Z. Physik 90, 209 (1934).

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tween two charges q

$$V(r) = \frac{qq'}{4\pi r}$$

Here $\alpha = e^2/\hbar c$ is the fine structure constant. If $r \ll \hbar/mc$, the form,

$$V(r) = \frac{qq'}{4\pi r} \left\{ 1 - \frac{1}{6} \alpha \frac{\hbar^2}{m^2 c^2 r^2} + \dots \right\}$$

where $\gamma \approx 1$ is the entire series making use of in virtue of ties are introduced renormalization physical arguments. A test body surrounding positrons; so same sign as $-\delta q$ in the test body. the body from see an effective renormalized charge. penetrate the body, the charge q_0 , clear, then, (1.2), must be Thus, using (

$$q_0 q_0' = qq' \left\{ 1 - \frac{1}{6} \alpha \frac{\hbar^2}{m^2 c^2 r^2} + \dots \right\}$$

where the integrals are taken logarithmically in our work.

Such divergences whenever observed

⁵ J. Schwinger

Summary of the Gell-Mann Low paper.

By fiddling around with dimensional analysis, they come up with certain "renormalization group equations" which are not of the differential "Callan Symansik" form but which instead are the global $Z = Z.Z$ form, where the cutoff parameter and momentum appears as arguments. See eg (3.9). A later section of this paper actually uses two separate cutoff parameters so the multiplication equations are a little messier.

You cannot completely solve these renorm group equations, but you can learn something about the form of the solution, in the sense of KNO scaling say. You learn (as seems reasonable from multiplication idea) that $S(p)$ propagator goes as a power $(p^2)^n$ but you cannot compute n (later I think AF boys shown how you might compute n). Just how you solve the R-group equations to get this general power form is discussed in an appendix.

Obviously, if you know this "form of solution", you know something about the large p^2 behavior of the function $Z(p)$. Since a Z scales e to e_0 , you in principle are learning about e_0 . In other words, given the physical charge e , by learning about the large p^2 behavior of $Z(p)$ you can learn about e_0 . That was the main interest of this paper: just exactly what is the "bare charge", the charge you see at small distances, high energy.

The "scale form of the solutions" can be reformulated as an integral statement (5.9). This statements contains functions Ψ which is later to be called the BETA function. Here is the important fact: if this function has a zero for finite argument, say x_0 , then the large k^2 behavior is described by a finite Z (called d), and hence the bare charge is finite. If there are no zeros of the function Ψ , then the bare charge is infinite.

The authors are not able to state which is in fact the case. I think even today 1978 the answer to this question is unknown.

So, to summarize, this paper has two important contents: first, it has global renorm group equations of the type you now see in Bogomilov and Bjorken Drell. Second, it shows the "solution" of these equations in terms of a BETA function whose zeros have the significance that the high-energy charge "(bare charge) can be finite and not infinite. Later we learn that in a NAGT you have asymptotic freedom which means that the high-energy charge is in fact zero.

This paper does not contain differential R-group equations of the CZ type, nor does it say anything about scale invariance or PCAC or conserve d currents etc. Their main interest was with QED and whether e_0 is or is not a finite number. After all, this was written in 1954.

Gellman-Low: "QED at small distances". 1954

Comment: This is the paper which lay dormant for 15 years and then led into asymptotic freedom via the renorm group equations. Authors are of course interested in QED only.

1. Introduction. What is the true potential between two charges? As we know, it is the FT of the photon propagator and will thus contain effects from the vacuum polarization graph. The finite parts of the vacpole effects are shown in 1.3 as a certain $5/6$ and so on. There is a divergent piece also, ie, as you take $r = 0$ you get a log divergent piece. The bracketed factor is then just Z_3^{-1} and this is its usual log divergence. In effect, (1.3) is a computation of the bare charge in terms of the (Observed and finite) renormalized charge. But only to a finite order in perturbation theory. The authors are wondering this: just what is the bare charge e_0 to all orders of perturbation theory? Is it really infinite? They have this nice way of visualizing your measurement of the bare charge; you probe small distances so you get through the polarization cloud. As you really get in there, you see the bare charge e_0 . Interesting how the word "bare" is used but, this picture not usually conjured up by me.

As an aside, they quote that QED renormalization was proved in 1949-1951 by Dyson, Salam and Ward.

2. REpresentations for the Propagators. Basically the various Z 's are defined. D_F' is the unrenormalized-but-full photon propagator, the thing that will have Z_3 in its numerator at the renormalization point which is $p^2 = 0$. Then D_{FC} is what we would call the renormalized propagator, see (2.2). Certain integral representations for Z_3^{-1} and Z_2^{-1} are given, I think these are called Kallen reps, no doubt based on unitarity somehow. A comment is made about photon mass renormalization and how you have to throw away the big quadratic divergent term.

3. QED ignoring photon propagator renormalization, ie, with $Z_3=1$. Suppose you compute Z_2 . You can use a Feynman cutoff on the photon prop so that electron self-energy parts are finite in the cutoff. Evaluating $\Sigma(p)$ with renormed charge e_1 , if you look near $p = m$ you can get Z_2 , and if you look for large p you get some result as shown in (3.5). Again, just a few terms in perturbation series for these quantities. Equation (3.9) shows how the numerators of the unrenormed and renormed ~~photon~~ electron props are related in a certain limit: cutoff.MGT.p.MGT.m. You notice that the renormalized propagators numerator S_0 is a function of p/m and does not have problems involving the cutoff λ . This thing is finite. Of course Z_2 is function of λ / m

and the un renormed prop is function of λ/e , where m is neglected. This equation (3.9) is the kind of renormalization-group ratio-style equation I have seen in BS and BD and elsewhere. Roughly it says that the UV divergence of the unrenorm prop is all contained in Z_2 . As you might guess, the only solution that equation (3.9) can have is a power solution (ie, a rep of the multiplication group). Thus, in this partial theory you find that Z_2 is a constant A times $(\lambda^2/m^2)^{-n}$ where n is some power. Ie, in perturbation theory you are only getting a few terms which are logs, but they seem to all add up to a power (a familiar happening). Then you conclude that the full electron propagator must behave as a power of p^2/m^2 , and then using the Ward identity you can learn the behavior of the vertex.

I think basically this is just dimensionality arguments at work. Equation (3.9) arises merely because you know what each function can depend on. So, using the renormalization group argument, you can in effect compute the form of the full propagator, although the little power n is not so-computed. That was probably the reason Gell-mann Low sat for a while: no way to compute n .

Notice, by the way, that there are no differential renormalization group equations in this paper. There seems as yet to be no mention of a "renormalization point" in the sense of μ of Pagels review. But there is a vague idea (look at (3.9) that you can compensate for a change in the cutoff λ by changing Z_2 (by changing the charge...). Ie, think of $s_C = s/Z_2$. This ratio has to be independent of the cutoff λ because it gives the finite, renormalized prop. I think it is this same idea which becomes the differential equation later on.

4. Ward's Method used as a cutoff. The vertex Dyson equation is written down (and Dyson given credit for defining the 1pi objects, I think). Then these equations are "subtracted" in a certain way which causes the unrenormed S_F and D_F propagators to be normalized (ie, unit residue) at points $p = \lambda'$ and $k^2 = \lambda'^2$. Thus, although they do not use the term "renormalization points", they are in effect choosing independent renormalization points for the two basic propagators of the theory. This then quickly leads to equations like (4.7). This equation relates the original divergent function $S_F'(e_0)$ to the $Z_2(\lambda, \lambda')$ and the $S_F(\lambda, \lambda')$. Ie, for any particular value of the pair λ, λ' this latter object is the finite renormalized propagator. There is sort of a double renormalization group here, two floating parameters. Now (4.7)' is just (4.7) with these parameters at their usual values (ie, the usual renorm points used in QED). Now if you divide (4.7) by (4.7)', then the original infinite or divergent function cancels out, and the ratio of the two Z_2 's is called z_2 , thus yielding equation (4.7)". This equation tells you in effect how you can deform away from the "standard" renormalized propagator S_{FC} with its charge e_1 to another prop that is a finite distance away, so to speak.

One might refer to this little triad of equations as the Renormalization Group equations for QED. Next, they go on to rewrite these equations in terms of the numerator functions only. Notice that z_2^{-1} as a number is the numerator of the standard finite function S_{FC} evaluated at the renorm point $\mu = \lambda$. Thus you end up with the set of functional equations (4.19) through (4.20). There seem to be 4 functions and two parameters e_1 .

5. Asymptotic Behavior of QED Propagators. Using dimensional arguments, our three renorm group equations are cast into the form (5.3) through (5.5). Now I have to go learn how to "solve" these things. This is appendix B so wait. For now just look at the solution: it is like KNO scaling: the function d_c = photon prop numerator = $F(z)$ where $z = (k^2/m^2) \times$ some unknown function of e_1^2 . As you take $k^2/m^2 \rightarrow \infty$ in (5.6), you are in effect computing the bare charge, as shown in (5.8). So, suppose $F(z) = \text{finite value}$ as $z \rightarrow \infty$; then the theory has a finite bare charge, which depends on the function F but not at all on the number e_1 . On the other hand perhaps $F(z) = \infty$ as $z \rightarrow \infty$, in which case there is an infinite bare charge, this is case (a), which we might now call UV Slavery (coupling grows without limit as you go to high energy). Perhaps as a special case of case (b) you will have e_0 approaches 0. This is Asymptotic Freedom. We now know that this does not happen in QED since it is abelian. Probably there is a finite UV coupling constant for QED, which may be found in new HE photon experiments. See Pagels review page 181.

In (5.9) they present an alternate form of (5.6) solution to the RG equations. Here the unknown functions are q and γ and it is claimed that you can compute these in perturbation theory (whereas you could not for F and ϕ of the other solution).

Look now at the form (5.9). Suppose the RHS diverges only as the upper endpoint goes to ∞ . This might be the case if function $\gamma(x)$ never goes through zero. In this case you would have that $k^2/m^2 \rightarrow \infty$ as $e_0 \rightarrow \infty$ so this is our friend Case (a). However, suppose $\gamma(x)$ has a zero somewhere at some finite $x = x_0$. Then you are getting $k^2/m^2 \rightarrow \infty$ corresponding to $e_0 = \text{some finite value}$, and this is case (b).

Thus, $\gamma(x)$ must be today's BETA function. Its zeros determine the positions of the UV values e might take. Somehow a future author is going to redo all this stuff using the differential equation approach.

Appendix A: Derives the Kallen-type representations for the Z's, not too relevant.

Appendix B: Shows how to solve the renormalization group equations. I will look at this discussion later if need be. It is all sitting there.