

Renormalization

Bogoliubov & Shirkov

Intro to the Theory of Quantized Fields

Chapt 4 S-Matrix

*AI in 2026: Introduction to the Theory of Quantized Fields (1959) by N.N. Bogoliubov and D.V. Shirkov is considered a classic and foundational text, but it is **outdated for learning modern Quantum Field Theory (QFT)** as a beginner. While its mathematical foundations and historical significance in the development of renormalization group methods remain highly regarded, modern curricula favor newer textbooks for covering contemporary techniques.*

I think for the first time I see how the finite corrections come into play. You start with a theory and you first make sure it is renormalizable, ie, that at worst $d_1 = 0$ for all interactions, taking proper account of vector particles. Next, you use Bogoloibov procedure to regulate all feynman graphs you might want to draw. This is a procedure for removing divergences. Basically, you only have to do this to the finite number of "renormalization parts" in your theory. Thus, eg, in QED you have certain objects like G = full electron propagator, D = full photon prop, and so on. Ie, when you write the symbol D you have in mind that the divergent piece has been subtracted out. Depending on how bad the superficial degree of divergence is for a particular renormalization part, that is how many subtractions you have to make;. These subtractions are done at your chosen renormalization point.

Thus, in QED you usually subtract the photon propagator at $k^2=0$ so that the photon pole stays at zero, and so that $Z_3 = 1$ for external on shell photon (ie, then there will be no external photon renormalization effects.)

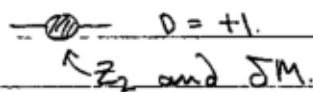
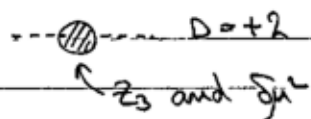
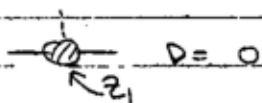
Once you have cleansed the renormalization parts of your theory, everything is just busy work from there. To compute a process, you must compute only the skeleton graphs. However, in these graphs you must include full vertices and full propagators. And note that these guys do include various corrections as you go off shell. Thus, the vacuum polarization is there, ie, there are corrections. Similarly there is a correction due to photon jumping and end vertex in Compton scattering, say. Of course formally that jump is included in the full vertex, but you still have to compute it as such. Ie, you have to compute its finite contribution. This is of course the subject of radiative corrections.

In QED, the electron self energy has $D=1$, so you have to do two subtractions, so the little derivative appears. Photon has $D=2$ so formally you have to subtract including second derivative. I think the idea of subtractions is finally sinking in.

933. The πN Field Theory

33.1 the theory. $\mathcal{L} = g : \bar{\psi} \gamma^5 \vec{\tau} \psi \cdot \vec{\phi}$ τ matrices etc. $D = -E\partial - \frac{3}{2}E\partial + Y$

33.2 divergent diagrams, etc. $\delta = \text{index} = 0 \Rightarrow \text{renormalizable.}$



Counterterms are:

$$(z_3 - 1) : g (\bar{\psi} \gamma^5 \psi) \phi :$$

$$(z_2 - 1) : \bar{\psi} (i \not{\partial} - m) \psi :$$

$$(z_1 - 1) g : \bar{\psi} \Gamma \psi \cdot \phi :$$

$$-\delta m : \bar{\psi} \psi :$$

$$-\delta \mu^2 : \phi \phi :$$

$$+ \delta : \phi^4 \leftarrow \text{a new one not in the orig Lag.}$$

$$\Gamma = \delta \tau \text{ say.}$$

33.3

You really have to add a term $h : \phi^4 :$ to original Lagrangian ("second change")
Then we can do systematically

$$(z_1 - 1) g : \bar{\psi} \delta \tau \psi \phi : \\ + (z_4 - 1) h : \phi^4 :$$

g is, both changes treated same way.

Then the renorm x fn is

$$G(m, M, g, h) \rightarrow z_2^{-1} G(m', M', g', h')$$

$$\Delta \rightarrow z_3^{-1}$$

$$\Gamma \rightarrow z_1$$

$$\square \rightarrow z_4 \square$$

$$m' = m + \delta m$$

$$M' = M + \delta M$$

$$g' = z_1 z_2^{-1} z_3^{-1} g$$

$$h' = z_4 z_1^{-1} h$$

where $\square =$

This renorm is same as $(m', g, M, h) \rightarrow (m', g', M', h')$

Note that you have to measure g and h by experiment!

Rev Sept 78

✓ Bogoliubov and Shirkov — partial contents

Chapter 4 : Removal of Divergences from the S-Matrix.

- ✓ 24. QED, Second Order (electron self mass, vacpol tensor)
- ✓ 25. QED, Third Order (the vertex, Ward, gauge invariance)
- ✓ 26. General Divergence Removal Rules (the R-operation)
- X 27. Analytic Properties of Coefficients (skipped)
- ✓ 28. Classification of Renormalizeable Theories. (the simple d_1 rule)

Chapter 5: Applications of the General Theory of Divergence Removal.

- ✓ 29. In ϕ^4 -theory (Hurst-Thirring)
- ✓ 30. QED, Part 1: General Form of Counterterms (Furry, gauge invar...)
- ✓ 31. QED, Part 2: Mass and Charge Renormalization. (Full Greens, multiplicative Renorm..)
- ✓ 32. QED, Part 3: Radiative Corrections, Second Order (Compton example)
- ✓ 33. The PI-N Theory
- 34. Schwinger's Equations for Greens functions.

Bogo + Shirk: Chapter 4 "Removal of Divergences from the S-matrix"

24.1 QED, second-order. The general notion of UV divergence is noted

24.1 The graph $\Sigma = \text{[diagram]}$. Notation $\hat{p}$ means p . n, m used for μ, ν . They note that $\Sigma \sim \int dk = \text{linearly divergent}$. What causes divergence? Product of singular functions, problem of some sort.

Regulates both D_0^C and S_0^C to $\text{reg}(D_0^C)$ and $\text{reg}(S_0^C)$ by subtraction:

$$\text{reg } D_0^C(k) = \left[-\frac{1}{k^2 + i\epsilon} + \frac{1}{k^2 - M^2 + i\epsilon} \right] \quad \text{reg } S_0^C(k) = \left[\frac{p+m}{m^2 - p^2 + i\epsilon} - \frac{p+m}{M^2 - p^2 + i\epsilon} \right]$$

Tricks used in integration are:

a) $1/(k^2 - m^2 + i\epsilon) = (-i) \int_0^\infty \exp[i\alpha(k^2 - m^2 + i\epsilon)] d\alpha$

b) $\int d^4k \exp[i(\alpha k^2 + b k)] = (\pi^2 / i \alpha^2) \exp[-i b^2 / 4\alpha]$ / derived in text.

Other nice ones also given.

24.2 Segregation of div part of Σ . The result of the above regulated calculation is

$$\text{reg } \Sigma(p) = (\text{a divergent term}) + (\text{a finite term})$$

BS then switches to coord. space. Note

that a constant in p -space $\rightarrow \delta(x)$ const in x -space. Claim: The above segregation of finite piece is not unique, example given.

If you use Feynman's cutoff method ($M^2/(M^2 - k^2)$), you get

$$\Sigma_F'(p) = \Sigma'(p)^{\text{old}} + (e^2 / 16\pi^2) (2m - \frac{3}{4}p)$$

Claim: The following object is independent of cutoff method:

$$\Sigma(p) - \Sigma(0) - \left(\frac{\partial \Sigma(p)}{\partial p^\mu} p^\mu \right) \Big|_{p=0}$$

I.e., $\Sigma(p)$ will contain a cutoff-dependent linear term of the form:

$$c_1(p-m) + c_2 m.$$

24.3 The graph [diagram] . $\Pi_{UV}(k)$ diverges quadratically this time. A long detailed calculation ensues, someday I may want to look at their method, now I don't care.

24.4 Segregation of div, + gauge invar. Result of calc is

$$\text{reg } \Pi_{\mu\nu}(k) = (\epsilon^2/8\pi^2) g_{\mu\nu} (M^2 - m^2) + \ln(M^2/m^2) \left(\dots \right) + \underbrace{\Pi'_{\mu\nu}(k)}_{\text{finite}}$$

↑
quadratic in k

Claim: $\Pi'_{\mu\nu}(k)$ is again not unique. In general

$$\Pi_{\mu\nu}(k) = \Pi'_{\mu\nu}(k) + (a + b \cdot k + c k^2)$$

contains finite & ∞ pieces, but g.i. saves only c.

Next, they note that $\Pi_{\mu\nu} = \bigcirc$

always appears between some A 's, eg, $\langle 0 | A A \bigcirc A A | 0 \rangle$

Thus, in order that you maintain

gauge invar under $A_\mu \rightarrow A_\mu + \partial_\mu \lambda$,

you need $k_\mu \Pi_{\mu\nu}(k) = 0$, which $\Rightarrow \Pi_{\mu\nu}(k) = (k_\mu k_\nu - g_{\mu\nu} k^2) \Pi(k^2)$.

Note: only the finite part of Π need respect this condition. Now look back at (24.35) to conclude that

$$\Pi'_{\mu\nu}(k) = (\epsilon^2/2\pi^2) (k_\mu k_\nu - g_{\mu\nu} k^2) \times (\text{integral}) + \text{quadratic}$$

but now we learn that the quadratic must be of the gauge invariant form.

Thus

$$\Pi'(k) = (\epsilon^2/2\pi^2) \left(\dots + c_3 \right) \text{ i.e., this term is not unique.}$$

(a fctn of k)

By the way, notice that in $\Pi'(k) - \Pi'(k_1)$ this c_3 thing goes away + there is no arbitrariness left

24.5 Conclusions for second order. I now have to input some of my own stuff.

They say

$$i^2 T(\mathcal{L}(x_1) \mathcal{L}(x_2)) = i^2 T'_M[\mathcal{L}(x_1) \mathcal{L}(x_2)] + \text{counterterms}$$

I.e., T'_M will be the order- α^2 S-matrix which $\rightarrow$ finite as $M \rightarrow \infty$. T'_M is the physically relevant part of the calc. All the counterterms contain $\delta(x_1 - x_2)$ + are divergent constants + some finite pieces too. The rule they take is this

Compute T'_M by changing Feynman rules like so

a) replace Δ with $\text{reg } \Delta$ for a propagator

b) replace Σ with Σ' , Π with Π' etc. for all loops.

↑ c_1, c_2 unknown

→ c^3 unknown

Then as $M \rightarrow \infty$, you are guaranteed a finite result.

So the amusing question which remains is: How do you say this using the counterterms, i.e., what are the counterterms?

$$\text{counterterms} = i\mathcal{L}_2(x_1, x_2) = T[\mathcal{L}(x_1)\mathcal{L}(x_2)] - T'_M[\mathcal{L}(x_1)\mathcal{L}(x_2)]$$

I have the answer. Consider the photon prop for a moment. You know that if you use the regular T you get this for a renormalized propagator:

$$D_F(q)_{\mu\nu} = \frac{-g_{\mu\nu}}{q^2 [1 + \pi(q^2)]}$$

where $\pi(q^2)$ is now in BS notation. Clearly, this makes a " Δ_3 ":

$$D_F(q \approx 0)_{\mu\nu} = -\frac{\Delta_3 g_{\mu\nu}}{q^2} \quad \text{where} \quad \Delta_3 = 1 + \pi(0)$$

But as BS show in (24.34,5): $\pi(0) = \underbrace{\frac{-e^2}{12\pi^2} \ln\left(\frac{M^2}{m^2}\right)}_{\text{a div piece}} + \underbrace{\pi'(0)}_{(24.35)} + \underbrace{c_3}_{\text{arb. term arising in other cutoff methods (may contain fin + o)}}$
 /note that $(M^2 - m^2)$ in this sentence in this g.i. form/

Thus we would normally generate a Δ_3 :

$$\Delta_3 = 1 + c_3 + \pi'(0) - (e^2/12\pi^2) \ln(M^2/m^2)$$

The idea is to have the counterterm "counter" the production of unwanted parts of this Δ_3 . So all you need do is make your counterterm alter the bare photon propagator. I.e., make the counterterm replace $A\Box A$ by $\Delta_3 A\Box A$. Then this little Δ_3 will cancel the generated Δ_3 in D_F . So counterterm should be

$$(\Delta_3 - 1)A\Box A \in i\mathcal{L}_2$$

(note that " D " is the inverse prop)

I.e., the $(A\Box A):1$ appears in T , + you want $(A\Box A):\Delta_3$ to appear in T' (i.e., with the modified rules). So actually

$$\text{counter} = T - T' = (1 - \Delta_3)A\Box A$$

$$\text{So } T = T' + (1 - \Delta_3)A\Box A.$$

So whenever c_3 arises, you counter term it away.

Note: $(1 - \Delta_3) = (1 - 1 - \pi(0)) = -\pi(0) = +\frac{e^2}{12\pi^2} \ln\left(\frac{M^2}{m^2}\right) - c_3 = "q_3"$ on p 299 BS!

Summarize: counterterm = $(1 - Z_3) \times (\text{kinetic term})$

yields the original

compensates for ignoring Z_3 your "charging the rule".

so $\underbrace{1 \cdot \text{prop}^{-1}}_{\text{the actual } Z} = \underbrace{Z_3 \cdot \text{prop}^{-1}}_{\text{the rule you use}} + \underbrace{(1 - Z_3) \text{prop}^{-1}}_{\text{the counterterm}}$

The photon mass counterterm works like this: add up series to get:

$$[q^2 g_{\mu\nu} - \Pi_{\mu\nu}(q^2)](P_F)_{\mu\nu} = -g_{\mu\nu}$$

$$= [q^2 g_{\mu\nu} - \underbrace{\left(\frac{e^2}{8\pi^2} g_{\mu\nu} [M^2 - m^2]\right)}_{\text{This is the non-gauge invariant part of } \Pi_{\mu\nu} \text{ which appears in (24.34) of their calc.}} + \underbrace{\Pi(q^2)}_{\text{gauge invariant}}] D = -1.$$

This is the non-gauge invariant part of $\Pi_{\mu\nu}$ which appears in (24.34) of their calc.

You know that unless $\Pi(q^2)$ has a pole at $q^2 = 0$ (it does not), it will cause no mass shift. Thus, that nongi piece is causing a mass shift.

But, your "operational rule" is to use the photon prop at its physical mass [just as you rule was to use it with its physical Z_3], so this happens to be zero. To fix that up you must add

$$\sim \sim = (\delta Z_3) A_\mu A^\mu \quad \text{where } \delta Z_3 = + \frac{e^2}{8\pi^2} \delta [M^2 - m^2].$$

So there is a photon mass counterterm just as for the electron.

But, probably if you were to use a gi cutoff method, such a term could never appear in $\Pi_{\mu\nu}$. That is why BD never need a photon mass counterterm.

By the way, the use of a $(1 - Z_3)$ counterterm is equivalent to only calculating skeleton graphs. One way to implement the rule

$Z_3(\text{prop})$, calculate all graphs, $\text{---} + \text{---} + \text{---}$

is to say instead

$1 \cdot (\text{prop})$, calc skeleton graphs only.

However you choose to implement the rule, the (1-23) counterterm is always implicitly there to give the difference between the "actual L " and the "operational L ".

(note) ~ I will finish this clip of BS later.

A summary of BS Renom Theory. (includes only P24)

- ① This section only discusses the graphs $\text{---}\text{---}\text{---} = \Sigma$, and $\text{---}\text{---}\text{---} = \Pi_{\mu\nu}$.
 When they go ahead and compute these integrals [allowing for possible variations in cutoff methods] they find this:

$$\text{Reg}(\Sigma(p)) = [\ln(M) \text{ term}] + \Sigma'(p)$$

$$\text{Reg}(\Pi_{\mu\nu}(k)) = [M^2_{\text{gauge viol piece}}] + [\ln(M^2)_{\text{gauge OK piece}}] + \Pi'_{\mu\nu}(k)$$

where

$$\Sigma'(p) = \frac{[\text{known finite form}]}{\Sigma''} + c_1(p-m) + c_2 m$$

$$\Pi'_{\mu\nu}(k) = (k_\mu k_\nu - g_{\mu\nu} k^2) \Pi'(k^2)$$

$$\Pi'(k^2) = \left[\frac{e^2}{2\pi^2} \dots \text{known finite} \right] + \frac{e^2}{2\pi^2} c_3$$

- ② By the way, $\text{Reg}(\Sigma(p))$ is what you get by doing subtractions in both S_F and D_F at large mass M^2 . You subtract the propagators, not $\Sigma(p)$ itself.

- ③ The renom idea is then this: in a feynman graph, replace all propagators with $\text{Reg}(\text{prop})$. Whenever you see a $\Sigma(p)$, replace it with Σ'' , and whenever you see $\Pi_{\mu\nu}$, replace with finite Π'' . Obviously, you are throwing out divergent pieces. I.e., you are changing the feynman rules.

The pieces you throw out are, if you like, recorded in the counterterms. But operationally you don't need the counterterms, you just need rules like $\Sigma \rightarrow \Sigma''$, etc.

In this §24, the counterterms are of the form

$$\pm (Z_3 - 1) F^{\mu\nu} F_{\mu\nu} \pm (Z_2 - 1) \bar{\psi}(\not{p} - m) \psi + \delta m \bar{\psi} \psi + \delta g^{\mu\nu} A A.$$

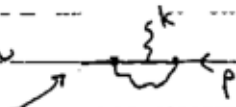
Notice that $(1 - Z_3) = "a_3" = (e^2/12\pi^2) \ln(M^2/m^2) - c_3$ may contain finite pieces also. And $\delta g^{\mu\nu} \sim (M^2 - m^2)$ is gauge violating.

- Keep in mind that really $\Sigma''(p)$ is uncertain by $c_1(p-m) + c_2 m$, i.e., only $[\Sigma''(p) - \Sigma''(0) = \Sigma''']$ is unambiguous. This is the idea of a Taylor Series of subtraction on Σ' itself which we may see in later paragraphs.

925: Removal of QED Divergences (3rd order)

● Draw some third order Feynman graphs. Many divergences can already be removed by our 2nd order analysis, eg

In fact the only new graph of interest is the vertex:



(25.1) The vertex. Here they calculate the vertex graph and find:

$$\text{reg } \Gamma^{\mu}(p, k) = \delta [\text{div } M^2 \text{ term}] + \Gamma^{\mu}(p, k) + c_4 \delta^{\mu}.$$

They choose $c_4 = 0$ to get $\Gamma^{\mu}(0, 0) = 0$, like choosing $c_1, c_2 = 0$ to get $\Sigma(p) = 0$ for $\Sigma(p)$ case.

(25.2) Segregation of Divergences + Gauge Invariance. Well, the segregation is shown above, obviously the new rule will be to replace $\text{Reg } P^{\mu}$ with P^{μ} , just like you replace $\text{Reg } \Sigma(p)$ with $\Sigma'(p)$.

● The interesting gauge invariance study here is that the sum (only) of three graphs is g_i :

$$\left(\text{---} \text{---} \text{---} + \text{---} \text{---} \text{---} + \text{---} \text{---} \text{---} \right)$$

Still true even when you Feynman regulate DF. Thus, you know that the finite parts in this sum are g_i . Interestingly, the divergent parts also sum to g_i , this is Ward Identity. By the way, here g_i means conserved current $\partial_{\mu} J^{\mu} = 0$.

(25.3) Ward Identity. Reverse the arg: the total is g_i formally: Ward ident $\Rightarrow$ divergent parts are g_i ; therefore the finite part of sum is g_i .

They go on to show that Ward Id. holds in either Feynman or their regularization procedure, I skip the details here.

(25.4) How to Renormalize at Third Order. Similar to earlier section in second-order discussion. For general 3rd order graph you render finite by first doing 2nd order subpieces as before, + now replacing also $\Gamma^{\mu} \rightarrow \Gamma^{\mu}$ for the vertex.

● The only new counterterm encountered is " $-e g_1 \bar{\psi} \gamma^{\mu} \psi$ ", i.e., a charge counterterm to remove divergent pieces of the vertex.

Section ends with comments about some spacetime "switching on" function $g(x)$.

9.26: General Rules to Remove Divergences from S-Matrix.

26.1 Formulation of Problem. Earlier in the book (1940), they show that the S-matrix greens functions can have a form like so

$$S_n(x_1, \dots, x_n) = T(\mathcal{L}(x_1) \dots \mathcal{L}(x_n)) + \sum P(x \dots) T(\Lambda(x \dots) \Lambda \dots \dots) + \Lambda_n(x_1 \dots x_n)$$

and still be covariant, unitary, etc, etc.

where the Λ^3 are called "quasilocal operators"

which exist only when, eg, $x_1 = x_2 = x_3 = \dots = x_n \Rightarrow \Lambda_n \neq 0$. These objects will here be neatly identified with subtraction terms.

You want the " Λ " to be of the same "operator type" as $T(\mathcal{L} \dots)$, so it is merely a question of shuffling coefficients.

Fine, OK so far. $\Lambda = \Delta$ modulo i^5 .

26.2 The R-operation. See added sheets. You sort of look at all possible subgraph groupings of a feynman graph. Come back later to this section.

26.3 Index $w(G)$.

$$\begin{aligned} L &= \# \text{ internal lines} & (\text{my } I) \\ n &= \# \text{ vertices} & (\text{my } V) \end{aligned}$$

Propagator is $\text{Reg } \Delta = \mathcal{L}(p) \left(\frac{1}{m^2 - p^2} - \text{sub} \right)$

polynomial like $p^2 + m$, highest power is $\textcircled{d_i}$.

So prop is $\dim(\text{Reg } \Delta) = (d_i - 2)$.

$$L - n + 1 = \text{my } l = \# \text{ indep loops.}$$

Any way, $w(G) = D = \text{superficial degree of diverg} = \text{independent of internal of graph!} = \text{"index"}$

Now suppose you do:



$$G = G_1 * G_2$$

Then $w(G) = w(G_1) + w(G_2) + \text{effect of new internal line}$

Look at a subgraph G_i . It needs a $d(G_i) = Z(G_i)$.

Claim: degree of poly $Z_{G_i}(p_1, \dots) = w(G_i)$.

OK, now put all propagators into " α rep" i.e. $\int_0^\infty e^{i\alpha(p^2 - m^2)} d\alpha$

Then,

$$\text{Feynman integral} = T_M(k) = \int d\alpha_1 \dots d\alpha_L$$

$$\times \int d^4 p_1 \dots \times \exp(i\alpha p^2 \dots)$$

Do the $d^4 p$ ints. Then

$$T_M(k) = \int d\alpha_1 \dots d\alpha_L \cdot F(k \dots \alpha \dots) e^{i \sum_{ab} A_{ab} k_a k_b}$$

↑ may have poles when $\alpha' \rightarrow 0$,
hence these are divergences

This is covered in ELOP Landau analysis.

26.4 Long mass on A_{ab}

26.5

Replace $T(\dots k) \rightarrow T(\dots k) - \text{McLaurin series up to power } w(G)$.

This stuff just is not clearly written!

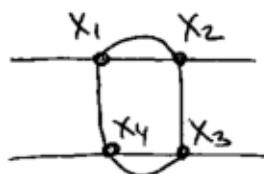
[26.2 $\rightarrow$ 26.5 are hazy sections about the R-operation]

Formula (26.9)

$$R(G) = 1 + \sum_{2 \leq m \leq n-1} \frac{\Delta(G_1) \dots \Delta(G_m)}{m!} + \Delta(G)$$

What does this mean? Take a particular Feynman graph with n vertices.

Example in ϕ^4 :

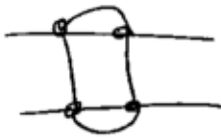


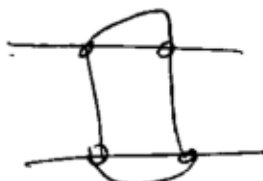
Define: a "generalized vertex" as what you get by shrinking some subgraph to a point. I.e., what you get by deleting some propagator. Eg, we might take the points x_1, x_2 + replace them by  which is a particular Δ a subtraction term, here $\Delta_2 = \text{shaded circle} = d_G(x_1, x_2) \phi^4$

Then $T(\Delta_2(x_1, x_2) \mathcal{L}(x_3) \mathcal{L}(x_4)) =$ 

The replacement  $\rightarrow$  $= d_G(x_1, x_2) \phi^4$ is called $\Delta(G)$

So here is what 26.9 says.

 $=$ a possible $\Delta(G_1) \Delta(G_2)$ on 

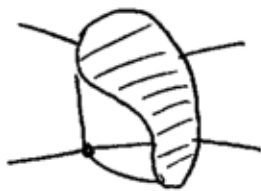
 $=$ $\Delta(G)$ on 
 $\uparrow$ full G

Other possibilities:



= a $\Delta(G_1) \Delta(G_2)$ type term.

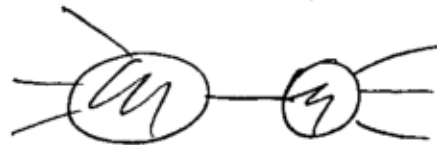
But,



seems unallowed because a gemlike $G \neq$ bare vertex.

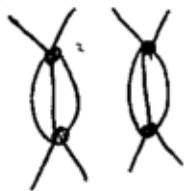
So you sort of sum over all subgraph groupings.

Connected "weakly":



Unconnected diagrams easy to handle.

So, for

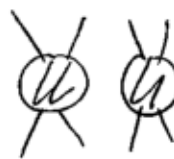


we have $\Delta(G) = 0$
 $\uparrow$
 full

i.e., this will not have a



counterpart.

 is all you get.

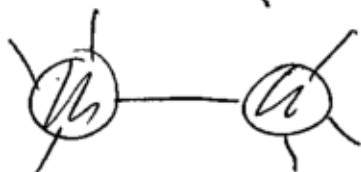
I.e., $R(G) = R(G_1) \cdot R(G_2)$

Next,



= weakly.

need only



, so $\Delta(G) = 0$ again.

Q27. Analytically of Conf Fctn? — skip

Q28. Classification of Renormalizable Theories

If $\delta_i \leq 0$, since $D = n_i \delta_i - E_S - 2E_F - 2E_W + 4$, the ~~at~~ largest possible ~~renom.~~ renom. part has 4 lines (all bosons). Any graph with > 4 ext lines is sup. convergent. Thus, # counterterms = finite

$\delta_i \leq 0 \Rightarrow$ finite # counterterms $\Rightarrow$ "renormalizable"

$\delta_i > 0 \Rightarrow \infty$ # counterterms $\Rightarrow$ "non-re"

The detail on vector bosons is noted.

Comment: a non-renormalizable theory having ∞ # counterterms is in effect non-local!

929: ϕ^3 Theory

29.1 The model $\mathcal{L} = \frac{1}{2}(\partial_\mu \phi \partial^\mu \phi - m^2 \phi^2) + \lambda \phi^3$ The simplest theory there is.

29.2 Divergent Diagrams There is one $\delta_i = 3 - 4 = -1$. (note: "vertex index") This we know theory is R.
Recall my own formula:

$$D = \underbrace{n\delta}_{\# \text{ vertices}} - E_B + 4 = -n - E_B + 4.$$

So among vac graphs:

 has $D = -2 + 4 = +2$

 has $D = -4 + 4 = 0$, also 

called
"Hust-Thinning"

Next, do one external line: (note: removes self-graphs)

 has $D = -3 - 1 + 4 = 0$. (requires a counterterm like $B \phi(x)$)

 has $D = -2 - 2 + 4 = 0$ ($D=0 \Rightarrow$ log divergence)
(usual mass counterterm needed)

Note that the linear $B\phi$ (tadpole) counterterm is removed by shifting the field just as in SSR.

930: QED, Part I: General Form of the Counterterms

30.1 Divergent Diagrams $\mathcal{L}_I = e: \bar{\psi} \not{A} \psi$ $\delta = 2 \cdot \frac{3}{2} + 1 - 4 = 0$. So,
 $\Rightarrow$ renormalizable.

$$D = 4 - E_B - \frac{3}{2} E_F.$$

The "divergent diagrams" (I call "renormalization parts") are carefully listed off.

Pause to prove Furry theorem: when you add up two closed loops going in opposite direction, sum = 0 if odd # photon attachments.
Proof is formal, but is extended to finite parts by doing the regulation.

There are only four renorm parts left:



30.2 Gauge Invariance. A long 14-page section, not too exciting.
First, (30.12) is regarded as basic statement of gauge invariance.
I think it says

$$k_\mu G^{(\mu)} \dots = 0$$

Prior to gauge invar consideration, you have A, B, C, D, E, F, G as undetermined counterterm coeffs. I.e., one Renorm Part may have >1 coefficient

Next, they want to show that, not only is the formal theory (with divergences) g_i , but the subtraction terms are also g_i , leaving you with a finite g_i theory. To do this, they make use of the "Pauli Villars" regularization method, big deal. Here is the long proof that subtracting the "quasi-local operators Λ " leaves a resultant g_i S-matrix. Skip it.

30.3 Counterterms. These are the usual ones:

$$-\delta m \bar{\psi} \psi + e Z_1 \bar{\psi} A \psi + (Z_2 - 1) [\bar{\psi} \psi] + (Z_3 - 1) [A \mu A].$$

and Ward says $Z_1 = Z_2$.

Since Pauli Villars method is G-I, you see no photon δm^2 mass term! This contrasts with the Q24 result we had earlier. So here all counter terms are g_i .

Q31. QED Part II: Mass and Charge Renorm.

31.1 About $k^\mu k^\nu$ terms. Since such terms in the "AA pairing", (α , D_F), have no effect via g_i , we can replace

$$\frac{g^{\mu\nu}}{k^2} \rightarrow \underbrace{\frac{1}{k^2} (g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2})}_{\text{transverse}} + \underbrace{\frac{d_F}{k^2} \frac{k^\mu k^\nu}{k^2}}_{\text{long part is gauge dependent.}}$$

31.2 Finite Renormalizations. This is akin to renorm group idea. Even after you remove the truly divergent things, you still can have a finite $(\delta m, Z_1, Z_3)$ type counterterm.

They show this little fact I did for myself: the $(Z_2 - 1) \bar{\psi} \psi$ counterterm is equivalent to letting $S(p) \rightarrow Z_2^{-1} S(p)$, and also $\psi \rightarrow Z_2^{-1} \psi$ for ext lines.

Similarly, the $(Z_3 - 1)$ causes a Z_3^{-1} renormalization of only the transverse part of $D_{\mu\nu}$.

However, δm cannot be interpreted in this way (i.e., as a multiplicative renorm of some causal functn).

31.3 Full Green's Etc. Here we have the usual sum of geometric series,

$$\Sigma = \text{circle with cross} = \text{wavy line} + \text{etc.}$$

$$G(p) = \text{line} + \text{line} - \text{circle} + \dots = \frac{1}{p - m - \Sigma} = \text{line} - \text{circle with cross}$$

being defined $G(p)$ and similarly D^h for photon, you can show that the little (z_1, z_2, z_3) business does this (ie, $1 - z_3$ etc)

$$\begin{aligned} G &\rightarrow \bar{z}_2 G \\ D^h &\rightarrow \bar{z}_3 D^h \\ m &\rightarrow m + \delta m \\ \Gamma &\rightarrow z_1 \Gamma \end{aligned}$$

More detail !!

$$\begin{aligned} G(e, m) &\rightarrow \bar{z}_2 G(e', m') & e' &= \left(\frac{z_1 \bar{z}_2}{\sqrt{z_3}} \right) e \\ D^h(e) &\rightarrow \bar{z}_3 D^h(e', m') \\ \Gamma(e, m) &\rightarrow z_1 \Gamma(e', m') & m' &= m + \delta m \end{aligned}$$

"Skeleton graph" = one having no renorm parts.

Rule: compute only skeleton graphs, but use full objects G, D^h, Γ .
Then no need to worry about subtracting because already done.

Comment: the (z_1, z_2, z_3) business is equivalent to an (e, m) transform.

Renorm group is noted, but discussion is deferred.

31.4 About infinite Renormalization. Dyson used these as a prescription for removing divergences. BtS note that often people do formal manipulations with divergent objects which can lead to contradictions (Pomeranchuk). They prefer their careful, regulated renorm approach.

31.5 Choice of Finite Constants. Instead of choosing $p=0$ as your renorm point, let

$$\Sigma_{\text{REG}}(p) \equiv \left[\Sigma(p) - \Sigma(p=m) - 2 \frac{\partial \Sigma(p)}{\partial p} (p-m) \right]$$

Then

$$\begin{aligned} \Sigma_{\text{REG}}(p=m) &= 0 \leftarrow \text{since } G(p) = \frac{1}{p - m - \Sigma(p)} \text{, this says } m = \text{experimental electron mass} \\ 2 \frac{\partial \Sigma_{\text{REG}}}{\partial p}(p=m) &= 0 \leftarrow \text{this? wait. No "external connected corrections"?} \end{aligned}$$

Photon: $\Pi_{reg}^{\mu\nu}(k)$ is chosen so

$$\Pi_{reg}^{\mu\nu}(0) = 0 \rightarrow \text{pole stays at } 0.$$

$$\frac{1}{k^2} \Pi_{reg}^{\mu\nu}(k=0) = 0 \rightarrow \text{no rad. correct to ext photon line}$$

Think: "no rad correct" means no Z 's for ext wavefunction.
I.e., we are choosing $Z_3 = 1$ at $k'=0$ and $Z_2 = 1$ at $p=m$.

Also, this means $e = \text{physical charge}$.

§32. QED, Part III "2nd Order Radiative Corrections".

32.1. Corrections to Photon Prop. In §24 we exactly calculated $\Pi_{\mu\nu}(k^2)$ to second-order + we identified the finite pieces. Thus, it is just math to compute

$$D_{\mu\nu}^{\text{ph}}(k) = -\frac{d(k^2)}{k^2} \left(g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2} \right)$$

where $d(k^2)$ computed to second order is

$$d(k^2) = \begin{cases} 1 - \frac{e^2}{60\pi^2} \frac{k^2}{m^2} & k^2 \ll m^2 \\ 1 + \frac{e^2}{12\pi^2} \ln \frac{k^2}{m^2} & k^2 \gg m^2 \end{cases}$$

Note that $d(0) = 1$, since we choose this as our renorm point.

32.2. Correction to Electron Prop. We want $\Sigma(p=m) = 0$ and $\partial \Sigma / \partial p \big|_{p=m} = 0$, but this second one will involve IR divergence, so put in a photon mass. The results are a bit messy:

$$G(p) = \frac{\not{p} a(p^2) + m b(p^2)}{p^2 - m^2}$$

The formulae involve $d_L^0 = \text{long term in photon prop.}$

If I set $d=0$ I get

$$a(p^2) = 1 + \frac{e^2}{16\pi^2} (A(p^2) - A(m^2)) [-6]$$

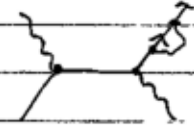
$$b(p^2) = 1 + \frac{e^2}{16\pi^2} \left(\frac{p^2 - m^2}{m^2} [-3A(p^2)] + [A(p^2) - A(m^2)] (-6) \right)$$

$$\text{where } A(p^2) = \begin{cases} \frac{m^2}{p^2} \ln \left| \frac{p^2 - m^2}{m^2} \right| & \text{when } |p^2 - m^2| \gg \lambda_0^2 \\ \frac{1}{2} \ln \frac{\lambda_0^2}{m^2} & \text{when } p^2 = m^2. \end{cases}$$

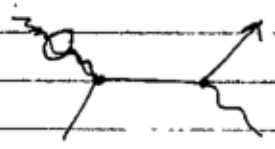
where λ_0 = that photon mass. Also, as $p \rightarrow \infty$.

$$G(p \rightarrow \infty) = -\frac{1}{p} \left(1 + \frac{e^2 d_0^0}{16\pi^2} \ln \left(\frac{p^2}{k_0^2} \right) \right).$$

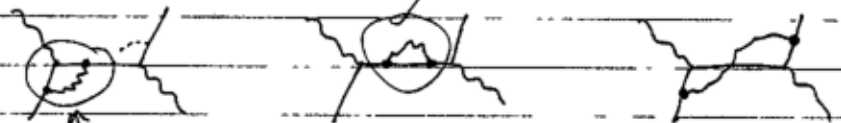
32.3 Vertex Corrections Here you must find the $\mathcal{O}^M$ extra term. They do it. No big deal.

32.4 Klein-Nishina Corrections Consider . We can ignore this one because our condition on $\partial \Sigma / \partial p = 0$ on shell. I.e., external particles are exactly on shell where $Z=1$, so no correction at all.

The same goes for because $\Pi^{\mu\nu}/k^2|_0 = 0$.

 again, need this

What do contribute are



that contribute as $\mathcal{O}^M$

(the vertex we already calculated!)

So: the finite parts of these graphs do affect physical scattering + change Compton amplitude. So you cannot just compute skeleton graphs. Of course these corrections are contained in G ,

ref: Born+Feynman

The IR problem of soft photons is then mentioned.

SELF.

Mass Renorm in Scalar QED.

① Start off with $\text{prop} = \text{---} \text{---} \text{---} + \text{---} \text{---} \text{---}$

$$= \left(\frac{+i}{p^2 - m_0^2 + i\epsilon} \right) + \left(\frac{+i}{p^2 - m_0^2 + i\epsilon} \right) \left(-i \Sigma^2(p, m_0) \right) \left(\frac{+i}{p^2 - m_0^2 + i\epsilon} \right)$$

where $[-i \Sigma^2(p, m_0)] \equiv \int d^4k \cdot [-ie_0 \not{A}_\mu] \left[\frac{-ig^{\mu\nu}}{k^2 - \lambda^2 + i\epsilon} \right] \left[\frac{+i}{(p-k)^2 - m_0^2 + i\epsilon} \right] [-ie_0 \not{A}_\nu]$

$$= -e_0^2 \cdot 4 \cdot p^2 \int d^4k \cdot \frac{1}{k^2 - \lambda^2} \cdot \frac{1}{(p-k)^2 - m_0^2}$$

If you were to add $\text{---} + \text{---} + \text{---} + \dots$,
you have

$$X + X(-i\Sigma^2)X + X\Sigma^2 X \Sigma^2 X + \dots$$

$$= X(1 + (-i\Sigma^2)X + (\Sigma^2 X)^2 + \dots) = X \frac{1}{1 - (-i\Sigma^2)X} = \frac{1}{X^{-1} + i\Sigma^2}$$

$$\text{So } (\text{prop}) = \frac{1}{\left[\frac{p^2 - m_0^2}{i} \right] + i \frac{\Sigma^2}{i}} = \frac{i}{p^2 - m_0^2 - \Sigma^2}$$

$$\boxed{\text{prop} = \frac{i}{p^2 - m_0^2 - \Sigma^2}}$$

similar to (19.3).

② Let's do the integral. Put in UV cutoff since it is linearly divergent near a at mass shell. ~~Maybe I should add no,~~
it is $dk/k = \log \text{div}$, so $\left(\frac{\Lambda^2}{k^2 + \Lambda^2} \right)$ is sufficient.

$$I \equiv \int d^4k \frac{1}{k^2 - \lambda^2} \frac{1}{(p-k)^2 - m_0^2} \cdot \left(\frac{\Lambda^2}{k^2 + \Lambda^2} \right)$$

Use Feynman trick.

Electron Mass Renormalization and the Concept of a Mass Counterterm.

1. Let m_0 be bare mass, and m be the renormalized mass to some order. Let us consider the renormalization of the electron line. The bare theory lagrangian is this:

$$\mathcal{L} = \bar{\Psi}(i\not{\partial} - m_0)\Psi - ie_0 \bar{\Psi} \not{A} \Psi \quad m_0, e_0.$$

Now, suppose we calculate the renormalized electron propagator only to lowest order, ie, to order α_0 . Let us stay close to $\not{p} = m$ so we neglect other terms in $\Sigma(p)$ other than the constant. Then you find that:

$$\begin{aligned} (\text{new prop}) &= \text{---}\xrightarrow{m_0}\text{---} + \text{---}\xrightarrow{m_0}\text{---} \text{ (loop with photon)} \text{---}\xrightarrow{m_0}\text{---} \\ &= \frac{1}{\not{p} - m_0} + \frac{1}{\not{p} - m_0} \Sigma \frac{1}{\not{p} - m_0} \end{aligned}$$

where $\Sigma = A^0 + B^0(\not{p} - m_0)$ = matrix in Dirac space.

where

$$A^0 = \frac{3\alpha_0 m_0}{4\pi} \ln \left[\frac{\Lambda^2}{m_0^2} \right] \quad B^0 = \frac{\alpha_0}{4\pi} \left(\ln \left(\frac{\Lambda^2}{m_0^2} \right) - 2 \ln \frac{m_0^2}{\lambda^2} \right)$$

Notice that A^0 and B^0 are functions of e_0 and m_0 . Here Λ is a UV cutoff and λ is a photon mass (regulator for IR problems).

2. Our problem now is to combine the two terms shown above. By ignoring any terms of higher power in α_0 , you can conclude that

$$(\text{new prop}) \doteq \frac{1}{\not{p} - m_0 - \Sigma}$$

You could then in effect look at your function $A^0(m_0)$ and choose m_0 such that $A^0 + m_0 = m$, the physical mass. Then in effect; you seem to have a new propagator propagating at a new mass. You could go on to show that the residue of the propagator is shifted because of $B^0(m_0)$.

In this approach, you see clearly that the mass is shifted by amount A_0 . However, the various functions like B_0 are functions of m_0 and not m . This is OK, but not very convenient. Notice that we have not used any "counterterms" in this approach.

Notice also that the (new prop) shown above is exactly equal to:

$$\frac{1}{\not{p} - m_0 - \Sigma} = \text{---}\xrightarrow{m_0}\text{---} + \text{---}\xrightarrow{m_0}\text{---} \text{ (loop)} \text{---}\xrightarrow{m_0}\text{---} + \text{---}\xrightarrow{m_0}\text{---} \text{ (2 loops)} \text{---}\xrightarrow{m_0}\text{---} + \text{etc}$$

Ie, if you include selected higher order corrections, you can arrange to get exactly Σ in the denominator. However, I don't think this is a useful way to think of things.

3. Let us now present a better way of doing what we have just done. Simply and trivially rewrite the lagrangian as follows:

$$\mathcal{L} = \bar{\Psi}(i\partial - m)\Psi - ie_0 \bar{\Psi} A \Psi + \delta m \bar{\Psi} \Psi \quad \delta m = m - m_0$$

We have now arranged that the bare propagator has physical mass m , even though it is a bare propagator. Also, we have added an "interaction" which, if you like, shifts propagator back again to bare mass.

Now, using this new lagrangian, let's go back and compute the lowest order shift in the prop. We know, by the way, that δm is order d_0 just like Σ , so do not forget to include the "counterterm" interaction:

$$\begin{aligned} (\text{new prop}) &= \text{---} \xrightarrow{m} \text{---} + \text{---} \text{---} \text{---} \xrightarrow{m} \text{---} + \text{---} \times \text{---} \xrightarrow{m} \text{---} \\ &= \frac{1}{\not{p} - m} + \frac{1}{\not{p} - m} \Sigma^m \frac{1}{\not{p} - m} + \frac{1}{\not{p} - m} \delta m \frac{1}{\not{p} - m} \end{aligned}$$

We have now slightly rearranged the theory, because now when we compute Σ we get:

$$\Sigma = A + B(\not{p} - m)$$

$$A = A^0(m_0 \rightarrow m) \quad B = B^0(m_0 \rightarrow m)$$

Although Σ has the same form as before, you see that A and B are now functions of the renormalized mass m . Also, the propagators are now $\not{p} - m$ instead of $\not{p} - m_0$. Of course both these facts must be compensated by the counterterm because we are computing exactly the same (new prop) object as before. We cannot get a different answer. What is nice is that now, in the counterterm method, A and especially B are evaluated at the mass m .

Also, the three terms are now very easy to add up:

$$\begin{aligned} (\text{new prop}) &= \frac{1}{\not{p} - m} \left[1 + [B(\not{p} - m) + A] \frac{1}{\not{p} - m} + \delta m \frac{1}{\not{p} - m} \right] \\ &= \frac{1}{\not{p} - m} \left[(A + \delta m) \frac{1}{\not{p} - m} + B \right] \end{aligned}$$

Now we choose m_0 such that $\delta m + A(m) = 0$. This is in fact not the same condition as before. But to order d_0 it is the same. You have to keep your cutoff in there so that a "term of order d_0 " is really of that order, and not logarithmically divergent, or worse.

Having so chosen the mass m_0 , we have eliminated m_0 from the theory and we find:

$$(\text{new prop}) = \frac{1+B}{\not{p} - m} \quad \text{so} \quad \Sigma_2 = 1+B$$

So, once again, I can add these higher terms without doing any harm to the order of interest. It makes it easy to sum up the series to get answer which must be correct to order of interest:

$$(\text{new prop}) = \frac{1}{p-m-\Sigma-\delta m}$$

Now the trick is to choose m_0 in such a way that δm cancels against the constant term in Σ . This will cause your mass to come out at m as desired. Consider:

$$\Sigma = A + B(p-m).$$

$$(\text{new prop}) = \frac{1}{p-m-\cancel{A}-B(p-m)-\cancel{\delta m}} = \frac{1}{(p-m)(1-B)} \approx \frac{1+B}{p-m}$$

So fine, we get the same result as before. It does not matter how you write something if it is the same either way.

Renormalization of QED to order α .

Once the counterterm idea is understood, the rest is relatively easy. Eg, we go off and compute the graph γ_m (with m everywhere) and we find that the regular bare vertex is enhanced from 1 to $(Z_1^{-1} - 1)$. But the Ward identity shows that $Z_1 = Z_2$ and we have already computed Z_2 . Thus, no need to compute Z_1 .

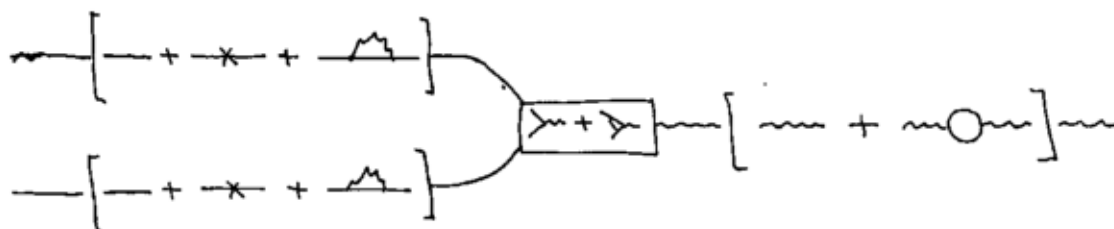
Now eventually you want to see the complete renormalization of the vertex. To do this, you simply add together all graphs of order α , as shown on page 168 of BD1. You trivially add all these things up and you find that the net effect is that the bare vertex is enhanced by $\sqrt{Z_3}$.

What was Z_3 ? Had to do with the Vac pol.

1. Vacuum pol graph. First of all, when you add the correction  to , (again with physical mass m on these electron lines), you find that the photon mass is not shifted away from the bare value of zero. This is so because the loop $\Pi_{\mu\nu}(q)$ has no pole in q^2 . This in turn follows from unitarity, because there are no massless intermediate states you can put in there (like Goldstones!). In fancy gauge theory, the goldstones generated by SSB are just those massless states needed to shift the "photon" mass away from zero (see Bernstein review notes).

So aside from the small q^2 correction (which yields the Uhling potential and 27mc worth of hydrogen Lamb shift), the effect of the vacpol thing is to renormalize the photon propagator to (new prop) = $Z_3 g^{\mu\nu}/q^2$. It is this $\sqrt{Z_3}$ taken to each adjacent vertex which causes the $\sqrt{Z_3}$ charge renormalization noted above.

2. A very easy way to look at things is this: Z_2 and Z_3 are the new electron and photon propagator residues. Consider this way of thinking about the renormalization of a vertex internal to a Feynman graph:



This symbolic figure represents the sum of $3 \times 3 \times 2 \times 2 = 36$ Feynman graphs. One is order α_0 (the bare vertex), 6 are of order α_0^2 , some are α_0^3 , and some are even order α_0^4 . The reason for writing this is that what we essentially have is:



I.e., the box which is $(\gamma + \delta)$ yields a multiplicative factor of Z_1^{-1} (this is the definition of Z_1); each of the line propagators yields its $\sqrt{Z_1}$ since you take ~~half the full~~ since you write $Z_1 = \sqrt{Z_1} \sqrt{Z_1}$ and take one factor in each direction.

By this method, we immediately get the result of (8.58) on page 169 of BD. You can then say that the fact that we added graphs of higher order does not change things because we are neglecting them anyway.

This approach has the advantage of applying to an arbitrary internal vertex. Notice that we have not computed the renormalization to order α_0^3 because, even though we included 36 graphs, we did not include such things as . Thus, our result is only good for this order α_0^2 .

In effect, we "divided" by $\sqrt{Z_1}$ for each electron propagator, because this is the factor that goes off and attaches to the "external" blobs which I did not show. Thus could just as well be external particles.

3. Important point: you must maintain and hold your cutoffs at small finite values while you do renormalization in any order. So far the only counterterm I have encountered is the simple mass counterterm.

Renormalization of the electron propagator to arbitrary order.

1. Need a good notation. I will assume from the start that we shifted to the physical mass in lagrangian and add counterterm δm . It will turn out that in any order of renormalization, the quantity δm will be of order α_0 , but we will compute it to higher and higher orders of accuracy. Now, here are some definitions:

$$\boxed{1} = \text{---} \times \text{---} + \text{---} \text{ (loop) } \text{---} = \Sigma'(1) + \delta m$$

$$\boxed{2} = \text{---} \text{ (loop) } \times \text{---} + \text{---} \text{ (loop) } \text{---} + \text{---} \text{ (loop) } \text{---} = \Sigma'(2)$$

$$\boxed{3} = \text{---} \text{ (loop) } \times \times \text{---} + \text{---} \text{ (loop) } \times \text{---} + \text{---} \text{ (loop) } \times \text{---} + \text{---} \text{ (loop) } \times \text{---} + \text{etc.}$$

$$\boxed{1,2} = \boxed{1} + \boxed{2} = \Sigma'(1) + \Sigma'(2) + \delta m = \Sigma'(1,2) + \delta m$$

$$\boxed{1,2..n} = \boxed{1} + \boxed{2} + \dots + \boxed{n} = \Sigma'(1,2..n) + \delta m$$

Obviously, Box(3) are all third order proper graphs contributing to the electron self-energy part. Box(1,2) are all graphs of 1st and second order. Notice that the simple counterterm graph is included in the Box(1) object — it is a proper graph.

2. Next, we can define the ~~fn~~ renormalized electron propagator. Notice that you are always allowed to inadvertently include graphs of higher order than you want without harming anything. Thus, we have:

$$\textcircled{1} = \text{---} + \boxed{1}$$

$$\textcircled{1,2} = \text{---} + \boxed{1,2} + \boxed{1,2} \text{---} \boxed{1,2}$$

$$\textcircled{1,2..n} = \text{---} + \boxed{1,2..n} + \boxed{1..n} \text{---} \boxed{1..n} + \dots$$

Notice that we never deal with Egg(3), but only Egg(1,2,3). Thus we can shorten the notation by using instead:

$$\textcircled{1} = \text{---} + \boxed{1}$$

$$\textcircled{2} = \text{---} + \boxed{1,2} + \boxed{1,2} \text{---} \boxed{1,2}$$

$$\textcircled{3} = \text{---} + \boxed{1,2,3} + \boxed{1,2,3} \text{---} \boxed{1,2,3} + \text{---} \text{---} \text{---}$$

Eg, in Egg(3). (in new notation), we must include Box(123) iterated 2 and 3 times so that we are sure to include all terms of third order. Obviously we accidentally include terms of higher order but that is OK.

In fact, since we are allowed to add such higher terms to our hearts content we can do the following:

$$\text{---} \textcircled{1} \text{---} = \text{---} + \text{---} \boxed{1} \text{---} + \text{---} \boxed{1} \boxed{1} \text{---} + \dots \text{fancy}$$

$$\text{---} \textcircled{12..n} \text{---} = \text{---} + \text{---} \boxed{12..n} \text{---} + \text{---} \boxed{12..n} \boxed{12..n} \text{---} + \text{fancy.}$$

By faking these things up into infinite series, we have made it easier to sum them up. Thus we have:

$$\text{---} \textcircled{12..n} \text{---} = \frac{1}{(-)^{-1} - \boxed{12..n}}$$

Or, in more familiar notation:

$$D(n) = \frac{1}{\not{p} - m - \Sigma(1,2..n) - \delta m}$$

3. Now a slight digression. Consider an arbitrary self energy graph that is part of some Box(12...n). Exactly on the mass shell this must be some number, a function of the mass of the electron. Expanding around this point, the lowest correction would be of the form $B(\not{p}-m)$, where we set $p^2 = m^2$ inside coefficient B. We could include a higher order correction $(\not{p}-m)^2$ term, but easy to show that this doesn't matter because we are interested in the final propagator very close to $\not{p}=m$.

Thus, in general we will find that:

$$\text{---} \boxed{n} \text{---} = A(n) + B(n)(\not{p}-m) = \Sigma'(n)$$

Adding things up we will find then that:

$$\text{---} \boxed{12..n} \text{---} = A(12..n) + B(12..n)(\not{p}-m) = \Sigma'(1,2..n) + \delta m$$

where $A(1,2..n) = A(1) + A(2) + \dots A(n)$.

Obviously $A(n)$ is of order n ; and $A(12...n)$ is of order 1, but is computed to accuracy of order n .

4. Insert this last result into the n -th order propagator:

$$D(n) = \frac{1}{\not{p} - m - A(12..n) - B(12..n)(\not{p}-m) - \delta m} = \frac{(1 + B(1,2...n))}{\not{p} - m}$$

$$\text{if } \delta m = -A(1,2...n).$$

Now the prescription is clear. If you are working to order n , you choose $\delta m = -A(1,2,\dots,n)$ and then $Z_2 = 1 + B(1,2,\dots,n)$.

5. Look what happens.

$$\begin{aligned}\delta m(n) &= -A(1,2,\dots,n) = -A(1,2,\dots,n-1) - A(n) \\ &= \delta m(n-1) - A(n).\end{aligned}$$

Each time we go from one order to the next higher order, we have to adjust δm by the amount $A(n)$ which is of order n . I.e., the adjustments get smaller and smaller. But each stage you do have to reset the counterterm mass δm .

Similarly, each term you improve the accuracy of Z_2 (ie, of $1+B$) to another order of n .

6. So, how do you do things to second order? First you compute $\text{Box}(1)$; this tells you $A(1)$ and $B(1)$. But $\delta m(1) = -A(1)$, so you now know the first order mass shift. Your residue is $1+B(1)$.

Next, you examine the graphs which make up $\text{Box}(2)$. You now know $A(2)$ and $B(2)$. But these are precisely the corrections needed to compute $\delta m(2)$ from $\delta m(1)$, and $Z_2(2)$ from $Z_2(1)$. Thus you are all set to second order.

In principle you keep going forever. Presumably the partial sums $A(1,2,\dots,n)$ and $B(1,2,\dots,n)$ converge! Obviously you have to hold all your regularizations and cutoffs fixed for the entire process. Things like $\delta m(n)$ will depend on these cutoffs. Only after the entire renorm process are you allowed to let these cutoffs go to inf.

7. So, we now know precisely how to compute $\delta m(n)$ and $Z_2(n)$ to arbitrary order n . Let us move on.

$$\Sigma(n) = A(n) + B(n)(\not{p} - m)$$

$$A(1,2,\dots,n) = A(1) + A(2) + \dots + A(n).$$

$$\delta m(n) = -A(1,2,\dots,n)$$

$$Z_2(n) = 1 + B(1,2,\dots,n).$$

$$D(n) = \frac{Z_2(n)}{\not{p} - m}$$

Renormalization and the General Notion of a Counterterm.

1. QED. Up to now we have encountered a mass counterterm only. We found that if you considered:  in QED you found by trivial power count that this thing diverges, ie, is cutoff-dependent. Thus we added a counterterm ~~*~~ chosen just right to cancel the cutoff-dependent constant = . Then we go to higher order and get graphs like . We add them all up in some order, and then we just readjust the original counterterm ~~$\delta m \bar{\psi} \psi$~~ to exactly cancel. Thus, to any given order we can arrange to have a finite propagator; more accurately, we can arrange for Σ to be finite when $p = m$. In fact it appears that we always make adjustments so that $\Sigma_{\text{eff}} = 0$. I'm not sure what happens to Z_2 .

Now, if you want you can look at the vertex  in the same way. You find that the graph  diverges, ie, is a function of cutoff; so you introduce a counterterm of the form ~~$\delta e \bar{\psi} \not{A} \psi$~~ to cancel this divergence. Thus, once you ~~xxx~~ have included this counterterm, you can ignore the graph  since you have cancelled it, and you only need worry about . Ie, you have adjusted the "charge counterterm" so that $\Lambda_p = 0$, in entire analogy to adjusting the mass counterterm so that $\Sigma = 0$.

As you go to higher order, you have to readjust δe to get this δe counterterm to cancel against Λ^μ computed to order of interest.

Are there any other counterterms needed in QED? Potentially, each n-point function may need a counterterm, so you should check them. Look at . Had this loop integration been infinite, you would need a counterterm of the form $\lambda \bar{\psi} \psi \bar{\psi} \psi$ in your lagrangian. Then QED would not be "renormalizeable". But as simple powercount shows $4/6$ yields convergent integral. Less obvious is the fact that Delbruck  does not require a counterterm.

So it is really the fact that, in QED, the only counterterms ever required are ones which are already in the lagrangian, that makes QED renormalizeable. Ie, to any order you can kill off divergences. There are of course subtleties like overlapping vertex insertions etc, but the basic point seems clear.

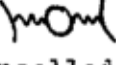
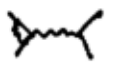
2. Now consider ϕ^n theories. Try some ϕ^3 . Do you need a mass counterterm?  Yes. Do you need a ϕ^3 counterterm?  = $4/6$ no! Higher terms are even better. Thus, in this theory I don't think anything is infinite except the mass stuff.

What about ϕ^4 ? Mass counterterm?  Each loop is $4/4 = \log$ divergent, so you will need a mass counterterm. There are no $n=3$ graphs. Look at this $n=4$ graph:  = $4/4 = \log \text{div}$. So you do need a $\delta \lambda \phi^4$ counterterm. Maybe you need a counterterm for $n=6$? Then theory is nonrenorm. The only graphs you might

draw is something like this: . But this contains subgraphs which have already been "counter-termed out". I.e., the graph just drawn is cancelled by one like this  where the dots indicate these are the counterterm ϕ^4 vertices. This leaves the regular graph  = $4/6$ which is finite UV. I can't find any ϕ^4 graphs with $n=6$ that require counterterms. Graph must be divergent and not contain reducible subgraphs. By the way,  contributes to mass ct-term.

Now we move on to ϕ^5 . Rather than look at the lower cases, let's go higher. Let $n=6$ and consider  = $4/4 = \text{logdiv}$. At once we need a ϕ^6 counterterm, but this is only ϕ^4 theory! Also there is  = $4/2$ so we need a ϕ^3 counterterm. This theory is terrible! In this theory, for any even number of external particles, or odd for that matter, there is a graph like ,  ie, an inscribed regular polygon with n sides. You cannot isolate counter-termed out factors, so you will need a counterterm of every order!

{ In general there is a rule which says, above ϕ^4 you will need infinite number of counterterms, and for ϕ^4 and less theory is renormalizable.

The notion of a graph being "counter-termed out": in QED, eg, you don't ever compute the graph  in ee scattering, even if you want order α^4 . This would have been cancelled by a mass counterterm graph, except it is a gauge theory so skip this example. You can omit  because left vertex has been adjusted to include this already. If you like, this is cancelled by a charge counter-term graph .

Of course it is the infinite piece which is cancelled out. I guess some finite piece might sometimes remain (Uehling potential etc. ()) yes!

The Meaning of a Renormalization Part.

1. Recall (AL Section 18) that any L_I you look at is characterized by a number $\delta = \dim(L) - 4$ where you ignore the dimensionality of constants of coupling. Recall that $\delta = N_b + (3/2)N_F + N_D - 4$, where N_b = number of bosons at this vertex, N_D is the number of derivatives in the coupling, etc.

Here is a powerful fact that is easily proved (AL 18.5): regardless of how complicated a graph you have, even if it involves several kinds of interaction lagrangians L_i with their various δ_i , then in two special cases we can relate the superficial divergence (ie, the power count) D of the graph to simply the number of external particles:

case 1) no derivatives case 2) If some L_i does have derivs, $\delta_i = 0$.
No, I said that wrong. The formula is this:

$$D = n_i \delta_i - E_B - (3/2)E_F + 4.$$

Often it happens that theory has $\delta_i = 0$ for all couplings; this corresponds to dimensionless coupling constants; for example, this is the case in ϕ^4 , QED, and gauge-ghost interactions. (n_i = # vertices of i^{th} type).

2. So what? Assume for the moment that $\delta_i = 0$. Then you can see that as you make larger and larger Greens functions, you improve the convergence of any diagram. Thus, if $\delta_i = 0$, only a small number of Greens functions will have $D > 0$.

Example: in QED only "renormalization parts" are  with $D = 2$; (gauge invar reduces this to $D=2$ see BD);  with $D=+1$ (but $=0$ by Furry);  $D=0$, but gauge makes $D=-4$; all higher photon parts have negative D at the start;  with $D = +1$; and  with $D=0$.

So in QED, after gauge invariance is accounted for, the only surviving renormalization parts are , , .

def: "renormalization part" is a proper graph with $D > 0$. In general, these are the only graphs for which you will need counterterms in your theory; these are the only n -point functions which diverge.

At once without looking at any diagrams, you know that  in QED does not require a counterterm; it is not a renormalization part; $D < 0$. By the way, $D < 0$ does not promise you that ~~the diagram~~ the diagram is finite; it could have a vertex insertion; but in the end such things are countertermed out, so $D < 0$ really does imply finite in the end.

Example: In ϕ^4 theory (again: $\delta_i = 0$),  has $D=+2$;  = $D=+1$ except $=0$. Then also  has $D=0$; in this theory, you will have only two counterterms.

↓

2. Suddenly now a fact becomes very obvious: any theory which contains an L_i with a ~~xxx~~ $\delta_i > 0$ must have an infinite number of renormalization parts! .
 The reason is that, no matter how many external particles you have (the more the better since they tend to reduce D) by adding enough vertices of type i with $\delta_i > 0$ you can always make a feynman diagram with $D > 0$. In fact there is no limit as to how high you can make D .

In such a theory (one having a $\delta_i > 0$) every n-point function is a renormalization part and you need counterterms of every order; (I wonder if non-polynomila interactions like $e^{a\phi}$ could help here?).

Example: ϕ^5 or more; I suddenly realize that equation (3.5) of Colemans notes is just what I have been saying here. Another example: weak interactions with $\bar{\psi}\psi\psi$ kind of stuff; here $\delta_i = +2$, you are dead.

3. Digression: why doesnt the W boson resuce weak interactions renorm, because $L = g \bar{\psi}(1+\gamma_5)\psi W^\mu$ has $\delta_i = 0$, dimensionless coupling, etc. Ie it lloks good.

The reason is this: the W-boson is coupled to a non-conserved current (in particular the axial part), so the W-boson does not give you E^{-2} for each propagator but rather E^0 . Thus we have to readjust all those formulas. I'll make new ones here for this case:

$I_W = \# \text{ internal "bad" vector boson lines.}$

$$\text{Then } D = 4I - 2I_B - 1I_F - 0I_W = 4I - 2I_B - I_F.$$

$$\text{Now } I = (I_B + I_W) + I_F - V + 1 \text{ so}$$

$$D = 4I_B + 4I_W + 4I_F - 4V + 4 - 2I_B - I_F = 2I_B + 4I_W + 3I_F - 4V + 4.$$

Now we also have:

$n_i b_i = \text{total regular boson lines emitted in digim by vertex} = E_B + 2I_B$

$$\text{Then } \left. \begin{aligned} n_i w_i &= E_W + 2I_W \\ n_i f_i &= E_F + 2I_F \end{aligned} \right] \checkmark$$

$$\text{So } D = (n_i b_i - E_B) + 2(n_i w_i - E_W) + 3(n_i f_i - E_F) - 4V + 4 + \sum n_i d_i$$

$$= \sum n_i [b_i + 2w_i + \frac{3}{2}f_i - 4] - E_B - 2E_W - \frac{3}{2}E_F - 4V + 4$$

$\uparrow$

$$= -4 \sum n_i$$

Thus, let

$$\delta_i = (b_i + 2w_i + \frac{3}{2}f_i + d_i) - 4 \quad \checkmark$$

$$\text{Then: } D = \sum n_i \delta_i - E_b - 2E_w - \frac{3}{2}E_f + 4 \quad \checkmark$$

Ouch, now you see the problem. A massive vector boson enters just like a regular boson, but each appearance in these equations is multiplied by 2! In particular, we now see that the $\bar{\psi}\psi W$ lagrangian really has $\epsilon = +1$; again you are dead! $\checkmark$ A good result to know.

4. Conclusion: when you see an interaction lagrangian, figure out ϵ . If $\epsilon > 0$ you can throw it out the window because it will have an infinite number of renormalization parts and counterterms.

Derivation of the D and δ formulas.

(taken from Ph.D. talk on 3.3.99)

- ① Consider a Feynman graph.
- B = scalars or massless vectors
- W = massive vectors
- F = spin- $\frac{1}{2}$ fermions.

then: $D = 4L - \sum_i [2 \cdot I_B^i + 1 \cdot I_F^i + 0 \cdot I_W^i] + \sum_i n_i d_i$

since these are powers for various propagators $d_i = \#$ derivatives in d_i .

② but $L = I - V + 1 = \sum_i [I_B^i + I_F^i + I_W^i] - \sum_i n_i + 1$.

③ Also, $\sum_i n_i b_i^x = E_B^x + 2 I_B^x$ + similarly for others.

④ Plug these in to find:

$$D = \sum_i n_i \delta_i + 4 - E_B - \frac{3}{2} E_F - 2 E_W$$

$$E_B = \sum_i E_B^i, \text{ etc.}$$

where $\delta_i = \left(\sum_x [1 \cdot b_i^x + 2 \cdot w_i^x + \frac{3}{2} f_i^x] + d_i \right) - 4$.

So this last is just a definition.

- ⑤ Examples:
- a) $\bar{\Psi} \gamma_\mu (1 + \gamma_5) \Psi \bar{\Psi} \gamma^\mu (1 + \gamma_5) \Psi$ $\delta = +2$
 - b) $\bar{\Psi} \gamma_\mu (1 + \gamma_5) \Psi W^\mu$ $\delta = +1$
 - c) $\bar{\Psi} \gamma_\mu \Psi A^\mu$ QED $\delta = 0$
 - d) $\bar{F}^{\mu\nu} g \vec{A} \times \vec{A}$ QCD $\delta = 0$
 - e) $g(\vec{A} \times \vec{A}) \cdot g(\vec{A} \times \vec{A})$ QCD 4gluon $\delta = 0$
 - f) $\lambda \phi^4$ $\delta = 0$

⑥ $D=2$ quad. Λ^2

$D=1$ linear Λ

$D=0$ log. $\ln \Lambda$

$D < 0$ conv.

Now you could examine:

$$\underbrace{\Pi \left[\frac{\Lambda^2}{q^2}, \frac{m_0^2}{q^2} \right]}_{f(q^2)} = \Pi \left[\frac{\Lambda^2}{m_0^2}, 1 \right] + (q^2 - m_0^2) f'(m_0^2)$$

$$f(q^2) = A^0(m_0) + (q^2 - m_0^2) B^0(m_0).$$

Then

$$iD'(q^2) = \frac{i}{(q^2 - m_0^2) + g_0^2 A^0(m_0) + (q^2 - m_0^2) g_0^2 B^0(m_0)}$$

$$= \frac{i (1 + g_0^2 B^0(m_0))^{-1}}{q^2 - m_0^2 + \left[\frac{g_0^2 A^0(m_0)}{(1 + g_0^2 B^0(m_0))} \right]}$$

Now, write

$$Z_1 = (1 + g_0^2 B^0(m_0))^{-1} \approx 1 - g_0^2 B^0(m_0)$$

$$m^2 = m_0^2 - \left[\frac{g_0^2 A^0(m_0)}{(1 + g_0^2 B^0(m_0))} \right]$$

Since you know A^0, B^0 you choose m_0 such that $m_0^2 - [\] = m^2$, where $m^2 =$ desired physical mass. Λ is held fixed. I.e., there is some function $m_0 = m_0(m)$ which solves $m_0^2 - [\] = m^2$. Then you have

$$iD'(q^2) = \frac{i [1 + g_0^2 B^0[m_0(m)]]^{-1}}{q^2 - m^2 + i\epsilon}$$

and this involves only knowing g_0 and m . This is the propagator renormalized only to first order.

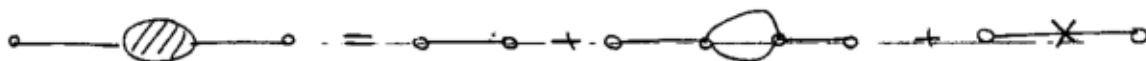
Counterterm Method. A little easier

$$\mathcal{L} = \frac{1}{2} (\partial_\mu \phi \partial^\mu \phi - m^2 \phi^2) + \frac{1}{2} (\delta m^2) \phi^2 - i g_0 \phi^3$$

where $\delta m^2 = m^2 - m_0^2$. Now 2 interactions. We have already seen that

$$\delta m^2 = \text{order } g_0^2, \text{ so:}$$

Now,



$$iD'(q^2) = iD(q^2) + iD(q^2) \left[ig_0^2 \Pi(q^2) + i \frac{1}{2} \delta m^2 \right] iD(q^2)$$

where now D, Π evaluated at m , not m_0 as before. Note i in Feynman rule for δm^2 . So add up again to get (also no $\frac{1}{2}$ because $\langle a | \phi \phi | a \rangle$ looks up two ways).

$$iD'(q^2) = \frac{i}{q^2 - m^2 + g_0^2 \Pi(q^2, m^2) + \delta m^2}$$

This time we compute

$$\Pi(q^2, m^2) = \Pi(m^2, m^2) + (q^2 - m^2) \Pi'(m^2, m^2)$$

Then

$$iD'(q^2) = \frac{i [1 + g_0^2 \Pi'(m^2, m^2)]^{-1}}{(q^2 - m^2 + i\epsilon) + [g_0^2 \Pi(m^2, m^2) + \delta m^2]}$$

Now we compute $\Pi(q^2, m^2)$ easily to deduce $\Pi(m^2, m^2)$ and $\Pi'(m^2, m^2)$. Then we simply choose δm^2 to cancel:

$$\delta m^2 \equiv -g_0^2 \Pi(m^2, m^2)$$

(In effect, this is like choosing m_0^2 .) The advantage of this method is that you now have Z with no further work:

$$iD'(q^2) = \frac{iZ}{q^2 - m^2 + i\epsilon} \quad Z = (1 + g_0^2 \Pi'(m^2, m^2))^{-1}$$

Counterterm method (1st order).

- 1) compute things with m not m_0
- 2) for Λ fixed, choose δm^2 to cancel $g_0^2 \Pi'$
- 3) Z is then known.

Dyson Equation. We know that there is a full self-energy part

$$\text{diagram with cross-hatched circle} = \text{diagram with cross} + \text{diagram with loop} + \dots$$

$$= \boxed{1, 2, \dots, \infty} \text{ in my older notation.}$$

Then

$$\text{diagram with diagonal lines} = \text{diagram with two dots} + \text{diagram with cross-hatched circle} + \text{diagram with two cross-hatched circles} + \dots \text{ etc.}$$

Let us now name things:

$$\text{diagram with diagonal lines} = iD'(q^2) = \text{the fully renormalized propagator.}$$

$$\text{diagram with cross-hatched circle} = i\Pi'(q^2) = \text{the full self-energy part.}$$

How are these related? Also

$$iD' = \frac{iD}{1 + \Pi'D}$$

$$\text{diagram with two dots} = iD(q^2) = \frac{i}{q^2 - m^2}$$

Assume that $\Pi'(q^2) = \Pi'(m^2) + (q^2 - m^2) \frac{d\Pi'}{dq^2} \Big|_{m^2}$. Then

$$iD'(q^2) = \frac{i}{(q^2 - m^2) + \Pi'(q^2)} = \frac{i}{(q^2 - m^2) + \Pi'(m^2) + (q^2 - m^2) \frac{d\Pi'}{dq^2} \Big|_{m^2}}$$

$$= \frac{i \left(1 + \frac{d\Pi'}{dq^2} \Big|_{m^2}\right)^{-1}}{q^2 - m^2 + \Pi'(m^2) \left(1 + \frac{d\Pi'}{dq^2} \Big|_{m^2}\right)^{-1}}$$

The condition that mass be m^2 is $\Pi'(m^2) = 0$. To first order this reads

$$g_0^2 \Pi^{(1)} + \delta m^2 = 0$$

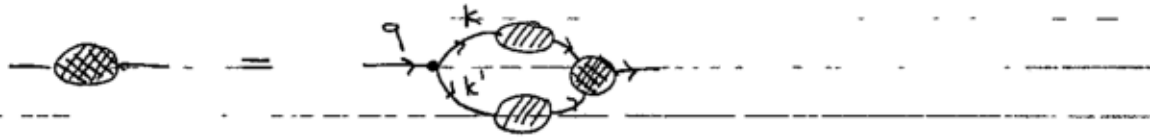
So here is result:

$$iD'(q^2) = \frac{iZ}{q^2 - m^2} \text{ near } q^2 = m^2.$$

$$\Pi'(m^2) = 0$$

$$Z^{-1} = \left(1 + \frac{d\Pi'(q^2)}{dq^2} \Big|_{m^2}\right)$$

Back to the Dyson: it reads:



$$i\Pi'(q^2) = \frac{1}{2} \int dk [i\Pi'(q, k, k')] [iD'(k^2)] [iD'(k'^2)] [i\Pi'(k, k', -q)]$$

where $i\Pi' = \text{diagram} = g_0$ as shown on first page. Comment: I have never proved this Dyson eqn. It is a topological thing. I will prove it later & elsewhere, accept it for now:

1) what does mass-renom say about dyson?

$$0 = \left(\frac{1}{2} \int dk i\Pi'(q, k, k') iD'(k^2) iD'(k'^2) i\Pi'(k, k', -q) \right)_{q^2 = m^2}$$

2) how do we relate Π' to unrenorm propagators:

$$D' = \frac{D}{1 + \Pi'D} \Rightarrow D'^{-1} = \frac{1 + \Pi'D}{D} = D^{-1} + \Pi'$$

Thus $\boxed{\Pi'(q^2) = D'(q^2)^{-1} - D^{-1}(q^2)}$ with no exponents.

3) General form of prop:

$$\boxed{D'(q^2) = \frac{Z(q^2)}{q^2 - m^2} = \text{most general form!}}$$

$$D(q^2) = \frac{1}{q^2 - m^2}$$

Dyson reads

$$\frac{(q^2 - m^2)}{Z(q^2)} - (q^2 - m^2) = g_0 \frac{1}{2} i \int dk \frac{Z(k^2)}{k^2 - m^2} \frac{Z(k'^2)}{k'^2 - m^2} [i\Pi'(k, k', -q)]$$

$$\Rightarrow \frac{1}{Z(q^2)} = 1 + \frac{1}{(q^2 - m^2)} \frac{1}{2} i g_0 \int dk \frac{Z(k^2) Z(k'^2)}{k^2 - m^2 k'^2 - m^2} f(k, q^2, k')$$

Since no gauge invariance [no gauge fields!] there is no Ward identity + I see no way to relate Γ' to propagator, so cannot solve this thing, as shown it is not a closed equation:

Renormalization in QED

● Lagrangian: $\mathcal{L} = \bar{\Psi}(i\gamma - m)\Psi - e\bar{\Psi}\gamma^\mu\Psi A_\mu + \frac{1}{2}m^2\bar{\Psi}\Psi$

Feynman Rules:

 $= -ie\gamma_\mu$

 $= iD_F(q)_{\mu\nu} = \frac{-ig_{\mu\nu}}{q^2} + \text{gauge}$

 $= iS_F(q^2) = \frac{i}{q - m}$

 $= i\delta m$

Photon Renormalization

 $=$  $+$  $+$  $+$ etc.

$iD_F'^{\mu\nu} = iD_F^{\mu\nu} + iD_F^{\mu\alpha} [ie^2\Pi_{\alpha\beta}] iD_F^{\beta\nu} + \text{etc}$

↑ put here only for convenience; (Πe^2) is computed to all orders.

Add up the series like so:

$\underline{iD_F'} = [1 + \underline{iD_F} ie^2\Pi + \dots] \underline{iD_F} = [1 + e^2 \underline{D_F} \Pi]^{-1} \underline{iD_F}$

$[1 + e^2 \underline{D_F} \Pi] \underline{iD_F'} = \underline{iD_F}$ Let $\underline{iD_F} = -ig_{\mu\nu}/q^2$

$\Rightarrow [1 + e^2(-\frac{1}{q^2})\Pi/q^2] \underline{D_F'} = -\frac{1}{q^2}$

$\Rightarrow [q^2 \underline{1} - e^2 \Pi] \underline{D_F'} = -\underline{1}$ where $\underline{1} = g_{\mu\nu}$.

This is BDT (19.6).

The object $\Pi(q)$: using Ward you can show that $q_\mu \Pi^{\mu\nu} = 0$, so you conclude that

$\Pi_{\mu\nu} = (q_\mu q_\nu - g_{\mu\nu} q^2) \Pi$

$\Rightarrow [q^2(1 + e^2 \Pi(q^2)) g_{\mu\lambda} - e^2 q_\mu q_\lambda \Pi(q^2)] D_F'(q^2)^{\lambda\nu} = -g_{\mu\nu}$

Solve this for D_F' to find [as in BDI, 1924]

$$D_F'(q)_{\mu\nu} = - \frac{g_{\mu\nu} \cdot [1 + e_0^2 \Pi(q^2)]^{-1}}{q^2} + (\text{gauge terms which you can give})$$

Pause here to compare to scalar theory:

scalar:
(massless)

$$D'(q^2) = \frac{1}{q^2 + g_0^2 \Pi(q^2)}$$

photon:

$$D_F'(q)_{\mu\nu} = \frac{-g_{\mu\nu}}{q^2 [1 + e_0^2 \Pi(q^2)]}$$

The two Π 's are not on same footing! In scalar-massless case, you get a mass-shift if $\Pi(0) \neq 0$, and then no-mass-shift condition is $\Pi(0) = 0$. But for photon case, $\Pi(0) \neq 0$ does not cause a mass shift. You would need a pole in $\Pi(q^2)$ at $q^2 = 0$ in order to get a photon mass shift; there is no such pole in $\Pi(q^2)$ (since $m_e \neq 0$).

scalar-massless: $\Pi(0) = 0 \Rightarrow$ no mass shift

photon: $\lim_{q \rightarrow 0} q^2 \Pi(q^2) = 0 \Rightarrow$ no mass shift

Back to $D_F'(q)_{\mu\nu}$: no

$$D_F'(q)_{\mu\nu} = - \frac{Z_3 g_{\mu\nu}}{q^2} \quad \text{near } q^2 = 0$$

$$\Rightarrow Z_3 = [1 + e_0^2 \Pi(0)]^{-1} \quad \text{or} \quad Z_3^{-1} - 1 = e_0^2 \Pi(0)$$

In second order,

$$\Pi(0) = \frac{1}{12\pi^2} \log\left(\frac{\Lambda^2}{m_e^2}\right)$$

Use of "renormalized objects":

At this point we know that

$$\underline{D_F'(q^2)}_{\mu\nu} = - \frac{\underline{Z_3} g_{\mu\nu}}{q^2} \quad \text{near } q^2=0$$

$$\underline{e_0} \quad \underline{S_F'(q^2)} = \frac{\underline{Z_2}}{q-m} \quad \text{near } q^2=m^2 \text{ and } q=m.$$

$$\underline{\Gamma(P, P', q)}_\mu = \underline{Z_1} \gamma_\mu \quad \text{near } q^2=0 \text{ and } p=m.$$

[(Note on $\underline{Z_1}$): result must be $\propto \gamma_\mu$ because only other vector is $p_\mu 1^{4 \times 4}$ which $= m \gamma_\mu$ inside $\pi(\cdot) \dots u(p)$, so not independent.]

We know that $\underline{Z_1}, \underline{Z_2}, \underline{Z_3}$ will look like $\ln(1/m^2)$, ie, they want to diverge as cutoff is removed.

By the way, $\underline{Z_i} = \underline{Z_i}(e_0, m)$ in this method, m is related to m_0 through the equivalent of my ϕ^3 result $\Pi(q^2=m^2, \delta m^2) = 0$, so $\underline{Z_i}(e_0, m_0)$ if you like. Also $\underline{Z_i}(e_0, m_0, \Lambda)$.

Claim: if you examine the integral equations of QED, or if you just think about what happens at a vertex, you would conclude that:

$$\text{transformation } T(z_1, z_2, z_3) \left\{ \begin{array}{l} D_F' \rightarrow \underline{z_3}^{-1} D_F' \equiv \tilde{D}_F' \\ S_F' \rightarrow \underline{z_2}^{-1} S_F' \equiv \tilde{S}_F' \\ \Gamma_\mu \rightarrow \underline{z_1} \Gamma_\mu \equiv \tilde{\Gamma}_\mu \\ e_0 \rightarrow \left(\frac{\underline{z_2} \sqrt{\underline{z_3}}}{\underline{z_1}} \right) e_0 \equiv e \end{array} \right.$$

is a transformation which leaves all integral equations invariant, ie, of the same form. Ie, any computed physical result is independent of this $T(z_1, z_2, z_3)$ transformation, where z_1, z_2, z_3 are arbitrary numbers!! However, you will need $z_1 = z_2$ to keep the Ward Identity invariant, hence $T(z_1, z_3)$.

In particular, if you choose $z_3 = Z_3$, etc, then if you are doing all calculating with e , $\tilde{\Gamma}^{(n)}$, and all those divergent Z_i 's have been made to go away! This is what "renormalization" is all about:

Divergences are isolated to the Z_i , and then you get rid of the Z_i by a transformation $T(Z_i)$!

Renormalized Integral Equations: when you convert all $\tilde{G}^{(n)}$ to $\tilde{G}^{(n)}$ and take e_0 to e , you get a set of integral equations and "boundary conditions" on the function!

$$\tilde{\Gamma}_\mu(p,p)|_{p=m} = \underbrace{\gamma_\mu}_{Z_1} = Z_1 \Gamma_\mu(p,p)|_{p=m}.$$

This tells you $Z_1 = Z_1(m, e, \Lambda)$. Similarly, recall that

$$e^2 = Z_3 e_0^2 \quad \text{and} \quad Z_3^{-1} - 1 = e_0^2 \Pi(0)$$

$$\Rightarrow Z_3 = 1 - e^2 \Pi(q^2=0; e, m)$$

This gives you $Z_3(m, e, \Lambda)$.

Catch 22: unfortunately, the new integral equations in terms of $\tilde{G}^{(n)}$ and e [see p311 II] still contain two Z_i 's; somehow you have to show that as these $Z_i \rightarrow \infty$, "the theory" is still finite.

At least you have rendered your $\tilde{G}^{(n)}$ finite at mass shell; this is a great improvement over using the $\tilde{G}^{(n)}$ with all those Z_i 's there!!

RenomGroup: assuming that "the two Z_i 's" don't really matter, we see that our arbitrary transformation $T(z_1, z_3)$ leaves QED invariant. Note that you are then free to choose arbitrarily how your $\tilde{D}_F$ and $\tilde{S}_F$ are "normalized". You could take the canonical choice + then choose:

$$\begin{aligned} z_3^{-1} = 2 & \Rightarrow \tilde{D}_F = -\frac{2g_{\mu\nu}}{q^2} & e_{\text{new}} = \sqrt{\frac{1}{2}} e \\ z_1^{-1} = 3 & \Rightarrow \tilde{S}_F = \frac{+3}{\not{q} - m} \end{aligned}$$

The notation Π_c :

$$\Pi_c(q^2) \equiv \Pi(q^2) - \Pi(0) \Rightarrow \Pi_c(0) = 0 \quad \text{Big deal.}$$

Recall from (12.24) as above that

$$D_F(q)_{\mu\nu} = -\frac{g_{\mu\nu}}{q^2} [1 + e_0^2 \Pi(q^2)]^{-1} \Rightarrow Z_3^{-1} = 1 + e_0^2 \Pi(0)$$

Now construct: $\tilde{D}_F \equiv D_F / Z_3 = -\frac{g_{\mu\nu}}{q^2} \frac{1}{Z_3 [1 + e_0^2 \Pi(0) + e_0^2 \Pi_c(q^2)]}$

$$= -\frac{g_{\mu\nu}}{q^2} \frac{1}{[1 + e^2 \Pi_c(q^2)]} + \text{gauge, since } Z_3 e_0^2 = e^2.$$

So in $\tilde{D}_F$ we have replaced e_0 with e , and $\Pi_c(0) = 0$, giving us the canonical normalization.

Use of e : once you have gone over to $\tilde{D}_F$, $\tilde{S}_F$, $\tilde{\Gamma}$ and e , you regard each object as calculated in terms of e ! Look at the integral equations on p 311:

old equa

$$\Gamma_\mu = \gamma_\mu - \int \Gamma_\mu S_F' S_F' K$$

$$S_F'^{-1}(p') = g_\mu \tilde{\Gamma}^\mu(p', p) |_{p=m}$$

$$D_F(q)_{\mu\nu} = -\frac{g_{\mu\nu}}{q^2 [1 + e_0^2 \Pi(q^2)]}$$

$$\Pi_{\mu\nu}(q^2) = i \int h[\gamma_\mu S_F' \tilde{\Gamma}_\nu S_F']$$

$$\tilde{\tau} = \sum_{S \in \text{shell}} \tilde{\tau}^S(S_F', D_F', \Gamma_\mu, e_0)$$

new equa

$$\tilde{\Gamma}_\mu = Z_1 \gamma_\mu - \int \tilde{\Gamma}_\mu \tilde{S}_F' \tilde{S}_F' \tilde{K}$$

$$\tilde{S}_F'^{-1}(p') = g_\mu \tilde{\Gamma}^\mu(p', p) |_{p=m}$$

$$\tilde{D}_F(q)_{\mu\nu} = -\frac{g_{\mu\nu}}{q^2 [1 + e^2 \Pi_c(q^2)]}$$

$$\tilde{\Pi}_{\mu\nu}(q^2) = i Z_1 \int h[\gamma_\mu \tilde{S}_F' \tilde{\Gamma}_\nu \tilde{S}_F']$$

$$\tilde{\tilde{\tau}} = \sum_{S \in \text{skel}} \tilde{\tilde{\tau}}^S(\tilde{S}_F', \tilde{D}_F', \tilde{\Gamma}_\mu, e)$$

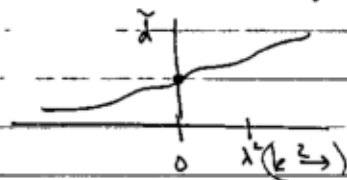
After you go to "new scheme", the rule as shown in last equa. is to always use e , not e_0 . Thus, eg, $K = \tilde{K}(e)$, $\tilde{\Gamma} = \tilde{\Gamma}(e)$, and so on. Now, e, e_0 are always related by Z_3 , and you can imagine $Z_3 = Z_3(e)$ since $Z_3^{-1} = 1 - e^2 \Pi(0)$ and $\Pi = \Pi(e)$. Note there is no $\tilde{\Pi}$. Then you know $e = e(e_0)$ and $e_0 = e(e)$.

Idea of a subtraction point: hold.

The idea of norm. point λ :

In the canonical norm, we say that $\tilde{D}^{F'} = -\frac{g_{\mu\nu}}{k^2} \tilde{d}(k^2)$

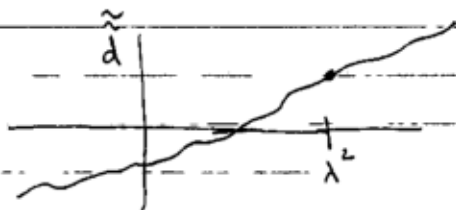
where $\tilde{d}(0) = 1$. Now, in this case $\tilde{d}(k^2)$ is a function that is uniquely determined by "the theory". Maybe it looks like this:



Now you can now do a $T(z, z_3)$ transformation on this canonical $\tilde{D}^{F'}$ to get

$$\tilde{D}^{F'} = -\frac{g_{\mu\nu}}{k^2} [\tilde{z}_3^{-1} \tilde{d}(k^2)] \quad \tilde{e}^2 = z_3 \tilde{e}^2.$$

This yields $\tilde{d}(k^2) = \tilde{z}_3^{-1} \tilde{d}(k^2)$. I.e., $d(k^2)$ is merely rescaled. Perhaps like this



Now, unless $\tilde{d}$ is ~~too~~, assuming $\tilde{d}$ monotonic for simplicity, we have

$$\tilde{d}(0) = \tilde{z}_3^{-1} \quad \text{and} \quad \tilde{d}(\lambda^2) = 1$$

So, by varying λ^2 in $\tilde{d}(\lambda^2) = 1$, you are merely exercising your ability to rescale a la R-group.

What points λ^2 are allowed? It must be true that $\tilde{d}(\lambda^2) = \text{real}, > 0$, so you can rescale to 1 with the positive real $\tilde{z}_3$,

$$z_3 = [\tilde{e}/\tilde{e}]^2 > 0, \text{ real.}$$

where $\tilde{d}$ = the canonical. Now, $\tilde{D}^{F'}(k^2)$ is real if you are below all thresholds. Thus $\lambda^2 < 0$ is a safe bet!

So regard $\lambda = \lambda(z_3)$ or $z_3 = z_3(\lambda)$, either way:

λ^2 as argument of d

Consider $\tilde{d}(m^2, e, k^2)$, $\tilde{d}(m^2, e, 0) = 1$
 $\tilde{\tilde{d}}(m^2, e, k^2)$, $\tilde{\tilde{d}}(m^2, e, \lambda^2) = 1$

Let's incorporate all possible $\tilde{d}^3$ by defining:

$$d(m^2, e, k^2; \lambda^2)$$

By specifying λ^2 , we are specifying z_3 because

$$\tilde{\tilde{d}}(\lambda^2) = 1 \Rightarrow \tilde{d}(\lambda^2) = z_3 \text{ and } \tilde{d} \text{ is the (known) canonical fct.}$$

$$\text{This } \Rightarrow z_3 = z_3(\lambda).$$

But specifying z_3 then completely specifies d away from $\tilde{d}$.
 Basically

$d(m^2, e, k^2; \lambda^2)$ is that d which does this: $d(m^2, e, \lambda^2; \lambda^2) = 1$

So $d(m^2, e, k^2; 0) = \tilde{d}^{\text{canon}}(m^2, e, k^2)$.

Dimensionless:

Since any d is dimensionless, write

$$d(m^2, e, k^2; \lambda^2) = \lambda^0 d\left(\frac{k^2}{\lambda^2}, \frac{m^2}{\lambda^2}, e^2\right)$$

where $\lambda^0 = 1$; note that e^2 is dim. in QED! So

$$d\left[1, \frac{m^2}{\lambda^2}, e^2\right] = 1 \quad \checkmark$$

Note that $z_3 = \tilde{d}(m^2, e^2, k^2 = \lambda^2)$ as shown above

$$= d(m^2, e^2, \lambda^2; 0)$$

Idea to get $z_3 = z_3(\text{ratio})$.

I have already shown that

$$z_3 = d(m^2, e^2, k^2; \lambda^2) \Big|_{\substack{\lambda^2=0 \\ k^2=\lambda^2}} = d(m^2, e^2, \lambda^2; 0)$$

Since z_3 is dimensionless, we can then write it as

$$z_3 = d(m^2, e^2, \lambda^2; 0) = \frac{d(\cancel{m^2/\lambda^2}, \cancel{e^2/\lambda^2})}{\cancel{\lambda^2}} \text{ where } e = \tilde{e} = \text{const.}$$

$$z_3\left(\frac{\lambda^2}{m^2}, e^2\right)$$

Note that, if $\lambda^2=0$, then $z_3=1$, so $z_3(0, e^2)=1$.

I am only trying to show that $z_3 = z_3(\lambda^2)$ which is obvious, and since dimensionless, $z_3 = z_3'(\lambda^2/m^2)$.

Uncouth Graphical View of Renormalization

14 Aug 78

- ① First Think of QED. For each "object" you choose a "renormalization point", i.e., a value of the momenta/masses to define the Z 's. Eg, usually in QED you do this

object	renorm point	Z_i
$D_F(q^2)_{\mu\nu}$	$q^2 = 0$	$D_F(q^2)_{\mu\nu} = -Z_3 g_{\mu\nu} / q^2$
$S_F(q)$	$\begin{cases} q^2 = m^2 \\ p = m \end{cases}$	$S_F(q) = \frac{Z_2}{p - m}$
$\Gamma'(p, p', q)_\mu$	$\begin{cases} q^2 = 0 \\ p = p' = m \end{cases}$	$\Gamma' = Z_1^{-1} \gamma_\mu$

- a) Notice that Z_1, Z_2, Z_3 are here regarded as constants, not functions of q^2 .
 b) We could have chosen different evaluation points & thereby defined different Z 's. Eg D_F at $q^2 = -\mu^2$.
 c) The Z_i arise from corrections like $\frac{m}{\Lambda}$ to bare objects. The charge is $e_0 = \text{bare charge}$.

d) The Ward says $\Gamma'(p, p', q)_\mu q^\mu = S_F^{-1}(p') - S_F^{-1}(p)$ (19.16)
 If we first go to $p^2 = p'^2 = m^2$ we get

$$\Gamma'(p, p', q)_\mu q^\mu = Z_2^{-1} [(p' - m) - (p - m)] = Z_2^{-1} [(p' - p)] = Z_2^{-1} \gamma_\mu q^\mu$$

If we then take $q^\mu \rightarrow 0$, we get $Z_1^{-1} \gamma_\mu q^\mu = Z_2^{-1} \gamma_\mu q^\mu$. I.e., Ward $\Rightarrow Z_1 = Z_2$

- e) notice that object Γ does not include the charge. It should be multiplied by the charge, here e_0 .

② Now I will show graphically why the Renom Group transformation leaves theory invariant.

(over)

Suppose:

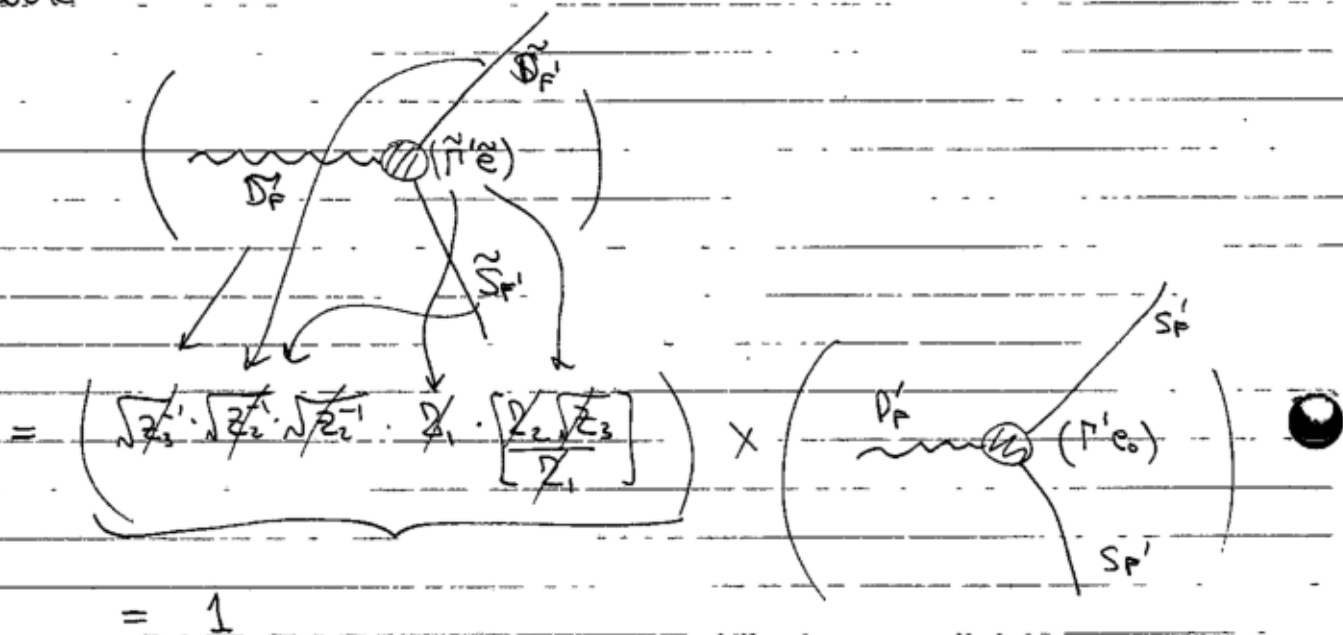
$$D_F' \rightarrow \tilde{D}_F' \equiv Z_3^{-1} D_F'$$

$$S_F' \rightarrow \tilde{S}_F' \equiv Z_2^{-1} S_F'$$

$$\Gamma' \rightarrow \tilde{\Gamma}' \equiv Z_1 \Gamma'$$

$$e_0 \rightarrow \tilde{e} \equiv \left[\frac{Z_2 \sqrt{Z_3}}{Z_1} \right] e_0$$

Then look:



$$= \left(\sqrt{Z_3^{-1}} \cdot \sqrt{Z_2^{-1}} \cdot \sqrt{Z_1^{-1}} \cdot Z_1 \cdot \left[\frac{Z_2 \sqrt{Z_3}}{Z_1} \right] \right) \times \left(\text{Diagram} \right)$$

$$= 1$$

I.e., by grouping $\sqrt{Z}$ to vertex for each prop, we see that vertex is unchanged! Moreover, the Z 's no longer appear in the theory because, eg,

$$\tilde{D}_F'(q^2 \neq 0) = -g_{\mu\nu}/q^2 \left(\frac{Z_2^{-1} Z_1}{Z_3} \right)$$

I.e., the Z 's were "renormalized away."

③ Now consider NACT. For simplicity, imagine that glue propagators have only one numerator term, A . Then we might say

object	renom. pt.	Δ :
$\Delta_F(q^2)_{\mu\nu}$	$q^2 = 0$	$\Delta_F(q^2)_{\mu\nu} = -\Delta_3 (g_{\mu\nu} - q_\mu q_\nu) / q^2$
$\Gamma^{(3)}(p_1, p_2, p_3)_{\mu\nu\lambda}$	$(p_1^2 = p_2^2 = p_3^2 = 0, p_3^\mu \approx 0)$	$\Gamma_{\mu\nu\lambda}^{(5)} = \Gamma^{(3)}(\dots) \cdot \Delta_1^{-1}$

where I am using axial gauge, $\Gamma^0 =$ bare vertex.

a) note that it is better to choose all $p_i^2 = -\mu^2$ as renom mass. And $\Gamma^{(5)}$ is sometimes chosen at a more symmetric renom pt.

b) Ward says that $i\Gamma_{\sigma\alpha\beta}(q_1, q_2, q_3)(q_3)_\alpha = \Pi_{\sigma\alpha}(q_1) - \Pi_{\sigma\alpha}(q_2)$

If we first go to $q_i^2 = -\mu^2$, we will find that

$\Gamma = -(q^2 g_{\mu\nu} - q_\mu q_\nu)$

$\Pi'_{\mu\nu}(q^2)$	$q^2 = -\mu^2$	$\Pi'_{\mu\nu}(q^2) = \Delta_3^{-1} \cdot \Pi_{\mu\nu}^0(q^2)$
----------------------	----------------	--

$$\begin{aligned}
 i\Gamma(\dots)q_3 &= \Delta_3^{-1} \{ \Pi_{\mu\nu}^0(q_1^2) - \Pi_{\mu\nu}^0(q_2^2) \} \\
 &= -\Delta_3^{-1} \{ (q_1^2 \delta_{\mu\nu} - q_{1\mu} q_{1\nu}) - (q_2^2 \delta_{\mu\nu} - q_{2\mu} q_{2\nu}) \} \quad q_i^2 = -\mu^2 \\
 &= -\Delta_3^{-1} \{ (q_2)_\mu (q_2)_\nu - (q_1)_\mu (q_1)_\nu \}
 \end{aligned}$$

Now take $q_3^\mu \rightarrow 0$ + get

$$\Delta_3^{-1} \{ (q_1 - q_2)_\mu \delta_{\nu\lambda} + (q_2 - q_3)_\nu \delta_{\lambda\mu} + (q_3 - q_1)_\lambda \delta_{\mu\nu} \} \Rightarrow \chi(q_3)^\lambda$$

After fiddling you will find that: $\text{Ward} \Rightarrow \Delta_3 = 2$.

c) again, $\Gamma^{(5)}$ does not include charge which is g_0 .

④ Now the graphical renormalization

$$\Delta' \rightarrow \tilde{\Delta}' = \Delta_3^{-1} \Delta'$$

$$\Pi' \rightarrow \tilde{\Pi}' = \Delta_1^{(3)} \Pi'$$

$$g_0 \rightarrow \tilde{g} = \left[\frac{(\Delta_3)^{3/2}}{\Delta_1} g_0 \right] \Rightarrow g_0 = \tilde{g} \Delta_1^{(3)} / \Delta_3^{3/2}$$

$$\left(\begin{array}{c} \tilde{\Delta}' \\ \Delta_P' \end{array} \right) \left(\tilde{g} \tilde{\Pi}' \right) = \underbrace{\left(\sqrt{\Delta_3^{-1}} \right)^3 \cdot \Delta_1 \cdot \left[\frac{(\Delta_3)^{3/2}}{\Delta_1} \right]}_{= 1} \times \left(\begin{array}{c} \Delta_P' \\ \Delta_P' \end{array} \right) \left(g_0 \Pi' \right)$$

Thus, no change in theory + you have eliminated the Δ_3 's!

⑤ Obviously we have also a $\Pi^{(4)}$ to worry about. It too has a Ward but so what.

$$\left(\begin{array}{c} \tilde{\Delta}' \\ \Delta_P' \end{array} \right) \left(\tilde{g}^2 \tilde{\Pi}^{(4)} \right) = \underbrace{\left(\sqrt{\Delta_3^{-1}} \right)^4 \cdot \Delta_1^{(4)} \cdot \left[\frac{(\Delta_3)^{3/2}}{\Delta_1^{1/2}} \right]^2}_{= 1} \left(\begin{array}{c} \text{bar} \\ \Delta_P' \end{array} \right) \left(g_0^2 \Pi^{(4)} \right)$$

$$\Rightarrow (\Delta_3)^{-4/2} (\Delta_3)^3 (\Delta_1)^{-2} \Delta_1^{(4)} = 1 \Rightarrow \Delta_1^{(4)} = (\Delta_1^2 / \Delta_3)^{?}$$

Hold on this.

2/18/97

A comment on the electron self-energy, compared to ϕ^3 .

The lowest order expression for this object is stated on page 162 of BD v1. Power count shows that $D=1$, ie, this thing is linearly divergent. Thus, even after you perform one subtraction, the once-subtracted object is still logarithmically divergent. That is why Z_2 is cutoff-dependent in QED. Details below:

$$\text{---} \bigcirc \text{---} = -i \Sigma(e_0^2, p^2, m^2, \Lambda^2, \lambda^2) \quad , \text{ includes } \text{---} \times \text{---} .$$

$\nwarrow$ bare ch. $\nwarrow$ phys mass $\nwarrow$ IR cutoff

$$\text{---} \bigcirc \text{---} = i S_F(e_0^2, p^2, m^2, \Lambda^2, \lambda^2)$$

$$\text{Dyson} \Rightarrow S_F(e_0^2, p^2, m^2, \Lambda^2, \lambda^2) = \frac{1}{\not{p} - m - \Sigma(e_0^2, p^2, m^2, \Lambda^2, \lambda^2)}$$

$$\text{Expand: } \Sigma(p^2) = \Sigma(m^2) + (\not{p} - m) \Sigma'(m^2) \text{ and } \ln(\Lambda)$$

$$\text{Then: } \Sigma(m^2) = 0 \text{ puts mass at } m, \text{ ie choose } \delta m(\Lambda) \text{ so } \Sigma(m^2) = 0$$

$\uparrow$ order Λ to any given order.

$$\text{Then: } S_F(\not{p} \approx m^2) = \frac{1}{(\not{p} - m) [1 - \Sigma'(m^2)]} \Rightarrow Z_2^{-1} = 1 - \Sigma'(m^2)$$

Here you see that Z_2 involves the first derivative of the self-energy wrt momentum, and thus Z_2 will contain $\log(\text{cutoff})$. To back this up, recall how in BS they had to subtract twice to get a finite object. *see other page for explicit*

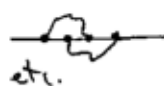
example 2; in ϕ^3 theory, power count shows that the self-energy is only log-divergent. Thus, in this theory the once-subtracted self-energy is finite. Thus, in ϕ^3 your Z_2 will be finite to any order, unlike QED. In this theory, the constant in your expansion for the self-energy is log divergent and is the thing to be cancelled by the mass counterterm, so you still had an infinite mass renormalization. In QED above, the constant is linearly divergent: in QED you have an infinite electron mass renorm, and an infinite charge renorm. In ϕ^3 there is only a finite charge renormalization in any order.

2/18/79.

Redo the electron self-energy business, stating all important facts.

I will be using the counterterm philosophy, so define:

$$\text{---}\bigcirc\text{---} = -i \left[\underbrace{\Sigma_1(e_0^2, p^2, m^2, \Lambda^2, \lambda^2)}_{\text{---}\bigcirc\text{---}} + \underbrace{\Sigma_2(e_0^2, p^2, m^2, \Lambda^2, \lambda^2, \delta m)}_{\text{---}\times\text{---} + \text{---}\bigcirc\text{---}} \right] = -i \Sigma$$

+  etc.

$$S_F(e_0^2, p^2, m^2, \Lambda^2, \lambda^2) = \frac{1}{\not{p} - m - \Sigma_1(e_0^2, p^2, m^2, \Lambda^2, \lambda^2) - \Sigma_2(e_0^2, p^2, m^2, \Lambda^2, \lambda^2, \delta m)}$$

Here I have separated the self energy into two obvious parts. All effects of the mass counterterm are put into the second part, so the first part is independent of the mass counterterm.

Now, go compute the first part of the self energy; power count shows it is linearly divergent, $D=1$, so contains $(\text{cutoff}) + \ln(\text{cutoff}) + \text{finite terms}$. Compute both objects through some definite order in e_0^2 .

To do mass renormalization, you simply adjust the parameter δm so that:

$$\Sigma_1(e_0^2, m^2, m^2, \Lambda^2, \lambda^2) = \text{linearly divergent} = -\Sigma_2(e_0^2, m^2, m^2, \Lambda^2, \lambda^2, \delta m)$$

↑

In lowest order, $\Sigma_2 = \delta m$ only, and then you see that $\delta m(\Lambda)$ is linear in Λ in order e_0^2 .

Thus, to any order you would expect that the mass shift δm required is linearly divergent in the cutoff, since this is even true in lowest order. Thus, you might say we are doing a linear mass renormalization for the electron. A linearly divergent mass renormalization. We are doing this by adding a counterterm to the lagrangian of the same form as a term already there, hence we are renormalizable in this regard.

Now, to our order of interest assume we have selected δm so propagator has physical mass. Having selected δm , we can now examine Z_2 (see last sheet)

$$Z_2^{-1} = 1 - \underbrace{\Sigma_1'(e_0^2, m^2, m^2, \Lambda^2, \lambda^2)}_{\log(\Lambda) \text{ div.}} - \underbrace{\Sigma_2'(e_0^2, m^2, m^2, \Lambda^2, \lambda^2, \delta m)}_{\log(\Lambda)} \quad \text{selected value.}$$

I am claiming that the first part of the electron self-energy looks like this:

$$\underbrace{\Sigma_1(e^2, p, m, \Lambda, \lambda)}_{> A \cdot \Lambda} = \underbrace{\Sigma_1(e^2, m, m, \Lambda, \lambda)}_{> A \cdot \Lambda \text{ also}} + (p - m) \Sigma_1'(\text{same})$$

Ie, I am claiming that the linear term has a coefficient which does not depend on momentum q . To see why this always happens (I think) consider the lowest order loop integral. To get the linear divergent piece, you take all k 's large and you get an integral which does not depend on q , so there. Thus, your selected dm will get rid of the mass shift for p near m as well as right at m . This is all to say that you can remove the leading divergence just by subtracting.

Now, if you look at the lowest graphs involving dm , you can conclude that the second part of the self-energy also can have its linearly divergent part removed by a simple subtraction. Therefore, the derivative will have only $\log(\text{cutoff})$. [Assured markings].

So you can conclude that in QED, the object Z_2 is really log divergent, so you have logarithmically divergent charge renormalization to any order. Ie, you would put in a counterterm looking like the charge interaction, and this generated a multiplicative renorm which can then remove this log divergent charge.

Summarize: in QED, the electron mass renormalization needs a linearly divergent counterterm, the electron charge renormalization needs a log divergent counterterm.

Now show how multiplicative renormalization removes cutoff dependence.

$$S_F(p) = \frac{1}{\not{p} - m - \Sigma(p)}$$

$$\Sigma(p) = \underbrace{\Sigma(m)}_{=0} + (\not{p} - m) \underbrace{\Sigma'(m)}_{=\log(\Lambda)} + \underbrace{\text{finite} \cdot (\not{p} - m)^2}_{\frac{\Sigma''(p)}{2}}$$

Rewrite as:

$$\Sigma(p) = (\not{p} - m) \left[\underbrace{\Sigma'(m)}_{\log \Lambda} + \underbrace{C(p)}_{\text{finite, and } C(m)=0, \text{ at least a full zero.}} \right]$$

$$S_0 \quad S_F(p) = \frac{1}{(\not{p} - m) [1 - \Sigma'(m) - C(p)]} \equiv \frac{h(p)}{\not{p} - m}$$

Note that $h(m) = [1 - \Sigma'(m)]^{-1} = \Sigma_2$.

Now do mult renorm:

$$\tilde{S}_F^{(n)}(p) \equiv \Sigma_2^{-1} S_F^{(n)}(p) = \frac{[1 - \Sigma'(m)]}{[1 - \Sigma'(m) - C(p)]} \left(\frac{1}{\not{p} - m} \right)^n$$

Now comes the trick: everything is computed to some order in e_0^2 . Even though the cutoff is some large parameter, we still have an expansion in the small charge. Thus, making errors only in the next highest order, we can shift the numerator into the denominator to get:

$$\tilde{S}_F^{(n)}(p) = \frac{1}{[1 - \underbrace{[\Sigma'(m) - \Sigma'(m)]}_{\log(\Lambda) \text{ is subtracted out!}} - C(p)]} \frac{1}{(\not{p} - m)} \int + \mathcal{O}(e_0^{n+1}) = \frac{1}{[1 - C(p)]} \left(\frac{1}{\not{p} - m} \right)^n$$

finite.

Of course as long as the cutoff is some finite value, $C(p)$ will depend on the cutoff. But the point is we can now take the cutoff to infinite and $C(p)$ is still finite and we have left a renormalized propagator which is some finite function of p . Of course you are now supposed to consider it a function of e and not e_0 in accordance with the multip renorm program. In effect, here is what we have done:

$$\tilde{S}_F(p) = \frac{1}{\not{p} - m - [\Sigma(p) - \underbrace{(\not{p} - m) \Sigma'(m)}_{\text{we subtracted this, } \Sigma(m) \text{ already } = 0}]}$$

Repeat the last few pages now for the photon propagator.

The essential difference now is the spinology. If you cutoff in a gauge invariant manner, the self-energy will have the usual gauge invariant form (covariant gauge) and you just don't get any mass renormalization! Well I better say this more correctly: there could be mass renorm if Π has a pole at $q^2=0$. But in QED from unitarity you know this is not the case because the first singularity in Π is the two-electron normal threshold. Note that in QCD the two-gluon threshold falls down to the origin (ie, the contribution to Π from two bare-gluons with zero bare-mass); thus in QCD you can't say a priori that Π doesn't have a pole at the origin or some other singularity, so in QCD there may be mass renormalization. (Maybe to infinite mass)

Notice the asymmetry between the roles of Σ and Π . Both are scalar functions; both are in effect $\log(\text{cutoff})$ divergent, once m is adjusted in Σ to kill off the linear divergent term. But, we require $\Sigma(m) = 0$ as condition that electron mass doesn't shift away from its physical value, and Z_2 is related to $\Sigma'(m)$. Due to its different spinology, for the photon the condition for no mass shift away from 0 is that $q^2 \Pi(q^2) = 0$ as $q \rightarrow 0$, ie no pole in Π . And Z_3 is connected with $\Pi(0)$, and nothing of interest is related to $\Pi'(0)$.

$$\text{wavy line with circle} = \underbrace{i \Pi_{\mu\nu}}_{D=2} = i (q^2 g_{\mu\nu} - q_\mu q_\nu) \underbrace{\Pi(q^2)}_{D=0} + \text{div't core.}$$

$$\text{wavy line with circle} = i D_F(q^2)_{\mu\nu}$$

$$D_F(q^2)_{\mu\nu} = - \frac{g_{\mu\nu}}{q^2} \frac{1}{[1 + \Pi(q^2)]} \Rightarrow Z_3^{-1} = 1 + \Pi(0) = \log. \text{div.}$$

$$\tilde{D}_F(q^2)_{\mu\nu} = Z_3^{-1} D_F(q^2)_{\mu\nu} = - \frac{g_{\mu\nu}}{q^2} \left[\frac{1 + \Pi(0)}{1 + \Pi(q^2)} \right] = - \frac{g_{\mu\nu}}{q^2} \left[\frac{1}{1 + \underbrace{[\Pi(q^2) - \Pi(0)]}_{\text{subtr.}}} \right]$$

As before, to order n , you see that the multiplicative renorm is equivalent to simply subtracting $\Pi(0)$ downstairs. One can then take the cutoff to infinite and have a finite renormalized propagator for the photon.

$$\Pi_s(q^2) \equiv \lim_{\Lambda \rightarrow \infty} [\Pi(q^2) - \Pi(0)] = \text{a finite fctn like } C(p) \text{ for edech.}$$

The Gauge Dyson in general Landau gauge

① Take this bare prop:

$$D^{\text{Bare}} = -\frac{1}{q^2} \left[g_{\mu\nu} - (1-\alpha) \frac{q_\mu q_\nu}{q^2} \right]$$

$$\Rightarrow D_{\mu\nu}^{\text{Bare}} = + \frac{T_{\mu\nu}}{q^4} - \frac{\alpha}{q^4} q_\mu q_\nu \quad \text{where } T_{\mu\nu} \equiv [q_\mu q_\nu - q^2 g_{\mu\nu}]$$

$$= \frac{(1-\alpha)}{q^4} T_{\mu\nu} - \frac{\alpha}{q^2} g_{\mu\nu} \Rightarrow \underline{D^{\text{Bare}}} = \frac{(1-\alpha)}{q^4} T - \frac{\alpha}{q^2} 1$$

If $\alpha=1$, get usual Feynman gauge result. Note that $T^2 = -q^2 T$ $q \cdot T = 0$

② Using convention (19.23) of BD,

$$T_{\mu\nu} = T_{\mu\nu} \cdot \Pi$$

③ Now also in their conv, except for B^i or Π ,

$$\text{---} \otimes \text{---} = \text{---} + \text{---} \otimes \text{---} + \dots$$

$$iD = iD^B + iD^B i\Pi iD^B + \dots$$

$$= [1 + (\Pi D^B) + (\Pi D^B)^2 + \dots] iD^B \quad \leftarrow \text{even 1 but no changes!}$$

$$= [1 + \Pi D^B]^{-1} iD^B$$

$$\Rightarrow [1 + \Pi D^B] D = D^B \Rightarrow [1 + \Pi T \left(\frac{(1-\alpha)}{q^4} T - \frac{\alpha}{q^2} 1 \right)] D = \frac{(1-\alpha)}{q^4} T - \frac{\alpha}{q^2} 1$$

Now write $D = AT + B1$, most general; then

$$\left[1 - (1-\alpha) \frac{\Pi T}{q^2} T - \frac{\alpha \Pi T}{q^2} T \right] = \left[1 - \frac{\Pi T}{q^2} T \right] \quad \checkmark \quad \text{correct form, now on since } [T, T] = [T, 1] = 0.$$

Then

$$\left[1 - \frac{\Pi T}{q^2} T \right] [AT + B1] = \left(A - \frac{B \Pi T}{q^2} \right) T + \Pi A T + B \cdot 1 = \left[A + \Pi A - \frac{B \Pi T}{q^2} \right] T + [B] 1$$

$$\Rightarrow B = -\frac{\alpha}{q^2} \Rightarrow \left[A + \Pi A + \frac{\alpha \Pi T}{q^4} \right] = \frac{(1-\alpha)}{q^4} \Rightarrow A(1+\Pi) = \frac{(1-\alpha)}{q^4} - \frac{\alpha \Pi}{q^4}$$

$$\Rightarrow A(1+\Pi) q^4 = (1-\alpha)(1+\Pi) \Rightarrow A = \frac{1-\alpha(1+\Pi)}{q^4(1+\Pi)}$$

Then we get

$$D = \left[\frac{1 - \alpha(1+\pi)}{1+\pi} \right] \frac{\pi}{q^4} - \frac{\alpha}{q^2} 1$$

You see that the longitudinal part was not "renormalized".

Summarize:

$$T_{\mu\nu} \equiv [q_\mu q_\nu - q^2 g_{\mu\nu}]$$

$$\Pi_{\mu\nu} \equiv T_{\mu\nu} \cdot \Pi$$

$$D^{\text{Bare}} = -\frac{1}{q^2} \left[g_{\mu\nu} - (1-\alpha) \frac{q_\mu q_\nu}{q^2} \right]$$

$$\Rightarrow (1 + \Pi^B) D = D^B$$

Then we have

general α

$\alpha = 1$ (Feyn)

$\alpha = 0$ (Transverse):

D^B

$$\frac{(1-\alpha)\pi}{q^4} - \frac{\alpha}{q^2} 1$$

$$-\frac{1}{q^2} g_{\mu\nu}$$

$$-\frac{1}{q^2} \left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \right)$$

D

$$\left[\frac{1 - \alpha(1+\pi)}{q^4(1+\pi)} \right] \pi - \frac{\alpha}{q^2} 1$$

$$-\frac{g_{\mu\nu}}{q^4(1+\pi)} - \frac{\pi}{(1+\pi)} \frac{q_\mu q_\nu}{q^4}$$

$$-\frac{1}{q^4(1+\pi)} \left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \right)$$

Other ways to write general result $\forall \alpha$:

$$D = \frac{1}{[1+\pi(q^2)]} \frac{(q_\mu q_\nu - g_{\mu\nu} q^2)}{q^4} - \frac{\alpha}{q^4} q_\mu q_\nu$$

Notation for ϕ^3 -Theory

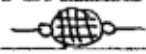

 $=$

 $+ \text{etc} = -i \Sigma^2 = \text{self energy}$


 $= i D^{\text{Bare}} \quad D^B = \frac{1}{p^2 - m_0^2}$

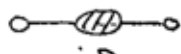

 $= [i D^B] [-i \Sigma^2] [i D^B]$


 $= i D$


 $=$

 $- [\text{tadpole}]^{-1} = \Gamma^{(2)}$

① The regular Dyson reads


 $=$

 $+ \text{tadpole} + \text{tadpole} + \text{tadpole} + \text{etc.}$

$$i D = i D^B + i D^B (-i \Sigma^2) i D^B + \dots$$

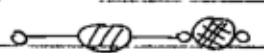
you get $D = D^B [1 + \Sigma^2 D^B + (\Sigma^2 D^B)^2 + \dots] = D^B [1 - \Sigma^2 D^B]^{-1}$

$$\Rightarrow D = \frac{1}{p^2 - (m_0^2 + \Sigma^2)}$$

② Relation between D and $\Gamma^{(2)}$ acc. to my older notes is:


 $= -1 \quad \text{i.e.,} \quad \text{tadpole} = - [\text{tadpole}]^{-1}$

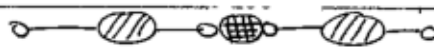
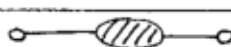
Let's see if consistent with Dyson:


 $- \text{tadpole} [\text{tadpole}]^{-1} = -1$

Says $i D [-i \Sigma^2] - i D [i D^B]^{-1} = -1$

$$\Rightarrow D [\Sigma^2 - (D^B)^{-1}] = -1 \Rightarrow D = \frac{1}{(D^B)^{-1} - \Sigma^2} \quad \checkmark$$

③ I have also checked that this works (graphs on both sides):


 $=$


Notation for pure Yang-Mills Theory, all gauges.

$$\text{---} \bigcirc \text{---} = i \Pi_{\mu\nu}$$

$$\text{---} \bigcirc \text{---} = i D_{\mu\nu}^B$$

$$\text{---} \bigcirc \text{---} = i D_{\mu\nu}$$

$$\text{---} \bigcirc \text{---} = \Gamma_{\mu\nu}^{(2)}$$

① The Dyson looks graphically the same as ϕ^3 . There are two ways to state result.

$$a) \quad [\delta_{\mu\lambda} + D_{\mu\lambda}^B \Pi_{\lambda\nu}] D_{\nu\mu} = D_{\mu\nu}^B$$

$$b) \quad D_{\mu\lambda} [\delta_{\lambda\nu} + \Pi_{\lambda\nu} D_{\nu\mu}^B] = D_{\mu\nu}^B$$

In general you cannot solve for D until you know the forms of $D_{\mu\nu}^B, \Pi_{\mu\nu}$.
I.e., gauge must be specified in order to solve.

② What about $\Gamma^{(2)}$ Using same proof as in notes, I'm sure that

$$\begin{cases} i D_{\mu\lambda} \Gamma_{\lambda\nu}^{(2)} = -\delta_{\mu\nu} = \Gamma_{\mu\lambda}^{(2)} (i D_{\lambda\nu}) \\ \text{---} \bigcirc \text{---} \bigcirc \text{---} = \text{---} \bigcirc \text{---} \bigcirc \text{---} = -\delta_{\mu\nu} \end{cases}$$

We may now use a) and b) above to find that

$$a) \quad (i D^B) \Gamma^{(2)} = -1 + (i D^B)(i \Pi) \Rightarrow \text{---} \bigcirc \text{---} \bigcirc \text{---} = -\delta_{\mu\nu} + \text{---} \bigcirc \text{---} \bigcirc \text{---}$$

$$b) \quad \Gamma^{(2)}(i D^B) = -1 + (i \Pi)(i D^B) \Rightarrow \bigcirc \text{---} \bigcirc \text{---} = -\delta_{\mu\nu} + \bigcirc \text{---} \bigcirc \text{---}$$

Either of these may be "closed" to yield

$$\begin{cases} (i D^B) \Gamma^{(2)}(i D^B) = -(i D^B) + (i D^B)(i \Pi)(i D^B) \\ \text{---} \bigcirc \text{---} \bigcirc \text{---} \bigcirc \text{---} = -\text{---} \bigcirc \text{---} \bigcirc \text{---} + \text{---} \bigcirc \text{---} \bigcirc \text{---} \bigcirc \text{---} \end{cases}$$

All these results are like ϕ^3 . They relate $\Gamma^{(2)}$ to Π , but again you cannot yet solve for one in terms of the other.

BD Chapt 19

Relativistic Quantum Fields
Bjorken & Drell
1965

Chapt 19: Renormalization

BD Chapter 19: Renormalization

19.1 Introduction Simply assume that perturbation expansion converges. Will build on vol 1 info: Will do induction proof.

19.2 Σ, Γ, K  =  +  + etc. = $\Gamma^{(2)}$ (proper) = $[-i\Sigma]$

$$[iS_F'] = \text{tadpole} = \frac{i}{p-m-\Sigma(p)}$$

$$\text{tadpole} = [ie^2 \Pi_{\mu\nu}] = \text{tadpole} + \text{etc.}$$

$$[iD_F'] = \text{tadpole} = \dots$$

$$\text{tadpole} = \Gamma_2 = \gamma_2 + \lambda_2$$

$$\text{tadpole} = K = \text{tadpole} + \dots$$

These are good, clear basic definitions.

19.3 Integral Equations. They are clearly written down; it is noted that K appearing in # [12] has very mild divergence problems, so its addition to the set Γ, S, D does not hurt too much.

19.4 Skeleton Graph Expansions. Claim: The only time you can have overlapping insertions is in self-energy graphs. Excluding such objects as D_F', S_F' , it is possible to write a skeleton-graph expansion for any greens fctn: [13] shows it for K , [15] for the vertex. You can do similarly for $\Gamma^{(n)}(p_1, \dots, p_n)$.

Then in each skeleton graph you add the "meat" by letting $\gamma \rightarrow \Gamma$, $S_F \rightarrow S_F'$ and $D_F \rightarrow D_F'$.

No meat so far.

19.5 A topological theorem The theorem here proved is this: "only vertex insertions within self-energy parts can overlap" Otherwise you can always box things off. Curiously, I will accept the theorem.

19.6 The Ward Identity Ward is credited with $\Lambda_n(p, p) = -\frac{2\Sigma(p)}{2p^\mu}$, and Takahashi with $q_\mu \Gamma^\mu = S_F^{-1} - S_F'^{-1}$ (1957!).

It is noted that via the Ward, S_F' is fully determined by Γ , whereas $\Gamma(p, p)$ is determined S_F' .

A simple + fast diagrammatic proof is stated.

Next, Ward + [10] is used to prove that $q_\mu \Pi^{\mu\nu} = 0$ and that therefore $\Pi_{\mu\nu} = (q_\mu q_\nu - g_{\mu\nu} q^2) \Pi(q^2)$ where

$\Pi(q^2)$ has no pole at $q^2=0$ Thus you get

$$\left. \begin{aligned} D_F'(q)_{\mu\nu} &= - \frac{g_{\mu\nu}}{q^2 [1 + e^2 \Pi(q^2)]} + q_\mu q_\nu \dots \\ S_F'(p) &= \frac{1}{\not{p} - m - \Sigma(p)} \end{aligned} \right\} \text{compare.}$$

19.7 Define Z_i 's + do Multiplicative Renorm

As usual, we choose $S_F' \rightarrow Z_2 / (\not{p} - m)$ for $\not{p} \rightarrow m$ where $Z_2 = \text{a constant}$. This means that $\Sigma(\not{p}=m, \delta m) = 0$ so we adjust this in any given order. So, there is a mass counterterm, but not $(1 - Z_2)(\not{p} - m)$ deal to remove this δZ .

$$\text{So: } S_F' \rightarrow \frac{Z_2}{\not{p} - m} \Rightarrow \Sigma(\not{p}=m, \delta m) = 0, \quad -\frac{\partial \Sigma(\not{p}=m, \delta m)}{\partial p} = Z_2^{-1} - 1$$

$$D_F' \rightarrow -\frac{Z_3 g_{\mu\nu}}{q^2} \Rightarrow e^2 \Pi(0) = Z_3^{-1} - 1, \quad q^2 \Pi(q^2)|^0 = 0$$

$$\Gamma_\mu' \rightarrow Z_1 \gamma_\mu, \quad Z_1 = Z_2 \text{ by Ward.}$$

Now rescale in usual way to $\tilde{S}_F', \tilde{D}_F', \tilde{\Gamma}$ and $\tilde{e}$

Observation: easy to show that, looking at skeleton expansion [13] for K .

$$K(e^2, D_F', S_F', \Gamma, m) = Z_2^{-2} K(e^2, \tilde{D}_F', \tilde{S}_F', \tilde{\Gamma}, m) \equiv \tilde{K}(\tilde{e}, \tilde{D}_F', \tilde{S}_F', \tilde{\Gamma})$$

$\uparrow$ original exp param $\uparrow$ new exp param

This is based on the "uncouth graphical analysis" of my notes. So clearly define $\tilde{K}$ as shown.

Observation: The new rule for computing S -matrix elements is this: do only skeleton graphs + shoot in $\tilde{D}_F', \tilde{S}_F', (\tilde{\Gamma}e)$. No more Z_i 's appear. If you know these basic objects + how to do loop momenta, you are all set.

Other facts:

$$\tilde{\Gamma}_\mu = Z_1 \gamma_\mu - \int \tilde{\Gamma} \tilde{S}_F' \tilde{S}_F' \tilde{K}$$

$$\tilde{\Gamma}(p, p) = \gamma_\mu \text{ det's } Z_1$$

$$\tilde{D}_F' = -\frac{g_{\mu\nu}}{q^2 (1 + e^2 \Pi(q^2))} \text{ where } \Pi_c = \Pi - \Pi(0).$$

19.8 Summary

Before renorm we have: (no need for [9])

- (a) skeleton exp. for any τ as series in ϵ_0
- (b) skeleton exp. for K
- (c) [12]
- (d) Ward gives S_F in terms of Γ
- (e) D_F in terms of $\Pi(q^2)$
- (f) [10]

After renorm we have same form equations, all $\text{Thygs.} \sim$ and $\epsilon_0 \rightarrow \epsilon$. But there are still a few floating 2 's left over!! These will be the main problem. Note that we now have an derivative Thygs. in ϵ !

19.9 Analytic Continuation + Intermediate Renormalization

Recall that Feynman graphs have Landau Singularities in (p^M) space, like eg normal thresholds. These appear via Cutkosky rules when $p^2 = m^2$ etc. If you Wick rotate to Euclidean p^2 , you avoid all these sings which are really irrelevant to the question of UV sings.

Having done this, you now have, $\bar{p}^2 < 0 \quad \forall \bar{p}^M$ so $p^2 = m^2$ are all avoided. But, now you must change your renormalization point from $p^2 = m^2$ to $p^2 = 0$ which is Euclidean accessible.

So define $\tilde{S}_F$ etc. and $\tilde{Z}_2$ etc. to be the multiplicative renormalization at $p^2 = 0$ ["intermed renorm"]. Show that $\tilde{S}_F$ etc is finite. Then it is easy to make the extra step from $\tilde{S}_F$ to S_F .

Notice that this method neatly avoids the IR problem.

19.10 Degree of Divergence. For no-derivative couplings, the superficial degree of divergence is $D = 4 - B - \frac{3}{2}F$ as I well know. Also, each "subtraction" you do on a Feynman integral lowers D by one.

D is computed for each QED renorm part, and our only concern will be $\sim \epsilon$ with $D_{\text{EFT}} = 0$ (g. down from 2) and $\sim \epsilon^2$ again with $D = 0$.

Obviously D is not enough to tell true convergence because inside a graph you can always replace γ_μ with $\gamma_\mu + \text{make it diverge without changing } D!$ (a vertex insertion)

Weinberg's Theorem (1960) gives the correct, complete convergence criterion for a Feynman graph.

We are now at * on page 322.

The rest of this section is a bit harder than the first part, but at last I think I understand what is going on here. Look at the Feynman graph in figure 19.32. If you let all momenta scale out together (you go out in some arbitrary direction in the momentum hyperspace), you will find that the convergence of the graph really is the superficial D you trivially compute, here $D=2$. BUT, suppose you hold 4-momentum l_1 fixed and small and look in large l_2 region. Then you lose convergence powers present in the l_1 loop and you find in this region or "tube" of the hyperspace that $D=0$. I.e., this superficially convergent graph has a divergent subgraph.

For this graph all the interesting tubes are shown in figure 19.35. Each tube is just a region where certain integration momenta are held fixed and only the rest (solid lines) of the graph is considered. You can see that the vertex subgraph and its tube are the only divergence problems this graph has.

Weinberg's theorem says the obvious: a graph is truly convergent if it converges superficially and also in all the special tubes, i.e., all subgraphs must converge. Thus, figure 19.32 flunks the test.

Now, look back at page 311. Notice the Z_1 sitting in the vertex equation. This divergent thing will be cancelled by a subtraction made in the second term on the right of this equation (d). I.e., eg, when you see a simple vertex of lowest order (having $D=0$), you subtract to get a resultant $D=-1$. Similarly the Z_1 which appears in the Vac Pol dyson equation (f) reminds you that since $D=0$ for vacpol PI, you also have to make a subtraction there.

Now, I gather that you handle vertices and PI vacpol subtractions in the same way, so they only mention the vertex. Look again at Figure 19.32. We have learned that although $D=2$ overall, effectively $D=0$ because of that vertex insertion. To fix this up we should take the little insertion and subtract within the graph. This "fixes up" the $D=0$ tube in figure 35 and changes that tube to $D=-1$. But, as shown on page 323, by adding a new term inside the graph which is essentially a constant with respect to all other loop momenta (like l_1), you may worsen the divergence in other tubes. To see the effect of a subtraction, you draw a dotted line around the vertex insertion and shrink it to a point and see what you get. Example: Fig 33A has $D=-3$ in the tube shown, but in Fig 34B the subtraction causes a new situation where $D=2$, i.e., less convergent (but still OK). Be very careful: Fig 34B which is the subtraction and has $D=0$, cancels 33B, so you are left with $D=-1$. Fig 33A is still there with $D=-3$, but the subtraction adds new situation 34A with $D=-2$. This subtraction does repair the graph of 19.32.

As one more example of the subtraction problem, look at page 326. On left side you see a tube marked * which has $D=-3$. But, when you put in a vertex subtraction to repair the vertex insertion tube and lower it from $D=0$ to $D=-1$, you accidentally bring up the * tube from $D=-3$ to $D=0$ as shown in 37B. This means that we have failed to repair this graph. Notice that this is an example of a subtraction within an overall vertex insertion.

A similar problem is encountered with figure 193B 38. See page 329. The tube (b) has $D=0$. If you try to fix this by a vertex subtraction, you generate graph (d) which causes another $D=0$ problem.

Thus, you see that when you have a subtraction made within either a vertex insertion or within a self-energy insertion, you have a special problem. I think this same problem carries the name: overlapping subgraphs.

This section merely draws attention to these two problems, but does not tell how to resolve them.

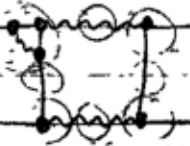
By the way, I think the idea of shrinking down to a point of a piece of a graph must be the same as B+S so-called generalized vertex G and their operation R . Their R -operation must go in and check each subgraph and subtract where needed. But they too must have the overlap problem, later solved I think by Zimmermann.

19.11 Proof that Renormalization Theory is finite. In earlier sections we have noted that after the multiplicative renormalization we have a well-defined iterative scheme for computing things in powers of not e_0 but physical e . In Volume 1 authors showed how you get finite results to order e^2 . To do this they had to do one vertex subtraction, and probably one PI subtraction.

The gist of this section is the following induction proof. Suppose you assume that the theory is "finite" to order e^{n-2} . What this means is you assume that for any graph of order $n-2$, that graph is convergent, ie, has no bad tubes with $D \geq 0$. Suppose this is assumed. Then you somehow construct graphs of order n , and you ask: in the process of adding two extra powers of e , do we somehow create divergent subgraphs somewhere? If not, then $n-2$ implies n and you have an induction proof that to any order n all Feynman graphs are finite, ie, have no divergent tubes.

First, and simplest, consider the kernel K . Pick one of the skeleton graphs in the skeleton expansion for K . (must be order less than n since we are now building a graph of order n). Now take this skeleton and put the meat on, ie, put on full vertices and propagators. Claim: since the skeleton is at least order e^2 , no piece of meat can be of order larger than $n-2$. Eg, no vertex you might add can be more than order $n-2$ in that vertex.

Now sit back and examine this K-contribution you have constructed. Eg, if $n=6$ a graph might be:



Here in fact the only meat I put on is at one vertex. Question: does the graph so constructed have any divergent subgraphs, ie, bad tubes? First of all, there are no problems within any of the meat pieces because by hypothesis these are "finite", since they must be order $n-2$ or less. In the above example, the vertex has been subtracted so has $D=-1$.

Then you must consider graphs which lie outside the meat insertions. But it is claimed all such subgraphs have negative D ; the only problem D graphs involve vertex or self energy insertions, and there are no such things in the skeleton expansion for K or for any higher Greens function. (why I don't know...) ✓

That concludes the induction proof as far as n -point functions with $n \geq 4$ are concerned. Next, we have to analyze the vertex. A term of order n in the vertex is generated from $n-2$ meaty parts via the vertex integral equation. Again the meaty insertions themselves cause no trouble by hypothesis, so you need only consider subgraphs which involve the linking fermion lines as shown top of page 332. Could there somehow be a subintegration involving these lines that will be a bad tube? By exhausting all potentially dangerous possibilities, authors show that there are not. There are some $D=0$ graphs, but these will be removed by the overall vertex subtraction in this order, eg, graphs of figure 41B will be $D=0$ prior to overall subtraction.

Finally you have to do the vac pol self energy case. This is the real mess because now you can have overlapping stuff. A huge amount of work is needed to do the iteration proof here. I haven't even read it. In the end they find that Π is OK (after overall subtraction of course), and ergo D_F is finite. Similarly since vertex Γ was finite by the above, S_F is too via Ward.

And that is the end of this induction proof of the finiteness of the ~~matrix~~ QED computed to as high an order as you want.

19.12 Example: Up to $\mathcal{O}^4$ order. Let integral equation find the term:

$$\text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3} + \text{Diagram 4} - \text{Diagram 5}$$

The circles indicate that you use previously renormalized objects. Thus,

$$\text{Diagram 1} \text{ means } [\Lambda_{(2)}^M(p, p+q) - \Lambda_{(2)}^M(p, p)] = \Lambda_{(2)}^M(p, p+q) - L^{(2)} \delta^M$$

Recall that

$$\Gamma^M(p, p+q) = \delta^M + \Lambda_{(1)}^M + \Lambda_{(2)}^M + \dots$$

$$\Gamma^M(p, p) = \delta^M + \Lambda_{(2)}^M(p, p) + \Lambda_{(3)}^M(p, p) + \dots$$

$$= Z_1 \delta^M = \delta^M + (Z_1 - 1) \delta^M$$

$$\Rightarrow (Z_1 - 1) \delta^M = \underbrace{\Lambda_{(2)}^M(p, p)}_{\substack{L^{(2)} \\ \delta^M}} + \underbrace{\Lambda_{(3)}^M(p, p)}_{\substack{L^{(3)} \\ \delta^M}} + \dots$$

So, the main term in one of the ~~Diagram 1~~ terms will cancel the extra term, so in effect you have

$$\Pi^{(2)} = 2 \left[\text{Diagram 1} \right] + \left[\text{Diagram 2} - 2 \left[\text{Diagram 3} \right] \right]$$

box $\Rightarrow$ only the subtraction itself.

First, it is easy to handle $[]_1$.

You subtract ~~Diagram 1~~ to get ~~Diagram 1~~ $\equiv$ SS. Then compute ~~Diagram 1~~ and make overall subtraction to get $[\Pi]_1$. Result is (19.98) for large q^2 .

Second, + difficult, is to show that the divergent subintegrations in ~~Diagram 2~~ cancel against the two $[]_2$ subtractions. To, pp 349-359 are all needed to show that $[]_2 = \text{finite}$.

The total order $\mathcal{O}^4$ result is (19.121) : $\alpha \Pi_c^{(4)} = -\frac{\alpha^2}{16\pi^3} \ln\left(-\frac{q^2}{m^2}\right)$
for large q^2 , + yields (19.122)
for D_F .

I am making these notes because I think that finally, after all these years, I understand what is going down in this section of BB.

The first 5 pages of the section is old hat by now: identifying the renormalization parts of the theory and so on. In previous sections the renormalized integral equations have been set up and the authors have finally come to face the problem of defining a convergent integral. This section is then good because it gives examples.

The first example is the graph shown in p 322. On page 323 you see the possible subintegrations. Each one is associated with a "tube" and I finally know what that means, I know how to count the power of any subgraph.

From my own notes on ϕ^3 , I finally understand why the symbol $\tilde{\Gamma}$ for the renormed vertex is actually the graph minus the subtraction at the symmetric point. Thus, I finally understand the origin of the equation on page 323. Since we are dealing with renormalized integral equations involving objects like $\tilde{\Gamma}$, each vertex is really a vertex graph minus a subtraction, the subtraction being a divergent constant. Thus, you have to consider in your analysis of divergent subintegrations those which arise from the subtraction itself. This is the idea of the dotted box, see page 324 figures x. In figure a the dotted box is imagined as a point (since just a constant) and you then power count the rest of the graph as shown. Figure b just sets aside the integration which is that constant. This figure b will then cancel subintegration (b) on the preceding page so that the resultant object is finite.

Although I have not gotten that far in my analysis, it seems that the rule for self energy is also a simple subtraction, but eg of the scalar function PI for photon self energy. Thus you should also worry about dotted boxes around self energy insertions. Again, the rule as discussed on page 324 is to shrink these away and pretend they aren't there.

Next example is the more complicated graph on page 326. I am not now worried about the disjoint cases. Basically there are three subgraphs to worry about, page 326 which I call there 1,2,3. Graphs 1 and 2 are cancelled by the subtraction graphs, one of which is shown on page 327 graph (c). One then worries about the new graphs present when the innermost vertex is considered as its subtraction constant, ie, dotted box on that inner vertex. Only (b) and (c) on page 327 interest me. Graph (b) cancels graph (a) on the preceding page. Graph (b) is more interesting: when you shrink the inner vertex to a point, you then expose the other, outer vertex divergence! Depending on what you are doing, this might be cancelled by a dotted box around the outer vertex, but that really depends on what equation you are dealing with, ie, does $\tilde{\Gamma}^{(4)}$ appear, or what.

The final example is the page 329 analysis of 4th order photon self energy. Somehow this appears in such a way that each side is supposed to be subtracted. Then graphs a,b,c show at once the divergence subintegrations. Then d and g show results from putting in separately a subtraction constant on each side, like my ln times born loop in ϕ^3 notes. Then e shows the actual subtraction integral which cancels b, f shows a similar thing which cancels c.

So much for the section. Now we go on to see how we are supposed to come up with such objects.

Showing that the vertex iterates up OK (part of section 19. 11)

Consider the usual integral equation for the vertex which you use to iterate up to power n from power $n-2$. You want to show that your order n vertex is finite, after its single subtraction overall is performed.

How could it not be finite? The overall ~~Box~~ $D=0$ will obviously be "tamed" by the overall subtraction, so the only problem can be divergent subintegrations of some sort. Luckily and conveniently, due to the iterative method, we know that the vertex and the kernel themselves cannot contain divergent subintegrations (logs) because by inductive assumption they are finite completely.

Therefore the only possible problematic subintegration must "run through" the connecting electron propagators as suggested by figure (a) page 332. First, consider how the line runs through the vertex on the left. It seems clear to me that the worst case on the left is the case of just a point vertex or a subtraction box like that shown in (b). In such case you get $D=0$ contribution from the left. Any other "path" through some vertex graph on the left makes negative D , like the $D = -2$ I show in picture on the bottom of the page. So let's take worst case on the left.

Similarly, I think I can argue that the worst case ~~for~~ path through $\tilde{K}$ on the right makes $D=-2$ (my picture in the left margin shows a case $D=-3$). Somehow there is a little theorem here that a subintegration path through some object cannot be worse in power count than the object's canonical D . I believe it.

So figure (b) shows some problem graphs, ie, these are subintegrations which amount to $D=0$. However, there is another theorem which I have yet to consider which claims this: if all your external momenta "flow through" a subgraph (see footnote on page 329), ie if your external momenta are all attached to the subgraph, then the divergence (if any) of that subgraph will be lowered by the overall subtraction.

Thus, the graphs of figure (b) page 332 fall into this category and are thus reduced from $D=0$ to $D=-1$ by the overall subtraction.

The same applies to any subgraph which involves all the lines inside some $\tilde{K}$. (Last sentence page 332). So, the only remaining problems are possible subgraphs contained within $\tilde{K}$ which do not connect out to the right as shown eg on page 333. The subgraph is supposed to be contained entirely within box A and is thus "insulated" from the external lines on the right, and is thus potentially dangerous since a $D=0$ so arising ~~might~~ not be tamable by overall subtraction, according to the just mentioned theorem.

However, the possibilities in this class are ruled out one by one. The last case is shown in figure page 333 bottom. But it is Delbruck and gets in effect an extra -4 , "rescued by four very welcome powers".

Thus, subintegrations which go through the electron lines are all OK and the overall subtraction removes all divergences, and iteration works for the vertex!!!

Showing that the photon self-energy iterates up OK. (part of section 19.11)

This section begins on page 334. The crucial starting step is to get rid of The Dyson equation which is asymmetrical and has that bare vertex at one end, and to replace it with the equation (19.67) which is sketched on page 335 by me.

Why is this step so important? In the asymmetrical equation, you knew that subdivergences at the right side were removed, but that there still remained buried inside subintegrations arising from lines on the right vertex moving over to the left vertex. I.e., these subintegrations were unexposed. The hope was that somehow they exactly cancelled with the logs in the overall Z_1 (or Z_5) factor sitting out front.

The advantage of writing the self-energy in the symmetric form is evident: now subtractions have been performed on both sides, so nothing has been left out. The only complicating is that you have overcounted your feynman graph, which is why you want to correct with the second term. In this second term also, all right and left subintegrations have been automatically accounted for, it turns out.

Of course, with this intuition, you have to go and really show that there are no divergent subintegrations. The trick for doing this is to expand each full vertex in a kernel-expansion, as discussed on page 337. Then you regroup the terms in your self-energy by the number of kernels involved, as shown in (19.77).

So consider the general term which involves n kernels. Some of the kernels are buried, so to speak, in the L 's, and one of the K 's is shown explicitly. You know of course that this entire object has overall $D=0$ which is removed by the overall subtraction, but the question is this: are there any divergent subintegrations of the type that cannot be removed by overall subtraction?

Any subintegrations that cause trouble must interlink more than one K , because you know by induction hypothesis that any particular K has no divergent subintegrations within it, to order $n-2$. The general n -kernel term is shown in the picture on page 339. The procedure is to examine all possible subintegrations and show they are not dangerous. First you look for ones involving only the pair of electron lines called l_1 , i.e., you look for a divergent subintegration which is contained in the first two kernels. You quickly deduce there is none. Then through tortuous consideration you finally conclude that no subset of the kernel groupings, say $l_r \dots l_s$ contains a divergent subintegration. As usual, you are free to ignore disjoint subgraphs. Then finally on page 344 you are done with this proof: the photon self-energy does iterate without introducing logs. Thus, the uncancelled subdivergences at the left vertex in the asymmetric Dyson equation must have exactly cancelled the divergences in that overall Z_1 factor. To prove this directly would be very messy! This was my first approach which led to a mess.

M. Baker & Choonkyu Lee
1977

Overlapping-divergence-free skeleton expansion in non-Abelian gauge theories*

M. Baker and Choonkyu Lee

Department of Physics, University of Washington, Seattle, Washington 98195

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We derive a skeleton expansion in Yang-Mills theory, which is completely free of overlapping divergences. This skeleton expansion provides a generalization of the Schwinger-Dyson equations of quantum electrodynamics to non-Abelian gauge theories. It yields a simple procedure for calculating renormalized perturbation theory integrals without using regulators and could serve as a starting point for nonperturbative approximations. In carrying out this analysis, we show that the Yang-Mills vector self-energy and the Yang-Mills vector three-point function are uniquely determined by the Ward-Takahashi identity in terms of the other three basic vertex functions.

I. INTRODUCTION

The problem of overlapping divergences has greatly complicated the theory of renormalization, even in the simplest case of quantum electrodynamics (QED). In QED, there are three basic vertex functions which are superficially divergent: the photon and electron self-energy functions and the electron-photon vertex. The first two functions have overlapping divergences. For the case of the electron self-energy, one may avoid the problem by invoking the Ward-Takahashi identity, but one still needs a detailed analysis for the photon self-energy.¹

Despite ever-increasing interest in non-Abelian gauge theories for unifying electromagnetic, weak, and strong interactions,² the skeleton expansion structure of Yang-Mills theory has not been investigated in detail because of the enormous complexity of this theory compared to QED. In pure Yang-Mills (YM)³ theory, there are five basic vertex functions (in the covariant gauge) which are superficially divergent: the vector self-energy (quadratically divergent), the ghost self-energy (linearly divergent), the vector three-point function (linearly divergent), the vector four-point function (logarithmically divergent), and the ghost-ghost-vector vertex function (logarithmically divergent). All of the above except the ghost-ghost-vector vertex function have complicated overlapping-divergence problems. Some typical examples are shown in Fig. 1. By taking appropriate derivatives with respect to external momenta, we have succeeded in systematically decoupling the overlapping divergences in the vector four-point function, the vector three-point function, and the ghost self-energy. For the vector self-energy, we use the Ward-Takahashi identity^{4,5} to obtain the vector self-energy directly from the vector three-point vertex, the ghost-ghost-vector vertex, and the

ghost self-energy. Then by further exploiting the Ward-Takahashi identity, we can determine both the vector self-energy and the vector three-point function in terms of the ghost self-energy, the ghost-ghost-vector vertex, and the vector four-point function. Additional simplifications occur in the Landau gauge, where the ghost-ghost-vector vertex function is superficially finite and the ghost self-energy does not have any overlapping divergences. This occurs because, in the Landau gauge, two momentum factors from the first and last vertices on the ghost line are really those of the external momenta of each subgraph.

The skeleton expansions obtained in this paper

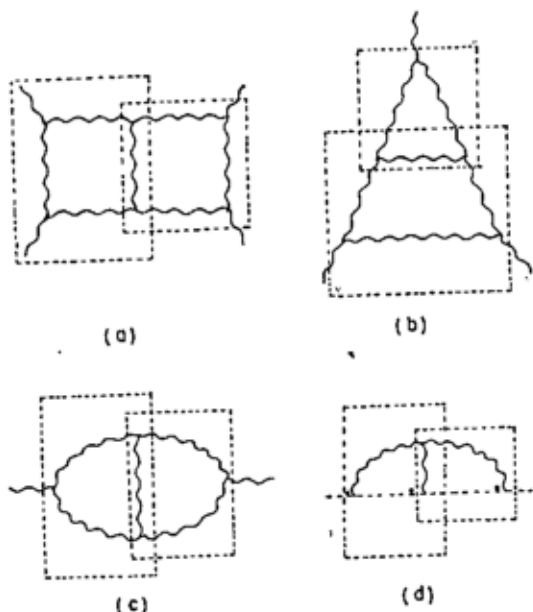


FIG. 1. Some typical examples of overlapping divergences in Yang-Mills theory. The wiggly lines represent the YM vector lines and the dotted lines represent the Faddeev-Popov ghost lines. The dotted boxes indicate various divergent subintegrations.

Baker + Lee

"Skeleton Expansion of NAGT with no overlapping divergences"
PRD⁵, 2201 (1977)

I. Intro: see later

I. QED case. Today is Feb 3, 1979. I am rereading this section for the second time.

Here is what is going on. First, recall from Bjorken Drell how you renormalized QED making use of the various integral equations. That is the "conventional" program.

Here, a fancier alternative is discussed that avoids overlapping divergences.

So how does it work? You want to be able to compute the basic objects Π , Σ , and Γ , the three interesting low n -point functions. The first step is to derive figure 9 which shows the first derivative of Π in terms of the other basic functions plus the kernel K (which is not a renormalization part, and which has a skeleton expansion). The advantage of figure 9 is that there are no overlaps on the RHS, although the RHS is now more complicated than the usual RHS of figure 2.

So, do the same for the derivative of Σ , see figure 11. Again, there are no overlaps. Notice that this figure involves another $n=4$ kernel, one with two electrons and two photons. Now, insert skeleton expansions for the two kinds of kernels and this of course means that $K = K(D, S, \Gamma)$, i.e., functionals of the basic functions of the theory.

Thus, you now have equations for Π' and Σ' which do not have overlaps. Their overall integrations have simple log divergences so you do a single subtraction.

Now we are ready to renormalize the theory in the usual multiplicative manner. Define Z_1, Z_2, Z_3 in the usual way and put a bar over the newly defined functions.

Note that the scalar function Π can be written in terms of a second derivative of Π' as in (20), and you can get the second derivative as in (15) by differentiating the first derivative of figure 9. The single subtraction needed in Π'' thus becomes a single subtraction needed in Π . Thus, define Π_R by subtraction off $\Pi(u^2)$, an arbitrary renorm. point. Notice from (21) and (19) that Z_2 is defined to make the photon propagator have unit numerator and the point $q^2 = u^2$. Notice the claim in (23) that $\Pi(0)$ can only be constant γ times g_{uv} . Then (24) gives Π_R in terms of our Π'' . The point of all this was to show how you get Π_R from Π_{uv}'' .

Now the pair of equations (22) and (24) tell you how to get $\bar{\Pi}$ (order $n+1$) if you know all the other functions to order n .

Next, the argument is repeated for the function $\bar{S}$ - electron propagator. And also for Γ . So you ended up with a way to get from order n to order $n+1$ using the renormed integral equations; the only divergences are the usual overall ones

which can be dumped by simple single subtractions. Now it's on to QCD!

III. Gluons + Ghosts We now consider QCD without matter fields (ie, "pure YMTh").

$$\begin{aligned}
 \rightarrow \text{---} \bigcirc \text{---} &= \text{Duv cont. } J(q^2) \\
 \rightarrow \text{---} \bigcirc \text{---} &= G(p) \\
 \rightarrow \text{---} \bigcirc \text{---} &= p^\lambda \Gamma_{3,\lambda\mu}^{abc} \\
 \rightarrow \text{---} \bigcirc \text{---} &= \Gamma_{3,\lambda\mu\nu}^{abc} \\
 \rightarrow \text{---} \bigcirc \text{---} &= \Gamma_{4,\lambda\mu\nu\sigma}^{abcd} \\
 \rightarrow \text{---} \bigcirc \text{---} &= p^\lambda \Gamma_{4,\lambda\mu\nu}^{abcd}
 \end{aligned}$$

These 4 objects related by one of Ben Lee's Ward Identities, see eqn (40).

ell $\sum 4$

These are related by super-messy W-T Ident (42) of this paper.

(44) relates $\Sigma^{(g)}(p)$ to $G(p)$.

So, these two WTI's are conditions on the theory. They will go on to find skeleton expansions such as

$$g^2 \Sigma^{(g)} = \text{fcn} (D, G, g_0 \Gamma_3, g_0 \Gamma_3^{(g)}, g_0^2 \Gamma_4) \text{ etc.}$$

$$\begin{aligned}
 \text{Denote as: } D &= \Sigma_V D & g_0 &= X g \text{ etc.} \\
 G &= \Sigma_G G
 \end{aligned}$$

so there are three constants. After doing this, the two WTI's look the same except all fcn's now boxed

$$\text{Lemma: } \underbrace{\Gamma_4(n) + \bar{G}(n) + \bar{P}_3^{(g)}(n)}_{\text{order}} + \bar{P}_2(n-1) + \bar{J}(n-1) \Rightarrow \bar{P}_3(n), \bar{J}(n)$$

knowledge of these

$\Rightarrow$ these $\uparrow$

This is proved using WTI's.

It then follows that

$$\bar{P}_4 + \bar{G} + \bar{P}_3^3 \Rightarrow \bar{P}_3, \bar{J} \text{ to all orders}$$

They then go on to show that $\bar{P}_4, \bar{G}, \bar{P}_3^3$ are determined uniquely by the skeleton expansions $\rightarrow$ by your choice of ... well, you require various facts at $p_i = \mu^2$ + these in turn determine X, Z_0 and Z_V . By requiring these norm conditions [I need them], you remove all amb from $\bar{P}_4, \bar{G}, \bar{P}_3^3$, and you set X, Z_0, Z_V .

Once, the renorm program is completed, just as with QED. They argue that matter fields are easily incorporated.

IV. Explicit construction of skeleton exp. for Ξ

This exp. was used as G_1 (45a) in the renorm proof. First, Ξ is written in terms of other objects

$$\Xi = f(T_{sta}^{sta}, T_c^s, G_4^u, D, P_3)$$

$\swarrow$ skel exp in Fig 22. $\searrow$ rel. to T_c^s Fig 16

So somehow you are supposed to end up with a skeleton exp for $\Xi = P_4$ which has no overlap problems; analysis similar to QED.

V. Ghost self-energy skel. exp. Again, this is to establish the existence of such a skeleton exp., since this was used as (45c) in the renorm proof.

Appendix A 14 page mess which gets a no-overlapping-div skel. exp for Ξ . Not needed in their program because P_3 determined by other objects via WTI, but they do it anyway. Zips!

I. Introduction. Recall the proof of renormalizability of QED as given in Bjorken and Drell. You write down a bunch of integral equations, and you make skeleton expansions for some basic objects. Then you multiplicatively renorm the basic functions and you end up with renormed integral equations with nearly no Z's showing at all. Putting pieces together, you get a way to compute things to order n if you know order $n-1$, and all integrals are finite, so theory can be renormalized. The main complications involved in the proof involve the occurrence of divergent subintegrals which cannot be disentangled from one another, ie, overlapping divergences. When you make a subtraction on one divergence, you hurt the other integration, not clear what to do. The nightmarish 16 pages section 19.11 of BD deals with showing that you can handle the overlaps.

Baker and Lee have found a better way to do this. They cleverly find a way to show that QED is renormalizable which simply avoids the overlap problem altogether. All they really do is construct skelton expansions for basic objects which, though more complicated, simply don't have the overlaps. Not really all that fancyh. So perhaps their method is the best proof of renorm of QED and would replace the chapter in BD.

They then go on to generalize their proof of renorm to QCD, and general NAGT.

In the general pure gauge theory there are all kinds of overlap problems, as shown in figure 1. In Section 3 of this paper, they show how to do it.

How? In QCD there are 6 basic functions to deal with, (including ghosts, this is not axial gauge). But you also have two Ward Tak identities to work with. These in effect allow you to determine two functions in terms of the other 4 and simplifies the problem. Then the usual renorm problem is stated and pushed through, assuming the existence of fancy overlap-free skeleton expansions for various objects. In sections 4 and 5 these needed skeleton expansions are computed, very messy indeed.

So I guess the main part of their work is to derive these overlap-free skeleton expansions. Certainly an amazing mess.

The appendix goes on to get one more skeleton expansion for the 3-gluon, one which was not needed in the main text because the 3-gluon was eliminated by the Ward's. But they think result might be useful anyway. This appendix is 15 of the papers 33 Phys Rev pages.

So despite the technical complexity, I think I get the main point: this paper gives a proof of renorm of NAGT. Makes use of the WT identities, and makes use of their own overlap-free skeleton expansions.

Multi-fields

Contents of this Section.

1. Comment: in the previous section I derived the Invariance Theorem for $[dA][\det M]$ first in covariant gauge, gauge fields only; and then in general gauge, but gauge fields only. Here I have generalized to multi-fields. I.e, you can have as many fields as you want along with the gauge fields, and you can choose any gauge you want, i.e, the gauge condition can be any function of all the fields. From the invariance proven here, you conclude that the generating functional is independent of a and this allows the "exponentiation of the delta" so you can get rid of your gauge surface delta and incorporate it, in effect, into your Feynman rules. Now I am equipped to study arbitrary gauges made by combining fields together.

Contents:

- Sec 1: compare gauge field transform to regular field, then invent a notation which allows both to be expressed as one.
- Sec 2: With many fields, redefine the object $M_{ab}(x,y)$ and do chain rule on it. This now involves type sums.
- Sec 3: Combine type index with component index to get supervector index i . Write things in this notation.
- Sec 4: Invent a generalized D_{ib} object to further shorten notation and to imitate the pure field case. For example, write $M^T = (-1/g) D^T R$ in full matrix sense. Note that some matrices no longer square. Define R and S matrices.
- Sec 5: Compute $dM_{ab}(x,y)/d\phi_i(z)$ functional derivative. Use this with the constraint to get a result for matrix $S = dg/d\phi$. Then use this to get expression for dJ , variation of the full, all-fields Jacobian.
- Sec 7: Get formula for dM^T . Insert generalized Jacobi Identity to get this into simple matrix form. Then compute $d[\ln(\det M)]$ and observe that it cancels $dJ!!!$
- Appendix 1: Attack.... here was my search for and derivation of generalized Jacobi
- Appendix 2: Faster Derivation of Jacobi Identity! Result compared to LZJ's incorrect result, All of their appendix.
- Appendix 3: Derivation of the Gamma Identity used in Appendix 2.

Composite Notation to include all types of fields at once.

1. Comment: I am going to try to imitate LZJ-4 as much as possible, but not to the point where I lose precision in the notation. First, from my own notes, I can write down infinitesimal gauge transformation on gauge fields or non-gauge fields:

$$\begin{aligned} A_{a,\mu}^{\gamma} &= A_{a,\mu}^{\gamma} + \left[-C_{af}^e A_{f,\mu} - \frac{1}{g} \delta_a^e \partial_{\mu} \right] \Theta_e & / \text{gauge} \\ f_i^{\gamma} &= f_i + \left[-i G_{ij}^e f_j \right] \Theta_e & / \text{regular.} \end{aligned}$$

In this last line, you are to understand that the objects G^e are matrices for the representation which describes the fields f_i . For the gauge fields, I am finally convinced that the above shows how they transform no matter what fields they are coupled to. Recall that at one time it appeared that the transformation might be γ -dependent, but that is not the case.

Let me first replace the C with $G^{(\text{adj})}$ in the first line (so it looks like the second), then I'll artificially enlarge the thing so I can treat the Lorentz and color index of the gauge field as a single index:

$$\begin{aligned} A_{[a,\mu]}^{\gamma} &= A_{[a,\mu]}^{\gamma} + \left[\left(-i G_{[a,\mu],[f,\nu]}^{(\text{adj})e} \right) A_{[f,\nu]} + \left(-\frac{1}{g} \delta_a^e \partial_{\mu} \right) \right] \Theta_e \\ f_i^{\gamma} &= f_i + \left[\left(-i G_i^e \right) f_j \right] \Theta_e \end{aligned}$$

where

$$\begin{aligned} \left(-i G_{[a,\mu],[f,\nu]}^{(\text{adj})e} \right) &= \delta_{\mu,\nu} \left(-i G_{af}^{(\text{adj})e} \right) \\ &= \delta_{\mu\nu} \left[-C_{af}^e \right] = \delta_{\mu\nu} \left[-C_{eaf} \right]. \end{aligned}$$

So my only "doctbring" is to dummy up some Lorentz indices on the adjoining group representation so I can treat the gauge fields as if they had one index.

Notice that there are no space integrations in the above. The spacetime argument x just sits in each function. Lee and ZJ go on to include the spacetime label as part of their single, composite field index, but that is where I get confused. That is the point whereat the notation ceases to aid me and begins to confuse me. So I aint gonna do it. Ill stick to the above. So summarize on next page:

$$\phi_i^g(x) = \phi_i(x) + \left[\Gamma_{ij}^e \phi_j(x) + \vec{\Lambda}_i^e(x) \right] \Theta_e(x)$$

object

regular field

gauge field

i, j

i, j

$[a, \mu], [b, \nu]$

ϕ_i

f_i

$A_{a, \mu}$

Γ_{ij}^e

$(-i G_{ij}^e)$

$(-i^{(a,b)} G_{a, \mu, b, \nu}^e)$

$\vec{\Lambda}_i^e$

0

$(-\frac{1}{g} \delta_a^e \vec{\partial}_\mu^e(x))$

1) Number of values index of type i, j takes depends on kind of field.

Example: for gauge field and $SU(3)$, index i takes on $4 \times 8 = 32$ values

2) However, index e in the above always takes on the usual 8 values in $SU(3)$.

Thus, you want to keep these index types different!

3) There are no implied space integrations !

(2) Next, let's find out how to write the FP determinant and object, as well as the object M , and so on.

To do this we shall start with a gauge function $F_e(\phi^g(x))$. The only new idea now is that this might be a function of all the fields, not just the gauge fields. But I don't think this really makes much change (e is a color index). As before we can do a change of variables:

$$F_{e,n} \equiv F_e(\phi^g(x_n))$$

$$\Rightarrow \prod_{e,n} [dF_{e,n}] = J \cdot \prod_{e,n} [dg_{e,n}]$$

$$g_{e,n} \equiv \Theta_e[x_n]$$

$$\text{where } J = \left| \frac{\nabla F_{e,n}}{\nabla \Theta_{f,n}} \right| = \det M[\phi^g]$$

$$\text{where } M_{ef}(\phi^g; x_n, x_m) = \frac{\nabla F_e(\phi^g(x_n))}{\nabla \Theta_f(x_m)}$$

Before writing this out with a chain rule, it might be useful to go back and introduce a field type index. For example, I might rewrite the above general form of a gauge transformation as follows:

$${}^{(t)}\phi_i^g(x) = {}^{(t)}\phi_i(x) + \left[{}^{(t)}\Gamma_{ij}^e {}^{(t)}\phi_j(x) + {}^{(t)}\vec{\Lambda}_i^e(x) \right] \Theta_e(x)$$

Essentially, this equation is "diagonal" in the type index so you don't really need it, though later it may be useful. Once you know the type t , you then know the nature of and the number of values of the indices i, j . You also know the Gamma object. Earlier, this type index was implicitly carried by the i, j indices. I think now that it is probably a good idea to explicitly display the type index always.

So, we can now do our chain rule as follows :

$$M_{ab}(\phi^g; x, y) = \frac{\nabla F_a(\phi^g(x))}{\nabla \Theta_b(y)} = \sum_{t,i} \int dz \left(\frac{\nabla F_a(\phi^g(x))}{\nabla {}^{(t)}\phi_i(z)} \right) \left(\frac{\nabla {}^{(t)}\phi_i(z)}{\nabla \Theta_b(y)} \right)$$

Now things are sensible: in the chain rule you sum over types of fields, because F may involve all the types, then you sum over field components i , and you integrate over spacetime as shown.

Now let's assume that g is very small in the above, then we can use the infinitesimal formula above, and we put $\phi^g = \phi$ in functional arguments. So:

$$M_{ab}(\phi; x, y) = \sum_{t,i} \int dz \cdot \left[\frac{\nabla F_a(\phi(x))}{\nabla {}^{(t)}\phi_i(z)} \right] \cdot \left[{}^{(t)}\Gamma_{ij}^e {}^{(t)}\phi_j(z) + {}^{(t)}\vec{\Lambda}_i^e(z) \right] \delta_{eb} \delta(z-y)$$

Next, this can be converted to spacematrix notation as follows:

$$[\underline{M}_{ab}(\phi)] = \sum_{t,i} \left[\frac{\nabla F_a(\phi)}{\nabla \phi_{t,i}} \right] \left[{}^{(t)}\Gamma_{ij}^b [\phi_{t,j}] + [{}^{(t)}\vec{\Lambda}_i^b] \right]$$

- ③ Now I must admit it does appear useful to put the type and component indices together. So let's define this:

$$\Gamma_{t_i, t_j}^e = \delta_{t_i, t_j} \left[{}^{(t)}\Gamma_{ij}^e \right]$$

Then we get:

$$\phi_{t_i}^g(x) = \phi_{t_i}(x) + \left[\Gamma_{t_i, t_j}^e \phi_{t_j}(x) + \vec{\Lambda}_{t_i}^e(x) \right] \Theta_e(x)$$

$$[\underline{\underline{M_{ab}(\phi)}}] = \sum_{t,i} \left[\frac{\nabla F_a(\phi)}{\underline{\underline{\nabla \phi_i}}} \right] \left[\Gamma_{t_i, t_j}^b [\underline{\underline{\phi_{t_j}}}] + [\underline{\underline{\Lambda_{t_i}^b}}] \right]$$

And now you see that everywhere you sum on i , you are also going to be summing over type index t . Thus you may as well combine them into a single index:

(t,i) now becomes just (i)

Then whenever you see a sum on i , you understand that you are supposed to sum over each component of each type of field. So finally I will rewrite the above:

$$\phi_i^g(x) = \phi_i(x) + \left[\Gamma_{ij}^e \phi_j(x) + \vec{\Lambda}_i^e(x) \right] \Theta_e(x)$$

$$[\underline{\underline{M_{ab}(\phi)}}] = \left[\frac{\nabla F_a(\phi)}{\underline{\underline{\nabla \phi_i}}} \right] \left[\frac{\nabla \phi_i^g}{\underline{\underline{\nabla \Theta_b}}} \right]$$

$$= \left[\frac{\nabla F_a(\phi)}{\underline{\underline{\nabla \phi_i}}} \right] \left[\Gamma_{ij}^b [\underline{\underline{\phi_j}}] + [\underline{\underline{\Lambda_{ij}^b}}] \right]$$

- 1) Each repeated i, j, k type index is to be summed over.
- 2) Each such index is to be understood as $i = (t_i, i)$ so you sum over field types, and for each field type you sum over components.
- 3) In this notation, all the fields are jammed together into one super-vector.

3. Now I think there is little point in writing this fancy object again and again, so I now propose to invent the following symbol to mimic the pure gauge-fields notation:

$$\begin{aligned} \underbrace{[\underline{\underline{\vec{D}_{ib}(\phi)}}]} &= (-g) \left\{ \Gamma_{ij}^b [\underline{\underline{\phi_j}}] + [\underline{\underline{\vec{\Lambda}_i^b}}] \right\} \\ &\Downarrow \\ \phi_i^a(x) &= \phi_i(x) - \frac{1}{g} \vec{D}_{ib}(x) \Theta_b(x) \\ [\underline{\underline{M_{ab}(\phi)}}] &= \left[\frac{\nabla F_a(\phi)}{\nabla \phi_i} \right] (-\frac{1}{g}) [\underline{\underline{D_{ib}(\phi)}}] \end{aligned}$$

4. Now we can even go one step further: you see that the only difference between our old notation and this notation is:

- a) the derivatives Λ are simply absent for $i \neq$ gauge index
- b) the first index on D etc takes on more than the color values.

It is my guess that these changes are so minor, that my old proof of invariance etc. can be carried through with little change. Lets write:

$$[\underline{\underline{M(\phi)}}] = -\frac{1}{g} \left[\frac{\nabla F}{\nabla \phi} \right] [\underline{\underline{D}}]$$

Now this looks exactly like our former expression when we had only gauge fields. Of course now it must be understood that matrices like D are no longer "square", ie, the matching "color index" in the above product is actually a super-vector index.

So lets run through our basic stuff in the invariance proof and generalize each statement. For example:

For starters, we can define matrices R and S as before (but now they are not square!):

$$\left[\frac{\nabla \theta}{\nabla \phi} \right]_{\alpha x, i y} = \frac{\nabla \theta_a(x)}{\nabla \phi_i(y)} = [S^T]_{\alpha x, i y} = S_{ai}^T(x, y) = S_{ia}(y, x)$$

$$\sim \left[\frac{\nabla \theta}{\nabla \phi} \right]^T \equiv \underline{\underline{[S]}}$$

Similarly:

$$\left[\frac{\nabla F}{\nabla \phi} \right]^T \equiv \underline{\underline{[R]}}$$

One:

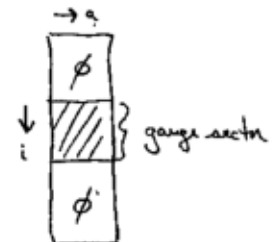
$$R_{ia}(y, x) = \frac{\nabla F_a(x)}{\nabla \phi_i(y)}$$

If you want to link back to the covariant gauge, where $F = d_u A^u$ depended only on the gauge field, then you get:

$$R_{ia}(y, x) = R_{(x, i), a}(y, x) = \delta_{i, \text{gauge}} \underbrace{R_{[\mu, b], a}(y, x)}_{-\delta_{ba} [\partial_y^\mu \delta(y-x)]}$$

$$\Rightarrow [\underline{\underline{R}}] = -\delta_{i, \text{gauge}} \cdot \delta_{b, a} [\underline{\underline{\partial}}^\mu]$$

Here I have sketched R as a non-square matrix. Then in the usual covariant gauge, it is zero except for a square portion which is the gauge sector.



Now we can go on to write M two ways: notice that M happens to be a square matrix, although it is made out of the product of two non-square ones. Note by the way that you can never have $D^T = -D$ now because of non-squareness:

$\underline{\underline{[M]}} = -\frac{1}{g} \left[\frac{\nabla F}{\nabla \phi} \right] \underline{\underline{[D]}}$ $\boxed{R^T} \quad \boxed{D}$	$\underline{\underline{[M^T]}} = -\frac{1}{g} \left[\underline{\underline{[D^T]}} \right] \underline{\underline{[R]}}$ $\boxed{D^T} \quad \boxed{R}$
--	---

⑤. Now using the expression above for M^T we can compute $dg/d\phi$ due to the constraint as follows:

$$\dot{M}_{ba}^T(y, x) = -\frac{1}{g} \int dt D_{bi}^T(y, t) R_{ia}(t, x)$$

$$\Rightarrow \frac{\nabla M_{ba}^T(y, x)}{\nabla \phi_j(z)} = -\frac{1}{g} \int dt \left\{ \left(\frac{\nabla D_{bi}^T(y, t)}{\nabla \phi_j(z)} \right) R_{ia}(t, x) + D_{bi}^T(y, t) \left(\frac{\nabla R_{ia}(t, x)}{\nabla \phi_j(z)} \right) \right\}$$

But,

$$\frac{\nabla D_{bi}^T(y, t)}{\nabla \phi_j(z)} = \frac{\nabla D_{ib}(t, y)}{\nabla \phi_j(z)} = (-g) \Gamma_{ik}^b \delta_{jk} \delta(t-z) \delta(t-y)$$

Here I use the fact that the Λ thing, which only exists for gauge fields, is never a function of the fields so yields nothing to the derivative here. Also, look back at formula for definition of D_{ib} to see how I got this, and note that Γ_{ij} is always "anti sym in lower indices (look back at definition!). Therefore the first term on the RHS above is:

$$-\delta(y-z) \Gamma_{ji}^b R_{ia}(z, x)$$

Second term:

$$-\frac{1}{g} \int dt D_{bi}^T(y, t) \frac{\nabla}{\nabla \phi_j(z)} \frac{\nabla R_{ia}(x)}{\nabla \phi_i(t)}$$

↔
switchaden.

$$= \left(-\frac{1}{g} \int dt D_{bi}^T(y, t) \frac{\nabla}{\nabla \phi_i(t)} \right) R_{ja}(z, x) \quad \begin{matrix} \uparrow \text{add } \delta_{ij} \\ \uparrow \text{a. do k} \end{matrix} \quad = \text{second term}$$

Now combine these terms to state the result in a manner similar to last time:

$$\boxed{\frac{\nabla M_{ab}(x, y)}{\nabla \phi_j(z)} = \left\{ -\delta(y-z) \Gamma_{ji}^b - \frac{1}{g} \delta_{ji} \overrightarrow{D_{bk}^T}(y) \frac{\nabla}{\nabla \phi_k(y)} \right\} R_{ia}(z, x)}$$

This result is now inserted into the constraint condition as follows:

$$\lambda_a = \int dy \Theta_b(y) M_{ba}^T(y, x)$$

$$\Rightarrow 0 = \int dy \Theta_b(y) \left[\frac{\nabla M_{ba}^T(y, x)}{\nabla \phi_j(z)} \right] + \int dy \left[\frac{\nabla \Theta_b(y)}{\nabla \phi_j(z)} \right] M_{ba}^T(y, x)$$

Result is :

$$0 = -\Theta_b(z) \Gamma_{ji}^b R_{ia}(z, x) - \vec{F}[R_{ja}(z, x)] + \int dy S_{jb}(z, y) M_{ba}^T(y, x)$$

Therefore we get

$$\boxed{\begin{aligned} \underline{[S]} \underline{[M^T]} &= \underline{[\Theta_b]} \underline{[\Gamma^b]} \underline{[R]} + \vec{F}(\underline{[R]}) \\ \text{where } \vec{F} &= + \left(\frac{1}{f} \int dy \Theta_b(y) \vec{D}_{bi}^T(y) \frac{\nabla}{\nabla \phi_i(y)} \right) \end{aligned}}$$

By the way,

$$\begin{cases} \vec{D}_{ib}(y) = (-g) [\Gamma_{ij}^b \phi_j(y) + \Lambda_i^b(y)] \\ \vec{D}_{bi}^T(y) = (-g) [\Gamma_{ij}^b \phi_j(y) - \Lambda_i^b(y)] \end{cases}$$

If you have only a gauge sector, then:

$$\begin{aligned} \vec{D}_{ab}(y) &= (-g) [\Gamma_{ac}^b \phi_c(y) + \Lambda_a^b(y)] \\ \vec{D}_{ba}^T(y) &= (-g) [\Gamma_{ac}^b \phi_c(y) - \Lambda_a^b(y)] \\ &= -(-g) [\Gamma_{bc}^a \phi_c(y) + \Lambda_b^a(y)] = -\vec{D}_{ba}(y) \end{aligned}$$

Thus, if you have only gauge fields, you can replace D^T by $-D$ without touching the color indices. Then this operator F matches the previous operator F exactly.

Note also that:

$$\Gamma_{ac}^b = -C_{ac}^b \text{ for gauge only.}$$

Thus, we have extended our previous result to multi-fields. In the limit of gauge fields only this result reduces to the previous result. Now as before, you can solve for $dg/d\phi$ as follows:

$$\boxed{[S]_{\underline{a}} = \left\{ [\underline{\Theta}_b] [\underline{\Gamma}^b] [\underline{R}] + (\underline{\sigma} [\underline{R}]) \right\} [\underline{M}^{-1}]_{\underline{b} \underline{a}}}$$

The transposed statement is obvious, no need to write it, see before results.

⑥ Now of course we are going to put this object into some trace to get the variation of the field Jacobian. I will now show that the usual form is maintained in that operation:

$$\frac{\nabla \widehat{D}_{ib}(x)}{\nabla \phi_j(y)} = (-g) \Gamma_{ij}^b \delta(x-y)$$

$$\Rightarrow \left[\frac{d\phi^j}{d\phi} \right] = \left[\underline{1} \right] - \left\{ \frac{1}{g} [\underline{D}] \left[\frac{\nabla \Theta}{\nabla \phi} \right] - [\underline{\Gamma}^b] [\underline{\Theta}_b] \right\}$$

$$\Rightarrow [d\phi^j] = J [d\phi] \quad \text{where} \quad \boxed{J \approx 1 - \frac{1}{g} \text{Tr} \left\{ [\underline{D}]_{\underline{a} \underline{b}} \left[\frac{\nabla \Theta}{\nabla \phi} \right]_{\underline{b} \underline{a}} \right\}}$$

The result looks the same as before, but of course now the trace is in the larger super-vector space since you are computing a Jacobian from all the fields to all the gauge-transformed fields! Even though objects in there are not square, you can still interchange order of traced matrix product. I am inclined to transpose both objects in there to get

$$\boxed{\delta J = -\frac{1}{g} \text{Tr} \left\{ [\underline{D}^T]_{\underline{a} \underline{i}} [\underline{S}]_{\underline{i} \underline{a}} \right\}}$$

$$\Rightarrow \boxed{\delta J = -\frac{1}{g} \text{Tr} \left([\underline{D}^T] \left\{ [\underline{\Theta}_b] [\underline{\Gamma}^b] [\underline{R}] + (\underline{\sigma} [\underline{R}]) \right\} [\underline{M}^{-1}] \right)}$$

7. We now turn to the computation of dM^T for use in the $\det M$ variation deal. To this end I find it useful to utilize the previously derived formula on bottom of page 7, except I want to switch the order of the functional derivatives in the second term (back to what it originally was). Thus:

$$\delta M_{ba}^T(y, x) = \int dz [\delta \phi_j(z)] \left[\frac{\nabla M_{ba}^T(y, x)}{\nabla \phi_j(z)} \right]$$

$$= \int dz \left[-\frac{1}{g} \int dt D_{jc}(z, t) \Theta_c(t) \right] \left[-\delta(y-z) \Gamma_{ji}^b R_{ia}(z, x) - \frac{1}{g} \overrightarrow{D}_{bk}^T(y) \frac{\nabla}{\nabla \phi_j(z)} R_{ka}(y, x) \right]$$

First term is:

$$+ \frac{1}{g} \int dt D_{jc}(y, t) \Theta_c(t) \Gamma_{ji}^b R_{ia}(y, x) = \left(\frac{1}{g} [\underline{D}\underline{\Theta}]_j \Gamma_{ji}^b [\underline{R}_{ia}] \right)_{yx}$$

Second term:

$$\frac{1}{g^2} \overrightarrow{D}_{bk}^T(y) \int dz dt \Theta_c(t) D_{cj}^T(t, z) \frac{\nabla}{\nabla \phi_j(z)} R_{ka}(y, x) = \left\{ \frac{1}{g} [\underline{D}_{bk}^T] (F[\underline{R}_{ka}]) \right\}_{yx}$$

$\Rightarrow$

$$\boxed{[\underline{\delta M}_{ba}^T] = \frac{1}{g} [\underline{D}_{bk}^T] (F[\underline{R}_{ka}]) + \frac{1}{g} [\underline{D}\underline{\Theta}]_j \Gamma_{ji}^b [\underline{R}_{ia}]}$$

Notice that, in the second term, the color index b is left hanging in the sky, so it is not yet possible to write this thing as a color-space matrix (for the pure gauge you could because you could just rotate indices). No matter. Obviously the next thing we will need is a generalization of the fancy Jacobi identity.

Easy to show that (since $\Lambda_i = \text{const} \times \text{deriv.}$):

$$\boxed{[\underline{D}_{ib}][\underline{\Theta}_c] - [\underline{\Theta}_c][\underline{D}_{ib}] = (-g) [\underline{\Lambda}_i^b \underline{\Theta}_c]}$$

Of course the RHS here vanishes generally. Only if i is in the gauge field sector, and even then, only if $i=b$, do you get something on the right.

In the section which follows this ("Attack on...") I show tortously that ~~after~~ the following identity is true:

$$[D_{0j}]_j^b = - \Gamma_{be}^c [\underline{0}_e] [\underline{D}_{ei}^T] + [\underline{D}_{bj}^T] [\underline{0}_e] \Gamma_j^c$$

Rewrite:

$$[D_{0j}]_j^b = \left([\underline{D}_{jj}^T] [\underline{0}_e] [\underline{\Gamma}_j^c] - [\underline{\Gamma}_j^c] [\underline{0}_e] [\underline{D}_{jj}^T] \right)_{bi}$$

My notation does not let me write the object on the left as a color matrix.

Now, we go ahead and compute $d[\ln(\det M)]$. Since we have not changed the dimensions of M since last time, the formula is exactly what it was, namely:

$$\delta[\ln(\det M)] = \text{Tr} \left[[\underline{\delta m}^T] [\underline{m}^{-1}] \right]$$

Moreover, we have shown that:

$$[\underline{\delta m}^T] = \frac{1}{g} [\underline{D}_{jj}^T] (\underline{\tilde{F}}[\underline{R}]) + \frac{1}{g} [\underline{D}_{jj}^T] [\underline{0}_e] [\underline{\Gamma}_j^c] [\underline{R}] - \frac{1}{g} [\underline{\Gamma}_j^c] [\underline{0}_e] [\underline{D}_{jj}^T] [\underline{R}]$$

$-g \underline{m}^T$

Repeat:

$$[\underline{\delta m}^T] = \frac{1}{g} [\underline{D}_{jj}^T] \left\{ (\underline{\tilde{F}}[\underline{R}]) + \frac{1}{g} [\underline{0}_e] [\underline{\Gamma}_j^c] [\underline{R}] \right\} + [\underline{\Gamma}_j^c] [\underline{0}_e] [\underline{m}^T]$$

Thus:

$$\delta[\ln(\det M)] = \frac{1}{g} \text{Tr} \left([\underline{D}_{jj}^T] \left\{ [\underline{0}_e] [\underline{\Gamma}_j^c] [\underline{R}] + (\underline{\tilde{F}}[\underline{R}]) \right\} [\underline{m}^{-1}] \right)$$

where $\text{Tr}([\underline{\Gamma}_j^c] [\underline{0}_e]) = 0$ by coln. So

$$\delta\mathcal{L} + \delta \ln \det M = 0.$$

Attack Strategy on Manipulation of Second term of dM^T

1. My quick kill efforts have failed. Here is what I have to show, but what I am not yet able to show:

$$\text{tr} \left\{ [\underline{D}\underline{\phi}]_j \Gamma_{ji}^b [\underline{R}_{ia}] [\underline{M}_{ab}^{T^{-1}}] \right\} = \text{tr} \left\{ [\underline{D}_{bj}^T] [\underline{\phi}_c] \Gamma_{ji}^c [\underline{R}_{ia}] [\underline{M}_{ab}^{T^{-1}}] \right\}$$

Ie, once I show this, I am done with the completely general invariance proof. What is stalling my efforts right now is this: the useful Jacobi Identity which made things go last time doesn't work here, and refuses to generalize! The problem is that the Jacobi identity I used last time seems to require a color label on the entity $[\underline{D}\phi]_c$, whereas in the above I am facing $[\underline{D}\phi]_j$ where j is a complete supervector label.

2. So here is a plan: take the first trace above and break it into two pieces. In the first piece, index j runs through the gauge field's sector, and in the second piece it runs through the rest of its supervector index values.

First, here is an expression for the terms which do not involve the color sector:

$$g \cdot \sum_{\substack{j \in \text{color} \\ \text{sector}}} \Gamma_{kj}^c \Gamma_{ji}^b \text{tr} \left\{ [\underline{\phi}_k] [\underline{\phi}_c] [\underline{R}_{ia}] [\underline{M}_{ab}^{T^{-1}}] \right\}$$

3. Now examine the j sum which is in the gauge field sector. If we put $j=c$, then the index i may as well be replaced with d since you get nothing if it is outside the gauge field sector. So this term yields:

$$\text{tr} \left\{ [\underline{D}\phi]_c \Gamma_{cd}^b [\underline{R}_{da}] [\underline{M}_{ab}^{T^{-1}}] \right\} = -\text{tr} \left\{ [\Gamma_{bd}^c] [\underline{D}\phi]_c [\underline{R}_{da}] [\underline{M}_{ab}^{T^{-1}}] \right\}$$

$\Gamma_{cd}^b = \Gamma_{db}^c = -\Gamma_{bd}^c \quad \nearrow$

But use this: $\Gamma_{cd}^b [\underline{D}\phi]_c = [\underline{D}] \Gamma_{cd}^b [\underline{\phi}_c] - [\underline{\phi}_c] \Gamma_{cd}^b [\underline{D}]$

Then even go ahead and transpose them!

$$= -[\underline{D}^T] \Gamma_{cd}^b [\underline{\phi}_c] + [\underline{\phi}_c] \Gamma_{cd}^b [\underline{D}^T]$$

Thus we get: $+ \text{tr} \left\{ [\underline{D}_{be}^T] \Gamma_{ed}^c [\underline{\phi}_c] - [\underline{\phi}_c] \Gamma_{be}^c [\underline{D}_{ed}^T] \right\} [\underline{R}_{da}] [\underline{M}_{ab}^{T^{-1}}]$

So far, then, we have taken the left hand side trace in the original equation and have converted it into these three terms:

$$a) \quad g \sum_{j \neq GF} \Gamma_{kj}^c \Gamma_{ji}^b + \text{tr} \left([\underline{\phi}_k] [\underline{0}_c] [\underline{R}_{ia}] [\underline{M}_{ab}^{T-1}] \right)$$

$$b) \quad \Gamma_{ed}^c + \text{tr} \left([\underline{D}_{be}^T] [\underline{0}_c] [\underline{R}_{da}] [\underline{M}_{ab}^{T-1}] \right)$$

$$c) \quad - \Gamma_{be}^c + \text{tr} \left([\underline{0}_c] [\underline{D}_{ed}^T] [\underline{R}_{da}] [\underline{M}_{ab}^{T-1}] \right)$$

Notice that we cannot yet simplify the last term because the correct formula for M^T in the presence of multi-fields is this:

$$(-g) [\underline{M}_{ea}^T] = \dots [\underline{D}_{ej}^T] [\underline{R}_{ja}]$$

So, an obvious idea that comes to mind is this: add to term c) above the terms that are missing to make this work, then subtract them out again. Thus, we shall replace term c) with $c1) + c2)$:

$$c) = c1) + c2)$$

$$c1) \quad - \Gamma_{be}^c + \text{tr} \left([\underline{0}_c] [\underline{D}_{ej}^T] [\underline{R}_{ja}] [\underline{M}_{ab}^{T-1}] \right)$$

$$c2) \quad + \Gamma_{be}^c \sum_{j \neq GF} \text{tr} \left([\underline{0}_c] [\underline{D}_{ej}^T] [\underline{R}_{ja}] [\underline{M}_{ab}^{T-1}] \right).$$

Then:

$$\begin{aligned} c1) &= - \Gamma_{be}^c + \text{tr} \left([\underline{0}_c] [\underline{M}_{ea}^T] [\underline{M}_{ab}^{T-1}] \right) \cdot (-g) \\ &= - \Gamma_{be}^c + \text{tr} \left([\underline{0}_c] \delta_{eb} [\underline{1}] \right) \cdot (-g) \\ &= 0 \end{aligned}$$

So I have forced this through, mimicking the pure gauge approach. We are now left with (a) + (b) + (c).

Next, lets try to force term b) into what we want to see in the result:

$$\begin{aligned}
 (b) &= \text{tr} \left\{ [\underline{D}_{be}^T] [\underline{O}_c] \Gamma_{ed}^c [\underline{R}_{da}] [\underline{M}_{ab}^{T^{-1}}] \right\} \\
 &= \text{tr} \left\{ [\underline{D}_{bj}^T] [\underline{O}_c] \Gamma_{ji}^c [\underline{R}_{ia}] [\underline{M}_{ab}^{T^{-1}}] \right\} \leftarrow \text{put in the rest} = (b1) \\
 &\quad - \sum_{\substack{i,j \\ \notin GF}} \text{tr} \left\{ [\underline{D}_{bj}^T] [\underline{O}_c] \Gamma_{ji}^c [\underline{R}_{ia}] [\underline{M}_{ab}^{T^{-1}}] \right\} \leftarrow \text{take rest back out.} = (b2)
 \end{aligned}$$

But term (b1) is our desired result. Therefore we have to show this:

$$(a) + (b2) + (c2) = 0$$

where:

$$(a) = g \sum_{j \in GF} \Gamma_{kj}^c \Gamma_{ji}^b + \text{tr} \left([\underline{\phi}_k \underline{O}_c] [\underline{R}_{ia}] [\underline{M}_{ab}^{T^{-1}}] \right)$$

$$(b2) = g \sum_{\substack{j \in GF \\ \text{NOT NEEDED} \rightarrow}} \Gamma_{jk}^b \Gamma_{ji}^c + \text{tr} \left([\underline{\phi}_k \underline{O}_c] [\underline{R}_{ia}] [\underline{M}_{ab}^{T^{-1}}] \right)$$

$$(c2) = -g \sum_{j \in GF} \Gamma_{be}^c \Gamma_{jk}^e + \text{tr} \left([\underline{\phi}_k \underline{O}_c] [\underline{R}_{ia}] [\underline{M}_{ab}^{T^{-1}}] \right)$$

Things are really confusing without type labels, but I am going to make a guess which I think is correct. First, in the first two terms above, change from sum on j restricted to sum-on-i restricted, effect is exactly the same. Then j is a complete sum. Add these two terms to get:

$$\begin{aligned}
 (a) + (b2) &= g \sum_{i \in GF} \left\{ \Gamma_{kj}^c \Gamma_{ji}^b - \Gamma_{kj}^b \Gamma_{ji}^c \right\} + \text{tr} \left([\underline{\phi}_k \underline{O}_c] [\underline{R}_{ia}] [\underline{M}_{ab}^{T^{-1}}] \right) \\
 &= C_{cbe} \Gamma_{ki}^e \quad / \leftarrow \text{my conjecture}
 \end{aligned}$$

$$= g \sum_{j \in GF} C_{be}^c \Gamma_{kj}^e + \text{tr} \left([\underline{\phi}_k \underline{O}_c] [\underline{R}_{ia}] [\underline{M}_{ab}^{T^{-1}}] \right)$$

But now you can see that these terms exactly cancel (c2) and we are done!

What about this commutation rule? Consider:

$$[\overset{(t)}{G^c}, \overset{(t)}{G^b}] = i C^{cbe} \overset{(t)}{G^e} \quad \text{within some field type.}$$

$\times (-i)^t$:

$$\rightarrow [(-i \overset{(t)}{G^c}), (-i \overset{(t)}{G^b})] = + C^{cbe} (-i \overset{(t)}{G^e})$$

$$\Rightarrow [\overset{(t)}{\Gamma^c}, \overset{(t)}{\Gamma^b}] = C^{cbe} \overset{(t)}{\Gamma^e} \quad \text{still within a type.}$$

This is a statement within Hilbert Space H^t , but you can extend this to a direct sum statement in the supervector space. This then gives the "conjecture". Finally, lets rewrite the last line on previous page to make sure you see how it cancels:

$$(a) + (b2) = g \sum_{j \in GF} [-\Gamma_{be}^c] [-\Gamma_{jk}^e] + (\dots) \quad \checkmark$$

Comments: this was a very tough theorem to show. Clearly there must be a more concise way to state it. Nevertheless, I am convinced that I proved it correctly.

Faster Proof of Generalized Jacobi-like Identity used in Multi-field Invariance Proof.

1. My first derivation of this was awful 4 page mess. Now that I know what I want, here is faster way to get it. Desired result is this:

$$[D_{jc} \theta_c] \Gamma_{ji}^b = - \Gamma_{be}^c [\theta_c] [D_{ei}^T] + [D_{bj}^T] [\theta_c] \Gamma_{ji}^c \quad (1)$$

To verify that this is indeed correct, take each D and expand it like so:

$$[D_{jc} \theta_c] = (-g) [\Gamma_{jk}^c [\phi_k \theta_c] + [\Lambda_j^c \theta_c]]$$

$$[D_{bj}^T] = (-g) [\Gamma_{jk}^b [\phi_k] - [\Lambda_j^b]]$$

Notice in this last thing that when you transpose with capital T, you switch color labels back to regular form, and the derivative changes sign. Insert all these things to get:

$$\left. \begin{aligned} & [\phi_k \theta_c] \left(\Gamma_{jk}^c \Gamma_{ji}^b + \Gamma_{be}^c \Gamma_{ik}^e + \Gamma_{kj}^b \Gamma_{ji}^c \right) \\ & + [\Lambda_j^c \theta_c] \Gamma_{ji}^b - \Gamma_{be}^c [\theta_c] [\Lambda_i^e] + \Gamma_{ji}^c [\Lambda_j^b] [\theta_c] \end{aligned} \right\} = 0 \quad (2).$$

If you can show equation (2) is true, then you have shown that (1) is true.

Now, in fact each line of equation (2) separately vanishes. The first line does so due to a Jacobi-like property of the Gamma objects which I will prove on a separate sheet. The more interesting problem is showing that the second line vanishes. The trick here is to realize that this line is really confined to the gauge field sector. For example, the first and third terms vanish unless index j = a color index. So write this index as d. Next, you observe that each term vanishes unless index i is also a color index. So put it in as f. Then the second line reads:

$$[\Lambda_d^c \theta_c] \Gamma_{df}^b - \Gamma_{be}^c [\theta_c] [\Lambda_f^e] + \Gamma_{df}^c [\Lambda_d^b] [\theta_c] = 0 ?$$

$$\Rightarrow \delta_{cd} \Gamma_{df}^b [\partial_\mu \theta_c] - \Gamma_{be}^c \delta_{ef} [\theta_c] [\partial_\mu] + \Gamma_{df}^c \delta_{db} [\partial_\mu] [\theta_c] = 0 ?$$

$$\Rightarrow \Gamma_{cf}^b [\partial_\mu \theta_c] + \Gamma_{cf}^b [\theta_c] [\partial_\mu] + \Gamma_{bf}^c [\partial_\mu] [\theta_c] = 0 ?$$

$$\Rightarrow \Gamma_{cf}^b \left\{ [\partial_\mu \theta_c] + [\theta_c] [\partial_\mu] - [\partial_\mu] [\theta_c] \right\} = 0 \quad \checkmark$$

2. Attempt to get this identity into the Lee ZinnJustin form. Obviously the thing is true for any color vector of functions Θ , so try to abstract them away. If I functionally differentiate each term with respect to parameter $\Theta_c(t)$, I get:

$$D_{jc}(x,t) \delta(x-y) \Gamma_{ji}^b \\ = - \Gamma_{be}^c \delta(x-t) D_{ei}^T(t,y) + D_{bj}^T(x,t) \delta(t-y) \Gamma_{ji}^c$$

If you multiply back by $\Theta(t)$ and integrate over t , you duplicate starting point.

Now take the above and just integrate over t . You get:

$$\delta(x-y) \Gamma_{ji}^b \underbrace{\int dt D_{jc}(x,t)}_{(-g) \Gamma_{jk}^c \phi_k(x)} = - \Gamma_{be}^c D_{ei}^T(x,y) + \Gamma_{ji}^c D_{bj}^T(x,y)$$

Rewrite:

$$\Gamma_{ji}^c D_{jb}(y,x) - \Gamma_{ji}^b \delta(x-y) \int dt D_{jc}(x,t) \\ = \Gamma_{be}^c D_{ie}(y,x)$$

Write out more detail [cancel of $-g$]:

$$\Gamma_{ji}^c [\Gamma_{jk}^b \phi_k(y) + \vec{\Lambda}_j^b(y)] \delta(y-x) - \Gamma_{ji}^b \delta(y-x) \int dt [\Gamma_{jk}^c \phi_k(x) + \vec{\Lambda}_j^c(x)] \delta(x-t) \\ = \Gamma_{be}^c [\Gamma_{ik}^e \phi_k(y) + \vec{\Lambda}_i^e(y)] \delta(y-x)$$

$$\Rightarrow \begin{array}{c} \delta \quad \delta \\ \textcircled{i} \quad \textcircled{j} \\ \textcircled{k} \quad \textcircled{i} \end{array} [\Gamma_{jk}^b \phi_k(y) + \vec{\Lambda}_j^b(y)] \delta(y-x) - \begin{array}{c} \delta \quad \delta \\ \textcircled{i} \quad \textcircled{j} \\ \textcircled{k} \quad \textcircled{i} \end{array} [\Gamma_{jk}^c \phi_k(x) + \vec{\Lambda}_j^c(x)] \delta(x-t) \\ = \begin{array}{c} \delta \quad \delta \\ \textcircled{i} \quad \textcircled{j} \\ \textcircled{k} \quad \textcircled{i} \end{array} [\Gamma_{ik}^e \phi_k(y) + \vec{\Lambda}_i^e(y)] \delta(y-x)$$

$$\Rightarrow \begin{array}{c} \delta \quad \delta \\ \textcircled{i} \quad \textcircled{j} \\ \textcircled{k} \quad \textcircled{i} \end{array} [\Gamma_{jk}^b \phi_k(y) + \vec{\Lambda}_j^b(y)] \delta(y-x) - \begin{array}{c} \delta \quad \delta \\ \textcircled{i} \quad \textcircled{j} \\ \textcircled{k} \quad \textcircled{i} \end{array} [\Gamma_{jk}^c \phi_k(x) + \vec{\Lambda}_j^c(x)] \delta(x-t) \\ = C_{\delta\epsilon}^{\gamma} [\Gamma_{kj}^{\gamma} \phi_j(y) + \vec{\Lambda}_k^{\gamma}(y)] \delta(y-x)$$

Now remove the deltas and make this an operator statement:

$$\Gamma_{ki}^{\gamma} \left[\Gamma_{ij}^{\delta} \phi_j + \vec{\Lambda}_i^{\delta} \right] - \Gamma_{ki}^{\delta} \left[\Gamma_{ij}^{\gamma} \phi_j + 0 \right] = C_{\gamma\delta\alpha} \left[\Gamma_{kj}^{\gamma} \phi_j + \vec{\Lambda}_k^{\alpha} \right]$$

This is the corrected form of Lee ZinnJustine equation A11 in their appendix. The whole validity hinges on a piece being missing as in the above. Their result is simply wrong!

Derivation of the Gamma Identity.

1. In the multi-field setup, the field vector ϕ_i is actually a direct sum of field vectors in various representations. We have labelled these representations by types. Thus, for example:

Now, each field type means a set of fields that belong to some representation of

$${}^{(t)}\phi_i = \begin{pmatrix} \phi \\ \vdots \\ \phi \end{pmatrix} \leftarrow \text{only non zero.}$$

the gauge group. Thus, we have, within a given representation, the Lie algebra:

$$\left[\Gamma_{bc}^{(t)}, \Gamma_{de}^{(t)} \right] = \Gamma_{bce}^{(t)} \Gamma_d^e \quad (\Gamma_{bce} = -C_{bce})$$

When you first write this, you only intend that the indices on the matrices go over their appropriate values. But, you can "extend" the matrix $\Gamma_{bc}^{(t)}$ to the full superspace (direct sum of Hilbert Spaces) by simply making it zero everywhere outside the correct region. Thus, for example:

$$\underline{\Gamma_{bc}^{(t)}} = \begin{bmatrix} \phi & & \phi \\ & \boxed{\text{diagonal}} & \\ \phi & & \phi \end{bmatrix}$$

This thing will be a square matrix somewhere on the diagonal. Obviously with things set up this way, the matrices going with different field multiplets commute.

Sometimes we just say: matrices in different Hilbert spaces commute, but really you have to look at the direct sum algebra to see how that comes out. Here we are explicitly implementing the direct sum algebra by extending all matrices to full size and putting in zeros.

Now we can define the "full gamma matrix" as the direct sum of the basic gamma matrices. If you look back on top of page 4 you see that that is precisely how we defined the full gammas: as direct sum of t-gammas. Thus, the full gamma matrix is in block form. It is then easy to verify our desired identity. Of course it follows at once from a theorem I already know: if operators satisfy a Lie Algebra, then direct sum operators also solve it, QED. But here is a proof anyway:

$$[\Gamma_{bc}^b, \Gamma_{de}^d] = \Gamma_{bce}^b \Gamma_d^e ? \Leftrightarrow \sum_t \sum_{t'} [\Gamma_{bc}^{b,t}, \Gamma_{de}^{d,t'}] = \Gamma_{bce}^b \sum_t \Gamma_d^{e,t} ?$$

$$\sum_t [\Gamma_{bc}^{b,t}, \Gamma_{de}^{d,t}] = \Gamma_{bce}^b \sum_t \Gamma_d^{e,t} \quad \checkmark \text{ QED.}$$

So:

$$\boxed{\Gamma_{ij}^b \Gamma_{jk}^c - \Gamma_{ij}^c \Gamma_{jk}^b = \Gamma_{ce}^b \Gamma_{ik}^e}$$

Old Notes

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Various Levels of a practical QM calculation.

1. Here is the problem: you have a non-rel QM problem involving a particle in a harmonic oscillator well. How do you analyze the problem? Compare the analysis with same particle in a Coulomb $1/r$ well. Think in terms of SE, propagators, field theory, any method you have ever heard of.
2. First go to non-rel SE QM. You have a potential which is known. You want to watch a packet bounce around in the potential. I.e., you start off with a wavefunction at time t , and any later time you want to know what the wavefunction has become. The most economical way to think of this is to construct the Greens function $G(x,t;x',t')$ by solving the differential equation. This is the "propagator for harmonic oscillator potential". The answer is given in Abers and Lee if you set $J=0$. If you then go on to set frequency $\omega = 0$, then you will get the famous free-particle propagator. Once you know the propagator, you are done. You could even get the thing by doing a path integral.
3. Now let's go to rel QM. Now you tend to leave the single particle sector because you can pair-produce and so on. Shift to the relativistic Hamiltonian. You could go on to compute the relativistic propagator, but I imagine it is rather messy. Has pieces which go both ways in time because negative mass states are now possible. This is not the free-particle scalar propagator because there is a HO potential on all the time, and not a weak one either. So, again, knowing the relativistic propagator and keeping energy low, you would get a good answer to your question of what happens with that little packet. Note that the Rel treatment of the hydrogen atom is like this, i.e., rel Ham plus the potential. Of course the concept of potential is only meaningful if energy is low.
4. Now move to second-quantized field theory. The rel Ham becomes the Klein-Gordon equation, except you have a potential $V(x)$, not $V(\phi)$, sitting in your lagrangian density. Problems are never done like this! However, the $1/r$ potential can be simulated by going to an interacting field theory and adding a source term, eg, having $\phi^3 + J\phi$. BUT, I know of no way to simulate a $\frac{1}{2}kr^2$ potential by allowing some kind of exchange. Is gluon exchange supposed to cause a kr potential in QCD? I know that the photon propagator $1/k^2$ causes the $1/r$ potential, but not at all clear how you could make some other potential.
5. So the point is this: you cannot do a harmonic oscillator potential problem in second-quantized field theory. (Do not confuse the HO expansion of a free field with the problem at hand!)

Gell-Mann/ Levy 1960

1. Introduction. For muon decay, answer works out right, Berman computed radiative corrections. For beta decay, however, you have C_V and C_A to deal with. CVC (1958) tells us that $C_V=1$. This leaves $G(\text{beta}) = .97 G(\text{muon})$ which GM and Levy find troubling. Later Cabibbo will explain this in 1963.

What about the axial hadronic weak current? They know that $C_A = 1.25$ and that this current cannot be conserved because pion $\mu \rightarrow e$ decay is observed and involves this current's divergence.

They would like to be able to say this: $\partial_\mu A^\mu = \frac{ia}{\sqrt{2}} \pi$ where a is pion decay constant. This is Eq (5).

This formula is significant because it yields the Goldberger-Treiman relation which is amazingly accurate. So authors are interested in theories which give rise to this Eq. (5).

2. Charged Pion Decay. By taking the $(0/\dots/\pi)$ and $(n/\dots/p)$ matrix elements of Eq 5, authors derive the GT relation as shown in (11). Their result differs from original GTR only in form factors; they make a big deal about these extra factors. Claim: any theory which allows pion decay will have such a GT relation with some form factor; the only question is how fast the form factor varies.

3. Noether theorem. We are reminded that if you have a lagrangian and some specified infinitesimal gauge transformation on all the fields, you get the Euler like equation

$$\frac{\partial \mathcal{L}}{\partial x} = \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial (\partial_\mu x)} \right]$$

Thus, if your lagrangian is invariant under some gauge transformation, then the thing in the square bracket will be the related conserved current. Various examples are given. Eg, in the "conventional PI-N model" the three conserved currents form the isotopic spin current.

The problem is finding a lagrangian and a gauge transformation so that you have a simple axial weak hadronic current. The vector current is just the isospin current by CVC. If you take the standard PI-N lagrangian and try some obvious axial gauge transformations, you always get unpleasant axial currents; nothing with divergence like Eq. 5 that you want.

4. Gradient model. So now let's examine some models which do yield equation 5, and which therefore imply the GT relation via this path. Gradient coupling model is obvious modification of standard (Bjorken-Drell standard) model. In this little model, the current divergence is given by the pion mass term and you do get Eq. 5.

Weaknesses of this model are: (1) the axial and vector gauge transformations seem unrelated (though strength same to 1.25, seems wrong; (2) such a theory is nonrenormalizable. I think this arises from gradient bring down k factors.

5. The σ model. This is the section that makes this paper famous. This model is combination of two earlier models, mainly Schwinger. The unshifted lagrangian is shown in (38) in terms of field ψ . Basically this is an $O(4)$ or $SU_2 \times SU_2$ chiral model, though such fancy language is not used here. Last term breaks symmetry and also generates your desired equation (5). This lagrangian shows no nucleon mass term, but the shifted lagrangian (36) does. They say nothing about why ~~the sigma field is shifted~~ the sigma field is shifted, but I know that it is to minimize the effective potential. Some axial gauge transformation is stated. The precise axial current is found. Eq. 5 is happy.

Strengths of this model are: it is renormalizable (no data given) and both currents vector and axial come in in the same way (over)

A drawback of the model is that you have to explain this new field ϕ , some scalar particle which could decay into two pions. Tadpole diagrams exist in this theory (has to do with superconducting vacuum of course. The fact that a nucleon mass is generated is noted as "curious".

6. Nonlinear model. Third and last model they consider. This is just a renovation of the sigma model in which the ϕ is not a new particle but is related to the pion somehow. You still get Eq. 5, but not renormalizable.

7. Operators. A vague effort to write generators as Lie algebra. Still no mention of chiral and $SU_2 \times SU_2$. I skip.

8. Conclusions. Seeking models which give $DJ = PI$ and thence GT relation. Three models presented, each has its problems. Want to extend to $dS=1$ part of hadronic current, want to learn how to renormalize their third version. Whole paper is based on that Eq. 5; making explicit model in which it is true.

AFFR on the Sigma Model.

1. What references do I have on this subject? First, papers and text in the Current Algebra book. Includes the original GellMann Levy paper. Second is this AFFR section in their chapter 5. Third there is stuff in Abers and Lee. My other current algebra refs don't seem to have much. The Bernstein gauge review has a little.

2. What is the significance of this model, what are the main points? First I will review what AFFR say and then I will answer that question.

3. Group Theory. You know that you are talking about the group $SU(2) \times SU(2)$ because this is the usual thing. I.e., this is isospin $\times$ axial isospin if you like and will be connected with vector and axial currents which of course are the hadronic weak interaction currents, and are also present in beta as well as pion decay.

This group is the same as $O(4)$, which is much like $SO(3,1)$, probably a few minus signs in the Lie algebra. Thus you can think of three rotation-like generators (the regular isospins, charges called Q), and there will also be three "boost" generators, these will be the axial charges $\bar{Q}$.

Since this group is so much like Lorentz group, you see that statement (5.4) really defines the transformation rule for a vector, i.e., for a $(\frac{1}{2}, \frac{1}{2})$ rep. The specific comrels for any vector are then given in (5.5) in terms of the vector and axial charges.

Now, the idea is to make the pion triplet a part of an $O(4)$ or "chiral" 4-vector. Basically, this lagrangian or model was designed to do PI-N physics; you have to put the pion into some multiplet, and the 4-vector is easiest place to put it. Notice that if you restrict to the isospin subgroup, the pion goes in the right way!

So clearly (5.6) gives you a chiral invariant lagrangian.

Where do you want to put the nucleons? In order to generate the desired vector and ~~weak~~ axial currents, you had better put into L and R doublets as shown. Since $\bar{\psi}\psi$ (albeit an isoscalar) is 4th component of a chiral 4-vector, you cannot have nucleon mass in lagrangian. The chiral invariant coupling to nucleons is shown in (5.14), includes usual PI-N-N thing as a part.

4. Currents. To this point we have constructed a completely chiral invariant lagrangian, including PI-N interaction. Recalling that the Noether transformation is always determined by commutator of charges with relevant fields, you can go ahead and use Noether to get your conserved currents. The vector current is the usual isospin current and $\bar{\psi}\psi$ does not appear at all. That is because $\bar{\psi}\psi$ has $I=0$ so has no isocurrent. The nucleon and pion contributions to this isocurrent are what I remember.

The conserved axial current has a familiar looking (by design) nucleon contribution, but the pion-sigma piece is new to me. Note that since current axial is conserved, pion decay cannot go in this theory. Note also that $C_A = 1$ in this theory. Also, pion and sigma have same mass as required by chiral invar.

5. As noted by Gellmann and Levy, you can convert this CAC theory to a PCAC theory simply by adding a term $a\bar{\psi}\psi$ to the lagrangian. Recall that $\bar{\psi}\psi$ is 4th component of chiral vector, not a scalar, hence you have broken chiral invar. This breaking term then causes CVC same as before, but PCAC in that $\partial A = -a\pi$, the famous equation (5) of GM-L paper. This PCAC equation of course leads to GT relation.

6. Tadpoles. We know from general ideas that going from CAC to PCAC causes allowed pion decay. How do we see this directly in this model. Answer: by adding a linear ϕ term to lagrangian, you allow possibility of ϕ decaying directly into vacuum. Thus, graphs which were not allowed before can be made allowed by adding tadpoles. Similar to idea of Coulomb potential allowing pair production etc. Example: there is no $\pi\pi\pi$ term in lagrangian (5.55) so ϕ cannot decay into two pions. But there is a $\pi\pi\phi$ term, so once you add the tadpole term, you allow graph like shown in Fig. 5.1 (b), and sigma can now decay.

So the tadpole graphs is a mechanical way to understand what the linear term does. Obviously such particles going into vacuum indicate some trouble with theory. Tadpoles are eliminated by shifting the sigma field so that potential is minimum. Then $(0/\phi'/0) = 0$ and no more tadpoles.

When you do such a shift and rewrite lagrangian, the earlier near-total chiral symmetry is now completely non-evident. Pion is now massless, ϕ has some mass, nucleons have acquired some mass. This is the amazing aspect of SSB. "Secret Symmetry" as Sidney says.

7. It is noted that you can add extra invariant terms to lagrangian to get C_A to be different from 1. This aspect is model dependent.

8. Other bonuses of this model: you get all regular results of current algebra, since this model is an explicit realization of current algebra. Eg, you get the $\pi\pi$ - $\pi\pi$ scattering threshold condition, and the $\pi\pi$ -N threshold condition also. I guess no-one doubts the validity of these conditions.

9. Now take an overview and see what this model does. :

The Sigma Model has these favorable aspects:

- a) realization of current algebra, hence soft pion theorems etc.
- b) is renormalizable since $d=4$ in unshifted lagrangian.
- c) gives the PCAC-Equation when trivial linear ϕ -term is added.
- d) gives vector and axial currents which look right for weak interaction
- e) that linear ϕ term requires field-shift to eliminate tadpoles.

Thus, this model is example of a field theory with SSB. The SSB has these fringe benefits:

1. Allows nucleons to acquire mass.
 2. Allows π and ϕ to have different masses
- f) in fact, the linear term breaks 3 generators, hence you get 3 Goldstone Bosons, the massless pions. Example of Gold. Theorem.

10. What does this model not show? It is not a gauge theory because they never try to insist on local chiral $O(4)$ invariance and thereby introduce gauge fields. There are no vector fields at all in this model for strong interactions. QCD on the other hand is a gauge theory model.

The recent interest is the combination of a SSB theory like the Sigma Model with a Gauge Theory like the Yang-Mills thing. Together, such a theory allows the undesired Goldstones to be adsorbed onto the massless gauge particles to get massive vector mesons. Higgs.

Gauge Invariance in terms of Finite Transformations.

- ① Consider $\mathcal{L}(\phi, \partial_\mu \phi)$. Global gauge invariance says that

$$\mathcal{L}(\phi', \partial_\mu \phi') = \mathcal{L}(\phi, \partial_\mu \phi) \quad \text{where } \phi' = U\phi \quad U \neq U(x).$$

- ② We can rewrite global invariance in a form that is still valid even if $U = U(x)$, i.e.,

$$\mathcal{L}(\phi', [\partial_\mu \phi]') = \mathcal{L}(\phi, \partial_\mu \phi)$$

or better yet

$$\mathcal{L}(U\phi, U[\partial_\mu \phi]) = \mathcal{L}(\phi, \partial_\mu \phi)$$

global invariance,
even when $U = U(x)$.
but $U = \text{unitary}$.

- ③ From the global invariance we therefore know that:

$$\mathcal{L}(U[\textcircled{1}], U[\textcircled{2}]) = \mathcal{L}([\textcircled{1}], [\textcircled{2}]) \quad (1)$$

where entries ① and ② are variables, anything you want.

- ④ Local invariance says this:

(2)

$$\mathcal{L}(\phi', (\partial_\mu - ig \vec{G} \cdot \vec{A}_\mu') \phi') = \mathcal{L}(\phi, [\partial_\mu - ig \vec{G} \cdot \vec{A}_\mu] \phi)$$

- ⑤ I.e., rotate all the fields + allow $U = U(x)$, no change, provided A_μ transforms in correct way.

- ⑤ Now apply global invariance to the RHS of (2) so that (2) now reads:

$$\mathcal{L}(\psi\phi, [\partial_\mu - ig \vec{G} \cdot \vec{A}'_\mu] \psi\phi) = \mathcal{L}(\psi\phi, \psi [\partial_\mu - ig \vec{G} \cdot \vec{A}_\mu] \phi) \quad (3)$$

- ⑥ By equating the second-entries in (3), we learn how $\vec{A}_\mu$ must transform:

$$\begin{aligned} (\partial_\mu \psi)\phi + \cancel{\psi(\partial_\mu \phi)} - ig \vec{G} \cdot \vec{A}'_\mu (\psi\phi) \\ = \cancel{\psi(\partial_\mu \phi)} - ig \psi \vec{G} \cdot \vec{A}_\mu (\psi\phi) \end{aligned}$$

$$(\partial_\mu \psi)\psi^\dagger - ig \vec{G} \cdot \vec{A}'_\mu = -ig (\psi \vec{G} \psi^\dagger) \cdot \vec{A}_\mu$$

Thus:

$$\boxed{\vec{G} \cdot \vec{A}'_\mu = (\psi \vec{G} \psi^\dagger) \cdot \vec{A}_\mu + \frac{1}{ig} (\partial_\mu \psi)\psi^\dagger} \quad (4)$$

- ⑦ Note: ϕ is a vector of, say, m dimensions. $\vec{G}$ are the various $m \times m$ generator matrices. ψ is an $m \times m$ finite trans. matrix. $\vec{A}$ indicates a vector with n components where $n = \# \text{ generators} = \text{order of Lie group}$, unrelated to m .

- ⑧ Line (2) which states local invariance can be restated as

$$\mathcal{L}(\phi, D_\mu \phi) = \mathcal{L}(\phi', D'_\mu \phi') \quad \text{where} \quad D_\mu \phi = [\partial_\mu - ig \vec{G} \cdot \vec{A}_\mu] \phi \quad (5)$$

$$\text{OR, } \mathcal{L}(\phi, \phi_{;\mu}) = \mathcal{L}(\phi', \phi'_{;\mu}) \quad \text{where} \quad \phi'_{;\mu} = \phi'_{;\mu} - ig \vec{G} \cdot \vec{A}'_\mu \phi \quad (6)$$

↑
analogous to gen rel.

Take limit of general result to get inf-mal change in ϕ and A .

① Sticking by my guns I would say $U(\Theta) = e^{-i\vec{\Theta}(x) \cdot \vec{G}}$ where $\vec{G}$ = gen. matrices. Thus, consider

$$\phi'_i(x) = [e^{-i\vec{\Theta}(x) \cdot \vec{G}}]_{ij} \phi_j(x) \quad \text{or} \quad \phi' \approx (1 - i\vec{\Theta} \cdot \vec{G}) \phi$$

$$\text{So } \phi' = \phi - i\vec{\Theta} \cdot \vec{G} \phi \quad \text{or} \quad \delta\phi = -i\vec{\Theta} \cdot \vec{G} \phi.$$

In more detail this is:

$$\boxed{\delta\phi_i(x) = -i \Theta_a(x) [G_a]_{ij} \phi_j(x)}$$

where $i=1,2,\dots,m$ = size of rep; $a=1,2,\dots,n$ = order of group.

② Look at (4) on previous sheet. Note that the μ indices seem not to get mixed up! First, consider

$$[G_a, G_b] = f_{abc} G_c$$

$$\begin{aligned} U G_a U^{-1} &= e^{-i\vec{\Theta} \cdot \vec{G}} G_a e^{+i\vec{\Theta} \cdot \vec{G}} = (1 - i\Theta_b G_b) G_a (1 + i\Theta_b G_b) \\ &= G_a - i\Theta_b [G_b, G_a] = G_a - i\Theta_b f_{bac} G_c \\ &= G_a + i f_{abc} \Theta_b G_c \quad \text{/get nice ordering.} \end{aligned}$$

$$\text{Also, } (\partial_\mu U) U^{-1} \approx \partial_\mu [1 - i\vec{\Theta} \cdot \vec{G}] \cdot 1 = -i(\partial_\mu \vec{\Theta}) \cdot \vec{G} = -i(\partial_\mu \Theta_a) G_a.$$

Now line (4) reads

$$\begin{aligned} \not{D}_a (A'_\mu)^a &= [G_a + i f_{abc} \Theta_b G_c] A_\mu^a + \frac{1}{ig} [-i(\partial_\mu \Theta_a) G_a] \\ &= \not{D}_a A_\mu^a + i f_{cba} \Theta_b \not{D}_a A_\mu^c - \frac{1}{g} (\partial_\mu \Theta_a) \not{D}_a \end{aligned}$$

$$\Rightarrow (A'_\mu)_a = (A_\mu)_a + i f_{cba} A_\mu^c \Theta_b - \frac{1}{g} (\partial_\mu \Theta_a)$$

Aside: all my non-Lorentz indices are "down"; I put them that way at the start.

$$(A_a^\mu)' = A_a^\mu + i f_{cba} A_c^\mu \Theta_b - \frac{1}{g} (\partial^\mu \Theta_a)$$

③ What are the conserved currents? Recall

$$\frac{\partial \mathcal{L}}{\partial \Theta_a} = \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial (\partial_\mu \Theta_a)} \right] = 0 \quad \text{so}$$

$$J_a^\mu = \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_i)} (G_a)_{ij} \phi_j$$

④ Show that covariant deriv. is covariant:

$$\delta \phi = -i \vec{\Theta} \cdot \vec{G} \phi \quad / \text{mxm space indices implicit}$$

$$\delta(\partial_\mu \phi) = \partial_\mu \delta \phi = -i \vec{\Theta} \cdot \vec{G} (\partial_\mu \phi) - i (\partial_\mu \vec{\Theta}) \cdot \vec{G} \phi$$

$$\text{Define:} \quad i f_{cba} A_c^\mu \Theta_b \equiv -[\vec{A}^\mu \times \vec{\Theta}]_a = +(\vec{\Theta} \times \vec{A}^\mu)_a$$

$$\text{Then:} \quad \delta \vec{A}^\mu = +\vec{\Theta} \times \vec{A}^\mu - \frac{1}{g} \partial^\mu \vec{\Theta}$$

↑
reason: for $SU(2)$

$$f_{cba} = i \epsilon_{cba} = i \epsilon_{acb}$$

$$\text{So} \quad i f_{cba} A_c^\mu \Theta_b = -\epsilon_{acb} A_c^\mu \Theta_b = -\vec{A}^\mu \times \vec{\Theta}$$

Cannot do this yet.

Generalized Pauli Identity

① Recall that, for Pauli matrices,

$$(\vec{\sigma} \cdot \vec{a})(\vec{\sigma} \cdot \vec{b}) - (\vec{\sigma} \cdot \vec{b})(\vec{\sigma} \cdot \vec{a}) = 2i \vec{\sigma} \cdot (\vec{a} \times \vec{b})$$

② Want to generalize this to any Lie group. Consider

$$\underline{[G_a, G_b] = f_{ab}^c G_c} \quad \text{as Lie algebra.}$$

$$\text{Then: } A_a^a [G_a, G_b] B_b^b = G_c f_{ab}^c A^a B^b$$

$$\underline{\text{Define: } [\vec{A} \times \vec{B}]^c \equiv -i f_{ab}^c A^a B^b}$$

Thus:

$$(\vec{A} \cdot \vec{G})(\vec{B} \cdot \vec{G}) - (\vec{B} \cdot \vec{G})(\vec{A} \cdot \vec{G}) = +i \vec{G} \cdot \vec{A} \times \vec{B}$$

③ Comments:

a) note that nothing is known about symmetry of f_{ab}^c except that $f_{ab}^c = -f_{ba}^c$.

b) for SU_2 , $-i f_{ab}^c = -i(i \epsilon_{abc}) = \epsilon_{abc} = \epsilon_{cab}$, so $(\vec{A} \times \vec{B})^c = \epsilon^{cab} A^a B^b$ as usual.

c) note that $\vec{A} \cdot (\vec{B} \times \vec{G}) \neq (\vec{A} \times \vec{B}) \cdot \vec{G}$ etc because these require extra symmetry!

Show that covariant derivative is covariant.

① Here is what we do know:

$$\delta\phi = -i\vec{\theta} \cdot \vec{G}\phi$$

$$\delta(\partial_\mu\phi) = -i\vec{\theta} \cdot \vec{G}(\partial_\mu\phi) - i(\partial_\mu\vec{\theta}) \cdot \vec{G}\phi$$

$$\delta\vec{A}_\mu = \vec{\theta} \times \vec{A}_\mu - \frac{1}{g}\partial_\mu\vec{\theta}$$

We wish to show that

$$\delta(D_\mu\phi) = -i\vec{\theta} \cdot \vec{G}(D_\mu\phi).$$

$$D_\mu\phi = (\partial_\mu - ig\vec{G} \cdot \vec{A}_\mu)\phi$$

$$\delta(D_\mu\phi) = \delta(\partial_\mu\phi) - ig\vec{G} \cdot [(\delta\vec{A}_\mu)\phi + \vec{A}_\mu(\delta\phi)]$$

$$= -i\vec{\theta} \cdot \vec{G}(\partial_\mu\phi) - i(\partial_\mu\vec{\theta}) \cdot \vec{G}\phi - ig\vec{G} \cdot [\vec{\theta} \times \vec{A}_\mu\phi - \frac{1}{g}\partial_\mu\vec{\theta}\phi] - ig\vec{G} \cdot \vec{A}_\mu [-i\vec{\theta} \cdot \vec{G}\phi]$$

$$= -i\vec{\theta} \cdot \vec{G}(\partial_\mu\phi) - g \underbrace{[i\vec{G} \cdot (\vec{\theta} \times \vec{A}_\mu)]\phi}_{\hookrightarrow [(\vec{\theta} \cdot \vec{G})(\vec{A}_\mu \cdot \vec{G}) - (\vec{A}_\mu \cdot \vec{G})(\vec{\theta} \cdot \vec{G})]} - g(\vec{G} \cdot \vec{A}_\mu)(\vec{G} \cdot \vec{\theta})\phi$$

$$= -i\vec{\theta} \cdot \vec{G}(\partial_\mu\phi) - g(\vec{\theta} \cdot \vec{G})(\vec{G} \cdot \vec{A}_\mu)\phi$$

$$= -i\vec{\theta} \cdot \vec{G} [\partial_\mu\phi - ig\vec{G} \cdot \vec{A}_\mu\phi] = -i\vec{\theta} \cdot \vec{G}(D_\mu\phi). \quad \boxed{\text{QED}}$$

2. Comments: this shows that, under local gauge transformations, the group vector $D_\mu\phi$ transforms exactly the same way as the group vector ϕ . Both are vectors in the same representation.

3. Notice that in the above stuff, you always keep the G and ϕ things in right order because their indices are suppressed.

3. Comments about local gauge stuff.

We start with a lagrangian $L(\phi, d_u \phi)$ and we presume it is globally invariant. This means that $L(U\phi, U(d_u \phi)) = L(\phi, d_u \phi)$. Ie, the objects ϕ and $d_u \phi$ both transform as group vectors in the same way under global gauge transformations. This means that L must be made out of dot products of these vectors so that L will be a global scalar.

If you consider instead local transformations, you will find that although ϕ is still a group vector, $(d_u \phi)$ is no longer a vector; for U element of local gauge group, the above invariance of L is no longer true. But then you repair it by saying: $L(u\phi, U(D_u \phi)) = L(\phi, D_u \phi)$ because now $D_u \phi$ is a group vector.

Rephrasing of The Finite Gauge Page.

- ① Suppose you want to construct a scalar from quantities $\vec{A}, \vec{B}$.
Clearly, you do dot products etc.

$$s = (\vec{A} \cdot \vec{B})^2 \vec{A} \cdot \vec{B} + (\vec{A} \cdot \vec{B})^3 \dots$$

Rotational invariance means $s(A, B) = s(A', B') = s(UA, UB)$.

- ② Now think of ϕ and $\partial_\mu \phi$ as (global) vectors of gauge group.
Then global gauge invariance says

$$\mathcal{L}(\phi, \partial_\mu \phi) = \mathcal{L}(U\phi, U(\partial_\mu \phi))$$

So globally, ϕ and $(\partial_\mu \phi)$ are both vectors of the global gauge group and $\mathcal{L}$ is made out of their dot products.

- ③ When we say ϕ is a "group vector", we mean:

$$\phi' = e^{-i\vec{\theta} \cdot \vec{G}} \phi \quad \text{or} \quad \delta\phi = -i\vec{\theta} \cdot \vec{G} \phi$$

For the global gauge group, we may prove that $(\partial_\mu \phi)$ is also a vector in this way:

$$U[\partial_\mu \phi] = \partial_\mu[U\phi] = \partial_\mu \phi' = [\partial_\mu \phi]'$$

$$\text{or } \underset{\uparrow}{\delta}[\underset{\uparrow}{\partial_\mu \phi}] = \underset{\uparrow}{\partial_\mu}[\underset{\uparrow}{\delta\phi}] = \underset{\uparrow}{\partial_\mu}[-i\vec{\theta} \cdot \vec{G} \underset{\uparrow}{\phi}] = -i\vec{\theta} \cdot \vec{G} (\underset{\uparrow}{\partial_\mu \phi})$$

- ④ Now consider a local gauge group. ϕ is still a group vector of G_L , but $\partial_\mu \phi$ is not!

$$\delta(\partial_\mu \phi) = \partial_\mu(\delta\phi) = \partial_\mu [-i\vec{\theta} \cdot \vec{G}\phi] = -i\vec{\theta} \cdot \vec{G}(\partial_\mu \phi) - \underbrace{i(\partial_\mu \vec{\theta}) \cdot \vec{G}\phi}_{\text{extra piece}}$$

- ⑤ We repair this by making a covariant deriv:

$$D_\mu \phi = \partial_\mu \phi - ig \vec{G} \cdot \vec{A}_\mu \phi$$

where $\delta \vec{A}_\mu = \vec{\theta} \times \vec{A}_\mu - \frac{1}{g} \partial_\mu \vec{\theta}$. As shown on sheet before, D_μ is covariant. Thus, local gauge invariance is

$$\mathcal{L}(U\phi, U(D_\mu \phi)) = \mathcal{L}(\phi, D_\mu \phi) \quad U \in G_{\text{LOCAL}}$$

- ⑥ Notice that $\vec{A}_\mu$ is not a true vector of the adjoint rep because it has extra piece.

- ⑦ To get transformation rule for $\vec{A}_\mu$, we need only require that $D_\mu \phi$ be a true vector:

$$U[D_\mu \phi] = D'_\mu \phi'$$

$$\text{i.e., } U[\partial_\mu \phi - ig \vec{G} \cdot \vec{A}_\mu \phi] = \partial_\mu \phi' - ig \vec{G} \cdot \vec{A}'_\mu \phi'$$

$$\text{a } U(\partial_\mu \phi) - U\vec{G} \cdot \vec{A}_\mu \phi ig = \partial_\mu(\phi') - ig \vec{G} \cdot \vec{A}'_\mu \phi'$$

$$\Rightarrow \text{after a few steps } [\vec{G} \cdot \vec{A}'_\mu = U\vec{G}U^\dagger \cdot \vec{A}_\mu + \frac{1}{ig}(\partial_\mu U)U^\dagger]$$

The finite trans rule for $\vec{A}$ comes from requiring that $D_\mu \phi$ = vector.

Finite transf. rule for $\vec{A}_\mu$:

① So far we know that $\vec{G} \cdot \vec{A}'_\mu = (V \vec{G} V^{-1}) \cdot \vec{A}_\mu + \frac{1}{ig} (\partial_\mu V) V^{-1}$.

② Suppose our Lie group is "special" so set $V=1$ and therefore trace $\vec{G}=0$. Suppose also that the generators have this property

$$\hbar [G_i G_j] = N \delta_{ij} \quad \hbar [G_i] = 0$$

[I don't know under what conditions this is true. N can depend on rep. chosen. Certainly consistent with Lie algebra and traceless G^b . This fact is true for Pauli σ_i as well as λ_i of SU_3 .

③ With the above assumptions, we can "solve" for $\vec{A}'_\mu$:

$$(A'_\mu)^b = \underbrace{\frac{1}{N} \hbar [G^b V G_a V^{-1}]}_{\text{rotation matrix}} A_\mu^a + \frac{1}{ig} \frac{1}{N} \hbar [G_b (\partial_\mu V) V^{-1}].$$

Obviously, this is the rotation matrix for the adjoint representation of this chosen (unitary) group.

$$R^b_a = \frac{1}{N} \hbar [G^b V G_a V^{-1}] = e^{-i \vec{\theta} \cdot (\vec{G})^b_a}$$

So we get:

$$\left[(A'_\mu)^b = R^b_a A_\mu^a + \underbrace{\frac{1}{ig} \frac{1}{N} \hbar [G_b (\partial_\mu V) V^{-1}]}_{\text{extra piece which}} \right]$$

$\Rightarrow A_\mu^b$ not a true group vector of local group.

Generalized Jacobi Cross Product Theorem

① For regular x , claim that $\vec{A} \times (\vec{B} \times \vec{C}) + \text{cyclic} = 0$.

Proof: $(\vec{A} \cdot \vec{C})\vec{B} - (\vec{A} \cdot \vec{B})\vec{C} + (\vec{B} \cdot \vec{A})\vec{C} - (\vec{B} \cdot \vec{C})\vec{A} + (\vec{C} \cdot \vec{B})\vec{A} - (\vec{C} \cdot \vec{A})\vec{B} = 0$ ✓

② Now show for a general Lie group. The structure constants do this:

$$f_{ab}^m f_{mc}^l + \text{cyclic}_{abc} = 0 \quad / \text{Hijima (1.3)}$$

So tack on some vectors:

$$f_{ab}^m f_{mc}^l A^a B^b C^c + f_{bc}^m f_{ma}^l A^a B^b C^c + f_{ca}^m f_{mb}^l A^a B^b C^c = 0$$

Rename labels:

$$f_{ab}^m f_{mc}^l A^a B^b C^c + \text{cyclic}(A, B, C) = 0.$$

③ But, $[\vec{A} \times (\vec{B} \times \vec{C})]^l = -i f_{mc}^l A^m [\vec{B} \times \vec{C}]^c$

$$= -i f_{mc}^l A^m [-i f_{ab}^c B^a C^b] = -f_{mc}^l f_{ab}^c A^m B^a C^b$$

$$\text{So } [\vec{A} \times (\vec{B} \times \vec{C})]^l = + f_{ab}^m f_{mc}^l A^a B^b C^c = f_{ab}^m f_{mc}^l B^a C^b A^c$$

↑ mod cyclic perm.

④ Conclude:

$$[(\vec{A} \times \vec{B}) \times \vec{C}]^l = f_{ab}^m f_{mc}^l A^a B^b C^c$$

$$(\vec{A} \times \vec{B}) \times \vec{C} + \text{cyclic} = 0$$

Explicit Generators for Adjoint Rep of a Lie Group

① Consider group of order n , so n generators:

$$[G_i, G_j] = f_{ij}^k G_k$$

② Claim: The $n \times n$ Lie algebra is realized by:

$$(G_i)^t_m = f_{im}^t$$

Verify: $G_i G_j - G_j G_i = f_{ij}^k G_k$ says L.A.

$$\text{So } (G_i)^t_m (G_j)^m_n - (G_j)^t_m (G_i)^m_n = f_{ij}^k (G_k)^t_n$$

$$\Rightarrow f_{im}^t f_{jn}^m + f_{mj}^t f_{in}^m + f_{mi}^t f_{jn}^m = 0$$

$$\Rightarrow f_{ij}^m f_{mi}^t + f_{in}^m f_{mj}^t + f_{ji}^m f_{mn}^t = 0$$

But this is Jacobi thing, QED.

③ Examples: for $SU(2)$.

$$(T_i)^t_m = f_{im}^t = i\epsilon_{iml}$$

Show that $\vec{F}_{\mu\nu}$ is a group vector.

① Assume that $\vec{F}_{\mu\nu} = \partial_\mu \vec{A}_\nu - \partial_\nu \vec{A}_\mu + g (\vec{A}_\mu \times \vec{A}_\nu)$.

Want to show that

$$\delta \vec{F}_{\mu\nu} = \vec{\Theta} \times \vec{F}_{\mu\nu}$$

since this is how a vector transforms. (vector $\in$ adjoint rep).

② Basic data are: $\delta \vec{A}_\mu = \vec{\Theta} \times \vec{A}_\mu - \frac{1}{g} \partial_\mu \vec{\Theta}$

Thus:

$$\delta(\partial_\nu \vec{A}_\mu) = \partial_\nu (\delta \vec{A}_\mu) = (\partial_\nu \vec{\Theta}) \times \vec{A}_\mu + \vec{\Theta} \times (\partial_\nu \vec{A}_\mu) - \frac{1}{g} \partial_\nu \partial_\mu \vec{\Theta}$$

So: $\delta \vec{F}_{\mu\nu} = \delta(\partial_\mu \vec{A}_\nu) - \delta(\partial_\nu \vec{A}_\mu) + g (\delta \vec{A}_\mu \times \vec{A}_\nu + g \vec{A}_\mu \times \delta \vec{A}_\nu)$

$$= \cancel{(\partial_\mu \vec{\Theta}) \times \vec{A}_\nu} + \vec{\Theta} \times (\partial_\mu \vec{A}_\nu) - \frac{1}{g} \cancel{\partial_{\mu\nu} \vec{\Theta}} - \cancel{(\partial_\nu \vec{\Theta}) \times \vec{A}_\mu} - \vec{\Theta} \times (\partial_\nu \vec{A}_\mu) + \frac{1}{g} \cancel{\partial_{\nu\mu} \vec{\Theta}} \\ + g [\vec{\Theta} \times \vec{A}_\mu - \frac{1}{g} \cancel{\partial_\mu \vec{\Theta}}] \times \vec{A}_\nu + g \vec{A}_\mu \times [\vec{\Theta} \times \vec{A}_\nu - \frac{1}{g} \cancel{\partial_\nu \vec{\Theta}}]$$

Desired result is: $= \vec{\Theta} \times [\partial_\mu \vec{A}_\nu - \partial_\nu \vec{A}_\mu + g (\vec{A}_\mu \times \vec{A}_\nu)]$.

Correct if:

$$g(\vec{\Theta} \times \vec{A}_\mu) \times \vec{A}_\nu + g \vec{A}_\mu \times (\vec{\Theta} \times \vec{A}_\nu) = \vec{\Theta} \times (\vec{A}_\mu \times \vec{A}_\nu) g$$

Is, if $\vec{\Theta} \times (\vec{A}_\mu \times \vec{A}_\nu) + \vec{A}_\nu \times (\vec{\Theta} \times \vec{A}_\mu) + \vec{A}_\mu \times (\vec{A}_\nu \times \vec{\Theta}) = 0$

But this is the Jacobi cross product rule! QED

③ Corollary: $\vec{F}_{\mu\nu} \cdot \vec{F}^{\mu\nu}$ is a group scalar as well as a Lorentz scalar.

↓ over.

④ Explain why $\delta \vec{A} = \vec{\theta} \times \vec{A}$ for an adjoint vector.

In general $\delta \vec{A} = -i \vec{\theta} \cdot \vec{G} A$. I.e.,

$$\delta A^i = -i \vec{\theta} \cdot (\vec{G})^i_j A^j = -i \theta_a (G_a)^i_j A^j$$

But $(G_a)^i_j = f_{aj}^i$ for adjoint rep, so:

$$\begin{aligned} \delta A^i &= -i \theta^a f_{aj}^i A^j = \cancel{G_a} [-i f_{aj}^i \theta^a A^j] \\ &= [\vec{\theta} \times \vec{A}]^i \quad \underline{\text{QED.}} \end{aligned}$$

More on Variational Notation.

1. For the time being I am going to start using two different notations for the two kinds of variational derivatives. The notation $\delta A(x)$ has always meant and will continue to mean a variation in the function $A(x)$. I.e., at each point there is some change in the function. If "f" is either a function or a functional of A, this variation will cause a change or variation in f which I call δf . If you compute δf in terms of δA you are quickly led to a derivative which I will now call $\delta f / \delta A$. I.e., you just divide both sides by δA . But this is not the "functional derivative" which I denote $\nabla f / \nabla A$ and which involves removing an integration. Here are some examples:

①

$$\begin{aligned} f(A) &= A^n \\ (\delta f) &= n A^{n-1} (\delta A) \\ &= \int dy n A^{n-1}(y) [\delta A(y)] \delta(x-y) \end{aligned}$$

$$\frac{\delta f(x)}{\delta A(x)} = n A^{n-1}(x)$$

$$\frac{\nabla f(x)}{\nabla A(y)} = n A^{n-1}(y) \delta(x-y)$$

②

$$\begin{aligned} f(A) &= 2_x A(x) \\ (\delta f) &= 2_x (\delta A) \\ &= \int dy 2_y (\delta A(y)) \delta(x-y) \\ &= - \int dy (\delta A(y)) 2_y \delta(x-y) \end{aligned}$$

$$\Rightarrow \frac{\delta f(x)}{\delta A(x)} = \frac{2_x [\delta A(x)]}{\delta A(x)}$$

$$\frac{\nabla f(x)}{\nabla A(y)} = -2_y \delta(x-y)$$

③

$$f[A] = \int dy A^n(y) h(y)$$

$$\delta f = \int dy n A^{n-1}(y) (\delta A(y)) h(y)$$

$$\Rightarrow \frac{\nabla f}{\nabla A(x)} = n A^{n-1}(x) h(x)$$

$$\frac{\delta f}{\delta A(x)} = \text{no meaning}.$$

In the first two cases, $f(A)$ is a function. In each of these cases the δ derivative is a function, but the ∇ derivative is a distribution. In the first case one derivative is just the other times a delta and both are determined by knowledge of A. In second case the δ derivative is not determined by A; it depends on which δA variation you choose. Eg, if you vary $A(x)$ by a constant shift, the δ derivative is identically zero. On the other hand, the ∇ derivative is determined completely without knowing what δA is.

In the third case, f is a functional. In this case the δ derivative is meaningless, and the ∇ derivative is as shown.

Footnote: notice that the ∇ derivative is the only kind of derivative which allows you to have different parameters like x and y .

2. We have commented on the δ and ∇ derivatives. What about the regular full and partial derivatives which we denote by d and ∂ ? How do these fit into the picture here?

$$f(A) = A^n$$

$$\frac{df}{dA} = \frac{\partial f}{\partial A} = n A^{n-1}$$

$$= \frac{\delta f}{\delta A}$$

$$f(A, \partial_x A) = \partial_x A$$

$$df = \left(\frac{\partial f}{\partial A} \right) dA + \left(\frac{\partial f}{\partial (\partial_x A)} \right) d[\partial_x A]$$

$$\frac{\partial f}{\partial A} = 0$$

$$\frac{\partial f}{\partial (\partial_x A)} = 1$$

$$\frac{df}{dA} = \text{meaningless}$$

$$f[A] = \int dy A''(y) h(y)$$

$$\frac{df}{dA} = \text{no meaning}$$

$$\frac{\partial f}{\partial A} = \text{no meaning}$$

$$\frac{\delta f}{\delta A} = \text{no meaning!}$$

The notation $dA(x)$ means that we change A at one point, namely x . This gives us no information at all on $d(d_x A)$. Thus $dA(x)$ does not give us enough information to compute df in example 2 above. In contrast, $\delta A(x)$ specifies a change in A at every x ; this is why δf is then computable.

So example 2 is very interesting. There are 5 kinds of derivatives you can define of f with respect to A or $d_x A$. The full derivative df/dA is undefined. The other four derivatives are all different!

In example 1, all derivatives are the same as shown, except the ∇ derivative has an extra delta function. In contrast, in example 3 only the ∇ derivative is even defined!

Moral: make sure you know what kind of derivative you are dealing with.

Application: whenever you take a derivative with respect to something that is x dependent, you should not use either δ or d , for reasons above. Consider the Field Equations and the Conserved Gauge Currents equations;

$$\textcircled{1} \quad \frac{\delta \mathcal{L}}{\delta \theta} = \partial_\mu \frac{\delta \mathcal{L}}{\delta (\partial_\mu \theta)} = \frac{\partial \mathcal{L}}{\partial \phi} \frac{\delta \phi}{\delta \theta} + \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \frac{\delta (\partial_\mu \phi)}{\delta \theta} = \frac{\nabla \mathcal{S}}{\nabla \theta} = \partial_\mu \frac{\nabla \mathcal{S}}{\nabla (\partial_\mu \theta)}$$

↖ do not use partial ∂ here!

$$\textcircled{5} \quad \frac{\partial \mathcal{L}}{\partial \phi} = \partial_\mu \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \Rightarrow \frac{\nabla \mathcal{S}}{\nabla \phi} = \int d^4x \frac{\partial \mathcal{L}}{\partial \phi} = 0.$$

↖ do not use variable ϕ here

Chain rules for various derivatives.

① For a regular fctn $f[A(x)]$ you have $\frac{df}{dx} = \frac{df}{dA} \cdot \frac{dA}{dx}$.
I.e., example ① type.

② For example ② type, you cannot define full derivative. So consider

$$f[A\{g(x)\}]$$

Claim:

$$\boxed{\frac{\delta f}{\delta g} = \frac{\delta f}{\delta A} \cdot \frac{\delta A}{\delta g}}$$

Proof: $\delta A = \frac{\delta A}{\delta g} \cdot \delta g$; $\delta f = \frac{\delta f}{\delta A} \cdot \delta A = \frac{\delta f}{\delta A} \frac{\delta A}{\delta g} \delta g$. QED.

③ For same case $f[A\{g(x)\}]$, claim that

$$\boxed{\frac{\nabla f(x)}{\nabla g(y)} = \int dz \frac{\nabla f(x)}{\nabla A(z)} \frac{\nabla A(z)}{\nabla g(y)}}$$

Proof: $\delta f(x) = \int dy \left[\frac{\nabla f(x)}{\nabla g(y)} \right] \delta g(y)$

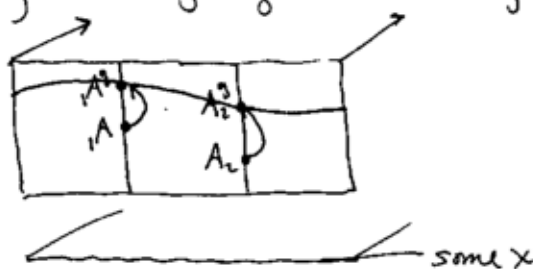
But $\delta f(x) = \int dz \left[\frac{\nabla f(x)}{\nabla A(z)} \right] \delta A(z) = \int dz \left[\frac{\nabla f(x)}{\nabla A(z)} \right] \left(\int dy \left[\frac{\nabla A(z)}{\nabla g(y)} \right] \delta g(y) \right)$
 $= \int dy \cdot \left\{ \int dz \frac{\nabla f(x)}{\nabla A(z)} \frac{\nabla A(z)}{\nabla g(y)} \right\} \delta g(y)$. QED

So $\frac{\nabla A(x)}{\nabla C(z)} = \int \frac{\nabla A(x)}{\nabla B(y)} \frac{\nabla B(y)}{\nabla C(z)} dy$ or

obs.

Development of the gauge delta function and its determinant.

- ① First consider these gauge conditions $f_a(A^g(x)) = 0$ $a=1, \dots, n$.
 What does this say? First, select a value of x . At that x , pick a field vector $A^b(x)$. Many other fields are connected to this chosen field by gauge transformations. Eg, $A^g = g \cdot A$. Only one such field, ie, only one such g , will satisfy gauge conditions. Thus, this condition should allow you to count only one gauge related field in each "fiber". Thus:



The gauge conditions

restrict you to a

cross section through

the bundle, instead of whole bundle. Fiber = orbit.

- ② You could solve $f_a(A^g(x)) = 0$ for g in terms of $A(x)$. Thus $g = g(x) = g(A(x))$. Here g = parameters g_a or $\underline{g_a}$. Thus,
- $$\prod_{a=1}^n df_a = \left| \frac{\partial f_a}{\partial g_b} \right| \prod dg_b \quad \text{where } \left| \frac{\partial f}{\partial g} \right| = n \times n \text{ determinant.}$$

Note that $J \equiv \left| \frac{\partial f}{\partial g} \right|$ will be fcn of $A(x)$, so $J = J(A(x))$. But also a fcn of g , so $J[A(x), g]$. But g is fcn of A , so no need to include it if you insist on gauge conditions. So

$$\prod_{a=1}^n df_a[A^g(x)] = J[A(x)] \cdot \prod_{a=1}^n dg_a(A(x)).$$

- ③ Now set $x = x_i$ and do $\prod_{x_i}$ on both sides, ie, make into

an appropriate $df[A(x)]$ for a path integral. But, $A =$ the coordinate, so we are not quite doing path integral.

- ④ Divide up spacetime into cubes centered at x_i . Then construct a new jacobian.

$$\prod_x \prod_a df_a(A(x)) = \prod_x \left| \frac{df}{dg} \right| \cdot \prod_x \prod_a dg_a(A(x))$$

where this is now a double determinant

a) in the $n \times n$ group space

b) in the $\infty \times \infty$ spacetime space.

Give this a name: $\left| \frac{df}{dg} \right| \equiv \Delta_f[A_\mu] = \det \left[\frac{\nabla f(x)}{\nabla g(y)} \right]$

This thing is now a functional of $\vec{A}_\mu$; it is no longer a function of x because we integrate over x in constructing this determinant. Thus, we have.

$$\prod_{x,a} df_a[A(x)] = \Delta_f[A_\mu] \prod_{x,a} dg_a(A(x)).$$

The corresponding delta form statement is:

$$\prod_{x,a} \delta[f_a(A(x))] = \Delta_f^{-1}[A_\mu] \cdot \prod_{x,a} \delta[g_a(A(x)) - g]$$

$\nwarrow$ "hit value" = 0
 $\nearrow$ hit value of g .

Now apply $\prod_{x,a} dg_a(A(x))$ to both sides. I.e., $\infty \times n$ integration variables. Then apply $\infty \times n$ integrations over the dg .

• You get

$$\prod_{x,a} \left\{ \int dg_a(x) \delta[f_a(A^g(x))] \right\} = \Delta_f^{-1}[A_\mu]$$

↑
one delta for each dg.

How many integrals? $N \times n$ where n = order of group and N = # of spacetime points.

⑤ Theorem: $\left[\Delta_f^{-1}[A_\mu^g] = f(g) \text{ is a gauge-group scalar!} \right]$

Proof: $f(g) \equiv \int dg' F((A^{g'})g') \dots$ write out

$$\begin{aligned} \Delta_f^{-1}[A_\mu^{g_1}] &= \prod_{x,a} \left\{ \int dg_a(x) \delta[f_a(A^{g_1 g}(x))] \right\} \\ &= \prod_x \left\{ \int_a [dg(x)] \cdot \prod_a \delta[f_a(A^{g_1 g}(x))] \right\} \\ &\quad \quad \quad \uparrow \\ &\quad \quad \quad = \prod_a dg_a = \text{Haar measure!} \\ &= \Delta_f^{-1}[A_\mu]. \end{aligned}$$

⑥ Now show the ∞ orbit volume deal.

$$\begin{aligned} \int [dA_\mu] e^{i \int d^4x \mathcal{L}[A_\mu]} &= \int [dA_\mu] \cdot \Delta_f[A_\mu] \prod_{x,a} \int dg_a(x) \delta[f_a(A^g(x))] e^{i S[A]} \\ &= \int [dA_\mu] \cdot \Delta_f[A_\mu] \cdot \prod_x \int [dg(x)] \cdot \prod_{x,a} \delta[f_a(A^g(x))] e^{i S[A]} \\ &= \prod_x \int [dg(x)] \cdot \int [dA] \Delta_f[A] \prod_{x,a} \delta[f_a(A^g(x))] e^{i S[A]}. \end{aligned}$$

↑ appears to depend on g.

⑦ Theorem: $\int [dA] f(A) = \int [dA] f(A^g)$
 $= \int [dA^g] f(A).$

It does not matter in what order we integrate over all fields. I.e., $[dA]$ had better be an invariant measure in this sense, like $\{dg\}$. I don't now know how to prove this:

⑧ So apply the above to result of 6:

$$= \left\{ \prod_{x,a} \int dg_a(x) \right\} \cdot \int [dA] \underset{\substack{\uparrow \\ \text{scalar}}}{\Delta_f[A^g]} \cdot \prod_{x,a} \delta[f_a(A^{g^i})] e^{iS[A^g]} \underset{\substack{\uparrow \\ \text{scalar}}}{e^{iS[A^g]}}$$

$$= \left\{ \prod_{x,a} \int dg_a(x) \right\} \cdot \int [dA] \Delta_f[A] \prod_{x,a} \delta[f_a(A)] e^{iS[A]} \\ = \int [dA] e^{iS[A]}.$$

What has been shown here? Whenever you have a lagrangian which is invariant under some internal symmetry group (gauge group), the path integral will always have this infinite factor. (Note: even if group G is compact, you have to integrate once for each spacetime point, so that makes it diverge). You can physically see where this factor comes from. In your dA integration, the integrand is constant when you integrate over certain directions in A -space (ie, when you integrate along group orbits).

Why did we have to do integration at each spacetime point? 1) this makes the determinant a functional, something which can appear inside a path integral; 2) the x -product will allow us to incorporate the delta into a modified action.

Calculation of the determinant.

① We have: $\Delta_f[A_\mu] = \det [M_{ab}(x,y)]$

where $M_{ab}(x,y) = \frac{\nabla f_a(x)}{\nabla g_b(y)}$. Calculate M first.

$$M_{ab}(x,y) = \frac{\nabla f_a(x)}{\nabla g_b(y)} = \int d^4z \frac{\nabla f_a(x)}{\nabla A_\mu^c(z)} \cdot \frac{\nabla A_\mu^c(z)}{\nabla \Theta_b(y)} \quad \text{/functional chain rule.}$$

② For any gauge theory, $\delta A_\mu^c(z) = -i f_{bd}^c \Theta^b(z) A_\mu^d(z) - \frac{1}{g} \partial_\mu \Theta^c(z)$

$$= \int d^4s \left[-i f_{bd}^c \Theta^b(s) A_\mu^d(s) - \frac{1}{g} \partial_\mu^s \Theta^c(s) \right] \delta^4(s-z)$$

● $= \int d^4s \{ \Theta^b(s) \} \left[-i f_{bd}^c A_\mu^d(s) - \frac{\delta^{bc}}{g} \partial_\mu^z \right] \delta^4(s-z)$

Since Θ small, $\delta\Theta = \Theta$ so we read off:

$$\frac{\nabla A_\mu^c(z)}{\nabla \Theta_b(y)} = \left[-i f_{bd}^c A_\mu^d(y) - \delta_b^c \frac{1}{g} \partial_\mu^z \right] \delta^4(y-z)$$

So for any gauge theory,

$$M_{ab}(x,y) = \int d^4z \left[-i f_{bd}^c A_\mu^d(y) \delta^4(y-z) - \delta_b^c \frac{1}{g} \partial_\mu^z \delta^4(y-z) \right] \cdot \frac{\nabla f_a(x)}{\nabla A_\mu^c(z)}.$$

Now shift ∂_μ^z to $\frac{\partial}{\partial}$ to get:

● $= \int d^4z \left[-i f_{bd}^c A_\mu^d(y) \delta^4(y-z) \frac{\nabla f_a(x)}{\nabla A_\mu^c(z)} + \delta_b^c \frac{1}{g} \delta^4(y-z) \partial_\mu^z \frac{\nabla f_a(x)}{\nabla A_\mu^c(z)} \right].$

Thus,

$$M_{ab}(x,y) = \left(-i f_{bd}^c A_\mu^d(y) + \delta_b^c \frac{1}{g} \partial_\mu^{(y)} \right) \frac{\nabla f_a(x)}{\nabla A_\mu^c(y)}$$

any gauge

③ Consider Landau gauge:

$$f_a(A_\mu) = \partial_\mu A_a^\mu = 0$$

$$\begin{aligned} \delta f_a &= \partial_\mu (\delta A_a^\mu) = \int d^4s \partial_\mu^s (\delta A_a^\mu(s)) \delta^4(x-s) \\ &= -\int d^4s [\delta A_a^\mu(s)] \partial_\mu^s \delta^4(x-s) \end{aligned}$$

$$\Rightarrow \left[\frac{\nabla f_a(x)}{\nabla A_c^\mu(y)} = -\delta_{a,c} \partial_\mu^{(y)} \delta^4(x-y) \quad \text{Landau} \right]$$

Combine with the above:

$$M_{ab}(x,y) = \left[-i f_{bd}^a A_\mu^d(y) \partial_\mu^{(y)} + \delta_{ba} \frac{1}{g} \square_{(y)}^2 \right] \delta^4(x-y)$$

Landau (14.15) A.L !

① For Coulomb gauge we get

$$\frac{\nabla f_a(x)}{\nabla A_c^\mu(y)} = \begin{cases} +\delta_{a,c} \partial_\mu^{(y)} \delta^4(x-y) & \mu=1,2,3 \\ 0 & \mu=0 \end{cases}$$

Leading to:

$$\left[M_{ab}(x,y) = \left[-i f_{bd}^a A_i^d(y) \partial_i^{(y)} + \delta_{ab} \frac{1}{g} \nabla_y^2 \right] \delta^4(x-y) \right]$$

Coulomb. (14.14) A.L.

Exercise for structure constants.

DBS

- ① Let C^a_{bc} be structure constants in Wybourne notation. Then we know that

$$C^{\tilde{n}}_{kl} C^p_{mn} + C^{\tilde{n}}_{lm} C^p_{kn} + C^{\tilde{n}}_{mk} C^p_{ln} = 0 \quad (4.21)$$

$$g_{\alpha\lambda} = C^{\tau}_{\alpha\beta} C^{\beta}_{\lambda\tau}. \quad (5.22)$$

- ② Now lets assume that the Killing form is diagonal and that

$$C^a_{bc} = C_{abc} = \text{totally anti sym.}$$

Then we get

$$\begin{aligned} k \cdot \delta_{\alpha\lambda} &= C_{\tau\alpha\beta} C_{\beta\lambda\tau} = C_{\alpha\beta\tau} C_{\lambda\tau\beta} \\ &= [G_\alpha]_{\beta\tau} [G_\lambda]_{\tau\beta} = \text{trace} [G_\alpha G_\lambda] \end{aligned}$$

as claimed by GG. So (4.21) says:

$$C_{kl} C_{gmn} + C_{lm} C_{pkn} + C_{mk} C_{pln} = 0.$$

To this now apply C_{lmj}

$$C_{lmj} C_{kl} C_{gmn} + C_{lmj} C_{lm} C_{pkn} + C_{lmj} C_{mk} C_{pln} = 0$$

But this is what we want.

$$\rightarrow \text{this term} = C_{ljm} C_{kl} C_{gmn} = C_{ljm} C_{mgn} C_{nkl}$$

③ $\equiv T^{(1)}_{jgk}$. Last term is

$$-C_{ljm} C_{nkm} C_{lpn} = +C_{mj l} C_{lpn} C_{nkm} = +T^{(1)}_{jpk}$$

Meanwhile the remaining term is

$$\begin{aligned} \underbrace{C_{mj} C_{nlm} C_{pkn}} &= -k \delta_{jn} C_{pkn} = -k C_{pkj} \\ &= -k C_{jpk} \\ \therefore C_{jlm} C_{nml} &= -C_{jlm} C_{nml} \\ &= - (C_j)_{lm} (C_n)_{ml} = -\text{trace} [C_j C_n] = -k \delta_{jn} \end{aligned}$$

Thus we have

$$2 T_{jpk}^{(1)} - k C_{jpk} = 0$$

$$\Rightarrow T_{abc}^{(1)} = \frac{k}{2} C_{abc}.$$

But also,

$$\begin{aligned} T_{abc}^{(1)} &= \underbrace{C_{mal} C_{lbk} C_{kcm}} \\ &= - (C_a)_{ml} (C_b)_{lk} (C_c)_{km} = -\text{trace} [C_a C_b C_c]. \end{aligned}$$

So here is our result:

$$\text{trace} [G_a G_b] = \overbrace{C_{alm} C_{bml}} = k \delta_{ab}$$

$$-\text{trace} [G_a G_b G_c] = \underbrace{C_{mal} C_{lbk} C_{kcm}} = T_{abc}^{(1)} = \frac{k}{2} C_{abc}$$

$$[G_a]_{bc} = C_{abc} = C_{bc}^a$$

Total Antisym of Structure Constants:

① So far we have used $[G_a, G_b] = f_{abc} G_c$ where $f_{abc} = f_{ab}^c$ if you like. Then we observed that, for the adjoint rep, this was true:

$$(G_a)^b_c = f_{ac}^b$$

Thus, for general case

$$\delta \phi_i = -i \theta_a (G_a)_{ij} \phi_j$$

but for adjoint case

$$\delta A_i = -i \theta_k f_{kj}^i \phi_j$$

$$\text{or } \delta A_i = -i f_{kj}^i \theta^k A^j \quad \text{or } \delta \vec{A} = [\vec{\theta} \times \vec{A}]$$

② Now I realize an added necessity:

$$\delta(\vec{A} \cdot \vec{A}) = 0 \text{ requires that } \vec{A} \cdot \delta \vec{A} = 0, \text{ or } \vec{A} \cdot \vec{\theta} \times \vec{A} = 0$$

$$\text{But this says } A^i [-i f_{kj}^i \theta^k A^j] = 0 = -i f_{kji} A^i A^j \theta^k = 0$$

This requires extra asym:

$$f_{abc} = -f_{bac} \text{ from Lie Algebra}$$

$$f_{abc} = -f_{acb} \text{ from } \vec{A} \cdot \vec{A} = \text{scalar.} \quad \leftarrow \text{extra requirement.}$$

$$\text{Thus } f_{abc} = -f_{bac} = +f_{bca} = -f_{cba}$$

$$\text{Thus } f_{abc} = -f_{cba}$$

The structure constants must be totally antisym!

$$f_{abc} = f_{bca}$$

$$f_{abc} = +f_{bca}$$

[cyclic sym]

③ Implications? Can treat f_{abc} exactly like ϵ_{abc} .

④ Note; for nice groups like $SU(n)$, $SO(n)$, the group metric tensor can be chosen $(+1, +1 \dots +1)$, so raise/lower does nothing.

$$⑤ [\vec{A} \times \vec{B}]_i = -i f_{ijk} A^j B^k = -i f_{jki} A_j B_k = -i f_{ijk} A_j B_k$$

↑
usual order.

So for cross products there you are.

⑥ Why not let $C_{ijk} = -i f_{ijk}$ so that

$$[G_i, G_j] = f_{ijk} G_k = i C_{ijk} G_k$$

Then $C_{ijk} = \epsilon_{ijk}$ in $SU(2)$ case.

⑦ Also, if $A_\mu = A_\mu^\dagger$, require $C_{ijk} = \text{real}$.

⑧ Rule: $f_{abc} = f_{abc} = i C_{abc}$

$$(\vec{A} \times \vec{B})_a = C_{abc} A^b B^c$$

$$\vec{A} \times \vec{B} \cdot \vec{C} = \vec{A} \cdot \vec{B} \times \vec{C} \text{ etc.}$$

$C_{abc} = \text{totally anti sym}$

$$k(G_i G_j) = \text{const } \delta_{ij} \quad \forall \text{ reps (GA footnote)}$$

$$-\frac{8}{3}$$

③ Other facts: $(G_a)_{bc} = f_{abc}$ / my result

$$\text{So } \boxed{(G_a)_{bc} = -f_{abc} = -i C_{abc}}$$

Note:

$$\begin{aligned} \frac{1}{2} [G_i G_j] &= \frac{1}{2} [(G_i)_{ab} (G_j)_{ba}] \\ &= (-i)^2 C_{iab} C_{jba} = -C_{iab} C_{jba} = +C_{iab} C_{jab} \end{aligned}$$

But look at (5.22) of Mybourn.

$$g_{ij} = \text{const. } C_{\alpha\beta\gamma} C_{\lambda\tau\delta}$$

If we assume that $g_{ij} = \text{const. } \delta_{ij}$ then result follows.

Summary of the $M_{ab}(x, y)$ calculation so far.

① For an arbitrary gauge I have shown that

$$M_{ab}(x, y) = \frac{1}{g} \left[\delta_{b,c} \partial_\mu^{(y)} + g C_{bdc} A_\mu^d(y) \right] \frac{\nabla f_a(x)}{\nabla A_\mu^c(y)}$$

② For the Landau gauge we had

$$= + \int d^4z [\partial_\mu^z A_\mu^a(z)] \delta(x-z) \delta_z$$

$$f_a[A_\mu] = \partial_\mu A_\mu^a = 0 \quad \Rightarrow \quad \partial_\mu^{(x)} A_\mu^a(x) = - \int d^4y A_\mu^a(y) \partial_\mu^{(y)} \delta^{(4)}(y-x)$$

$$\Rightarrow \frac{\nabla f_a[A_\mu(x)]}{\nabla A_\mu^c(y)} = - \partial_\mu^{(y)} \delta^{(4)}(x-y) \delta_{ac}$$

Then:

$$M_{ab}(x, y) = \frac{1}{g} \left[\delta_{bc} \partial_\mu^{(y)} + g C_{bdc} A_\mu^d(y) \right] \left\{ - \delta_{ac} \partial_\mu^{(y)} \delta^{(4)}(x-y) \right\}$$

$$= - \frac{1}{g} \left[\delta_{a,b} \partial^2 + g C_{bda} A_\mu^d(y) \partial_\mu^{(y)} \right] \delta^{(4)}(x-y)$$

$$= - \frac{1}{g} \left[\delta_{a,b} \partial_{(x)}^2 - g C_{abd} A_\mu^d(y) \partial_\mu^{(x)} \right] \delta^{(4)}(x-y)$$

Finally:

$$M_{ab}^L(x, y) = - \frac{1}{g} \left[\delta_{ab} \square_{(x)}^2 - g C_{abc} A_\mu^c(x) \partial_\mu^{(x)} \right] \delta^{(4)}(x-y).$$

my sign comes out -. There is +.

Note: since $M_{ab}(x, y) = \frac{\nabla f_a(x)}{\nabla A_\mu^b(y)}$, you change the sign of M

by changing sign of f_a . Obviously no significance at all!

Understanding Feynman Rules in W[J] formalism.

- ① Assume no gauge symmetry so no $\Delta_F = \det M$ problems. The basic idea is this

$$W[J] = \int [d\phi] e^{i \int d^4x [\mathcal{L}_0 + \mathcal{L}_I + J\phi]} = \int [d\phi] e^{i \int d^4x \mathcal{L}_I \left(\frac{1}{i} \frac{\nabla}{\nabla J(x)} \right)} e^{i \int d^4x [\mathcal{L}_0 + J\phi]}$$

This step is based on the mean $f\left(\frac{\nabla}{\nabla J(x)}\right) e^{\int dy A(y) H(y)} = f(H(y)) e^{\int dy A(y) H(y)}$ which is true. This follows from the mean $f(\partial/\partial A) e^{AH} = f(H) e^{AH}$.
Now, slide path integral to the right + have

$$W[J] = e^{i \int d^4x \mathcal{L}_I \left(\frac{1}{i} \frac{\nabla}{\nabla J(x)} \right)} W_0[J]$$

where $W_0[J] = \int [d\phi] e^{i \int d^4x [\mathcal{L}_0 + J\phi]} = e^{-\frac{i}{2} \int d^4x d^4y J(x) \Delta^0(x-y) J(y)}$

where Δ^0 is propagator of the free theory. Thus the final result is

$$W[J] = e^{i \int d^4x \mathcal{L}_I \left(\frac{1}{i} \frac{\nabla}{\nabla J(x)} \right)} e^{-\frac{i}{2} \int d^4x d^4y J(x) \Delta^0(x-y) J(y)}$$

Comments: I have shown elsewhere why it is that $W_0(J)$ can be written as shown in terms of the free propagator. The real question at this point is to explore in detail exactly how it is that this boxed expression gives you Feynman rules for your theory. I know that the only right way to do this is to start with examples. There are indeed other "statements" of the Feynman rules, but at the moment I want to understand this statement.

Example 1: take no derivatives of W , you should get vacuum bubbles. Also, take ϕ^2 theory.

$$W[J] = \sum_{n=0}^{\infty} \frac{g^n}{n!} \left[\int dx \mathcal{L}_I \left(\frac{1}{i} \frac{\delta}{\delta J(x)} \right) \right]^n e^{-\frac{1}{2} i \int dx dy \delta(x) \Delta(x-y) J(y)}$$

a) Start by looking at order zero. You get $W[0] \big|_{\text{order 0}} = 1$. Not sure how to interpret this.

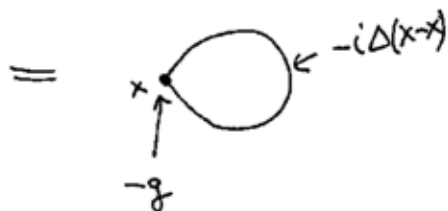
b) Next, do order one: $g \left[\int dx \mathcal{L}_I \left(\frac{1}{i} \frac{\delta}{\delta J(x)} \right) \right] e^{-\frac{1}{2} i \int dx dy \delta(x) \Delta(x-y) J(y)}$

Comment: recall that there was a little theorem for a $f(d/dx) e^{ax}$. There is no such simple theorem for $f(d/dx) e^{ax^2}$ or in general $f(d/dx) e^{g(x)}$. That theorem only worked when exponent was linear in x . Here we have a quadratic exponent in J ; the only way you can figure out the derivatives is to expand the exponential!

$$= g \left[\int dx (-i)^2 \frac{\delta}{\delta J(x)} \frac{\delta}{\delta J(x)} \right] \left[-\frac{1}{2} i \int dx dy \delta(x) \Delta(x-y) J(y) \right]$$

$$= g (-i)^2 \int dx [(-i) \Delta(x-x)]$$

$\uparrow$ goes with vertex $\uparrow$ seems to go with prop.



We have thus produced the order 1 tadpole vacuum graph.

c) look at order 2:

$$\frac{g^2}{2!} \int dx \left[\frac{1}{i} \frac{\delta}{\delta J(x)} \right]^2 \cdot \int dy \left[\frac{1}{i} \frac{\delta}{\delta J(y)} \right]^2$$

$$\approx \frac{\left[-\frac{1}{2} i \int dx dy \delta(x) \Delta(x-y) J(y) \right]^2}{2!}$$

$$= \frac{(-g)^2}{2!} \int dx_1 dx_2 \frac{\partial}{\partial J(x_1)} \frac{\partial}{\partial J(x_1)} \frac{\partial}{\partial J(x_2)} \frac{\partial}{\partial J(x_2)} \cdot \left(\frac{-\frac{1}{2}i}{2!} \left[\int da db dc dd \mathcal{T}(a) \mathcal{T}(b) \mathcal{T}(c) \mathcal{T}(d) \Delta(a-b) \Delta(c-d) \right] \right)$$

In general you have to choose the term in the rightside exponential which has the same number of J's as you have J-derivatives acting; other terms give 0 either because too few J's (deriv of const = 0) or too many J's (set J=0 at the end).

$$\text{Now } \frac{\partial}{\partial J(x_2)} [] = 4 \int da db dc \mathcal{T}(a) \mathcal{T}(b) \mathcal{T}(c) \Delta(a-b) \Delta(c-x_2)$$

$$\frac{\partial}{\partial J(x_2)} \frac{\partial}{\partial J(x_1)} [] = 4 \cdot 2 \int da dc \mathcal{T}(a) \mathcal{T}(c) \Delta(a-x_2) \Delta(c-x_2) + 4 \int da db \mathcal{T}(a) \mathcal{T}(b) \Delta(a-b) \Delta(x_2-x_2)$$

$$\frac{\partial}{\partial J(x_1)} [] = 4 \cdot 2 \cdot 2 \int da \mathcal{T}(a) \Delta(a-x_2) \Delta(x_1-x_2) + 4 \cdot 2 \int da \mathcal{T}(a) \Delta(a-x_1) \Delta(x_2-x_2)$$

$$\frac{\partial}{\partial J(x_1)} [] = 4 \cdot 2 \cdot 2 \cdot \Delta(x_1-x_2) \Delta(x_1-x_2) + 4 \cdot 2 \cdot 1 \cdot \Delta(x_1-x_1) \Delta(x_2-x_2)$$

$$\text{So } W[0] = \int dx_1 dx_2 \frac{(-g)^2}{2 \cdot 2^2 \cdot 2} (-i)^2 \left[4 \cdot 2 \cdot 2 \Delta(x_1-x_2) \Delta(x_1-x_2) + 4 \cdot 2 \Delta(x_1-x_1) \Delta(x_2-x_2) \right]$$

$$= (-g)^2 (-i)^2 \int dx_1 dx_2 [\Delta(x_1-x_2)]^2 + (-g)^2 (-i)^2 \frac{1}{2} \left[\int dx_1 \Delta(x_1-x_1) \right]^2$$

$$= \left[\text{Diagram 1} \right] + \frac{1}{2} \left[\text{Diagram 2} \times \text{Diagram 3} \right]$$

In this theory, there are only two ~~graphs with~~ vacuum graphs of order two. Second is product of two of loops found above. Note $\frac{1}{2}$ factor!

Now look at order 3:

$$\frac{(-g)^3}{3!} \int dx_1 dx_2 dx_3 \left[\frac{\nabla}{\partial J(x_1)} \right]^2 \left[\frac{\nabla}{\partial J(x_2)} \right]^2 \left[\frac{\nabla}{\partial J(x_3)} \right]^2$$

$$\frac{(-\frac{1}{2}i)^3}{3!} \left[\int da db dc dd de df \mathcal{T}(a) \mathcal{T}(b) \mathcal{T}(c) \mathcal{T}(d) \mathcal{T}(e) \mathcal{T}(f) \right] \\ \Delta(a-b) \Delta(c-d) \Delta(e-f)$$

Lots of combinatorics here:

$$\frac{\nabla}{\partial J(x_1)} [\] \rightarrow 6 \cdot \int da db dc dd de \mathcal{T}(a) \mathcal{T}(b) \mathcal{T}(c) \mathcal{T}(d) \mathcal{T}(e) \\ \Delta(a-b) \Delta(c-d) \Delta(e-x_1)$$

$$\frac{\nabla}{\partial J(x_1)} [\] \rightarrow 6 \cdot 4 \int da db dc dd \mathcal{T}(a) \mathcal{T}(b) \mathcal{T}(c) \mathcal{T}(d) \\ \Delta(a-b) \Delta(c-x_1) \Delta(d-x_1)$$

$$+ 6 \cdot \int da db dc dd \mathcal{T}(a) \mathcal{T}(b) \mathcal{T}(c) \mathcal{T}(d) \\ \Delta(a-b) \Delta(c-d) \Delta(x_1-x_1)$$

$$\frac{\nabla}{\partial J(x_2)} [\] = 6 \cdot 4 \cdot 2 \int da db dc \mathcal{T}(a) \mathcal{T}(b) \mathcal{T}(c) \\ \Delta(a-x_2) \Delta(b-x_1) \Delta(c-x_1) \\ + 6 \cdot 2 \int da db dc \mathcal{T}(a) \mathcal{T}(b) \mathcal{T}(c) \\ \Delta(a-b) \Delta(x_2-x_1) \Delta(c-x_1) \\ + 6 \cdot 4 \cdot \int da db dc \mathcal{T}(a) \mathcal{T}(b) \mathcal{T}(c) \\ \Delta(a-b) \Delta(c-x_2) \Delta(x_1-x_1)$$

Obviously we need a better notation. At start we had:

$$\underbrace{\frac{(\frac{1}{2}i)^3}{3!}}_{\text{hold til the end}} \underbrace{\frac{(-g)^3}{3!}}_{\text{hold. also.}} \underbrace{\int dx_1 dx_2 dx_3}_{\text{hold. also.}} \partial_1 \partial_1 \partial_2 \partial_2 \partial_3 \partial_3 \left[\int dy_1 dy_2 dy_3 \dots dy_6 \mathcal{T}(y_1) \dots \mathcal{T}(y_6) \Delta(y_1 - y_2) \Delta(y_3 - y_4) \Delta(y_5 - y_6) \right].$$

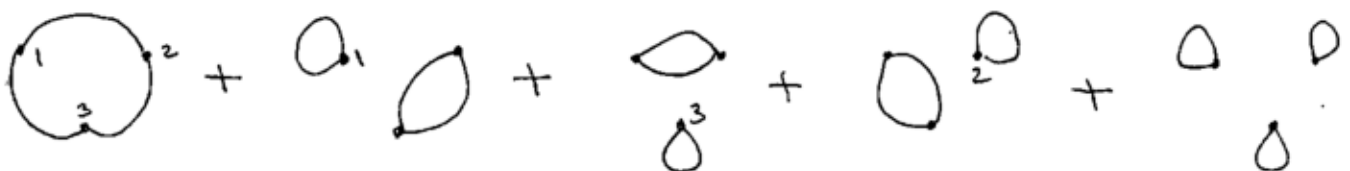
$$\text{Now: } \partial_3 [] = \int dy_1 \dots dy_5 \mathcal{T}(y_1) \dots \mathcal{T}(y_5) \times \left\{ 6 \Delta(y_1 - y_2) \Delta(y_3 - y_4) \Delta(y_5 - X_3) \right\}$$

$$\partial_3 \partial_3 [] = \int dy_1 \dots dy_4 \mathcal{T}(y_1) \dots \mathcal{T}(y_4) 6 \left[\Delta(y_1 - y_2) \Delta(y_3 - y_4) \Delta(X_3 - X_3) + 4 \Delta(y_1 - y_2) \Delta(y_3 - X_3) \Delta(y_4 - X_3) \right]$$

$$\partial_2 \partial_3 \partial_3 [] = \int dy_1 \dots dy_3 \mathcal{T}(y_1) \dots \mathcal{T}(y_3) \left[6 \cdot 4 \Delta(y_1 - y_2) \Delta(y_3 - X_2) \Delta(X_3 - X_3) + 4 \cdot 2 \Delta(y_1 - y_2) \Delta(y_3 - X_3) \Delta(X_2 - X_3) + 4 \cdot 2 \Delta(y_1 - X_2) \Delta(y_2 - X_3) \Delta(y_3 - X_3) \right]$$

$$\partial_2^2 \partial_3^2 [] = \int dy_1 dy_2 \mathcal{T}(y_1) \mathcal{T}(y_2) \left[6 \cdot 4 \cdot 2 \Delta(y_1 - X_2) \Delta(y_2 - X_2) \Delta(X_3 - X_3) + 6 \cdot 4 \Delta(y_1 - y_2) \Delta(X_2 - X_2) \Delta(X_3 - X_3) + 4 \cdot 2 \Delta(y_1 - y_2) \Delta(X_2 - X_3) \Delta(X_2 - X_3) + 4 \cdot 2 \cdot 2 \Delta(y_1 - X_2) \Delta(y_2 - X_3) \Delta(X_2 - X_3) + 4 \cdot 2 \cdot 2 \Delta(y_1 - X_2) \Delta(y_2 - X_3) \Delta(X_2 - X_3) + 4 \cdot 2 \Delta(X_2 - X_2) \Delta(y_2 - X_3) \Delta(y_2 - X_3) \right]$$

You can see all the possibilities being generated.



The problem is how to "count" each graph.

$$2_1 2_2 2_2 2_3 2_3 \int dy_1 \dots y_6 \mathcal{I}(y_1) \dots \mathcal{I}(y_6) \Delta(y_1 - y_2) \Delta(y_3 - y_4) \Delta(y_5 - y_6)$$

Count . Start with $\frac{(-\frac{1}{2})^3}{3!} \frac{(-g)^3}{3!}$. Then

pick up ~~6~~

$$\begin{aligned} & 6 \quad [\Delta(y_1 - y_2) \Delta(y_3 - y_4) \Delta(y_5 - X_3)] \\ & \times 4 \quad [\Delta(y_1 - y_2) \Delta(y_3 - X_3) \Delta(y_5 - X_3)] \end{aligned}$$

$$\begin{aligned} & \times 2 \quad [\Delta(y_1 - y_2) \Delta(y_3 - X_3) \Delta(X_2 - X_3)] \\ & \times 2 \quad [\Delta(y_1 - X_2) \Delta(y_3 - X_3) \Delta(y_5 - X_3)] \end{aligned}$$

$$\left\{ \begin{array}{l} \times 2 \quad [\Delta(y_1 - X_2) \Delta(y_3 - X_3) \Delta(X_2 - X_3)] \\ \times 4 \quad [\Delta(y_1 - y_2) \Delta(X_2 - X_3) \Delta(X_2 - X_3)] \\ \times 2 \quad [\Delta(y_1 - X_2) \Delta(y_3 - X_3) \Delta(X_2 - X_3)] \end{array} \right\} \rightarrow \text{many graphs}$$

$$6 \cdot 4 \cdot 2 = \frac{64}{6 \cdot 6 \cdot 8}$$

$$\begin{aligned} & = 4 \quad [\Delta(y_1 - X_2) \Delta(y_2 - X_3) \Delta(X_2 - X_3)] \\ & \times 2 \quad \Delta(y_1 - X_2) \Delta(X_1 - X_3) \Delta(X_2 - X_3) \\ & \times 1 \quad \Delta(X_1 - X_2) \Delta(X_1 - X_3) \Delta(X_2 - X_3) \end{aligned}$$

So factor is $6 \cdot 4 \cdot 2 \cdot 2 \cdot 2 \cdot 2$ so $\frac{6 \cdot 4 \cdot 4 \cdot 4 \cdot 4 \cdot 4}{3! \cdot 3! \cdot 3! \cdot 3! \cdot 3! \cdot 3!} = \frac{64}{48}$

I need a better way to get factor.

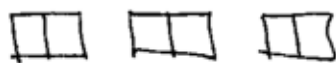
so get

$$\frac{6 \cdot 4 \cdot 4 \cdot 4}{8 \cdot 6 \cdot 6 \cdot 6} = \frac{4}{3}$$

this is the "count"

I now have a tentative method to count each graph. Consider each propagator to be a box. The number you get from applying all your J-derivatives is really the number of ways to put X's in the boxes. In our examples we were doing ϕ^2 theory, so each coordinate appears exactly twice. If third order graph, then three coordinates X_1, X_2 and X_3 . You keep a particular graph in mind and then you ask how many ways you can make that graph.

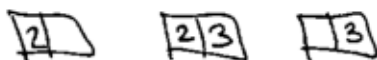
Example:  graph specifies theory and order, # props



a) how many ways to put in a pair of X_3 's? 24.
[X_3 's are different order].



b) how many ways to put pair of X_2 's: 8



c) X_1 pair? 2.

Thus count = $24 \times 8 \times 2$. But correct with $\frac{1}{3!} \frac{1}{3!} \left(\frac{1}{2}\right)^3$

Thus, get
$$\frac{24 \cdot 8 \cdot 2}{6 \cdot 6 \cdot 8} = \frac{48}{36} = \frac{8}{6} = \left(\frac{4}{3}\right)$$

Suppose you have theory which is order n, ie, ϕ^n theory, and V = number of vertices. You will at once get factor $1/V!$ from each exponential (?).

$V = \# \text{ vertices}$
$$\left[\left(\frac{\delta}{\delta J} \right)^n \right]^V$$

 $\uparrow$ nV derivs.
$$e^{iJ\phi} = \sum \frac{(iJ\phi)^m}{m!}$$

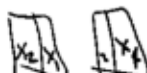
Need. $2m = nV$ so $m = \frac{nV}{2}$ so
$$\frac{1}{V!} \frac{1}{(nV/2)!} \left(\frac{1}{2}\right)^{\frac{nV}{2}}$$

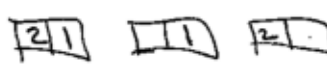
 $\uparrow$ (from $\delta/\delta J$) $\uparrow$ (from Δ_{exp}) $\uparrow$ $\frac{1}{2}$ factor

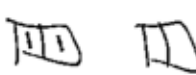
Obviously the $\frac{1}{2}$ factor always corrects for fact that you can put an X in a given prop in two ways, so now drop this fact and the $\frac{1}{2}$. Note that m = number of propagators.

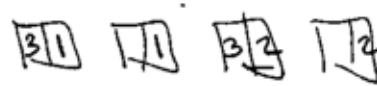
$$\frac{\text{count}}{V! m!} = \text{weight of graph. (vac.)}$$

I think I now know the weight of any vacuum bubble. Try out in ϕ^2 theory.

  $\frac{4 \cdot 1}{2! 2!} = \underline{\underline{1}} \checkmark$

  $\frac{8 \cdot 4 \cdot 1}{3! 3!} = \underline{\underline{\frac{4}{3}}} \checkmark$

$[\text{O} \times \text{O}]$  $\frac{2 \cdot 1}{2! 2!} = \underline{\underline{\frac{1}{2}}} \checkmark$

  $\frac{\overset{3}{12} \cdot \overset{3}{12} \cdot 4 \cdot 1}{\underset{6}{4!} \cdot \underset{6}{4!}} = \underline{\underline{1}}.$

So what have I done so far? Considered only vacuum graphs in ϕ^n theory.

To see Feynman rules, you have to

- 1) take only that term in the derivative expo which corresponds to V vertices
- 2) take only that term in second expo that has same number of ~~first~~ J 's.
- 3) then do derivatives in all ways and you will generate your graphs.

What about graphs with external lines? Here, the difference is that you take the term in second expo that has a number of extra J 's equal to number of external particles. Lets do example: ϕ^3 theory, two externals; two vertices:

$\leftarrow \partial_3 \partial_4 \text{ from } G_2 = \frac{\partial^2 W}{\partial_3 \partial_4}$

$$\left[\frac{(g/i^3)^2}{2!} \int dy_1 dy_2 (\partial_1 \partial_1 \partial_1)(\partial_2 \partial_2 \partial_2) \cdot \left[\frac{(-i/2)^4}{4!} \int dx_1 \dots dx_8 \delta(x_1) \dots \delta(x_8) \right. \right.$$

$$\left. \Delta(x_1 - x_2) \Delta(x_3 - x_4) \Delta(x_5 - x_6) \Delta(x_7 - x_8) \right].$$

 $\leftarrow$ 4 propagators, includes those of external lines. Only integrate on internal y 's though.

All derivs on same footing!

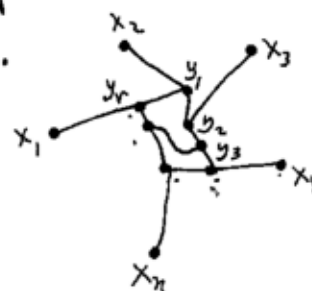
Let: $V = \# \text{ vertices (internal)} = \text{order of interaction.}$
 $m = \# \text{ propagations including external props}$
 $n = \# \text{ external particles.}$
 $N = \text{order of theory } \phi^N.$

$m =$

Show

$$N G_n^V(x_1, \dots, x_n) = \underbrace{\partial_1^* \partial_2^* \dots \partial_n^*}_n \cdot \frac{(g/i^N)^V}{V!} \int dy_1 \dots dy_V$$

$$\underbrace{(\partial_1^* \dots \partial_1^*)}_N \underbrace{(\partial_2^* \dots \partial_2^*)}_N \dots \underbrace{(\partial_V^* \dots \partial_V^*)}_N$$



$$\frac{(-i)^m}{m!} \int dz_1 dz_2 \dots dz_m \underbrace{J(z_1) \dots J(z_m)}_{2m} \underbrace{\Delta(z_1, z_2) \Delta(z_3, z_4) \dots \Delta(\dots)}_m$$

Clearly, $2m = n + VN$, to balance ∂_i with J 's.

So in a theory without derivative couplings and one kind of field and no gauge group, I know how to compute any graph from the above. I do not yet have a really concise formula for the combinatorics but by brute force I could tell you amplitude for any graph in such a theory. This has to be right because we derived this. I.e, it must agree with Wick method, I don't have to check. Also, only applies to boson fields; need extra work if fields anticommute!

maybe OK.

The exponentiation of the delta (version 3)

① First a simple example to illustrate the point:

$$\int dx g(x) \uparrow = \int dx dy g(x) \delta(y-a) \uparrow = \int dx dy g(x) da G(a) \delta(y-a)$$

$$\left\{ \int dy \delta(y-a) = 1 \right\} \quad \left\{ \int da G(a) = 1 \right\}$$

$$= \int dx dy g(x) G(y) \leftarrow$$

$$= \int dx dy g(x) \delta(y) \leftarrow$$

It is so trivial that you have to look carefully. Here, you are allowed to replace the delta function with an arbitrary but normalized function $G(y)$. The reason of course is that nothing else in the integral depends on y .

2. Now take above example and construct a fancier example simply by allowing $y = y(s)$. Then the above equality becomes:

$$\int dx ds g(x) \left| \frac{dy}{ds} \right| \underbrace{G(y(s))} = \int dx ds g(x) \left| \frac{dy}{ds} \right| \underbrace{\delta[y(s)]}.$$

Again, the replacement of the delta by arbitrary normalized function G is trivial. You are not saying the delta = G . You are just saying that here the replacement is allowed in these particular integrals.

3. Now lets move onto the actual case of interest:

$$\int d\psi dA \tilde{f}(\psi, A) \uparrow \quad \text{not same } f$$

$$\left\{ \int dg \left| \frac{df(A(g))}{dg} \right| \delta[f(A(g)) - a] = 1 \right\} \uparrow \quad \left\{ \int da G(a) \right\}$$

$$= \int dg d\psi dA \tilde{f}(\psi, A) \underbrace{G\{f[A(g)]\}} \left| \frac{df[A(g)]}{dg} \right| \leftarrow \text{LHS}$$

$$= \int dg d\psi dA \tilde{f}(\psi, A) \underbrace{\delta\{f[A(g)]\}} \left| \frac{df[A(g)]}{dg} \right| \leftarrow \text{RHS}$$

Here again you see the replacement of the delta with an arbitrary but normalized function G . Again it is trivial because you just do the g integration and you see that nothing else depends on g .

4. The only subtlety here is that on both sides we can identify the g integration with a multiplicative constant and thus cancel it on both sides. After you do this, the equality is then less obvious!

To show that you have a multiplicative constant, watch:

Lemma: A means $A(e)$; in general you have $A(g)$. Recall the reason that dx is a scalar:

$$x^{\mu'} = a^{\mu'}_{\nu} x^{\nu} \Rightarrow dx^{\mu'} = a^{\mu'}_{\nu} dx^{\nu} \Rightarrow J = \left| \frac{dx'}{dx} \right| = \det a = 1.$$

Similarly, dA means $dA = dA_1 dA_2 \dots dA_n$. Since $A(g) = U(g)A(e)$ and $|U|=1$ since unimodular group, you get $dA = dA(e) = dA(g)$. Even if not unimodular, you just get a constant for $|U(g)|$.

Lemma: $\tilde{f}(\psi, A)$ is $e^{iS[\psi, A]} = e^{iS[\psi(g), A(g)]}$ i.e., the action is a gauge group scalar. So is the object $\left| \frac{df[A(g)]}{dg} \right|$.

$$\text{LHS} = \int dg d\psi dA \tilde{f}(\psi, A) G\{f(A(g))\} \left| \frac{df[A(g)]}{dg} \right|$$

Take $A \rightarrow U(g')A$. Then $\tilde{f}[U(g')\psi, U(g')A] = \tilde{f}[\psi, A]$.
 $\psi \rightarrow \tilde{U}(g')\psi$ $G\{f(A(g))\} \Rightarrow G\{f(A)\}$ i.e. $A(e)$.
 $d[U(g')A] = dA$ etc. same for ψ .

Thus $\text{LHS} = \left(\int dg \right) d\psi dA \tilde{f}(\psi, A) G\{f(A)\} \left| \frac{df[A(g)]}{dg} \right|$
 $\nwarrow$ infinite constant when path integral aspect included.

Meanwhile

$$RHS = \left(\int dg \right) d\tau dA \tilde{f}(\tau, A) \delta\{f(A)\} \left| \frac{df(A)}{dg} \right|.$$

Now remove the $(\int dg)$ from both sides since a constant:

$$\int d\tau dA \tilde{f}(\tau, A) \underbrace{\delta\{f(A)\}}_{\Delta_f(A)} = \int d\tau dA \tilde{f}(\tau, A) \Delta_f(A) \underbrace{\delta\{f(A)\}}_{G\{f(A)\}}.$$

Now you see why you can replace the delta $\delta(f)$ with an arbitrary function $G(f)$. It is not obvious here because the determinant is $|df/dg|$ and that makes you wonder because you don't see any g anywhere. Just put the g back in and it becomes obvious again.

4. Everything we have done here really does not hinge on the presence of the path integral. I.e., all true for regular integrals here. How do we need to modify the above to get to path integral formalism? All integrals (including the $dg(x)$) are path integrals, and all delta functions are path-integral deltas (including the intermediate integration da which was used to insert $\mathcal{Q}(a)$). The path integralization of all integrals of course renders the jacobian to be a rather complicated object, but we have studied it elsewhere (ie, we studied how to calculate it, and how it yields ghosts).

We noted that $G(a)$ had to be normalized, or at least normalizeable. If da is a path integral, then we might take G to be a gaussian since this is the one path integral we think we can go.

I forgot to say that in path integralization, another change is that all the above functions become functionals, so in particular $G(a)$ is a functional. We take this gaussian form:

$$\tilde{f}(\tau, A) \rightarrow e^{iS[\tau, A]} \quad G[f(A)] = e^{-\frac{i}{2\alpha} \int d^4x (f[A(x)])^2}$$

This is the exact form taken by Coleman. AL take the other sign and insist that you take limit $\alpha \rightarrow 0$. I see no need for such a limit.

5. Thus, the net effect of the exponentiation of the delta is this: you add to your action a correction:

$$S_{\text{delta}}[A] = -\frac{1}{2\alpha} f_a(A) f_a(A)$$

$\alpha = \text{arbitrary (real) number.}$

Looking back at the original origin of delta (f) , you see that there are as many f 's as there are gauge group parameters; thus you get the sum as shown over a .

How Fadeev-Popov rescues QED.

1. First, write down the generating function for some photon-particle theory. For the moment ignore the matter fields, since the problem of interest has only to do with the photon propagator. The usual start-off is:

$$\begin{aligned} S &= \int d^4x \left[-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right] = \int d^4x \left[-\frac{1}{2} \partial_\mu A_\nu F^{\mu\nu} \right] = \int d^4x \left[+\frac{1}{2} A_\nu \partial_\mu F^{\mu\nu} \right] \\ &= \int d^4x \frac{1}{2} A_\nu \left[\square A^\nu - \partial_\mu \partial^\mu A^\nu \right] = \frac{1}{2} \int d^4x A_\mu(x) \left[g_{\mu\nu} \square - \partial_\mu \partial_\nu \right] A_\nu(x). \end{aligned}$$

I have not yet specified a gauge choice. The next step is to add the Fadeev Popov correction. Since this is an abelian gauge theory, the FP determinant is independent of A and can be ignored. However, you do have to include the expo-or-the-delta correction, which is:

$$S_{\text{corr}} = -\frac{1}{2\alpha} \int d^4x \left[f(A) \right]^2 \quad f(A) = 0 \text{ is gauge condition}$$

2. Now for starters, let's go into the Lorentz or Landau gauge. Notice that once you have made the replacement of the gauge delta with this gaussian factor, the delta is gone so you may not set $d_u A^u = 0$ wherever you see it!

Notice that the last term in the original action above is $+\frac{1}{2}(d_u A^u)(d_v A^v)$. In particular you do not want to throw this away.. well, maybe you could. It is really in the gaussian correction that you are not allowed to dump it. But for now keep in both places. Then with the delta correction we have:

$$S + S_{\text{delta}} = \frac{1}{2} \int d^4x A_\mu(x) \left[g_{\mu\nu} \square - (1-\alpha) \partial_\mu \partial_\nu \right] A_\nu(x)$$

Recall that α is any finite real number. Now the operator is invertible; in k -space you have:

$$-M_{\mu\nu} = g_{\mu\nu} k^2 - (1-\alpha) k_\mu k_\nu$$

$$-(M^{-1})_{\mu\nu} = \frac{1}{k^2} \left[g_{\mu\nu} - (1-\alpha) \frac{k_\mu k_\nu}{k^2} \right]$$

signs and is correct
6.4.79.

That this is so is trivial, multiply them together. Once you have an invertible M , you know from the path integral formalism that the inverse operator has the interpretation of the particle propagator. Thus, in the " α -gauge" of the Lorentz gauge we have:

$$\langle 0 | T(A_\mu(x) A_\nu(y)) | 0 \rangle = i D_{\mu\nu}(x-y) = \text{F.T. } i D_{\mu\nu}(k) \quad \alpha\text{-Landau gauge.}$$

where

$$i D_{\mu\nu}^{(\alpha)}(k) = \frac{(-i)}{k^2} \left[g_{\mu\nu} - (1-\alpha) \frac{k_\mu k_\nu}{k^2} \right].$$

3. Now, looking back, I see that I could have dropped the $(d_\mu A^\mu)^2$ factor in the original action; in the end this just corresponds to a redefining of the parameter α .

Note: $k_\mu D_{\mu\nu}(k) = 0$ only when $\alpha = 0$. Is this the transverse gauge? Something fishy here.

see
later.

More on the gauge-fixing correction (delta).

- ① In our simple example we knew we could replace $\delta(y) \rightarrow G(y)$ with no effect on the result of the double integral. Of course, with $\delta(y)$ you pin y at 0, whereas not so with $G(y)$.

- ② Let $G(y) = \left(\frac{e^{-R^2 y^2}}{\sqrt{\pi}} \right)$. This has unit area $\int_{-\infty}^{\infty} dy G(y) = 1$, as required. And, as $R \rightarrow \infty$, $G(y) \equiv \delta(y)$ precisely, so here we can smoothly return to the delta as a limit.

- ③ A similar thing happens in the gauge case. We start with $\delta[F]$, so $f=0$ throughout the $[dA]$ path integration, eg, $\eta_\mu A^\mu = 0$ a whatever. Now we replace $\delta[F]$ by

$$\delta[F] \rightarrow \exp \left[-\frac{1}{2\alpha} \int_E dx (f[A(x)])^2 \right] \frac{1}{\sqrt{2\pi\alpha}} \equiv G[F]$$

But we ignored the $\frac{1}{\sqrt{2\pi\alpha}}$ factor since it was a constant.

- Now we have a choice

a) keep α finite. Then $f(A) \neq 0$ throughout the integration, though no harm done. I.e., $f(A)$ may be peaked at 0 and speak, both other values of f contribute in $[dA]$.

b) Let $\alpha \rightarrow 0$. Then $G[F] \rightarrow \delta[F]$ and $f=0$ again!

- ④ Thus, only in case (b) will you really have $\eta_\mu A^\mu = 0$. For example,

$$\eta^\mu \langle 0 | T(A_\mu(x) A_\nu(y)) | 0 \rangle = (\text{diag}) \int [d\eta] \int [dA] e^{iS[A]} G[F] \left| \frac{df}{d\eta} \right| \eta^\mu A_\mu(x) A_\nu(y)$$

This vanishes in the axial gauge only if you really have $\delta[\eta_\mu A^\mu]$ there. If $\exp(-\int \eta_\mu A^\mu)$, no longer zero!!

Thus, the "axial gauge" really does require $\alpha \rightarrow 0$, as Kummer does.

- ⑤ Similarly, ~~($\eta_\mu A^\mu$)~~ consider $\langle \eta \rangle$

$$\partial_\mu D_{\mu\nu}(x-y) = \dots \int [dA] G[\partial_\mu A^\mu] \partial_\mu A^\mu(x) A^\nu(y)$$

= 0 only if $\alpha = 0$ so get delta.

I can check this directly of course!

Conclusion: if you mean by "axial gauge" that propagator is perpendicular to n^u , then you must take the limit $\alpha = 0$. This is what Kummer did. And if you mean by Lorentz gauge or Landau gauge or whatever that q^u is perp to propagator, then again you need the limit $\alpha = 0$.

Thus, α is a temporary tool which allows you to invert the object M to get the propagator.

A review of the Delta paradox.

- ① Let's make an analogy. Instead of using a field A and $[dA]$, just do this:

$$\begin{aligned}\int dg_1 e^{iS(g_1)} &= \int dg_1 e^{iS(g_1)} \times \underbrace{\int dg_2 \delta(g_2 - e)}_{=1} \\ &= \int dg_2 \underbrace{\int dg_1 e^{iS(g_1)} \delta(g_1 - g_2)}_{= \int dg_2 e^{iS(g_2)} \delta(g_2 - e)} \quad \text{if } S(g) = \text{scalar!}\end{aligned}$$

Thus we conclude that:

$$\int dg_1 e^{iS(g_1)} = [\int dg] \times \left[\int dg_2 e^{iS(g_2)} \delta(g_2 - e) \right]$$

and lets do compact group so $\int dg = \text{finite}$

- ② On the other hand, we could have inserted $1 = \int dg G(g)$ where $G(g)$ is some fcn which integrates to 1. Then you would get

$$\int dg_1 e^{iS(g_1)} = [\int dg] \times \left[\int dg_2 e^{iS(g_2)} G(g_2) \right]$$

These are equal

- ③ Now, whether or not we cancel the $\int dg$, these objects are equal. But, if S had not been a scalar, results would not be equal. Thus, suppose we turn on a source. Then

$$\int dg_2 e^{i[S(g_2) + \vec{j} \cdot \vec{g}_2]} \delta(g_2 - e) \neq \int dg_2 e^{i[S(g_2) + \vec{j} \cdot \vec{g}_2]} G(g_2)$$

And, taking derivatives:

$$\int dg_2 e^{iS(g_2)} \underbrace{(g_2^a g_2^b)}_{\text{not a scalar}} \delta(g_2 - e) \neq \int dg_2 e^{iS(g_2)} (g_2^a g_2^b) G(g_2)$$

- ④ Thus, the propagator you get with $\delta[f]$ differs from the one you get with $G[f]$! The fact that $(\int dg) = \infty$ has nothing to do with this difference!

Lets now put this discussion into words. If you include the action without source, it is true that you can replace the delta function by a gaussian G and the result is not changed, whether or not you drop the overall group volume factor.

However, if you then "turn on" your source J and compute up some greens functions, your greens functions will be different! The reason is simple: with the source on, you can not trace back to get an equality. The presence of the source breaks the gauge invariance of the action. So it matters then which you do: delta or gaussian.

This then explains why $n_u D_{uv} = 0$ only in the special case $\alpha = 0$. It is acceptable to use general α , but your greens functions are then α dependent and are only orthogonal to n^u in the special case $\alpha = 0$. So finally I have resolved this problem.

Why is the source term gauge breaking???

$$\vec{J} = \text{regular vector, so} \quad \delta \vec{J} = \vec{\Theta} \times \vec{J}$$

$$\vec{A} = \text{nonregular, so} \quad \delta \vec{A}_\mu = \vec{\Theta} \times \vec{A}_\mu - \frac{1}{g} \partial_\mu \vec{\Theta}$$

Ohue, $\vec{J} \cdot \vec{A}$ is not a gauge scalar!

QED in the Coulomb Gauge, via Fadeev Popov.

1. We start as on last sheet, changing only the gauge. Since still ~~non~~abelian theory, no det correction. So just do it:

$$\mathcal{L} = \underbrace{\frac{1}{2} A^\mu [g_{\mu\nu} \square - \partial_\mu \partial_\nu] A^\nu}_{\text{same as before}} - \frac{1}{2\alpha} [(\partial_i A^i)^2] \quad \leftarrow \text{coulomb gauge.}$$

Now rewrite the delta correction in pseudo-covariant notation like so:

$$-\partial_i A^i = \partial_\mu A^\mu - \eta_\mu \partial^\mu (\eta_\nu A^\nu) \quad \text{where} \quad \eta_\mu = (1, 0, 0, 0) \quad \eta \cdot \eta = \eta^2 = +1.$$

↑ note - sign for my metric.

So examine this correction term:

$$\begin{aligned} (\vec{\nabla} \cdot \vec{A})^2 &= [\partial_\mu A^\mu - \eta_\mu \partial^\mu (\eta_\nu A^\nu)]^2 = \overset{(1)}{(\partial_\mu A^\mu)^2} - 2 \overset{(2)}{(\partial_\mu A^\mu)(\eta_\alpha \partial^\alpha [\eta_\nu A^\nu])} \\ &\quad + \eta_\mu \partial^\mu [\eta_\nu A^\nu] \cdot \eta_\alpha \partial^\alpha [\eta_\beta A^\beta] \\ &= A^\mu \left\{ \overset{(1)}{-\partial_\mu \partial_\nu} + \overset{(2)}{2 \eta^\alpha \eta_\nu \partial_\mu \partial_\alpha} - \overset{(3)}{\eta^\alpha \eta^\beta \eta_\mu \eta_\nu \partial_\alpha \partial_\beta} \right\} A^\nu \\ &\quad = \overset{(2)}{2 \eta^\alpha \eta_\mu \partial_\nu \partial_\alpha} \\ &\quad = \eta^\alpha \partial_\alpha [\eta_\mu \partial_\nu + \eta_\nu \partial_\mu] \end{aligned}$$

This last change makes the correction terms each symmetric. Write now as:

$$-\frac{1}{2\alpha} A^\mu \left[-\partial_\mu \partial_\nu + [\eta_\mu \partial_\nu + \eta_\nu \partial_\mu] (\eta \cdot \partial) - \eta_\mu \eta_\nu (\eta \cdot \partial)(\eta \cdot \partial) \right] A^\nu$$

Therefore,

$$\mathcal{L}_{\text{eff}} = \frac{1}{2} A^\mu \left\{ g_{\mu\nu} (\partial \cdot \partial) - [1 - \frac{1}{2}] \partial_\mu \partial_\nu - \frac{1}{2} (\eta_\mu \partial_\nu + \eta_\nu \partial_\mu) (\eta \cdot \partial) + \frac{1}{2} \eta_\mu \eta_\nu (\eta \cdot \partial)^2 \right\} A^\nu$$

So

$$\eta_{\mu\nu}(k) = k^2 g_{\mu\nu} - (1 - \frac{1}{2}) k_\mu k_\nu - \frac{1}{2} (\eta_\mu k_\nu + \eta_\nu k_\mu) (\eta \cdot k) + \frac{1}{2} \eta_\mu \eta_\nu (\eta \cdot k)^2$$

Or we can write as

$$M_{\mu\nu} = (k^2 g_{\mu\nu} - k_\mu k_\nu) - \frac{1}{\alpha} (k_\mu - \eta_\mu(\eta \cdot k))(k_\nu - \eta_\nu(\eta \cdot k))$$

So for the moment, define

$$p_\mu = k_\mu - \eta_\mu(\eta \cdot k)$$

Then
$$[M_{\mu\nu} = k^2 g_{\mu\nu} - k_\mu k_\nu - \frac{1}{\alpha} p_\mu p_\nu]$$

Facts of interest:

$$(p \cdot k) = k^2 - (k \cdot \eta)^2 = (p \cdot p) = p^2$$

$$(p \cdot \eta) = 0$$

$$k_\mu p_\nu + k_\nu p_\mu = 2 k_\mu k_\nu - (\eta \cdot k) [k_\mu \eta_\nu + k_\nu \eta_\mu].$$

The game is now to solve for the inverse operator and this will be the photon propagator in the d-Coulomb gauge.

What do we know about the inverse operator?

- a) must be symmetric
- b) only independent vectors around are k and p (p has replaced η).

Thus:

$$(M^{-1})_{\mu\nu} = \frac{1}{k^2} g_{\mu\nu} + A k_\mu k_\nu + B p_\mu p_\nu + C [k_\mu p_\nu + k_\nu p_\mu].$$

We now have the problem of finding A, B and C . Brute force time has arrived:

We have now found our inverse propagator:

$$\begin{aligned}
 (M^{-1})_{\mu\nu} &= \frac{1}{k^2} g_{\mu\nu} + \frac{1}{k^2 p^2} \left\{ \left(1 - \alpha \frac{k^2}{p^2}\right) k_\mu k_\nu - (k_\mu p_\nu + k_\nu p_\mu) \right\} \\
 &= \frac{1}{k^2} \left[g_{\mu\nu} + \frac{1}{p^2} \left\{ - \left(1 + \alpha \frac{k^2}{p^2}\right) k_\mu k_\nu + (\eta \cdot k) (\eta_\mu k_\nu + \eta_\nu k_\mu) \right\} \right]
 \end{aligned}$$

where $p^2 = k^2 - (\mathbf{k} \cdot \boldsymbol{\eta})^2$. Rewrite once more

α -Coulomb gauge

$$= \stackrel{(-i)}{k^2} \left[-g_{\mu\nu} - \frac{\left(1 + \alpha \frac{k^2}{p^2}\right)}{(-p^2)} k_\mu k_\nu + \frac{(\eta \cdot k)}{(-p^2)} [\eta_\mu k_\nu + \eta_\nu k_\mu] \right] = D_{\mu\nu}(k)$$

= photon prop.

If I set $\alpha=0$ this duplicates exactly the BD page 79 result, except I don't get the Coulomb term. I.e., in this method it seems that the cancellation has already taken place.

Comments: in the canonical quantization procedure, you only quantize the transverse A_i fields, and you leave A_0 unquantized. The Coulomb interaction cannot be carried by the transverse fields, so you have to add that in separately.

In the path integral approach, you never make this artificial separation, and you find that Coulomb has already been included. good.

Also, $\alpha \rightarrow 0$ then gives true Coul gauge, see note elsewhere.

$$m_{\mu}^{\cdot\lambda} (m^{\cdot\lambda})_{\lambda\nu} = g_{\mu\nu}$$

$$= [k^2 \delta_{\mu}^{\lambda} - k_{\mu} k^{\lambda} - \frac{1}{2} p_{\mu} p^{\lambda}] [\frac{1}{k^2} g_{\lambda\nu} + A k_{\lambda} k_{\nu} + B p_{\lambda} p_{\nu} + C \{k_{\mu} p_{\lambda} + k_{\lambda} p_{\mu}\}]$$

Must write out all 12 terms:

$$\begin{aligned} g_{\mu\nu} &- \frac{1}{k^2} k_{\mu} k_{\nu} - \frac{1}{2k^2} p_{\mu} p_{\nu} + A k^2 k_{\mu} k_{\nu} + B k^2 p_{\mu} p_{\nu} + C \{k_{\mu} p_{\nu} + k_{\nu} p_{\mu}\} k^2 \\ &- A k^2 k_{\mu} k_{\nu} - B p^2 k_{\mu} p_{\nu} - C p^2 k_{\mu} k_{\nu} - C k^2 k_{\mu} p_{\nu} \\ &- \frac{A}{2} p^2 p_{\mu} k_{\nu} - \frac{B}{2} p^2 p_{\mu} p_{\nu} - \frac{C}{2} p^2 p_{\mu} k_{\nu} - \frac{C}{2} p^2 p_{\mu} p_{\nu} \end{aligned}$$

Regroup to get tensor coefficients:

$$\begin{aligned} g_{\mu\nu} [1] &+ k_{\mu} k_{\nu} [-\frac{1}{k^2} + A k^2 - A k^2 - C p^2] \\ &+ p_{\mu} p_{\nu} [-\frac{1}{2k^2} + B k^2 - \frac{B p^2}{2} - \frac{C p^2}{2}] \\ &+ k_{\mu} p_{\nu} [\cancel{k^2} - B p^2 - \cancel{2k^2}] + k_{\nu} p_{\mu} [C k^2 - \frac{A p^2}{2} - \frac{C p^2}{2}] \end{aligned}$$

$\Rightarrow B=0$

$$\begin{aligned} &\Rightarrow C = \frac{-1}{k^2 p^2} \\ = g_{\mu\nu} [1] &+ k_{\mu} k_{\nu} [-\frac{1}{k^2} - C p^2] + p_{\mu} p_{\nu} [-\frac{1}{2k^2} - \frac{C p^2}{2}] \\ &+ k_{\nu} p_{\mu} [C (k^2 - \frac{p^2}{2}) - A \frac{p^2}{2}] \\ = g_{\mu\nu} [1] &+ k_{\nu} p_{\mu} [(\frac{-1}{k^2 p^2})(k^2 - \frac{p^2}{2}) - A \frac{p^2}{2}] \Rightarrow A = \frac{-\alpha}{p^2} (\frac{1}{k^2 p^2})(k^2 - \frac{p^2}{2}) \end{aligned}$$

Conclusions:

$$\begin{cases} A = -\frac{\alpha}{k^2 p^4} (k^2 - \frac{p^2}{2}) = \frac{1}{k^2 p^2} (1 - \alpha \frac{k^2}{p^2}) \\ B = 0 \\ C = -\frac{1}{k^2 p^2} \end{cases}$$

Gauge and Ghost Propagators and Vertices (Landau gauge)

$$i_1 \text{ --- } \overrightarrow{\text{---}} \text{---} \dot{\text{---}} i_2 = (-i) \frac{\delta_{i_1 i_2}}{k^2} \quad \text{ghost propagator}$$

$$i_1 \text{ --- } \overrightarrow{\text{---}} \text{---} \dot{\text{---}} i_2 = +g C_{i_1 i_2 i_3} k'_{\mu_3} \quad \text{ghost-gauge vertex}$$

i_3, μ_3

$$i_1, \mu_1 \text{ ~~~~~ } i_2, \mu_2 = \frac{(-i) \delta_{i_1 i_2}}{k^2} \left[g_{\mu_1 \mu_2} - (1-\alpha) \frac{k_{\mu_1} k_{\mu_2}}{k^2} \right] \quad \text{gauge propagator}$$

$$i_1, \mu_1 \text{ ~~~~~ } i_2, \mu_2 = +g C_{i_1 i_2 i_3} \left[g_{\mu_1 \mu_2} (k' - k^2)_{\mu_3} + \text{cyclic} \right] \quad \text{triple gauge vertex}$$

i_3, μ_3

$$i_1, \mu_1 \text{ ~~~~~ } i_2, \mu_2 = -ig^2 \left\{ C_{i_1 i_2 k} C_{i_3 i_4 k} \left[g_{\mu_1 \mu_3} g_{\mu_2 \mu_4} - g_{\mu_2 \mu_3} g_{\mu_1 \mu_4} \right] + (12, 34) + (13, 24) + (14, 23) \right\} \quad \text{4-gauge vertex}$$

i_3, μ_3 i_4, μ_4

Indices like i_2 denote group-label (color); can take n values for n param. group.
Indices like μ_2 denote Lorentz vector index.

Structure constants are $f_{ab}^c = f_{abc} = i C_{abc}$ = totally antisymmetric.

g = gauge coupling via $d_u - ig G \cdot A_u$ prescription.

Ghost is scalar fermion, so (-1) for closed loop and no double-dotted propagator, also no double-undotted ghost prop

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$\downarrow \quad \equiv (12,34)$

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Comment on the effect of gauge-choice on the feynman rules.

1. Your choice of gauge has two effects. First, it affects the FP determinant, ie, the object M . Since M sets the ghost rules, you see that any feynman rules involving the ghosts will be gauge-dependent. I guess that both the ghost propagator and the ghost-gluon couplings are gauge dependent.

Second, the gauge choice affects the gauge delta which in turn affects the gauge-fixing lagrangian correction. Since this is usually quadratic in A^μ for linear gauges (which is what you usually find), you see that the gauge choice will affect the form of the gluon propagator also.

2. On the other hand, unless you use some wierd non-linear gauge, your gauge choice cannot affect the triple-gluon and four-gluon couplings, since the gauge-fixing terms will then only be quadratic in A . So for linear gauges the 3-gluon and 4-gluon couplings are always the same.

3. In the axial gauge, M is independent of A . You can do either of two things, they are the same:

a) you can keep the ghosts, but there will be no ghost-gluon coupling, so in effect the ghosts completely decouple from the gluons (and matter fields). Thus, you can then just ignore the ghosts since they cant affect any digrams you are interested in.

b) you dont even bother to "create" the ghosts, you just throw out the $\det M$ right at the start since it passes out of the functional integral since it doesnt depend on A . Ie, it is just an (infinite) constant and you always drop such constants.