

Triple Glue Vertex

$S_n, X_i^i \dots$

data

PPP

Ward 10

Ward 6

15 x 18

5 x 6 Operator

$Fk|^{ab}$

B Vectors

Complete Sets

Lost Ward

typed

Misc

PPP + DP Sector

(no go)

IRREDUCIBLE

the n's

n - oldway

Part 1 of 2

(4 July)

Flow Chart for Tensor Problem

It would be nice to see exactly why I "paused" when I did at various points. I.e., went down various alleys and hit brick walls. Are those walls still walls?

Pause 1: Here the last thing I did was to write a huge table showing 10 constraints on the set of 10 S_3 functions including four pairs, 3 S's and #3 A functions. Table entries are d_1, d_2 and d_3 the three dot products. Things are all completely analyzed in S_3 fractions. At this point I didn't know what to do about the "pairs". Somehow you can't just treat the two members of each pair as independent functions. There are in effect four extra constraints, namely, that each pair is really a pair. I think my current feeling on this charge is to express the pair "entries" using the X^1_2 operator so that you have 10 operator-constraints on 10 functions. This is where I plan to continue. I stopped because a) the table was so large and foreboding; b) Jim swore there were fewer tensors so I thought I had counted wrong and was therefore discouraged with the 10-function approach.

Pause 2: I showed how to reduce to 6 functions from 10 (albeit not S_3 symmetric ones). I rewrote the Ward but ran into the same kind of problem as with the pairs above. Here you can extend from 5 Ward equations to 15 by adding in the cyclics, so then you have in effect $6 \times 3 = 18$ "variables". But this approach is not too good because you then have to somehow add IN the constraints (12 of them) that the cyclic functions really are cyclically related. So 15 equations in 18 variables really scared me, so I stopped on this because too foreboding; later I returned to it (see below) for another try.

It then occurred to me that I could replace some of the pp products with the F-tensors and this might be useful because Marshall's vertex was in this form and because these tensors have great orthogonality properties to vectors. I came up with a good general vertex form, then I wrote down the Ward but did not Lorentz-tensor parse it.

Some of the contracted and double dotted equation here were quite simple, but still I didn't know what to do with them: the same $6 \times 3 = 18$ function problem. Too much coupling between functions.

I showed how Marshall's solution really does fit into this general form, then I just gave up on the approach.

Pause 3: Here, realizing that the B-vectors give you a very simply Ward null term, and convinced that finding Ward nulls was the entire deal, I tried converting to B from p.

I gave up when I could not make the three B's add up to zero, hence could not take over the independent tensor forms analysis.

Pause 4: I then returned to the 6-function earlier general ppp forms and the analysis of the 5 Ward equations. I did some elimination and reduce d it all to three operator equations in 6 functions; I decided it was probgly better to have symmetry operators like [e] and [12] sitting in your equations, rather than having $6 \times 3 = 18$ functions. The operator approach seems better because

- a) it keeps your number of functions down to 6, not 18
- b) no need to add in those extra 12 cyclicity constraints.

Then I considered the problem of solving these operator-like equations. I found that the operators appearing tended to be non-invertable, projection operators. This made life difficult because then a long story about how to solve a "double projection" equation. I finally gave up on that, though the operator technique should prove useful when I take up the thread again elsewhere.

The last burst: Showed how you can slide a "complete basis" into various forms, put it in the F-tensor basis which has two Ward Null Terms. Complete analysis of how you fit any object into any basis.

Then Ward analysis on the general F-tensor form. Result speaks for itself.

Method of Ward analysis here wasy this:

- a) obtain the 5 separate equations A,B,C,D,E
- b) replace A and D with A+D and A-D
- c) fractionate each equation into two equations using S_3 analysis.
- d) solve by hand the resultant set of 10 equations in 10 unknowns.

Retrospective Comments./

Look back at the original Ward Table which was obtained from the Standard Basis. There were 10 constraints on 14 variables. Thus, 4 degrees of freedom in the solution. Now I know where those four degrees are going:

2 go into a pair associated with the $F_{12}^{123} B_3$ coefficient function $S_1(12)$, and 1 goes into the totally symmetric piece of this function and 1 goes into $S(123)$, the Ward null BBB coefficient~~x~~.

Thus, if I had known what I was doing, once I showed those 10 constraints were linearly independent, I would have known right then that there were four constraints. I would have known that the two known Ward Null Terms ate up four degrees, so I would have known without doing anything else that there were no more Ward null terms! To show linear independence of the constraints, I could have deleted the columns of pairs and put the 10x10 thing into Reduce and computed the determinant. If $\text{Det} \neq 0$ then I know independent and can then add back the four columns and still will be independent.

Knowing Marshall's solution and knowing the Ward Nulls from the start, it was just a matter of counting.

BUT: The whole question is how do you count degrees of freedom! Apparently you must count "irreducible components" as degrees of freedom.

Why should you count both members of a function pair? You know you can get one from the other by an operator of symmetry. But as functions they are linearly independent. I.e., one is not a multiple of the other. But you can't change one without also changing the other.

So the question is: how do you know that if you solve all those equations, the functions that are supposed to be pairs really come out pairs? What guarantees this?? I think it is the symmetry; when you Clebsh up, you input the fact that the functions are pairs when you do your 12 combinations. So it must be that they couldn't come out any other way.

Similarly, in my Final Solution the objects R_1 and R_2 were treated as independent objects, but in the solution they were indeed pair-like. Again, the Clebshing built this in.

By the way, I could have had reduce solve these 10 equation in ten unknowns (again deleting the columns) . NO, that is wrong. Reduce cannot handle ~~11~~ 10 equations in 14 unknowns. So I don't know how I would have applied those constraints.

REduce could have solved my 6x6 problem in the final solution, but I did it by hand since most was so simple.

The real bitch problem is getting those extra degrees out of the problem so you have something you can solve. The best way to do this of course was to use the known Ward null terms!!!!!!

START 1

$X^2 - Y^2 \in \text{Reps of groups.}$

the "GOT" and projectors X_{ij}

Lichtenberg Notes

"the Matrix Problem"
(orthonormality)

Theremans added

Theremans on the X_{ij}

Problem Self Energy

Decompose of $f(123)$ into 6 pieces

Clebsch of $f(123)g(123)$

numerical notes

S_3 Basic Data

① The 36 ppp and S_3 tensor Forms.

② Impose $p_1 + p_2 + p_3 = 0$

③ The ppp vertex Terms

may + Choices

④ The $p\delta$ vertex Terms.

Result Sheet + Summary, 10 Terms

The 5 Ward Conditions
on the Ten Terms.

Clebsch Error

- Erroneous Sym. Fractionation

- Corrected Sym. Fractionation
of each Term.

Summary Table & Chart.

Page 1

START 5

B vectors to replace p ?
(false)

How to relate different
"Complete Sets"

a) standard
irreducible basis

b) F-tensor basis.

The [BBB] example.

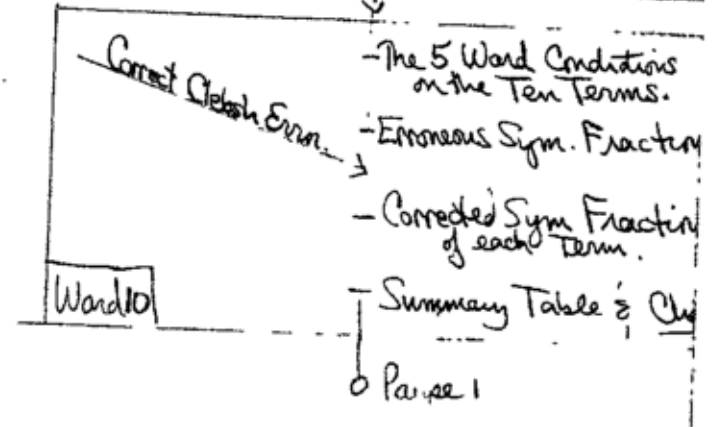
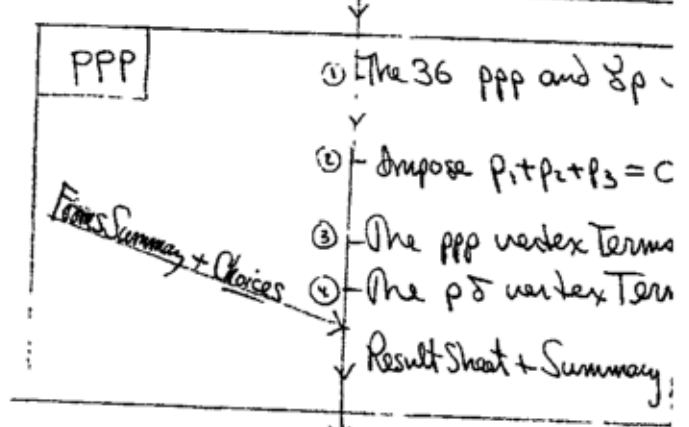
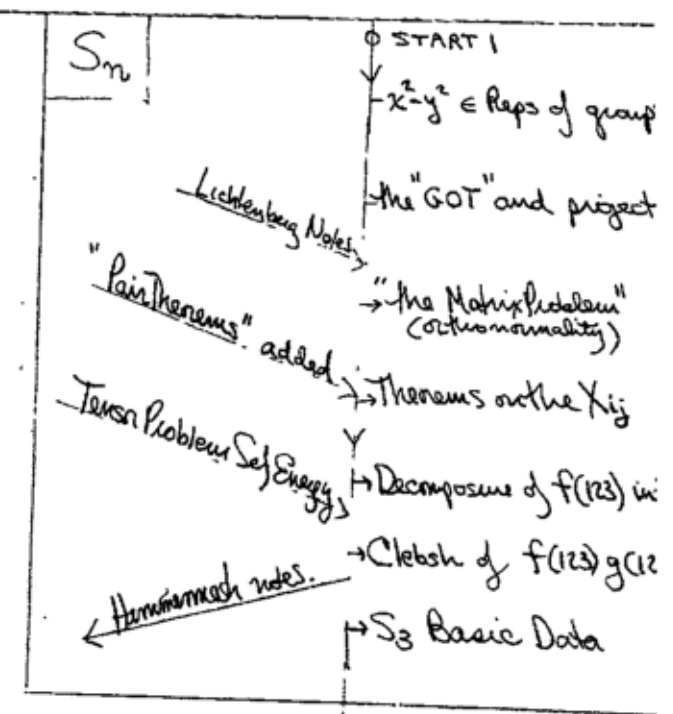
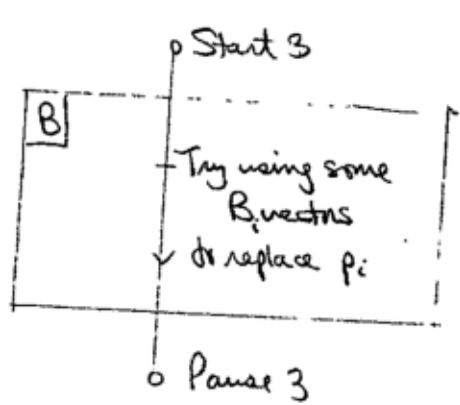
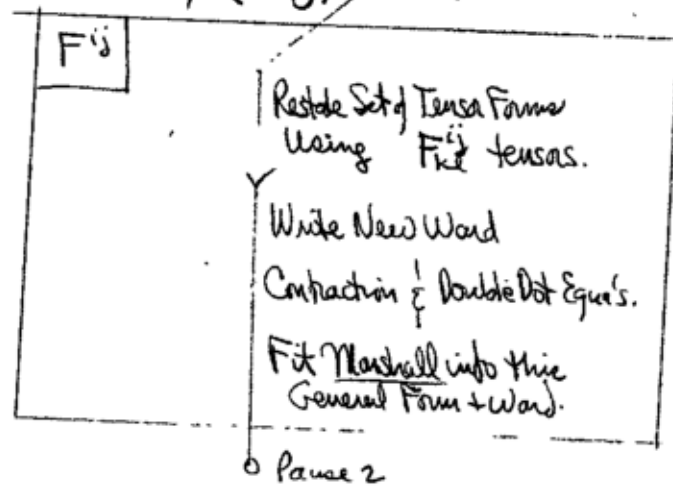
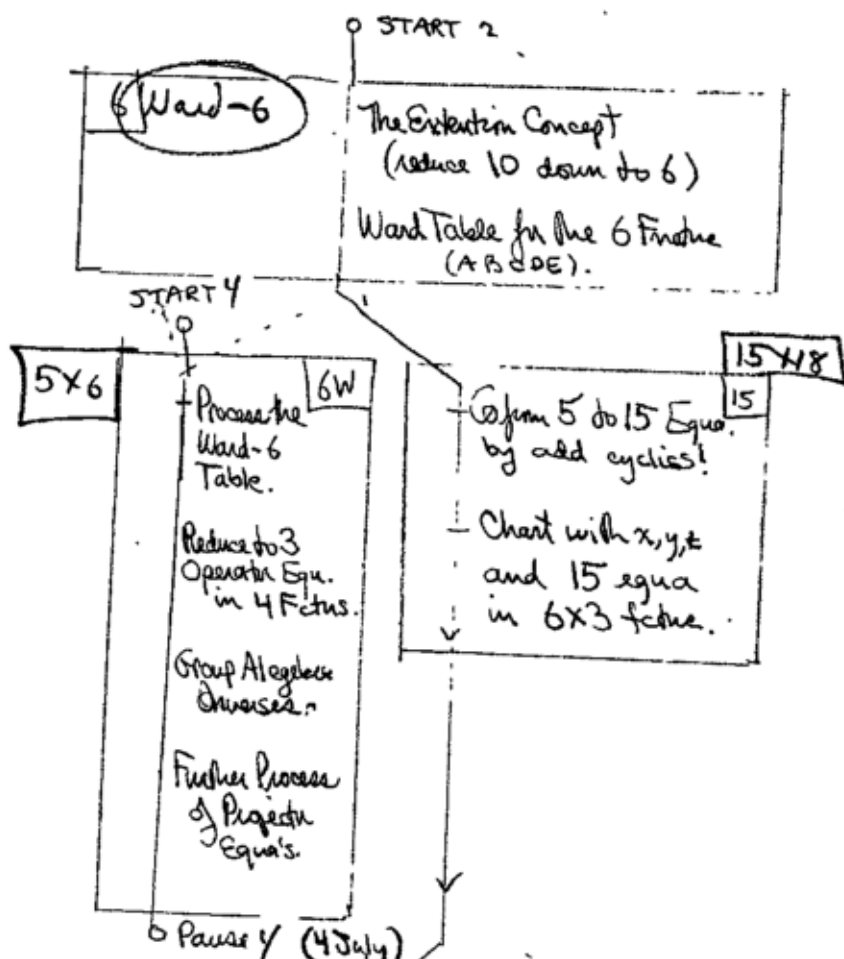
Nice Most General Form.

$A(1c), S(1c)$ analysis

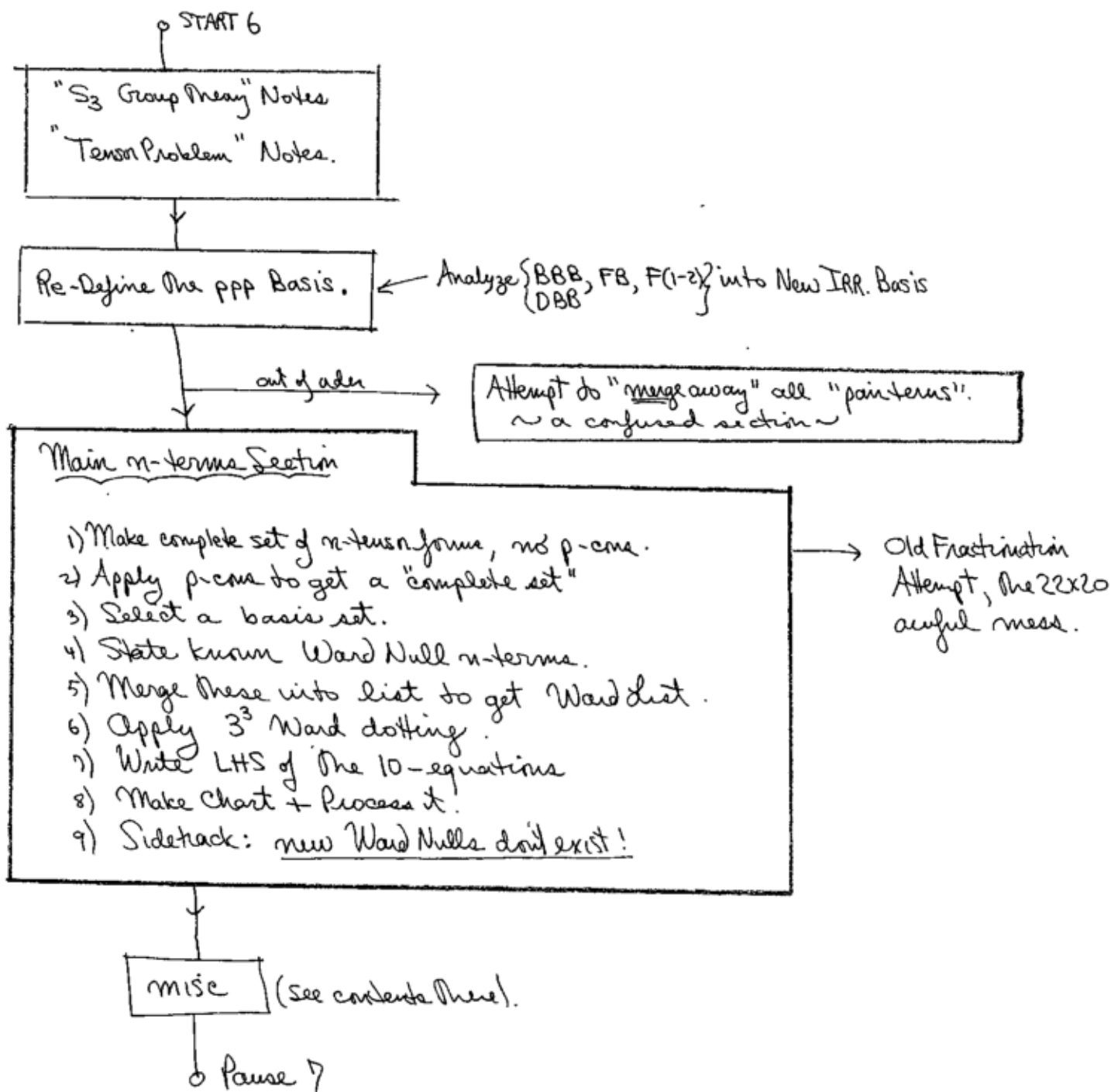
The Last Ward Analysis

One Result!

END



July 16, 1979.



$$x = (1, 2)$$

$$x + z = -(1, 1)$$

$$y = (2, 3)$$

$$y + x = -(2, 2)$$

$$z = (3, 1)$$

$$z + y = -(3, 3)$$

$$F_{12}^{12} = (1, 2) \delta^{12} - 2 \cdot 1^2$$

$$B_1^1 =$$

$$B_2^2 =$$

$$B_3^3 = (3, 2) 1^3 - (3, 1) 2^3$$

$$D_1^1 = (1, 1) 1^1 - (1, 1) 1^1$$

$$D_2^2 = (1, 2) 2^2 - (2, 2) 1^2$$

$$D_3^3 = (1, 3) 3^3 - (3, 3) 1^3$$

$S_n, X_i \dots$

Contents:

1. How to associate functions like $x^2 - y^2$ with UIR's of $SU(2)$ and discrete groups.
2. Interpretation of the Great Orthogonality Theorem:
 - a. the notion of X_{ij} "projectors"
 - b. example of S_3 group.
3. Left over details on Lichtenberg basis functions.
4. Solution to a Matrix Problem. (linearly combining the X_{11} etc to get orthogonal basis)
5. Theorems relating to projector construction.
 - a. $X_{ij}^+ = X_{ji}$
 - b. How to make new idempotents that are not projection operators.
 - c. How to interpret the Y and Y' operators of Hammermesh and others.
6. Earlier effort at decomposing a function into Irreducible pieces (only 4 pieces here)
7. The tensor problem for self-energy (ie, really only 2/1 momenta).
8. Counting triple glue amplitudes.
9. About multiplication of $f(123)$ by $g(123)$ and classification of the 36 pieces.
10. Basic S_3 data:
 - a. definitions of the X_{ij} projectors, their (vectors).
 - b. complete X multiplication table
 - c. the group multiplication table
 - d. The explicit 2×2 representation matrices
 - e. Table showing $R X_{ij}$ multiplications.
 - f. Decomposition of arbitrary element of group algebra.
 - g. the reason you really have six functions formable from $f(123)$, what they do.
 - h. Verification that Hamermesh's Clebsch table works for making symmetric combos.
 - i. more explicit $R X_2^1$ type multiplications.
 - j. how to generate M -pairs, mixed symmetry
11. The triple-glue tensor problem.
 1. classify raw tensor forms into symmetry classes
 2. Put $3 = -1 - 2$ and count independent tensor forms
 3. Write general triple-momenta vertex tensor forms
 4. Do the same for the g_{uv} forms.

summary of all tensor forms.

What does it mean to say that the function $f(x,y,z) = x$ "belongs" to the $L=1$ representation of $SU(2)$?

1. First, when I say that the QM operator X is a "vector operator", I mean this:

$$X'_i \equiv \hat{R}(g) X_i \hat{R}(g) = R_{ij}(g) X_j$$

If I sandwich this operator between two group elements as shown, I get a linear combination of the X 's back, and R_{ij} is the 3-dimensional rep of the rotation group in the xyz sense. This is what you mean when you say that X_i belongs to the $L=1$ representation.

2. A ket $|jm\rangle$ certainly "belongs" to the representation j with label m and by this you mean that:

$$\hat{R}(g) |jm\rangle = \sum_{m'} D_{m'm}^j(g) |jm'\rangle$$

If I were to close this with $\langle r|$ you would have:

$$\hat{R}(g) \langle \vec{r} | jm \rangle = \sum_{m'} D_{m'm}^j(g) \langle \vec{r} | jm' \rangle$$

So lets call this function by the name $f_{jm}(r)$. Then we have:

$$\hat{R}(g) f_{jm}(\vec{r}) = \sum_{m'} D_{m'm}^j(g) f_{jm'}(\vec{r})$$

A function which behaves under rotations in this way is said to be a function belonging to the " m^{th} row" of the representation j . Fine and dandy. On the other hand, we know that for any function the following is true:

$$\hat{R}(g) f(\vec{r}) = f(\hat{R}^{-1}(g) \vec{r})$$

3. Now, how might we search for functions $f_{jm}(r)$ that "belong" to various rep's. That's easy, here is the answer:

$$f_{jm}(\vec{r}) = F(r^2) Y_{jm}(\theta, \phi)$$

As you let F range over arbitrary functions of r , you exhaust all functions $f_{jm}(r)$.

4. Now for $j=0,1,2$ I will make a list of functions that "belong" to representations.

$j=0$: Any $F(r^2)$ will do. For example, r, r^2, e^{-r} all belong to the rep $j=0$.

$j=1$: Here are the relevant Y_{lm} 's: Let $F(r^2) = r$.

$$\left\{ \begin{array}{l} Y_{11} = -\sqrt{\frac{3}{8\pi}} \sin\theta e^{i\phi} \\ Y_{10} = \sqrt{\frac{3}{4\pi}} \cos\theta \\ Y_{1,-1} = +\sqrt{\frac{3}{8\pi}} \sin\theta e^{-i\phi} \end{array} \right\} \Rightarrow \begin{array}{l} f_{11}(\vec{r}) = [x+iy] \left(-\sqrt{\frac{3}{8\pi}}\right) \\ f_{10}(\vec{r}) = z \left(\sqrt{\frac{3}{4\pi}}\right) \\ f_{1,-1}(\vec{r}) = [x-iy] \left(+\sqrt{\frac{3}{8\pi}}\right) \end{array}$$

Of course we can change overall scale. This then is the usual set of functions which "belong" to the $L=1$ representation:

$$\left\{ \begin{array}{l} f_{11}(\vec{r}) = +\frac{1}{\sqrt{2}}(x+iy) \\ f_{10}(\vec{r}) = +z \\ f_{1,-1}(\vec{r}) = +\frac{1}{\sqrt{2}}(x-iy) \end{array} \right\} \Rightarrow x, y, z \text{ all } \in L=1.$$

$j=2$: Again, first we express the Y_{jm} 's. Now take r^2 as overall:

$$\left\{ \begin{array}{ll} Y_{22} = \frac{1}{4}\sqrt{\frac{15}{2\pi}} \sin^2\theta e^{2i\phi} \rightarrow \left(\frac{1}{4}\sqrt{\frac{15}{2\pi}}\right) [x^2-y^2+2ixy] & x^2-y^2 \\ Y_{21} = -\sqrt{\frac{15}{8\pi}} \sin\theta \cos\theta e^{i\phi} \rightarrow \left(-\sqrt{\frac{15}{8\pi}}\right) z[x+iy] & xy \\ Y_{20} = \sqrt{\frac{5}{4\pi}} \left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right) \rightarrow \left(\sqrt{\frac{5}{4\pi}}\right) \left[\frac{3}{2}z^2 - \frac{1}{2}r^2\right] & z^2 - r^2 \\ Y_{2,-1} = -Y_{21}^* \rightarrow \left(+\sqrt{\frac{15}{8\pi}}\right) z[x-iy] & zy \\ Y_{2,-2} = Y_{22}^* \rightarrow \left(\frac{1}{4}\sqrt{\frac{15}{2\pi}}\right) [x^2-y^2-2ixy] & 3z^2-r^2 \end{array} \right.$$

The five functions shown on the right are obtained by linear combination, so each of these 5 functions may be said to "belong" to $L=2$. The function x^2-y^2 is a combination of $|2,2\rangle$ and $|2,-2\rangle$, and $3z^2-r^2$ is entirely $|2,0\rangle$. Of course by combining you can get the other obvious functions:

$$\begin{array}{ll} z^2-y^2 & 3x^2-r^2 \\ z^2-x^2 & 3y^2-r^2 \end{array}$$

Notice of course that no linear combination can ever give you r^2 . That function does not belong to $L=2$, it belongs to $L=0$.

5. By the way, consider this direct product : $F_{ij} = x_i x_j$. You are combining two $L=1$ objects here, so you know you must get some $L=0, 1$ and 2 . But there is no $L=1$ because this tensor is symmetric, so only $L=0$ and $L=2$. For example the function x^2 is a mixture of $L=0$ and $L=2$ functions. You then have to linear combine to get functions that are completely $L=2$ such as $x^2 - y^2$.

Let me correct this: any function belonging to $L=1$ is proportional to Y_{1m} and any such function must change sign under parity. Obviously F_{ij} above does not change sign under parity, so no linear combination of the F_{ij} could give you an $L=1$. Forget the symmetry comment.

Consider $F_{ijk} = x_i x_j x_k$. In principle you could get $L=3, 2, 1, 0$. But parity will now only allow $L=3, 1$.

6. Suppose you are given an arbitrary function $f(r)$. How would you "project it" into its irreducible function components? First, for each L we can choose a set of basis functions like so:

$$f_{lm}(\vec{r}) = \langle \vec{r} | lm \rangle = F_L(r) Y_{lm}(\theta, \phi)$$

Ie, for each different L you can choose any $F_L(r)$ function you like, or you might set them all equal to 1, or perhaps r^L . Any way, then here is the decomposition:

$$f(\vec{r}) = \langle \vec{r} | f \rangle = \sum_{lm} \langle \vec{r} | lm \rangle \langle lm | f \rangle = \sum_{lm} f_{lm} f_{lm}(\vec{r})$$

where

$$\begin{aligned} f_{lm} &= \langle lm | f \rangle = \int d^3\vec{r} \langle lm | \vec{r} \rangle \langle \vec{r} | f \rangle \\ &= \int d^3\vec{r} \cdot f_{lm}^*(\vec{r}) f(\vec{r}). \end{aligned}$$

Having selected a set of basis functions, we can then get the weights f_{lm} . Then the original arbitrary function can be expanded in terms of functions which belong to UIRs.

7. Construction of a projection operator . First consider this:

$$\hat{R}(g) f_{jm}(\vec{r}) = \sum_{m'} D_{m'm}^j(g) f_{jm'}(\vec{r}).$$

$$\begin{aligned} \Rightarrow \int dg D_{n'n}^{j'}(g)^* \hat{R}(g) f_{jm}(\vec{r}) &= \sum_{m'} f_{jm'}(\vec{r}) \int dg D_{n'n}^{j'}(g) D_{m'm}^j(g) \\ &= \sum_{m'} f_{jm'}(\vec{r}) \cdot \frac{8\pi^2}{(2j+1)} \delta_{jj'} \delta_{n'n'} \delta_{mn} \\ &= 8\pi^2 (2j+1)^{-1} f_{jm'}(\vec{r}) \delta_{jj'} \delta_{mn} \end{aligned}$$

Now look what happens when you integrate against an arbitrary function:

$$\begin{aligned} &\int dg D_{n'n}^{j'}(g)^* \hat{R}(g) f(\vec{r}) dg \\ &= \int dg \sum_{jm} f_{jm} D_{n'n}^{j'}(g)^* \hat{R}(g) f_{jm}(\vec{r}) \\ &= \sum_{jm} f_{jm} 8\pi^2 (2j+1)^{-1} f_{jn'}(\vec{r}) \delta_{jj'} \delta_{nn} \\ &= \frac{8\pi^2}{(2j+1)} f_{jn} f_{jn'}(\vec{r}) \end{aligned}$$

Repeat: $\int dg D_{mm}^j(g)^* \hat{R}(g) f(\vec{r}) = \frac{8\pi^2}{(2j+1)} f_{jm} f_{jm}(\vec{r})$

So

$$P^{jm} = \frac{(2j+1)}{8\pi^2} \int dg D_{mm}^j(g)^* \hat{R}(g)$$

Obviously this is the projection operator which, when applied to any function, projects out the portion which belongs to jm . You need to know the diagonal matrix elements and the operators.

8. If you take the preceding projection operator and sum over m , you get a new projection operator which does this: it projects out all functions which belong to any row of the representation. So if you don't care about connecting functions with particular "rows", you can use this:

$$P^j \equiv \frac{(2j+1)}{8\pi^2} \int dg \left[\sum_m D_{mm}^{j*}(g) \right] \hat{R}(g) = \frac{(2j+1)}{8\pi^2} \int dg \chi^j(g) \hat{R}(g)$$

Here we have identified the trace of the representation as the character. I don't know the rotation group characters except for special cases of g , so perhaps this is not too useful for continuous groups.

9. Extension of these ideas to Discrete Groups. I can just quote the projection operators etc from Tinkham, all obvious once you know about orthogonality. We have:

R = group element, replaces label g used above.

P_R = the transformation operator, replaces $R(g)$ used before. I.e.,

$$P_R \cdot f(r) = f(R^{-1}r)$$

This operation can be a rotation or reflection or whatever.

$\Gamma^j(R)_{mm}$ = element of representation, called $D_{mm}^j(g)$ in the above.

Now here are the basic facts.

(a) Arbitrary function expanded in terms of basis functions:

$$f(\vec{r}) = \sum_{jm} F_{jm}(\vec{r}) \quad (3-33)$$

(b) Transformation rule for basis functions:

$$P_R F_{jm}(\vec{r}) = \sum_{m'} \Gamma_{mm'}^{(j)}(R) F_{jm'}(\vec{r}) \quad (3-29)$$

(c) the projection operators:

$$P^{jm} = \left(\frac{l_j}{h} \right) \sum_R \Gamma^{(j)}(R)_{mm}^* P_R \quad (3-31)$$

$$P^j = \left(\frac{l_j}{h} \right) \sum_R \chi^j(R)^* P_R \quad (3-37)$$

Here l_j is dimensionality of the rep, and h is order of the group. Note that $(2j+1)$ does in fact appear for the continuous group as well, but not h .

Interpretation of the Great Orthogonality Theorem.

1. For finite groups, here is what the theorem says:

$$\sum_R D_{il}^{(\mu)}(R) D_{jm}^{(\nu)*}(R) = \frac{h}{l_\mu} \delta_{\mu\nu} \delta_{ij} \delta_{lm}.$$

Now, let's define some elements of the group algebra:

$$\begin{aligned} X_{il}^\mu &\equiv \left(\sum_R D_{il}^{(\mu)}(R) R \right) \frac{l_\mu}{h} & \langle X_{il}^\mu, X_{jm}^\nu \rangle &= \sum_R D_{il}^{(\mu)}(R) D_{jm}^{(\nu)*}(R) \frac{l_\mu}{h} \frac{l_\nu}{h} \\ X_{jm}^\nu &\equiv \left(\sum_R D_{jm}^{(\nu)}(R) R \right) \frac{l_\nu}{h} & &= \frac{l_\mu}{h} \delta_{\mu\nu} \delta_{ij} \delta_{lm}. \end{aligned}$$

Clearly, these two vectors in the group algebra are orthogonal unless all three sets of indices are equal. See Pozzi (2.10) for definition of group algebra scalar product. Think of the various R as unit vectors and the coefficients as components of vector. Then the line above states precisely that these two vectors are orthogonal.

2. Thus, each UIR of a finite group provides n vectors which are (1) orthogonal to each other, and (2) orthogonal to the vectors of all the other UIR's. Clearly these are the spanning vectors of the regular representation!

3. Example: I am now going to construct the 4 spanning vectors which are related to the two-dim UIR's of the group S_3 . First, here are the matrices:

$$\begin{aligned} D_{ij}^{(2)}(e) &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & D_{ij}^{(2)}((13)) &= \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} & D_{ij}^{(1)}((123)) &= \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} & (11) &= \\ & & & & & (11)(11) &= AB = 0 \\ D_{ij}^{(2)}((12)) &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} & D_{ij}^{(2)}((23)) &= \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} & D_{ij}^{(1)}(132) &= \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} \end{aligned}$$

I have read these from page 8 Tinkham, but I could have gotten them from H's large table. Now I will construct the 4 basis vectors!

$$X_{11}^{(4)} = e - \frac{1}{2}(13) + \frac{1}{2}(123) + (12) - \frac{1}{2}(23) - \frac{1}{2}(132)$$

$$X_{12}^{(2)} = -\frac{\sqrt{3}}{2}(13) + \frac{\sqrt{3}}{2}(123) + \frac{\sqrt{3}}{2}(23) - \frac{\sqrt{3}}{2}(132)$$

etc.

Restate these with shortest form:

	fac.	e	(12)	(13)	(23)	(123)	(132)	
<i>first row</i> $X_{11}^{(3)}$	$\frac{1}{3} \times \frac{1}{2}$	2	2	-1	-1	-1	-1	$(Y_1 + Y'_1)$
$X_{12}^{(2)}$	$\frac{1}{3} \times \frac{1}{2}$	0	0	$-\sqrt{3}$	$\sqrt{3}$	$\sqrt{3}$	$-\sqrt{3}$	
<i>second row</i> $X_{21}^{(2)}$	$\frac{1}{3} \times \frac{1}{2}$	0	0	$-\sqrt{3}$	$\sqrt{3}$	$-\sqrt{3}$	$\sqrt{3}$	$(Y_1 - Y'_1)$
$X_{22}^{(1)}$	$\frac{1}{3} \times \frac{1}{2}$	2	-2	1	1	-1	-1	

You can see by inspection that these four vectors are orthogonal in the group algebra. If we multiply each of these by $1/\sqrt{3}$, they will be orthonormal.

Warning: Just because two vectors in the group algebra are orthogonal, that does not mean that if you group-multiply the two vectors you will get zero! You can sort of imagine the group multiplication as a cross product. $\hat{x} \cdot \hat{y} = 0 \Rightarrow \hat{x} \times \hat{y} = 0!$

4. Now forget that ~~part 3~~ above and consider this: the first object above, X_{11} , may be regarded as a 'first row' object. Why? Consider:

$$\begin{aligned}
 R X_{ij}^{\mu} &= R \sum_s D_{ij}^{\mu*}(s) S = \sum_s D_{ij}^{\mu*}(s) R S = \sum_s D_{ij}^{\mu*}(\bar{R} R S) R S \\
 &= \sum_T D_{ij}^{\mu*}(\bar{R} T) T = \sum_T \sum_K D_{ik}^{\mu*}(\bar{R}) D_{kj}^{\mu*}(T) T \\
 &= \sum_K D_{ik}^{\mu*}(\bar{R}) \cdot \sum_T D_{kj}^{\mu*}(T) T = \sum_K D_{ik}^{\mu*}(\bar{R}) X_{kj}^{\mu} \\
 &= \sum_K D_{ki}^{\mu*}(R) X_{kj}^{\mu}
 \end{aligned}$$

note that R does not mix copies!!

Thus, we can regard our X_{ij} objects in this way:

$$X_{ij}^{\mu} = X_i^{(\mu, j)} = \text{object in the } i^{\text{th}} \text{ row of the } j^{\text{th}} \text{ copy of VIR } \mu.$$

5. Now we can apply page 40 of Tinckham. Probably I should renormalize the X_{ij} objects so that they are exactly the objects of Tinckham (3131). Done.

$$P_{ij}^{(\mu)} = X_{ij}^{\mu}$$

$$g_k^{(8)} = X_k^{(8, j)}$$

Using Tinckham 3-30 applied here we get:

$$P_{ij}^{(\mu)} \phi_k^{(\nu)} = \delta_{\mu\nu} \cdot \delta_{jk} \phi_i^{(\mu)}$$

If differing IUR's you get zero. If the same UIR, then consider P_{ij} operator. If this acts on a function belonging to the row k , you get zero unless $k=j$. In that case you get precisely ϕ_i .

Now translate this to our present notation and situation:

$$X_{ij}^{(\mu)} X_{kl}^{(\nu, l)} = \delta_{\mu\nu} \delta_{jk} X_i^{(\nu, l)}$$

$$\Rightarrow X_{ij}^{(\mu)} X_{kl}^{(\mu)} = \delta_{jk} X_i^{(\mu)} \quad \forall i, j, k, l.$$

$$\Rightarrow \begin{cases} X_{11}^{(\mu)} X_{11}^{(\mu)} = X_{11}^{(\mu)} & X_{22} X_{11} = 0 \\ X_{11}^{(\mu)} X_{12}^{(\mu)} = X_{12}^{(\mu)} & X_{22} X_{12} = 0 \\ X_{11}^{(\mu)} X_{21}^{(\mu)} = 0 & X_{22} X_{21} = X_{21} \\ X_{11}^{(\mu)} X_{22}^{(\mu)} = 0 & X_{22} X_{22} = X_{22}. \end{cases}$$

This shows that X_{11} and X_{22} truly are projection operators.

Notice that, although the group algebra vectors X_{11} and X_{12} are orthogonal, you do not get zero when you multiply these two objects. Don't make that mistake.

Continuing:

$$\begin{array}{ll} \text{now } X_{12} X_{11} = 0 & X_{21} X_{11} = X_{21} \\ X_{12} X_{12} = 0 & X_{21} X_{12} = X_{22} \\ X_{12} X_{21} = X_{11} & X_{21} X_{21} = 0 \\ X_{12} X_{22} = X_{12} & X_{21} X_{22} = 0 \end{array}$$

Existence of Hermitian Adjoints of Group Algebra elements. Let us assume that we have a scalar product of some sort which is invariant in this sense:

$$(f, g) = (Rf, Rg) \quad \forall R \in \text{group}.$$

Obviously, the operators R are unitary with respect to this scalar product, $R^+ = R^{-1}$. This serves really to define R^+ , since the R^{-1} are known just from the group data.

Now consider an element of the group algebra,

$$\begin{aligned} A = \sum_R a(R) R &\Rightarrow A^+ = \sum_R \overline{a(R)} R^+ = \sum_R \overline{a(R)} R^{-1} \\ &= \sum_R \overline{a(R^{-1})} R = A^* \text{ of Pozzi (involution).} \end{aligned}$$

Thus, the Hermitian Adjoint of an element of the group algebra is precisely its involution, as defined by Pozzi. Now:

Example: here are some Hermitian conjugates:

$$R = (ij) \rightarrow \tilde{R} = (ij)$$

$$R = (ijk) = (ij)(jk) \Rightarrow \tilde{R} = (jk)(ji) = (kji) = (ikj)$$

$$\left\{ \begin{aligned} X_{11}^+ &= X_{11} \\ X_{22}^+ &= X_{22} \\ X_{12}^+ &= \frac{1}{6} [-\sqrt{3}(13) + \sqrt{3}(23) - \sqrt{3}(123) + \sqrt{3}(132)] = X_{21} \\ X_{21}^+ &= X_{12}. \end{aligned} \right.$$

Armed with this information, we can now investigate the orthogonality or lack thereof of basis functions.

Copy 1

Copy 2



line = orthog.

$$(X_{11}f, X_{21}f) = (f, X_{11}X_{21}f) = 0$$

These results tell us that within each copy, the basis functions are orthogonal.

$$(X_{12}f, X_{22}f) = (f, X_{12}X_{22}f) = 0$$

$$(X_{11}f, X_{22}f) = (f, X_{11}X_{22}f) = 0$$

And these tell us that across the two copies, different rows are orthogonal.

$$(X_{12}f, X_{21}f) = (f, X_{21}X_{12}f) = 0$$

$$(X_{11}f, X_{12}f) = (f, X_{11}X_{12}f) = (f, X_{12}f) \neq 0 \dots = (X_{21}f, X_{22}f)$$

This last result says, however, that the two row-1 functions of the two copies need not be orthogonal!

Theorem: The fact that the projection operators X_{ij}^u are orthogonal in the group algebra implies that the functions which they project out of some carrier space are also orthogonal, provided the carrier scalar product is "invariant", in the carrier space.

Proof:

$$\begin{aligned}
 (X_{ie}^u f, X_{jm}^v f) &\stackrel{\text{carrier sc. prod.}}{=} \frac{\delta_{\mu\nu}}{h^2} \sum_{RS} D_{ie}^{(\mu)}(R) D_{jm}^{(\nu)}(S)^* \underbrace{(Rf, Sf)}_{= (\bar{S}^i R f, f)} \\
 &= \frac{\delta_{\mu\nu}}{h^2} \sum_{RS} D_{ie}^{(\mu)}(S \bar{S}^i R) D_{jm}^{(\nu)}(S)^* (\bar{S}^i R f, f) \\
 &= \frac{\delta_{\mu\nu}}{h^2} \sum_{ST} D_{ie}^{(\mu)}(ST) D_{jm}^{(\nu)}(S)^* (Tf, f) \\
 &= \frac{\delta_{\mu\nu}}{h^2} \sum_{ST} D_{im}^{(\mu)}(S) D_{el}^{(\mu)}(T) D_{jm}^{(\nu)}(S)^* (Tf, f) \\
 &= \frac{\delta_{\mu\nu}}{h^2} \sum_T D_{el}^{(\mu)}(T) (Tf, f) \cdot \underbrace{\left\{ \sum_S D_{im}^{(\mu)}(S) D_{jm}^{(\nu)}(S)^* \right\}}_{\frac{h}{\delta_{\mu}} \delta_{\mu\nu} \delta_{ij} \delta_{nm}} \\
 &= \underbrace{\frac{\delta_{\mu\nu}}{h} \delta_{\mu\nu} \delta_{ij} \delta_{nm}}_{\langle X_{im}^u, X_{jm}^v \rangle} \cdot \underbrace{\sum_T D_{el}^{(\mu)}(T) (Tf, f)}_{(X_{ml}^u f, f)} \\
 &= \frac{\delta_{\mu\nu}}{h} \delta_{\mu\nu} \delta_{ij} (X_{ml}^u f, f)
 \end{aligned}$$

Ouch. This says you are guaranteed orthogonality only if your ~~projector~~ projection-like operators are in different rows. If they correspond to the same row, but different copies, of the same UIR, you are not guaranteed orthogonality of the functions, even though the projectors are orthogonal in the group algebra.

Mixing Table (Licht)

Lichtenberg information, save

	e	(12)	(13)	(23)	(123)	(132)
$X_{11} =$	$\frac{1}{3}$	$\frac{1}{3}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{1}{6}$
$X_{12} =$	0	0	$-\frac{1}{2\sqrt{3}}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{2\sqrt{3}}$	$-\frac{1}{2\sqrt{3}}$
$X_{21} =$	0	0	$-\frac{1}{2\sqrt{3}}$	$\frac{1}{2\sqrt{3}}$	$-\frac{1}{2\sqrt{3}}$	$+\frac{1}{2\sqrt{3}}$
$X_{22} =$	$\frac{1}{3}$	$-\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{1}{6}$

$$\psi_1 = 1 \quad 1 \quad -1 \quad 0 \quad -1 \quad 0 \quad \checkmark = 3X_{11} + \sqrt{3}X_{21}$$

$$\psi_2 = 1 \quad 1 \quad 0 \quad -1 \quad 0 \quad -1 \quad \checkmark = 3X_{11} - \sqrt{3}X_{21}$$

$$\psi_3 = 1 \quad -1 \quad 1 \quad 0 \quad 0 \quad -1 \quad \checkmark = 3X_{22} - \sqrt{3}X_{21}$$

$$\psi_4 = 1 \quad 0 \quad 1 \quad -1 \quad -1 \quad 0 \quad = \frac{3}{2}(X_{11} + X_{22})$$

$$3X_{11} = 1 \quad 1 \quad -\frac{1}{2} \quad -\frac{1}{2} \quad -\frac{1}{2} \quad -\frac{1}{2}$$

$$\sqrt{3}X_{12} = 0 \quad 0 \quad -\frac{1}{2} \quad \frac{1}{2} \quad \frac{1}{2} \quad -\frac{1}{2}$$

$$\sqrt{3}X_{21} = 0 \quad 0 \quad -\frac{1}{2} \quad \frac{1}{2} \quad -\frac{1}{2} \quad \frac{1}{2}$$

$$3X_{22} = 1 \quad -1 \quad \frac{1}{2} \quad \frac{1}{2} \quad -\frac{1}{2} \quad -\frac{1}{2}$$

$$-\frac{3\sqrt{3}}{2}X_{12} - \frac{\sqrt{3}}{2}X_{21}$$

$$\frac{3X_{11} + 3X_{22}}{2} = 1 \quad 0 \quad 0 \quad 0 \quad -\frac{1}{2} \quad -\frac{1}{2}$$

$$\Delta = 0 \quad 0 \quad 1 \quad -1 \quad -\frac{1}{2} \quad \frac{1}{2}$$

$$\psi = \psi_1 - 2\psi_2 = -3X_{11} + 3\sqrt{3}X_{21}$$

$$\phi = 2\psi_3 + \frac{1}{2}\psi_1 + \frac{1}{2}\psi = 6X_{22}$$

$$\chi = -\frac{6}{\sqrt{3}}X_{12}$$

Relativ Licht.

$$\psi_1 = 3X_{11} + \sqrt{3}X_{21} = (11-10-10)$$

$$\psi_2 = 3X_{11} - \sqrt{3}X_{21} = (110-10-1)$$

$$\psi_3 = 3X_{22} - \sqrt{3}X_{21} = (1-1100-1)$$

$$\psi_4 = \frac{3}{2}X_{11} + \frac{3}{2}X_{22} - \frac{3}{2}\sqrt{3}X_{12} - \frac{1}{2}\sqrt{3}X_{21} = (101-1-10)$$

$$\begin{aligned}\psi_1\psi_1 &= (3X_{11} + \sqrt{3}X_{21})(3X_{11} + \sqrt{3}X_{21}) \\ &= 9X_{11} + 3\sqrt{3}X_{21} = 3(3X_{11} + \sqrt{3}X_{21}) = 3\psi_1\end{aligned}$$

$$\begin{aligned}\psi_1\psi_2 &= (3X_{11} + \sqrt{3}X_{21})(3X_{11} - \sqrt{3}X_{21}) = \\ &= 9X_{11} + 3\sqrt{3}X_{21} = 3(3X_{11} + \sqrt{3}X_{21}) = 3\psi_1\end{aligned}$$

$$\begin{aligned}\psi_1\psi_3 &= (3X_{11} + \sqrt{3}X_{21})(3X_{22} - \sqrt{3}X_{21}) \\ &= 0 + 0 + 0 + 0 = 0.\end{aligned}$$

$$\begin{aligned}\psi_1\psi_4 &= (3X_{11} + \sqrt{3}X_{21})\left(\frac{3}{2}X_{11} + \frac{3}{2}X_{22} - \frac{3}{2}\sqrt{3}X_{12} - \frac{1}{2}\sqrt{3}X_{21}\right) \\ &= \frac{9}{2}X_{11} + \frac{9}{2}\sqrt{3}X_{12} + \frac{3}{2}\sqrt{3}X_{21} + \frac{9}{2}X_{22} - \frac{9}{2}\sqrt{3}X_{12} - \frac{3}{2}\sqrt{3}X_{21} \\ &\quad + \frac{3}{2}\sqrt{3}X_{21} + \frac{3}{2}\sqrt{3}X_{22} - \frac{9}{2}\sqrt{3}X_{12} - \frac{3}{2}\sqrt{3}X_{21}\end{aligned}$$

1. Here is a statement of the problem: you have a 4-dimensional vector space which is spanned by four vectors called e_1, e_2, e_3, e_4 . These vectors are not orthogonal nor are they unit vectors. In fact, their dot products are described by this matrix:

$$\begin{pmatrix} \vec{e}_1 \\ \vec{e}_2 \\ \vec{e}_3 \\ \vec{e}_4 \end{pmatrix} \cdot (\vec{e}_1, \vec{e}_2, \vec{e}_3, \vec{e}_4) = \begin{pmatrix} \vec{e}_1 \cdot \vec{e}_1 & \vec{e}_1 \cdot \vec{e}_2 & & \\ & \text{etc} & & \end{pmatrix} = \begin{pmatrix} a & 0 & c & 0 \\ 0 & a & 0 & c \\ c & 0 & b & 0 \\ 0 & c & 0 & b \end{pmatrix} = \left(\begin{array}{cc|cc} a & 1 & c & 1 \\ c & 1 & b & 1 \end{array} \right)$$

Before continuing, here is the correspondence between these symbols and earlier work:

$$\begin{aligned} \vec{e}_1 &= |f_{11}\rangle \equiv X_{11}|f\rangle & \vec{e}_1 \cdot \vec{e}_1 &= \langle X_{11}f | X_{11}f \rangle = \langle f | X_{11} | f \rangle \equiv \langle X_{11} \rangle \equiv a \\ \vec{e}_2 &= |f_{21}\rangle \equiv X_{21}|f\rangle & \vec{e}_2 \cdot \vec{e}_2 &= \langle X_{21}f | X_{21}f \rangle = \langle X_{11} \rangle = a \\ \vec{e}_3 &= |f_{12}\rangle \equiv X_{12}|f\rangle & \vec{e}_3 \cdot \vec{e}_3 &= \langle X_{22} \rangle = b = \vec{e}_4 \cdot \vec{e}_4 \\ \vec{e}_4 &= |f_{22}\rangle \equiv X_{22}|f\rangle & \vec{e}_1 \cdot \vec{e}_3 &= \langle X_{12} \rangle = c = \vec{e}_2 \cdot \vec{e}_4 = \langle X_{21} \rangle \end{aligned}$$

In addition, the vectors e_1 and e_2 span an invariant subspace of the four-dim space, and the vectors e_3, e_4 span another invariant subspace. In fact, the action of group elements on these two subspaces is the same, namely:

$$R \begin{pmatrix} \vec{e}_1 \\ \vec{e}_2 \\ \vec{e}_3 \\ \vec{e}_4 \end{pmatrix} = \begin{pmatrix} \Gamma(R) & 0 \\ 0 & \Gamma(R) \end{pmatrix} \begin{pmatrix} \vec{e}_1 \\ \vec{e}_2 \\ \vec{e}_3 \\ \vec{e}_4 \end{pmatrix}$$

Here, $\Gamma(R)$ is the two-dim representation of S_3 we have been using. On page 2 of earlier notes, I showed that the action of group element R in the objects X_{ij} was such that the action is exactly the same on either "copy". In new notation, e_1, e_2 span the first copy (subspace), and e_3, e_4 span the second copy.

Now here is the problem: Is it possible to do a transformation on the basis vectors to get some new basis vectors such that

- (a) the new basis vectors are mutually orthogonal, and
- (b) the action of R like that above but on new basis is still in block form.

2. Discussion of the situation: As we shall see below, a solution must exist to the problem of finding an orthogonal basis in this 4-dim space. This is completely obvious, and will be shown to be just the problem of diagonalizing a real-symmetric matrix.

The real question is: once you have obtained an orthogonal basis, how do you know your group action will have block form? Certainly the two invariant subspaces will still exist, but not obvious that the vectors you obtain by orthogonalizing will be the same vectors which, in pairs, span the subspace. In fact, it is not obvious that you can have block form and orthogonality at the same time. Lets return to this discussion at the end. It turns out that you can have both at the same time, though now I don't know why based on general principles, only based on this calculation.

3. The orthogonality calculation. Here is how I propose to transform the basis vectors:

$$\begin{pmatrix} \vec{e}_1' \\ \vec{e}_2' \\ \vec{e}_3' \\ \vec{e}_4' \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \vec{e}_1 \\ \vec{e}_2 \\ \vec{e}_3 \\ \vec{e}_4 \end{pmatrix}$$

$$\begin{pmatrix} \vec{e}_1' \\ \vec{e}_2' \\ \vec{e}_3' \\ \vec{e}_4' \end{pmatrix} \cdot (\vec{e}_1' \vec{e}_2' \vec{e}_3' \vec{e}_4') = \begin{pmatrix} h_1 & h_2 & 0 \\ 0 & h_3 & h_4 \\ 0 & h_4 & h_1 \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} a1 & c1 \\ c1 & b1 \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix}^T$$

It is our goal to find most general form of the matrix ABCD. You can see that it is just the problem of diagonalizing a real, symmetric matrix. You know therefore that the matrix $M = ABCD$ must be real orthogonal so that $M^T = M^{-1}$. Thus, we get:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} a1 & c1 \\ c1 & b1 \end{pmatrix} = \begin{pmatrix} h_+ & 0 \\ 0 & h_- \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

$$\Rightarrow \begin{aligned} 1. \quad A &= \left[\frac{-c}{(a-h_+)} \right] B & 3. \quad C &= \left[\frac{-c}{(a-h_-)} \right] D \\ 2. \quad A &= \left[\frac{-(b-h_+)}{c} \right] B & 4. \quad C &= \left[\frac{-(b-h_-)}{c} \right] D \end{aligned}$$

These four equations arise just by multiplying out the matrices shown. I have anticipated the fact that the eigenvalues are equal in pairs, which I will show in next section.

4. Determination of Eigenvalues. Just outline it as follows;

$$\begin{vmatrix} a-h & 0 & c & 0 \\ 0 & a-h & 0 & c \\ c & 0 & b-h & 0 \\ 0 & c & 0 & b-h \end{vmatrix} = 0$$

$$h_+ + h_- = a + b$$

$$h_+ h_- = ab - c^2$$

$$\Rightarrow x[x - 2c^2] + c^4 = 0 \text{ where } x \equiv (a-h)(b-h).$$

$$\Downarrow x^2 - 2xc^2 + c^4 = 0 \Rightarrow (x - c^2)^2 = 0 \Rightarrow x = c^2 \text{ twice.}$$

$$\Rightarrow (h-a)(h-b) = c^2 \Rightarrow h_{\pm} = \frac{1}{2}[(a+b) \pm \sqrt{a^2 + b^2 + 4c^2}]$$

Off hand I don't know how to deduce that the eigenvalues are equal in pairs without doing it; there must be some fact about the matrix which makes it true by inspection/.

Now we define some auxiliary symbols and state relations between them:

$$\alpha_{\pm} \equiv \frac{-c}{(a-h_{\pm})} = \frac{(b-h_{\pm})}{-c}$$

$$(\alpha_+ \alpha_- + 1) = 0$$

$$\text{since } (a-h_+)(a-h_-) + c^2 = 0.$$

$$\beta_{\pm} \equiv \frac{1}{\sqrt{\alpha_{\pm}^2 + 1}}$$

$$(\alpha_+ + \alpha_-) = \frac{a-b}{c}$$

In the first line, the equality tells us that equations 2. and 4. on page 2 are redundant, so our only conditions so far on ABCD are:

$$5. A = \alpha_+ B \quad 6. C = \alpha_- D$$

5. But, we have in the above already assumed that M was real orthogonal. We used this fact to right multiply and equation by M. This fact implies further conditions on ABCD as follows:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} A^T & C^T \\ B^T & D^T \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow \begin{matrix} 7. AA^T + BB^T = 1 & AC^T + BD^T = 0 \end{matrix} \quad 9.$$

$$8. CC^T + DD^T = 1 \quad (CA^T + DB^T = 0) \quad 10. \text{redundant.}$$

So, equations 7., 8., 9. are the conditions that the 2x2 matrices A, B, C, D must satisfy in order that matrix M be real orthogonal. Our goal now is to find a set of matrices which satisfy 5, 6, 7, 8, 9. Then we are done.

Inserting 5 and 6 into 7, 8, 9 we get:

$$11. \quad BB^T = \beta_+^2 \quad 12. \quad DD^T = \beta_-^2 \quad 13. \quad ((\alpha_+ \alpha_- + 1) BD^T = 0)_{0=0}.$$

This tells us that matrices B and D are "almost real orthogonal". Lets then define new versions:

$$\begin{aligned} \tilde{B} &\equiv B/\beta_+ & \Rightarrow & \tilde{B}\tilde{B}^T = 1 & A &= \alpha_+ \beta_+ \tilde{B} \\ \tilde{D} &\equiv D/\beta_- & \Rightarrow & \tilde{D}\tilde{D}^T = 1 & C &= \alpha_- \beta_- \tilde{D} \end{aligned}$$

Thus, here is our most general solution to the orthogonalization problem:

$$\begin{aligned} M &= \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right) = \left(\begin{array}{c|c} \alpha_+ \beta_+ \tilde{A}_+ & \beta_+ \tilde{A}_+ \\ \hline \alpha_- \beta_- \tilde{A}_- & \beta_- \tilde{A}_- \end{array} \right) \quad \text{where } \tilde{A}_+, \tilde{A}_- \text{ are arbitrary} \\ & \quad \text{real, orthog matrices.} \\ &= \left(\begin{array}{cc} A_+ & 0 \\ 0 & A_- \end{array} \right) \left(\begin{array}{c|c} \alpha_+ \beta_+ 1 & \beta_+ 1 \\ \hline \alpha_- \beta_- 1 & \beta_- 1 \end{array} \right). \end{aligned}$$

Here you see that the matrix on the right solves the problem giving orthogonal basis vectors. Then A_+ and A_- separately rotate basis vectors within the two subspaces. Obviously these extra arbitrary rotations cannot change the orthogonality.

6. What about the invariant subspaces? Consider the effect of matrix on the right above. We get:

$$\begin{aligned} R \begin{pmatrix} e_1' \\ e_2' \\ e_3' \\ e_4' \end{pmatrix} &= M R \begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{pmatrix} = \left[M \begin{pmatrix} \Gamma & 0 \\ 0 & \Gamma \end{pmatrix} M^T \right] \begin{pmatrix} e_1' \\ e_2' \\ e_3' \\ e_4' \end{pmatrix} \\ &= \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right) \begin{pmatrix} \Gamma & 0 \\ 0 & \Gamma \end{pmatrix} \begin{pmatrix} A^T & C^T \\ B^T & D^T \end{pmatrix} = \begin{pmatrix} A\Gamma A^T + B\Gamma B^T & A\Gamma C^T + B\Gamma D^T \\ C\Gamma A^T + D\Gamma B^T & C\Gamma C^T + D\Gamma D^T \end{pmatrix} \\ &= \left(\begin{array}{c|c} A_+ \Gamma A_+^T & 0 \\ \hline 0 & A_- \Gamma A_-^T \end{array} \right) \end{aligned}$$

Thus, you see that block form is maintained: vectors e_1' and e_2' span one subspace, and the other two span the other subspace. The effect of applying A_+ and A_- is imply to convert to equivalent new two-dimensional representation of S_3 .

I see no reason to bother with A_+ and A_- . I will remain aware that within each subspace you can rotate to your hearts content and get equivalent representations.

Here then is the simplest solution:

$$\vec{e}_1' = \beta_+ (\alpha_+ \vec{e}_1 + \vec{e}_3)$$

$$\vec{e}_2' = \beta_+ (\alpha_+ \vec{e}_2 + \vec{e}_4)$$

$$\vec{e}_3' = \beta_- (\alpha_- \vec{e}_1 + \vec{e}_3)$$

$$\vec{e}_4' = \beta_- (\alpha_- \vec{e}_2 + \vec{e}_4)$$

$$\vec{e}_1' \cdot \vec{e}_1' = \vec{e}_2' \cdot \vec{e}_2' = h_+$$

$$\vec{e}_3' \cdot \vec{e}_3' = \vec{e}_4' \cdot \vec{e}_4' = h_-$$

7. Comment: notice that the new "row 1" vectors e_1' and e_3' are linear combinations of the original row-1 vectors e_1 and e_3 . Thus, the problem was solved by recombining the row-1 functions seaparately from the row-2 functions. I though I had tried this!

My other attempt was doomed to failure, the one where I tried to solve this by rotating within each Copy. I now see how stupid that approach was: if you get an orthogonal solution, clearly separate independent rotations within the two subspaces can only lead to overall orthogonality. No possible rotations or transformations of any kind within the separate spaces can ever connect you from overall orthogonality to our starting non-orthogonla position.

8. Lichtenberg is wrong. I claim that the linear combinations necessary to solve this problem are non-trivial. We know that:

$$\alpha_{\pm} = \frac{1}{2c} [a - b \pm \sqrt{a^2 + b^2 + 4c^2}]$$

$$\text{where } a = \langle X_{11} \rangle = \frac{1}{3} \langle e \rangle + \frac{1}{3} \langle 12 \rangle - \frac{1}{6} \langle 13 \rangle - \frac{1}{6} \langle 23 \rangle - \frac{1}{3} \langle 123 \rangle$$

$$b = \langle X_{12} \rangle = \frac{1}{3} \langle e \rangle - \frac{1}{3} \langle 12 \rangle + \frac{1}{6} \langle 13 \rangle + \frac{1}{6} \langle 23 \rangle - \frac{1}{3} \langle 123 \rangle$$

$$c = \langle X_{12} \rangle = \langle X_{21} \rangle = -\frac{\sqrt{3}}{6} \langle 13 \rangle + \frac{\sqrt{3}}{6} \langle 23 \rangle$$

$$\text{where, eg, } \langle 12 \rangle \text{ means } \int_a^b dx_1 \int_a^b dx_2 \int_a^b dx_3 f(x_1, x_2, x_3) f(x_2, x_1, x_3)$$

Suppose you are trying to analyze a particular function $f(x_1, x_2, x_3)$. The numbers a, b, c are certain overlap integrals of this function. I have assumed that each integration has the same range so I could use a symmetry to simplify slightly the forms for a, b, c given on last page. Since the numbers a, b, c are uncorrelated, (ie, for different functions you will get different a, b, c and they have no constant relation, like sum to zero or something), α_+ is dependent on the function f .

Now look at the general solution:

$$\frac{1}{\beta_+} \vec{e}_1' = S_{\theta_+} (\alpha_+ \vec{e}_2 + \vec{e}_4) + C_{\theta_+} (\alpha_+ \vec{e}_1 + \vec{e}_3)$$

$$\frac{1}{\beta_+} \vec{e}_2' = C_{\theta_+} (\alpha_+ \vec{e}_2 + \vec{e}_4) - S_{\theta_+} (\alpha_+ \vec{e}_1 + \vec{e}_3) \quad \text{similarly for } \vec{e}_3', \vec{e}_4'$$

I see no way you could select angle θ_+ so that the linear combinations are f -independent.

9. Caveat: However, suppose $f(1,2,3)$ has the following very special form:

$$f(1,2,3) = a(1)b(2)c(3)$$

where a, b, c are mutually orthogonal functions of a single variable. If that is the case, then all exchange expectation values like $(f/(12))/f$ vanish and you get in particular that number $c=0$. In that case, your original basis $e_1 \dots e_4$ was already orthogonal and there is then no need to do anything.

According to Licht's definitions, we have:

$$\psi_1 = 3e_1 + \sqrt{3}e_2 = Y^{Ham.}$$

$$\phi = 6e_4$$

$$\psi = -3e_1 + 3\sqrt{3}e_2$$

$$\chi = -\frac{6}{\sqrt{3}}e_3$$

$$\begin{pmatrix} \psi_1 \\ \psi \end{pmatrix} = \begin{pmatrix} 3 & \sqrt{3} \\ -3 & 3\sqrt{3} \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}.$$

Recall that the most general subspace-transformation which leaves basis vectors orthogonal is the matrix $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ where the only condition is $AC+BD=0$. The reason it is more general than a rotation is due to the extra facts implied by the group algebra. You see that Licht's choice of basis functions uses just such a transformation on my $e_1 e_2$ basis! His second pair of vectors are proportional to my $e_3 e_4$. He makes no claims about normalizing his functions. So, he is correct, with this caveat that a, b, c are orthogonal functions!!!!

Some Theorems about Constructing Projectors.

Theorem 1: If your group elements R are represented by unitary operators R , and if your representation matrices are unitary then $X_{ij}^+ = X_{ji}$. I.e, then hermitian conjugation with respect to the Hilbert Space R acts on is the same as transposing indices for the projection operators.

Proof: Start with definition:

$$X_{il}^{(u)} \equiv \left(\sum_R D_{il}^{(u)}(R) R \right) \frac{1}{h}.$$

$$\begin{aligned} \Rightarrow X_{il}^{(u)\dagger} &= \frac{1}{h} \left(\sum_R D_{il}^{(u)}(R) R^\dagger \right) = \frac{1}{h} \left(\sum_{R^{-1}} D_{il}^{(u)}(R) R^{-1} \right) \\ &= \frac{1}{h} \left(\sum_R D_{il}^{(u)}(R^{-1}) R \right) \quad / \text{by rearrangement.} \\ &= \frac{1}{h} \left(\sum_R D_{li}^{(u)*}(R) R \right) \quad / \text{rep. is unitary.} \\ &= X_{li}^{(u)} \quad \underline{\text{QED.}} \end{aligned}$$

I had to assume the operators were unitary in order to go from $R^\dagger$ to R^{-1} . Only then can you use the rearrangement theorem. Note: we have already seen special cases of this theorem, like $X_{11}^+ = X_{11}$ for S_3 , etc.

Theorem 2: The object formed by adding to the standard projector X_{rr} an arbitrary linear combination of X_{ri} operators (ie, ones for same row but different copies) is a new idempotent. I.e: mix

$$\tilde{X}_{rr} \equiv \left[X_{rr} + \sum_{i \neq r} A_i X_{ri} \right] \Rightarrow (\tilde{X}_{rr})^2 = \tilde{X}_{rr}$$

Proof: We make use of the basic fact about the projectors (ie, the GOT):

$$X_{ij} X_{kl} = \delta_{jk} X_{il}$$

So, just write things out:

$$\begin{aligned} \tilde{X}_{rr}^2 &= \frac{(X_{rr})^2}{X_{rr}} + \sum_{i \neq r} A_i^r \left(\frac{X_{rr} X_{ri}}{X_{ri}} + \frac{X_{ri} X_{rr}}{0, i \neq r} \right) + \sum_{i, j \neq r} A_i^r A_j^r \frac{X_{ri} X_{rj}}{0, i \neq r} \\ &= X_{rr} + \sum_{i \neq r} A_i^r X_{ri} = (\tilde{X}_{rr}) \quad \text{QED.} \end{aligned}$$

Theorem 3: These new idempotents are orthogonal in the following sense: $\tilde{X}_{rr}^+ \tilde{X}_{ss} = 0$.
In other words, $(\tilde{X}_{rr} f, \tilde{X}_{ss} f) = 0$.

Proof:

$$(\tilde{X}_{rr}^+)(\tilde{X}_{ss}) = [X_{rr} + \sum_{i \neq r} A_i^r X_{ir}] [X_{ss} + \sum_{j \neq s} A_j^s X_{sj}]$$

$$t_1 = \sum_{i \neq r} A_i^r X_{ir} X_{ss} = 0 \text{ since } r \neq s$$

$$t_2 = \sum_{j \neq s} A_j^s X_{rr} X_{sj} = 0 \text{ since } r \neq s$$

$$t_3 = \sum_{i \neq r} \sum_{j \neq s} A_i^r A_j^s X_{ir} X_{sj} = 0 \quad \text{QED.}$$

The proof here is really obvious by inspection: the operators in first bracket have r as second index; in the second bracket have s as first index. Since $r \neq s$, all products vanish.

Theorem 4: The Hermitian adjoints of the $\tilde{X}_{rr}$ are also idempotent.

Proof: $\tilde{X}_{rr} \tilde{X}_{rr} = \tilde{X}_{rr} \Rightarrow \tilde{X}_{rr}^+ \tilde{X}_{rr}^+ = \tilde{X}_{rr}^+ \quad \text{QED.}$

Comment: Although these new ~~projectors~~ idempotents $\tilde{X}_{rr}$ "do the right thing" in the sense that the chunks of functions which they project out are orthogonal, as in theorem 3, these operators are not hermitian and are therefore not "projection operators" in the usual seince. Ie, we do not have $\tilde{X}_{rr} \tilde{X}_{ss} = 0$ for $r \neq s$. Also, $\tilde{X}_{rr}^+ \tilde{X}_{rr} \neq \tilde{X}_{rr}$. Unless you have honest-to-goodness projection operators, you cannot "decompose" the identity into sum of projectors, even if they are indempotents.

Exception: In the special case of the two copies of the two-dim rep of S_3 , we can consider these two objects to be acceptable projectors:

$$\left. \begin{aligned} P_1 &= \tilde{X}_{11}^+ \\ P_2 &= \tilde{X}_{22} \end{aligned} \right\} \Rightarrow \begin{aligned} P_1^2 &= P_1 \\ P_2^2 &= P_2 \end{aligned} \quad P_1 P_2 = P_2 P_1 = 0.$$

Thus, the following are acceptable projectors:

$$P_1 = X_{11} + A X_{21}$$

$$P_2 = X_{22} + B X_{21}$$

Specific Example: If you compute the projectors using the method of Hammermesh, see notes above, you get this:

$$P_1 = \frac{1}{3} Y = X_{11} + \sqrt{3} X_{21} \quad \left(= \frac{1}{3} \psi_1^{\text{Lidd}} \right) = \frac{1}{3} [e + (12) - (13) - (123)] = \psi_1$$

$$P_2 = \frac{1}{3} Y' = X_{22} - \sqrt{3} X_{21} \quad \left(= \frac{1}{3} \psi_3^{\text{Lidd}} \right) = \frac{1}{3} [e + (13) - (12) - (132)] = \psi_3$$

(11-10-10)
(1-11 0 0-1)

Thus, we see that this is a particular case of the above most general form of the projectors for the S_3 gwo-dim reps.

Comment: In more complicated cases involving n occurrences of an n -dimensional representation for $n=3,4,\dots$ you do not have the freedom discussed above. For example, in $n=3$ case you cannot arrange X_{11}, X_{22}, X_{33} to get three projectors in the way done above because there are just too many. Ie, if you choose P_1 and P_2 as on bottom of last page, there is no way to define B_3 except maybe $P_3 = X_{33}$ with no mixture.

Probably that is the way it works: you can shuffle two of the basic projectors, but all the remaining ones have to be "basic projectors".

Summary: In general, there are two methods I have to compute the projectors.

(a) If I know the actual representation matrices (they are given in Hammermesh for up through S_5) I know how to compute the "basic projectors" X_{rr} using the general formula: ie, its a linear combination of group elements where the weights are just the rr entries of those matrices. An alternative method is:

(b) Use Hammermesh or Messiah method: for each standard arrangement in a given pattern, construct the object called $Y = QP$. The only problem here is that you have to then work out the normalization by hand, even if you include those Messiah factors.

Theorem 5: The symmetrizers $X_i^{(1)}$ form a "pair", and so do the symmetrizers X_i^2 .

Proof: From their definitions we know that

$$R X_i^{(j)} = \sum_k D_{ki}(R) X_k^{(j)} \quad (j=1 \text{ or } 2).$$

$$\text{or } R \vec{X}^{(j)} = \underline{D(R)} \vec{X}^{(j)}$$

A vector of two objects which behaves this way under the action of R is called a "pair".

Theorem 6: Any linear combination of the above symmetrizers across copies also forms a "pair".

Proof: Let $\tilde{X}_i \equiv \sum_j \alpha^j X_i^j$.

$$\text{Then } R \tilde{X}_i = \sum_j \alpha^j \sum_k D_{ki}(R) X_k^j = \sum_k D_{ki}(R) \tilde{X}_k$$

$$\Rightarrow R \vec{\tilde{X}}_i = \underline{D(R)} \vec{\tilde{X}}. \quad \text{QED.}$$

Example:

$$\begin{aligned} \tilde{X}_1 &= X_1^1 + X_1^2 \\ \tilde{X}_2 &= X_2^1 + X_2^2. \end{aligned}$$

In general:

$$\begin{aligned} \tilde{X}_1 &= a X_1^1 + b X_1^2 \\ \tilde{X}_2 &= a X_2^1 + b X_2^2. \end{aligned}$$

Theorem 7: If you have one member of a general pair (the above pair with any a, b), you can generate the other member of the pair by applying either X_2^1 or X_2^2 .

Proof:

$$X_2^1 \tilde{X}_1 = a(X_2^1 X_1^1) + b(X_2^1 X_1^2) = a X_2^1 + b X_2^2 = \tilde{X}_2 \quad \checkmark$$

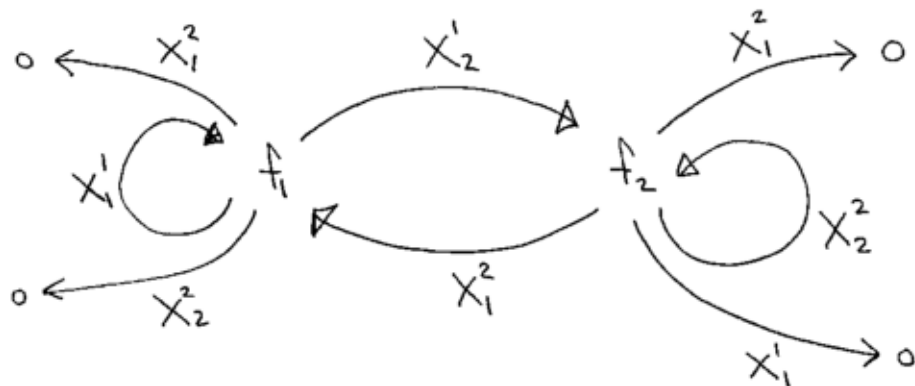
$$X_1^2 \tilde{X}_2 = a(X_1^2 X_2^1) + b(X_1^2 X_2^2) = a X_1^1 + b X_1^2 = \tilde{X}_1 \quad \checkmark$$

By the way, if you do it wrong:

$$\begin{cases} X_1^2 \tilde{X}_1 = 0 \\ X_2^1 \tilde{X}_2 = 0 \end{cases}$$

Let (f_1, f_2) be a "pair". Eg, $f_1 = \tilde{X}_1 f(123)$ $f_2 = \tilde{X}_2 f(123)$.

Then this picture summarizes the above stuff



Proofs that the six symmetrizers are linearly independent.

a) brute force

$$\begin{array}{r} \alpha(1 \ 1 \ 1 \ 1 \ 1 \ 1) \\ + \beta(1 \ -1 \ -1 \ -1 \ 1 \ 1) \\ \hline 2 \ 2 \ -1 \ -1 \ -1 \end{array} \Rightarrow X_s, A_s, X'_1 \text{ all lin indep}$$

$$\begin{array}{r} \alpha(1 \ 1 \ 1 \ 1 \ 1 \ 1) \\ + \beta(1 \ -1 \ -1 \ -1 \ 1 \ 1) \\ + \gamma(2 \ 2 \ -1 \ -1 \ -1 \ -1) \\ \hline 2 \ -2 \ 1 \ 1 \ -1 \ -1 \end{array} \Rightarrow \begin{array}{l} \alpha + \beta + 2\gamma = 2 \\ \alpha - \beta + 2\gamma = -2 \\ \Rightarrow (\alpha + 2\gamma) = 2 - \beta \\ (\alpha + 2\gamma) = -2 + \beta \end{array}$$

$$\Rightarrow (\alpha + 2\gamma) = 0 = (2 - \beta) \Rightarrow \beta = 2$$

$$\#3: (\alpha - \beta - \gamma = 1) \Rightarrow \alpha - \gamma = 1 + \beta = 3. \quad \text{So}$$

$$\#5: (\alpha + \beta - \gamma = -1)$$

$$0 - 2\beta + 0 = 2 \Rightarrow \beta = -1 \quad \text{CONTRA!}$$

$$\alpha + 2\gamma = 0$$

$$\alpha - \gamma = 3$$

$$\Rightarrow 3\gamma = -3 \Rightarrow \gamma = -1$$

$$\alpha = 2$$

$$\alpha \quad 2 + 2 + 1 = -1 \quad \times$$

Next:

$$\begin{array}{r} \alpha(1 \ 1 \ 1 \ 1 \ 1 \ 1) \\ \beta(1 \ -1 \ -1 \ -1 \ 1 \ 1) \\ \gamma(2 \ 2 \ -1 \ -1 \ -1 \ -1) \\ \delta(2 \ -2 \ 1 \ 1 \ -1 \ -1) \\ \hline 0 \ 0 \ -1 \ 1 \ 1 \ 1 \end{array} \quad \text{NO GO.}$$

$$\begin{array}{r} \alpha(1 \ 1 \ 1 \ 1 \ 1 \ 1) \\ \beta(1 \ -1 \ -1 \ -1 \ 1 \ 1) \\ \gamma(2 \ 2 \ -1 \ -1 \ -1 \ -1) \\ \delta(2 \ -2 \ 1 \ 1 \ -1 \ -1) \\ \hline \epsilon(0 \ 0 \ -1 \ 1 \ 1 \ 1) \\ (0 \ 0 \ -1 \ 1 \ -1 \ 1) \end{array}$$

So X_s, A_s, X'_1, X'_2 and X'_3 are indep. Lastly

$$\begin{array}{l} \alpha + \beta + 2\gamma + 2\delta = 0 \\ \alpha - \beta + 2\gamma - 2\delta = 0 \end{array} \Rightarrow \begin{array}{l} \alpha + 2\gamma = \beta + 2\delta = 0 \end{array}$$

$$\alpha = -2\gamma$$

$$\beta = -2\delta$$

$$\Rightarrow \delta - [-2\delta] = 1$$

$$3\delta = 1$$

$$\delta = \frac{1}{3}$$

$$\#3: \alpha - \beta - \gamma + \delta - \epsilon = -1 \Rightarrow -2\epsilon = -2 \Rightarrow \epsilon = 1$$

$$\#4: \alpha = \beta - \gamma + \delta + \epsilon = +1 \Rightarrow \alpha - \beta - \gamma + \delta = 0$$

$$\#5: \alpha + \beta - \gamma - \delta + \epsilon = -1 \Rightarrow \alpha + \beta - \gamma - \delta = -2$$

$$\#6: \alpha + \beta - \gamma - \delta - \epsilon = 1 \Rightarrow -2\beta + 2\gamma = 2 \Rightarrow \delta - \beta = 1$$

$$\Rightarrow (-\frac{2}{3}) + (-\frac{2}{3}) - (\frac{1}{3}) - (\frac{1}{3}) - (1) = 1 \quad 2\alpha - 2\gamma = -2$$

$$\Rightarrow \alpha - \gamma = -1 \Rightarrow -2\gamma = -1$$

$$\Rightarrow \gamma = \frac{1}{2}$$

$$\alpha = -\frac{1}{2}$$

$$-\frac{1}{3} - \frac{1}{3} - \frac{1}{3} - \frac{1}{3} - 1 = 1 \quad \text{NO}$$

So all six are indep!!

b) How do you test n vectors, $\dim = n$, for indep?

I can test the entire set at once like so.

$$\begin{array}{c|cccccc}
 & 1 & 2 & 3 & 4 & 5 & 6 \\
 \hline
 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
 2 & 1 & -1 & -1 & -1 & 1 & 1 \\
 3 & 2 & 2 & -1 & -1 & -1 & -1 \\
 4 & 0 & 0 & -1 & 1 & 1 & -1 \\
 5 & 0 & 0 & -1 & 1 & -1 & 1 \\
 6 & 2 & -2 & 1 & 1 & -1 & -1
 \end{array}
 \rightarrow
 \begin{pmatrix}
 2 & 1 & 1 & 1 & 1 & 1 \\
 0 & -1 & -1 & -1 & 1 & 1 \\
 4 & 2 & -1 & -1 & -1 & -1 \\
 0 & 0 & -1 & 1 & 1 & -1 \\
 0 & 0 & -1 & 1 & -1 & 1 \\
 0 & -2 & 1 & 1 & -1 & -1
 \end{pmatrix}$$

Add col 2 to col 1 to get

Step. Start w. add row 4 to 5:

$$\begin{pmatrix}
 1 & 1 & 1 & 1 & 1 & 1 \\
 1 & -1 & -1 & -1 & 1 & 1 \\
 2 & 2 & -1 & -1 & -1 & -1 \\
 0 & 0 & -1 & 1 & 1 & -1 \\
 0 & 0 & -2 & 2 & 0 & 0 \\
 2 & -2 & 1 & 1 & -1 & -1
 \end{pmatrix}$$

Next,

$$\begin{array}{cccccc}
 2 & 2 & 0 & 0 & 0 & 0 \\
 -1 & -3 & 0 & 0 & 0 & 0 \\
 \rightarrow & -4 & -4 & +2 & 0 & 0 \\
 0 & 0 & +1 & -2 & 0 & 0 \\
 0 & 0 & -1 & 2 & -1 & 0 \\
 2 & -2 & 1 & 0 & -1 & -1
 \end{array}$$

$F = +2$

$$\begin{array}{cccccc}
 3 & 3 & 0 & 0 & 0 & 0 \\
 3 & 1 & -2 & -2 & 0 & 0
 \end{array}$$

$$\det = (+2) \cdot 2(-3)(2)(-2)(-1)(-1) \neq 0$$

So here is my second verification that the six symmetrizers are linearly independent: (in the group algebra space). Systematically reduce the matrix to triangular form. If there is a zero on the diagonal anywhere, $\det=0$ and set is lin dep. But here there were no zeros, so all are linearly independent!!

c) fastest proof. These 6 vectors are orthogonal (in group alg hence here) Q.E.D.

About Decomposing Functions $f(123)$ into S_3 irreducible components.

1. Imagine an object $f(123)$ such that the six objects you get by permuting this thing are linearly independent. Such would be the case, eg, if $f(123) = a(1)b(2)c(3)$ where a, b, c are three different functions. Or more generally, if $f(123)$ is some arbitrary function of its arguments that possesses no particular symmetry.

If the pieces $Rf(123)$ are linearly independent, then the 6 objects you get by applying the six symmetrizers are also linearly independent. These 6 objects are:

$$\begin{aligned} f_s(123) &= (1/6) [f(123) + f(213) + f(321) + f(132) + f(231) + f(312)] \\ f_a(123) &= (1/6) [f(123) - f(213) - f(321) - f(132) + f(231) + f(312)] \\ \left\{ \begin{aligned} f_1^1(123) &= (1/6) [2f(123) + 2f(213) - f(321) - f(132) - f(231) - f(312)] \\ f_2^1(123) &= (\sqrt{3}/6) [- f(321) + f(132) - f(231) + f(312)] \end{aligned} \right. \\ \left\{ \begin{aligned} f_1^2(123) &= (\sqrt{3}/6) [- f(321) + f(132) + f(231) - f(312)] \\ f_2^2(123) &= (1/6) [2f(123) - 2f(213) + f(321) + f(132) - f(231) - f(312)] \end{aligned} \right. \end{aligned}$$

The brackets show pairs of functions which are "pairs" in the sense that they behave under action of R as a basis for the M-rep of S_3 . Here, then, we have two linearly independent pairs.

Recall that the "group algebra" space is essentially the regular representation space, and by the "celebrated theorem" this reduces to $S + A + 2M$ reps, so you expect to see two pairs for the M-reps.

2. First special case: suppose $f(123) = f(12)$ only. For arbitrary function $f(12)$ with no symmetry this is no restriction; there are still 6 independent objects and the above stuff still lists them, just delete the last argument. The 6 objects are of course:

$$f(12), f(21), f(13), f(31), f(23), f(32)$$

3. Symmetry: if your $f(123)$ has some sort of symmetry, like the tensor form $1^1_3 2^2_3$, you may find that the two pairs are degenerate; the copy 2 pair may just be multiples of the copy one pair. That is exactly what happens there. In this example the symmetry is this: $f(123) = f(132)$.

4. Second Special Case: $f(123) = f(1)$ only. In this case there are clearly only three independent objects; the totally symmetric, and one pair. In fact:

$$f_s(1) = (1/3) [f(1) + f(2) + f(3)]$$

$$f_a(1) = 0$$

$$\left\{ \begin{array}{l} f_1^1(1) = (1/6) [f(1) + f(2) - 2 f(3)] \\ f_2^1(1) = (\sqrt{3}/6) [f(1) - f(2)] \end{array} \right.$$

$$\left\{ \begin{array}{l} f_1^2(1) = (\sqrt{3}/6) [f(1) + f(2) - 2 f(3)] \\ f_2^2(1) = (1/2) [f(1) - f(2)] \end{array} \right.$$

Here you see an example of how the two pairs become degenerate; the top pair is related to the lower pair by $\text{SQRT}(3)$.

The tensor problem for self-energy.

1. Consider a self energy with momenta p_1 and p_2 entering. Ignore the constraint that $p_1 + p_2 = 0$. What is the most general tensor form involving two momenta that you can make?

Answer: this sum:

$$T^{\mu_1 \mu_2}(p_1^2, p_2^2) = \sum_{i,j=1}^2 A_{ij}(p_1^2, p_2^2) (p_i)^{\mu_1} (p_j)^{\mu_2}$$

This is only four terms, easy to write them out. Now, given four functions and thus an explicit form of the above, what is the "switched" tensor? Answer:

$$T^{\mu_2 \mu_1}(p_2^2, p_1^2) = \sum_{i,j=1}^2 A_{ij}(p_2^2, p_1^2) (p_i)^{\mu_2} (p_j)^{\mu_1}$$

Here, the notation $\bar{i}$ means if i was 1, then $\bar{i} = 2$. Maybe I should write that $\bar{i} = 3 - i$. You see that the actual momenta of the tensor form get switched, as well as the arguments of the functions. Their indices stay put of course, and the Lorentz indices on the momenta also switch. Now, rename the summation indices to get:

$$T^{\mu_2 \mu_1}(p_2^2, p_1^2) = \sum_{i,j} A_{\bar{j} \bar{i}}(p_2^2, p_1^2) (p_i)^{\mu_1} (p_j)^{\mu_2}$$

2. Now, what are the conditions on the A_{ij} if you are trying to construct an antisymmetric tensor? Since no momentum constraint, each of the four tensors in the sum is independent, so you conclude :

$$\begin{aligned} T^{\mu_1 \mu_2}(p_1^2, p_2^2) + T^{\mu_2 \mu_1}(p_2^2, p_1^2) &= 0 \\ &= \sum_{i,j} [A_{ij}(p_1^2, p_2^2) + A_{\bar{j} \bar{i}}(p_2^2, p_1^2)] (p_i)^{\mu_1} (p_j)^{\mu_2} \end{aligned}$$

$\Rightarrow$

$$A_{ij}(p_1^2, p_2^2) = -A_{\bar{j} \bar{i}}(p_2^2, p_1^2) \quad \Rightarrow \quad \begin{cases} A_{11}(1,2) = -A_{22}(2,1) \\ A_{12}(1,2) = -A_{12}(2,1) \\ A_{21}(1,2) = -A_{21}(2,1) \\ A_{22}(1,2) = -A_{11}(2,1) \end{cases}$$

You can think of there being originally 8 functions, the sym and anti sym parts of each of the 4 functions.

These conditions then reduce you to 4 functions: notice

that a self-condition removes only one function, whereas a condition relating two different functions removes two functions, in the sense of there being 8 originally.

So here then are the most general antisymmetric (and symmetric) tensors you could have:

$$\begin{aligned}
 AS^{T^{\mu_1 \mu_2}}(p_1^2, p_2^2) = & A_{11}^S(1,2) [(p_1)^{\mu_1} (p_1)^{\mu_2} - (p_2)^{\mu_1} (p_2)^{\mu_2}] \\
 & + A_{11}^{AS}(1,2) [(p_1)^{\mu_1} (p_1)^{\mu_2} + (p_2)^{\mu_1} (p_2)^{\mu_2}] \\
 & + A_{12}^{AS}(1,2) [(p_1)^{\mu_1} (p_2)^{\mu_2}] \\
 & + A_{21}^{AS}(1,2) [(p_2)^{\mu_1} (p_1)^{\mu_2}] .
 \end{aligned}$$

To get S^T , switch $S \leftrightarrow AS$ on your labels all.

Thus, you might argue that there are really 4 different "tensors" existing in this object, each requiring a multiplying scalar function of some definite symmetry.

2. Now let's remove the constraint and let $p_1 + p_2 = 0$. What does this do? Replace p_2 with $-p_1$.

The first tensor vanishes identically, the last three tensors are all proportional to one another. But since $p_1^2 = p_2^2$, it is now impossible to make an antisymmetric function to multiply. Thus, there are no forms whatsoever left! (for the anti sym tensor). For the sym tensor where S and AS labels are all switched, you have only one form, which is the confluence of the last three above. Thus:

$$\begin{aligned}
 AS^{T^{\mu_1 \mu_2}}(p_1^2) &= 0 \\
 S^{T^{\mu_1 \mu_2}}(p_1^2) &= \underbrace{A_{11}(p_1^2, p_1^2)}_{f(p_1^2)} (p_1)^{\mu_1} (p_1)^{\mu_2} + \text{of course } g(p_1^2) g^{\mu_1 \mu_2}
 \end{aligned}$$

of two momenta.

3. So, to summarize: we first hunted for all the tensor forms ^{of two momenta} without the constraint.

We got some general answers, each having four functions arbitraire of definite symmetry.

Then we put the constraint back in with two effects: 1) some of the tensor forms identically vanished; 2) it was no longer possible to make functions that were AS .

4. I now want to see how this goes for the case of a 3-vertex. Something tells me that the group S_3 and its representations might somehow be useful!

A. Review of the general construction of tensor forms with two momenta.

1. Question: what are the tensor forms you can construct from p_1 and p_2 . I have already figured this out but here is another way to do it.

2. Solution: for the group S_2 there are only two representations and so only two projectors, call them S and A. Write down all the basic tensor forms and then apply projectors to get forms which are associated with certain representations. Then at the end apply scalar functions.

So here we go:

	<u>S</u>	<u>A</u>
$(p_1)^{\mu_1} (p_1)^{\mu_2}$	$\frac{1}{2} [(p_1)^{\mu_1} (p_1)^{\mu_2} + p_2^{\mu_2} p_1^{\mu_1}]$	$\frac{1}{2} [p_1^{\mu_1} p_1^{\mu_2} - p_2^{\mu_2} p_1^{\mu_1}]$
$(p_1)^{\mu_1} p_2^{\mu_2}$	$p_1^{\mu_1} p_2^{\mu_2}$	0
$(p_2)^{\mu_1} p_1^{\mu_2}$	$p_2^{\mu_1} p_1^{\mu_2}$	0
$p_2^{\mu_1} p_2^{\mu_2}$	$\frac{1}{2} [p_2^{\mu_1} p_2^{\mu_2} + p_1^{\mu_2} p_1^{\mu_1}]$	$\frac{1}{2} [p_2^{\mu_1} p_2^{\mu_2} - p_1^{\mu_2} p_1^{\mu_1}]$

Thus, by applying our projectors to a list of basis functions, we have generated a list of new basis functions which have definite symmetry. Namely, we have:

Symmetries:

$$T_1^{\mu_1 \mu_2} = p_1^{\mu_1} p_2^{\mu_2}$$

$$T_2^{\mu_1 \mu_2} = p_2^{\mu_1} p_1^{\mu_2}$$

$$T_3^{\mu_1 \mu_2} = \frac{1}{2} [p_1^{\mu_1} p_1^{\mu_2} + p_2^{\mu_2} p_2^{\mu_1}]$$

antisym:

$$T_4^{\mu_1 \mu_2} = \frac{1}{2} [p_1^{\mu_1} p_1^{\mu_2} - p_2^{\mu_2} p_2^{\mu_1}]$$

Then here are the most general sym and anti sym tensors we can construct:

$$T_{\text{sym}}^{\mu_1 \mu_2} = f_1^S(p_1^2, p_2^2) T_1^{\mu_1 \mu_2} + f_2^S T_2^{\mu_1 \mu_2} + f_3^S T_3^{\mu_1 \mu_2} + f_4^{AS} T_4^{\mu_1 \mu_2}$$

Ie, you are allowed 4 scalar functions and each must have the correct symmetry. Similarly for T_{antisym} .

Count Amplitudes in Triple Glue Coupling by LS Coupling.

1. Imagine that you are in a rest frame of a gluon; it decays into two final state gluons. How many independent amplitudes are there?

The pair of final gluons can have $S=0,1,2$.

For $S=0$, to get $J=1$ you can have $L=1$ and that all.

For $S=1$, you get $J=1$ with $L=0,1,2$.

For $S=2$ you get $J=1$ with $L=1,2,3$.

It would seem then that there are 7 amplitudes:

$S=0$	$L=1$	1
$S=1$	$L=0,1,2$	3
$S=2$	$L=1,2,3$	<u>3</u>
		7.

Is this number further restricted by bose symmetry or parity or something I am not thinking of??

The problem of rearranging the 36 objects of the form $f(123)g(123)$ into tensors of definite overall symmetry.

1. Here is how you do it: first, take each of the two systems and decompose into 6 pieces of definite symmetry. Then combine the systems using direct product to get resultant combined forms of definite symmetry.

Namely:

$\square\square$ represents two functions.

$$(\square\square + \square\square + \square\square_1 + \square\square_2) \times (\square\square + \square\square + \square\square_1 + \square\square_2)$$

$$= (\square\square + \square\square) \times (\square\square \times \square\square)$$

$$+ (\square\square + \square\square) \times (\square\square) \cdot 4$$

$$+ (\square\square) \times (\square\square) \cdot 4$$

$$= (2 \cdot \square\square + 2 \cdot \square\square)$$

$$(+ 8 \cdot \square\square)$$

$$(+ 4 \cdot \square\square + 4 \cdot \square\square + 4 \cdot \square\square)$$

$$= 6 \cdot \square\square + 6 \cdot \square\square + 12 \cdot \square\square$$

Note that we get $6+6+24 = 36$ objects in the end. In particular, you should be able to make six different totally symmetric tensors in this way. Two arise in the obvious way, and 4 arise from the mixed symmetry combinations.

	x_1^1	x_2^1	x_1^2	x_2^2
x_1^1	x_1^1	0	x_1^2	0
x_2^1	x_2^1	0	x_2^2	0
x_1^2	0	x_1^1	0	x_1^2
x_2^2	0	x_2^1	0	x_2^2

	x_1^1	x_1^2	x_2^1	x_2^2
x_1^1	x_1^1	x_1^2	0	0
x_1^2	0	0	x_1^1	x_1^2
x_2^1	x_2^1	x_2^2	0	0
x_2^2	0	0	x_2^1	x_2^2

S_3 Expansions in the Standard Basis.

Basic S_3 data

1. Here are the 6 basis vectors of the group algebra, and their multiplication table:

$$\begin{aligned}
 6X_A &= \begin{matrix} (e) \\ 1 \end{matrix} \quad \begin{matrix} (12) \\ -1 \end{matrix} \quad \begin{matrix} (13) \\ -1 \end{matrix} \quad \begin{matrix} (23) \\ -1 \end{matrix} \quad \begin{matrix} (123) \\ 1 \end{matrix} \quad \begin{matrix} (132) \\ 1 \end{matrix} = 6X_A \\
 6X_S &= 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 = 6X_S \\
 6X_{11} &= 2 \quad 2 \quad -1 \quad -1 \quad -1 \quad -1 = 6X_1^1 \\
 6X_{12} &= 0 \quad 0 \quad -\sqrt{3} \quad \sqrt{3} \quad \sqrt{3} \quad -\sqrt{3} = 6X_1^2 \\
 6X_{21} &= 0 \quad 0 \quad -\sqrt{3} \quad \sqrt{3} \quad -\sqrt{3} \quad \sqrt{3} = 6X_2^1 \\
 6X_{22} &= 2 \quad -2 \quad 1 \quad 1 \quad -1 \quad -1 = 6X_2^2
 \end{aligned}$$

	X_A	X_S	X_{11}	X_{12}	X_{21}	X_{22}
X_A	X_A	0	0	0	0	0
X_S	0	X_S	0	0	0	0
X_{11}	0	0	X_{11}	X_{12}	0	0
X_{12}	0	0	0	0	X_{11}	X_{12}
X_{21}	0	0	X_{21}	X_{22}	0	0
X_{22}	0	0	0	0	X_{21}	X_{22}

These X objects are supposed to provide a complete, albeit not orthonormal, basis for the group algebra. Thus, any elements of the group algebra can be uniquely expanded in these basis vectors like so:

$$A = a_A X_A + a_S X_S + a_{11} X_{11} + a_{12} X_{12} + a_{21} X_{21} + a_{22} X_{22}.$$

Given the group algebra vector A , how can you deduce the 6 coefficients? Here are some useful facts that follow from the X multiplication table:

$$AX_A = X_A A = a_A X_A$$

$$X_{11} A = a_{11} X_{11} + a_{12} X_{12}$$

$$AX_S = X_S A = a_S X_S$$

$$X_{22} A = a_{21} X_{21} + a_{22} X_{22}$$

$$X_{11} A X_{11} = a_{11} X_{11}$$

$$X_{22} A X_{11} = a_{21} X_{21}$$

$$X_{11} A X_{22} = a_{12} X_{12}$$

$$X_{22} A X_{22} = a_{22} X_{22}.$$

For example, if you apply X_{11} on the left of A , and X_{22} on the right of that result, you get a result which is proportional to X_{12} and the coefficient will be a_{12} . So this is a way in which you could compute the 6 coefficients, given A .

Group Multiplication Table:

	e	(12)	(13)	(23)	(123)	(132)
e	e	12	13	23	123	132
(12)	12	e	132	123	23	13
(13)	13	123	e	132	12	23
(23)	23	132	123	e	13	12
(123)	123	13	23	12	132	e
(132)	132	23	12	13	e	123

Rules:

$$(ab)(bc) = (abc)$$

$$(ab)(abc) = (bc)$$

$$(ab)^2 = e$$

$$(cab)(ab) = (ca)$$

2. Action of group elements on basis symmetrizers: First make chart of $D^T(R)$:

$$D^T(e) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$D^T(23) = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2}\sqrt{3} \\ \frac{1}{2}\sqrt{3} & \frac{1}{2} \end{pmatrix}$$

$$D^T((12)) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$D^T(123) = \begin{pmatrix} -\frac{1}{2} & -\frac{1}{2}\sqrt{3} \\ \frac{1}{2}\sqrt{3} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} X_1^i \\ X_2^i \end{pmatrix}$$

$$D^T((13)) = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$$

$$D^T(132) = \begin{pmatrix} -\frac{1}{2} & +\frac{1}{2}\sqrt{3} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$$

Notice that these are in standard order reading down. The general rule for action of R on a basis symmetrizer is this:

$$RX_i^k = \sum_j D_{ij}^T(R) X_j^k \quad \text{where} \quad X_i^k = X_i^{\downarrow k \leftarrow \text{copy}}$$

For the one-dim reps the group actions are fairly trivial. Using the above matrices we can then generate this complete chart:

	X_S	X_A	X_1^i	X_2^i	
e	X_S	X_A	X_1^i	X_2^i	
(12)	X_S	$-X_A$	X_1^i	$-X_2^i$	$(13)X_2^i = -(123)X_2^i$ $(23)X_2^i = -(132)X_2^i$
(13)	X_S	$-X_A$	$-\frac{1}{2}X_1^i - \frac{1}{2}\sqrt{3}X_2^i$	$-\frac{\sqrt{3}}{2}X_1^i + \frac{1}{2}X_2^i$	$(12)X_j^i = (-1)^{j+1}X_j^i$
(23)	X_S	$-X_A$	$-\frac{1}{2}X_1^i + \frac{1}{2}\sqrt{3}X_2^i$	$+\frac{1}{2}\sqrt{3}X_1^i + \frac{1}{2}X_2^i$	$(13)X_1^i = (123)X_1^i$ $(23)X_1^i = (132)X_1^i$
(123)	X_S	X_A	$-\frac{1}{2}X_1^i - \frac{1}{2}\sqrt{3}X_2^i$	$+\frac{1}{2}\sqrt{3}X_1^i - \frac{1}{2}X_2^i$	
(132)	X_S	X_A	$-\frac{1}{2}X_1^i + \frac{1}{2}\sqrt{3}X_2^i$	$-\frac{1}{2}\sqrt{3}X_1^i - \frac{1}{2}X_2^i$	

3. By the way, I think you can check by doing hermitian conjugation that: right action by group elements mixes copies rather than rows like so: (check if needed)

$$X_k^i R = \sum_j D_{ij}^T(R) X_k^j$$

4. Now here is a table which will below enable us to write each group element as a sum of symmetrizers. The table is read easily from table on page 3:

<u>R</u>	<u>$X_1^1 R X_1^1$</u>	<u>$X_2^2 R X_1^1$</u>	<u>$X_1^1 R X_2^2$</u>	<u>$X_2^2 R X_2^2$</u>
e	X_1^1	0	0	X_2^2
(12)	X_1^1	0	0	$-X_2^2$
(13)	$-\frac{1}{2} X_1^1$	$-\frac{1}{2}\sqrt{3} X_2^1$	$-\frac{1}{2}\sqrt{3} X_1^2$	$+\frac{1}{2} X_2^2$
(23)	$-\frac{1}{2} X_1^1$	$+\frac{1}{2}\sqrt{3} X_2^1$	$+\frac{1}{2}\sqrt{3} X_1^2$	$+\frac{1}{2} X_2^2$
(123)	$-\frac{1}{2} X_1^1$	$-\frac{1}{2}\sqrt{3} X_2^1$	$+\frac{1}{2}\sqrt{3} X_1^2$	$-\frac{1}{2} X_2^2$
(132)	$-\frac{1}{2} X_1^1$	$+\frac{1}{2}\sqrt{3} X_2^1$	$-\frac{1}{2}\sqrt{3} X_1^2$	$-\frac{1}{2} X_2^2$

Remember that X_1^1 and X_2^2 are row projectors, hence above table is easy to see. Now recall the general decomposition of any element of the group algebra (here take $A = R$):

$$\downarrow \quad R = R X_A + R X_S + X_1^1 R X_1^1 + X_2^2 R X_1^1 + X_1^1 R X_2^2 + X_2^2 R X_2^2 \quad !$$

	<u>X_A</u>	<u>X_S</u>	<u>X_1^1</u>	<u>X_2^1</u>	<u>X_1^2</u>	<u>X_2^2</u>
e	1	1	1	0	0	1
(12)	-1	1	1	0	0	-1
(13)	-1	1	$-\frac{1}{2}$	$-\frac{1}{2}\sqrt{3}$	$-\frac{1}{2}\sqrt{3}$	$\frac{1}{2}$
(23)	-1	1	$-\frac{1}{2}$	$+\frac{1}{2}\sqrt{3}$	$+\frac{1}{2}\sqrt{3}$	$\frac{1}{2}$
(123)	1	1	$-\frac{1}{2}$	$-\frac{1}{2}\sqrt{3}$	$+\frac{1}{2}\sqrt{3}$	$-\frac{1}{2}$
(132)	1	1	$-\frac{1}{2}$	$+\frac{1}{2}\sqrt{3}$	$-\frac{1}{2}\sqrt{3}$	$-\frac{1}{2}$

$$\frac{1}{3} [(e) + (123) + (132)] = X_A + X_S$$

5. Identification of the 6 functions which can be made from $f(123)$:

Back on page 3 I have written how the X_i^k transform under group action. This implies that the projections of a function will transform in the same way:

$$R(X_i^k | f) = \sum_j D_{ji}(R) (X_j^k | f)$$

Let $f_i^k = X_i^k f$. Then.

$$\begin{array}{|c|c|c|} \hline & & \\ \hline \end{array} \quad f_s = X_s f$$

$$\begin{array}{|c|} \hline \\ \hline \\ \hline \end{array} \quad f_A = X_A f$$

$$\begin{array}{|c|c|} \hline & \\ \hline \end{array} \quad f_1^1 = X_1^1 f \quad \text{and} \quad f_2^1 = X_2^1 f \quad / \text{copy 1}$$

$$\begin{array}{|c|c|} \hline & \\ \hline \end{array} \quad \underbrace{f_1^2 = X_1^2 f}_{\text{row 1}} \quad \text{and} \quad \underbrace{f_2^2 = X_2^2 f}_{\text{row 2}} \quad / \text{copy 2}$$

$\begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline \end{array}$

$\begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline \end{array}$

Thus, these are the six functions of definite symmetry that you can make from the 6 permutations of $f(123)$. Of course it happens that only four of them are needed if you want to partition $f = \text{sum of functions}$.

If you look at table on bottom page page 4 and read across the top as the six functions, you see that the full set of six functions spans the 6-dimensional representation of the group S_3 . It just happens that the first row of that table involves only the true projectors.

6. Now, if Hammermesh's Clebsch table is correct, here are the four totally symmetric functions I can make:

row-1 x row-1 row-2 x row-2

$$F_s^1 = f_1^1 g_1^1 + f_2^1 g_2^1$$

$$F_s^3 = f_1^2 g_1^1 + f_2^2 g_2^1$$

$$F_s^2 = f_1^1 g_1^2 + f_2^1 g_2^2$$

$$F_s^4 = f_1^2 g_1^2 + f_2^2 g_2^2$$

Any choice of a row-1 x row-1 + row-2 x row-2 should be totally symmetric, according to the Clebsch table. Now let's check. Perhaps here is a faster notation to account for all of them:

$$F_s^{(ij)} = f_1^i g_1^j + f_2^i g_2^j \quad i=1,2 \quad j=1,2.$$

Use table on page 3 to investigate the action of various group elements on this F^{ij} :

$$e: \rightarrow [f_1^i][g_1^j] + [f_2^i][g_2^j] = \checkmark$$

$$(12) \rightarrow [f_1^i][g_1^j] + [-f_2^i][-g_2^j] = \checkmark$$

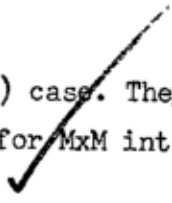
$$\begin{aligned} (13) \rightarrow & [-\frac{1}{2}f_1^i - \frac{1}{2}\sqrt{3}f_2^i][-\frac{1}{2}g_1^i - \frac{1}{2}\sqrt{3}g_2^i] + [-\frac{\sqrt{3}}{2}f_1^i + \frac{1}{2}f_2^i][-\frac{\sqrt{3}}{2}g_1^i + \frac{1}{2}g_2^i] \\ & = (\frac{1}{4} + \frac{3}{4})f_1^i g_1^i + (\frac{3}{4} + \frac{1}{4})f_2^i g_2^i + 0 = \checkmark \end{aligned}$$

$$(13) \rightarrow [- \quad -][- \quad -] + [- \quad +][- \quad +]$$

$$(23) \rightarrow [- \quad +][- \quad +] + [+ \quad +][+ \quad +] \quad \checkmark$$

$$(123) \rightarrow [- \quad -][- \quad -] + [+ \quad -][+ \quad -]$$

$$(321) \rightarrow [- \quad +][- \quad +] + [- \quad -][- \quad -]$$

The last three elements are just sign changes on (13) case. They all work. Therefore, I have shown in detail that the Clebsch's are correct for MxM into the TS representation. Probably then all the other ones are right too. 

Action of group elements on the projectors.

$$(13) X_1^1 = -\frac{1}{2} (X_1^1 + \sqrt{3} X_2^1) = -\frac{1}{12} [6X_1^1 + \sqrt{3} X_2^1]$$

$$= -\frac{1}{12} (2 \quad 2 \quad -4 \quad 2 \quad -4 \quad 2) = +\frac{1}{6} (-1 \quad -1 \quad 2 \quad -1 \quad 2 \quad -1)$$

$$(12) X_1^1 = X_1^1$$

Stop. The easiest way to visualize this is the (13) or R in general acts on the heading and rearranges the headings. When you unshuffle, you find that the integers have simply been rearranged! For example, X_1^1 has two 2's and four -1's. Then $(13)X_1^1$ also has two 2's and four -1's. In fact any object like $X_1^1 R$ or $R X_1^1$ must contain two 2's and four -1's. That set of integers is in this sense "invariant" under group multiplication on either side.

However, there are 30 ways to have two 2's and four -1's. Of these, only 6 are accessible, so to speak. Eg:

$$X_1^1 \rightarrow \left\{ \begin{array}{cccccc} 2 & 2 & -1 & -1 & -1 & -1 \\ (e) & (12) & (13) & (23) & (123) & (132) \end{array} \right\} = (22-1-1-1-1)$$

$$(13) \rightarrow (13) \quad (123) \quad (e) \quad (132) \quad (12) \quad (23) \Rightarrow (-1-1 \quad 2-1 \quad 2-1)$$

$$(23) \rightarrow (23) \quad (132) \quad (123) \quad (e) \quad (13) \quad (12) \Rightarrow (-1-1-1 \quad 2-1 \quad 2)$$

$$(123) \rightarrow (123) \quad (13) \quad (23) \quad (12) \quad (132) \quad (e) \Rightarrow (-1-1 \quad 2-1 \quad 2-1)$$

$$(132) \rightarrow (132) \quad (23) \quad (12) \quad (13) \quad (e) \quad (123) \Rightarrow (-1-1-1 \quad 2-1 \quad 2)$$

Thus:

$$\checkmark X_1^1 = (2 \quad 2 \quad -1 \quad -1 \quad -1 \quad -1)$$

$$\checkmark (12) X_1^1 = (2 \quad 2 \quad -1 \quad -1 \quad -1 \quad -1) = X_1^1$$

$$(13) X_1^1 = (-1 \quad -1 \quad 2 \quad -1 \quad 2 \quad -1) = X_1^1 (13)$$

$$(23) X_1^1 = (-1 \quad -1 \quad -1 \quad 2 \quad -1 \quad 2) = X_1^1 (23)$$

$$(123) X_1^1 = (-1 \quad -1 \quad 2 \quad -1 \quad 2 \quad -1) = X_1^1 (13)$$

$$(132) X_1^1 = (-1 \quad -1 \quad -1 \quad 2 \quad -1 \quad 2) = X_1^1 (23)$$

In our vector notation, note that

$$(a \ b \ c \ d \ e \ f)^t = (a \ b \ c \ d \ \underline{f \ e}) \quad \text{if all real.}$$

So table also says:

$$\chi_1^1(12) = (22 \ -1 \ -1 \ -1 \ -1)$$

$$\chi_1^1(13) = (-1 \ -1 \ 2 \ -1 \ -12)$$

$$\chi_1^1(23) = (-1 \ -1 \ -1 \ 22 \ -1)$$

$$\chi_1^1(132) = (-1 \ -1 \ 2 \ -1 \ -12)$$

$$\chi_1^1(123) = (-1 \ -1 \ -1 \ 22 \ -1)$$

Therefore:

$$(22 \ -1 \ -1 \ -1 \ -1) = \chi_1^1 = (12)\chi_1^1 = \chi_1^1(12)$$

$$(-1 \ -1 \ 2 \ -1 \ -12) = (13)\chi_1^1 = (123)\chi_1^1$$

$$(-1 \ -1 \ -1 \ 22 \ -1) = (23)\chi_1^1 = (132)\chi_1^1$$

$$(-1 \ -12 \ -1 \ -12) = \chi_1^1(13) = \chi_1^1(132)$$

$$(-1 \ -1 \ -1 \ 22 \ -1) = \chi_1^1(23) = \chi_1^1(123)$$

Next, consider

χ_1^2

0 0 -1 1 1 -1

(12) →

A fast way to do these things: reorder integers according to new order of group elements in the group multiplication table.

For example, let's redo the X_1^1 thing in this way:

$$(12) X_1^1 = 2 \ 2 \ -1 \ -1 \ -1 \ -1$$

$$(13) X_1^1 = -1 \ -1 \ 2 \ -1 \ 2 \ -1 \quad \checkmark$$

$$(23) X_1^1 = -1 \ -1 \ -1 \ 2 \ -1 \ 2 \quad \checkmark$$

What I am doing is this: take $2 \ 2 \ -1 \ -1 \ -1 \ -1$ and imagine it as a heading on the group mult table. Then look at each row to see how they are reordered. So:

$$X_1^2 = \underline{0 \ 0 \ -1 \ 1 \ 1 \ -1}$$

$$(12) X_1^2 = (0 \ 0 \ -1 \ 1 \ 1 \ -1)$$

$$(13) X_1^2 = (-1 \ 1 \ 0 \ -1 \ 0 \ 1)$$

$$(23) X_1^2 = (1 \ -1 \ 1 \ 0 \ -1 \ 0)$$

$$(123) X_1^2 = (-1 \ 1 \ 0 \ -1 \ 0 \ 1)$$

$$(132) X_1^2 = (1 \ -1 \ 1 \ 0 \ -1 \ 0)$$

$$X_2^1 = \underline{(0 \ 0 \ -1 \ 1 \ -1 \ 1)}$$

$$(12) X_2^1 = (0 \ 0 \ 1 \ -1 \ 1 \ -1)$$

$$(13) X_2^1 = (-1 \ -1 \ 1 \ 0 \ 1)$$

$$(23) X_2^1 = (1 \ 1 \ -1 \ 0 \ -1 \ 0) = Y_1 = \eta_1$$

$$(123) X_2^1 = (1 \ 1 \ 0 \ -1 \ 0 \ -1) = \eta_2$$

$$(132) X_2^1 = (-1 \ -1 \ 1 \ 0 \ 1 \ 0) = -Y_1$$

daggers

$$(00-111-1) = X_2^1(12)$$

$$\rightarrow (-110-110) = X_2^1(13)$$

$$(1-1100-1) = X_2^1(23) = Y_2$$

$$(-110-110) = X_2^1(132)$$

$$(1-1100-1) = X_2^1(123) = Y_2$$

$$(001-1-11) = X_1^2(12)$$

$$(-1-10110) = X_1^2(13)$$

$$(11-100-1) = X_1^2(23)$$

$$(110-1-10) = X_1^2(132)$$

$$(-1-11001) = X_1^2(123)$$

Ways to generate functions which are "pairs" for mixed symmetry.

1. In general, we know how to get a set of partners for a rep of any dimension for arbitrary group S_r . You act on a function with the set of projectors X_i^k for any fixed copy index k , letting i go over the rows.

2. Any linear combination across the copies also works. For example, consider;

$$T_i \equiv \sum_k C_k X_i^k$$

$$\begin{aligned} \Rightarrow RT_i &= \sum_k C_k R X_i^k = \sum_k C_k \sum_j D_{ji}(R) X_j^k \\ &= \sum_j D_{ji}(R) T_j \end{aligned}$$

Since any such linear combination gives a set of acceptable "irreducible functions" or symmetrizers, you might just as well choose the simplest ones. Notice of course that you have to add the copy elements with the same weights for all i .

3. Example:

	<u>e</u>	<u>12</u>	<u>13</u>	<u>23</u>	<u>123</u>	<u>132</u>	
$X_1^1 =$	2	2	-1	-1	-1	-1	$T_1 = X_1^1 + \overset{-1}{(\sqrt{3})} X_1^2$
$X_1^2 =$	0	0	$-\sqrt{3}$	$\sqrt{3}$	$+\sqrt{3}$	$-\sqrt{3}$	$T_2 = X_2^1 + \frac{1}{\sqrt{3}} X_2^2$
$X_2^1 =$	0	0	$-\sqrt{3}$	$\sqrt{3}$	$-\sqrt{3}$	$\sqrt{3}$	
$X_2^2 =$	2	-2	1	1	-1	-1	
$T_1 =$	2	2	-2	0	0	-2	
$T_2 =$	2	-2	a mess				

So, contrary to my first impression, you cannot define T_1 and T_2 here such that each one involves only 4 group elements. Nor do Y_1 and Y_2 give you an acceptable pair.

~~6~~

4. Lets check out Y and Y' and show (I think) that they dont generate an acceptable pair:

$$\frac{1}{3}Y = P_1 = X_1' + \sqrt{3}X_2' \quad / \text{mixes rows!}$$

$$\frac{1}{3}Y' = P_2 = X_2' - \sqrt{3}X_1'$$

Well. $RP_1 = RX_1' + \sqrt{3}RX_2' = \sum_j \underline{D_{j1}(R)} X_j' + \sqrt{3} \sum_j \underline{D_{j2}(R)} X_j'$
 $= \text{you are stuck!}$

So, although these P_1 and P_2 are row and column projectors, they dont transform in the standard way and thus they wont generate a correct partner. You need a "correct pair" if you are going to utilize standard Clebsch coefficients.

5. So I seem to be stuck with these operators as my pair generators:

$$X_1' = (2 \ 2 \ -1 \ -1 \ -1 \ -1) \frac{1}{6}$$

$$X_2' = (0 \ 0 \ \sqrt{3} \ \sqrt{3} \ -\sqrt{3} \ \sqrt{3}) \frac{1}{6} :$$

6. Example: The simplest non-trivial case of generating a pair of functions like this is to assume that $f(123) = f(1)$ only. Then you find:

$$f_1' = f(1) + f(2) - 2f(3)$$

$$f_2' = \sqrt{3} (f(1) - f(2)).$$

Comment: If you are trying to make a pair of functions, you can always premultiply on the right by some fixed group element. See later.

① First, get all tensors ignoring momentum conservation

Irreducible Components of Tensor Forms.

1. Lets consider two momenta the same. In particular, take $1^{123} = p_1^{u_1} p_1^{u_2} p_2^{u_3}$. This is an example of a $f(123)$ object. I can form six irreducible objects by applying my six basic X_{ij} symmetrizers. Lets do it:

First, compute the action of all group elements on this $f(123)$ object. You should do it in two stages or however you like. When I leave the superscripts off, it means they have been ordered in "standard order" meaning $u_1 u_2 u_3$. So I find:

	<u>e</u>	<u>12</u>	<u>13</u>	<u>23</u>	<u>123</u>	<u>132</u>
112	112	221	233	131	322	313

Here by the way you see the six tensor forms that get mixed together. The complete set of "two momenta the same" raw tensor forms really has 18 elements; its like three separate functions $f(123)$ which do not mix. Lets keep to this set of 6 for now.

Now use table on page 1 of "expansions". Set the root threes equal to 1, I dont see any need to keep them, and ignore the overall factor of 6's. Then here are the six functions:

$$(112)_A = [112 - 221 - 233 - 131 + 322 + 313]$$

$$(112)_S = [112 + 221 + 233 + 131 + 322 + 313]$$

$$(112)_1 = [2 \cdot 112 + 2 \cdot 221 - 233 - 131 - 322 - 313]$$

$$(112)_1^2 = [-233 + 131 + 322 - 313] \sqrt{3}$$

$$(112)_2 = [-233 + 131 - 322 + 313] \sqrt{3}$$

$$(112)_2^2 = [2 \cdot 112 - 2 \cdot 221 + 233 + 131 - 322 - 313]$$

The only work is getting the table headings. The coefficients here are just those of the symmetrizers. So, we could replicate this information like so:

$$\begin{matrix} \sqrt{121} \\ -(112) \end{matrix}$$

	<u>121</u>	<u>122</u>	<u>323</u>	<u>113</u>	<u>223</u>	<u>133</u>
	<u>112</u>	<u>221</u>	<u>233</u>	<u>131</u>	<u>322</u>	<u>313</u>
$\begin{matrix} (121)_A \\ (112)_A \end{matrix}$	1	-1	-1	-1	1	1
$\begin{matrix} (121)_S \\ (112)_S \end{matrix}$	1	1	1	1	1	1
$\begin{matrix} (121)_1 \\ (112)_1 \end{matrix}$	2	2	-1	-1	-1	-1
$\begin{matrix} (121)_2 \\ (112)_1 \end{matrix}$			-1	+1	+1	-1 $\times \sqrt{3}$
$\begin{matrix} (141)_1 \\ (112)_2 \end{matrix}$			-1	+1	-1	+1 $\times \sqrt{3}$
$\begin{matrix} (141)_2 \\ (112)_2 \end{matrix}$	2	-2	+1	+1	-1	-1

To handle the other 12 functions in this class, we simply change the labels on the table; the entries are just the same. The facts we need are:

	<u>12</u>	<u>12</u>	<u>13</u>	<u>23</u>	<u>123</u>	<u>132</u>
112	112	221	233	131	322	313
121	121	-122	-323	-113	223	133
211	211	212	332	311	232	331

I have shown in red how you would change the headings to get the second set of 6 irreducible tensors.

Conclusion: Having done very little work, I have classified the 18 raw tensor forms which have two momenta the same into 18 irreducible combinations. Much faster than the last time i did this!!!

2. Now lets do the same thing for "all momenta different". Again, its just a question of getting the headings. We have:

	<u>e</u>	<u>12</u>	<u>13</u>	<u>23</u>	<u>123</u>	<u>132</u>	
<u>123</u>	123	123	123	123	123	123	/done earlier
<u>132</u>	132	<u>321</u>	<u>213</u>	132	321	213	/"
<u>231</u>	<u>231</u>	<u>312</u>	312	312	231	231	/"
213	213						
312	312						
321	321						

Maybe I don't have to do these last three. I am starting as usual with a set of 6 raw tensors, if I get 6 tensors of definite symmetry I know I am done.

So lets take each row as a heading and see where it leads:

$$\begin{array}{lll}
 (123)_A = 0 & (132)_A = 0 & (231)_A \stackrel{N}{=} 3 \cdot [231 - 312] \checkmark \\
 (123)_S \stackrel{N}{=} 6 \cdot 123 \checkmark & (132)_S \stackrel{N}{=} 2 [132 + 321 + 213] \checkmark & (231)_S \stackrel{N}{=} 3 [231 + 312] \checkmark \\
 (123)_1' = 0 & (132)_1' \stackrel{N}{=} \frac{[132 + 321 - 2 \cdot 213]}{\sqrt{3}} & (231)_1' = 0 \\
 (123)_1^2 = 0 & (132)_1^2 \stackrel{N}{=} (132)_1' \uparrow \sqrt{3} & (231)_1^2 = 0 \\
 (123)_2' = 0 & (132)_2' \stackrel{N}{=} \frac{[132 - 321] \cdot \sqrt{3}}{3} & (231)_2' = 0 \\
 (123)_2^2 = 0 & (132)_2^2 \stackrel{N}{=} 3 \cdot [132 - 321] \uparrow & (231)_2^2 = 0 \dots \dots
 \end{array}$$

Since we do get 6 tensors, we know we are golden. Also, notice that it is all right for the same tensore to appear in two places as shown: same row idea.

Next, lets do "all momenta the same" :

	<u>1</u>	<u>12</u>	<u>13</u>	<u>23</u>	<u>123</u>	<u>122</u>
111	111	222	333	111	222	333

a b c | a b c

222

333

$$(111)_A = 0$$

$$(111)_1^2 = \sqrt{3} (111)_1^1 \times \sqrt{3}$$

$$(111)_S = \frac{\sqrt{3}}{2} [111 + 222 + 333]$$

$$(111)_2^1 = \frac{\sqrt{3}}{2} [111 - 222] \cdot \sqrt{3}$$

$$(111)_1^1 = \frac{\sqrt{3}}{2} [111 + 222 - 2 \cdot 333]$$

$$(111)_2^2 = \frac{\sqrt{3}}{2} 3 [111 - 222]$$

Finally, lets combine one momenta with a kronecker delta.

	<u>1</u>	<u>12</u>	<u>13</u>	<u>23</u>	<u>123</u>	<u>122</u>
$\delta^{12} \underline{1}^3$	$\delta^{12} \underline{1}^3$	$\delta^{21} \underline{2}^3$	$\delta^{32} \underline{3}^1$	$\delta^{13} \underline{1}^2$	$\delta^{23} \underline{2}^1$	$\delta^{31} \underline{3}^2$
$\delta^{12} \underline{3}^3$	$\delta^{12} \underline{3}^3$	$\delta^{21} \underline{3}^3$	$\delta^{32} \underline{1}^1$	$\delta^{13} \underline{2}^2$	$\delta^{23} \underline{1}^1$	$\delta^{31} \underline{2}^2$

We should generate 9 tensors here. What are they?

$$(\delta^{12} \underline{1}^3)_A = \delta^{12} (1-2)^3 + \delta^{23} (2-3)^1 + \delta^{31} (3-1)^2$$

$$(\delta^{12} \underline{1}^3)_S = \delta^{12} (1+2)^3 + \delta^{23} (2+3)^1 + \delta^{31} (3+1)^2$$

$$(\delta^{12} \underline{1}^3)_1^1 = 2 \delta^{12} (1+2)^3 - \delta^{23} (2+3)^1 - \delta^{31} (3+1)^2$$

$$(\delta^{12} \underline{1}^3)_1^2 = [\delta^{13} (1-3)^2 + \delta^{23} (2-3)^1] \sqrt{3}$$

$$(\delta^{12} \underline{1}^3)_2^1 = [\delta^{13} (1+3)^2 - \delta^{23} (2+3)^1] \sqrt{3}$$

$$(\delta^{12} \underline{1}^3)_2^2 = 2 \delta^{12} (1-2)^3 + \delta^{23} (3-2)^1 + \delta^{13} (1-3)^2$$

The second header gives these tensors:

$$(\delta_{12}^3)_A = 0$$

$$(\delta_{33}^{12})_S = 2 [\delta_{33}^{12} + \delta_{11}^{23} + \delta_{22}^{31}]$$

$$(\delta_{33}^{12})'_1 = 2 \cancel{1} [2 \cdot \delta_{33}^{12} - \delta_{11}^{23} - \delta_{22}^{31}]$$

$$(\delta_{33}^{12})_1^2 = 0$$

$$(\delta_{33}^{12})'_2 = 2 \cancel{1} [\delta_{22}^{31} - \delta_{11}^{23}] : \sqrt{3}$$

$$(\delta_{33}^{12})_2^2 = 0$$

② Now set $3 = -1 - 2 +$ count independent tensors of PPP class.

$$(112)A = b - g - \cancel{e} - \cancel{f} - g - h + a + c - d - h + a + b + \cancel{e} + \cancel{f}$$

$$= 2a + 2b + c - d - 2g - 2h$$

	111	112	121	122	211	212	221	222
	a	b	c	d	e	f	g	h
111	+							
112		+						
113	-	-						
121			+					
122				+				
123			-	-				
131	-		-					
132		-		-				
133	+	+	+	+				
211					+			
212						+		
213					-	-		
221							+	
222								+
223							-	-
231					-	-	-	-
232						-	-	-
233					+	+	+	+
311	-				-			
312		-				-		
313	+	+			+	+		
321			-				-	
322				-				-
323			+	+			+	+
331	+		+		+		+	
332		+		+		+		+
333	-	-	-	-	-	-	-	-

Each row of this table tells you how each of the 27 momenta objects reduces to a linear combination of the remaining 8 independent tensor forms.

So go to your set of interest, a set involving 6 objects. Then look up each entry in this table and enter its 8-fold vector of numbers in a table with 6 rows.

Then add these 6 rows in the different ways to get your results. I.e, add the rows as if you were adding the six objects according to your rules for producing irreducible objects.

112.

$$0 - \sqrt{(12)} A$$

								1
	+						-	2
							-- --	3
+ +		+						4
			-				-	5
+ +					+ +			6 ✓
2 2	1 -1	0 0	-2 -2					

$$\sqrt{(112)s}$$

+						1
				+		2
		++		++		3
-	-					4
		-			-	5
++			++			6
0	2	-1	-1	2	2	0 ✓

$$f(112)_1^+$$

2				1
			2	2
		--	--	3
+	+			4
		+		5
--		--		6
0	1	1	1	-2 -2 1 0 ✓

$$\sqrt{(112)_1^2}$$

		— — — —	3
—	—		4
	—		5
—	—	— —	6
-2-1	-1-1	-2-2	-1-2 ✓

$$f(112)_2^+$$

				-	-	-	3
-		-					4
			+			+	5
+	+			++			6
0	1	-1	1	0	0	-1	0

$$\sqrt{(112)_2}^2$$

1	2					
2						
3					++	++
4	-	-				
5			+			+
6	-	-			--	
7	-2	1	-1	1	0 0	2
						-1

121

(121)_A

1.	+			-	+	
2.		+		-		
3.	++		++	-	--	--
4.	--			-	++	
5.			--	-	00	22
6.	++	++		1	++	++

$(121)_3 = 00 \ 33 \ 00 \ 00 \quad 22 \ 11 \ 11 \ 11 = (121)_A$

$(121)_1$

1.	+2		
2.	+2		
3.	--	--	
4.	++		
5.		++	
6.	--	--	

$(121)_1 = 00 \ 00 \ 00 \ 00$

$(121)_2$

1.			
2.			
3.	--	--	--
4.	--		
5.			--
6.	--	--	

$(121)_2 = -2-2 \ -2-2 \ 00 \ -2-2$

$(112)_2$

1.			
2.			
3.	--	--	
4.	--		
5.		++	
6.	++	++	

$(112)_2 = 00 \ 00 \ 00 \ 00$

$(112)_1$

1.	2		
2.	-2		
3.	++	++	
4.	--		
5.		++	
6.	--	--	

$(112)_1 = -2-2 \ 2-2 \ 00 \ 22$

211

1		2	
2		3	
3	-	-	-
4	+	+	
5		+	+
6	-	-	-

1				
2				
3	-	-	-	-
4	-		-	
5			-	-
6	-	-	-	-

1				
2				
3	-	-	-	-
4	-		-	
5			+	+
6	+	+	+	+

1		2
2		-2
3	+	+
4	-	-
5		+
6	-	-

$$(211)_2 = \begin{array}{ccc|cc} -2 & 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 & 0 \end{array}$$

(123)

$$\sqrt{(123)}_S = 6 \begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} = 00, -6, -6, 00, 00$$

132

$$\sqrt{(132)}_S = \begin{pmatrix} \begin{array}{c|c|c|c|c|c} 1. & - & - & & & \\ 2. & & - & & - & \\ 3. & & & - & - & \end{array} \\ \hline 0-1 & -1-1 & -1-1 & -1-0 \end{pmatrix} \times 2$$

$$\sqrt{(132)}'_1 = \begin{pmatrix} \begin{array}{c|c|c|c|c|c} 1. & - & - & & & \\ 2. & & - & & - & \\ 3. & & & 22 & & \end{array} \\ \hline 0-1 & -1-1 & 22 & -1-0 \end{pmatrix} \checkmark$$

$$\sqrt{(132)}'_2 = \begin{pmatrix} \begin{array}{c|c|c|c|c|c} 1. & - & - & & & \\ 2. & & + & & + & \\ 3. & & & & & \end{array} \\ \hline 0-1 & -1-1 & 00 & 10 \end{pmatrix} \sqrt{3}$$

231

$$\sqrt{(231)}_S = \begin{pmatrix} \begin{array}{c|c|c|c|c|c} 1. & - & - & & & \\ 2. & - & & - & & \\ 3. & & & & - & \end{array} \\ \hline 0-1 & 00 & -1-1 & -1-0 \end{pmatrix} 3 \quad \sqrt{(231)}'_1 = \begin{pmatrix} \begin{array}{c|c|c|c|c|c} 1. & & & & & \\ 2. & & & & & \\ 3. & & & & & \end{array} \\ \hline 01 & 00 & -1 & 1 & -1-0 \end{pmatrix} 3$$

111

$$\sqrt{(111)}_S = \begin{pmatrix} \begin{array}{c|c|c|c|c|c} 1. & + & & & & \\ 2. & & & & + & \\ 3. & - & - & - & - & \end{array} \\ \hline 0-1 & -1-1 & -1-1 & -1-0 \end{pmatrix} \times 2 \quad \sqrt{(111)}'_1 = \begin{pmatrix} \begin{array}{c|c|c|c|c|c} 1. & + & & & & \\ 2. & & & & & \\ 3. & & & & & \end{array} \\ \hline 10 & 00 & 00 & 0-1 \end{pmatrix} 3$$

$$\sqrt{(111)}'_2 = \begin{pmatrix} \begin{array}{c|c|c|c|c|c} 1. & + & & & & \\ 2. & & & & & \\ 3. & & & & & \end{array} \\ \hline 22 & 22 & 22 & 22 & 22 \\ 32 & 22 & 22 & 23 \end{pmatrix} \checkmark$$

A: $\checkmark (11\bar{2})_A = 22, 1-1, 00, -2, -2 \leftarrow$ all are indep.
 ~~$(1\bar{2}1)_A = 22, 1-1, 00, -2, -2$~~
 ~~$\rightarrow (211)_A = 22, 1-1, 3-3, 1-2$~~
 $\checkmark (231)_A = 01, 00, -11, -10 \leftarrow$

S: ~~$\times (11\bar{2})_S = 02, 1-1, 22, 20$~~
 $\checkmark (1\bar{2}1)_S = 00, 33, 00, 00 \leftarrow$
 ~~$\times (211)_S = 01, 11, 11, 10$~~
 $\checkmark \times (1\bar{2}3)_S = 00, 6-6, 00, 00$
 ~~$\times (132)_S = 01, 1-1, 1-1, 10$~~
 $\checkmark (231)_S = 0-1, 00, -1-1, -10 \leftarrow$
 ~~$\times (111)_S = 0-1, 1-1, 1-1, 10$~~

(1) $\checkmark (11\bar{2})_1 = 01, 11, -2-2, 10$
 ~~$\times (1\bar{2}1)_1 = 00, 00, 00, 00$~~
 ~~$\times (211)_1 = 0-1, 1-1, 22, 10$~~
 ~~$\times (132)_1 = 0-1, 1-1, 22, 10$~~
 ~~$\times (111)_1 = 32, 22, 22, 23$~~

(2) ~~$\times (11\bar{2})_2 = -2-2, 1-1, -2-2, 1-2$~~
 $\checkmark (1\bar{2}1)_2 = -2-2, -2-2, 00, -2-2 = 33, 33, 00, 33$
 ~~$\times (211)_2 = -2-1, 1-1, -2-2, 1-2$~~

(1/2) $\checkmark (11\bar{2})_{1/2} = 01, -11, 00, -10$
 ~~$\times (1\bar{2}1)_{1/2} = 00, 00, 00, 00$~~
 ~~$\times (211)_{1/2} = 0-1, 1-1, 00, 10$~~
 ~~$\times (132)_{1/2} = 0-1, 1-1, 00, 10$~~

(2/2) ~~$\rightarrow (11\bar{2})_2 = -21, 11, 00, 32$~~
 ~~$\rightarrow (1\bar{2}1)_2 = -2-2, 2-2, 00, 22$~~
 ~~$\rightarrow (211)_2 = -21, 11, 00, 12$~~
 $\checkmark (111)_2 = 10, 00, 00, 0-1$

Summary to this Point.

1. I figured out that X_{11} X_{12} X_{21} X_{22} really are the correct projector objects to use to get functions which belong to the two M-reps.
2. Next, I took the 27 triple-momenta tensor forms and I completely classified them according to their symmetry. These 27 forms grouped like so:

	<u>all momenta different</u>	<u>two momenta same</u>	<u>three momenta same</u>	<u>total</u>
A	1	3	0	4
S	3	3	1	7
X_1	1	6	1	8
X_2	1	6	1	8
				<u>27</u>

3. Next, I classified the 9 forms involving g_{uv} .

4. Next, I replaced $3 = -1-2$ and figured out how many tensors were left and in what categories. Here are those results: We know there are 8 raw tensors available, this means we expect to get 8 linear combinations grouped by symmetry. It turns out that there are ~~xxxxxx~~ two forms in each of the four groups:

	<u>111</u>	<u>112</u>	<u>121</u>	<u>122</u>	<u>211</u>	<u>212</u>	<u>221</u>	<u>222</u>	
f_{A1}	2	2	1	-1	0	0	-2	-2	$\doteq (112)_A$
f_{A2}	0	1	0	0	-1	1	-1	0	$\doteq (231)_A$
f_{S1}	0	0	1	1	0	0	0	0	$\doteq (121)_S$
f_{S2}	0	+1	0	0	1	1	1	0	$\doteq (231)_S$
f_1^{M1}	0	1	1	1	-2	-2	1	0	$\doteq (112)_1^1$
f_1^{M2}	1	1	1	1	0	0	1	1	$\doteq (121)_1^2$
f_2^{M1}	0	1	-1	1	0	0	-1	0	$\doteq (112)_2^1$
f_2^{M2}	1	0	0	0	0	0	0	-1	$\doteq (111)_2^2$

(3) General P form, ppp class only.

Choosing 8 Basic Tensors for triple glue. (3-mmta)

1. Once it has been established that there are precisely two tensors in each category, you can choose the pairs within each symmetry category in any way you want, providing you don't accidentally pick the same one or providing one doesn't vanish. I will try to choose these forms which are easiest to write down. So here is my latest choice:

$$(123)_S = 123$$

$$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$$

$$(231)_S = 231 + 312$$

$$(231)_A = 231 - 312$$

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$(112)_A = 112 - 221 - 233 - 131 + 322 + 313$$

$$(132)_1 \stackrel{N}{=} 132 + 321 - 2 \cdot 213$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$$

$$(132)_2 \stackrel{N}{=} (132 - 321) \cdot \sqrt{3}$$

$$\begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}$$

$$(111)_1 \stackrel{N}{=} 111 + 222 - 2 \cdot 333$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$$

$$(111)_2 \stackrel{N}{=} (111 - 222) \cdot \sqrt{3}$$

$$\begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}$$

Comments: Notice that, concerning the "pairs" of functions for the mixed rep, the pairs really do have to be pairs in this sense: they transform among themselves exactly as prescribed by the D matrices. This means that they have to be X_j^i projections of the same object. You cannot just take an arbitrary "first row" function and expect it to be a pair to some unrelated "second row function". They have to be an honest pair.

Also, in order to transform correctly they have to have relatively correct normalization! I have made sure that, in the projection process, I have not dropped any factors of 2 or 3 or signs (red N's). Of course this normalization question does not arise for the one dim reps A and S.

2. General form of the vertex so far:

$$\begin{aligned}
 A \Gamma(123) = & f^{S1} \cdot (231)_A + f^{S2} \cdot (112)_A \\
 & + f^{A1} \cdot (123)_S + f^{A2} \cdot (231)_S \\
 & + \frac{1}{\sqrt{2}} \left[f_1^{M1} (132)_2' - f_2^{M1} (132)_1' \right] \\
 & + \frac{1}{\sqrt{2}} \left[f_1^{M2} (111)_2' - f_2^{M2} (111)_1' \right]
 \end{aligned}$$

I think you could argue that there are in fact six arbitrary functions involved here. For example, you could say:

$$\begin{aligned}
 f^{S1} &= X_S f_1(123) & f^{A1} &= X_A f_3(123) \\
 f^{S2} &= X_S f_2(123) & f^{A2} &= X_A f_4(123) \\
 \begin{cases} f_1^{M1} \\ f_2^{M1} \end{cases} &= X_1' f_5(123) & \begin{cases} f_1^{M2} \\ f_2^{M2} \end{cases} &= X_1' f_6(123) \\
 & & & f_2^{M2} = X_2' f_6(123).
 \end{aligned}$$

In its most general form, this amplitude involves six different and unrelated functions! I have shown a possible way to generate function-pairs using X_{11} and X_{21} acting on arbitrary function. Could also have used the other pair of projectors, no matter.

3. Now let's examine the mixed-terms in light of result of next section: I.e, suppose we restrict our interest to functions of a single permutative variable like $f(1)$ instead of $f(123)$. Then we at once get the basis functions as shown in next section. Putting these functions in we get:

$$f_5(123) = f(1) \Rightarrow f_6(123) = g(1)$$

$$\begin{aligned} (\text{mixed term \#1}) &\doteq [1\bar{3}2] (f(2) - f(3)) \\ &+ [3\bar{2}1] (f(3) - f(1)) \\ &+ [2\bar{1}3] (f(1) - f(2)) \end{aligned}$$

$$\begin{aligned} (\text{mixed term \#2}) &\doteq [1\bar{1}3] (g(2) - g(3)) \\ &+ [2\bar{2}1] (g(3) - g(1)) \\ &+ [3\bar{3}2] (g(1) - g(2)) \end{aligned}$$

You can check that each of these terms is cyclically symmetric. That must be the case since each term should be totally antisymmetric.

4. General Case for mixed terms: If you just multiply things out keeping arbitrary functions f_5 and f_6 here is what you get:

$$\text{mixed term \#1} = [1\bar{3}2] (f_5(123) + f_5(213) - f_5(132) - f_5(312)) + \text{cyclic}$$

$$\text{mixed term \#2} = [1\bar{1}3] (f_6(123) + f_6(213) - f_6(132) - f_6(312)) + \text{cyclic}$$

be careful.

5. Here then is my most general form except for the G_{uv} terms:

$$\begin{aligned}
 \Gamma_A^{\mu_1 \mu_2 \mu_3}(123) = & [P_2^{\mu_1 \mu_2 \mu_3} P_3^{\mu_1 \mu_2 \mu_3} - P_3^{\mu_1 \mu_2 \mu_3} P_2^{\mu_1 \mu_2 \mu_3}] P_S f_1(123) \\
 & + [(P_1^{\mu_1 \mu_2 \mu_3} P_2^{\mu_1 \mu_2 \mu_3} - P_2^{\mu_1 \mu_2 \mu_3} P_1^{\mu_1 \mu_2 \mu_3}) + \text{cyclic}] P_S f_2(123) \\
 & + [P_1^{\mu_1 \mu_2 \mu_3} P_2^{\mu_1 \mu_2 \mu_3} P_3^{\mu_1 \mu_2 \mu_3}] P_A f_3(123) \\
 & + [P_2^{\mu_1 \mu_2 \mu_3} P_3^{\mu_1 \mu_2 \mu_3} P_1^{\mu_1 \mu_2 \mu_3} + P_3^{\mu_1 \mu_2 \mu_3} P_1^{\mu_1 \mu_2 \mu_3} P_2^{\mu_1 \mu_2 \mu_3}] P_A f_4(123) \\
 & + \{ [P_1^{\mu_1 \mu_2 \mu_3} P_3^{\mu_1 \mu_2 \mu_3} P_2^{\mu_1 \mu_2 \mu_3}] P_M f_5(123) + \text{cyclic} \} \\
 & + \{ [P_1^{\mu_1 \mu_2 \mu_3} P_2^{\mu_1 \mu_2 \mu_3} P_3^{\mu_1 \mu_2 \mu_3}] P_M f_6(123) + \text{cyclic} \}
 \end{aligned}$$

OK but see later.

where

$$\begin{cases}
 P_S \equiv e + (12) + (13) + (23) + (123) + (132) \\
 P_A \equiv e - (12) - (13) - (23) + (123) + (132) \\
 P_M \equiv e + (12) - (23) - (132) = (123) \chi_2^1
 \end{cases}$$

Thus, there are six independent, unrelated scalar functions which I have called $f_1 \dots f_6$. Notice that the symmetrizers do not act on the function names like the 1 of f_1 . The above form is the most general possible form the triple vertex could have (I am assuming a color tensor of the usual $C_{a_1 a_2 a_3}$ to the above is really the most general antisymmetric tensor you can make using triple momentum terms.

Next step is to add the G_{uv} terms and then think about implications of the Ward Identity.

$$\overbrace{[132+321-2\cdot 213]}^{nw1} \cdot \overbrace{(0\ 0\ -1\ 1\ -1\ 1)}^{nw2} \cdot \sqrt[3]{3} \cdot \frac{1}{6} - 4a -$$

$$\bullet - [132-321] \cdot \sqrt[3]{3} \underbrace{(2\ 2\ -1\ -1\ -1\ -1)}_{\sim P_M} \cdot \frac{1}{6}$$

$$= [132] (-2\ -2\ 0\ 2\ 0\ 2) \quad (11\ 0\ -1\ 0\ -1) = P_M.$$

$$+ [321] (2\ 2\ -2\ 0\ -2\ 0)$$

$$+ [213] \underbrace{(0\ 0\ 2\ -2\ 2\ -2)}_{\sim X'_2 =}$$

$$\equiv [213] X'_2 f.$$

So either

a) $[132] P_M f + \text{cyclic.}$

b) $[213] X'_2 f + \text{cyclic.}$

Repeat all this for
[111] stuff

or

a) $[111] P_M f + \dots$

b) $[333] X'_2 f + \text{cyc.}$

④ Analysis of the $g_{\mu\nu}$ tensor forms when momentum conserved.

①

	δ_1^{123}	δ_2^{123}	δ_{-1}^{231}	δ_2^{231}	δ_1^{312}	δ_2^{312}	
--	------------------	------------------	---------------------	------------------	------------------	------------------	--

$$(\delta^{123})_A \quad 1 \quad -1 \quad 1 \quad 2 \quad -2 \quad -1 \quad \checkmark$$

$$(\delta^{123})_S \quad 1 \quad 1 \quad -1 \quad 0 \quad 0 \quad -1 \quad \checkmark$$

$$(\delta^{123})_1^1 \quad 2 \quad 2 \quad 1 \quad 0 \quad 0 \quad 1 \quad = \sqrt{\frac{1}{2}} (\delta^{123})_1^1$$

$$(\delta^{123})_1^2 \quad \left(\begin{array}{cccccc} 0 & 0 & 1 & 2 & 2 & 1 \end{array} \right) \times \sqrt{3}$$

$$(\delta^{123})_2^1 \quad \left(\begin{array}{cccccc} 0 & 0 & 1 & 0 & 0 & -1 \end{array} \right) \times \sqrt{3} (\delta^{123})_2^1$$

$$(\delta^{123})_2^2 \quad 2 \quad -2 \quad -1 \quad -2 \quad 2 \quad 1 \quad \checkmark$$

$$(\delta^{123})_S \quad -2 \quad -2 \quad 2 \quad 0 \quad 0 \quad 2 \quad = -(\delta^{123})_S$$

$$(\delta^{123})_1^1 \quad -4 \quad -4 \quad -2 \quad 0 \quad 0 \quad -2 \quad \checkmark$$

$$(\delta^{123})_2^1 \quad \left(\begin{array}{cccccc} 0 & 0 & -2 & 0 & 0 & 2 \end{array} \right) \times \sqrt{3}$$

$$(\delta^{123})_A$$

$$(\delta^{123})_S$$

$$\left\{ \begin{array}{l} (\delta^{123})_1^2 \\ (\delta^{123})_2^2 \end{array} \right.$$

$$\left\{ \begin{array}{l} (\delta^{123})_1^1 \\ (\delta^{123})_2^1 \end{array} \right.$$

$$(\delta^{12} 1^3)_1 = [\delta^{13} (1-3)^2 + \delta^{23} (2-3)^1] \sqrt{3}$$

$$\odot (\delta^{12} 1^3)_2 = 2 \cdot \delta^{12} (1-2)^3 + \delta^{23} (3-2)^1 + \delta^{13} (1-3)^2$$

Combine with a "pair":

$$\frac{1}{\sqrt{3}} [2 \cdot \delta^{12} (1-2)^3 + \delta^{23} (3-2)^1 + \delta^{13} (1-3)^2] \begin{pmatrix} 2 & 2 & -1 & -1 & -1 & -1 \end{pmatrix} \frac{1}{6} \quad \text{row 2}$$

$$\frac{1}{\sqrt{3}} [\delta^{13} (1-3)^2 + \delta^{23} (2-3)^1] \cdot \sqrt{3} \begin{pmatrix} 0 & 0 & -1 & 1 & -1 & 1 \end{pmatrix} \frac{1}{6} \quad \text{row 1}$$

$$= \delta^{12} (1-2)^3 (44 - 2 - 2 - 2 - 2) = \delta^{12} (1-2)^3 \cdot 2 \cdot X_1^1$$

$$+ \delta^{23} (2-3)^1 (-2 - 2 + 4 - 2 + 2) = \delta^{23} (2-3)^1 \cdot 2 \cdot (123) X_1^1$$

$$+ \delta^{31} (3-1)^2 (-2 - 2 + 4 - 2 + 4) = \delta^{31} (3-1)^2 \cdot 2 \cdot (132) X_1^1$$

$$= \delta^{12} (1-2)^3 \cdot 2 X_1^1 + \text{cyc.}$$

$$= \delta^{12} (1-2)^3 (44 - 2 - 2 - 2 - 2) + \text{cyclic} \quad \text{row 1} \quad X_1^1$$

Question: suppose you combined these functions with the other function pair implied by the other projectors X_1^2 and X_2^2 . Would you get a different answer here?

$$[2 \cdot \delta^{12} (1-2)^3 + \delta^{23} (3-2)^1 + \delta^{13} (1-3)^2] \begin{pmatrix} 0 & 0 & -1 & 1 & -1 & -1 \end{pmatrix} \frac{1}{6} \sqrt{3} \cdot \frac{1}{6}$$

$$\frac{1}{\sqrt{3}} [\delta^{13} (1-3)^2 + \delta^{23} (2-3)^1] \cdot \sqrt{3} \cdot \begin{pmatrix} 2 & -2 & 1 & -1 & -1 & 1 \end{pmatrix} \frac{1}{6}$$

$$= \delta^{12} (1-2)^3 \begin{pmatrix} 0 & 0 & -2 & 2 & 2 & -2 \end{pmatrix} \quad (123) X_1^2 \checkmark$$

$$+ \delta^{23} (2-3)^1 \begin{pmatrix} -2 & 2 & 0 & 0 & 0 & 0 \end{pmatrix} \quad (132) X_1^2 \checkmark$$

$$+ \delta^{31} (3-1)^2 \begin{pmatrix} 2 & -2 & 0 & 0 & 0 & 0 \end{pmatrix} \quad X_1^2$$

$$= \delta^{12} (1-2)^3 X_1^2 + \text{cyclic} !!$$

$$\rightarrow \begin{pmatrix} 1 & -1 & 1 & 0 & -1 & 0 \end{pmatrix} \equiv P_M^1$$

$$\Rightarrow \delta^{12} (1-2)^3 X_1^2 + \text{cyclic}$$

$$\delta^{23} (2-3)^1 P_M^1 + \text{cyclic}$$

$$\delta^{13} (1-3)^2 X_1^2 + \text{cyclic}$$

$$(\delta^{12} 3^3)_1 = [\delta^{23} 1^1 + \delta^{31} 2^2 - 2 \cdot \delta^{12} 3^3]$$

$$(\delta^{12} 3^3)_2 = [\delta^{23} 1^1 - \delta^{31} 2^2] \cdot \sqrt{3}$$

It seems to me that these guys combine just the way the [123] triple momenta forms did. Replace things in the obvious way. Result will be:

$$(\delta^{23} 1^1) \rho_M f + \text{cyclic}, \quad \text{or} \quad (\delta^{12} 3^3) X_2^1 f + \text{cyc.}$$

So here are the results: The g_{uv} "mixed" terms look like this:

$$\left. \begin{aligned} & [\delta^{12} (1-2)^3 X_1^1 f + \text{cyclic}] \\ & [\delta^{12} (1-2)^3 X_1^2 f' + \text{cyclic}] \end{aligned} \right\} \text{ equivalent. Let } f' = X_2^1 f$$

I am still not sure whether or not to count both of these possibilities separately, or are they essentially the same? There really are two different ways to project a first-row function out of an arbitrary function.

For any function f you use in the first form, I can use $X_2^1 f$ for my f' and the second form will duplicate it. Alternatively, if you give me an f' , and use the second form, I can use $f = X_1^2 f'$ and duplicate your result from the first form. So if f or f' is allowed to be arbitrary, then really one form will do.

Other one:

$$\begin{array}{c} [\delta^{12} 3^3 X_2^1 f + \text{cyclic}] \\ \uparrow \\ X_2^1 \end{array}$$

Comment: Recall that the matrix for 12 is $D(12) = \text{diag}(1, -1)$. This tells you that if you multiply any "first row" object by (12), you get same thing back times +1, whereas if you take a "second row" object and multiply (on the left) by (12), you get same thing back but with a minus sign. In particular, we have:

$$(12) X_1^i = + X_1^i$$

$$(12) X_2^i = - X_2^i$$

We can use this observation to verify that our g_{uv} tensor forms are in fact totally antisymmetric. As shown above, they are clearly cyclically symmetric, so all you have to do is show that thing negates under (12). Lets try it for first form:

$$(12) \{ \delta^{12} (1-2)^3 X_1^1 f \} = \delta^{21} (2-1)^3 \underbrace{(12) X_1^1}_{X_1^1} f = - \{ \}.$$

So the first term is 12 antisymmetric. You dont have to even look at the other terms because they will automatically be 12 antisymmetric. Here is a proof:

$$\begin{aligned} \{ f(123) + \text{cyclic} \} &= [e + (123) + (132)] f(123) \\ (12) \{ \} &= (12) [e + (123) + (132)] (12)^{-1} \underbrace{(12) f(123)}_{-f(123)} \\ &= - \left[(12) [e + (123) + (132)] (12)^{-1} f(123) \right] \end{aligned}$$

But recall that any class is just rearranged by conjugation with any group element. In this case the class containing the cyclic and anticyclic permatotors has its elements switched. QED.

So showing that the first function $f(123)$ was 12 anti sym is sufficient together with the cyclic symmetry to show that the entire thing is 12 antisymmetric. But of course this implies anti sym on any other pair, etc. Eg:

$$(13) \{ f(123) + \text{cyc} \} = (12)(132) \{ \} = (12) \{ \} = - \{ \} \checkmark.$$

Similarly, it is clear that the second form is also antisymmetric.

Restate the two mixed forms:

$$\{ \delta^{12} (1-2)^3 X_1^2 f + \text{cyclic} \}$$

$$\{ \delta^{12} 3^3 X_2^1 g + \text{cyclic} \}$$

$$\{ \delta^{12} (1-2)^3 + \text{cyclic} \} = (\delta^{12} 1^3) / \Lambda$$

$$\{ \delta^{12} 3^3 + \text{cyclic} \} = (\delta^{12} 3^3) / S$$

So here is our most general tensor form for vertex with only g_{uv} terms:

$$\begin{aligned}
 p^{\mu_1 \mu_2 \mu_3}(123) = & [(\delta^{\mu_1 \mu_2} [p_1 - p_2]^{\mu_3} + \text{cyclic})] P_5 \cdot f_9(123) \\
 & + [(\delta^{\mu_1 \mu_2} p_3^{\mu_3} + \text{cyclic})] P_A \cdot f_8(123) \\
 & + \{ \delta^{\mu_1 \mu_2} (p_1 - p_2)^{\mu_3} X_1^2 f_9(123) + \text{cyclic} \} \\
 & + \{ \delta^{\mu_1 \mu_2} p_3^{\mu_3} X_2^1 f_{10}(123) + \text{cyclic} \}
 \end{aligned}$$

Of course we can replace the X's with slightly simpler permeters as follows:

$$X_1^2 f_9 = \underbrace{X_1^2(23)(23)}_{\tilde{P}_{M1}} f_9 = \underbrace{(11-100-1)}_{e+(12)-(13)-(132)} \tilde{f}_9 \equiv \tilde{P}_{M1}$$

$$X_2^1 f_{10} = \underbrace{X_2^1(23)(23)}_{\tilde{P}_{M2}} f_{10} = \underbrace{(1-1100-1)}_{e-(12)+(13)-(132)} \tilde{f}_{10} \equiv \tilde{P}_{M2}$$

Actually, I'd rather do this in a way that yields the forward cyclic and not the reverse cyclic, just to avoid later confusion. So:

$$X_1^2 f_9 = X_1^2(132)(123) f_9 = \underbrace{(110-1-10)}_{e+(12)-(23)-(123)} f_9' \equiv P_{M1}$$

$$X_2^1 f_{10} = X_2^1(132)(123) f_{10} = \underbrace{(1-101-10)}_{e-(12)+(23)-(123)} f_{10}' \equiv P_{M2}$$

absol.

A summary of independent tensor forms (no η 's).

all correctly normalized.

① First, here is a list of all forms same moment:

$$(112)_S = [+112 + 221 + 233 + 131 + 322 + 313]$$

$$02 \ -1 \ -1 \ 22 \ 20$$

$$(112)_A = [+112 - 221 - 233 - 131 + 322 + 313]$$

$$22 \ +1 \ -1 \ 00 \ -2 \ -2$$

$$\{(112)_1^1 = [+2 \cdot 112 + 2 \cdot 221 - 233 - 131 - 322 - 313]$$

$$01 \ 11 \ -2 \ -2 \ 10$$

$$\{(112)_2^1 = \sqrt{3}[-233 + 131 - 322 + 313]$$

$$\sqrt{3}(01 \ -11 \ 00 \ -10)$$

$$\{(112)_1^2 = \sqrt{3}[-233 + 131 + 322 - 313]$$

$$\sqrt{3}(-2 \ -1 \ -1 \ -2 \ -1 \ -2)$$

$$\{(112)_2^2 = [2 \cdot 112 - 2 \cdot 221 + 233 + 131 - 322 - 313]$$

$$-21 \ -11 \ 00 \ -12$$

$$(121)_S = [+121 + 122 + 323 + 113 + 223 + 133]$$

$$00 \ 33 \ 00 \ 00$$

$$(121)_A = [+121 - 122 - 323 - 113 + 223 + 133]$$

$$22 \ 1 \ -1 \ 00 \ -2 \ -2$$

$$\{(121)_1^1 = [+2 \cdot 121 + 2 \cdot 122 - 323 - 113 - 223 - 133]$$

$$00 \ 00 \ 00 \ 00$$

$$\{(121)_2^1 = \sqrt{3}[-323 + 113 - 223 + 133]$$

$$\sqrt{3}(00 \ 00 \ 00 \ 00)$$

$$\{(121)_1^2 = \sqrt{3}[-323 + 113 + 223 - 133]$$

$$\sqrt{3}(-2 \ -2 \ -2 \ 00 \ -2 \ -2)$$

$$\{(121)_2^2 = [+2 \cdot 121 - 2 \cdot 122 + 323 + 113 - 223 - 133]$$

$$-2 \ -2 \ 2 \ -2 \ 00 \ 22$$

$$(211)_S = [+211 + 212 + 332 + 311 + 232 + 331]$$

$$01 \ 11 \ 11 \ 10$$

$$(211)_A = [+211 - 212 - 332 - 311 + 232 + 331]$$

$$2 \ -1 \ -1 \ 3 \ -3 \ 1 \ -2$$

$$\{(211)_1^1 = [+2 \cdot 211 + 2 \cdot 212 - 332 - 311 - 232 - 331]$$

$$0 \ -1 \ -1 \ 22 \ -10$$

$$\{(211)_2^1 = \sqrt{3}[-332 + 311 - 232 + 331]$$

$$\sqrt{3}(0 \ -1 \ -1 \ 00 \ 10)$$

$$\{(211)_1^2 = \sqrt{3}[-332 + 311 + 232 - 331]$$

$$\sqrt{3}(-2 \ -1 \ -1 \ -2 \ -2 \ -1 \ -2)$$

$$\{(211)_2^2 = [+2 \cdot 211 - 2 \cdot 212 + 332 + 311 - 232 - 331]$$

$$-21 \ -11 \ 00 \ -12$$

$$(123)_S = 6[123]$$

$$00 \ +6 \ +6 \ 00 \ 00$$

$$(132)_S = 2[132 + 321 + 213]$$

$$2(0 \ -1 \ -1 \ -1 \ -1 \ -10)$$

$$\{(132)_1^1 = [132 + 321 - 2 \cdot 213]$$

$$0 \ -1 \ -1 \ -1 \ 22 \ -10$$

$$\{(132)_2^1 = \sqrt{3}[132 - 321]$$

$$0 \ -1 \ -1 \ 00 \ 10$$

$$(231)_S = 3[231 + 312]$$

$$3(0 \ -1 \ 00 \ -1 \ -1 \ -10)$$

$$(231)_A = 3[231 - 312]$$

$$3(01 \ 00 \ -1 \ -1 \ -10)$$

$$(111)_S = 2[111 + 222 + 333]$$

$$2(0 \ -1 \ -1 \ -1 \ -1 \ -10)$$

$$\{(111)_1^1 = [111 + 222 - 2 \cdot 333]$$

$$32 \ 22 \ 22 \ 23$$

$$\{(111)_2^1 = \sqrt{3}[111 - 222]$$

$$\sqrt{3}(10 \ 00 \ 00 \ 0 \ -1)$$

$$\begin{aligned}
 (\delta^{12} 1^3)_S &= [\delta^{12} (1+2)^3 + \delta^{23} (2+3)^1 + \delta^{31} (3+1)^2] & 11 & -10 & 0 & -1 \\
 (\delta^{12} 1^3)_A &= [\delta^{12} (1-2)^3 + \delta^{23} (2-3)^1 + \delta^{31} (3-1)^2] & 1 & -1 & 12 & -2 & -1 \\
 \{(\delta^{12} 1^3)_1\} &= [2\delta^{12} (1+2)^3 - \delta^{23} (2+3)^1 - \delta^{31} (3+1)^2] & 22 & 10 & 0 & 1 \\
 \{(\delta^{12} 1^3)_2\} &= \sqrt{3} [-\delta^{23} (2+3)^1 + \delta^{31} (3+1)^2] & \sqrt{3} & (00 & 10 & 0 & -1) \\
 \{(\delta^{12} 1^3)_3\} &= \sqrt{3} [\delta^{23} (2-3)^1 - \delta^{31} (3-1)^2] & \sqrt{3} & (00 & 12 & 2 & 1) \\
 (\delta^{12} 1^3)_2 &= [2\delta^{12} (1-2)^3 - \delta^{23} (2-3)^1 - \delta^{31} (3-1)^2] & 2 & -2 & -1 & -2 & 2 & 1 \\
 \\
 (\delta^{12} 3^3)_S &= [\delta^{12} 3^3 + \delta^{23} 1^1 + \delta^{31} 2^2] & -2 & -2 & 20 & 0 & 2 \\
 \{(\delta^{12} 3^3)_1\} &= [2 \cdot \delta^{12} 3^3 - \delta^{23} 1^1 - \delta^{31} 2^2] & -4 & -4 & -20 & 0 & -2 \\
 (\delta^{12} 3^3)_2 &= \sqrt{3} [-\delta^{23} 1^1 + \delta^{31} 2^2] & \sqrt{3} & (00 & -20 & 0 & 2)
 \end{aligned}$$

② There are 36 entries in this list. There are 9S, 5A and 11 pairs.

If we now let $3 = -1-2$, there are only 8 independent tensor forms involving three momenta, and there are only 6 involving one momenta and a d_{uv} . Thus, each of the above forms can be represented by a 8 or 6 component vector with order defined here:

triple momenta: 111 112 121 122 211 212 221 222

single momenta $d^{12} 1^3$ $d^{12} 2^3$ $d^{23} 1^1$ $d^{23} 2^1$ $d^{31} 1^2$ $d^{31} 2^2$

In the above listing, the vector for each form is listed at the right.

Now there can be at most 8 independent forms in the triple p group, and 6 in the d_{uv} group. Here are my choices, including their vectors:

$$\begin{aligned}
 (123)_S &= 123 & 00 & 11 & 00 & 00 & \times \\
 (231)_S &= 231 + 312 & 01 & 00 & 11 & 10 & \times \\
 (231)_A &= 231 - 312 & 01 & 00 & -11 & -10 & \times \\
 (112)_A &= 112 - 221 - 233 - 131 + 322 + 313 & 22 & 14 & 00 & -2 & -2 & \times \\
 \{(\delta^{12} 1^3)_1\} &= 132 + 321 - 2 \cdot 213 & 0 & -1 & -1 & 22 & -10 & \times \\
 \{(\delta^{12} 1^3)_2\} &= \sqrt{3} [132 - 321] & 0 & -1 & -1 & 00 & 10 & \times \\
 \{(\delta^{12} 1^3)_3\} &= 111 + 222 - 2 \cdot 333 & 32 & 22 & 22 & 23 & \times \\
 \{(\delta^{12} 1^3)_4\} &= \sqrt{3} [111 - 222] & 10 & 00 & 00 & 0 & -1 & \times \\
 \\
 (\delta^{12} 3^3)_S &= \delta^{12} (1+2)^3 + \delta^{23} (2+3)^1 + \delta^{31} (3+1)^2 & 11 & -10 & 0 & -1 & \\
 (\delta^{12} 3^3)_A &= \delta^{12} (1-2)^3 + \delta^{23} (2-3)^1 + \delta^{31} (3-1)^2 & 1 & -1 & 12 & -2 & -1 \\
 \{(\delta^{12} 3^3)_1\} &= \sqrt{3} [\delta^{23} (2-3)^1 - \delta^{31} (3-1)^2] & 00 & 12 & 2 & 1 \\
 \{(\delta^{12} 3^3)_2\} &= 2\delta^{12} (1-2)^3 - \delta^{23} (2-3)^1 - \delta^{31} (3-1)^2 & 2 & -2 & -1 & -2 & 2 & 1 \\
 \{(\delta^{12} 3^3)_3\} &= 2 \cdot \delta^{12} 3^3 - \delta^{23} 1^1 - \delta^{31} 2^2 & +2 & +2 & 10 & 0 & 1 \\
 \{(\delta^{12} 3^3)_4\} &= \sqrt{3} [-\delta^{23} 1^1 + \delta^{31} 2^2] & 00 & -10 & 0 & 1
 \end{aligned}$$

3. We can check the independence of the 8 vectors in the first set, and the 6 vectors in the second set, merely by making sure that $\det \neq 0$. I.e, bring each det to triangular form and make sure no zeros on the diagonal.

First set:

$$\begin{vmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -6 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 10 & 0 & 0 & 4 & 0 & 0 & 0 \\ 0 & -1 & -1 & -1 & 2 & 4 & 2 & 0 \\ 0 & 4 & 4 & 4 & 4 & -4 & 4 & 6 \end{vmatrix}$$

This is a difficult det to get; should be more systematic and show what needs not be considered. Better to let reduce do this stuff!

4. Actually there is no need to do all this; you need only check linear independence within each row of each rep. A symmetric function can never be written as a sum of functions of other symmetries. This is easy to check and you conclude that the above tensors are indeed independent.

5. What to do with the mixed rep pairs. If you have two pairs, you can make ta object like so (sed the Clebsh's):

$$\begin{aligned} \begin{cases} f_1 \\ f_2 \end{cases} \quad \begin{cases} g_1 \\ g_2 \end{cases} & \rightarrow \begin{aligned} F_3 &= f_1 g_1 + f_2 g_2 \\ F_A &= f_1 g_2 - f_2 g_1 \end{aligned} \\ & \begin{aligned} F_1 &= f_1 g_1 - f_2 g_2 \\ F_2 &= -f_1 g_2 - f_2 g_1 \end{aligned} \end{aligned}$$

So we take each of our tensor form pairs and combine with an arbitrary function pair to get full list of acceptable triple glue vertex tensor forms with functtions:

$$(123)_S f_A$$

$$(231)_S f_A$$

$$(231)_A f_S$$

$$(112)_A f_S$$

$$(132)_1 f_2 - (132)_2 f_1$$

$$(111)_1 f_2 - (111)_2 f_1$$

$$(\delta^{12} 1^3)_S f_A$$

$$(\delta^{12} 1^3)_A f_S$$

$$(\delta^{12} 1^3)_1^2 f_2 - (\delta^{12} 1^3)_2^2 f_1$$

$$(\delta^{12} 3^3)_1 f_2 - (\delta^{12} 3^3)_2 f_1$$

These then are your ten basic forms .

3. It might be nicer to write each form as something plus cyclic. This is done as follows:
for example, :

$$[(\delta^{12} 1^3)_1^2 X_2' - (\delta^{12} 1^3)_2^2 X_1'] =$$

$$[\sqrt{3} \delta^{23} (2-3)^1 - \sqrt{3} \delta^{31} (3-1)^2] X_2' - [2\delta^{12} (1-2)^3 - \delta^{23} (2-3)^1 - \delta^{31} (3-1)^2] X_1'$$

$$= -2\delta^{12} (1-2)^3 [X_1']$$

$$-2\delta^{23} (2-3)^1 [-\frac{1}{2}\sqrt{3}X_2' + \frac{1}{2}X_1']$$

$$-2\delta^{31} (3-1)^2 [+ \frac{1}{2}\sqrt{3}X_2' - \frac{1}{2}X_1']$$

$$= -2\{\delta^{12} (1-2)^3 [X_1'] + \delta^{23} (2-3)^1 [(123)X_1'] + \delta^{31} (3-1)^2 [(132)X_1']\}$$

$$\text{So } \{\delta^{12} (1-2)^3 X_1' + \text{cyclic}\}.$$

In this way I generated the 10 forms on the signal sheet.

Summary of Choices in Each Sym Category

Symmetries

$$(112)_S = 02 \ -1-1 \ 22 \ 20 = 2S_2 - S_1 \quad \checkmark$$

$$(121)_S = 00 \ 33 \ 00 \ 00 \equiv S_1 \quad \checkmark$$

$$(211)_S = 01 \ 11 \ 11 \ 10 \equiv S_2 \quad \checkmark$$

$$(123)_S = 00 \ -6-6 \ 00 \ 00 = -2S_1 \quad \checkmark$$

$$(132)_{S/\sqrt{2}} = 0-1 \ -1-1 \ -1-1-10 = -S_2$$

$$(231)_{S/\sqrt{3}} = 0-1 \ 00 \ -1-1-10 = -S_2 + \frac{1}{3}S_1$$

$$(111)_{S/\sqrt{2}} = 0-1 \ -1-1 \ -1-1-10 = -S_2$$

Anti Syms

$$(112)_A = 22 \ 1-1 \ 00 \ -2-2 \equiv A_1 \quad \checkmark$$

$$(121)_A = 22 \ 1-1 \ 00 \ -2-2 \equiv A_1 \quad \checkmark$$

$$(211)_A = 2-1 \ 1-1 \ 3-3 \ 1-2 \equiv A_1 - 3A_2 \quad \checkmark$$

$$(231)_{A/\sqrt{3}} = 01 \ 00 \ -11 \ -10 \equiv A_2$$

Pairs (row-1 only)

$$(112)_1^1 = 01 \ 11 \ -2-2 \ 10 \equiv ONE_1 = O_1 \quad \checkmark$$

$$(112)_1^2 = (-2-1 \ -1-1 \ -2-2 \ -1-2)\sqrt{3} \equiv O_2 \quad \checkmark$$

$$(121)_1^1 = 00 \ 00 \ 00 \ 00 \equiv \emptyset \quad \checkmark$$

$$(121)_1^2 = (-2-2 \ -2-2 \ 00 \ -2-2)\sqrt{3} = O_2 - \sqrt{3}O_1 \quad \checkmark$$

$$(211)_1^1 = 0-1 \ -1-1 \ 22 \ -10 = -O_1 \quad \checkmark$$

$$(211)_1^2 = (-2-1 \ -1-1 \ -2-2 \ -1-2)\sqrt{3} = O_2 \quad \checkmark$$

$$(132)_1^1 = 0-1 \ -1-1 \ 22 \ -10 = -O_1 \quad \checkmark$$

$$(111)_1^1 = 32 \ 22 \ 22 \ 23 = \frac{1}{2}O_1 - \frac{3}{2\sqrt{3}}O_2 \quad \checkmark$$

↑
note all named OK

Summary Combining the Triple Momenta forms with the g_{uv} single momenta forms:

1. First, here are all the "mixed symmetry" forms; ie, the forms which are of course totally antisymmetric, but which got that way via the mixed-symmetry reps interacting with each other:

$$\checkmark 1. [2^1 1^2 3^3] X_2^1 f_1 + \text{cyclic}$$

$$\checkmark 2. [3^1 3^2 3^3] X_2^1 f_2 + \text{cyclic}$$

$$\checkmark 3. [\delta^{12} 3^3] X_2^1 f_3 + \text{cyclic}$$

$$\checkmark 4. [\delta^{12} (1-2)^3] X_1^2 f_4 + \text{cyclic}$$

$\times$ OK!

Notice that the first three terms have a 12 symmetric bracket, whereas the last has a 12 anti sym bracket. That is why its X projector is different from the others.

2. Now the symmetric and antisymmetric:

$$\checkmark 5. [2^1 3^1 1^3 - 3^1 1^2 2^3] X_S f_5$$

$$\checkmark 6. [(1^1 1^2 2^3 - 2^1 2^2 1^3) + \text{cyclic}] X_S f_6$$

$$\checkmark 7. [\delta^{12} (1-2)^3 + \text{cyclic}] X_S f_7$$

$$8. [1^1 2^2 3^3] X_A f_8$$

$$\checkmark 9. [2^1 3^1 1^3 + 3^1 1^2 2^3] X_A f_9$$

$$10. [\delta^{12} 3^3 + \text{cyclic}] X_A f_{10}$$

3. Thus, there are 10 independent tensors forms!! If you like, you can replace the mixed projectors as follows:

$$X_2^1 \rightarrow P_{M2} \equiv e - (12) + (23) - (123) = X_2^1 \cdot (132)$$

$$X_1^2 \rightarrow P_{M1} \equiv e + (12) - (23) - (123) = X_1^2 \cdot (132)$$

June 26 '79

Check list of tensor forms.

1. First, lets review how they arrived. Prior to going on ~~mass~~ momentum conservation we had:

$$4A + 7S + 8 \text{ pairs} = 19 \text{ tensors of form } p_i p_j p_k$$

$$1A + 2S + 3 \text{ pairs} = \underline{6} \text{ tensors of } d_{ij} p_k \text{ form}$$

25 total

Next we set $3=-1-2$ and then counted only the independent tensor forms. We got:

$$2A + 2S + 2 \text{ pairs} = 6 \text{ tensors of form } p_i p_j p_k$$

$$1A + 1S + 2 \text{ pairs} = \underline{4} \text{ tensors of form } d_{ij} p_k$$

10 total

The main problem here was counting and constructing the tensors. Once you know there are ten of them, if you have a list you can check them for antisymmetry. They have already been checked for independence. This check is just to make sure no typos.

2. So lets inspect the list. In terms that are + cyclic, I have shown that you need only show 12 anti sym in the exposed term. For the first four terms in the list, this inspection is easy once you know the effect of (12) operation on the X_j^i objects: second row objects change sign, first row objects don't. That completes the verification that terms 1 through 4 are symmetric! Just look at them.

Term 5 is obvious from inspection.

Term 6 is also obvious (cyclic commutes with X_s and X_a). Term 7 obvious.

Term 8: the [...] is clearly totally symmetric: it is 12 sym and cyclic sym.

Term 9: again, bracket is cyclically sym and 12 sym

Term 10: cyclic so just check the 12, looks find.

3. I have now explicitly checked that my 10 tensor forms are completely antisymmetric. I believe this is the complete set of independent forms.

Ward 10

July 1, 1979

Contents.

Comment: This binder contains an attempt to start with the general tensor form of the triple glue vertex and see how the Ward identity restricts that form. However, I think this is all obsolete because I really assumed too many terms; there are in fact fewer than ten independent terms, right now (July 1) I am not sure how many, and how they should be written. So I have to go back and see,

1. This section leads up to the Table at the end of the section. This table has 5 columns which arise by balancing the tensor terms in the Ward identity. ~~several~~ The rows correspond to the supposed 10 terms in the vertex.

2. It seemed reasonable to take each column and project it into its symmetry components; thus each column might actually be 4 identities rather than one. In order to project each column-identity into its symmetry fractions, I had to do a certain Clebsching of products of objects. In my first version of this calculation I did that wrong. This section contains residual pages from that wrong calculation.

3. Here I correct this Clebsch error and redo the computation. The final result here is a table which lists the 10 constraints imposed by the Ward identity. I have never done anything with this table because I am about to simplify my "most general form" of the vertex.

Constraints on the General Vertex Imposed by the Ward Identity.

1. I am not yet ready to do this in generality. Right now I have not yet figured out the n -dependent terms in the vertex. Later I will have to add them. For now, I am going to ignore n -dependent terms in the vertex. Thus, no n -dependent (gauge variant, if you like) terms will be generated by the LHS of the Ward. Therefore, on the RHS I will set the function $c = 0$ (which is the simple ansatz), and then there will also be no n terms on the RHS of the Ward. I will then balance the Lorentz tensor forms and the symmetry "fractions" on both sides. If all goes well, I will obtain the following object: the Most General Form of the Vertex that contains no n -terms and which is consistent with the simple ansatz. I presume that I will get something like Marshall's result plus a set of Ward Null terms. However, I don't even want to guess or look at an answer: I want the machinery to grind out the result.

If this works out all right, I will reinsert the function c , figure out all the n tensors, and re-balance the whole thing. This is a longer problem, I prefer to do the shorter one first. Of course information from the shorter problem will be helpful in the full version of the problem.

2. There are of course three separate Ward identities on the vertex, one for each momentum. But if you arrange to have the vertex satisfy one of these Wards, it will satisfy the others since it is cyclically symmetric. I.e., the Ward in most general form now is C-symmetric for any choice of the 10 arbitrary functions. The fact that one of the Ward's puts restrictions on those 10 functions does not destroy cyclicity.

3. In passing, one wonders if there are any other constraints on the triple-gluon vertex. Bose symmetry and Ward identity in axial gauge have been considered. What is the effect of charge conjugation and parity? Also: what about gauge invariance? I.e., you could do some kind of Wigner Eckart analysis with gluons being octet members and so on. I presume that all this is contained in the Ward identity, but should check that!

On this Wigner thing: how many times does the adjoint appear in adjoint $\times$ adjoint? Let's continue this line some other time. I think it is important to know how many amplitudes you have, and so on.

4. So here is the Ward identity and its RHS:

set = 0 now.

$$3 \Gamma^{123} = \pi^{12}(1) - \pi^{12}(2) ; \quad -\pi^{12}(1) = \left[(1 \cdot 1) \delta^{12} - 1^1 1^2 \right] b(1) + Q_1^1 Q_1^2 c(1)$$

$$\Rightarrow \text{RHS Ward} = - \left[(1 \cdot 1) \delta^{12} - 1^1 1^2 \right] b(1) + \left[(2 \cdot 2) \delta^{12} - 2^1 2^2 \right] b(2)$$

S:

$$RHS_{\text{Ward}} = b(1) 1^1 1^2 - b(2) 2^1 2^2 + \delta^{12} [(2 \cdot 2) b(2) - (1 \cdot 1) b(1)]$$

Now let's attack the ten general terms, one at a time:

Term 1: In order to dot 3^3 into this thing, you really have to write out the cyclic terms completely. Suppose P is a projector of some sort acting on a function f . If you see Pf appearing in the first term, then you have to remember that cyclic implies action on this vector (Pf), it becomes $(123)Pf$ in the next term. Thus, here is what we want:

$$3^3 \{ [2^1 2^3 3^1] [X_2^1 f_1] + [3^2 2^3 1^1] [(123) X_2^1 f_1] + [1^3 3^1 2^2] [(132) X_2^1 f_1] \}$$

My general procedure will be this: dot in the 3^3 and leave dot products in raw form. But for uncontracted momenta always replace $3 = -1-2$ and then group things in a standard order in terms of the ~~XXX~~ 5 possible tensor forms. That standard order will be:

For example, here is what the above yields (done on scratch):

$$\begin{array}{ccccccc} & 1^1 1^2 & 1^1 2^2 & 2^1 1^2 & 2^1 2^1 & d^{12} & \\ & \swarrow & \downarrow & \swarrow & \swarrow & \swarrow & \\ \{ - (3 \cdot 2) \cdot (123) \}^{\checkmark} & \{ - (3 \cdot 2) (123) - (3 \cdot 1) (132) \}^{\checkmark} & \{ (3 \cdot 3) \}^{\checkmark} & \{ - (3 \cdot 1) (132) \}^{\checkmark} & \{ \phi \}^{\checkmark} & \rightarrow [X_2^1 f_1] \end{array}$$

The notation here is this: each curly bracket is a coefficient of one of the standard order tensors shown: each bracket really contains the factor way over on the right, no need to write it 4 times. The red checks mean that this version of the calculation agrees with my previous, independent calculation. So far so good.

Term 2:

$$3^3 \{ [3^1 3^3 3^3] (e) + [1^2 1^3 1^1] (123) + [2^3 2^1 2^2] (132) \} \rightarrow X_2^1 f_2$$

$$\Rightarrow \{ (3 \cdot 3) + (3 \cdot 1) (123) \}^{\checkmark} \{ (3 \cdot 3) \}^{\checkmark} \{ (3 \cdot 3) \}^{\checkmark} \{ (3 \cdot 3) + (3 \cdot 2) (132) \}^{\checkmark} \{ \phi \}^{\checkmark} \rightarrow X_2^1 f_2$$

Again, red checks indicate agreement with last calculation. So far so good again...

Term 3:

$$3^3 \{ \delta^{12} 3 + \delta^{23} 1' (123) + \delta^{31} 2^2 (132) \} \rightarrow X_2^1 f_3$$

$$\Rightarrow \left. \begin{array}{l} \checkmark 1. - (123) \\ \checkmark 2. - (123) - (132) \\ \checkmark 3. 0 \\ \checkmark 4. - (132) \\ \checkmark 5. (3 \cdot 3) \end{array} \right\} \propto X_2^1 f_3$$

Obviously the numbers 1 thru 5 refer to tensor forms in stanrd order. Easier to list them off this way. Agrees.

Term 4:

$$3^3 \{ \delta^{12} (1-2)^3 + \delta^{23} (2-3)^1 (123) + \delta^{31} (3-1)^2 (132) \} \rightarrow X_1^2 f_4$$

$$\Rightarrow \left. \begin{array}{l} \checkmark 1. - (123) + 2 \cdot (132) \\ \checkmark 2. - (123) + (132) \\ \checkmark 3. - 2 \cdot (123) + 2 \cdot (132) \\ \checkmark 4. - 2 \cdot (123) + (132) \\ \checkmark 5. [(1 \cdot 3) - (2 \cdot 3)] \end{array} \right\} X_1^2 f_4$$

Term 5:

$$3^3 \{ 2^1 3^2 1^3 - 3^1 1^2 2^3 \} \rightarrow X_5 f_5$$

$$\Rightarrow \left. \begin{array}{l} \checkmark 1. (3 \cdot 2) \\ \checkmark 2. 0 \\ \checkmark 3. (3 \cdot 2) - (3 \cdot 1) \\ \checkmark 4. - (3 \cdot 1) \\ \checkmark 5. 0 \end{array} \right\} X_5 f_5$$

Term 6:

$$3^3 \{ 1^1 2^2 3^3 - 2^1 2^1 3^3 + 2^2 2^3 1^1 - 3^2 3^2 1^1 + 3^3 3^1 1^2 - 1^1 1^3 3^2 \} \rightarrow X_5 f_6$$

$$\Rightarrow \left. \begin{array}{l} \checkmark 1. - 2(3 \cdot 3) \\ \checkmark 2. (3 \cdot 1) - (3 \cdot 2) \\ \checkmark 3. 0 \\ \checkmark 4. + 2(3 \cdot 3) \\ \checkmark 5. 0 \end{array} \right\} X_5 f_6$$

Only four more terms to go. So far no errors in former calculation:....

Term 7:

$$3^3 \{ \delta^{12} (1-2)^3 + \delta^{23} (2-3)^1 + \delta^{31} (3-1)^2 \} \rightarrow X_S f_7$$

$\Rightarrow$

$$\left. \begin{array}{l} \checkmark 1. +1 \\ \checkmark 2. 0 \\ \checkmark 3. 0 \\ \checkmark 4. -1 \\ \checkmark 5. (3 \cdot 1) - (3 \cdot 2) \end{array} \right\} X_S f_7$$

$$\left. \begin{array}{l} \checkmark 1. 0 \\ \checkmark 2. (3 \cdot 3) \\ \checkmark 3. 0 \\ \checkmark 4. 0 \\ \checkmark 5. 0 \end{array} \right\} X_A f_8$$

Term 8:

$$3^3 \{ 1^2 3^3 \} \rightarrow X_A f_8 \Rightarrow$$

Term 9:

$$3^3 \{ 2^1 3^2 1^3 + 3^1 2^2 3^3 \} \rightarrow X_A f_9$$

$$\Rightarrow \left. \begin{array}{l} \checkmark 1. -(2,3) \\ \checkmark 2. 0 \\ \checkmark 3. -(1,3) - (2,3) = +(3,3) \\ \checkmark 4. -(1,3) \\ \checkmark 5. 0 \end{array} \right\} X_A f_9$$

Aha! An error in the previous calculation. I found the error, this is the correct result for this term. I have done this one three times to make sure. Not much to do, really.

Term 10:

$$3^3 \{ \delta^{12} 3 + \delta^{23} 1 + \delta^{31} 2^2 \} \rightarrow X_A f_{10}$$

$$\Rightarrow \left. \begin{array}{l} \checkmark 1. -1 \\ \checkmark 2. -2 \\ \checkmark 3. 0 \\ \checkmark 4. -1 \\ \checkmark 5. (3 \cdot 3) \end{array} \right\} X_A f_{10}$$

Now that I have done each term twice, I 'll throw out the original rough calculations and retain only this. Original summary sheet follows:

	A $1'1^2$	B $1'2^2$	C $2'1^2$	D $2'2^2$	E $3'1^2$	form.
1.	$-(3 \cdot 2)(123)'$	$-[(3 \cdot 2)(123) + (3 \cdot 1)(132)]'$	$(3 \cdot 3)'$	$-(3 \cdot 1)(132)'$	$0'$	$X_2' f_1'$
2.	$(3 \cdot 3) + (3 \cdot 1)(123)'$	$(3 \cdot 3)'$	$(3 \cdot 3)'$	$(3 \cdot 3) + (3 \cdot 2)(132)'$	$0'$	$X_2' f_2'$
3.	$-(123)'$	$-[(123) + (132)]'$	$0'$	$-(132)'$	$(3 \cdot 3)'$	$X_2' f_3'$
4.	$-(123) + 2 \cdot (132)'$	$-(123) + (132)'$	$-2(123) + 2(132)'$	$-2 \cdot (123) + (132)'$	$[(1 \cdot 3) - (2 \cdot 3)]'$	$X_1^2 f_4'$
5.	$(3 \cdot 2)'$	$0'$	$[(3 \cdot 2) - (3 \cdot 1)]'$	$-(3 \cdot 1)'$	$0'$	$X_3 f_5'$
6.	$+2(3 \cdot 3)'$	$[(3 \cdot 1) - (3 \cdot 2)]'$	$0'$	$+2(3 \cdot 3)'$	$0'$	$X_3 f_6'$
7.	$1'$	$0'$	$0'$	$-1'$	$[(3 \cdot 1) - (3 \cdot 2)]'$	$X_3 f_7'$
8.	$0'$	$(3 \cdot 3)'$	$0'$	$0'$	$0'$	$X_A f_8'$
9.	$-(2 \cdot 3)'$	$0'$	$+(3 \cdot 3)'$	$-(1 \cdot 3)'$	$0'$	$X_A f_9'$
10.	$-1'$	$-2'$	$0'$	$-1'$	$(3 \cdot 3)'$	$X_A f_{10}'$

A

B

C

D

E

 $1^1 1^2$ $1^1 2^2$ $2^1 1^2$ $2^1 2^2$ δ^{12}

1. ✓

$-(3.2) P_2^+ f_1$

$-(3.2) P_2^+ f_1 - (3.1) P_2^+ f_1$

$(3.3) P_2 f_1$

$-(3.1) P_2^+ f_1$

0

2. ✓

$(3.3) P_2 f_2 + (3.1) P_2^+ f_2$

$(3.3) P_2 f_2$

$(3.3) P_2 f_2$

$(3.3) P_2 f_2 + (3.2) P_2^+ f_2$

0

3. ✓

$-P_2^+ f_3$

$-P_2^+ f_3 - P_2^+ f_3$

0

$-P_2^+ f_3$

$(3.3) P_2 f_3$

4. ✓

$-P_1^+ f_4 + 2P_1^+ f_4$

$-P_1^+ f_4 + P_1^+ f_4$

$-2P_1^+ f_4 + 2P_1^+ f_4$

$-2P_1^+ f_4 + P_1^+ f_4$

$(3(1-2)) P_1 f_4$

5. ✓

$(2.3) X_5 f_5$

0

$-(3)(1-2) X_5 f_5$

$-(1.3) X_5 f_5$

0

6. ✓

$-2(3.3) X_5 f_6$

$3(1-2) X_5 f_6$

0

$2(3.3) X_5 f_6$

0

7. ✓

$X_5 f_7$

0

0

$-X_5 f_7$

$[(3)(1-2)] X_5 f_7$

8. ✓

0

$(3.3) X_A f_8$

0

0

0

9. ✓

$-(3.2) X_A f_9$

0

$+(3.3) X_A f_9$

$-(1.3) X_A f_9$

0

10. ✓

$-X_A f_{10}$

$-2X_A f_{10}$

0

$-X_A f_{10}$

$(3.3) X_A f_{10}$

RHS

$b(1)'$

0

0

$-b(2)'$

$(2.2)b(2) - (1.1)b(1)'$

Summary of Information Deleted at this Point:

- 1 . Idea that each Ward equation is a vector equation in the (s,1,2,a) sense.
2. Idea that $R[fg] = Rf \cdot Rg$ where R = group element, but not true for R replaced by X , some element of the group algebra.
3. Explanation of why only a,2 components appear in most Ward equations. (Obvious)
4. Expansion of Ward table terms into symmetry fractions. Did this wrong originally because had Clebsching wrong, later repaired.
5. Derivation of facts stated on "Main Data Sheet" see below.
6. In general, this section contained the work needed to setup the 10-equation Ward Chart which I have retained below. I am quite sure this chart is correct, but I never used it for anything. Later of course I solved the entire problem very quickly using the cyclic operator method of solution. That is my main reason for dumping stuff out of this section.

Main Data Sheet for Analyzing Component Terms into Symmetry Fractions.

1. General Clebsch Formula:

$$F \times G =$$

$$\begin{aligned} & \hat{S} [F_S G_S + F_A G_A + \frac{1}{2}(F_1 G_1 + \tilde{F}_1 \tilde{G}_1) + \frac{1}{2}(F_2 G_2 + \tilde{F}_2 \tilde{G}_2)] \\ & + \hat{1} [F_S G_1 + F_1 G_S + F_A G_2 + F_2 G_A + \frac{1}{2}(F_1 G_1 - \tilde{F}_1 \tilde{G}_1) + \frac{1}{2}(F_2 G_2 - \tilde{F}_2 \tilde{G}_2)] \\ & + \hat{2} [F_S G_2 + F_2 G_S + F_A G_1 + F_1 G_A + \frac{1}{2}(F_1 G_2 + \tilde{F}_1 \tilde{G}_2) + \frac{1}{2}(F_2 G_1 + \tilde{F}_2 \tilde{G}_1)] \\ & + \hat{a} [F_S G_A + F_A G_S + \dots + \frac{1}{2}(F_1 G_2 - \tilde{F}_1 \tilde{G}_2) + \frac{1}{2}(F_2 G_1 - \tilde{F}_2 \tilde{G}_1)] \end{aligned}$$

2. Diagonal dot products:

$$\begin{aligned} (1.1) &= e_s \hat{S} + e_1 \hat{1} + e_2 \hat{2} \\ (2.2) &= e_s \hat{S} + e_1 \hat{1} - e_2 \hat{2} \\ (3.3) &= e_s \hat{S} - 2e_1 \hat{1} \end{aligned}$$

$$\tilde{e}_1 = \frac{1}{\sqrt{3}} e_2$$

$$\tilde{e}_2 = \sqrt{3} e_1$$

see below.

3. Off-diagonal dot products:

$$\begin{aligned} y = (2.3) &= d_s \hat{S} + d_1 \hat{1} + d_2 \hat{2} \\ z = (1.3) &= d_s \hat{S} + d_1 \hat{1} - d_2 \hat{2} \\ x = (1.2) &= d_s \hat{S} - 2d_1 \hat{1} \end{aligned}$$

$$\tilde{d}_1 = \frac{1}{\sqrt{3}} d_2$$

$$\tilde{d}_2 = \sqrt{3} d_1$$

4. Relations implied by momentum conservation:

$$(2.1) + (2.2) + (2.3) = 0$$

$$\begin{aligned} \hat{S} & d_s + e_s + d_s = 0 \\ \hat{1} & -2d_1 + e_1 + d_1 = 0 \\ \hat{2} & 0 - e_2 + d_2 = 0 \end{aligned}$$

$\Rightarrow$

$$e_s = -2d_s$$

$$e_1 = d_1$$

$$e_2 = d_2$$

$$\begin{aligned} (1.1) &= -2d_s \hat{S} + d_1 \hat{1} + d_2 \hat{2} \\ (2.2) &= -2d_s \hat{S} + d_1 \hat{1} - d_2 \hat{2} \\ (3.3) &= -2d_s \hat{S} - 2d_1 \hat{1} \end{aligned}$$

Base Cases, sum'd on "Main Sheet"

Let

$$\begin{cases} (1)_S \equiv \frac{1}{3}[(1)+(2)+(3)] \\ (1)_1 \equiv \frac{1}{6}[(1)+(2)-2(3)] \\ (1)_2 \equiv \frac{1}{2}[(1)-(2)] \end{cases}$$

Then:

$$(1) = (1)_S \hat{S} + (1)_1 \hat{1} + (1)_2 \hat{2}$$

$$(2) = (1)_S \hat{S} + (1)_1 \hat{1} - (1)_2 \hat{2}$$

$$(3) = (1)_S \hat{S} - 2(1)_1 \hat{1}$$

$$(\tilde{1})_1 = X_2^1(1)_1 = X_2^1(1) = \frac{1}{\sqrt{3}}(1)_2$$

$$(\tilde{2})_1 = X_2^1(2)_1 = X_2^1(2) = \frac{1}{\sqrt{3}}(1)_2$$

$$(\tilde{3})_1 = X_2^1(3)_1 = X_2^1(3) = -\frac{2}{\sqrt{3}}(1)_2$$

$$(\tilde{1})_2 = X_1^2(1)_2 = X_1^2(1) = \sqrt{3}(1)_1$$

$$(\tilde{2})_2 = X_1^2(2)_2 = X_1^2(2) = -\sqrt{3}(1)_1$$

$$(\tilde{3})_2 = X_1^2(3)_2 = X_1^2(3) = 0$$

	$f_1^{(1)}$	$f_1^{(2)}$	$f_2^{(1)}$	$f_2^{(2)}$	$f_3^{(1)}$	$f_3^{(2)}$	$f_4^{(1)}$	$f_4^{(2)}$
A_5	$-\frac{\sqrt{3}}{2}d_1$	$\frac{1}{2}d_2$	$\frac{\sqrt{3}}{2}d_1$	$\frac{1}{2}d_2$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
A_1	$-\frac{\sqrt{3}}{2}d_5$	$\emptyset$	$\frac{\sqrt{3}}{2}d_5$	$\emptyset$	$-\frac{\sqrt{3}}{2}$	$\emptyset$	$-\frac{1}{2}$	$\emptyset$
A_2	$-\frac{1}{\sqrt{3}}d_2$	$\frac{1}{2}d_5+d_1$	$-\frac{2}{\sqrt{3}}d_2$	$-\frac{5}{2}d_5-2d_1$	$\emptyset$	$\frac{1}{2}$	$\emptyset$	$\frac{3\sqrt{3}}{2}$
A_a	$-\frac{1}{2\sqrt{3}}d_2$	$-\frac{1}{2}d_1$	$+\frac{1}{2\sqrt{3}}d_2$	$-\frac{1}{2}d_1$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
B_2	$-\frac{2}{\sqrt{3}}d_2$	d_5+2d_1	$-\frac{1}{\sqrt{3}}d_2$	$-2d_5-d_1$	$\emptyset$	1	$\emptyset$	$\sqrt{3}$
B_a	$-\frac{1}{\sqrt{3}}d_2$	$-d_1$	$+\frac{1}{\sqrt{3}}d_2$	$-d_1$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
C_2	$-\frac{1}{\sqrt{3}}d_2$	$-2d_5-d_1$	$-\frac{1}{\sqrt{3}}d_2$	$-2d_5-d_1$	$\emptyset$	$\emptyset$	$\emptyset$	$2\sqrt{3}$
C_a	$+\frac{1}{\sqrt{3}}d_2$	$-d_1$	$+\frac{1}{\sqrt{3}}d_2$	$-d_1$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
E_2	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$-\frac{1}{\sqrt{3}}d_2$	$-2d_5-d_1$	$-d_2$	$-\sqrt{3}d_1$
E_a	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$+\frac{1}{\sqrt{3}}d_2$	$-d_1$	$-d_2$	$+\sqrt{3}d_1$

$f_5^{(s)}$	$f_6^{(s)}$	$f_7^{(s)}$	$f_8^{(a)}$	$f_9^{(a)}$	$f_{10}^{(a)}$	$b(1)_s$	$b(1)_1$	$b(1)_2$
d_s	$4d_s$	1	ϕ	ϕ	ϕ	-1	ϕ	ϕ
d_1	$4d_1$	ϕ	ϕ	$-d_2$	ϕ	ϕ	-1	ϕ
d_2	ϕ	ϕ	ϕ	$-d_1$	ϕ	ϕ	ϕ	-1
ϕ	ϕ	ϕ	ϕ	$-d_s$	-1	ϕ	ϕ	ϕ
ϕ	$-2d_2$	ϕ	$-2d_1$	ϕ	ϕ	ϕ	ϕ	ϕ
ϕ	ϕ	ϕ	$-2d_s$	ϕ	-2	ϕ	ϕ	ϕ
$2d_2$	ϕ	ϕ	ϕ	$-2d_1$	ϕ	ϕ	ϕ	ϕ
ϕ	ϕ	ϕ	ϕ	$-2d_s$	ϕ	ϕ	ϕ	ϕ
ϕ	ϕ	$-2d_2$	ϕ	ϕ	$-2d_1$	$+2d_2$	$+2d_2$	$-4d_s$ $+2d_1$
ϕ	ϕ	ϕ	ϕ	ϕ	$-2d_s$	ϕ	ϕ	ϕ

Ward 6

July 1

Status Report.

1. I did indeed find 14 "linearly independent tensor forms". There were 3 symmetric, 3 antisymmetric, and 4 mixed "pairs" of forms. It took a lot of work for me to be convinced that this was the complete set of independent tensor forms.
2. Having found these "tensor forms", I then considered how to construct "vertex terms". I did this in an obvious way and I thought I had thereby constructed 10 independent "vertex terms". BUT, I then discovered that some of these terms contained others so that some could be removed from the set.
3. I now wish to consider the "most general form" problem in terms of these "vertex terms". This is really ~~ex~~ a new problem in reduction. I think I know that I have generated "vertex terms" from my complete set of "tensor forms" in all possible ways. I think, therefore, that nothing has been omitted. It is a problem now of reducing this set to its minimum somehow.

Material Deleted From this Section.

1. This was where I reduced my list of 10 tensor forms to a list of 6 forms using the "extension idea". This led to the Most General Form of the vertex given below as six terms (non-n of course). I then took that ~~ixx~~ form of the vertex and constructed a new Ward table, see below.

At this point I tried to "solve" this Ward table but I failed because I was not yet using the cyl,ci-operator method. I have delted pages of how this went since I no longer care about it.

Most General Form of Triple Glue Vertex Prior to Ward Identity Restrictions.

1. How many totally antisymmetric, linearly independent tensor forms can be made from p_1, p_2 and g_{uv} ? The answer is 10. Here they are:

$$8. [1^1 2^2 3^3] f_8^A(123)$$

$$9. [2^1 3^2 1^3 + 3^1 2^2 1^3] f_9^A(123)$$

$$5. [2^1 3^2 1^3 - 3^1 2^2 1^3] f_5^S(123)$$

$$1. [2^1 1^2 3^3] f_1^A(12) + \text{cyclic}]$$

$$2. [3^1 3^2 3^3] f_2^A(12) + \text{cyclic}]$$

$$6. [(1^1 1^2 2^3 - 2^1 2^2 1^3) + \text{cyclic}] f_6^S(123)$$

$$7. [\delta^{12}(1-2)^3 + \text{cyclic}] f_7^S(123)$$

$$10. [\delta^{12} 3^3 + \text{cyclic}] f_{10}^A(123)$$

$$4. [\delta^{12}(1-2)^3] f_4^S(12) + \text{cyclic}]$$

$$3. [\delta^{12} 3^3] f_3^A(12) + \text{cyclic}]$$

However, I now see that some of these "vertex forms" are contained in others! Although my original tensor forms were truly linearly independent, when I combine them with functions to make overall antisymmetric, some vertex forms become contained in others.

In particular, it appears that the forms arising through the "mixed symmetry" route are somehow more general than the other forms.

1. Consider form 1. above. As a special case of this form we might consider f_1 to be a totally antisymmetric function, instead of just 12 anti-sym. Then you get:

$$1. \Rightarrow [2^1 1^2 3^3 + 3^2 2^1 1^3 + 1^3 3^2 2^1] f_1^{TA}(123)$$

$$= (213 + 132 + 321) f_1^A(123)$$

$$= (132)_S f_1^A(123).$$

But this symmetric tensor is a linear combinations of forms 8 and 9. Thus, we should rule out either 8 or 9 as being an independent form.

2. Lets do the same thing here. We then get:

$$[333 + 111 + 222] f_2^A(123) = \underbrace{(111)}_S f_2^A(123) \\ \doteq (132)_S$$

Here we get essentially the same form as

before, so we don't have to knock out two. Only 8 or 9 must go. Lets knock out 9 since it is more complicated.

10. This case is contained in 3. by same idea, so kill 10.

7. This case is contained in 4 so kill 7.

So our list has now shrink to the following:

8. $[1' 2' 3'] f_8^A(123)$

5. $[2' 3' 1' - 3' 1' 2'] f_5^S(123)$

6. $[(1' 1' 2' - 2' 2' 1') + \text{cyclic}] f_6^S(123)$

1. $[2' 1' 3' f_1^A(12)] + \text{cyclic}$

2. $[3' 3' 3' f_2^A(11)] + \text{cyclic}$

4. $[8^{12} (1-2)^3 f_4^S(12)] + \text{cyclic}$

3. $[8^{12} 3^3 f_3^A(12)] + \text{cyclic}$

Lets see if further reductions can't be found. Can I make 1 and 2 coincide??

-3-

$$[2^1 3^2 f_1^A(12)] + [3^2 2^1 f_1^A(23)] + [3^1 2^2 f_1^A(31)]$$

$$=? = [333 f_2(12)] + [111 f_2(23)] + [222 f_2(31)] ?$$

Reduce both sides to $p_1 p_2$ tensor forms and see if the functions can be related.

$$\text{LHS} = f^A(12) 213 + f^A(23) 132 + f^A(31) 321$$

$$= f^A(12) 21 [-1-2] + f(23) 1 [-1-2] 2 + f(31) [-1-2] 21$$

$$= -f(12) [211 + 212] - f(23) [112 + 122] - f(31) [121 + 221]$$

$$\text{RHS} = -(1+2)(1+2)(1+2)^2 g(12) + 111 g(23) + 222 g(31)$$

$$= 111 [g(23) - g(12)] + 222 [g(31) - g(12)]$$

$$+ g(12) [112 + 121 + 211 + 122 + 212 + 221].$$

To make these equal you need:

$$g(23) = g(12)$$

$$g(31) = g(12)$$

$$f(12) = g(12)$$

$$f(23) = g(12)$$

$$f(31) = g(12)$$

Stop. Have to consider how you might make a function like $g(23)$, for example.

A scalar function can only depend on p_1^2 , p_2^2 , $p_1 \cdot p_2$. All other scalars like $p_2 \cdot p_3$ can be reduced to these. So;

$$f^A(12) = F^-(p_1^2 - p_2^2) \quad \text{most general form, odd parity function of one argument.}$$

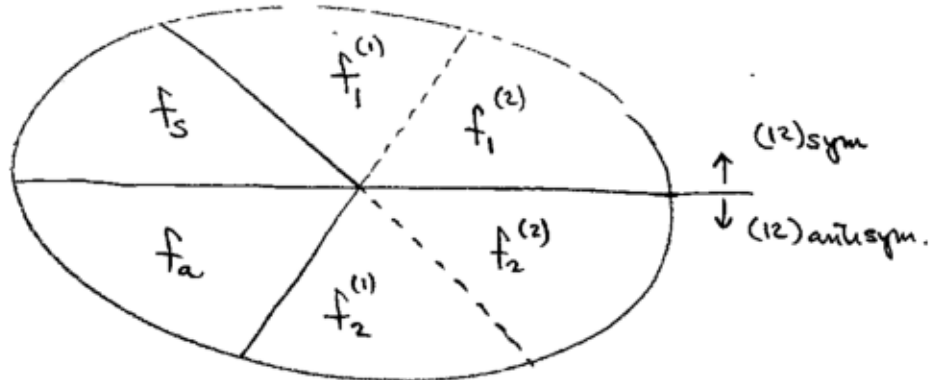
A special case of this general form would be

$$Q^-(p_1^2 - p_2^2) + Q^-(p_2^2 - p_3^2) + Q^-(p_3^2 - p_1^2)$$

which is totally antisymmetric.

Comments about the Vertex Forms

1. I now see that in fact my 10 vertex terms do not overlap as I thought earlier. To see how this works, let's classify the space of functions into 6 or four pieces like so:



This is supposed to illustrate that the totally symmetric functions together with the row-1 functions comprise the set of 12 symmetric functions, whereas the lower half of the diagram represents the 12-antisymmetric functions.

Now, my tensor vertex term #4 represents combinations of that tensor form only with row-1 functions, whereas my vertex term #7 represents the same tensor form combined with totally symmetric functions. You could combine these two "vertex terms" into one by just saying you are going to combine with an arbitrary (12)-symmetric function.

But maybe there is some advantage to keeping these categories separated. The set of functions which are (12)-symmetric is not really an irreducible set relative to S_3 ; it is the union of two such sets.

2. Similarly, consider tensor form 1. If I were to extend the row-2 function there to include the class of other 12-antisymmetric functions, that would bring in the totally antisymmetric functions, then term 1 overlaps with 8 and 9.

3. So here is the major result here.

$$[12\text{-symmetric functions}] = [\text{row-1 functions}] \cup [\text{totally S functions}]$$

$$[12\text{-antisym functions}] = [\text{row-2 functions}] \cup [\text{totally A functions}]$$

Extensions and Removal of some Terms.

1. I have shown that the extension of term 4 equals term 7, so if you use the extended term 4 then you can eliminate term 7 altogether.

Similarly, the extension of term 3 equals term 10, so you can drop 10 and use an extended term 3.

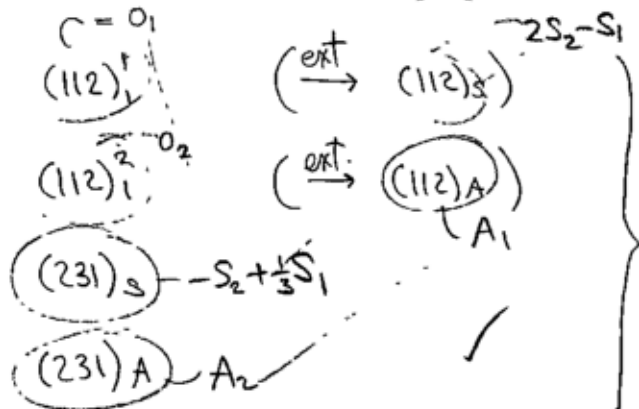
2. The extension of term 1 gives a linear combination of 8 and 9. Thus, if you use an extended term 1, you can drop 8 or 9. Specifically, the extension of term 1 gives the form $(132)_S = \text{combination of } (123)_S \text{ and } (231)_S$.

3. Interestingly, the extension of term 2 also gives $(132)_S$. Thus, if you use an extended term 1 and an extended term 2, you have in effect "covered" $(132)_S$ twice.

Perhaps it would be better to take a different pair at the start to remove this double overlap?

4. OK, I have relisted the choices we have for various tensors in the ppp classes. And I have looked at their "reductions" when you "extend". (see attached page). Eg, if you start with $(112)_1^1$ as your chosen "pair", then when you extend this thing you pick up $(112)_S$. The thing you pick up is always in the class you started with.

So here is a nice proposed list:



So, if we include extensions, our list contains only six entries. Here they are:

- 1.) $(112 + 221) f_A^{(1)}(12) + \text{cyclic}$ A_1
- 2.) $(112 - 221) f_S^{(2)}(12) + \text{cyclic}$ S_1
- 3.) $(231 + 312) f_A^{(3)}(123)$ A_1
- 4.) $(231 - 312) f_S^{(4)}(123)$ S_1
- 5.) $-\delta^{12} (1+2) f_A^{(5)}(12) + \text{cyclic } S_2$ A_2
- 6.) $\delta^{12} (1-2) f_S^{(6)}(12) + \text{cyclic}$

first 6-member list.

Full Table A

1

1 1

B

1 2

C

2 1

D

2 2

E

2 2

1)

$$\begin{aligned} &+(3.2) f_A(12) \\ &-(3.3) f_A(31) \\ &-(3.1) f_A(31) \end{aligned}$$

$$\begin{aligned} &-(3.2) f_A(23) \\ &-(3.1) f_A(31) \end{aligned}$$

$$\begin{aligned} &-(3.3) f_A(23) \\ &-(3.3) f_A(31) \end{aligned}$$

$$\begin{aligned} &(3.1) f_A(12) \\ &-(3.2) f_A(23) \\ &-(3.3) f_A(23) \end{aligned}$$

2)

$$\begin{aligned} &+(3.2) f_B(12) \\ &-(3.3) f_B(31) \\ &+(3.1) f_B(31) \end{aligned}$$

$$\begin{aligned} &-(3.2) f_B(23) \\ &+(3.1) f_B(31) \end{aligned}$$

$$\begin{aligned} &+(3.3) f_B(23) \\ &-(3.3) f_B(31) \end{aligned}$$

$$\begin{aligned} &-(3.1) f_B(12) \\ &-(3.2) f_B(23) \\ &+(3.3) f_B(23) \end{aligned}$$

3)

$$-(2.3) f_A(123)$$

$$\begin{aligned} &-(1.3) f_A(123) \\ &-(2.3) f_A(123) \end{aligned}$$

$$-(1.3) f_A(123)$$

4)

$$+(2.3) f_B(123)$$

$$\begin{aligned} &-(1.3) f_B(123) \\ &+(2.3) f_B(123) \end{aligned}$$

$$-(1.3) f_B(123)$$

5)

$$-f_A(23)$$

$$\begin{aligned} &-f_A(23) \\ &-f_A(31) \end{aligned}$$

$$-f_A(31)$$

$$(3.3) f_A(12)$$

6)

$$\begin{aligned} &-f_B(23) \\ &+2 f_B(31) \end{aligned}$$

$$\begin{aligned} &-f_B(23) \\ &+f_B(31) \end{aligned}$$

$$\begin{aligned} &-2 f_B(23) \\ &+2 f_B(31) \end{aligned}$$

$$\begin{aligned} &-2 f_B(23) \\ &+f_B(31) \end{aligned}$$

$$[(3.1) - (3.2)] f_B(12)$$

RHS

$$b(1)$$

$$0$$

$$0$$

$$-b(2)$$

$$(2.2) b(2) - (1.1) b(1)$$

Check

- (1) Each coeff in B, C, E col is (12) antisym ✓
- (2) D-coeff is $-(12)A$ ✓

2

15 x 18

A plan for solving the Ward Identity

1. The problem was this: how do you convey to your equation system that $f(12)$ is somehow related to $f(23)$ and so on? I now see that the way to do this is to take each equation and state its cyclic and reverse cyclic duplicate equations. In effect this says you are forcing the other two Ward Identities, but that is just an interpretation. In fact you can apply any R of S_3 to any equation to get a new equation, so the question is now: how many different equations are there?

2. Here are some facts we do know:

$$(12) B = -B$$

$$(12) C = -C$$

$$(12) E = -E$$

Thus, read as" application of permutation (12) to equation "B" yields $-B$. In other words, does not yield a new equation. Same for C and E. We also know:

$$(12) A = -D$$

3. Conjecture: I bet that the two cyclics on each of the above equations exhausts the entire set of equations. Lets try it and see:

$$(123)B = B^+ \text{ def.}$$

$$(132)B = B^- \text{ def. and so on for the others.}$$

$$\left. \begin{array}{l} (e) B = B \\ (12) B = -B \\ (13) B = (123)(12)B = -(123)B = -B^+ \\ (23) B = (132)(12)B = -(132)B = -B^- \\ (123)B = B^+ \\ (132)B = B^- \end{array} \right\} \text{ so } \{B, B^+, B^-\} = \text{complete set of } B\text{-equations}$$

Same goes for C, E.

$$\left. \begin{array}{l} (e) A = A \\ (12) A = -D \\ (13) A = (123)(12)A = -(123)D = -D^+ \\ (23) A = (132)(12)A = -(132)D = -D^- \\ (123)A = A^+ \\ (132)A = A^- \end{array} \right\} \quad \left. \begin{array}{l} (e) D = D \\ (12) D = -A \\ (13) D = -A^+ \\ (23) D = -A^- \\ (123)D = D^+ \\ (132)D = D^- \end{array} \right\} \Rightarrow \{A, A^+, A^-, D, D^+, D^-\} = \text{full set.}$$

Thus, the complete set of equations includes $3+3+3+3+3 = 15$ equations! I mean there is no other equation you can get by doing a permutation. So lets make a list.

4. Suggested notation for the dot products:

$$\begin{array}{ll} (1.2) = ,x & (1.1) = -x-z \\ (2.3) = y & (2.2) = -y-x \\ (3.1) = z & (3.3) = -z-y \end{array}$$

Note that x, y, z are independent since there are three independent dot products. Note also that these things "cyclic" in the obvious way.

5. Well, I have made the table and there are 15 constraints on 14 functions. But the function sets are so strongly coupled I don't know what to do! In principle this sytem could be solved, but practically it is too much of a mess.

6. What about this idea of finding a solution to the Ward identity and then adding null terms to that solution. How do you know there isnot some other non-trivial sol'n?

Fact: Any two solutions of the Ward must differ by a Ward null term. If you have a solution and if you have exhausted all possible Ward null terms, then you have exhausted all solutions!

7. I think it would help to have a more intelligent set of starting tensor forms!

	"A"	"(123)A"	"(132)A"	"A"	"(123)A"	"(132)A"
$A_1(12)$	(2,3)	(3,1)	(1,2)	y	z	x
$A_1(23)$	.	(3,1)	(1,2)	.	z	x
$A_1(31)$	(2,3)	.	(1,2)	y	.	x
$S_1(12)$	(2,3)	(1,2)-(1,1)	.	y	2x+z	.
$S_1(23)$	.	(3,1)	(2,3)-(2,2)	.	z	2y+x
$S_1(31)$	(3,1)-(3,3)	.	(1,2)	2z+y	.	x
$A(123)$	-(2,3)	-(3,1)	-(1,2)	-y	-z	-x
$S(123)$	+(2,3)	+(3,1)	+(1,2)	y	z	x
$A_2(12)$	.	.	-1	.	.	-1
$A_2(23)$	-1	.	.	-1	.	.
$A_2(31)$	.	-1	.	.	-1	.
$S_2(12)$	.	2	-1	.	2	-1
$S_2(23)$	-1	.	2	-1	.	2
$S_2(31)$	+2	-1	.	2	-1	.
$b(1)$	-1	.	.	-1	.	.
$b(2)$	.	-1	.	.	-1	.
$b(3)$	.	.	-1	.	.	-1
	=0	=0	=0	=0	=0	=0

Let $f(12)$ be abbr. for some arbitrary $f(123)$.

Let f^+ and f^{++} denote obvious cyclics.

Assume an equation:

$$\alpha f + \beta f^+ + \gamma f^{++} = F$$

Assume the cyclics of this equation are also true.

Then the system becomes:

$$\begin{bmatrix} \alpha & \beta & \gamma \\ \gamma^+ & \alpha^+ & \beta^+ \\ \beta^{++} & \gamma^{++} & \alpha^{++} \end{bmatrix} \begin{bmatrix} f \\ f^+ \\ f^{++} \end{bmatrix} = \begin{bmatrix} F \\ F^+ \\ F^{++} \end{bmatrix}$$

The Cramer's-Rule solution is:

$$\therefore \dots \dots \dots f = \frac{\begin{vmatrix} F & \beta & \gamma \\ F^+ & \alpha^+ & \beta^+ \\ F^{++} & \gamma^{++} & \alpha^{++} \end{vmatrix}}{\begin{vmatrix} \alpha & \beta & \gamma \\ \gamma^+ & \alpha^+ & \beta^+ \\ \beta^{++} & \gamma^{++} & \alpha^{++} \end{vmatrix}}$$

Then get f^+, f^{++} by cyclics. Of course sol only exists if $\text{denn } \det = 0$.

	A	A ⁺	A ⁻	B	B ⁺	B ⁻	C	C ⁺
A ₁ (12)	y	z			-x	-x		+x+z
A ₁ (23)		z	x	-y		-y	+z+y	
A ₁ (31)	y		x	-z	-z		+z+y	+x+z
S ₁ (12)	y	2x+z			x	-x		x+z
S ₁ (23)		z	2y+x	-y		y	-z-y	
S ₁ (31)	2z+y		x	z	-z		z+y	-x-z
A(123)	-y	-z	-x				-z-y	-x-z
S(123)	y	z	x				-z+y	-x+z
A ₂ (12)			-1		-1	-1		
A ₂ (23)	-1			-1		-1		
A ₂ (31)		-1		-1	-1			
S ₂ (12)		2	-1		1	-1		2
S ₂ (23)	-1		2	-1		1	-2	
S ₂ (31)	2	-1		1	-1		+2	-2
b(1)	-1							
b(2)		-1						
b(3)			-1					
	=0	=0	=0	=0	=0	=0	=0	=0
	✓	/	/	/	/	/	✓	/

C^-	D	D^+	D^-	E	E^+	E^-	$A+D$
$+y+x$	z		y				$y+z$
$+y+x$	z	x					z
		x	y				y
$-y-x$	$-z$		$-2x-y$				$y-z$
$y+x$	$-2y-z$	$-x$					$-2y-z$
		$-2z-x$	$-y$				$2z+y$
$-y-x$	$-z$	$-x$	$-y$				
$-y+x$	$-z$	$-x$	$-y$				
		-1		$-z-y$			
			-1		$-x-z$		
	-1					$-y-x$	
-2		1	-2	$z-y$			
2	-2		1		$x-z$		
	$+1$	-2				$y-x$	
			1	$-x-z$		$x+z$	
	$+1$			$y+x$	$-y-x$		
		1			$z+y$	$-z-y$	
$=0$	$=0$	$=0$	$=0$	$=0$	$=0$	$=0$	
$/$	$\checkmark$	$.$	$/$	$\checkmark$	$\checkmark$	$/$	

5 x 6 Operator

Material Deleted from this Section:

1. Back in section called "Ward-6" I made a Ward-Table using the list of 6 terms for the vertex. Here is one of my attempts to solve that Table.

This was the first time I thought about using operator equations, but my implementation was no good. Later I learned how to do it.

So I am dumping this entire section except the page showing how to invert operators involving (e) and (12).

Damned Operator Equation Method!

~~Reconsider the B equation:~~

Think of this as $f(S_1, S_1^+, S_2, S_2^+) = 0$. A function of four variables. However, we have some other implicit conditions, namely $S_1^+ = (123)S_1$. For any "form" assumed by S_1 , this tells you what S_1^+ is. Thus, it says $F(S_1, S_1^+) = 0$! This is as much a "condition" as any other condition/. So we might insert both these auxiliary conditions into our original equation to get $f(S_1, (123)S_1, S_2, (123)S_2) = 0$.

So try using operator eqns.

↗
But projection
operator so
cannot invert!

Group Algebra Problem: do elements of a group algebra have inverses?

1. The answer is: it depends what elements you are looking at; not all have inverses. Here is the poop:

$$A \equiv \sum_R a(R) R$$

$$\text{Find } B = \sum_S b(S) S \text{ such that } BA = E$$

$$\begin{aligned} \text{So find } b(S) \text{ such that } \sum_{R,S} a(R) b(S) SR &= \sum_{R,S} a(\tilde{S}R) b(S) R \\ &= \sum_{R,S} a(SR) b(\tilde{S}^{-1}) R = \sum_R \left[\sum_S a(SR) b(\tilde{S}^{-1}) \right] R = E \end{aligned}$$

Thus we want:

$$\sum_S a(SR) b(\tilde{S}^{-1}) = \delta_{R,E}$$

$$\sum_R a(RS) b(\tilde{R}^{-1}) = \delta_{S,E} \quad R_1 = E, R_2, \dots, R_h.$$

$$\begin{pmatrix} a(R_1) & a(R_2) & a(R_3) & \dots & a(R_h) \\ a(R_1 R_2) & a(R_2 R_2) & a(R_3 R_2) & \dots & a(R_h R_2) \\ a(R_1 R_3) & a(R_2 R_3) & a(R_3 R_3) & \dots & a(R_h R_3) \\ \vdots & \vdots & \vdots & \ddots & \vdots \end{pmatrix} \begin{pmatrix} b(\tilde{R}_1^{-1}) \\ b(\tilde{R}_2^{-1}) \\ b(\tilde{R}_3^{-1}) \\ \vdots \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix}$$

$$\text{Requires: } \begin{vmatrix} a(R_1) & a(R_2) & a(R_3) & \dots & a(R_h) \\ a(R_1') & a(R_2') & a(R_3') & \dots & a(R_h') \\ a(R_1'') & a(R_2'') & a(R_3'') & \dots & a(R_h'') \\ \vdots & \vdots & \vdots & \ddots & \vdots \end{vmatrix} \neq 0$$

So in order for element A of the group algebra to have an inverse, this determinant must not vanish. Notice that the arguments in each row are a rearrangement of the first row, so this is like a regular rep representation. This says that the h rearrangements of the vector $a(R)$ must be linearly independent! Obviously this works for $a(R) = \delta_{R,S}$; this just verifies that the elements of the group have inverses.

So given a "vector" $A(R)$, how do you know if its h rearrangements are linearly independent?

Think of the coefficients $a(R)$ as a vector in h -dimensional space. The condition that $\det = 0$ is one condition on h - numbers (ie, on the $a(R)$). Only these h numbers appear in the $\det$ because the rows are just rearrangements.

Therefore, the "surface" upon which $\det=0$ is one dimensional down from the whole space (codimension 1?). Thus, we expect that $\det=0$ on some $h-1$ dimensional manifold within the h -dimensional space.

For example, if $h=2$ we know that $\det=0$ only if $a(E)=a(R)$. Ie, on a line at 45° through the origin in the plane. Every "vector" in that plane is "invertible" except vectors on the line. So "most" vectors are invertible.

If $h=3$, again there will be a surface on which vectors are non-invertible. There will be some two-dimensional surface on which vectors vanish (not a plane because the $\det=0$ condition is non-linear). In fact, if the vector is (x,y,z) , you know that the 3 rearrangements are just the cyclics so you get this surface:

$$x^3 + y^3 + z^3 - 3xyz = 0$$

Some kind of surface in 3-space.

Obviously if you choose $(x,y,z) = (1,1,1)$ you get non-invertibility. This obviously extends to the general case.

By the way, note that for $n=4,5..$ and so on the arrangements are not just the cyclics because you can arrange things moving everything in other ways. Eg, with $n=4$ you can rearrange like this $(abcd)$ and also like this $(1b)(cd)$. So you don't know without naming the group which rearrangements occur in the $\det$. So the invertibility of a 4 -dim vector depends on which 4 -dim group you are using.

Conclusion: For arbitrary vectors in a group algebra, the probability is overwhelming that the vector is invertible, because the chances are always great that a vector does not lie on a surface within a space. Of course projection operators correspond to group algebra vectors which are not invertible, as do operators where all coefficients are the same.

A Group Algebra Problem in S_2 .

Motivation: I think this has something to do with solving my constraint equations for the Ward Problem.

1. Consider this equation:

$$(a(12)[e] + b(12)[12]) F(12) = G(12)$$

Here a, b, F, G are all functions of two arguments, ex, $a(x_1, x_2)$, and $[...]$ denotes a permutation group element. The problem here is, given a, b, G , how do you solve for $F(12)$?!

In effect, then, we want to invert the bracket $(\dots)$ on the left. What is this object? It is an element of the group algebra but with coefficients which are functions of 12, not just constants.

Earlier we showed that elements of group algebras (with constant coefficients) are in general invertible. Here things are a little different because the coefficients here are functions which feel the action of group elements. However, I feel that the general result is the same: generally there will exist an inverse.

For this particular problem, we can explicitly construct an inverse like so:

$$(A[e] + B[12])(a(12)[e] + b(12)[12]) = [e]$$

$$A(a(12)[e] + b(12)[12]) + B(a(21)[12] + b(21)[e]) = [e].$$

$$\Rightarrow \begin{aligned} A a(12) + B b(21) &= 1 \\ A b(12) + B a(21) &= 0 \end{aligned} \Rightarrow \begin{pmatrix} a(12) & b(21) \\ b(12) & a(21) \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} A \\ B \end{pmatrix} = \frac{1}{\det} \begin{pmatrix} a(21) & -b(21) \\ -b(12) & a(12) \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\Delta} \begin{pmatrix} a(21) \\ -b(12) \end{pmatrix}$$

$$\Rightarrow \begin{aligned} A &= a(21)/\Delta \\ B &= -b(12)/\Delta \end{aligned} \quad \text{where} \quad \Delta = \begin{vmatrix} a(12) & b(21) \\ b(12) & a(21) \end{vmatrix} = a(12)a(21) - b(12)b(21)$$

$$\Rightarrow \begin{aligned} A(12) &= a(21)/\Delta(12) \\ B(12) &= -b(12)/\Delta(12) \end{aligned}$$

$$\Delta(12) = a(12)a(21) - b(12)b(21).$$

So here below is the general result showing the explicit construction of the inverse. Of course if the det happens to vanish, you are sunk.

$$\Rightarrow \boxed{(a(21)[e] - b(12)[12])(a(12)[e] + b(12)[12]) = \Delta(12)[e].}$$

So, solution to question posed is -

$$F(z) = \Delta(z)^{-1} [a(z)[e] - b(z)[z]] G(z).$$

3. Suppose the determinant does vanish. Then what do you have? You cannot invert g . Consider:

$$(a(z)[e] + b(z)[z])^2 = (a(z)[e] + b(z)[z])(a(z)[e] + b(z)[z])$$

$$= a(z) \{ a(z)[e] + b(z)[z] \} + b(z) [a(z)[z] + b(z)[e]]$$

$$= [a(z)^2 + \underbrace{b(z)b(z)}_{= a(z)a(z)}] [e] + [a(z)b(z) + b(z)a(z)] [z]$$

$$= (a(z) + a(z)) (a(z)[e] + b(z)[z])$$

$$(P^2 = \alpha P \Rightarrow \frac{P}{\alpha} = \text{p.o.})$$

Thus:

$$\left\{ \frac{a(z)[e] + b(z)[z]}{a(z) + a(z)} \right\} = \text{a projection operator!}$$

So, if $\det=0$, then our operator is projectionlike : So consider $PF=G$ where P is a projectionlike operator. Either G is or is not in the range of P . If it is not, then there is no F that makes $PF=G$. So assume that G is in range. Then we can "solve" the equation for F like so:

$P^2 = \alpha P$ for projection like operator.

$PF=G$ with G in range implies $F = G/\alpha + F_{\text{null}}$

where F_{null} is an arbitrary function lying in the nullspace of P .

Fkl^{ab}

Material Deleted From this SECTION.

1. Once I got the form given below with some F's in it (later I backtracked and pulled the F's back out of the third and fourth terms!), I applied 3^3 and tried to "solve" the remaining equations. I considered the double-dot and trace of GAMMA equations instead of the tensor Ward table equations; ie, I never constructed a Ward Table for these six terms. But I could not solve, so pages now axed.

2. Also axed: pages showing that Marshall's Vertex does in fact satisfy the Ward. This proof is contained implicitly in Reduce program I later did with the n terms back in; there is no question but that Marshall's vertex works; I did it by hand a long time ago anyway.

1/. I have obtained general tensor form for the vertex. Now I am considering the general Ward problem. I have written down a method of solution which, although well-defined, is not very practical. Now is the time to start making use of other people work on this subject. Eg, it will probably help if I pick my list of tensor forms in a clever way, compared to an arbitrary way. Like choosing good coordinates!

2. So for openers, lets see how the "Marshall Vertex" fits into my list of tensor vertex terms. Start by rewriting the RHS Ward like so:

$$RHS = g(1) \left[\delta^{12} - \frac{1^1 1^2}{(1 \cdot 1)} \right] - g(2) \left[\delta^{12} - \frac{2^1 2^2}{(2 \cdot 2)} \right] \checkmark$$

$$\text{where } g(1) = -b(1)(1 \cdot 1) = -(1 \cdot 1) \Delta(1) \Rightarrow b(1) = \Delta(1) \checkmark \checkmark$$

Here you see certain rank-2 tensors that occur "naturally" on the RHS Ward. Perhaps these should replace less natural tensors, like g_{uv} all by itsdle.

In fact, here are some tensors that always seem to be useful:

$$A_{ji}^{\mu_1 \mu_2} = \left[\delta^{\mu_1 \mu_2} - \frac{p_i^{\mu_1} p_j^{\mu_2}}{(p_i \cdot p_j)} \right] \quad \text{OR} \quad A_{ji}^{12} = \left[\delta^{12} - \frac{j^1 j^2}{(i \cdot j)} \right]$$

For example,

$$RHSW = g(1) A_{11}^{12} - g(2) A_{22}^{12}$$

Note that

$$p_j^{\mu_1} A_{ji}^{\mu_1 \mu_2} = \left[p_j^{\mu_2} - \frac{(p_j \cdot p_i) p_i^{\mu_2}}{(p_i \cdot p_j)} \right] = 0$$

$$A_{ji}^{\mu_1 \mu_2} p_i^{\mu_2} = p_i^{\mu_1} - \frac{(p_j \cdot p_i) p_i^{\mu_1}}{(p_i \cdot p_j)} = 0$$

Let

$$A_{ij}^{12} = \left[\delta^{12} - \frac{j^1 i^2}{(j \cdot i)} \right]$$

$$\Rightarrow i^1 A_{ij}^{12} = A_{ij}^{12} j^2 = 0$$

$$i^2 A_{ij}^{12} = i^1 - \left[\frac{(i \cdot i)}{(i \cdot j)} \right] j^1$$

$$j^1 A_{ij}^{12} = j^2 - \left[\frac{(j \cdot j)}{(i \cdot j)} \right] i^2$$

$$A_{ij}^{12} = A_{ji}^{21}$$

$$i^1 A_{im}^{12} = 0$$

$$j^k A_{km}^{12} = 0$$

We should really be comparing this tensor A_{ij}^{12} with the tensor ~~$j^1 i^2$~~ $j^1 i^2$ which it can be regarded as replacing. I.e., products of two momenta. It seems reasonable that use of A will simplify things because A is orthogonal to i^1 and j^2 , whereas the tensor $j^1 i^2$ is not orthogonal to any single momenta. This orthogonality will be useful because Ward identity involves dot with single momenta!!!

3. So how can we recast our "complete list of vertex terms" using the tensor A above???
One idea is to take the original list and replace $j^1 i^2$ everywhere it occurs by this:

$$j^1 i^2 = (i \cdot j) \left[\delta^{12} - A_{ij}^{12} \right]$$

If I do this in the first four terms of our general formula, then group all the deltas together into the last term, here is what I get:

$$\begin{aligned} 1)' & - \left\{ (1 \cdot 1) A_{11}^{12} 2^3 + (2 \cdot 2) A_{22}^{12} 1^3 \right\} A_1(12) + c. \\ 2)' & - \left\{ (1 \cdot 1) A_{11}^{12} 2^3 - (2 \cdot 2) A_{22}^{12} 1^3 \right\} S_1(12) + c. \\ 3)' & - \left\{ (2 \cdot 3) A_{32}^{12} 1^3 + (1 \cdot 3) A_{31}^{21} 2^3 \right\} A(123) + c. \\ 4)' & - \left\{ (2 \cdot 3) A_{32}^{12} 1^3 - (1 \cdot 3) A_{31}^{21} 2^3 \right\} S(123) + c. \\ 5)' & + \delta^{12} (1+2)^3 \tilde{A}_2(12) + c. \\ 6)' & + \delta^{12} (1-2)^3 \tilde{S}_2(12) + c. \end{aligned}$$

Contemplate this new list: all triple momenta occurrences are now replaced with A p occurrences. The functions are all the same except the last two functions are linear combinations of the old set, modified by dot products to get the right symmetry. Each curly bracket has a definite 12 symmetry either A or S and this can be seen by eyeball.

Note also that I had to fake up terms 2 and 3 by artificially adding cyclic. Then when you convert to A's, you now really need the "+cyclic" in these terms because the delta parts are not done correctly if you leave that off.

4. Here is a minor technical detail: go back and look at terms 3 and 4 in the original deal. Let $A(123)$ be an arbitrary TA function. I claim that any such function can be decomposed in the following way:

$$A(123) = [A(12) + A(23) + A(31)]$$

Namely,

$$A(123) = \left[\frac{1}{3} A(\underline{123}) + \frac{1}{3} A(\underline{132}) + \frac{1}{3} A(\underline{213}) \right]$$

By the way, the notation $A(12)$ means a function of 1,2,3 which is anti sym in 12.

Therefore, I can replace terms .. wait. Suppose you write term 3 as stuff + cyclic. Then the original function no longer need be TA: an arbitrary $A(12)$ will do, because the $A(123)$ so generated by the sum will still be TA.

The question is: are these "classes" equal or does one contain the other:

a) $[231+312] A(123)$

b) $(231+312) A(12) + \text{cyclic} = [231+312](A(12)+A(23)+A(31))$

Can an arbitrary $A(123)$ be generated from $A(12)$ functions? Yes because you can certainly take $A(12)=\underline{A(123)}$. On the other hand, given $A(123)$, can you find $A(12)$? Same solution. So these ways of writing things are the same. So we can add + cyclic and get rid of $A(123)$ in favor of $A(12)$. Thus, I will now rewrite the last like so:

- 1) $[(1.1) A_{11}^{12} 2^3 + (2.2) A_{22}^{21} 1^3] A_1(12) + \text{cyclic}$
- 2) $[(1.1) A_{11}^{12} 2^3 - (2.2) A_{22}^{21} 1^3] S_1(12) + \text{cyclic}$
- 3) $[(2.3) A_{32}^{12} 1^3 + (1.3) A_{31}^{21} 2^3] A_2(12) + \text{cyclic}$
- 4) $[(2.3) A_{32}^{12} 1^3 - (1.3) A_{31}^{21} 2^3] S_2(12) + \text{cyclic}$
- 5) $[\delta^{12} (1+2)^3] A_3(12) + \text{cyclic}$
- 6) $[\delta^{12} (1-2)^3] S_3(12) + \text{cyclic}$

I claim this is still the most general form for writing the vertex, prior to the Ward restrictions. Now there are 6 functions that are undetermined: three are A, and three are S, in only one pair of variables. All terms are + cyclic and must be so. The symmetry or antisymmetry of each bracket can be seen easily, so no question but that all these tensor terms are TA. The important fact is that this is all there are!

5. I wonder if I should attempt to replace the single momenta vectors with my fancy B vectors? Maybe first I will try to apply the Wards to the above!

6. Pause and make observation: it would certainly be nicer if we had A_{12}^{12} type objects appearing instead of those A_{11}^{12} . The reason is that, in the Ward, the A_{12}^{12} would be annihilated by either 1^1 or by 2^2 ! I am further reinforced in this view by the fact that the Marshall vertex does contain the A_{12}^{12} object. So lets go back and see if this cannot be worked out.



How were these chosen? The mixed pairs chosen from obvious desire to get the A_{12}^{12} objects. These are then "extended" as was discussed earlier. I then had to pick one more A tensor. My desire to remove the S(123) in favor of S(12) by procedure mentioned above forces me to take on of the "short forms" and $(231)_A$ is the only one available. Luckily this A tensor is independent of the $(211)_A$ which comes out of the extension.

Now lets make an "original table" like so using the above forms:

$$\begin{aligned}
 & (211^3 + 212^3) A_1(12) + \text{cyclic} \\
 & (211^3 - 212^3) S_1(12) + \text{cyclic} \\
 & (213^3 + 312^3) A_2(12) + \text{cyclic} \\
 & (213^3 - 312^3) S_2(12) + \text{cyclic} \\
 & \sigma^{12} (1+2)^3 A_3(12) + \text{cyc.} \\
 & \sigma^{12} (1-2)^3 S_3(12) + \text{cyc.}
 \end{aligned}$$

2. Now I want to replace the first pair of momenta with A objects in the first four lines. First I want to argue clearly that the delta terms so created can for sure be absorbed into the last two lines by a simple redefinition of the A_2 and S_2 functions. The delta's picked up from the first two terms are OK because first term gives $(1.2)(d^{12})(1+2)^3 A_1(12)$ which clearly can be absorbed by the second last term. Etc. So the third and fourth terms are the only ones that worry me so lets do them out:

$$k'l^2 = (k.l) [\delta^{12} - A_{lk}^{12}] = (k.l) \delta^{12} - (k.l) A_{lk}^{12}$$

$$\begin{aligned} 3 \rightarrow & [(2.3) \delta^{12} 1^3 + (3.1) \delta^{12} 2^3] A_2(12) \\ &= \frac{1}{2} [(2.3) [(1^3 + 2^3) + (1^3 - 2^3)] + (3.1) [(2^3 + 1^3) - (2^3 - 1^3)]] \delta^{12} A_2(12) \\ &= \delta^{12} (1+2)^3 \cdot \left\{ \frac{1}{2} (2.3) + \frac{1}{2} (3.1) \right\} A_2(12) = \delta^{12} (1+2)^3 \tilde{A}_2(12) \\ &+ \delta^{12} (1-2)^3 \cdot \left\{ \frac{1}{2} (2.3) - \frac{1}{2} (3.1) \right\} A_2(12) = \delta^{12} (1-2)^3 \tilde{S}_2(12) \checkmark \end{aligned}$$

This shows that the term-3 deltas do get absorbed, and term-4 is then obvious, just a few sign changes and A goes to S etc. So lets now rewrite our general form in terms of the A's:

$$\begin{aligned} & [-(2.1) A_{12}^{12} (1^3 + 2^3)] A_1(12) + \text{cyclic} \\ & [-(2.1) A_{12}^{12} (1^3 - 2^3)] S_1(12) + \text{cyclic} \\ & [-(2.3) A_{32}^{12} 1^3 - (1.3) A_{31}^{21} 2^3] A_2(12) + c. \\ & [-(2.3) A_{32}^{12} 1^3 + (1.3) A_{31}^{21} 2^3] S_2(12) + c. \end{aligned}$$

$$k'l^a = (k.l) \delta^{ab} - F_{lk}^{ab}$$

$$\begin{aligned} \Rightarrow & (1.2) A_{12}^{12} 3^3 A_1(12) \\ & (1.2) A_{12}^{12} (1-2)^3 S_1(12) \\ & [(2.3) A_{32}^{12} 1^3 + (1.3) A_{31}^{21} 2^3] A_2(12) \\ & [(2.3) A_{32}^{12} 1^3 - (1.3) A_{31}^{21} 2^3] S_2(12) \\ & (1.2) \delta^{12} \end{aligned}$$

I am not yet sure what is the best way to display these terms. I could give them all the same dimension. Perhaps I should redefine my A object to absorb all those dot products. Lets try it:

$$(k \cdot l) A_{kl}^{ab} = (k \cdot l) \left[\delta^{ab} - \frac{l^a k^b}{(k \cdot l)} \right] = [(k \cdot l) \delta^{ab} - l^a k^b] \equiv F_{kl}^{ab} = (k \cdot l) A_{kl}^{ab}.$$

$$\begin{aligned} \vec{k} \cdot \vec{l} F_{kl}^{ab} &= 0 \\ F_{lm}^{ab} &= F_{ml}^{ba}. \end{aligned}$$

$$F_{kl}^{aa} = 3(k \cdot l)$$

$$\begin{aligned} \text{Lorentz} = & F_{12}^{12} 3^3 A_1(12) + \text{cy.} \\ & F_{12}^{12} (1-2)^3 S_1(12) + \text{cy.} \\ & [F_{32}^{12} 1^3 + F_{31}^{21} 2^3] A_2(12) + \text{cy.} \\ & [F_{32}^{12} 1^3 - F_{31}^{21} 2^3] S_2(12) + \text{cy.} \\ & \delta^{12} 3^3 A_3(12) + \text{cy.} \\ & \delta^{12} (1-2)^3 S_3(12) + \text{cy.} \end{aligned}$$

$$\begin{aligned} 3^2 3^3 F_{12}^{32} &= 3^2 3^3 F_{21}^{32} \\ &= (1 \cdot 2)(3 \cdot 3) - (1 \cdot 3)(2 \cdot 3) \end{aligned}$$

$$3^3 F_{21}^{31} = (1 \cdot 2) 3^1 - (1 \cdot 3) 2^1$$

$$2^2 F_{11}^{12} = (1 \cdot 1) 2^1 - (1 \cdot 2) 1^1$$

$$3^3 F_{12}^{32} = (1 \cdot 2) 3^2 - (3 \cdot 2) 1^2$$

This looks much nicer. I am ready to start dotting for the Ward.

Deatall: Notice that F is linear in either of its momenta. It follows then that:

$$F_{1l}^{ab} + F_{2l}^{ab} + F_{3l}^{ab} = 0$$

Show how Marshall's Vertex fits into my Form.

$$\Gamma^{123}(123) = \delta^{12} \left[-\frac{g(1)}{(1.1)} 1^3 + \frac{g(2)}{(2.2)} 2^3 \right] \checkmark + \frac{\left(\frac{-g(1)}{(1.1)} + \frac{g(2)}{(2.2)} \right)}{(1.1) - (2.2)} \left[F_{12}^{12} \right] (2-1)^3 \checkmark + c.$$

$$= \delta^{12} \left[b(1) 1^3 - b(2) 2^3 \right] + \frac{(b(1) - b(2))}{(1.1) - (2.2)} F_{12}^{12} (2-1)^3 + c. \checkmark$$

$$1 = \delta^{12} \left[b(1) \frac{1}{2} \{ (1+2)^3 + (1-2)^3 \} - b(2) \frac{1}{2} \{ (1+2)^3 - (1-2)^3 \} \right] \checkmark$$

$$= \delta^{12} (1+2)^3 \left[\frac{1}{2} b(1) - \frac{1}{2} b(2) \right] \checkmark + \delta^{12} (1-2)^3 \left[\frac{1}{2} (b(1) + b(2)) \right] \checkmark$$

S.

$$\begin{aligned} p_{\text{marshall}} = & + \phi \\ & + F_{12}^{12} (1-2)^3 S_1^M(12) + c. \\ & + \phi \\ & + \phi \\ & + \delta^{12} 3^3 A_3^M(12) + c. \\ & + \delta^{12} (1-2)^3 S_3^M(12) + c. \end{aligned}$$

S.

$$\begin{aligned} S_1^M(12) &= - \left[\frac{b(2) - b(1)}{(2.2) - (1.1)} \right] \checkmark \\ A_3^M(12) &= \frac{1}{2} [b(2) - b(1)] \checkmark \\ S_3^M(12) &= \frac{1}{2} [b(2) + b(1)] \checkmark \end{aligned}$$

B Vectors

Material Deleted from this Section.

1. My idea was to replace p with B everywhere throughout the tensor analysis. I invented some B -vectors (later definition changed!) which had these properties:

$$a) \quad (12)B_1 = B_2 \quad (12)B_3 = B_3$$

The idea, for example, would have been to just replace 123 tensor form with $B_1 B_2 B_3$ etc, and just take over all that stuff.

But then I found that these B 's could not be made to sum to zero so I could not just "take over" all that tensor forms stuff.

In retrospect, I don't think it really very helpful to put everything into such B -vectors anyway. So this is just an idea that never got anywhere.

$$B_1^{\wedge} = \left[\frac{2^{\wedge}}{(1.2)} - \frac{3^{\wedge}}{(1.3)} \right] \times [(1.2) - (1.3)]$$

↑ here is what the B 's were.

Complete Sets

Point of Departure.

1. At the moment I am favoring the F-tensor way of doing things on several grounds:
 - a) I think the Ward Table will be simple because F's are so orthogonal
 - b) Marshall's solution fits easily into this form
 - c) there are only 6 functions to play with, not 10.

2. Once the Ward Table is constructed, there will be only four independent operator equations in six functions. This sounds like a tractable problem. The plan is to use the operator-inversion method wherever possible and just solve the whole thing.

3. The two equations obtained from the double-dot $2^2_3 G^{123}$ and the contraction equation G^{112} must be three of the four equations. I guess the other is contained in $1^2_3 G^{123}$. I think it is probably better to just go ahead and decomposed things into the raw tensor forms like 1^1_2 and so on.

Lets see what our four equations look like, compare to earlier formulation to see if this F-tensor deal does or does not simplify things. By the way, do not compute A+D type things; this is really $[e] - [12]$ on A and that is a projector. Well, I guess its OK to take sym and antisym parts, but lets get the straight table first.

I think I will first check the Ward identity to make sure it is right before doing this table.

Material Deleted from this Section

1. Here for the first time I considered the problem of how you "fit" a given tensor into a claimed "complete set". I concluded that you should take your candidate tensor, break it down into its "irreducible components" using some kind of lookup table, and then fit those irreducibles into your extended form: X
2. I made a table using reduce to analyze ppp forms into irreducible components. However, later I made a more comprehensive table so this one has been deleted. The names of the Standard Basis tensors were later changed to conform more to the F-tensor basis.
3. Deleted: sheet showing irreducible basis for ppp sector.
4. Analysis of BBB into irreducibles. See later work in ppp section on this.
5. Connection between irreducible basis and F-tensor basis. Since I have changed my irreducible basis to a more useful form, I no longer care about this connection, so axe.

Question: What does it mean for a bunch of Vertex Terms to form a "Complete Set"

1. Consider this so-called complete set:

1. $(211+212)A_1(12) + \text{cyclic}$
2. $(211-212)S_1(12) + \text{cyclic}$
3. $(231+312)A_2(12) + \text{cyclic}$
4. $(231-312)S_2(12) + \text{cyclic}$

F-tensor basis

In the ppp sector, we claim this is a complete set of vertex terms. What we mean, of course, or are claiming, is that there is no totally antisymmetric ppp-related vertex term which cannot be written or decomposed into the above forms.

2. Example: Consider the vertex term $A(123)123$. How does this "fit" into the above complete set? Ie, tell me exactly what functions A_1S_1 and so on "absorb" this new vertex term.

To do this, I think I first have to "undo" the complete set and write it in a different way: Ie, I have to undo the 'extension' and say that the set is:

1. $\begin{cases} (211)_S A_3(123) \\ (211)_1^1 F_1^{(1)}(123) - (211)_2^1 F_1^{(1)}(123) \end{cases}$
2. $\begin{cases} (211)_A S_3(123) \\ (211)_2^2 F_2^{(1)}(123) - (211)_1^2 F_2^{(1)}(123) \end{cases}$
3. $(231)_S A_4(123)$
4. $(231)_A S_4(123)$

I could find expressions for $A_1(12)$ in terms of $A_3(123)$ and F_1 and so on. Lets come back later to do this.

Now lets consider the term under the spotlight: since $(123)_S = (123)_S$ we look up to find:

$$(211)_S = S_2 \quad (231)_S = [-S_2 + (1/3)S_1]3 \quad (123)_S = -2S_1$$

Therefore:

$$3(211)_S + (231)_S = 3(1/3)S_1 = (-1/2)(123)_S$$

Thus: $(123)_S = (-2/3)[3(211)_S + (231)_S]$

Therefore this term can be "absorbed:" by setting:

$$A_3(123) = -6A(123)$$

$$A_4(123) = -\frac{1}{2}A(123)$$

Now go back:

$$(211)_1' F^{(2)} - (211)_2' F^{(1)} = (211 + 212) [2 F^{(2)}] + \text{cyclic}$$

$$\Rightarrow \Delta A_1(12) = 2 F_1^{(2)}(123)$$

Also:

$$(231 + 312) A_4(123) = (231 + 312) \left[\frac{1}{3} A_4(123) \right] + \text{cyclic}$$

$$\Rightarrow \Delta A_2(12) = \frac{1}{3} A_4(123)$$

$$\Rightarrow A_1(12) = 2 F_1^{(2)}(123) + A_3(123)$$

Next:

$$(211)_5 A_3(123) = (211 + 212) A_3(123) + \text{cyclic}$$

$$\Rightarrow \Delta A_1(12) = A_3(123)$$

$$A_2(12) = \frac{1}{3} A_4(123)$$

Thus:

$$\left[A(123)(123)_S \Rightarrow \begin{cases} A_1(12) = -6 A(123) \\ A_2(12) = -2 A(123) \\ \text{else} = 0 \end{cases} \right.$$

So this shows exactly how our new term "registers" in the complete set. In retrospect, obvious that only the A_1 and A_2 terms are active here because those terms are where the totally symmetric tensor forms reside.

end of example

3. Could I have done this example in some other way? My method above was to first express the complete set in terms of "irreducible objects". Since my "new" tensor vertex term was itself "irreducible", it was easy to see how it fit in.

If I sit here and try to "absorb" this new term directly I find it nearly impossible to do. I really have to go via the irreducible channel. Eg, you see that the new term 123 contains only 121 and 122 pieces, so it at first seems unlikely that the A_2 gets involved since

it contains neither of these desired pieces. It turns out that you need this A_2 term to cancel off pieces of the A_1 term to make it work.

So it is incredibly easier to go through the IR channel!!!!

4. Example: Now suppose someone hands you this: $(121+122) A_5(12) + \text{cyclic}$. How can we fit this one into our complete set? This example is a little harder because this object itself is not irreducible.

It would probably be nice to be able to decompose the raw tensor forms like 112 into their irreducible pieces. Have I ever done that?

This example is still easy:

$$(121 + 122) = -123 = -\frac{1}{6}(123)_S$$

$$\begin{aligned} S_0 &= -\frac{1}{6}(123)_S [A_5(12) + A_5(23) + A_5(31)] \\ &= S_1 \left\{ \frac{1}{3} [A_5(12) + \text{cyc}] \right\} = [3(211)_S + (231)_S] \left\{ \right\} \end{aligned}$$

But $S_1 = 3(211)_S + (231)_S$ as noted earlier,

$$\begin{aligned} \Rightarrow A_4(123) &= \frac{1}{3} [A_5(12) + \text{cyc}] \\ A_3(123) &= \frac{2}{3} [A_5(12) + \text{cyc}] \end{aligned}$$

General Procedure: Take any candidate totally antisymmetric tensor. It is a linear combination of functions times raw tensor forms. Analyze each tensor form into its irreducible components. Then you know the amount of S_1, A_1 etc in this vertex term. Then expand back out to get to desired basis.

Symmetry Analysis Table for Basic raw tensor forms:

1. This table resulted from simply inverting the 8x8 matrix as described on last few pages. We find:

	$\vec{S}_1$ (121) _S	$\vec{S}_2$ (211) _S	$\vec{A}_1$ (112) _A	$\vec{A}_2$ $\frac{1}{3}(231)_A$	$\vec{O}_1$ (112) ₁	$+\frac{1}{\sqrt{3}}\vec{O}_2$ $+\frac{1}{\sqrt{3}}(112)_2$	$\vec{T}_2$ (112) ₂	$\frac{1}{\sqrt{3}}\vec{T}_1$ $\frac{1}{\sqrt{3}}(112)_1$
111	0	$-\frac{1}{3}$	0	0	$\frac{1}{12}$	$-\frac{1}{4}$	$-\frac{1}{4}$	$\frac{1}{4}$
112	$-\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{6}$	0	$\frac{1}{6}$	0	$\frac{1}{6}$	0
121	$\frac{1}{6}$	0	$\frac{1}{6}$	0	0	0	$\frac{1}{6}$	$-\frac{1}{2}$
122	$\frac{1}{6}$	0	$-\frac{1}{6}$	0	0	0	$-\frac{1}{6}$	$\frac{1}{2}$
211	0	$\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{2}$	$-\frac{1}{6}$	0	$\frac{1}{6}$	0
212	0	$\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{2}$	$-\frac{1}{6}$	0	$-\frac{1}{6}$	0
221	$-\frac{1}{6}$	$\frac{1}{3}$	$-\frac{1}{6}$	0	$\frac{1}{6}$	0	$-\frac{1}{6}$	0
222	0	$-\frac{1}{3}$	0	0	$\frac{1}{12}$	$-\frac{1}{4}$	$\frac{1}{4}$	$-\frac{1}{4}$

This stuff was read directly from Reduce output, so hopefully no errors. Note that you read this table across, not down. (Theorem: you cannot invert a matrix by looking at it).

Example of Table Useage:

$$(123)_S = 6 \cdot 123 = (-6)[121 + 122] \xrightarrow{\text{look up.}} (-6)\left[\frac{2}{6}S_1\right] = -2S_1 \quad \checkmark$$

A Spillover Theorem for the Manipulation of Complete Tensor Sets.

1. Suppose you start off with what you know is a complete tensor set, eg,

$$T_1 f_1 + T_2 f_2 + \dots + T_n f_n$$

Or for that matter, why just treat them as single objects, tensor vertex terms:

$$T_1 + T_2 + T_3 + \dots + T_n$$

You know that any tensor of certain criteria can be dissected into these forms.

Now someone comes along and hands you some new form call it F1. You analyze F1 and you find that:

$$F_1 = a T_2 + b T_3 + c T_4$$

You can replace any of those forms which appears in F1 by F1; eg, if you replace T2 by F1, you have added a new form (namely, f1), deleted one form (namely T2), and you have "spillover" into the other forms T3 and T4. Thus, your new "list" is:

$$T_1 + F_1 + T_3 + \dots + T_n$$

There is no need to indicate that T3 and T4 were modified by spillover since they are "arbitrary amounts of the tensor form indicated" anyway. So, in short, because T2 appeared in the list of F1, you could simply replace T2 by F1 in the list.

2. Now someone hands you another form, call it F2. You analyze F2 wrt the original set and you find:

$$f_2 = T_1 + T_2 + T_4 + T_8 \text{ with some coefficients}$$

Now you have to be a little careful. Can you simply replace say T4 with F2? In the new, once-modified list? What you really should do is first eliminate T2 from the F2 list, then re'ask the question. Eg, suppose upon elimination of T2 you find that:

$$F_2 = t_1 + T_8 + F_1$$

Then obviously you cannot replace T4 by F2 since T4 no longer appears in the list.

3. Moral: For each new substitution, you have to analyze in the n-1th basis.

① Start with the F-Tensor Basis:

$(211+212) A_1(12) + c$	$T1$
$(211-212) S_1(12) + c$	$T2$
$(231+312) A_2(12) + c$	$T3$
$(231-312) S_2(12) + c$	$T4$

Analyze BBB to learn:

$$BBB = "F1" = T1 + T2 + T4.$$

Thus, replace say $T4$ by BBB , let $T1, T2$ gain spillans.

New list:	$(211+212) \times A_1(12) + c$	$T1$
	$(211-212) \times S_1(12) + c$	$T2$
	$(231+312) \times A_2(12) + c$	$T3$
	$(231-312) BBB \times S(123)$	$F1.$

② Next, consider $(123)_S A(123)$ as $F2$. Analyze:

$$F2 = -2 \vec{S}_1 \Rightarrow F2 = -2 A(123) \vec{S}_1$$

So $\Delta A_1(123) = -2 A(123)$ in uned. basis

$$\Rightarrow \Delta A_1(12) = 3 \Delta A_1(123) = -6 A(123)$$

$$\Delta A_2(12) = \Delta A_1(123) = -2 A(123)$$

Thus, $F2 = T1 + T3$, so to speak. So replace $T3$ by $F2$, no wavy.

New list:	$(211+212) A_1(12) + c$	$T1$
	$(211-212) S_1(12) + c$	$T2$
	$(123) A(123)$	$F2$
	$BBB S(123)$	$F1.$

Re-write:

$(-213) A_1(12) + c$	$T1$
$(211-212) S_2(12) + c$	$T2$
$(123) A(123)$	$F2$
$BBB S(123)$	$F1.$

③ Now we want to replace ~~well~~, new term of interest
is $21^2 B_3$

Maybe let's first consider:

$$F_{21}^{ab} = (1.2) \delta^{ab} - 1.2$$

S_2

$$F_{12}^{12} 3^3 = (1.2) \delta^{12} 3^3 - \frac{1.2}{21} 3^3$$

$$\frac{F_{12}^{12} 3^3}{F_3} = \frac{(1.2) \delta^{12} 3^3}{T_5} - \frac{213}{T_1}$$

We can replace T_1 by F_3 , let spill over into T_5 δ term.
Also.

$$\frac{F_{12}^{12} (1.2)^3}{F_4} = \underbrace{(1.2) \delta^{12} (1.2)^3}_{T_6} - \underbrace{[211 \delta^{12}]}_{T_2}$$

Again ok. So new list is:

$F_{12}^{12} 3^3 A_1(1.2) + c.$	F_3
$F_{12}^{12} (1.2)^3 S_2(1.2) + c.$	F_4
$(1.23) A(1.23)$	F_2
$BBB S(1.23)$	F_1

④ Now, consider

$$F_{12}^{12} B_3 \equiv F_5$$

Well, $B_3 = \alpha(1.2) + \beta(1.2)$ certainly.

$$B_3^3 = \alpha 3^3 + \beta (1.2)^3$$

$$\alpha, \beta \neq 0!$$

$$\alpha = \alpha_A(1.2)$$

$$\beta = \beta_B(1.2)$$

S_2

$$F_{12}^{12} B_3^3 \alpha_A(1.2) + c = \alpha F_{12}^{12} 3^3 \left(\frac{1.5}{3} \alpha + c \right)$$

$$+ F_{12}^{12} (1.2)^3 \left(\frac{\beta}{6} \alpha + c \right)$$

$(1.23) \rightarrow B_3$
 $\rightarrow B_3$

straight

Oh yes $F_5 = F_3 + F_4$! So replace F_3 :

Result: I am convinced that the following is a complete tensor set:

$$F_{12}^{12} B_3^3 S_1(12) + c. \quad \text{Ward Null}$$

$$F_{12}^{12} (1-2)^3 S_2(12) + c.$$

$$(123)^3 A(123)$$

$$B_1^1 B_2^2 B_3^3 S(123) \quad \text{Ward Null.}$$

$$\delta^{12} 3^3 A_1(12) + c.$$

$$\delta^{12} (1-2)^3 S_3(12) + c.$$

where $B_1^u = (1.3) 2^u - (1.2) 3^u = z 2^u - x 3^u$ etc cyclic

$$F_{12}^{12} = (1.2) \delta^{12} - 2^1 1^2$$

Lost Ward

Projection Analysis of S(12) and A(12) type objects.

$$\begin{aligned} S(12) &= (3) - \text{like} \\ S(23) &= (1) \\ S(31) &= (1) \end{aligned}$$

1. This keeps recurring and I don't have the full thing down so lets just do it now. Remember that these are really S(12,3) eg. Turn to that page where the general function f(123) is decomposed. Then you get:

$$S(12)_S = \frac{1}{3} [S(12) + S(23) + S(31)]$$

$$S(12)_A = 0$$

$$S(12)_1' = \frac{1}{3} [2S(12) - S(23) - S(31)] \equiv S(12)_1 = -2S(23)_1$$

$$S(12)_2^2 = 0$$

$$S(23)_S = S(12)_S$$

$$S(23)_A = 0$$

$$S(23)_1' = -\frac{1}{6} [2S(12) - S(23) - S(31)] \equiv S(23)_1$$

$$S(23)_2^2 = \frac{1}{2} [S(23) - S(13)] \equiv S(23)_2$$

So chart:

Same analysis
as f(1)
or (2,3)



	$\hat{S}$	$\hat{1}$	$\hat{2}$	
S(23)	$S(23)_S$	$S(23)_1$	$S(23)_2$	✓
S(31)	$S(23)_S$	$S(23)_1$	$-S(23)_2$	✓
S(12)	$S(23)_S$	$-2S(23)_1$	$\emptyset$	✓
where	$S(23)_S = \frac{1}{3} [S(23) + \text{cyclic}]$			✓
	$S(23)_1 = -\frac{1}{6} [2S(12) - S(23) - S(31)]$			✓
	$S(23)_2 = \frac{1}{2} [S(23) - S(13)]$			✓

Now do anti's

$$A(23)_S = 0 \quad \checkmark$$

$$\begin{aligned} A(23)_A &= [A(23) - A(13) - A(21) - A(32) + A(31) + A(12)] \frac{1}{6} \checkmark \\ &= \frac{1}{3} [A(23) + \text{cyclic}] \checkmark \equiv A(23)_a \end{aligned}$$

$$\begin{aligned} A(23)_1' &= \frac{1}{6} [2A(23) + 2A(13) - A(21) - A(32) - A(31) - A(12)] \checkmark \\ &= \frac{1}{2} [A(23) - A(31)] \checkmark \equiv A(23)_1 \checkmark \end{aligned}$$

$$\begin{aligned} A(23)_2^2 &= \frac{1}{6} [2A(23) - 2A(13) + A(21) + A(32) - A(31) - A(12)] \checkmark \\ &= \frac{1}{6} [A(23) + A(31) - 2A(12)] \checkmark = -\frac{1}{6} [2A(12) - A(23) - A(31)] \checkmark \equiv A(23)_2 \checkmark \end{aligned}$$

$$\begin{aligned} A(31)_1' &= \frac{1}{6} [2A(31) + 2A(32) - A(13) - A(21) - A(12) - A(23)] \checkmark \\ &= \frac{1}{2} [A(31) - A(23)] \checkmark = -\frac{1}{2} [A(23) - A(31)] = -A(23)_1 \checkmark \end{aligned}$$

$$\begin{aligned} A(31)_2^2 &= \frac{1}{6} [2A(31) - 2A(32) + A(13) + A(21) - A(12) - A(23)] \checkmark \\ &= \frac{1}{6} [A(23) + A(31) - 2A(12)] = A(23)_2 \checkmark \end{aligned}$$

	$\hat{a}$	$\hat{1}$	$\hat{2}$	
$A(23)$	$A(23)_a$	$A(23)_1$	$A(23)_2$	$\checkmark$
$A(31)$	$A(23)_a$	$-A(23)_1$	$A(23)_2$	$\checkmark$
$A(12)$	$A(23)_a$	$\emptyset$	$-2A(23)_2$	$\checkmark$
where	$A(23)_a = \frac{1}{3} [A(23) + \text{cyclic}] \checkmark$ $A(23)_1 = \frac{1}{2} [A(23) - A(31)] \checkmark$ $A(23)_2 = -\frac{1}{6} [2A(12) - A(23) - A(31)] \checkmark$			

$$\begin{cases} X_1^2 S(12) = S(12)_1^2 = \frac{\sqrt{3}}{6} [-S(32) + S(13) + S(23) - S(31)] = 0 \checkmark \\ X_2^1 S(12) = S(12)_2^1 = \frac{\sqrt{3}}{6} [-S(32) + S(13) - S(23) + S(31)] \end{cases}$$

$$= \frac{\sqrt{3}}{3} [S(13) - S(23)] = \widetilde{S(12)}_1$$

$$\begin{cases} X_1^2 S(23) = S(23)_1^2 = \frac{\sqrt{3}}{6} [-S(21) + S(32) + S(31) - S(12)] \\ = \sqrt{3} \cdot \left(-\frac{1}{6}\right) [2S(12) - S(23) - S(31)] = \widetilde{S(23)}_2 \end{cases}$$

$$X_2^1 S(23) = \frac{\sqrt{3}}{6} [-S(21) + S(32) - S(31) + S(12)]$$

$$= \frac{\sqrt{3}}{6} [S(32) - S(31)] = \widetilde{S(23)}_1$$

$$X_1^2 S(31) = S(31)_1^2 = \frac{\sqrt{3}}{6} [-S(13) + S(21) + S(12) - S(23)]$$

$$= \sqrt{3} \cdot \frac{1}{6} [2S(12) - S(13) - S(23)] = \widetilde{S(31)}_2$$

$$X_2^1 S(31) = S(31)_2^1 = \frac{\sqrt{3}}{6} [-S(13) + S(21) - S(12) + S(23)]$$

$$= \frac{\sqrt{3}}{6} [S(23) - S(13)] = \widetilde{S(31)}_1$$

Summary :

$\widetilde{S(12)}_1 = -\frac{2}{\sqrt{3}} S(23)_2$	✓
$\widetilde{S(12)}_2 = 0$	✓
$\widetilde{S(23)}_1 = \frac{1}{\sqrt{3}} S(23)_2$	✓
$\widetilde{S(23)}_2 = +\sqrt{3} S(23)_1$	✓
$\widetilde{S(31)}_1 = +\frac{1}{\sqrt{3}} S(23)_2$	✓
$\widetilde{S(31)}_2 = -\sqrt{3} S(23)_1$	✓

↑ agree with $(1,2)$, etc.

$$X_1^2 A(12) = \frac{\sqrt{3}}{6} [2A(23) - 2A(31)] = \frac{2\sqrt{3}}{3 \cdot 2} [A(23) - A(31)] = \frac{2}{\sqrt{3}} A(23)_1 = \widetilde{A(12)}_2$$

$$X_2^1 A(12) = \frac{\sqrt{3}}{6} [0] = 0 = \widetilde{A(12)}_1$$

$$X_1^2 A(23) = \frac{\sqrt{3}}{6} [-A(12) + A(31)] = -\frac{\sqrt{3}}{3 \cdot 2} [\dots] = -\frac{\sqrt{3}}{3} A(23)_1 = -\frac{1}{\sqrt{3}} A(23)_1 = \widetilde{A(23)}_2$$

$$X_2^1 A(23) = \frac{\sqrt{3}}{6} [2A(12) - A(23) - A(31)] = -\sqrt{3} A(23)_2 = \widetilde{A(23)}_1$$

$$X_1^2 A(31) = \frac{\sqrt{3}}{6} [-A(23) + A(31)] = -\frac{\sqrt{3}}{3 \cdot 2} [A(23) - A(31)] = -\frac{1}{\sqrt{3}} A(23)_1 = \widetilde{A(31)}_2$$

$$X_2^1 A(31) = \frac{\sqrt{3}}{6} [-2S(12) + S(23) + S(31)] = \sqrt{3} \cdot (-\frac{1}{6}) [\dots] = \sqrt{3} A(23)_2 = \widetilde{A(31)}_1$$

So:

$\widetilde{A(12)}_2 = \frac{2}{\sqrt{3}} A(23)_1$	✓
$\widetilde{A(12)}_1 = 0$	✓
$\widetilde{A(23)}_2 = -\frac{1}{\sqrt{3}} A(23)_1$	✓
$\widetilde{A(23)}_1 = -\sqrt{3} A(23)_2$	✓
$\widetilde{A(31)}_2 = -\frac{1}{\sqrt{3}} A(23)_1$	✓
$\widetilde{A(31)}_1 = \sqrt{3} A(23)_2$	✓

This list is related to the S-function list in a simple way: replace partner labels 1 interchange 2, and then change all signs. The fact that it is so regular suggests to me that both lists are correct, no mistakes.

$$X_1^2 A(12) = \frac{2}{\sqrt{3}} A(23)_1 = X_1^2 A(12)_2 = -2X_1^2 A(23)_2$$

$$\Rightarrow \boxed{X_1^2 A(23)_2 = -\frac{1}{\sqrt{3}} A(23)_1}$$

$$X_2^1 A(23) = -\sqrt{3} A(23)_2 = X_2^1 A(23)_1$$

$$\Rightarrow \boxed{X_2^1 A(23)_1 = -\sqrt{3} A(23)_2}$$

Construct New Ward Table

- ① First, consider the delta terms. Both were already analyzed in Ward-6.
We can later just copy those into our new table via this relation.

$$f_A^{(5)}(12) = A_1(12) \quad f_S^{(6)}(12) = S_3(12).$$

- ② The F_{12} term yields

$$\begin{aligned} 3^3 [F_{12}^{12} (1-2)^3 S_2(12) + c] &= [(3.1) - (3.2)] F_{12}^{12} S_2(12) = (z-y) F_{12}^{12} S_2 \\ &= \{[(3.1) - (3.2)] S_2(12) (1.2)\} \delta^{12} = \{(z-y) \times S_2(12)\} \delta^{12} \\ &\quad + \{[(3.1) - (3.2)] S_2(12) (-1)\} 2'1^2 = \{(y-z) S_2(12)\} 2'1^2 \end{aligned}$$

- ③ Finally we have:

$$3^3 [1'2^2 3^3 A(123)] = \{(3.3) A(123)\} 1'2^2 = \{-(z+y) A(123)\} 1'2^2$$

That's it!!

	A	B	C	D	E
	$1'1^2$	$1'2^2$	$2'1^2$	$2'2^2$	δ^{12}
(1)	ϕ	ϕ	ϕ	ϕ	ϕ
(2)	ϕ	ϕ	$(y-z)S_2(12)✓$	ϕ	$x(z-y)S_2(12)✓$
(3)	ϕ	$-(y+z)A(123)✓$	ϕ	ϕ	ϕ
(4)	ϕ	ϕ	ϕ	ϕ	ϕ
(5)	$-A_1(23)✓$	$-A_1(23) - A_1(31)✓$	ϕ	$-A_1(31)✓$	$-(y+z)A_1(12)✓$
(6)	$2S_3(31) - S_3(23)✓$	$S_3(31) - S_3(23)✓$	$2S_3(31) - 2S_3(23)✓$	$S_3(31) - 2S_3(23)✓$	$(z-y)S_2(12)✓$

This table is just phenomenally more simple than anything I have had up to now.

The use of the 123 tensor simplifies row (3) a lot (compare to earlier use of (231+312) which spread itself out into four columns!) The advantage of using the Ard null terms is obvious. So lets get on with it!

(7)	$-b(1)$	ϕ	ϕ	$+b(2)$	$-(x+z)b(1) + (x+y)b(2)✓$
Σ	ϕ	ϕ	ϕ	ϕ	ϕ
	(3)	(3)	(3)	ϕ	(3)

A+D
11 + 22

A-D
11 - 22

B
12

C
21

E
2

(2)

ϕ

ϕ

ϕ

$(y-z) S_2(12)$

$x(z-y) S_2(12)$

(3)

ϕ

ϕ

$-(y+z) A(123)$

ϕ

ϕ

(5)

$-A_1(23) - A_1(31)$

$-A_1(23) + A_1(31)$

$-A_1(23) - A_1(31)$

ϕ

$-(y+z) A_1(12)$

(6)

$3[S_3(31) - S_3(23)]$

$S_3(31) + S_3(23)$

$S_3(31) - S_3(23)$

$2[S_3(31) - S_3(23)]$

$(z-y) S_3(12)$

(7)

$-b(1) + b(2)$

$-b(1) - b(2)$

ϕ

ϕ

$-(x+z)b(1) + (x+y)b(2)$

Material Deleted from this Section

1. This was my first solution to the non-n sector Ward Table problem. I fractionated the terms of the Ward table and got the 10 equations stated below. I solved these equations and got Marshall's Vertex exactly.

I have deleted all the fractionation work that was needed to get these ten equations since the cyclic operator method is much more efficient (and starts from the same Ward table given in this section)

3. So this is very convenient. I have then two things to do:

- 1) resolve the non-n problem with the slight modification underlined
- 2) write a most general form for the n-terms vertex
- 3) balance the n-terms vertex against the n-terms RHS.

4. Lets set about the first task seince it seems easiest,. This will be quite easy.

Look at section "Last Ward" and rewrite the 10 equations including the change:

Raw equations:

SAVE

✓ ① $-2R_a = 0$

✓ ② $-2R_2 - 6T_2 = +2B_2 - \underline{2C_2}$

③ $-2R_1 + 2T_1 = +2B_1 - \underline{2C_1}$

✓ ④ $2T_5 = +2B_5 - \underline{2C_5}$

✓ ⑤ $-2d_5V - 2R_a = 0$

✓ ⑥ $-2d_1V - 2R_2 - 2T_2 = 0$

✓ ⑦ $-2[d_2S_1 - d_1S_2] = 0$

⑧ $-2[d_2S_1 + d_1S_2] + 2d_2S_5 - 4T_2 = 0$

⑨ $-x(z-y)S_1 + (xy+xz-2zy)S_2$

$-2d_5R_a + 2d_2T_1 - 2d_1T_2 + 2d_1R_2 + \frac{2}{3}d_2R_1 = 0$

⑩ $-x(z-y)S_1 + x(z-y)S_5 - (xy+xz-2zy)S_2$

$-2d_1R_a + 2d_2T_1 + 2d_1T_2 - 2d_2T_5 + (4d_5+2d_1)R_2 - \frac{2}{3}d_2R_1$

$= -2d_2B_5 - 2d_2B_1 - (2d_1-4d_5)B_2$

$A_1 \leftrightarrow R$

$S_2 \leftrightarrow S$

$S_3 \leftrightarrow T$

$A \leftrightarrow V$

$b(1) \leftrightarrow B$

These are the raw 10 equations with no substitutions made, all terms present. You can see the extent of the change shown in underline. We could just feed this into Reduce but might as well simplify by hand first:

possibly useful junk.

Comments: Lets think of these equations. We know there are really only four equations sitting there, we can ignore the column D. I would feel much better if each of these four equations were submitted to a complete S_3 symmetry fractionation analysis. Remember that $S(12)$ means a function of three variables, I might have written it like $S(12,3)$. So lets analyze every term into symmetry pieces.

Lets use d_1, d_2, d_3 as our basis.

$$\left\{ \begin{array}{l} d_3 \stackrel{\vee}{=} (2.3)_s = \frac{1}{3}[x+y+z] \checkmark \\ d_1 \stackrel{\vee}{=} (2.3)_1 = \frac{1}{6}[z+y]-2x \checkmark \\ d_2 \stackrel{\vee}{=} (2.3)_2 = \frac{1}{2}[y-z] \checkmark \end{array} \right.$$

$$\begin{array}{l} y = d_3 \hat{s} + d_1 \hat{i} + d_2 \hat{z} = (2.3) \checkmark \\ z = d_3 \hat{s} + d_1 \hat{i} - d_2 \hat{z} = (1.3) \checkmark \\ x = d_3 \hat{s} - 2d_1 \hat{i} = (1.2) \checkmark \end{array}$$

$$\text{So } (y-z) = 2d_2 \hat{z} \checkmark$$

$$(y+z) = 2d_3 \hat{s} + 2d_1 \hat{i} \checkmark$$

$$\left\{ \begin{array}{l} (y-z) = (2d_2) \hat{z} \checkmark = (y-z) \hat{z} \checkmark \\ (y+z) = (2d_3) \hat{s} + (2d_1) \hat{i} \checkmark \\ x(z-y) = 2d_2(2d_1-d_3) \hat{z} = x(z-y) \hat{z} \checkmark \end{array} \right.$$

$$\begin{aligned} \text{What about } -x(z-y) &= [d_3 \hat{s} - 2d_1 \hat{i}] [2d_2 \hat{z}] = \hat{s} [\emptyset] \\ &= \hat{z} [(d_3)(2d_2) + \frac{1}{2}((-2d_1)(2d_2) + (-2\tilde{d}_1)(2\tilde{d}_2))] \\ &\quad + \hat{a} [\frac{1}{2}((-2d_1)(2d_2) - (-2\tilde{d}_1)(2\tilde{d}_2))] \end{aligned}$$

$$\begin{aligned} &= \hat{a} [2d_2 d_3 - 2d_1 d_2 - 2\tilde{d}_1 \tilde{d}_2] = \hat{a} [2d_2 d_3 - 2(d_1 d_2 + \tilde{d}_1 \tilde{d}_2)] \\ &\quad + \hat{a} [-2\tilde{d}_1 d_2 + 2\tilde{d}_1 \tilde{d}_2] + \hat{a} [-2(\tilde{d}_1 d_2 - \tilde{d}_1 \tilde{d}_2)] \end{aligned}$$

$$= \hat{a} [2d_2 d_3 - 4d_1 d_2] = 2d_2 [d_3 - 2d_1] \hat{a} = (y-z)x \hat{z} \checkmark$$

So this grouping of dot products is entirely 2-like. See appended sheet for another way to conclude the same thing.

confirmed on next page

So here comes the bottom line :

Question: what is the form of the most general solution to the Ward Identity?

Answer: first, you can have any Ward Null terms you want. Then if you impose the Ward on the remaining terms as parametrized in the purple box, here is what you find:

$$\left. \begin{aligned} A_1(12) &= -B_2 &= \frac{1}{2} [b(2) - b(1)] \\ S_3(12) &= B_5 + B_1 &= \frac{1}{2} [b(2) + b(1)] \\ S_2(12) &= -B_2/d_2 &= \frac{\frac{1}{2} [b(2) - b(1)]}{\frac{1}{2} [(1,1) - (2,2)]} \end{aligned} \right\}$$

$$S_1(12) = \text{anything}$$

$$S(123) = \text{anything}$$

$$A(123) = 0.$$

But the solution I have generated is precisely Marshall's solution!

Conclusion: Assuming the simple ansatz that $c=0$ so only one function b on the RHS of the Ward identity, and assuming that there are no n -terms among the tensor forms, we are now sure that: The most general form of a solution to the Ward Identity is Marshall's solutions plus arbitrary linear combinations of the two Ward Null Terms. !

Conversion of Marshall's Solution to My Variables

See earlier sheet for details

$$\begin{aligned} \Gamma^{\text{Marshall}} &= F_{12}^{12} (1-z)^3 S_2(1z) + \text{cyc.} \\ &+ \delta^{12} 3^3 A_1(1z) + \text{cyc.} \\ &+ \delta^{12} (1-z)^3 S_3(1z) + \text{cyc.} \end{aligned}$$

where $S_2(1z) = - \left[\frac{b(z) - b(1)}{(2,2) - (1,1)} \right] \checkmark$

$$A_1(1z) = \frac{1}{2} [b(z) - b(1)] \checkmark$$

$$S_3(1z) = \frac{1}{2} [b(z) + b(1)] \checkmark$$

$$\Rightarrow A_1(1z) = -b(1)_2 \checkmark = -B_2 \checkmark$$

$$S_3(1z) = b(1)_s + b(1)_l \checkmark = B_s + B_l \checkmark$$

$$S_2(1z) = \frac{-2}{[(2,2) - (1,1)]} A_1(1z) \checkmark = \frac{(-2)(-B_2)}{(-2d_2)} = -\frac{B_2}{d_2} \checkmark$$

$$\Rightarrow R = -B_2 = R_a - 2R_2 \checkmark$$

$$T = B_s + B_l = T_s - 2T_l$$

$$S = -B_2/d_2 = \frac{S_s - 2S_l}{d_1 S_2}$$

$$\Rightarrow d_2 S_s - 2(d_2 S_l) = -B_2 \checkmark = \text{Eq. \# 8}$$

$$\Rightarrow \left(\frac{B_s}{d_2} + \frac{B_l}{d_2} \right) = \frac{B_2}{d_2} \Rightarrow S_2 = 0$$