

Triple Glue Vertex

$S_n, X_1^i, \dots$

data

PPP

Ward 10

Ward 6

15 x 18

5 x 6 Operator

Fkl^{ab}

B Vectors

Complete Sets

Lost Ward

typed

Misc

PPP + DP Sector

(no go)

IRREDUCIBLE

the n's

n - oldway

Part 2 of 2

typed

S_3 Group Theory

1. The group elements. S_3 is the group of permutations of three objects. The group has six elements = $3!$ which can be denoted by:

$$(e) \quad (12) \quad (13) \quad (23) \quad (123) \quad (132)$$

We shall always enumerate the group elements in this order. The meaning of these elements is clear from these examples:

$$(12) f(123) = f(213) \quad / \text{ transpose 1 and 2}$$

$$(123)f(123) = f(231) \quad / \text{ forward cyclic}$$

2. Irreducible Representations. Here they are:

<u>Young symbol</u>	<u>letter symbol</u>	<u>name</u>	<u>dimensionality</u>
	S	symmetric	1
	M	mixed	2
	A	antisymmetric	1

You see that the sum of the squares of the dimensionalities of the irreducible representations equals the order of the group, a general fact of finite groups.

definition: a pair is a set of functions (or objects of some sort) which belong to the M representation. The pair has a row-1 member and a row-2 member.

3. Group Algebra. The six elements of S_3 may be regarded as forming a basis in a 6-dimensional space. Elements of this space (vectors) are just linear combinations of group elements, eg, $5(12) - 3(123)$. Since you can multiply any two elements of this 6-dimensional space in the obvious way [since you know how to multiply the group elements themselves], the elements of this space form an algebra called the group algebra. Since vectors in the group algebra mix among themselves under action of group elements, any six linearly independent vectors in the group algebra form the basis for a 6-dimensional group representation called the regular representation.

This regular representation is reducible and contains each irreducible representation a number of times equal to the dimensionality of that representation. (Another general fact of finite groups). Thus, schematically,

$$(\text{group algebra}) = (\text{regular rep}) = \begin{array}{c} \begin{array}{ccccccc} \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array} & + & \begin{array}{|c|} \hline \square \\ \square \\ \hline \end{array} & + & \begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array} & + & \begin{array}{|c|} \hline \square \\ \hline \end{array} \\ S & + & A & + & \underbrace{\text{two copies of } M} \end{array} \end{array}$$

4. Analysis of functions. The abstract statements about the group algebra become more real when elements of the group algebra are applied to functions. Let $f(123)$ be a function of three variables [eg, $f(p_1, p_2, p_3)$] which has no particular permutation symmetry. Then in general the following six functions are linearly independent:

$$f(123), f(213), f(321), f(132), f(231), f(312)$$

This list of permuted functions was obtained by applying the elements of S_3 to $f(123)$ in the order in which they appear on top of page 1.

Since these functions only mix among themselves under application of group elements, they form a basis for the 6-dimensional regular representation. It is the same as before, except now we apply everything to the function $f(123)$.

We shall now show how these six functions above can be shuffled (linearly combined) into six new functions which belong to the irreducible representations which appear in the decomposition of the regular representation shown on bottom of last page. We expect to find one S-type function, one A-type function, and two pairs of M-type functions (see def. of "pair" above).

At this point it is very convenient to define six very special elements of the group algebra which we shall call " the X's ". These X's will tell us how to find the six reshuffled functions mentioned above. Here are the X's:

	<u>(e)</u>	<u>(12)</u>	<u>(13)</u>	<u>(23)</u>	<u>(123)</u>	<u>(132)</u>	<u>factor</u>
X_s	1	1	1	1	1	1	(1/6)
X_a	1	-1	-1	-1	1	1	(1/6)
X_1^1	2	2	-1	-1	-1	-1	(1/6)
X_2^1	0	0	-1	1	-1	1	($\sqrt{3}/6$)
X_1^2	0	0	-1	1	1	-1	($\sqrt{3}/6$)
X_2^2	2	-2	1	1	-1	-1	(1/6)

For example, X_s is (1/6) times the sum of the six group elements.

Now we return to the ~~stuff~~ shuffling problem above. The six reshuffled functions are:

$\begin{array}{c} S \\ \hline \square \square \square \end{array}$	$\begin{array}{c} A \\ \hline \square \end{array}$	$\begin{array}{c} M^1 \\ \hline \square \square \end{array}$	$\begin{array}{c} M^2 \\ \hline \square \square \end{array}$
$X_s f(123)$	$X_a f(123)$	$\begin{cases} X_1^1 f(123) \\ X_2^1 f(123) \end{cases}$	$\begin{cases} X_1^2 f(123) \\ X_2^2 f(123) \end{cases}$

Clearly, the S-function $X_s f(123)$ is totally symmetric, whereas the A-function $X_a f(123)$ is totally antisymmetric.

Looking at the two pairs of M type functions, one sees that the lower subscript on the X operator tells the row of the member of the pair (row-1 or row-2), whereas the upper superscript tells which copy of the M-rep the function belongs to (there are two copies which we labelled M^1 and M^2).

A characteristic of the two functions (the pair) which belong to the M^1 copy is that both functions are even under (12); each member of the M^2 pair is odd under (12).

In case the use of the table above is not clear, here explicitly is the row-1 member of the pair of functions which belong to the M^2 copy of the M representation:

$$f_1^2(123) \equiv X_1^2 f(123) = \frac{\sqrt{3}}{6} [-f(321) + f(132) + f(231) - f(312)]$$

Similarly we could write down the other member of this pair, f_2^2 . Under application of group elements, the two functions f_1^2 and f_2^2 mix among themselves; they form a basis for an M-representation of S_3 .

In summary, we have shown how to linearly combine the six functions $f(123), f(213)$, etc. to get new functions which have definite symmetry, ie, which belong to irreducible reps of S_3 . We got one totally symmetric function $f_s(123)$, one totally antisymmetric function $f_a(123)$, and two pairs of mixed-symmetry functions:

$$\begin{array}{ll} \text{pair-1:} & \begin{cases} f_1^1(123) \\ f_2^1(123) \end{cases} & \text{pair-2:} & \begin{cases} f_1^2(123) \\ f_2^2(123) \end{cases} \end{array}$$

Example: Consider the special case $f(123) = F(1)$, ie, a function of only one variable. [Equivalently, we could consider $f(123) = S(23)$, a function symmetric in 2 and 3 interchange only; but let's deal with $F(1)$ instead.] In this special case we find:

$$\begin{aligned} F(1)_s &\equiv X_s F(1) = \frac{1}{3} [F(1) + F(2) + F(3)] \\ \left\{ \begin{aligned} F(1)_1' &\equiv X_1^1 F(1) = \frac{1}{6} [F(1) + F(2) - 2F(3)] \\ F(1)_2' &\equiv X_2^1 F(1) = \frac{\sqrt{3}}{6} [F(1) - F(2)] \end{aligned} \right\} && \text{example of a "pair"} \end{aligned}$$

Obviously if $f(123)=F(1)$, the six functions $f(123), f(213)...$ etc are not all linearly independent. Only three are independent. It turns out that the second function pair $F(1)_1^2, F(1)_2^2$ is proportional to the first pair shown, and $F(1)_a = 0$. So the three functions shown on bottom of last page are all there are.

5. More about the X's. Knowing the group multiplication table of S_3 , eg $(12)(23)=(123)$, one can construct a multiplication table for the X's. Here it is:

	X_S	X_A	X_1^1	X_2^1	X_1^2	X_2^2
X_S	X_S	0	0	0	0	0
X_A	0	X_A	0	0	0	0
X_1^1	0	0	X_1^1	0	X_1^2	0
X_2^1	0	0	X_2^1	0	X_2^2	0
X_1^2	0	0	0	X_1^1	0	X_1^2
X_2^2	0	0	0	X_2^1	0	X_2^2

This table leads to several interesting observations:

- 1) The four operators X_S, X_A, X_1^1, X_2^2 form a set of projection operators. Ie, if you multiply together two operators in this set you get zero if the operators are different and you get the operator back again if the operators are the same.
- 2) Look at the 4×4 subtable in the above table. When you left-multiply something by operator X_1^1 you get zero if that operator is a row-2 operator (like X_2^2), but you get "transparency" if X_1^1 hits a row-1 object. In other words, X_1^1 is the row-1 projection operator. Similarly, X_2^2 is the row-2 projection operator.
- 3) What about the operators X_2^1 and X_1^2 ? What do they do? When X_2^1 hits a row-2 object, you get zero. But when it hits a row-1 object it generates the partner to that object, ie, the other member of the pair. Repeat: when X_1^2 hits a row-1 member of a pair, it generates the row-2 member of the pair. Very convenient! Similarly X_1^2 gives you the row-1 member from the row-2 member.

Examples:

$$X_1^2 f_2^2(123) = f_1^2(123)$$

$$\text{/proof: } X_1^2 f_2^2 = X_1^2 X_2^2 f = X_1^2 f = f_1^2$$

$$X_1^2 f_2^1(123) = f_1^1(123)$$

Thus, if you are given either member of a "pair" of functions, you know at once how to compute the other member of the pair. [However, the two members of the pair are still linearly independent, one is not a multiple of the other!]

6. The Projection Operators. Since X_s, X_a, X_1^1 and X_2^2 are a complete set of projection operators, we know that:

$$(e) \quad = X_s + X_a + X_1^1 + X_2^2$$

as can be verified by adding the rows of the X-definition table on page 2. [Notice that both sides of the above equation are in the group algebra].

Applying the above to a function $f(123)$ we get this symmetry fractionation of a function:

$$f(123) = f_s(123) + f_a(123) + f_1^1(123) + f_2^2(123)$$

In other words, any function $f(123)$ has an S-component, an A-component, and a row-1 component and a row-2 component. This decomposition or "fractionation" is unique; you get each of the four pieces ~~xy~~ by applying the appropriate X-operator.

Obviously this is the generalization of the S_2 decomposition of an arbitrary function $f(12)$ into its irreducible pieces, in that case the symmetric and antisymmetric pieces. In S_3 you have the extra complication of the mixed-symmetry pieces.

Notice that f_1^1 and f_2^2 are not a pair because they are from different copies.

To simplify notation, lets write f_1^1 just as f_1 , and f_2^2 just as f_2 . To emphasize the idea that we are fractionating a function into its four components, we imagine $f(123)$ as a vector in a fourdimensional space whose axes are labelled by s,a,1,2. Then we write, just schematically:

$$f(123) = f_s(123) \hat{s} + f_a(123) \hat{a} + f_1(123) \hat{1} + f_2(123) \hat{2}$$

The main reason for this notation is this: if you claim that two functions are equal, say $f(123) = g(123)$, they must be equal component for component. Said another way, the single equation $f(123)=g(123)$ is really four independent equations! This is why you always want to break functions into their symmetry fractions.

7. The Clebsch Problem. From the last section we know how to decompose a function $f(123)$ into its symmetry fractions. The result is simply this:

$$f = f_s \hat{s} + f_1 \hat{1} + f_2 \hat{2} + f_a \hat{a} = (f_s, f_1, f_2, f_a) = \text{vector}$$

Now we ask: how do you decompose the product of two functions, say $f(123)g(123)$, into symmetry fractions? Like any function this product (call it $h(123)$) must be analyzable into its symmetry fractions.

Here is the answer: first take each function $f(123)$ and $g(123)$ and analyze into components, so you have (f_s, f_1, f_2, f_a) and (g_s, g_1, g_2, g_a) . Then the result is this:

$$f(123) g(123) =$$

$$\begin{aligned} & \hat{s} [f_s g_s + f_a g_a + \frac{1}{2}(f_1 g_1 + \tilde{f}_1 \tilde{g}_1) + \frac{1}{2}(f_2 g_2 + \tilde{f}_2 \tilde{g}_2)] \\ & + \hat{1} [f_s g_1 + f_1 g_s + f_a g_2 + f_2 g_a + \frac{1}{2}(f_1 g_1 - \tilde{f}_1 \tilde{g}_1) + \frac{1}{2}(f_2 g_2 - \tilde{f}_2 \tilde{g}_2)] \\ & + \hat{2} [f_s g_2 + f_2 g_s + f_a g_1 + f_1 g_a + \frac{1}{2}(f_1 g_2 + \tilde{f}_1 \tilde{g}_2) + \frac{1}{2}(f_2 g_1 + \tilde{f}_2 \tilde{g}_1)] \\ & + \hat{a} [f_s g_a + f_a g_s + \frac{1}{2}(f_1 g_2 - \tilde{f}_1 \tilde{g}_2) + \frac{1}{2}(f_2 g_1 - \tilde{f}_2 \tilde{g}_1)] \end{aligned}$$

Here a twiddle means "partner of". Thus, $\tilde{f}_1 = X_2^1 f_1$ (ie, $f_2^1 = X_2^1 f_1^1$). We already know how to compute partners from top of page 5.

How does one interpret the above result? Essentially you are combining two regular representations together and decomposing the result into irreducibles. Implicit in the above organization of the 16 terms in $f(123)g(123)$ are the Clebsch Gordan coefficients for S_3 . Eg, the combination $f_a g_1$ behaves as a row-2 function, $f_1 g_1 + \tilde{f}_1 \tilde{g}_1$ behaves as a S type function. Apart from the obvious cases (like $f_s g_s$ and $f_a g_a$ are symmetric), you really have to look up the Clebschs (or compute them) to make the above decomposition.

8. Construction of Totally Antisymmetric Objects. Given $f(123)$ and $g(123)$, there are in general four different totally antisymmetric combinations you can make. These appear as the last line in our equation of page 6. They are:

- | | |
|--------------|-------------------------|
| 1. $f_s g_a$ | 3. $f_1 g_2$ - partners |
| 2. $f_a g_s$ | 4. $f_2 g_1$ - partners |

Combinations 1. and 2. are rather obvious ways to generate totally antisymmetric functions. The 3rd and 4th ways are not quite so obvious.

Here we want only to recast the combinations 3. and 4. into cyclic notation. For example:

$$3. = f_1 g_2 - \tilde{f}_1 \tilde{g}_2 = f_1' g_2' - f_2' g_1' = \frac{1}{3} [f(123) + f(213)] g_2 + \text{cyclic}.$$

$$4. = f_2 g_1 - \tilde{f}_2 \tilde{g}_1 = f_2' g_1' - f_1' g_2' = \frac{1}{3} [f(123) - f(213)] g_1 + \text{cyclic}.$$

You can verify these re-writings using the X table on page 2. When written in this way it is more obvious that the combinations are indeed totally antisymmetric: in line 3. the bracket [...] is clearly (12) symmetric, but g_2 is a row-2 function and is therefore (12) antisymmetric, as was noted earlier. Then the + cyclic gives total antisymmetry in the usual way. Since g_1 is (12) symmetric, being a row-1 function, line 4. is seen to be also totally antisymmetric.

Tensor Problem

(this section assumes knowledge of section " S_3 Group Theory")

definition: a tensor form is a linear combination raw tensor forms.

definition: a raw tensor form is something like this:

$$123 = 1^1 2^2 3^3 = (p_1)^{u_1} (p_2)^{u_2} (p_3)^{u_3} \quad / \text{ first class}$$

$$d^{12} 3^3 = g^{u_1 u_2} (p_3)^3 \quad / \text{ second class}$$

definition: a vertex term is some combination of a tensor form and a function that is totally antisymmetric and therefore a candidate for the non-color part of the triple glue vertex.

1. Statement of Problem and Outline of Solution. We want to construct a "complete set" of vertex terms for the triple glue vertex. If someone hands us a vertex term out of the blue, we should be able to show how it fits into our complete set of vertex terms. Once we have our complete set, we can impose the axial gauge Ward Identity, eg, to get a most general Ward-respecting form for the vertex. At this point we are only considering raw tensor forms of the classes noted above (ie, no gauge vectors n^u , yet!).

The method is as follows: first, we shuffle the raw tensor forms into tensor forms of definite symmetry, without regard to momentum conservation. Thus we will have $3 \times 3 \times 3 = 27$ tensor forms of definite symmetry in the first class (ie, triple momenta products), and $3 \times 3 = 9$ tensor forms of definite symmetry in the second class (one momenta and a g_{uv}). Then we impose momentum conservation, $1+2+3=0$. When this is done we find that of the 27 tensor forms in the first class, only 8 are linearly independent; of the 9 tensor forms in the second class, only 6 are linearly independent. These numbers 8 and 6 are obvious since there are only 8 and 6 raw tensor forms in the respective classes after momentum conservation, ie:

class 1: 111, 112, 121, 122, 211, 212, 221, 222

class 2: $d^{12} 1^3$, $d^{12} 2^3$, $d^{23} 1^1$, $d^{23} 2^1$, $d^{31} 1^2$, $d^{31} 2^2$

We will then have $8+6 = 14$ tensor forms of definite S_3 symmetry which we can play with. To generate totally antisymmetric vertex terms, we will combine these tensor forms with functions of definite symmetry. In section on " S_3 group theory" we showed exactly all the ways there are to make an totally antisym object out of two objects $f(123)$ [here, functions] and $g(123)$ [here tensor forms].

2. Tensor Forms of the First Class. It is a simple matter to reshuffle the 27 triple-mometa tensor forms into tensor forms of definite symmetry: we just pick forms at random and apply X-operators until we have 27 objects. This procedure is really much simpler than it sounds. For example, here immediately are 6 of the 27 tensor forms:

$$\begin{aligned}(112)_a &= 6X_s \cdot 112 = (112+221) + \text{cyclic} \\ (112)_s &= 6X_a \cdot 112 = (112-221) + \text{cyclic} \\ (112)_1^1 &= 6X_1^1 \cdot 112 = 2 \cdot 112 + 2 \cdot 221 - 233 - 131 - 322 - 313 \\ (112)_2^2 &= 6X_2^2 \cdot 112 = 2 \cdot 112 - 2 \cdot 221 + 233 + 131 - 322 - 313 \\ &\text{etc.}\end{aligned}$$

By the way, one must be careful with the notation "+ cyclic" in our present notation. Since we always put the momenta into standard order $i^1 j^2 k^3$, things are not always what they seem. For example:

$$\begin{aligned}(12) 123 &= (12) 1^1 2^2 3^3 = 2^2 1^1 3^3 = 123 \\ (112+221) + \text{cyclic} &= 112 + 221 + 233 + 131 + 322 + 313\end{aligned}$$

Once we have our set of 27 forms we apply momentum conservation to see which ones are linearly independent. It turns out that there are two S-type, two A-type, and two "pairs" of M-type forms. (eight in all, as noted). These can be chosen in many ways, here is one we which we shall use as a reference:

$$\begin{aligned}\vec{S}_1 &= (121)_s = [(121+122) + \text{cyclic}] \\ \vec{S}_2 &= (211)_s = [(211+212) + \text{cyclic}] \\ \vec{A}_1 &= (112)_A = [(112-221) + \text{cyclic}] \\ \frac{1}{3}\vec{A}_2 &= \frac{1}{3}(231)_A = [231-312] \\ O_1 = \vec{M}_1^1 &= (112)_1^1 = [2 \cdot 112 + 2 \cdot 221 - 233 - 131 - 322 - 313] \\ T_1 = \vec{M}_2^1 &= (112)_2^1 = \sqrt{3} [-233 + 131 - 322 + 313] \\ O_2 = \vec{M}_1^2 &= (112)_1^2 = \sqrt{3} [-233 + 131 + 322 - 313] \\ T_2 = \vec{M}_2^2 &= (112)_2^2 = [2 \cdot 112 - 2 \cdot 221 + 233 + 131 - 322 - 313]\end{aligned}$$

Here is a table showing how all the tensor forms in the first class can be written in terms of the forms we have selected on the previous page:

		$\vec{S}_1$	$\vec{S}_2$	$\vec{A}_1$	$\frac{1}{3}\vec{A}_2$	$\vec{M}_1$	$\vec{M}_2$	$\vec{T}_1$	$\vec{T}_2$
$\vec{S}$	$(112)_S$	-1	2	0	0	0	0	0	0
	$(121)_S$	1	0	0	0	0	0	0	0
	$(211)_S$	0	1	0	0	0	0	0	0
	$(123)_S$	-2	0	0	0	0	0	0	0
	$(132)_S$	0	-2	0	0	0	0	0	0
	$(231)_S$	1	-3	0	0	0	0	0	0
	$(111)_S$	0	-2	0	0	0	0	0	0
$\vec{A}$	$(112)_A$	0	0	1	0	0	0	0	0
	$(121)_A$	0	0	1	0	0	0	0	0
	$(211)_A$	0	0	1	-3	0	0	0	0
	$(231)_A$	0	0	0	3	0	0	0	0
$\vec{Q}$	$(112)_1$	0	0	0	0	1	0	0	0
	$(112)_2$	0	0	0	0	0	1	0	0
	$(121)_1$	0	0	0	0	0	0	0	0
	$(121)_2$	0	0	0	0	$-\sqrt{3}$	1	0	0
	$(211)_1$	0	0	0	0	-1	0	0	0
	$(211)_2$	0	0	0	0	0	1	0	0
	$(132)_1$	0	0	0	0	-1	0	0	0
	$(111)_1$	0	0	0	0	$\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	0	0
row 2	$(112)_2^1$	0	0	0	0	0	0	1	0
	$(112)_2^2$	0	0	0	0	0	0	0	1
	$(121)_2$	0	0	0	0	0	0	0	0
	$(121)_2^2$	0	0	0	0	0	0	$-\sqrt{3}$	1
	$(211)_2$	0	0	0	0	0	0	-1	0
	$(211)_2^1$	0	0	0	0	0	0	0	1
	$(132)_2$	0	0	0	0	0	0	-1	0
	$(111)_2$	0	0	0	0	0	0	$\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$

Table 1

Repeat: the table on the last page shows exactly how each of the 27 tensor forms of the first class can be expressed in terms of the 8 linearly independent tensor forms we chose as our "standard" basis.

While we are putting down tables, here is another one with the same heading. This table shows how each "raw tensor form" can be expressed in terms of the 8 basic tensors. :

	$\vec{S}_1$	$\vec{S}_2$	$\vec{A}_1$	$\frac{1}{3}\vec{A}_2$	$\vec{M}_1$	$\vec{M}_1^2$	$\vec{M}_2$	$\vec{M}_2^2$
111	0	$-\frac{1}{3}$	0	0	$\frac{1}{12}$	$-\frac{1}{4\sqrt{3}}$	$\frac{1}{4\sqrt{3}}$	$-\frac{1}{4}$
112	$-\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{6}$	0	$\frac{1}{6}$	0	0	$\frac{1}{6}$
121	$\frac{1}{6}$	0	$\frac{1}{6}$	0	0	0	$-\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$
122	$\frac{1}{6}$	0	$-\frac{1}{6}$	0	0	0	$\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$
211	0	$\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{2}$	$-\frac{1}{6}$	0	0	$\frac{1}{6}$
212	0	$\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{2}$	$-\frac{1}{6}$	0	0	$-\frac{1}{6}$
221	$-\frac{1}{6}$	$\frac{1}{3}$	$-\frac{1}{6}$	0	$\frac{1}{6}$	0	0	$-\frac{1}{6}$
222	0	$-\frac{1}{3}$	0	0	$\frac{1}{12}$	$-\frac{1}{4\sqrt{3}}$	$-\frac{1}{4\sqrt{3}}$	$-\frac{1}{4}$

Table 2

This table was generated by inverting the 8 definitions of the objects S_1 , etc which are shown on page 2. This inversion was done in 10 seconds by REDUCE.

Later we shall have a use for this table.

3. Tensor Forms of the Second Class. We just repeat the same analysis for tensor forms in this class. For this class there is one S-type, one A-type, and two pairs of M-type tensor forms. Again here is a way of choosing them for reference:

$$\vec{S}_3 = \frac{1}{2}(\delta^{12} 3^3)_S = [\delta^{12} 3^3 + \text{cyclic}]$$

$$\vec{A}_3 = (\delta^{12} 1^3)_A = [\delta^{12} (1-2)^3 + \text{cyclic}] = \text{the famous one!}$$

$$\vec{M}_1^3 = \frac{1}{2}(\delta^{12} 3^3)_1 = [2\delta^{12} 3^3 - \delta^{23} 1^1 - \delta^{31} 2^2]$$

$$\vec{M}_2^3 = \frac{1}{2}(\delta^{12} 3^3)_2 = \sqrt{3} [\delta^{31} 2^2 - \delta^{23} 1^1]$$

$$\vec{M}_1^4 = (\delta^{12} 1^3)_1 = -\sqrt{3} [\delta^{31} (3-1)^2 - \delta^{23} (2-3)^1]$$

$$\vec{M}_2^4 = (\delta^{12} 1^3)_2 = [2\delta^{12} (1-2)^3 - \delta^{23} (2-3)^1 - \delta^{31} (3-1)^2]$$

added

We have started using overarrows to indicate the tensor forms of definite symmetry. Also, note that a factor of 6 is present everywhere as on top of page 2.

4. Most General Vertex Term in Standard Reference Form. We simply combine our definite symmetry tensor forms with definite symmetry functions as discussed in previous notes. We use the comments of paragraph 8 of those notes (S_3 Group Theory) to express the M-rep terms in cyclic notation. Our result is:

$$\begin{aligned} \Gamma(123) = & \vec{S}_1 A_1(123) + \vec{S}_2 A_2(123) + \vec{S}_3 A_3(123) \\ & + \vec{A}_1 S_1(123) + \vec{A}_2 S_2(123) + \vec{A}_3 S_3(123) \\ & + \vec{M}_1^1 M_2^1(123) - \vec{M}_2^1 M_1^1(123) \\ & + \vec{M}_1^2 M_2^2(123) - \vec{M}_2^2 M_1^2(123) \\ & + \vec{M}_1^3 M_2^3(123) - \vec{M}_2^3 M_1^3(123) \\ & + \vec{M}_1^4 M_2^4(123) - \vec{M}_2^4 M_1^4(123) \end{aligned}$$

Standard Basis

The notation we now use is the following:

$A_i(123) =$ totally antisymmetric function

$S_i(123) =$ totally symmetric function

$M_1^i(123) =$ a row-1 function

$M_2^i(123) =$ a row-2 function

Since the functions M_1^i and M_2^i form a "pair" ($i=1,2,3,4$), we know how to compute one from the other (see paragraph 5 3) of previous notes):

$$X_2^1 M_1^i = M_2^i \quad \text{and} \quad X_1^2 M_2^i = M_1^i$$

Taking this into account, we can conclude that:

There are 10 functions of definite symmetry in the general form of the vertex: three S_i , three A_i , and four M_1^i (or M_2^i).

In fact, using the results of paragraph 8 (previous notes) we can rewrite the general vertex in a form which shows only 10 functions. Eg, we know that:

$$\begin{aligned} [\vec{M}_1^1 M_2^1 - \vec{M}_2^1 M_1^1] &= (112 + 221) [2 M_2^1] + \text{cyclic} \\ [\vec{M}_2^2 M_1^2 - \vec{M}_1^2 M_2^2] &= (112 - 221) [2 M_1^2] + \text{cyclic} \\ [\vec{M}_1^3 M_2^3 - \vec{M}_2^3 M_1^3] &= (\delta^{12} 3^3) [2 M_2^3] + \text{cyclic} \\ [\vec{M}_2^4 M_1^4 - \vec{M}_1^4 M_2^4] &= [\delta^{12} (1-2)^3] [2 M_1^4] + \text{cyclic} \end{aligned}$$

5. How to "fit" an arbitrary vertex term into the Standard Basis. If someone hands you a vertex term out of the blue, you can dissect it into its 10 irreducible functions (of the standard basis) in the following way: Any vertex term either is, or can be quickly reduced to, a linear combination of raw tensor forms like 112 with coefficients which are functions. The trick is to simply substitute for each raw tensor form using Table 2 (page 2b) [and a similar, less complicated table for the class-two tensor forms which I have not bothered to write down], then regroup and identify the coefficients of the standard tensor forms like $\vec{S}_1$. That is all there is to it.

Example: Consider the tensor form 123. How does this "fit" into the standard irreducible bases? First reduce it to raw tensor forms: $123 = -121 - 122$. Second, negate then add rows 3 and 4 of Table 2 to find: $123 = -(1/3) \vec{S}_1$. Done! To check the result note that:

$$123 = (1/6) (123)_S = (1/6) [(-2) \vec{S}_1] \quad \text{according to Table 1, QED.}$$

Thus, if someone asks you to dissect the vertex term $A(123) 123$, your dissection is this:

$$\begin{cases} A_1(123) = -\frac{1}{3} A(123) \\ A_2(123) = 0 \\ \text{etc, all others} = 0 \end{cases}$$

6. The Six-Function form for the Vertex. If one insists on having functions of definite S_3 symmetry, the general vertex term has 10 components as noted above. However, if one is willing to have functions were are merely symmetric or antisymmetric in a single pair of the three variables, the number of arbitrary functions in the vertex can be reduced to six.

definition: $S(12)$ is a function $f(123)$ which is symmetric in arguments 1 and 2 (and has unknown symmetry with respect to other pairs of arguments) Similarly, $A(12)$ is antisymmetric in 1 and 2.

As we shall verify below, the general vertex term can be written in terms of six of these $?(12)$ type functions as follows:

$$\begin{aligned} \Gamma(123) = & (211+212) A_1(12) + \text{cyclic} \\ & + (211-212) S_1(12) + \text{cyclic} \\ & + (231+312) A_2(12) + \text{cyclic} \\ & + (231-312) S_2(12) + \text{cyclic} \\ & + (\delta^{12} 3^3) A_3(12) + \text{cyclic} \\ & + [\delta^{12} (1-2)^3] S_3(12) + \text{cyclic} \end{aligned}$$

Six-Function Basis

The claim now is that any vertex term can be dissected into the six functions $A_1, A_2, A_3, S_1, S_2, S_3$ shown in this box. If we can express these six functions in terms of the 10 irreducible functions of the standard basis, then we know that the most general form of the vertex can be fitted into these six functions (since we know it can be fitted into the 10 functions).

This is actually quite simple to do. First we fractionate our six functions into their irreducible symmetry fractions using the projection X operators. One finds that:

$$A_i(12) = A_i(12)_a \hat{a} + A_i(12)_2 \hat{2}$$

$$S_i(12) = S_i(12)_s \hat{s} + S_i(12)_r \hat{r}$$

where, eg, $S(12)_s = (1/3) [S(12) + S(23) + S(31)]$. Each of these functions of "partial symmetry" contains two functions of definite symmetry. For example, the sum of any A-function and any 2-function gives you a function which is 12 antisymmetric. Put in other words, the space of 12-antisymmetric functions partitions into the sum of two spaces: the A-functions and the row-2 functions.

Having thus analyzed the $A(12)$ and $S(12)$ functions into their symmetry components, we may now consider each of the six terms in turn and dissect them into their irreducible components.

Example: Consider the first term which is $(211+212)A_1(12) + \text{cyclic}$. Being a valid vertex term, this thing is totally antisymmetric. Since only the A-piece will survive the cyclicization, let us compute this object:

$$[(211+212) A_1(12)]_a \quad / \text{ ie, the A-component of this product.}$$

In our previous notes we showed how to analyze a product of two objects into its symmetry fractions. Here we are only interested in the A-fraction, so we ~~read~~ first analyze each of our two factors into fractions and then use the bottom line of our Blebsch equation on page 6 of previous notes:

$$(211+212) = \left(\frac{1}{3} \vec{S}_2\right) \hat{s} + \left(-\frac{1}{3} \vec{M}_1\right) \hat{1} \quad / \text{ Table 2}$$

$$A_1(12) = A_1(12)_a \hat{a} + A_1(12)_2 \hat{2} \quad / \text{ above.}$$

$$\begin{aligned}
 \text{Thus, } & [(211+212)A_1(12)]_a \\
 &= \hat{a} \left\{ \left(\frac{1}{3}\vec{S}_2\right)A_1(12)_a + \frac{1}{2} \left(\left(-\frac{1}{3}\vec{M}_1'\right)A_1(12)_2 - \text{partners} \right) \right\} \\
 &= \left(\frac{1}{3}A_1(12)_a\right)\vec{S}_2 + \left[\left(-\frac{1}{6}A_1(12)_2\right)\vec{M}_1' - \text{partners} \right]
 \end{aligned}$$

Comparison to the Standard Basis on page 3 shows that this first term makes the following contributions to the 10 basis functions:

$$\begin{cases}
 A_2(123) = A_1(12)_a = \frac{1}{3}[A_1(12) + \text{cyclic}] \\
 M_2'(123) = -\frac{1}{2}A_1(12)_2 = \left(-\frac{1}{6}\right)[2A_1(12) - A_1(23) - A_1(31)] \\
 \text{else} = 0
 \end{cases}$$

Here we have illustrated how one performs a non-trivial dissection of a vertex term into its irreducible components. The other 5 terms of our candidate general tensor form on page 5 can be similarly analyzed. Here is how the whole thing "fits" into the irreducible standard basis:

$$\begin{aligned}
 S_1(123) &= S_1(12)_s &= S_1(23)_s \\
 S_2(123) &= 3S_2(12)_s - 3S_1(12)_s &= 3S_2(23)_s - 3S_1(23)_s \\
 S_3(123) &= S_3(12)_s &= S_3(23)_s \\
 A_1(123) &= A_2(12)_a &= A_2(23)_a \\
 A_2(123) &= A_1(12)_a - 3A_2(12)_a &= A_1(23)_a - 3A_2(23)_a \\
 A_3(123) &= 6A_3(12)_a &= 6A_3(23)_a \\
 M_2'(123) &= -\frac{1}{2}A_1(12)_2 &= A_1(23)_2 \\
 M_2^2(123) &= \frac{1}{\sqrt{3}}S_1(23)_2 &= \frac{1}{\sqrt{3}}S_1(23)_2 \\
 M_2^3(123) &= \frac{3}{2}A_3(12)_2 &= -3A_3(23)_2 \\
 M_1^4(123) &= -\frac{1}{2}S_3(12)_1 &= S_3(23)_1
 \end{aligned}$$

Inversion of these equations yields

$$S_1(23)_S = S_1(123)$$

$$S_1(23)_2 = \sqrt{3} M_2^2(123)$$

$$S_2(23)_S = S_1(123) + \frac{1}{3} S_2(123)$$

$$S_3(23)_S = S_3(123)$$

$$S_3(23)_1 = M_1^4(123)$$

$$A_1(23)_a = A_1(123) + \frac{1}{3} A_2(123)$$

$$A_1(23)_2 = M_2^1(123)$$

$$A_2(23)_a = A_1(123)$$

$$A_3(23)_a = \frac{1}{6} A_3(123)$$

$$A_3(23)_2 = -\frac{1}{3} M_2^3(123)$$

$$\Rightarrow \begin{cases} S_1(12) = S_1(123) \\ S_2(23) = S_1(123) + \frac{1}{3} S_2(123) \\ S_3(23) = S_3(123) - 2 M_1^4(123) \\ A_1(12) = A_1(123) + \frac{1}{3} A_2(123) - 2 M_2^1(123) \\ A_2(12) = A_1(123) \\ A_3(12) = \frac{1}{6} A_3(123) - \frac{2}{3} M_2^3(123) \end{cases}$$

Summary: We know that any vertex term can be fitted into the Standard Basis back on page 3. We can dissect any candidate vertex term out of the blue into its 10 irreducible basis functions. But then the above formulas tell us how to then put that tensor term into the 6-Function Basis; after all, right there you have explicit formulas for construction of the 6 Basis functions given the 10 Standard basis functions.

7. The F-tensor Basis. Starting from the Six-Function Basis given on page 5, we can gradually make one change at a time, always without losing the generality of the form of the vertex, and we then end up with the following very useful version of the general vertex term:

define : $F_{12}^{12} \equiv [(1.2) \delta^{12} - 2^1 1^2] = \text{"the F-tensor"}$

define : $B_1^i = (1.3) 2^i - (1.2) 3^i$ and cyclically

The F-tensor is a useful replacement for the tensor $2^1 1^2$ because F_{12}^{12} is orthogonal to 1^1 and 2^2 . Also, B_1^1 is orthogonal to 1^1 and cyclically.

Our new general form of the vertex is this:

$$\begin{aligned}\Gamma^{123}(123) = & F_{12}^{12} B_3^3 S_1(12) + \text{cyclic} \\ & + F_{12}^{12} (1-2)^3 S_2(12) + \text{cyclic} \\ & + 1^1 2^2 3^3 A(123) \\ & + B_1^1 B_2^2 B_3^3 S(123) \\ & + \delta^{12} 3^3 A_1(12) + \text{cyclic} \\ & + \delta^{12} (1-2)^3 S_3(12) + \text{cyclic}.\end{aligned}$$

F-Tensor Basis

Any vertex term can be fitted into the six functions S_1, S_2, A, S, A_1, S_3 . This basis is just as complete as any of our earlier bases.

8. Ward Identity and The Bottom Line. The usefulness of the F-tensor basis is very clear: the tensor forms $F_{12}^{12} B_3^3$ and its cyclics, as well as the BBB form are Ward null. Thus the functions S_1 and S are completely unconstrained by the Ward identity.

When the Ward identity is imposed on the remaining four functions here is what you get: You are forced to Marshal Bakers "solution" exactly! [Of course we are assuming that $c=0$ on the RHS of the Ward.] Thus, there are no "hidden" Ward null terms. The only Ward null vertex terms are linear combinations of the two obvious ones appearing in the above expression for the vertex.

In more detail here is how the Ward thing works: you expand both sides of the Ward identity into the 5 tensor forms $d^{12}, a_1, 12, 21, 22$. Thus you have five equations that must be satisfied. However, when these equations are symmetry fractionated you find that there are really 10 equations. There are also 10 unknowns which can be taken to be:

$S_2(23)_s$	$S_2(23)_1$	$S_2(23)_2$
$S_3(23)_s$	$S_3(23)_1$	$S_3(23)_2$
$A_1(23)_a$	$A_1(23)_1$	$A_1(23)_2$
$A(123)$		

If the 10 equations in the 10 unknowns had not been linearly independent, there would have been other Ward null terms! It turns out the 10 equations are linearly dependent

although I see no intuitive way to know this .

Just for fun, here is the solution expressed in the functions above:

$$A_1(12) = \frac{1}{2} [b(2) - b(1)]$$

$$A(123) = 0$$

$$S_2(12) = \frac{b(2) - b(1)}{(1 \cdot 1) - (2 \cdot 2)}$$

$$S_3(12) = \frac{1}{2} [b(2) + b(1)].$$

When these functions are plugged back in you get -- as was just noted -- exactly Marshall's solution. If I hadn't known his solution, I would have been led to it.

Misc

Contents of Miscellany Section.

Comment: I have tried to store here sheets that I thought would be "lost" if they were stored in their original historical location.

1. The RHS of the Ward Identity in terms of $c(1)$ and $b(1)$.
2. Symmetry analysis of the ten Ward equations.
3. The Operator Inversion Problem (cyclics)
4. "Why work with Irreducible Components"
5. Count independent functions in $A(12)$.
6. The Group C_3 .
7. The counting problem.
8. The Operator Solution to the non-n Vertex Problem !!!
9. Write Marshall's null term as linear combination of ours.
10. How many vectors are perp to p_1 ?(Jim).
11. Restatement of Jim Vertex General Form
12. REDUCE progrzm to check vertex Ward validity.

Inclusion of the n-terms, general "solution" to the Ward Identity.

1. First let's review what has already been done. In the simple ansatz you set $c=0$ on the RHS of the Ward. When c is included along with b , the RHS of the Ward looks like this:

$$\text{extra terms} = -Q_1' Q_1^2 c(1) + Q_2' Q_2^2 c(2)$$

$$Q_1' = 1' - n' \frac{(1 \cdot 1)}{(1 \cdot n)}$$

$$Q_1' 1' = 0$$

$$Q_1^2 = 1^2 - n^2 \frac{(1 \cdot 1)}{(1 \cdot n)}$$

$$Q_1^2 1^2 = 0$$

$$Q_2' = 2' - n' \frac{(2 \cdot 2)}{(2 \cdot n)}$$

$$Q_2' 2' = 0$$

$$Q_2^2 = 2^2 - n^2 \frac{(2 \cdot 2)}{(2 \cdot n)}$$

$$Q_2^2 2^2 = 0$$

$$\Rightarrow -Q_1' Q_1^2 c(1) = - \left[1' - n' \frac{(1 \cdot 1)}{(1 \cdot n)} \right] \left[1^2 - n^2 \frac{(1 \cdot 1)}{(1 \cdot n)} \right] c(1)$$

$$= 1' 1^2 \{ -c(1) \} + 1' n^2 \left\{ c(1) \frac{(1 \cdot 1)}{(1 \cdot n)} \right\} - n' n^2 \left\{ c(1) \frac{(1 \cdot 1)^2}{(1 \cdot n)^2} \right\} - n' 1^2 \left\{ c(1) \frac{(1 \cdot 1)}{(1 \cdot n)} \right\}$$

So RHS = this $\uparrow$ - (1 $\leftrightarrow$ 2) $\Rightarrow$

$$\text{RHS} = 1' 1^2 \{ -c(1) \} \checkmark$$

$$+ 2' 2^2 \{ +c(2) \} \checkmark$$

$$+ \left[n' 1^2 + 1' n^2 \right] \left\{ \frac{c(1)(1 \cdot 1)}{(1 \cdot n)} \right\} \checkmark$$

$$+ \left[n' 2^2 + 2' n^2 \right] \left\{ \frac{-c(2)(2 \cdot 2)}{(2 \cdot n)} \right\} \checkmark$$

$$+ n' n^2 \left\{ \frac{-c(1)(1 \cdot 1)^2}{(1 \cdot n)^2} + \frac{c(2)(2 \cdot 2)^2}{(2 \cdot n)^2} \right\} \checkmark$$

2. What comments can we make at the start here?

a) first, we are going to have to add new tensor forms to the vertex to generate the new forms we are observing on the right. These forms will look like:

$$\begin{aligned} \text{single } n: & \quad \sqrt{g^{ij}n^k}, \quad p^i p^j n^k \\ \text{double } n: & \quad \sqrt{p^i n^j n^k} \\ \text{triple } n: & \quad \sqrt{n^i n^j n^k} \end{aligned}$$

b) there will be "coupling". Ie, when a single-n term is dotted into 3^3 , if the 3^3 dots into an n^3 you get $(n.3)$ times non-n tensors! But if you include the single-n terms on the left in your vertex, they will also make single-n terms on the right.

Thus, you have to include single-n terms on the RHS. But these RHS single-n terms can also be generated from double-n terms on the left so include them. But they in turn can generate double-n on the right so keep them. But these in turn can be made from triple n on the left so keep them.

So, the problem is completely coupled! We have to solve the whole shebang at once.

3. So here are the steps to solution:

a) make up a complete set of tensor forms for the n terms on the left. (Try to make this compatible with the F-tensor basis!)

b) construct a new Ward table and solve!!!

Analysis of the Ward RHS with n terms included.

1. Here is a list of the raw tensor forms you can get. Recall that anything on the RHS must be overall (12) antisym since the vertex is. Thus, if you classify your forms into sym and anti sym, you know what their partner functions look like:

$$\begin{aligned} \text{RHS}_{\text{Ward}} = & (11+22) \cdot A(12) \\ & (11-22) \cdot S(12) + 12 \cdot A(12) + 21 \cdot A(12) + 3 \cdot A(12) \\ & + (n1+2n) \cdot A(12) + (n2+1n) \cdot A(12) + nn \cdot A(12) \\ & + (n1-2n) \cdot S(12) + (n2-1n) \cdot S(12) \end{aligned}$$

Ie, this is what the general form of our mess will be when we put in all our terms and do 3^3 . First, you see the old non-n terms and the second group is the new terms. Since each A or S function has two symmetry components, I seem to get 20 equations.

2. This seems strange. How does this count work again? Each A(12) like column in our Ward table will have an a-equation and a 2-equation. Etc. There simply cannot be more than 20 equations! ~~OH, I think I see it. Look back at the vertex general form.~~

~~These two functions are the same: $A_5(12)_a$ and $A_6(12)_a$. Ie, these terms collapse onto the same antisymmetric tensor. That lowers us from 22 to 21.~~

3. Maybe this is a hint that I substituted for the wrong term! If I kept $(n22+1n1)$ I could get rid of another function and get us down to 20 equations in 20 functions! I guess I have to go face that problem now. "BUM STEER"

4. After that is done, here is the work that remains: first, I don't have to re-analyze the non-n terms into symmetry classes, that is already done. Ie, I will make full use of my equations 1. through 20 that I got there; its just that I will have to add some extra terms to these equations. However, I will have to symmetry fractionate (and first compute) the various n-terms. All straightforward if tedious.

The Operator Inversion Problem

① Let

$$M = \begin{bmatrix} a & d & b \\ b & a & d \\ d & b & a \end{bmatrix} \checkmark$$

$$\begin{aligned} \text{Now: } \det M &= a[a^2 - bd] - b[da - b^2] + d[d^2 - ab] \\ &= a^3 + b^3 + d^3 - 3adb \end{aligned}$$

Next:

$$\text{cof } M = \begin{bmatrix} a^2 - bd & d^2 - ba & b^2 - ad \\ b^2 - ad & \vdots & \vdots \\ d^2 - ab & \vdots & \vdots \end{bmatrix}$$

$$\text{Then: } M^{-1} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = (\det M)^{-1} \cdot \widetilde{\text{cof } M} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = (\det M)^{-1} \cdot \begin{bmatrix} a^2 - bd & b^2 - ad & d^2 - ab \\ d^2 - ba & \vdots & \vdots \\ b^2 - ad & \vdots & \vdots \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow$

$$\begin{aligned} \alpha &= (a^2 - bd) / \det \\ \beta &= (d^2 - ba) / \det \\ \gamma &= (b^2 - ad) / \det \end{aligned}$$

$$\det = a^3 + b^3 + d^3 - 3adb$$

$$[\alpha e + \beta c^+ + \gamma c^-][ae + bc^+ + dc^-] = e$$

Tested $\checkmark$

$$[2\bar{c} - c^+]^{-1} = \frac{1}{7}(2e + 4c^+ + \bar{c}) \checkmark$$

$$[c^+ - c^-]^{-1} = [c^+ - e]^{-1} = [c - e]^{-1} = \text{no existe} \quad (1-30)$$

Why work with Irreducible Components?

I am having trouble putting this into words but I think there is a reason for this that helps you out. Why bother-- you say -- decompose the 5 Ward identity equations (non-n sector) into their irreducible pieces or components? Why not just use them and solve them as they are?

First off, in the form "as they are" you get not just $S(12)$ but also $S(23)$ and $S(31)$, and similarly for other functions. Since these three functions are linearly independent (though functionally they are all determined by one of them), you cannot do any eliminations. So you are stuck with 5 equations in 10 variables where now I include all the cyclic permutations as variables. Obviously this won't do.

But, you say, you can then include all the cyclic equations to get 15 equations in 10 variables. Now the problem is the other way: too many equations. Well, no doubt you would find that only 10 of those 15 equations are linearly independent. (However, by inspection I am unable to tell which ones these are!)

Now I would insert an old question: suppose in fact only 10 are linearly independent and you solve the thing on Reduce, say. How do you know that the solution you get, say for $S(12), S(23), S(31)$ treated as independent variables -- how do you know that $S(12)$ and $S(23)$ of the solution will in fact be cyclically related? Go off to appendix and work on this question.

I now have an answer to that question: imagine that you solved the problem by the operator method and you thereby got $S(12) = \dots$. Then you would generate your $S(23)$ by applying (123) to $S(12)$, so of course by definition $S(23)$ would be cyclic on $S(12)$! See operator notes somewhere.

Suddenly I am wondering if that cyclic-operator method isn't a very efficient method after all! For the full case I will get 10 operator equations in 8 variables. Isn't that simpler to handle than 20 equations in 22 variables !???

Review Once Again this Count of How Many Independent Functions there are in $A(12)$.

1. Let $(TF)_s$ be some tensor form, symmetric projection with the 6 as usual.

For example, TF could be 112 or d3 or 12n or whatever. Then here are the the two antisymmetric vertex terms you can make from $(TF)_s$ and the pair of tensor forms $(TF)_1^1$ and $(TF)_2^1$:

$$\#1) \quad A(123) \cdot (TF)_s$$

$$\#2) \quad F_2(123) \cdot (TF)_1^1 - \tilde{F}_2(123) (TF)_2^1$$

where twiddle means partner. If you write that function as F_1 , you must remember that F_1 is the partner to F_2 .

2. Now we can simply rewrite these two tensors, or vertex terms, as follows:

$$\#1) \quad [TF + (12)TF] A(123) + \text{cyclic}$$

$$\#2) \quad [TF + (12)TF] 2F_2(123) + \text{cyclic}$$

3. Now finally we can rewrite both these terms as one term as follows:

$$\#1) + \#2) = [TF + (12)TF] A(12) + \text{cyclic}$$

$$\text{where } A(12) = A(123) + 2F_2(123).$$

4. Now let us symmetry analyze the function $A(12)$ constructed in this way. We have:

$$A(12) = A(123) \hat{a} + 2F_2(123) \hat{2}$$

$$\Rightarrow A(23)_a = A(123)$$

$$A(23)_2 = -F_2(123)$$

$$A(23)_1 = -\sqrt{3} A(23)_2 = +\sqrt{3} \tilde{F}_2(123)$$

The function $A(12)$ is completely general. Such a function only has two functionally independent components, not three!!! I.e, $A(23)_1$ and $A(23)_2$ are always "partnerlike" ie partners apart from constants. The three functions are, however, all linearly independent.

The group C_3 and its relevance to things.

1. I have been playing with equations and operators E, C^+, C^- . These three operators, the identity, cyclic, reverse cyclic, form an abelian group. The group is called C_3 and its representations are shown on page 324 of Tinkham. There are three one-dimensional representations. Thus, I expect there to be a projection operator for each of these representations. First, here is the representations (same as character of course):

	E	C^+	$\bar{C} = (C^+)^2$	
$\rho^{(1)}$	1	1	1	
$\rho^{(2)}$	1	ω	ω^2	$i\pi(\frac{2}{3})$
$\rho^{(3)}$	1	ω^2	ω	$\omega = e$

$$\omega^* = \omega^2$$

$$X^{(1)} = \frac{1}{3} [E + C^+ + \bar{C}]$$

$$X^{(2)} = \frac{1}{3} [E + \bar{\omega} C^+ + \omega \bar{C}]$$

$$X^{(3)} = \frac{1}{3} [E + \omega C^+ + \bar{\omega} \bar{C}]$$

$$X^{(2)} + X^{(3)} = \frac{1}{3} [2E - \bar{C} - C^+] \equiv X^+$$

$$X^{(2)} - X^{(3)} = \frac{i}{\sqrt{3}} [\bar{C} - C^+] \equiv X^-$$

	X^1	X^2	X^3
X^1	X^1	0	0
X^2	0	X^2	0
X^3	0	0	X^3

	X^1	X^+	X^-
X^1	X^1	0	0
X^+	0	X^+	X^-
X^-	0	X^-	X^+

2. Now consider the relation between the groups S_3 and C_3 . First, recall how the octet of $SU(3)$ decomposes into several reps of $SU(2)$, the subgroup. Here we expect the reps of S_3 to decompose into several reps of C_3 .

For example, let's work with functions. We can say:

$$X_S f(123) = \frac{1}{6} \underbrace{[f(123) + \text{cyclic}]}_{\in \rho^{(1)}} + \frac{1}{6} \underbrace{[f(213) + \text{cyclic}]}_{\in \rho^{(1)}} = \frac{3X^{(1)}}{6} f(123) + \frac{3X^{(1)}}{6} f(213)$$

$$X_A f(123) = \frac{1}{6} \underbrace{[f(123) + \text{cyclic}]}_{\in \rho^{(1)}} - \frac{1}{6} \underbrace{[f(213) + \text{cyclic}]}_{\in \rho^{(1)}} = \frac{3X^{(1)}}{6} f(123) - \frac{3X^{(1)}}{6} f(213)$$

$$X_1' f(123) = \frac{1}{6} \underbrace{[2f(123) - f(231) - f(312)]}_{\in \rho^{(1)}} + \frac{1}{6} \underbrace{[2f(213) - f(321) - f(132)]}_{\in \rho^{(1)}}$$

$$= \frac{3X^{+}}{6} f(123) + \frac{3X^{+}}{6} f(213)$$

$$X_2^2 f(123) = \frac{3X^{+}}{6} f(123) - \frac{3X^{+}}{6} f(213).$$

Repeat:

$$X_S f(123) = \frac{1}{2} X^{(1)} [f(123) + f(213)]$$

$$X_A f(123) = \frac{1}{2} X^{(1)} [f(123) - f(213)]$$

$$X_1' f(123) = \frac{1}{2} X^{+} [f(123) + f(213)]$$

$$X_2^2 f(123) = \frac{1}{2} X^{+} [f(123) - f(213)]$$

$$X_2' f(123) = -\frac{i}{2} X^{-} [f(123) + f(213)]$$

$$X_1^2 f(123) = +\frac{i}{2} X^{-} [f(123) - f(213)].$$

Attempt to Count the number of equations really present in (21) equation.

$$X^+ [(n_3)(C^+ + \bar{C})] = \frac{1}{3} (2E - C^+ - \bar{C}) n_3 (C^+ + \bar{C})$$

$$= \frac{1}{3} [2n_3(C^+ + \bar{C}) - n_3^+ - n_3^- C^- - n_3^+ C^+ - n_3^-]$$

$$= \frac{1}{3} [(-n_1 - n_2)E + (2n_3 - n_1)C^+ + (2n_3 - n_2)C^-]$$

$$X^1 [(n_3)(C^+ + \bar{C})] = \frac{1}{3} (E + C^+ + \bar{C}) n_3 (C^+ + \bar{C})$$

$$= \frac{1}{3} [(n_1 + n_2)E + (n_3 + n_1)C^+ + (n_3 + n_2)C^-]$$

$$= -\frac{1}{3} [n_3E + n_2C^+ + n_1C^-]$$

$$X^- [(n_3)(C^+ + \bar{C})] = \frac{1}{\sqrt{3}} (C^- - C^+) n_3 (C^+ + \bar{C})$$

$$= \frac{1}{\sqrt{3}} \{ n_3 C^- + n_3 - n_3^+ - n_3^+ C^+ \} = \frac{i}{\sqrt{3}} [(n_2 - n_1)E + (-n_1)C^+ + (n_2)C^-]$$

So here are three projections of the original 21) equation

$$X^+: \left[\frac{2\sqrt{3}}{i} X^- \right] T + \frac{1}{3} [n_3E + (2n_3 - n_1)C^+ + (2n_3 - n_2)C^-] K = 0$$

$$X^-: \left[\frac{2\sqrt{3}}{i} X^+ \right] T + \frac{i}{\sqrt{3}} [(n_2 - n_1)E - n_1C^+ + n_2C^-] K = 0$$

$$X^1: \emptyset - \frac{1}{3} [n_3E + n_2C^+ + n_1C^-] K = 0.$$

Well here are the three C_3 projections of this equation. They seem to me to be linearly independent still, so no conclusion:

Theorem: to count the equations you really have to S_3 analyze the ~~RN~~ Ward equations; this is the fine-tooth comb. S_3 tells me: less or equal to 20 equations on 22 variables. C_3 is not as good: all I can glean from C_3 is 30 equations or less on 22 variables. BUT, I think the C_3 method is better to compute with!

The Counting Problem

1. I have lost faith in my ability to count ward-null degrees of freedom. Here is why:

Example: Suppose you have two equations in three unknowns, the unknowns or variables being $f_1(123)$, $f_2(123)$ and $f_3(123)$. Normally, this means there is one degree of freedom. So perhaps these are the equations: $f_1=0$, $f_2=0$. So f_3 remains unspecified, and represents your degree of freedom.

However, suppose in addition to the two equations you also have an "operator constraint" which says that $f_3(123) = C^+ f_2(123)$. Then when you combine the above two equations with this constraint, you end up with no degrees of freedom: $f_1=f_2=f_3=0$ is the only system solution. So in effect, the operator constraint does play a role here. However, you cannot identify it -- in any way I can tell -- with one or a set of linear equations. In this context it behaves as one equation, so you get three in three homo meaning only trivial solution if $\det \neq 0$; but it is not really "one" equation.

2. This example shows that, when you are counting degrees, you may have to include the operator constraints. So, when I claimed to have 20-or-less equations in 22 unknowns, that does not include the operator constraints. In fact in the J-sector I found what I would call two equations in three unknowns, but the only solution is $J=0$, all pieces; just like the example above. So the operator constraint acted to lower you by one. So without J you now have $20 - 2 = 18$ equations in $19 = 22-3$ variables. So now there is only one Ward null degree seemingly left. But it too may be knocked out by an operator constraint, and you may find no degrees!

3. So we have a general problem that I don't know the solution of: suppose you are given a set of equations in some variables plus some operator constraints. How do you count true degrees of freedom? Should you count each operator constraint as if it were another equation? In that case, in my non-n analysis I had 10 equations in 10 variables but also there are three "pair operator constraints". But if you count all of these you get 13 equations on ten variables which is overdetermined. So given a set of equations and a set of constraints, how do you tell how many are "independent", and what does independent mean?

One thing is certain: the operator constraints can only lower your degrees of freedom. In my 10x10 problem of the non-n sector, it must be that the operator constraints had no further effect than the ten equations. Once the problem is "determined" ie is n-in-n, you can have no further constraints I guess? My 10x10 problem had this solution: everything = 0 for ward null. Thus adding the constraints operatorice could not produce ward nulls where there were none. !

4. Now lets get back to the n-term full problem. One of my raw equations, the nn, says that $A(12)=0$. Applying my operator constraints, I then learn that $A(23)=0$ and $A(31)=0$. So all three components vanish.

But when I took this equation, which was $(3.3)A(12)=0$, and I fractioned it into its symmetry components 2 and 1, this simple fact was hidden! I got:

$$\begin{array}{l|l} -2d_s J_a + 2d_1 J_2 + \frac{2}{3} d_2 J_1 = 0 & J_2 = X_2' J_1 \left(-\frac{1}{\sqrt{3}}\right) \\ 4d_s J_2 - 2d_1 J_a + 2d_1 J_2 - \frac{2}{3} d_2 J_1 = 0 & = -\frac{1}{\sqrt{3}} J_1 \end{array}$$

If you only had the two equations on the left, there is no way you could show that $J_a=J_1=J_2=0$ because to show this you need the operator constraint shown on the right! If you implement the constraint you get:

$$-2d_s J_a + 2d_1 \left(-\frac{1}{\sqrt{3}} X_2'\right) J_1 + \frac{2}{3} d_2 J_1 = 0$$

$$-2d_1 J_a + (4d_s \left(-\frac{1}{\sqrt{3}} X_2'\right) + 2d_1) J_1 - \frac{2}{3} d_2 J_1 = 0$$

$$\Rightarrow (-2d_s) J_a + \left(\frac{2}{3} d_2 - \frac{2}{3} d_1 \hat{X}_2'\right) J_1 = 0 \quad \text{a-like}$$

$$(-2d_1) J_a + \left(-\frac{2}{3} d_2 + (4d_s + 2d_1) \left(-\frac{1}{\sqrt{3}}\right) \hat{X}_2'\right) J_1 = 0 \quad \text{2-like}$$

Now we have two operator equations in two variables. Still not obvious that only the trivial solution exists. Suppose we combine them to get rid of J_a . Then:

$$\left[\frac{2}{3} (d_2 d_1 + d_2 d_s) \hat{e} + \frac{2}{\sqrt{3}} (2d_s^2 + d_1 d_s - d_1^2) \hat{X}_2' \right] J_1 = 0$$

You would then probably find that this operator (element of group algebra with function coefficients) is invertible! Thus you get $J_1=0$, hence $J_2=0$, hence $J_a=0$.

Notice how this simple fact gets completely obscured by takeing the symmetry fractions of this equation!

I take this as a strong hing that it does make a difference which way you do things!

(This is the fast, nm-n complete solution!!)

Examine the WARDNULL non-n problem using operator methods.

1. I already have this table with RHS, so now set RHS=0. You get:

	<u>A(123)</u>	<u>A₁(12)</u>	<u>S₃(12)</u>	<u>S₂(12)</u>	<u>RHS</u>
A)	.	$-c^+$	$2\bar{c}-c^+$	.	$\emptyset$ $b(1)$
B)	$-(y+z)$	$-c^+-c^-$	$\bar{c}-c^+$	.	$\emptyset$ $\emptyset$
C)	.	.	$2\bar{c}-2c^+$	$(y-z)$	$\emptyset$ $\emptyset$
D)	.	$-c^-$	$\bar{c}-2c^+$	.	$\emptyset$ $-b(2)$
E)	.	$-(y+z)$	$(z-y)$	$x(z-y)$	$\emptyset$ $(x+z)b(1) - (x+y)b(2)$

Take it in careful steps:

1) regard B) as telling you A(123); so get rid of B) from dynamic equations.

2) replace E) by E)' = E) + xG). Then

3) regard C) as telling you S₂.

4) new table then has A,D,E' and variables A₁(12) and S₃(12):

	<u>A₁(12)</u>	<u>S₃(12)</u>	<u>RHS</u>
A)	$-c^+$	$2\bar{c}-c^+$	$\emptyset$ $b(1)$
D)	$-c^-$	$\bar{c}-2c^+$	$\emptyset$ $-b(2)$
E)'	$-(y+z)$	$(z-y) + 2x(\bar{c}-c^+)$	$\emptyset$ $(x+z)b(1) - (x+y)b(2)$

5) Now, replace A) by A)' = (y+z) C⁻ A).

6) Also, replace D) by D)' = (y+z) C⁺ D). Make new table:

			<u>RHS</u>
A)'	$-(y+z)$	$(y+z)[2\bar{c}-E]$	$\emptyset$ $(y+z)b(3)$
D)'	$-(y+z)$	$(y+z)[E-2\bar{c}]$	$\emptyset$ $-(y+z)b(3)$
E)'	$-(y+z)$	$(z-y) + 2x(\bar{c}-c^+)$	$\emptyset$ $(x+z)b(1) - (x+y)b(2)$

- 7) Now, replace $E)'$ by $E)'' = E)' - A)'$.
- 8) Replace $D)'$ by $D)'' = D)' - A)'$.
- 9) Regard $A)'$ as telling you $A_1(12)$, remove from dynamics.
- 10) New chart is $D)''$ and $E)''$ in variable $S_3(12)$. Draw it:

	$S_3(12)$	RHS
$D)''$	$(y+z)(E-2C^-) - (y+z)(2C^+-E)$	$\phi \quad -2(y+z)b(3)$
$E)''$	$(z-y) + 2x(C^- - C^+) - (y+z)(2C^+-E)$	$\phi \quad (x+z)b(1) - (x+y)b(2) - (y+z)b(3)$

ie:

$$D)'' \Rightarrow 2(y+z)[E - C^+ - C^-] S_3(12) = 0$$

$$E)'' \Rightarrow \{(2z)E - 2(x+y+z)C^+ + (2x)C^-\} S_3(12) = 0$$

We have reached the end of the line. Cannot eliminate any more functions since we only have one, namely S_3 . It must satisfy both these equations! If either of these operators is invertible, or if any linear combination of them is invertible, then you get $S_3=0$. In fact, for the first operator $\det = 1-1-1-3 = -4$, so it can be inverted and yields $S_3 = 0$ regardless of the second operator.

2. Conclusion: Thus $S_3=0$. Then $A)'$ tells you that $A_1(12) = 0$. And $C)$ tells you that $S_2(12) = 0$. And $B)$ tells you that $A(123) = 0$. Thus all functions are forced to equal zero; there is no wardnull term possible!!!!

This is my best method to date for showing no ward-null terms!

3. But why stop there! This method is so powerful I bet I can solve the inhomogeneous problem! Since all I did was add and subtract rows, I can go back and reinserit the true RHS and follow them through each step. See pencil above on the right! Now when you invert the $D)''$ equation here is what you get!!!!

$$[E - C^+ - C^-]^{-1} = -\frac{1}{2}(C^+ + C^-) \Rightarrow S_3(12) = \frac{1}{2}[b(1) + b(2)] \checkmark$$

$$\text{Then } A) \Rightarrow A_1(12) = \frac{1}{2}[b(2) - b(1)] \checkmark$$

$$\text{Then } C) \Rightarrow S_2(12) = \frac{b(2) - b(1)}{y - z} = \frac{b(2) - b(1)}{y+x-z-x} = \frac{b(2) - b(1)}{(1.1) - (2.2)} \checkmark$$

$$\text{Then } B) \Rightarrow A(123) = 0. \checkmark$$

3. VERY SIGNIFICANT!!!!!!!!!!!!!!

Using the operator method, I am able to take the raw non-n Ward Table including the RHS and in about one page completely solve it to get the Marshall solution!! No need to fractionate anything! Just totally trivial calculations!

Compare this C_3 group algebra method to the old method. There I had to

- 1) take each term in the Ward table and fractionate it into s,1,2,a
- 2) write 10 equations in 10 unknowns and solve, which took lots of algebra.

In contrast, the C_3 method requires one page of trivial algebra!!!

4. Therefore, is there any question as to which method to use for the full problem!!!

Try to compare Marshall's Null term to mine just putting everything in row.

$$\begin{aligned} \textcircled{1} \quad \Gamma_M &= \left\{ \delta^{12} B_3^3 + \frac{1}{3} [312-231] \right\} + \text{cyclic} \\ &= \left\{ S(123) B_1^1 B_2^2 B_3^3 + S_1(12) F_{12}^{12} B_3^3 \right\} + \text{cyclic} \end{aligned} \quad \left. \begin{array}{l} \text{assume} \\ (1) \end{array} \right\}$$

where

$$\begin{cases} B_3^3 = [(3.2)1^3 - (3.1)2^3] \\ F_{12}^{12} = (1.2)\delta^{12} - 2^1 1^2 \end{cases}$$

Since my BBB term is entirely class one, I will have to generate Marshall's delta or class-2 term with my second term. This should determine the function $S_1(12)$.

Do it:

$$\begin{aligned} & S_1(12) [(1.2)\delta^{12} - 2^1 1^2] B_3^3 + \text{cy.} \\ &= \left([S_1(12)(1.2)] \delta^{12} B_3^3 + \text{cyclic} \right) - \left(S_1(12) 2^1 1^2 B_3^3 + \text{cyclic} \right) \end{aligned}$$

Thus:

$$S_1(12) = (112)^{-1}$$

② Our problem is now Ans:

$$\frac{1}{3} [312-231] + \text{cyclic}$$

$$= \left(S(123) B_1^1 B_2^2 - (1.2)^{-1} 2^1 1^2 \right) B_3^3 + \text{cyclic}$$

To, we want to show that

$$\left(\frac{1}{3} [312-231] \right)_a = \left[\left(S(123) B_1^1 B_2^2 - \frac{2^1 1^2}{(1.2)} \right) B_3^3 \right]_a$$

for some $S(123)$. The left side here we know is A-type function so no need to project it. But finding the A-part of the RHS is more work! The first term is exactly A-like but not the second. Rewrite:

$$\frac{1}{3} [312-231] = S(123) B_1^1 B_2^2 B_3^3 \mp \left[\frac{2^1 1^2 B_3^3}{(1.2)} \right]_a$$

3. So how are we going to get that A-part of the last term? Let's try direct use of the X_A projection operator.

$$X_A \left[\frac{2^1 2^2 B_3^3}{(1.2)} \right] = \frac{1}{6} \left\{ \frac{2^1 2^2 B_3^3}{(1.2)} - \frac{2^1 2^2 B_3^3}{(1.2)} - \frac{2^3 2^2 B_1^1}{(3.2)} - \frac{3^1 3^2 B_2^2}{(1.3)} \right. \\ \left. + \frac{3^1 2^3 B_1^1}{(2.3)} + \frac{1^3 3^1 B_2^2}{(3.1)} \right\}$$

This is of course obvious (now!) because the thing you are projecting is 12 symmetric and is thus a $S(12)$ object which cannot have an A piece. So we are left with the amazing conjecture:

$$\text{There exists an } S \text{ such that } \frac{1}{3} [312 - 231] = S(123) B_1^1 B_2^2 B_3^3$$

This seems unlikely.

4. To check the feasibility here, expand everything down to raw tensor forms:

$$312 - 231 = [-1-2]12 + 2[1+2]1 = -112 - 212 + 211 + 211$$

$$B_1^1 B_2^2 B_3^3 = -(xy)(x+y)111 + \text{etc.}$$

$$\text{So fasteners we need: } -(xy)(x+y) S(123) = 0 \Rightarrow S(123) = 0^+$$

Thus we are led to a contradiction! Lets:

5. Reconsider the $S_1(12)$ solution:

$$S_1(12)(1.2) \delta^{12} B_3^3 + \text{cyclic} = \delta^{12} B_3^3 + \text{cyclic.}$$

What is the most general solution to this pup? Absorb (1.2) into S and ask again:

$$S_1(12)(1.2) \equiv S(12)$$

$$\text{RHS} = \text{like } S(12)$$

$$\text{RHS} = 3 (\delta^{12} B_3^3)_2$$

where

$$(\delta^{12} B_3^3)_a = \frac{1}{3} \delta^{12} B_3^3 + \text{cyclic}.$$

So

$$S(12)(\delta^{12} B_3^3) + \text{cyclic} = 3(\delta^{12} B_3^3)_a$$

Find general solution to this. How???

6. Question: do all components "survive" cyclicization. I.e, consider a general vector with all four components active. What happens when you cyclic it? The A and S pieces we know survive in an obvious way. Consider the other pieces:

$$\begin{aligned} f_1 \uparrow + \text{cyclic} &= f_1 + (123)f_1 + (132)f_1 \\ &= f_1 + \left\{ -\frac{1}{2}f_1 - \frac{1}{2}\sqrt{3}f_1 \right\} + \left\{ -\frac{1}{2}f_1 + \frac{1}{2}\sqrt{3}f_1 \right\} \\ &= 0 \end{aligned}$$

Or.

$$[(e) + (123) + (132)] f_1 = 0$$

why so?

$(e) + (123) + (132) =$	X_A	X_S	X_1^1	X_2^1	X_1^2	X_2^2
	1	1	1	0	0	1
	1	1	$-\frac{1}{2}$	$-\frac{1}{2}\sqrt{3}$	$\frac{1}{2}\sqrt{3}$	$-\frac{1}{2}$
	1	1	$-\frac{1}{2}$	$+\frac{1}{2}\sqrt{3}$	$-\frac{1}{2}\sqrt{3}$	$-\frac{1}{2}$

$$\Rightarrow (e) + (123) + (132) = 3X_A + 3X_S$$

Ans: on any row, $2 \rightarrow 0$.

So it's true: only A and S type functions can survive cyclicization. For some reason I failed to notice that earlier. Thus lets continue to analyze the above:

$$\begin{aligned} \text{LHS} &= \underbrace{\frac{1}{3}[(e) + (123) + (132)]}_{X_A + X_S} \cdot (3S(12)(\delta^{12} B_3^3)) \\ &= 3[S(12)(\delta^{12} B_3^3)]_a + 3[X(12)(\delta^{12} B_3^3)]_s \end{aligned}$$

Thus our conditions on the function $S(12)$ may be written simply as:

$$\begin{cases} (S(12) [\delta^{12} B_3^3])_a = (\delta^{12} B_3^3)_a = A(23)_a \\ (S(12) [\delta^{12} B_3^3])_s = 0 \end{cases}$$

Now let's find the a and s pieces of the object on the left in usual way:

Let $\delta^{12} B_3^3 = A(12)$ for shorthand. Then:

$$S(12) A(12) = [S(23)_s \hat{s} - 2 S(23)_1 \hat{1}] [A(23)_a \hat{a} - 2 A(23)_2 \hat{2}]$$

$$= \hat{a} \left\{ S(23)_s A(23)_a + \frac{1}{2} [S(23)_1 A(23)_2 - S(23)_2 A(23)_1] (-4) \right\}$$

$$+ \hat{s} \{ \phi \}$$

+ discrete

$$= -2 [S(23)_1 A(23)_2 - \left(\frac{1}{\sqrt{3}} S(23)_2\right) \left(-\frac{1}{\sqrt{3}} A(23)_1\right)]$$

$$= -2 [S(23)_1 A(23)_2 + \frac{1}{3} S(23)_2 A(23)_1]$$

$$\Rightarrow S(23)_s A(23)_a - 2 S(23)_1 A(23)_2 - \frac{2}{3} S(23)_2 A(23)_1 = A(23)_a$$

This is the first of the above equations; the second is satisfied for any $S(12)$.

Balancing the various terms in this above equation takes a little extra:

$$A(23)_a = \frac{1}{3} [\delta^{12} B_3^3 + \text{cyclic}] = \text{obviously} \neq 0$$

$$A(23)_1 = \frac{1}{2} [\delta^{23} B_1^1 - \delta^{31} B_2^2] \neq 0 \text{ etc.}$$

Since none of the A projections vanish, we get

$$S(23)_s = 1$$

$$S(23)_1 = 0$$

$$S(23)_2 = 0$$

$\Rightarrow$

$$\boxed{S(12) = 1}$$

Thus our guessed solution (the obvious one) was in fact the only solution!

7. I have found my mistake. Go back to paragraph 2 and rewrite:

$$[312-231] = 3S(123) B_1^1 B_2^2 B_3^3 + \underbrace{[(e) + (123) + (132)]}_{3X_S + 3X_A} (- (1 \cdot 2)^{-1} 2^1 1^2 B_3^3)$$

$$= -3 \left[\frac{2^1 1^2 B_3^3}{(1 \cdot 2)} \right]_S - 3 \left[\frac{2^1 1^2 B_3^3}{(1 \cdot 2)} \right]_A$$

Define: $A(12) = \frac{2^1 1^2 B_3^3}{(1 \cdot 2)}$ /anti!!! that was the goof.

Clearly $A(12)_S = 0$ so we get

$$\frac{1}{3} [312-231] = S(123) B_1^1 B_2^2 B_3^3 - A(12)_A$$

The correction is this: the last term does have an a-piece. In fact that piece is just this:

$$A(12)_A = \frac{1}{3} A(12) + \text{cyclic} = \frac{1}{3} \left[\frac{2^1 1^2 B_3^3}{(1 \cdot 2)} + \frac{3^2 2^1 B_1^1}{(2 \cdot 3)} + \frac{1^3 3^1 B_2^2}{(3 \cdot 1)} \right]$$

We have to show that all 8 tensor coefficients balance on both sides; ie, the same function S has to work for everything. Lets try to further process this thing:

$$A(12)_A = \frac{1}{3} \left[\frac{2^1 1^2}{(1 \cdot 2)} \{ (3 \cdot 2) 1^3 - (3 \cdot 1) 2^3 \} + \frac{3^2 2^1}{(2 \cdot 3)} \{ (1 \cdot 3) 2^1 - (1 \cdot 2) 3^1 \} \right]$$

$$+ \frac{1^3 3^1}{(3 \cdot 1)} \{ (2 \cdot 1) 3^2 - (2 \cdot 3) 1^2 \}]$$

$$= \frac{1}{3} \left[\frac{2^1 1^2}{x} \{ y 1^3 - z 2^3 \} + \frac{3^2 2^1}{y} \{ z 2^1 - x 3^1 \} + \frac{1^3 3^1}{z} \{ x 3^2 - y 1^2 \} \right]$$

$$= \frac{1}{3xyz} [211 y^2 z - 212 \cdot z^2 y + 232 z^2 x - 332 x^2 z + 331 x y^2 - 311 y^2 x]$$

⊕

8. Now, I want to write this as an 8-vector. Plug in for each term using the blue ink table in black binder:

	<u>111</u>	<u>112</u>	<u>121</u>	<u>122</u>	<u>211</u>	<u>212</u>	<u>221</u>	<u>222</u>
211	0	0	0	0	$y^2 z$	0	0	0
212	0	0	0	0	0	$-z^2 y$	0	0
232	0	0	0	0	0	$-z^2 x$	0	$-z^2 x$
332	0	$-x^2 z$	0	$-x^2 z$	0	$-x^2 z$	0	$-x^2 z$
331	$x^2 y$	0	$x^2 y$	0	$x^2 y$	0	$x^2 y$	0
311	$y^2 x$	0	0	0	$y^2 x$	0	0	0

$$Q = \begin{bmatrix} xy(x+y) & -x^2 z & x^2 y & -x^2 z & xy(x+y) & z(-x^2 - xz - zy) & x^2 y & -zx(x+z) \end{bmatrix}$$

$+ y^2 z$

So we now have:

$$xyz [312 - 231] = 3xyz S(123) BBB - Q = \text{LHS}$$

Now LHS =

$$\begin{bmatrix} 0 & -xyz & 0 & 0 & 0 & -xyz & 0 & 0 \\ 0 & 0 & 0 & 0 & xyz & 0 & xyz & 0 \end{bmatrix}$$

$$\Rightarrow \text{LHS} = \begin{bmatrix} 0 & -xyz & 0 & 0 & xyz & -xyz & xyz & 0 \end{bmatrix}$$

$\rightarrow \text{LHS} + Q =$

$$\begin{bmatrix} xy(x+y) & -xz(x+y) & x^2 y & -x^2 z & xy(x+y+z) & & xy(x+z) & -zx(x+z) \end{bmatrix}$$

$$- BBB = \begin{bmatrix} +xy(x+y) & -xz(x+y) & x^2 y & -x^2 z & +y(x+y)(x+z) & & xy(x+z) & -zx(x+z) \end{bmatrix}$$

$- z(x+y)(x+z)$

So we seem to have a solution with $3xyz S(123) = -1$. I have however to check two of the entries which don't quite seem right:

$$\begin{aligned} 211 \text{ of } (-888) \text{ is } y(x+y)(x+z) &= y(x^2+yx+xz+yz) \\ &= yx^2+y^2x+y^2z+xyz \end{aligned} \quad \left. \vphantom{\begin{aligned} 211 \text{ of } (-888) \text{ is } y(x+y)(x+z) \\ &= yx^2+y^2x+y^2z+xyz \end{aligned}} \right\} \text{A-OK.}$$

$$211 \text{ of } (LHS+0) \text{ is: } xyz + = \checkmark$$

$$212: -z(x+y)(x+z) = -z(x^2+yx+xz+yz) = -zx^2 - zx - z^2y - xyz.$$

$$212: -x^2z - xz^2 - z^2y - xyz: \checkmark$$

9. Hurray, I have made it work. The solution functions are:

$$S(123) = -(3xyz)^{-1}$$

$$S_1(12) = (1.2)^{-1} = x^{-1}$$

gives Jim

$$\Gamma_{\text{null}}^{AB} \equiv \{ \delta^{12} B_3^3 + \text{cyclic} \} + \{ 312^3 - 231^3 \}$$

$$= \left\{ - \frac{B_1^3 B_2^3 B_3^3}{(1.2)(2.3)(3.1)} \right\} + \left\{ \frac{F_{12}^{12} B_3^3}{(1.2)} + \text{cyclic} \right\}$$

$$\text{where } B_3^3 \equiv [(3.2)1^3 - (3.1)2^3]$$

$$F_{12}^{12} \equiv [(1.2)\delta^{12} - 2^1 1^2]$$

This shows how Marshall's null term is a linear combination of the obvious null terms.

Consider the Possibility of Making New Ward Null Terms from the vector $E = n \cdot F$
as Jim suggested.

1. Here, then, are three vectors which are perpendicular to 1^1 :

$$B_1' = (1 \cdot 3) 2^1 - (1 \cdot 2) 3^1$$

$$D_1' = (n \cdot 1) 1^1 - (1 \cdot 1) n^1$$

$$E_1' = n^2 F_{12}^{12} = (1 \cdot 2) n^1 - (1 \cdot n) 2^1$$

2. To make an arbitrary vector you do this:

$$\vec{V} = \alpha(\vec{n}) + \beta(\vec{p}_1) + \gamma(\vec{p}_2)$$

For fixed Lorentz index, any vector V^u has three components as shown. V^u is a vector in a three dimensional space. In this space, at most two vectors can be perpendicular to p_1 ?

Start over: these vectors are in a 4-dim space. Let p_1 point in its direction. Then there should really be three vectors perpendicular to p_1 !!! I.e., three independent ones. The above three don't do it because they are related:

$$(1 \cdot 1) E_1' = (1 \cdot n) B_1' - (1 \cdot 2) D_1'$$

3. Consider this simpler problem!: Let n and p be two 3-dimensional vectors. Each exists in 3-space. Imagine p along the z axis. Then in principle there are two vectors in this space perp to p . However, since you only have available n and p , you can never construct a vector lying outside the n - p plane. Thus, you can only make one vector perp to p . In fact, suppose n and p are vectors in N -dimensional space. Then there are in principle $N-1$ vectors perp to p . BUT, you can only construct vectors in the n - p plane, so there is still only one vector perp to p that you can make:

$$(\alpha \vec{p} + \beta \vec{n}) \cdot \vec{p} = 0 \Rightarrow \alpha(p \cdot p) + \beta(n \cdot p) = 0 \Rightarrow \alpha = -\beta \left(\frac{n \cdot p}{p \cdot p} \right)$$

$$\text{Let } \beta = (p \cdot p). \text{ Then } \alpha = -(n \cdot p) \Rightarrow \vec{V}_\perp = [(n \cdot p) \vec{p} - (p \cdot p) \vec{n}]$$

No others even in 26 dim space.

4. Now back to Jim's thing. If we have available n, p_1 and p_2 , then the only vectors we can make lie in a 3-dim subspace of the whole space, assuming N.G.T.3 ($N=4$). In this subspace, there are only two vectors you can make which are perp to p_1 . They may be taken to be B and D .

Someone comes along with another one, it must be a linear combination of these. Case in point: E_1 is linear combination of B_1 and D_1 .

5. However, now let's consider making Ward nulls:

Let $M'_1 = \alpha_1 B'_1 + \beta_1 D'_1 = \text{general vector } \perp \text{ to } l.$

Then consider:

$$\begin{aligned} M_1 M_2 M_3 &= (\alpha_1 B_1 + \beta_1 D_1)(\alpha_2 B_2 + \beta_2 D_2)(\alpha_3 B_3 + \beta_3 D_3) \\ &= \alpha_1 \alpha_2 \alpha_3 (B_1 B_2 B_3) + \beta_1 \beta_2 \beta_3 (D_1 D_2 D_3) \\ &\quad + \alpha_1 \alpha_2 \beta_3 (B_1 B_2 D_3) + \text{cyclic} (1 \leftrightarrow 2, \alpha_2 \alpha_3 \beta_1 (D_1 B_2 B_3) \\ &\quad + \alpha_3 \alpha_1 \beta_2 (B_1 D_2 B_3)) \\ &\quad + \alpha_1 \beta_2 \beta_3 (B_1 D_2 D_3) + \text{cyclic} \end{aligned}$$

But these four terms are each Ward null and have already been counted.

6. In fact, let me consider two arbitrary vectors combined in this way:

$$M'_1 = \alpha_1 B_1 + \beta_1 D_1$$

$$N_1 = \gamma_1 B_1 + \delta_1 D_1$$

Then $\left\{ \begin{array}{l} M_1 M_2 M_3 \rightarrow \text{done above, now new} \\ N_1 N_2 N_3 \rightarrow \text{done above, no need} \\ M_1 M_2 N_3 \pm c \rightarrow \text{consider next} \\ M_1 N_2 N_3 \pm c \rightarrow \text{same as } \uparrow \end{array} \right\}$

So: $M_1 M_2 N_3 + \text{cyclic}$

Really $M_1 M_2 N_3 \circlearrowleft (123) + \text{cyclic} = \text{candidate for new ward null?}$

We don't know sym: need $N_3^3 \rightarrow \pm N_3^3$ under $(1 \leftrightarrow 2)$.

Actually, we need $N_3^3 F(123) \rightarrow -N_3^3 F(123)$ under 123.

Assume the Dirac case. Then

$$\cancel{M_1^1} M_1^1 M_2^2 N_3^3 F(123) + \text{cyclic}$$

$$= (\alpha_1 B_1 + \beta_1 D_1)(\alpha_2 B_2 + \beta_2 D_2)(\delta_3 B_3 + \gamma_3 D_3) F(123) + c.$$

$$= (\alpha_1 B_1)(\alpha_2 B_2)(\delta_3 B_3) F(123) + c = B_1 B_2 B_3 [\alpha_1 \alpha_2 \delta_3 F(123) + c].$$

$$+ (\beta_1 D_1)(\beta_2 D_2)(\gamma_3 D_3) F(123) + c = D_1 D_2 D_3 [\beta_1 \beta_2 \gamma_3 F(123) + c].$$

$$+ B_1 B_2 D_3 [\alpha_1 \alpha_2 \gamma_3 F(123)] + \text{cyclic} \text{ etc.}$$

4. Conclusion: There is no way to make any new Ward null terms in this way. You can make any new vectors you want that are perp to p_1 ; there are only two independent ones which I call B and D. Any Ward nulls you try to make with three vectors reduce to these.

Express Jim's General Vertex in my Standard Form.

1. Standard form refers to a list in "the n's" section somewhere.

First term: $\delta^{12} \left[\frac{1^3 g(1)}{(1 \cdot 3)} - \frac{2^3 g(2)}{(2 \cdot 3)} \right] + c.$

$$= \delta^{12} (1+2)^3 \left\{ \frac{1}{2} \left[\frac{g(1)}{(1 \cdot 3)} - \frac{g(2)}{(2 \cdot 3)} \right] \right\} + c.$$

$$+ \delta^{12} (1-2)^3 \left\{ \frac{1}{2} \left[\frac{g(1)}{(1 \cdot 3)} + \frac{g(2)}{(2 \cdot 3)} \right] \right\} + c.$$

Second Term: $211 \left(\frac{g(1)}{(1 \cdot 1)(1 \cdot 3)} \right) + c$

$$+ 311 \left(\frac{-g(1)}{(1 \cdot 1)(1 \cdot 2)} \right) + c.$$

Cyclic forward once the second term to get

$$211 \left(\frac{g(1)}{(1 \cdot 1)(1 \cdot 3)} \right) + c$$

$$+ 212 \left(\frac{-g(2)}{(2 \cdot 2)(2 \cdot 3)} \right) + c.$$

Now replace:

$$21 = (1 \cdot 2) \delta^{12} - F_{12}^{12}$$

$$1 = \frac{1}{2} (1+2) + \frac{1}{2} (1-2)$$

$$2 = \frac{1}{2} (1+2) - \frac{1}{2} (1-2)$$

to get

$$= F_{12}^{12} (1+2)^3 \left\{ \frac{1}{2} \left[\frac{g(2)}{(2 \cdot 2)(2 \cdot 3)} - \frac{g(1)}{(1 \cdot 1)(1 \cdot 3)} \right] \right\}$$

$$+ F_{12}^{12} (1-2)^3 \left\{ -\frac{1}{2} \left[\frac{g(1)}{(1 \cdot 1)(1 \cdot 3)} + \frac{g(2)}{(2 \cdot 2)(2 \cdot 3)} \right] \right\}$$

$$+ \delta^{12} (1+2)^3 \left\{ \frac{1}{2} (1 \cdot 2) \left[\frac{g(1)}{(1 \cdot 1)(1 \cdot 3)} - \frac{g(2)}{(2 \cdot 2)(2 \cdot 3)} \right] \right\}$$

$$+ \delta^{12} (1-2)^3 \left\{ \frac{1}{2} (1 \cdot 2) \left[\frac{-g(1)}{(1 \cdot 1)(1 \cdot 3)} + \frac{g(2)}{(2 \cdot 2)(2 \cdot 3)} \right] \right\}$$

Third Term:

$$\delta^{12} n^3 \left[\frac{f(2) - f(1)}{(n-3)} \right] + c.$$

/ already in std form.

Fourth Term:

$$nn2 \left[\frac{1}{(n-1)} \left(\frac{f_2}{n_2} + \frac{f_3}{n_3} \right) \right] + c.$$

$$- nn1 \left[\frac{1}{n_2} \left(\frac{f_1}{n_1} + \frac{f_3}{n_3} \right) \right] + c.$$

Let

$$1 = \frac{1}{2}(1+2) + \frac{1}{2}(1-2)$$

$$2 = \frac{1}{2}(1+2) - \frac{1}{2}(1-2)$$

$$\Rightarrow = nn(1+2) \left\{ \frac{1}{2n_1 n_2 n_3} \left[n_3 (f_2 - f_1) + (n_2 - n_1) f_3 \right] \right\} + c.$$

$$+ nn(1-2) \left\{ \frac{1}{2n_1 n_2 n_3} \left[n_3 (f_2 + f_1) + (n_2 + n_1) f_3 \right] \right\} + c.$$

Fifth Term:

$$nnn \cdot \left\{ \frac{f_1}{n_1^2} \left(\frac{(1-2)}{n_3} - \frac{(1-3)}{n_2} \right) \right\} + c.$$

Combine deltas:

$$\delta^{12} (1+2)^3 \left\{ \frac{1}{2} \left[\frac{g(2)}{(2-2)} - \frac{g(1)}{(1-1)} \right] \right\} + c.$$

$$\delta^{12} (1-2)^3 \left\{ \frac{1}{2} \left[\frac{-g(2)}{(2-2)} - \frac{g(1)}{(1-1)} \right] \right\} + c.$$

Now you can show that

$$(1+z)^3 = \frac{2}{(y-z)} B_3^3 - \left(\frac{y+z}{y-z} \right) (1-z)^3$$

So we convert that term to get:

$$F_{12}^{12} B_3^3 \left\{ \left[\frac{g(z)}{(2.2)(2.3)} - \frac{g(1)}{(1.1)(1.3)} \right] \frac{1}{(y-z)} \right\}$$

$$+ F_{12}^{12} (1-z)^3 \left\{ -\frac{1}{2} \left(\frac{y+z}{y-z} \right) \left[\frac{g(z)}{(2.2)(2.3)} - \frac{g(1)}{(1.1)(1.3)} \right] \right\}$$

Combine this with term already there to get:

$$- F_{12}^{12} (1-z)^3 \left\{ +\frac{1}{2} \left[\frac{g(1)}{(1.1)(1.3)} + \frac{g(z)}{(2.2)(2.3)} \right] + \frac{1}{2} \left(\frac{y+z}{y-z} \right) \left[\frac{g(z)}{(2.2)(2.3)} - \frac{g(1)}{(1.1)(1.3)} \right] \right\}$$

Thus we get:

$$F_{12}^{12} B_3^3 \left\{ \frac{1}{(y-z)} \left[\frac{g(z)}{(2.2)(2.3)} - \frac{g(1)}{(1.1)(1.3)} \right] \right\} + c.$$

$$+ F_{12}^{12} (1-z)^3 \left\{ \frac{1}{(y-z)} \left[\frac{g(1)}{(1.1)} - \frac{g(z)}{(2.2)} \right] \right\} + c.$$

Now define:

$$h(1) = \frac{-g(1)}{(1.1)} = \text{red} (b-c).$$

Then:

$$\Gamma_{B2P}^{JIM} = \delta^{12} n^3 \left\{ \frac{f_2 - f_1}{n_3} \right\} + c.$$

$$+ n n (1+2) \left\{ \frac{1}{2 n_1 n_2 n_3} [n_3 (f_2 - f_1) + (n_2 - n_1) f_3] \right\} + c.$$

$$+ n n (1-2) \left\{ \frac{1}{2 n_1 n_2 n_3} [n_3 (f_2 + f_1) + (n_2 + n_1) f_3] \right\} + c.$$

$$+ n n n \left\{ \frac{f_1}{n_1^2} \left[\frac{(1-2)}{n_3} - \frac{(1-3)}{n_2} \right] \right\} + c. \quad \leftarrow \text{re-write!}$$

$$+ \delta^{12} (1+2)^3 \left\{ \frac{1}{2} [h_1 - h_2] \right\} + c.$$

$$+ \delta^{12} (1-2)^3 \left\{ \frac{1}{2} [h_1 + h_2] \right\} + c.$$

$$+ F_{12}^{12} (1-2)^3 \left\{ \frac{1}{(y-z)} [h_2 - h_1] \right\} + c.$$

$$+ F_{12}^{12} B_3^3 \left\{ \frac{1}{(y-z)} \left[\frac{h_1}{(1 \cdot 3)} - \frac{h_2}{(2 \cdot 3)} \right] \right\} + c.$$

So here is an interesting fact: if you set $c=0$ so that $f=0$, then Jim's vertex is exactly the same as Marshall's except for the FB ward null term!

$$h_1 = -g_1 / (1 \cdot 1) = b - e$$

$$y - z = y + x - z - x = (1 \cdot 1) - (2 \cdot 2)$$

$$f_1 = (1 \cdot 1) c(1)$$

$$\frac{f_1}{n_1} = \frac{(1 \cdot 1)}{(n \cdot 1)} c(1)$$

Recast some terms:

$$nn(1+2) \left\{ \frac{1}{2} \left[\left(\frac{f_2 + f_1}{n_1 n_2} \right) + \left(\frac{n_2 n_1}{n_1 n_2} \right) \frac{f_3}{n_3} \right] \right\}$$

$n_1 \rightarrow 0$ when $(1.1) \rightarrow 0$ in Euclidean, $\Rightarrow p_2 = p_3 \Rightarrow n_2 = n_3$.

$$\rightarrow nn(2) \left\{ \frac{1}{2} \left[\frac{f_2}{n_2 n_1} + \frac{n_1}{n_1 n_2} \frac{f_3}{n_3} \right] \right\} = \left\{ \frac{f_2}{n_1 n_2} + \frac{f_3}{n_1 n_3} \right\} = \left(\frac{2f_2}{n_2} \right) \frac{1}{n_1} \checkmark$$

But dimension does this:

$$-nn(2) \left\{ \frac{1}{2} \left[\frac{f_2}{n_1 n_2} + \frac{f_3}{n_1 n_3} \right] \right\} = \text{same with } n_1, \text{ so } = \text{OK.}$$

Moral: writing the nnp terms as two separate terms tends to obscure the fact that they have no singularity as $n_1=0$. Let me write them back in Jim's form again:

$$nn2 \left[\frac{1}{n_1} \left(\frac{f_2}{n_2} + \frac{f_3}{n_3} \right) \right] + \text{cyclic}$$

$$-n3n \left[\frac{1}{n_1} \left(\frac{f_3}{n_3} + \frac{f_2}{n_2} \right) \right] + \text{cyclic.}$$

no good, need p in same position!

Look at "4th Term" of Jim

As $(1.1)=0$ you get $(n.1)=0$. The first term is then singular. Even if $3=2$, no help. The $-nn1$ term in fact has a zero so it cannot help. The terms in "cyclic" are different tensors so they cannot help.

Conclusion: the nnp terms in Jim's vertex have singularities as $(1.1)=0$ which are kinematic; ie, if you assume that b and c do nothing awful, the vertex terms are singular. Clearly, then, you need to add some DDB Ward null to this to get OK.

About Clearing Sings from the nnn term.

$$DDD \cdot A(123) =$$

$$[n_1 1' - (1 \cdot 1) n_1'] [n_2 2' - (2 \cdot 2) n_2'] [n_3 3' - (3 \cdot 3) n_3']$$

$$= n_1 n_2 n_3 (123)$$

$$- n_1 n_2 (3 \cdot 3) (12n) + \text{cyclic}$$

$$(n23) = -(n21 + n22)$$

$$+ n_1 (2 \cdot 2) (3 \cdot 3) (1nn) + \text{cyclic}$$

$$(n2n) \rightarrow (nn3)$$

$$- (1 \cdot 1) (2 \cdot 2) (3 \cdot 3) (nnn)$$

Write in terms of "linked" tensors:

$$DDD \cdot A(123) = nnn \cdot \{ - (1 \cdot 1) (2 \cdot 2) (3 \cdot 3) A(123) \}$$

$$+ 123 \cdot \{ A(123) \}$$

$$+ n n 3 \cdot \{ n_3 (1 \cdot 1) (2 \cdot 2) + \text{cyclic} \} A(123)$$

$$+ (n21 + n22) \{ n_2 n_3 (1 \cdot 1) + \text{cyclic} \} A(123)$$

Last line can be also written as:

$$\frac{1}{2} (n21 + n22) \{ n_2 n_3 (1 \cdot 1) + \text{c} \} A(123) \\ + \frac{1}{2} (n22 + n21) \{ n_2 n_3 (1 \cdot 1) + \text{c} \} A(123)$$

$$= \frac{1}{6} [(n21)_S + (n22)_S] \{ \} A$$

So this term fits itself into 5 terms of my "List-A".

Now: what $A(123)$ cures our problem:

$$\underline{\text{Jim:}} \quad \text{nnn} \left\{ \frac{f_1}{n_1^2} \left(\frac{1.2}{n_3} - \frac{1.3}{n_2} \right) + \frac{f_2}{n_2^2} \left(\frac{2.3}{n_1} - \frac{2.1}{n_3} \right) + \frac{f_3}{n_3^2} \left(\frac{3.1}{n_2} - \frac{3.2}{n_1} \right) \right\}$$

$$\underline{\text{DDP:}} \quad \text{nnn} \left\{ -(1.1)(2.2)(3.3) A(123) \right\}.$$

Examine the problem as $(1.1) \rightarrow 0$. In that case, $2 \rightarrow 3$ so $A(0, 22) = 0!$
You are stuck!

Re-examine Jim

$$\text{Jim (as } 1.1 \rightarrow 0) = \text{nnn} \left\{ \text{finite} + \frac{1}{n_1} \left[\frac{f_2 \cdot (2.3)}{n_2^2} - \frac{f_3 \cdot (3.2)}{n_3^2} \right] \right\} = \text{OK after all!}$$

Perhaps rewrite:

$$\text{nnn} \left\{ \frac{(3.2)}{n_1} \left[\frac{f_2}{n_2^2} - \frac{f_3}{n_3^2} \right] + \text{cyclic} \right\} = \text{Jim}$$

At first I thought this term was singular, but now it appears to not be so: -- as

$(1.1) = 0$ you have $n_1 = (n.1) = 0$, but difference vanishes so OK. This term

also OK on $n_2=0$ since f_2 contains a 2.2 in numerator. So this baby is OK!!!!

Examine DDB $S(12)$ + cyclic.

$$\dim S = -6$$

①

$$[(n \cdot 1)1 - (1 \cdot 1)n'][(n \cdot 2)2^2 - (2 \cdot 2)n^2][(3 \cdot 2)1^3 - (3 \cdot 1)2^3] S(12) + c.$$

I have already analyzed the nmp piece of this and found:

$$nn(1+2) \left\{ \frac{1}{2}(1 \cdot 1)(2 \cdot 2)[y-z] \right\} S(12) + c.$$

$$+ nn(1-2) \left\{ \frac{1}{2}(1 \cdot 1)(2 \cdot 2)[y+z] \right\} S(12) + c.$$

Seems better not to symmetrize? Then you get:

$$nn1 \left\{ (1 \cdot 1)(2 \cdot 2)(3 \cdot 2) S(12) \right\} + c$$

$$- nn2 \left\{ (1 \cdot 1)(2 \cdot 2)(3 \cdot 1) S(12) \right\} + c.$$

So here is the question: what is the most unsingular thing you can let $S(12)$ be such that Jim's sing at $n_1=0$ goes away? You don't want to cause sings into the other terms that DDB will generate, what's why you would like the least harmful thing here. Here is what we need:

$$\left[\frac{1}{n_1} \left(\frac{f_2}{n_2} + \frac{f_3}{n_3} \right) - (1 \cdot 1)(2 \cdot 2)(3 \cdot 1) S(12) \right] = \text{finite as } (1 \cdot 1) \rightarrow 0.$$

$$+ (1 \cdot 1)(2 \cdot 2)(3 \cdot 1) S(12) = = \frac{A}{n_1 n_2} \left(\frac{f_2}{n_2} + \frac{f_1}{n_1} \right) \Rightarrow \left[S(12) = \frac{2(f_1 + f_2)}{n_1 n_2 (1 \cdot 1)(2 \cdot 2)(3 \cdot 1)} \right] \dim = -6 \checkmark$$

$$\Rightarrow \left[\frac{1}{n_1} \left(\frac{f_2}{n_2} + \frac{f_3}{n_3} \right) - \frac{2(f_1 + f_2)}{n_1 n_2} \right] nn2 + c.$$

$$= \left[\frac{1}{n_1} \left(-\frac{f_2}{n_2} + \frac{f_3}{n_3} \right) - \frac{2}{n_2 n_1} \right] nn2 + c. \quad \text{OK as } n_1 \rightarrow 0, n_2 \rightarrow 0, n_3 \rightarrow 0 \checkmark$$

This is really my first experience with this type of thing. I expect it to fail, but lets go over what we have:

If you added some DDB in the amount I show, you will cancel out the singularities in Jims vertex. Of course you also add other terms and they probably will now be singular, but lets forget that just for a moment of mini-glory here: if I choose the DDB-term coefficient $S(12)$ as on the preceding page, I get some nnp terms which I can add to Jims. The resultant non-singular complete nnp term in the vertex is then:

$$\text{Add: DDB} \cdot \left[\frac{2(f_1 + f_2)}{n_1 n_2 (1.1)(2.1)(1.3)} \right] + \text{cyclic}$$

Then the nnp terms are:

$$\begin{aligned} & n n^2 \left\{ \frac{1}{n_1} \left(\frac{f_3}{n_3} - \frac{f_2}{n_2} \right) - \frac{2}{n_2} \frac{f_1}{n_1} \right\} + \text{cyclic} \\ & - n n^1 \left\{ \frac{1}{n_2} \left(\frac{f_3}{n_3} - \frac{f_1}{n_1} \right) - \frac{2}{n_1} \frac{f_2}{n_2} \right\} + \text{cyclic} \end{aligned}$$

2. By the way, I should have checked Jim's nnn term first because if I had had to add some DDD, that would have made a contribution here to the nnp sector. Luckily, however, Jim's nnn term is OK. OR IS IT? It's ok as $n_1=0$ ie as $(1.1)=0$, but not as $n_3=0$. Cyclic it ahead and look again:

$$n n n \left\{ \frac{f_2}{n_2^2} \left[\frac{(2.3)}{n_1} - \frac{(2.1)}{n_2} \right] \right\}$$

As $(1.1) \rightarrow 0$, this $\rightarrow 0$, pole at n_1 !!

So we have to repair this term first,. Maybe if we are lucky the repair to this will also repair the other nnp terms. By the way, only the nnn and nnp terms have problems!

(it was OK after all, don't forget all terms of the cyclic !!

3. Now that we have found a way to "cure" the nnp terms by adding some DDB, let's examine the other pieces of DDB and see what "damage" has been done:

$$\begin{aligned} \text{DDB} \cdot S(12) + c. = & 123 \left\{ \frac{(n_1)(n_2)(3.2)}{(1.1)(2.2)(3.1)} S(12) \right\} + c. \\ & + 122 \left\{ \frac{n_1 n_2 (3.1)}{(1.1)(2.2)(3.1)} S(12) \right\} + c. \\ & - n_{21} \left\{ \frac{(1.1) n_2 (3.2)}{(1.1)(2.2)(3.1)} S(12) \right\} + c. \\ & + n_{22} \left\{ \frac{(1.1) n_2 (3.1)}{(1.1)(2.2)(3.1)} S(12) \right\} + c. \\ & - 1n_1 \left\{ \frac{n_1 (2.2)(3.2)}{(1.1)(2.2)(3.1)} S(12) \right\} + c. \\ & + 1n_2 \left\{ \frac{n_1 (2.2)(3.1)}{(1.1)(2.2)(3.1)} S(12) \right\} + c. \\ \text{These are the ones just considered} \leftarrow & + n_{11} \left\{ \frac{(1.1)(2.2)(3.2)}{(1.1)(2.2)(3.1)} S(12) \right\} + c. \\ & - n_{12} \left\{ \frac{(1.1)(2.2)(3.1)}{(1.1)(2.2)(3.1)} S(12) \right\} + c. \end{aligned}$$

Now set $S(12) = \frac{2(f_1 + f_2)}{n_1 n_2 (1.1)(2.2)(3.1)}$. But, we have to reduce to stand'd form (1st).

First, tho, we can check for trouble:

$$\begin{cases} = 123 \left\{ \frac{(3.2) 2(f_1 + f_2)}{(1.1)(2.2)(3.1)} \right\} + c. \\ - 122 \left\{ \frac{2(f_1 + f_2)}{(1.1)(2.2)} \right\} + c. \end{cases}$$

These certainly look troublesome. Let's

$$\begin{aligned} = & (123) \left\{ \frac{2(3.2)(f_1 + f_2)}{(1.1)(2.2)(3.1)} + \frac{2(1.3)(f_2 + f_3)}{(2.2)(3.3)(1.1)} + \frac{2(2.1)(f_3 + f_1)}{(3.3)(1.1)(2.3)} \right\} \\ & - 122 \left\{ \frac{2(f_1 + f_2)}{(1.1)(2.2)} \right\} - 323 \left\{ \frac{2(f_2 + f_3)}{(2.2)(3.3)} \right\} - 113 \left\{ \frac{2(f_3 + f_1)}{(3.3)(1.1)} \right\} \end{aligned}$$

Now let $(1.1) \rightarrow 0$, keep singulars, set $2=3$.

$$= 122 \left\{ \underbrace{\frac{2(3.3)f_2}{(1.1)(3.3)(3.1)}}_{\text{triple}} + \underbrace{\frac{2(2.1)f_2}{(3.3)(1.1)(2.3)}}_{\text{single}} - \underbrace{\frac{2f_2}{(1.1)(2.2)}}_{\text{double}} \right\}$$

So you pick-up awful new terms.

① Here is Jim's vertex with the FB term dropped, the $n\bar{n}$ written in nicer way, and the $n\bar{n}$ terms not symmetrized:

$$\Gamma = \delta^{12} 1^3 h_1 + c.$$

$$- \delta^{12} 2^3 h_2 + c.$$

$$+ F_{12}^{12} (1-2)^3 \left[\frac{h_2 - h_1}{(1 \cdot 1) - (2 \cdot 2)} \right] + c.$$

"Marshall Vertex", but with $b \rightarrow b - c \equiv h$

$$+ \delta^{12} n^3 \left[\frac{f_2 - f_1}{n_3} \right] + c.$$

$$+ n^1 n^2 n^3 \left\{ \frac{(3 \cdot 2)}{n_1} \left[\frac{f_2}{n_2^2} - \frac{f_3}{n_3^2} \right] \right\} + c.$$

$$+ n^1 n^2 2^3 \left\{ \frac{1}{n_1} \left[\frac{f_2}{n_2} + \frac{f_3}{n_3} \right] \right\} + c.$$

$$- n^1 n^2 1^3 \left\{ \frac{1}{n_2} \left[\frac{f_1}{n_1} + \frac{f_3}{n_3} \right] \right\} + c.$$

where:

$$n_i = (n \cdot i) \text{ etc.}$$

$$h_i = b_i - c_i = -g_i / (1 \cdot i)$$

$$f_i = (1 \cdot i) c_i$$

1) I have checked with Reduce that this satisfies the Ward!!

2) the only kin sing. are in the $n\bar{n}$ terms (eg, when $(1 \cdot 1) \rightarrow 0$ so $n_1 \rightarrow 0$)

$$F_{12}^{12} \equiv (1 \cdot 2) \delta^{12} - 2 1^2$$

To check Dui's work with my changes

```

0100  VECTOR U1,U2,U3,H,P1,P2,P3
0200
0300  H1:=N.P1$
0400  H1:=B1-C1$
0500  F1:=P1.P1+C1$
0600  F12:=R1.P2+U1.U2+P2.U1+P1.U2$
0700  TA1:=U1.U2+P1.U3+H1$
0800  TB1:=-U1.U2+P2.U3+H2$
0900  TC1:=F12*(P1-P2).U3*(H2-H1)/(P1.P1-P2.P2)$
1000  TD1:=U1.U2+N.U3*(F2-F1)/N3$
1100  TE1:=N.U1+N.U2+N.U3+P3.P2/N1*(F2/N2+N2-F3/N3+N3)$
1200  TF1:=N.U1+N.U2+P2.U3/N1*(F2/N2+F3/N3)$
1300  TG1:=-N.U1+N.U2+P1.U3/N2*(F1/N1+F3/N3)$
1400
1500  N2:=N.P2$
1600  H2:=B2-C2$
1700  F2:=P2.P2+C2$
1800  F23:=P2.P3+U2.U3-P3.U2+P2.U3$
1900  TA2:=U2.U3+P2.U1+H2$
2000  TB2:=-U2.U3+P3.U1+H3$
2100  TC2:=F23*(P2-P3).U1*(H3-H2)/(P2.P2-P3.P3)$
2200  TD2:=U2.U3+N.U1*(F3-F2)/N1$
2300  TE2:=N.U2+N.U3+N.U1+P1.P3/N2*(F3/N3+N3-F1/N1+N1)$
2400  TF2:=N.U2+N.U3+P3.U1/N2*(F3/N3+F1/N1)$
2500  TG2:=-N.U2+N.U3+P2.U1/N3*(F2/N2+F1/N1)$
2600
2700  N3:=N.P3$
2800  H3:=B3-C3$
2900  F3:=P3.P3+C3$
3000  F31:=P3.P1+U3.U1-P1.U3+P3.U1$
3100  TA3:=U3.U1+P3.U2+H3$
3200  TB3:=-U3.U1+P1.U2+H1$
3300  TC3:=F31*(P3-P1).U2*(H1-H3)/(P3.P3-P1.P1)$
3400  TD3:=U3.U1+N.U2*(F1-F3)/N2$
3500  TE3:=N.U3+N.U1+N.U2+P2.P1/N3*(F1/N1+N1-F2/N2+N2)$
3600  TF3:=N.U3+N.U1+P1.U2/N3*(F1/N1+F2/N2)$
3700  TG3:=-N.U3+N.U1+P3.U2/N1*(F3/N3+F2/N2)$
3800
3900  P3:=-P1-P2$
4000  ON RAT;
4100  FACTOR N.P1 , N.P2;
4200  ON GCD;
4300  TA:=TA1+TA2+TA3$
4400  TB:=TB1+TB2+TB3$
4500  TC:=TC1+TC2+TC3$
4600  TD:=TD1+TD2+TD3$
4700  TE:=TE1+TE2+TE3$
4800  TF:=TF1+TF2+TF3$
4900  TG:=TG1+TG2+TG3$
5000
5100  INDEX U3;
5200  LHSa:=P3.U3+Ta$
5300  LHSb:=P3.U3+Tb$
5400  LHSc:=P3.U3+Tc$
5500  LHSD:=P3.U3+Td$
5600  LHSe:=P3.U3+Te$
5700  LHSf:=P3.U3+Tf$
5800  LHSg:=P3.U3+Tg$
5900
6000  LHS:=LHSa+LHSb+LHSc+LHSD+LHSe+LHSf+LHSg$

```

LHS = 0

$$-b_2 z_1^2 + c_2 z_2^2 = \frac{c(2)(2.2)}{(2.n)} [2^1 n^1 +$$

$$+ c(2) \frac{(2.2)^2}{(2.n)^2} n^1 n^2 + \frac{c(1)(1.1)}{(1.2)} [1^2 n^1 +$$

$$- c(1) \frac{(1.1)^2}{(1.n)^2} n^1 n^2$$

EMRAT;

EMFAC U1.U2;

EMFACTOR P1.N,P2.N;

EMGCD;

EMHS;

$$\begin{aligned} & \leftarrow B1 \cdot P1 \cdot P1 \cdot U1 \cdot U2 + B1 \cdot U2 \cdot P1 \cdot U1 \cdot P1 - C1 \cdot U2 \cdot P1 \cdot U1 \cdot P1 + U1 \cdot U2 \cdot P2 \cdot P2 \\ & \leftarrow U1 \cdot P2 \cdot B2 \cdot U2 \cdot P2 + U1 \cdot P2 \cdot C2 \cdot U2 \cdot P2 - (P2 \cdot P2 \cdot C2) \cdot (U1 \cdot P2 \cdot U2 \cdot N + U1 \cdot N \cdot U2 \cdot P2) \\ & \leftarrow \frac{P2}{P2 \cdot N} + \frac{(P2 \cdot P2 \cdot U1 \cdot N \cdot U2 \cdot N \cdot C2)}{P2 \cdot N^2} + C1 \cdot P1 \cdot P1 \cdot (U2 \cdot P1 \cdot U1 \cdot N) \\ & \leftarrow \frac{U2 \cdot N \cdot U1 \cdot P1}{P1 \cdot N} + (-C1 \cdot P1 \cdot P1 \cdot U1 \cdot N \cdot U2 \cdot N) / P1 \cdot N^2 \end{aligned}$$

This is the correct answer!

2. Assumptions. In this section we list a number of specific forms and simplifying assumptions used in making the calculations. It will be evident that most of the significant qualifications can be removed with increased care in making the calculations and with improved data on hadronic reactions.

(a) We assume the validity of the scaling hypothesis. In particular, for the reaction

$$A + B \rightarrow C + \text{anything} \quad (1)$$

the invariant cross section approaches an asymptotic limit

$$E_C \frac{d^3\sigma}{dp^3} = f(p_T, x) \quad (2)$$

where p_T is the transverse momentum of particle C and x is its longitudinal momentum in units of its maximum possible value. (Specifically, $x = 2p_L/\sqrt{s}$, where $\sqrt{s}$ is the total c.m. energy of the system.)

The hypothesis is on firm experimental ground for the reaction $p + p \rightarrow \pi^\pm + \text{anything}$ in the proton laboratory energy range 12-1500 GeV, at least for large x (Bali et al., 1970; Ratner et al., 1971; Jacob, 1972). Violations may exist for small x , but this kinematic range does not contribute substantially to muon production. Data for kaon production is sparse. Data for heavy target nuclei exists only for energies below 24 GeV (Allaby et al., 1970; Eichten et al., 1972), but by analogy with the $p + p \rightarrow \pi + \text{anything}$ reaction no difficulty is expected on this account.

(b) The particular form of $f(p_T, x)$ for the reactions $p + N$ (or O) $\rightarrow \pi^\pm + \text{anything}$ is obtained from Mason and Elbert (1973) and is discussed elsewhere in this conference. It is perhaps worth noting that preliminary calculations using the simpler fits of Bali et al. (1970) for the $p + p$ reaction yielded very similar results, with median primary proton energies about 15% lower.

(c) We have neglected kaon contributions and the effect of heavy primaries, pending the availability of better data.

(d) In considering muons arising from mesons produced after the first collision, we have neglected pion induced interactions. In part, the cross section data are not available, and in part the calculation is complicated by pion decay. Indications are that perhaps 14% of the observed muons arise from the meson induced parts of the cascade, about 26% from the baryon induced parts, and the remainder from the primary collision. Our inclusion of subsequent baryon interactions increased the median energy of the primary protons by about 10%, so perhaps a 6% further increase would be produced by correct inclusion of pion effects.

(e) In calculating the effect of subsequent baryon interactions, we have assumed that the cross section $f(p_T, x)/E_C$ is independent of x , where particle C is a daughter nucleon. This assumption is roughly borne out by the sparse data available (Jacob, 1972) and has no large effect upon the results.

(f) Muon decay has been neglected, inducing small errors at low energies (e.g., those of Elliot et al., 1969, and Jacklyn, 1965) and at large zenith angles (relevant to the planned experiments of M. Bercovitch).

(g) The discussion assumes a fixed energy threshold, although in reality the usual circumstance is a fixed depth. In the range of primary interest ($E_{p0} < 100$ GeV) straggling is unimportant, while at greater depths its

PPP + DP Sector

tensor	old name	new name
--------	----------	----------

$$(112)_S = 2S_2 - S_1 = -L_1 - L_2$$

$$(121)_S = S_1 = 3L_1 + L_2$$

$$\rightarrow (211)_S = S_2 = L_1$$

$$(123)_S = -2S_1 = -6L_1 + 3L_2$$

$$(132)_S = -2S_2 = -2L_1$$

$$\rightarrow (231)_S = S_1 - 3S_2 = L_2$$

$$(111)_S = -2S_2 = -2L_1$$

tensor	old name	new name
--------	----------	----------

$$(112)_A = A_1 = G_1 + G_2$$

$$(121)_A = A_1 = G_1 + G_2$$

$$\rightarrow (211)_A = A_1 - 3A_2 = G_1$$

$$\rightarrow (231)_A = 3A_2 = G_2$$

tensor	old	new
--------	-----	-----

$$(112)_1^1 = O_1 = -Q_1$$

$$(112)_1^2 = O_2 = Q_2$$

$$(121)_1^1 = \emptyset = \emptyset$$

$$(121)_1^2 = O_2 - \sqrt{3}O_1 = Q_2 + \sqrt{3}Q_1$$

$$\rightarrow (211)_1^1 = -O_1 = Q_1$$

$$\rightarrow (211)_1^2 = O_2 = Q_2$$

$$(132)_1^1 = -O_1 = Q_1$$

$$(111)_1^1 = \frac{1}{2}O_1 - \frac{\sqrt{3}}{2}O_2 = -\frac{1}{2}Q_1 - \frac{\sqrt{3}}{2}Q_2$$

$$\begin{bmatrix} L_1 \\ L_2 \\ G_1 \\ G_2 \\ Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & -3 & 0 & 0 \\ 2 & -1 & 1 & -1 \\ 0 & 3 & 0 & 0 \\ 0 & -1 & -1 & -1 \\ -2\sqrt{3} & -\sqrt{3} & -\sqrt{3} & -\sqrt{3} \\ 0 & -\sqrt{3} & \sqrt{3} & -\sqrt{3} \\ -2 & 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 0 \\ -3 & -3 & -3 & 0 \\ 3 & -3 & 1 & -2 \\ -3 & 3 & -3 & 0 \\ 2 & 2 & -1 & 0 \\ -2\sqrt{3} & -2\sqrt{3} & -\sqrt{3} & -2\sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \\ 0 & 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 111 \\ 112 \\ 121 \\ 122 \\ 211 \\ 212 \\ 221 \\ 222 \end{bmatrix} \begin{matrix} \checkmark \\ \checkmark \\ \checkmark \\ \checkmark \\ \checkmark \\ \checkmark \\ \checkmark \\ \checkmark \end{matrix}$$

Reduce invert:

	L_1	L_2	G_1	G_2	Q_1	Q_2	$\tilde{Q}_1$	$\tilde{Q}_2$
111	$-\frac{1}{3}$	0	0	0	$-\frac{1}{12}$	$-\frac{1}{4\sqrt{3}}$	$-\frac{1}{4\sqrt{3}}$	$-\frac{1}{4}$
112	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{6}$	0	0	$\frac{1}{6}$
121	$\frac{1}{2}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	0	0	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$
122	$\frac{1}{2}$	$\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{1}{6}$	0	0	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$
211	$\frac{1}{6}$	0	$\frac{1}{6}$	0	$\frac{1}{6}$	0	0	$\frac{1}{6}$
212	$\frac{1}{6}$	0	$-\frac{1}{6}$	0	$\frac{1}{6}$	0	0	$-\frac{1}{6}$
221	$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{1}{6}$	0	0	$-\frac{1}{6}$
222	$-\frac{1}{3}$	0	0	0	$-\frac{1}{12}$	$-\frac{1}{4\sqrt{3}}$	$\frac{1}{4\sqrt{3}}$	$\frac{1}{4}$

```

00100 Y:=3+*(1/2)$
00200 MATRIX A(8,8),AINV(8,8)$
00300 A:=MAT(
00400 (0,1,1,1,1,1,1,0),
00500 (0,-3,0,0,-3,-3,-3,0),
00600 (2,-1,1,-1,3,-3,-3,0),
00700 (8,3,0,0,-3,3,3,0),
00800 (0,1,1,1,1,1,1,0),
00900 (0,-3,0,0,-3,-3,-3,0),
01000 (2,-1,1,-1,3,-3,-3,0),
01100 (8,3,0,0,-3,3,3,0),
01200 AINV:=LAP(

```

• paar

• ader

• anzahl

$$113 =$$

$$123 =$$

$$131 =$$

$$\begin{pmatrix} T(1) \\ \vdots \\ T(27) \end{pmatrix} = \begin{pmatrix} 8 \\ \vdots \\ 27 \end{pmatrix}$$

$$\begin{pmatrix} A_{inv} \end{pmatrix} \begin{pmatrix} L_1 \\ \vdots \\ L_7 \end{pmatrix}$$

$$= \begin{pmatrix} 8 \\ \vdots \\ 27 \end{pmatrix} \begin{pmatrix} 8 \\ \vdots \\ 8 \end{pmatrix} = \begin{pmatrix} 8 \\ \vdots \\ 27 \end{pmatrix}$$

$$B(27, 8)$$

$$A_{inv}(8, 8)$$

$$C(27, 8)$$

$$V(2, 1)$$

$$\begin{pmatrix} L_1 \\ L_2 \end{pmatrix}$$

$$\begin{pmatrix} D \end{pmatrix} = \begin{pmatrix} C \end{pmatrix} \begin{pmatrix} V \end{pmatrix}$$

$$V(8, 1)$$

$$V(8, 1)$$

$$D(27, 1)$$

Analysis of PPP terms into irreducible pieces.

$$\text{111} \quad (1,1) := L1 \cdot (-1)/3 + Q1 \cdot (-1)/12 + Q2 \cdot (-1)/(4 \cdot 3^{(1/2)}) + Q1T \cdot (-1)/(4 \cdot 3^{(1/2)}) + Q2T \cdot (-1)/4$$

$$\text{112} \quad \text{MAT}(2,1) := L1 \cdot (-1)/6 + L2 \cdot (-1)/6 + G1/6 + G2/6 + Q1 \cdot (-1)/6 + Q2T/6$$

$$\text{113} \quad \text{MAT}(3,1) := L1/2 + L2/6 + G1 \cdot (-1)/6 + G2 \cdot (-1)/6 + Q1/4 + Q2/(4 \cdot 3^{(1/2)}) + Q1T/(4 \cdot 3^{(1/2)}) + Q2T/12$$

$$\text{121} \quad \text{MAT}(4,1) := L1/2 + L2/6 + G1/6 + G2/6 + Q1T/(2 \cdot 3^{(1/2)}) + Q2T/6$$

$$\text{122} \quad \text{MAT}(5,1) := L1/2 + L2/6 + G1 \cdot (-1)/6 + G2 \cdot (-1)/6 + Q1T \cdot (-1)/(2 \cdot 3^{(1/2)}) + Q2T \cdot (-1)/6$$

$$\text{123} \quad \text{MAT}(6,1) := -L1 + L2 \cdot (-1)/3$$

$$\text{131} \quad \text{MAT}(7,1) := L1 \cdot (-1)/6 + L2 \cdot (-1)/6 + G1 \cdot (-1)/6 + G2 \cdot (-1)/6 + Q1/12 + Q2/(4 \cdot 3^{(1/2)}) + Q1T \cdot (-1)/(4 \cdot 3^{(1/2)}) + Q2T/12$$

$$\text{132} \quad \text{MAT}(8,1) := L1 \cdot (-1)/3 + Q1/6 + Q1T/(2 \cdot 3^{(1/2)})$$

$$\text{133} \quad \text{MAT}(9,1) := L1/2 + L2/6 + G1/6 + G2/6 + Q1 \cdot (-1)/4 + Q2 \cdot (-1)/(4 \cdot 3^{(1/2)}) + Q1T \cdot (-1)/(4 \cdot 3^{(1/2)}) + Q2T \cdot (-1)/12$$

$$\text{211} \quad \text{MAT}(10,1) := L1/6 + G1/6 + Q1/6 + Q2T/6$$

$$\text{212} \quad \text{MAT}(11,1) := L1/6 + G1 \cdot (-1)/6 + Q1/6 + Q2T \cdot (-1)/6$$

$$\text{213} \quad \text{MAT}(12,1) := L1 \cdot (-1)/3 + Q1 \cdot (-1)/3$$

$$\text{221} \quad \text{MAT}(13,1) := L1 \cdot (-1)/6 + L2 \cdot (-1)/6 + G1 \cdot (-1)/6 + G2 \cdot (-1)/6 + Q1 \cdot (-1)/6 + Q2T \cdot (-1)/6$$

$$\text{222} \quad \text{MAT}(14,1) := L1 \cdot (-1)/3 + Q1 \cdot (-1)/12 + Q2 \cdot (-1)/(4 \cdot 3^{(1/2)}) + Q1T \cdot (-1)/(4 \cdot 3^{(1/2)}) + Q2T/4$$

$$\text{223} \quad (15,1) := L1/2 + L2/6 + G1/6 + G2/6 + Q1/4 + Q2/(4 \cdot 3^{(1/2)}) + Q1T \cdot (-1)/(4 \cdot 3^{(1/2)}) + Q2T \cdot (-1)/12$$

$$\text{231} \quad \text{MAT}(16,1) := L2/6 + G2/6$$

232

$$\text{MAT}(17,1) := L1/6 + G1/6 + Q1 \cdot (-1)/12 + Q2/(4 \cdot 3^{(1/2)}) + Q1T \cdot (-1)/12$$

233

$$\text{MAT}(18,1) := L1 \cdot (-1)/6 + L2 \cdot (-1)/6 + G1 \cdot (-1)/6 + G2 \cdot (-1)/6 + Q1/12 + Q2 \cdot (-1)/(4 \cdot 3^{(1/2)}) + Q1T/(4 \cdot 3^{(1/2)}) + Q2T/12$$

311

$$\text{MAT}(19,1) := L1/6 + G1 \cdot (-1)/6 + Q1 \cdot (-1)/12 + Q2/(4 \cdot 3^{(1/2)}) + Q1T \cdot (-1)/12 + Q2T/12$$

312

$$\text{MAT}(20,1) := L2/6 + G2 \cdot (-1)/6$$

$$\text{MAT}(21,1) := L1 \cdot (-1)/6 + L2 \cdot (-1)/6 + G1/6 + G2/6 + Q1/12 + Q2 \cdot (-1)/(4 \cdot 3^{(1/2)}) + Q1T \cdot (-1)/(4 \cdot 3^{(1/2)}) + Q2T \cdot (-1)/12$$

313

$$\text{MAT}(22,1) := L1 \cdot (-1)/3 + Q1/6 + Q1T \cdot (-1)/(2 \cdot 3^{(1/2)})$$

321

$$\text{MAT}(23,1) := L1 \cdot (-1)/6 + L2 \cdot (-1)/6 + G1/6 + G2/6 + Q1/12 + Q2/(4 \cdot 3^{(1/2)}) + Q1T/(4 \cdot 3^{(1/2)}) + Q2T \cdot (-1)/12$$

322

$$\text{MAT}(24,1) := L1/2 + L2/6 + G1 \cdot (-1)/6 + G2 \cdot (-1)/6 + Q1 \cdot (-1)/4 + Q2 \cdot (-1)/(4 \cdot 3^{(1/2)}) + Q1T/(4 \cdot 3^{(1/2)}) + Q2T/12$$

323

$$\text{MAT}(25,1) := L1/6 + G1/6 + Q1 \cdot (-1)/12 + Q2 \cdot (-1)/(4 \cdot 3^{(1/2)}) + Q1T \cdot (-1)/12 + Q2T/12$$

331

$$\text{MAT}(26,1) := L1/6 + G1 \cdot (-1)/6 + Q1 \cdot (-1)/12 + Q2 \cdot (-1)/(4 \cdot 3^{(1/2)}) + Q1T \cdot (-1)/12 + Q2T/12$$

332

$$\text{MAT}(27,1) := L1/6 + G1 \cdot (-1)/6 + Q1 \cdot (-1)/12 + Q2 \cdot (-1)/(4 \cdot 3^{(1/2)}) + Q1T \cdot (-1)/(4 \cdot 3^{(1/2)}) + Q2T/12$$

333

$$\text{MAT}(27,1) := L1 \cdot (-1)/3 + Q1/6 + Q2/(2 \cdot 3^{(1/2)})$$

*OFF NAT;

OUT MAT27*.RED;

*D:=D;

SHUT MAT27.RED;

MAT27.PEN;

```

00300  V:=3*(1/2)$
00300  MATRIX A(8,8):=AINV(8,8)$
00300  MATRIX B(27,8),C(27,8)$
00400  MATRIX V(8,1),D(27,1)$
00500  V:=TP(MAT( (L1,L2,G1,G2,Q1,Q2,Q1T,Q2T) ) )$
00600  A:=MAT(
00700  (0,1,1,1,1,1,1,0),
00800  (0,-3,0,0,-3,-3,-3,0),
00900  (2,-1,1,-1,3,-3,1,-2),
01000  (0,3,0,0,-3,3,-3,0),
01100  (0,-1,-1,-1,2,2,-1,0),
01200  (-2*Y,-Y,-Y,-Y,-2*Y,-2*Y,-Y,-2*Y),
01300  (0,-Y,Y,-Y,0,0,Y,0),
01400  (-2,1,-1,1,0,0,-1,2) )$
01500  AINV:=1/A$
01600  B:=MAT(
01700  (1,0,0,0,0,0,0,0),
01800  (0,1,0,0,0,0,0,0),
01900  (-1,-1,0,0,0,0,0,0),
02000  (0,0,1,0,0,0,0,0),
02100  (0,0,0,1,0,0,0,0),
02200  (0,0,-1,-1,0,0,0,0),
02300  (-1,0,-1,0,0,0,0,0),
02400  (0,-1,0,-1,0,0,0,0),
02500  (1,1,1,1,0,0,0,0),
02600  (0,0,0,0,1,0,0,0),
02700  (0,0,0,0,0,1,0,0),
02800  (0,0,0,0,-1,-1,0,0),
02900  (0,0,0,0,0,0,1,0),
03000  (0,0,0,0,0,0,0,1),
03100  (0,0,0,0,0,0,-1,-1),
03200  (0,0,0,0,-1,0,-1,0),
03300  (0,0,0,0,0,-1,0,-1),
03400  (0,0,0,0,1,1,1,1),
03500  (-1,0,0,0,-1,0,0,0),
03600  (0,-1,0,0,0,-1,0,0),
03700  (1,1,0,0,1,1,0,0),
03800  (0,0,-1,0,0,0,-1,0),
03900  (0,0,0,-1,0,0,0,-1),
04000  (0,0,1,1,0,0,1,1),
04100  (1,0,1,0,1,0,1,0),
04200  (0,1,0,1,0,1,0,1),
04300  (-1,-1,-1,-1,-1,-1,-1,-1) )$
04400  ORDER L1,L2,G1,G2,Q1,Q2,Q1T,Q2T;
04500  ON RAT;
04600  D:=B*AINV*V;

```

factor L1,...

Set up a new dp sector basis and tables.

1. First we define our standard elements:

new name	old name		new name
$(\delta^{12} 3^3)_S$	$= 2S_3$	$=$	L
$(\delta^{12} 3^3)_A$	$= A_3$	$=$	G
$(\delta^{12} 3^3)_1$	$= 2M_1^3$	$=$	Q_1
$(\delta^{12} 3^3)_2$	$= M_1^3$	$=$	Q_2

```

001500 MATRIX A(6,6)
002000 Y1=2+4+6+8+10+12
003000 A:=MAT(
004000 (-2,-2,2,0,0,0)
005000 (1,-1,1,2,-2,1)
006000 (-4,-4,-2,8,-8,-2)
007000 (0,0,Y,2+Y,2+Y,2+Y)
008000 (0,0,-2+Y,0,-2+Y,0)
009000 (2,-2,-1,-2,2,1)
AINV:=1/A;

```

2. Next, we set up the matrix form inversion :

two checks.

L	-2	-2	2	0	0	2	$\delta^{12} 3^3$	✓
G	1	-1	1	2	-2	-1	$\delta^{12} 2^3$	✓
Q_1	-4	-4	-2	0	0	-2	$\delta^{23} 1^1$	✓
Q_2	0	0	$\sqrt{3}$	$2\sqrt{3}$	$2\sqrt{3}$	$-\sqrt{3}$	$\delta^{23} 2^1$	✓
$\tilde{Q}_1$	0	0	$-2\sqrt{3}$	0	0	$2\sqrt{3}$	$\delta^{12} 1^2$	✓
$\tilde{Q}_2$	2	-2	-1	-2	2	1	$\delta^{12} 2^2$	✓

	δ^p L	δ^p G	δ^p Q_1	δ^p Q_2	$\tilde{\delta}^p$ $\tilde{Q}_1$	$\tilde{\delta}^p$ $\tilde{Q}_2$
$\delta^{12} 3^3$	$-\frac{1}{12}$	$\frac{1}{6}$	$-\frac{1}{12}$	0	0	$\frac{1}{6}$
$\delta^{12} 2^3$	$-\frac{1}{12}$	$-\frac{1}{6}$	$-\frac{1}{12}$	0	0	$-\frac{1}{6}$
$\delta^{12} 3^3$	$\frac{1}{6}$	0	$\frac{1}{6}$	0	0	0
$\delta^{12} 1^1$	$\frac{1}{6}$	0	$-\frac{1}{12}$	0	$-\frac{1}{4\sqrt{3}}$	0
$\delta^{12} 2^1$	$-\frac{1}{12}$	$\frac{1}{6}$	$\frac{1}{24}$	$\frac{1}{4\sqrt{3}}$	$\frac{1}{8\sqrt{3}}$	$-\frac{1}{12}$
$\delta^{12} 3^1$	$-\frac{1}{12}$	$-\frac{1}{6}$	$\frac{1}{24}$	$-\frac{1}{4\sqrt{3}}$	$\frac{1}{8\sqrt{3}}$	$\frac{1}{12}$
$\delta^{12} 1^2$	$-\frac{1}{12}$	$-\frac{1}{6}$	$\frac{1}{24}$	$\frac{1}{4\sqrt{3}}$	$-\frac{1}{8\sqrt{3}}$	$\frac{1}{12}$
$\delta^{12} 2^2$	$\frac{1}{6}$	0	$-\frac{1}{12}$	0	$\frac{1}{4\sqrt{3}}$	0
$\delta^{12} 3^2$	$-\frac{1}{12}$	$\frac{1}{6}$	$\frac{1}{24}$	$-\frac{1}{4\sqrt{3}}$	$-\frac{1}{8\sqrt{3}}$	$-\frac{1}{12}$

Analyze the BBB term by Reduce Table Lookup.

1. I am just experimenting to find the fastest way to compute things. I have already done the BBB several times, here it is again:

$$\begin{aligned} \text{BBB} = & xyz(231-312) \\ & + [yz^2 \cdot 212 + zx^2 \cdot 332 + xy^2 \cdot 311] \\ & - [zy^2 \cdot 211 + xz^2 \cdot 232 + yx^2 \cdot 331] \end{aligned}$$

	<u>G₁</u>	<u>L₂</u>	<u>G₁</u>	<u>G₂</u>	<u>Q₁</u>	<u>Q₂</u>	<u>Q₁</u>	<u>Q₂</u>
231	0	$\frac{1}{6}xyz$	0	$\frac{1}{6}xyz$	0	0	0	0
312	0	$-\frac{1}{6}xyz$	0	$+\frac{1}{6}xyz$	0	0	0	0
212	$\frac{1}{6}yz^2$	0	$-\frac{1}{6}yz^2$	0	$\frac{1}{6}yz^2$	0	0	$-\frac{1}{6}yz^2$
332	$\frac{1}{6}zx^2$	0	$-\frac{1}{6}zx^2$	0	$-\frac{1}{12}zx^2$	$-\frac{1}{\sqrt{3}}zx^2$	$-\frac{1}{\sqrt{3}}zx^2$	$\frac{1}{12}zx^2$
311	$\frac{1}{6}xy^2$	0	$-\frac{1}{6}xy^2$	0	$-\frac{1}{12}xy^2$	$\frac{1}{\sqrt{3}}xy^2$	$\frac{1}{\sqrt{3}}xy^2$	$\frac{1}{12}xy^2$
211	$-\frac{1}{6}zy^2$	0	$-\frac{1}{6}zy^2$	0	$-\frac{1}{6}zy^2$	0	0	$-\frac{1}{6}zy^2$
232	$-\frac{1}{6}xz^2$		$-\frac{1}{6}xz^2$		$+\frac{1}{12}xz^2$	$-\frac{1}{\sqrt{3}}xz^2$	$+\frac{1}{\sqrt{3}}xz^2$	$+\frac{1}{12}xz^2$
331	$-\frac{1}{6}yx^2$		$-\frac{1}{6}yx^2$		$+\frac{1}{12}yx^2$	$+\frac{1}{\sqrt{3}}yx^2$	$-\frac{1}{\sqrt{3}}yx^2$	$+\frac{1}{12}yx^2$

$$\begin{aligned} \Rightarrow \text{BBB} = & {}^{3p}L_1 \left\{ \frac{1}{6}(xy^2 + yz^2 + zx^2 - xz^2 - yx^2 - zy^2) \right\} \quad \checkmark \\ & + {}^{3p}G_1 \left\{ -\frac{1}{6}(xy^2 + yz^2 + zx^2 + xz^2 + yx^2 + zy^2) \right\} \quad \checkmark \\ & + {}^{3p}G_2 \left\{ \frac{1}{3}xyz \right\} \quad \checkmark \\ & + {}^{3p}Q_1 \left\{ \frac{1}{12}(2yz^2 - zx^2 - xy^2) - \frac{1}{12}(2zy^2 - xz^2 - yx^2) \right\} \quad - \text{partners.} \\ & + {}^{3p}Q_2 \left\{ \frac{1}{\sqrt{3}}(xy^2 - zx^2 + yz^2 - xz^2) \right\} \quad - \text{partners.} \end{aligned}$$

Comments: I think this method is best for several reasons:

- 1) you get the explicit functions in detail without figuring them out.
- 2) Direct lookup minimizes errors.

Analysis of the FB null term into irreducibles using Reduce Table Lookups.

1. My fractionating method was too messy for this, so now comes the lookup method.

$$F_{12}^{12} B_3^3 S(12) + \text{cyclic}$$

$$= \{xz S(12) (\delta 2^3) - x^2 S(12) (\delta 3^3) - z S(12) (212) + x S(12) (213)\} + \text{cyclic}.$$

Break this into two pieces, one in the pp p sector and the other in the dp sector.

2. First, do the dp sector:

$$[xz S(12)] (\delta 2^3) + [yx S(23)] (\delta 3^1) + [zy S(31)] (\delta 1^2) \\ - [x^2 S(12)] (\delta 3^3) - [y^2 S(23)] (\delta 1^1) - [z^2 S(31)] (\delta 2^2)$$

Lookup:

	<u>L</u>	<u>G</u>	<u>Q₁</u>	<u>Q₂</u>
$\delta 2^3$	$-\frac{1}{12} xz S(12)$	$-\frac{1}{6} xz S(12)$	$-\frac{1}{12} xz S(12)$	0
$\delta 3^1$	$-\frac{1}{12} yx S(23)$	$-\frac{1}{6} yx S(23)$	$\frac{1}{24} yx S(23)$	$-\frac{1}{4\sqrt{3}} yx S(23)$
$\delta 1^2$	$-\frac{1}{12} zy S(31)$	$-\frac{1}{6} zy S(31)$	$\frac{1}{24} zy S(31)$	$\frac{1}{4\sqrt{3}} zy S(31)$
$\delta 3^3$	$-\frac{1}{6} x^2 S(12)$	0	$-\frac{1}{6} x^2 S(12)$	0
$\delta 1^1$	$-\frac{1}{6} y^2 S(23)$	0	$+\frac{1}{12} y^2 S(23)$	0
$\delta 2^2$	$-\frac{1}{6} z^2 S(31)$	0	$+\frac{1}{12} z^2 S(31)$	0

$$= \overset{\delta p}{L} \left\{ -\frac{1}{12} (xz S(12) + yx S(23) + zy S(31)) - \frac{1}{6} (x^2 S(12) + y^2 S(23) + z^2 S(31)) \right\} \\ + \overset{\delta p}{G} \left\{ -\frac{1}{6} (xz S(12) + yx S(23) + zy S(31)) \right\} \\ + \overset{\delta p}{Q_1} \left\{ -\frac{1}{24} (2xz S(12) - yx S(23) - zy S(31)) - \frac{1}{12} (2x^2 S(12) - y^2 S(23) - z^2 S(31)) \right\} - \text{partner} \\ + \overset{\delta p}{Q_2} \left\{ -\frac{1}{4\sqrt{3}} (yx S(23) - zy S(31)) \right\} - \text{partner}.$$

3. Now do the ppp sector component:

$$x S(12) (213) + y S(23) (132) + z S(31) (321)$$

$$- z S(12) (212) - x S(23) (332) - y S(31) (311)$$

Lookup:

	L_1	L_2	G_1	G_2	Q_1	Q_2
213	$-\frac{1}{3} x S(12)$	0	0	0	$-\frac{1}{3} x S(12)$	0
132	$-\frac{1}{3} y S(23)$	0	0	0	$\frac{1}{6} y S(23)$	0
321	$-\frac{1}{3} z S(31)$	0	0	0	$\frac{1}{6} z S(31)$	0
212	$-\frac{1}{6} z S(12)$	0	$+\frac{1}{6} z S(12)$	0	$-\frac{1}{6} z S(12)$	0
³³² x	$-\frac{1}{6} x S(23)$	0	$+\frac{1}{6} x S(23)$	0	$+\frac{1}{12} x S(23)$	$+\frac{1}{\sqrt{3}} x S(23)$
311	$-\frac{1}{6} y S(31)$	0	$+\frac{1}{6} y S(31)$	0	$+\frac{1}{12} y S(31)$	$-\frac{1}{\sqrt{3}} y S(31)$

$$= {}^{3p}L_1 \left\{ -\frac{1}{3} (x S(12) + y S(23) + z S(31)) - \frac{1}{6} (z S(12) + x S(23) + y S(31)) \right\}$$

$$+ {}^{3p}G_1 \left\{ \frac{1}{6} (z S(12) + x S(23) + y S(31)) \right\}$$

$$+ {}^{3p}Q_1 \left\{ -\frac{1}{6} (2x S(12) - y S(23) - z S(31)) - \frac{1}{12} (2z S(12) - x S(23) - y S(31)) \right\} - \text{partner}$$

$$+ {}^{3p}Q_2 \left\{ \frac{1}{\sqrt{3}} (x S(23) - y S(31)) \right\} - \text{partner}$$

Analyze the $F_{12}(1-2)^3$ term using Reduce Table Lookup.

1. Want to merge this into my list, so must know its content:

$$F_{12}(1-2)^3 S(12) + \text{cyclic} = x S(12) [(\delta 1^3) - (\delta 2^3)] + c.$$

$$- S(12) [211 - 212] + c.$$

$$= x S(12) [(\delta 1^3) - (\delta 2^3)] + y S(23) [(\delta 2^1) - (\delta 3^1)] + z S(31) [(\delta 3^2) - (\delta 1^2)]$$

$$- S(12) [211 - 212] - S(23) [232 - 332] - S(31) [331 - 311]$$

2. First do the dp sector piece:

	<u>L</u>	<u>G</u>	<u>Q₁</u>	<u>Q₂</u>
$(\delta 1^3)$	$-\frac{1}{12} x S(12)$	$\frac{1}{6} x S(12)$	$-\frac{1}{12} x S(12)$	0
$(\delta 2^3)$	$+\frac{1}{12} x S(12)$	$+\frac{1}{6} x S(12)$	$+\frac{1}{12} x S(12)$	0
$(\delta 2^1)$	$-\frac{1}{12} y S(23)$	$\frac{1}{6} y S(23)$	$\frac{1}{24} y S(23)$	$\frac{1}{4\sqrt{3}} y S(23)$
$(\delta 3^1)$	$+\frac{1}{12} y S(23)$	$+\frac{1}{6} y S(23)$	$-\frac{1}{24} y S(23)$	$+\frac{1}{4\sqrt{3}} y S(23)$
$(\delta 3^2)$	$-\frac{1}{12} z S(31)$	$\frac{1}{6} z S(31)$	$\frac{1}{24} z S(31)$	$-\frac{1}{4\sqrt{3}} z S(31)$
$(\delta 1^2)$	$+\frac{1}{12} z S(31)$	$+\frac{1}{6} z S(31)$	$-\frac{1}{24} z S(31)$	$-\frac{1}{4\sqrt{3}} z S(31)$

$$= \overset{SP}{G} \left\{ \frac{1}{3} (x S(12) + y S(23) + z S(31)) \right\}$$

$$+ \overset{SP}{Q_2} \left\{ \frac{1}{2\sqrt{3}} (y S(23) - z S(31)) \right\} - \text{partner}$$

	<u>L_1</u>	<u>L_2</u>	<u>G_1</u>	<u>G_2</u>	<u>Q_1</u>	<u>Q_2</u>
211	$-\frac{1}{6} S(12)$	0	$-\frac{1}{6} S(12)$	0	$-\frac{1}{6} S(12)$	0
212	$\frac{1}{6} S(12)$	0	$-\frac{1}{6} S(12)$	0	$\frac{1}{6} S(12)$	0
232	$-\frac{1}{6} S(23)$	0	$-\frac{1}{6} S(23)$	0	$+\frac{1}{12} S(23)$	$-\frac{1}{4\sqrt{3}} S(23)$
332	$\frac{1}{6} S(23)$	0	$-\frac{1}{6} S(23)$	0	$-\frac{1}{12} S(23)$	$-\frac{1}{4\sqrt{3}} S(23)$
331	$-\frac{1}{6} S(31)$	0	$-\frac{1}{6} S(31)$	0	$+\frac{1}{12} S(31)$	$+\frac{1}{4\sqrt{3}} S(31)$
311	$\frac{1}{6} S(31)$	0	$-\frac{1}{6} S(31)$	0	$-\frac{1}{12} S(31)$	$\frac{1}{4\sqrt{3}} S(31)$

$$= {}^{3p}G_1 \left\{ -\frac{1}{3} (S(12) + S(23) + S(31)) \right\}$$

$$+ {}^{3p}Q_2 \left\{ -\frac{1}{2\sqrt{3}} (S(23) - S(31)) \right\} - \text{partner}$$

Analysis of certain $S(12)$ /function Combinations into irreducible components.

$$xz S(12) + c. = S_5 \{xz + yx + zy\} - S_1 \{2xz - yx - zy\} + S_2 \{yx - zy\}$$

$$x^2 S(12) + c. = S_5 \{x^2 + y^2 + z^2\} - S_1 \{2x^2 - y^2 - z^2\} + S_2 \{y^2 - z^2\}$$

$$x S(12) + c. = S_5 \{x + y + z\} - S_1 \{2x - y - z\} + S_2 \{y - z\}$$

$$z S(12) + c. = S_5 \{z + x + y\} - S_1 \{2z - x - y\} + S_2 \{x - y\}$$

$$S(12) + c. = S_5 \{3\}$$

$$2x S(12) - y S(23) - z S(31) = S_5 \{2x - y - z\} - S_1 \{4x + y + z\} - S_2 \{y - z\}$$

$$2z S(12) - x S(23) - y S(31) = S_5 \{2z - x - y\} - S_1 \{4z + x + y\} - S_2 \{x - y\}$$

subtract
↓

$$x S(23) - y S(31) = S_5 \{x - y\} + S_1 \{x - y\} + S_2 \{x + y\}$$

$$y S(23) - z S(31) = S_5 \{y - z\} + S_1 \{y - z\} + S_2 \{y + z\}$$

$$S(23) - S(31) = 2 \cdot S_2$$

$$yx S(23) - zy S(31) = S_5 \{yx - zy\} + S_1 \{yx - zy\} + S_2 \{yx + zy\}$$

Note:

$$S_5 = S(23)_5$$

$$S_1 = S(23)_1$$

$$S_2 = S(23)_2$$

$$2xz S(12) - yx S(23) - zy S(31) = S_5 \{2xz - yx - zy\} - S_1 \{4xz + yx + zy\} - S_2 \{yx - zy\}$$

$$2x^2 S(12) - y^2 S(23) - z^2 S(31) = S_5 \{2x^2 - y^2 - z^2\} - S_1 \{4x^2 + y^2 + z^2\} - S_2 \{y^2 - z^2\}$$

(Attempt to see where to substitute the DBB null tensor form.)

How do you analyze raw npp terms into terms in my list?

1. I have no trouble taking a term, like n_{11} or n_{22} , and analyzing it into its irreducible objects like $(n_{11})_2^2$ etc. I think my problem is more one of converting those basic components back into the list.

2. Maybe I don't really know what "the list" is. In the npp sector there only four terms in my list, after absorption. I guess I should examine these terms before and after absorption.

$$\begin{aligned} & A(123) (n_{22})_5 + (n_{22})_1' F_2(123) - (n_{22})_2' \tilde{F}_2(123) = \text{one of 4 terms} \\ & \{[n_{22} + 1n_1] (A(123)) + c\} = \{2 [n_{22} + 1n_1] F_2(123) + \text{cyclic}\} \\ & = [n_{22} + 1n_1] \{ A(123) + 2 F_2(123) \} + \text{cyclic} \end{aligned}$$

So there really seems to be no problem. If you know the irreducible form, you should know $A(123)$ and $F_2(123)$; then you add them with factors as shown to get what you want!

$$\begin{aligned} & A(123) (n_{12})_5 + (n_{12})_1' F_2(123) - (n_{22})_2' \tilde{F}_2(123) \\ & = [n_{12} + 2n_1] \{ A(123) + 2 F_2(123) \} \end{aligned}$$

$$\begin{aligned} & A(123) (n_{21})_5 + (n_{21})_1' F_2(123) - (n_{21})_2' \tilde{F}_2(123) \\ & = [n_{21} + 1n_2] \{ A(123) + 2 F_2(123) \} \end{aligned}$$

$$\begin{aligned} & S(123) (n_{21})_A + (n_{21})_2^2 F_1 - (n_{21})_1^2 \tilde{F}_1 \\ & = [n_{21} - 1n_2] \{ S(123) + 2 F_1(123) \} \end{aligned}$$

This last one is just slightly different.

3. So here is how you do it: take your npp term of interest, reduce it to basics, convert those to ones in the list, then do the above.

Example:

$$n_{11} = [X_S + X_A + X'_1 + X'^2_2](n_{11}) = \frac{1}{6} [(n_{11})_S + (n_{11})_A + (n_{11})'_1 + (n_{11})^2_2]$$

$$= \frac{1}{6} [(-L_2 - L_3) + (\phi) + (-Q_1 - \frac{1}{2}Q_3 - \frac{\sqrt{3}}{2}Q_4) + (-\sqrt{3}\tilde{Q}_1 - \frac{\sqrt{3}}{2}\tilde{Q}_3 - \frac{3}{2}\tilde{Q}_4)]$$

irreducible $\rightarrow$ LIST

But,

$$A(123)L_2 + Q_2 F_2(123) - \tilde{Q}_2 \tilde{F}_2(123) = [n_{12} + 2n_1] \{ A + 2F_2 \} + c_y$$

$$A(123)L_3 + Q_3 F_2(123) - \tilde{Q}_3 \tilde{F}_2(123) = [n_{21} + n_{22}] \{ A + 2F_2 \} + c_y$$

$$A(123)L_1 + Q_1 F_2(123) - \tilde{Q}_1 \tilde{F}_2(123) = [n_{22} + n_{21}] \{ A + 2F_2 \} + c_y$$

$$S(123)A_1 + \tilde{Q}_4 F_1(123) - Q_4 \tilde{F}_1(123) = [n_{21} - n_{22}] \{ S + 2F_1 \} + c_y$$

Now we have:

$$6 \cdot n_{11} = -L_2 - L_3 - Q_1 - \frac{1}{2}Q_3 - \frac{\sqrt{3}}{2}Q_4 - \sqrt{3}\tilde{Q}_1 - \frac{\sqrt{3}}{2}\tilde{Q}_3 - \frac{3}{2}\tilde{Q}_4$$

The question is: what do we do with this thing now? This term by itself with its coefficient $-(1.1)(2.2)(3.2)$ will not be totally antisymmetric of course, so it cannot be fitted into the above form. I guess you reallyh want to analyze all the four terms first:

$$6 \cdot n_{22} = (n_{22})_S + (n_{22})_A + (n_{22})'_1 + (n_{22})^2_2$$

$$= L_1 - A_1 + Q_1 + (\sqrt{3}\tilde{Q}_1 + \sqrt{3}\tilde{Q}_3 - \tilde{Q}_4)$$

$$6 \cdot n_{12} = (n_{12})_S + (n_{12})_A + (n_{12})'_1 + (n_{12})^2_2$$

$$= L_2 + A_1 + Q_2 + (\sqrt{3}\tilde{Q}_2 - \sqrt{3}\tilde{Q}_3 + \tilde{Q}_4)$$

$$6 \cdot n_{21} = (n_{21})_S + (n_{21})_A + (n_{21})'_1 + (n_{21})^2_2$$

$$= L_3 + A_1 + Q_3 + (\tilde{Q}_4)$$

Now lets get to the term of interest:

$$\begin{aligned} & \left\{ -\frac{1}{6}(1.1)(2.2)(3.2) \left[-\check{L}_2 - \check{L}_3 - Q_1 - \frac{1}{2}Q_3 - \frac{\sqrt{3}}{2}Q_4 - \sqrt{3}\tilde{Q}_1 - \frac{\sqrt{3}}{2}\tilde{Q}_3 - \frac{3}{2}\tilde{Q}_4 \right] \right. \\ & - \frac{1}{6}(1.1)(1.2)(3.1) \left[+\check{L}_1 - A_1 + Q_1 + \sqrt{3}\tilde{Q}_1 + \sqrt{3}\tilde{Q}_3 - \tilde{Q}_4 \right] \\ & + \frac{1}{6}(1.1)(2.2)(3.1) \left[+\check{L}_2 + A_1 + Q_2 + \sqrt{3}\tilde{Q}_2 - \sqrt{3}\tilde{Q}_3 + \tilde{Q}_4 \right] \\ & \left. + \frac{1}{6}(1.1)(1.2)(3.2) \left[+\check{L}_3 + A_1 + Q_3 + \tilde{Q}_4 \right] \right\} A(23) + c. \end{aligned}$$

This is the npp of DBB.

$$\Rightarrow \vec{L}_2 \left\{ \frac{1}{6}(1.1)(2.2) \left[(3.1) + (3.2) \right] \right\} A(23) + c.$$

$$= \vec{L}_2 \left\{ \underbrace{-\frac{1}{6}(1.1)(2.2)(3.3)}_{(32)\text{sym}} A(23) + c. \right\}$$

$$\Rightarrow \vec{L}_1 \left\{ \underbrace{-\frac{1}{6}(1.1)(1.2)(1.3)}_{(23)\text{sym}} A(23) + c. \right\}$$

$$= \vec{L}_1 \left\{ \underbrace{-\frac{1}{6}(1.1)(1.2)(1.3)}_{(23)\text{sym}} A(123) + c. \right\}$$

$$\Rightarrow \vec{L}_3 \left\{ \frac{1}{6}(1.1)(3.2) \left[(-3.2) \right] \right\} A(23) + c.$$

$$= \vec{L}_3 \left\{ \underbrace{-\frac{1}{6}(1.1)(3.2)(3.2)}_{A(123)} A(23) + c. \right\}$$

So, we are for the moment "feeding" into 3 of our "list terms" via these A's. Notice that each A(123) really is an A-type function, which is a check on errors. However, these A contributions could be cancelled by the mixed symmetry pieces, so no conclusions yet... Keep going.

$$\vec{Q}_1 \left\{ -\frac{1}{6}(1.1) \left[(1.2)(3.1) - (3.2)(2.2) \right] \right\} A(23) \quad \checkmark$$

$$+ \vec{Q}_1 \left\{ -\frac{1}{6}(2.2)(1.3) \left[-(2.3) - \frac{1}{2}(3.3) \right] \right\} A(31) \quad \checkmark \quad \checkmark$$

$$+ \vec{Q}_1 \left\{ \frac{1}{6}(3.3)(1.1)(2.3) \right\} A(12) \quad \checkmark$$

To get this full Q1 coefficient I had to actually do the cyclic! Very messy.

4. Now I have the awful problem of trying to extract the Q_1 coefficient from the above form. You have to remember that under "cyclic" Q_1 mixes into Q_1 and $\tilde{Q}_1$ and $\tilde{Q}_1$ mixes also into these two. So let's see what we've got in the way of Q_1 and twiddle material in there:

$$\frac{1}{6} (1.1) [Q_1 + \sqrt{3} \tilde{Q}_1] \{ (2.2)(3.2) - (2.1)(3.1) \} A(23) + \text{cyclic.}$$

However, since I am only interested in learning whether this npp term of DBB "feeds" or "does not feed" into various places, maybe I can only look at the $A(23)$ coefficient. Then I get:

$$\begin{aligned} F_2 = \text{coeff } Q_1 &= \frac{1}{6} [-(x+z)] \{ (-y-x)y - xz \} A(23) + \text{other.} \\ &= \frac{1}{6} (1.1) [(2.2)(3.2) - (2.1)(3.1)] A(23) + \text{other.} \end{aligned}$$

$$A = \text{coeff } L_1 = -\frac{1}{6} (1.1)(1.2)(1.3) A(23) + \text{cyclic}$$

$$\begin{aligned} \Rightarrow [A + 2F_2] &= \frac{1}{6} (1.1) A(23) \{ - (1.2)(1.3) + 2(2.2)(3.2) - 2(2.1)(3.1) \} + \text{other.} \\ &= \frac{1}{6} (1.1) A(23) \{ -3(1.2)(1.3) + 2(2.2)(3.2) \} + \text{other.} \\ &= -3xz + 2(-y-x)y \neq 0. \end{aligned}$$

Our logic now says: since the $A(23)$ piece of this coefficient does not vanish, the entire coefficient cannot vanish because the three $A(ij)$ objects may be regarded as independent. (linearly) Therefore our npp term of interest does feed into the $(L_1 - Q_1)$ term of our "list", and that is what we wanted to know. No need to compute other pieces of it.

5. Now let us see if there is "feed" into the $S(123)$ term. We have:

$$\vec{A}_1 \left\{ \frac{1}{6} (1.1) A(23) [(1.2)(3.2) + (2.2)(3.1) + (1.2)(3.1)] + \text{cyclic} \right\}$$

$S(123)$

$$[] = xy - (x+y)z + xz = xy - yz = y(x-z)$$

$$\text{so } (1.1)[] = -y(x^2 - z^2) \Rightarrow \vec{A}_1 \left\{ \frac{1}{6} A(23) y(x^2 - z^2) + \text{cyclic} \right\}$$

25 anti ✓

Again we will consider the $A(23)$ piece only. We want the Q_4 -twid coefficient, so just read it off:

$$\begin{aligned}
 F_1 = \text{coeff } \tilde{Q}_4 &= \frac{1}{6} (1.1) \left[\frac{3}{2} (2.2)(3.2) + (1.2)(3.1) + (2.2)(3.1) + (1.2)(3.2) \right] A(23) \\
 &= \frac{1}{12} (1.1) \left[(2.2)(3.2) \cdot 3 + 2(1.2)(3.1) + 2(2.2)(3.1) + 2(1.2)(3.2) \right] A(23) + \text{other} \\
 &= \frac{1}{12} (1.1) \left[(3y+2z)(-y-x) + 2xz + 2xy \right] A(23) + \text{other} \\
 &\quad - 3y^2 - 3xy - 2zy - 2zx + 2xz + 2xy = -3y^2 - 2zy - xy \\
 &= \frac{1}{12} A(23) (1.1) (-y) [3y+2z+x].
 \end{aligned}$$

So:

$$\begin{aligned}
 [S+2F_1] &= -\frac{1}{6} \left[y(x^2-z^2) + y(-x-z)(3y+2z+x) \right] A(23) + \text{other} \\
 &= -\frac{1}{6} y A(23) \left[x^2-z^2 - (x+z)(x+2z+3y) \right] + \text{other} \\
 &\quad \cancel{x^2-z^2} - \cancel{x^2} - 2xz - 3yx - zx - 2z^2 - 3zy \\
 &\quad \neq 0.
 \end{aligned}$$

So, we conclude here that our npp of DBB term does feed into the $(n_{21}-ln_2)$ term also.

6. My guess is that it feeds into all four of those terms. I have calculated for sure (modulo algebraic errors) that it feeds into two of them. My inclination now is to replace the $(n_{21}-ln_2)$ term with the DBB so I can see quickly that we are 20 in 20.

(no go) ⁻
IRREDUCIBLE

Question: What blocks you from always doing everything "in irreducibles"?

1. After all, you start with them, then you embed, then in effect you split them up again. Why bother with the embedding in the first place?

2. Eg, consider the list of 22 n-term tensor forms (irreducible forms). Suppose you gave each of these guys a coefficient function. For example, you would have:

$$A(123) (n32)_S + \left[(n32)_1 F_2(123) - (n32)_2 \widetilde{F}_2(123) \right] + \text{etc.}$$

$$= \left\{ 2[(e) + (12)] (n32) F_2(123) + \text{cyclic} \right\}$$

The "mixed pair" entity here is really a single amplitude, totally antisymmetric, characterized by the function $F_2(123)$. There are not three amplitudes, just two!!!

If you are classifying terms or whatever, you will never get $(n32)_1$ in a thing like this without its partner! Because you are always dealing exclusively with terms that are totally antisymmetric. So why bother to carry around extra functions!

Example: Suppose I go and analyze the $DDD.A(123)$ term in its npp sector. I find that it can be fitted into the above coefficients like so:

$$\begin{cases} "A(123)" = \left(\frac{1}{2} A(123) \cdot [-(1.1)(n.2)(n.3)]_S \right) \hat{a} \\ "F_2(123)" = \left(A(123) [-(1.1)(n.2)(n.3)]_1 \right) \hat{z} \end{cases}$$

$$DDD_3.A(123) \stackrel{\text{mly npp}}{=} "A(123)" (n32)_S + ["F_2" (n32)_1 - p]$$

Thus, the DDD term hits on these two amplitudes of the npp sector. Of course it hits on amplitudes in other sectors also. Still, if you wanted you could replace the $(n32)_1, (n32)_2$ pair with the DDD term!

3. Consider the effect on the Ward dotting process of removing one of these terms instead of the other. If you have the $(n32)_S$ left over your Ward will say:

$$"A(123)" \cdot 3^3 [n^3 3^2 + 3 n^2 1^2 + 2 n^2 1^2]$$

$$= "A(123)" \left\{ (3.2) n^3 3^2 + (3.1) 3 n^2 + (3.n) 2 1^2 \right\} = \text{very simple}$$

How would you dot the other term?

$$3^3 \left[F_2 (n_{32})_1 - p \right]$$

$$= 3^3 \left\{ 2 [n_{32} + 3n_1] F_2(123) + \text{cyclic} \right\}$$

If you blindly do this (as I have been doing) by using the cyclic form, its just a real mess because you have to go look up $(123)F_2$ matrix elements and so on. The first form is probably much easier:

$$F_2 \cdot 3^3 (n_{32})_1 - \tilde{F}_2 \cdot 3^3 (n_{32})_2$$

$$= F_2 \cdot 3^3 \left[\frac{1}{\sqrt{3}} (n_{32})_1 \right] - \tilde{F}_2 \cdot 3^3 \left[\frac{1}{\sqrt{3}} (n_{32})_2 \right]$$

$$= F_2 \cdot 3^3 \left\{ n_{32} + 3n_1 - 2 \cdot 2 \cdot n \right\} - \sqrt{3} \tilde{F}_2 \cdot 3^3 \left\{ n_{32} - 3n_1 \right\}$$

$$= F_2 \cdot \left\{ n^1 3^2 (2 \cdot 3) + 3 n^2 (3 \cdot 1) - 2 \cdot 2 \cdot 1^2 (3 \cdot n) \right\} - \sqrt{3} \tilde{F}_2 \cdot \left\{ n^1 3^2 (3 \cdot 2) - 3 n^2 (3 \cdot 1) \right\}$$

I dont even know what this means!

4. So, restate this: the tensors $(n_{32})_1^1$ and $(n_{32})_2^1$ occur in the vertex only in a certain combination! We analyze the DDD tensor and we find:

$$\text{DDD tensor} = (\text{the } n_{32}_s \text{ tensor}) + (\text{the } n_{32}\text{-pair combination}).$$

Obviously this is your golden opportunity to get rid of that pair combination entirely! Who needs it?! When you start putting such pair-combination tensors into the 3^3 dot, it seems that the single entity gets fractured into two entities which you must then track both of.

Question: what is the npp content of DDD.

1. And what is the best way to characterize it/? We need efficiently heered.

$$DDD = [(n.1)1' - (1.1)n'][(n.2)2' - (2.2)n''][(n.3)3' - (3.3)n''']$$

$$= n'2^2 3^3 \{ \underbrace{-(1.1)(n.2)(n.3)}_{23 \text{ sym.}} \} + \text{cyclic.}$$

$$+ 1' n^2 3^3 \{ \text{etc} \}$$

$$S_0: \langle npp | DDD \rangle = n23 \cdot \{ \underbrace{-(1.1)(n.2)(n.3)}_{S(23)} \} + \text{cyclic}$$

Suppose I were to fractionate this thing right here and now. Treat the bracket as an $S(23)$ object. Then:

$$\frac{1}{6} [(n23)_S + (n23)_1 + (n23)_2] [S(23)_S + S(23)_1 + S(23)_2]$$

$$= \hat{S} \{ (n23)_S S(23)_S + \frac{1}{2} \left((n23)_1 S(23)_1 + p \right) + \frac{1}{2} \left((n23)_2 S(23)_2 + p \right) \}$$

$$+ \hat{1} \{ (n23)_S S(23)_1 + (n23)_1 S(23)_S + \frac{1}{2} \left((n23)_1 S(23)_1 - p \right) + \frac{1}{2} \left((n23)_2 S(23)_2 - p \right) \}$$

$$+ \hat{2} \{ (n23)_S S(23)_2 + (n23)_2 S(23)_S + \frac{1}{2} \left((n23)_1 S(23)_2 + p \right) + \frac{1}{2} \left((n23)_2 S(23)_1 + p \right) \}$$

$$+ \hat{a} \{ \quad \quad \quad + \frac{1}{2} \left(\quad - p \right) \quad \quad \quad - p \cdot \quad \}$$

Well:

$$\left\{ \begin{aligned} (n23)_1 S(23)_1 &= \frac{1}{\sqrt{3}} (n23)_1^2 S(23)_1 \end{aligned} \right.$$

$$\text{partner} = \frac{1}{\sqrt{3}} (n23)_2^2 \frac{1}{\sqrt{3}} S(23)_2 = \frac{1}{3} (n23)_2^2 S(23)_2$$

So:

$$(n23)_1 S(23)_1 \pm p = (n23)_1 S(23)_1 \pm \frac{1}{3} (n23)_2^2 S(23)_2$$

$$+ (n23)_2^2 S(23)_2 \pm p = (n23)_2^2 S(23)_2 \pm 3(n23)_1 S(23)_1$$

$$= (n23)_1 S(23)_1 [1 \pm 3] + (n23)_2^2 S(23)_2 [1 \pm \frac{1}{3}]$$

Now:

$$\begin{aligned} & \left((n_{23})'_1 S(23)_2 - p \right) + \left((n_{23})'_2 S(23)_1 - p \right) = \hat{2} p \\ & = \left[(n_{23})'_1 S(23)_2 - p \right] + \left[(n_{23})'_2 S(23)_1 - p \right] = 0 \end{aligned}$$

$$\overbrace{(n_{23})'_2 S(23)_1} = \overbrace{(n_{23})'_2} \cdot \overbrace{S(23)_1} = \sqrt{3} (n_{23})'_1 \frac{1}{\sqrt{3}} S(23)_2 = (n_{23})'_1 S(23)_2$$

$$\begin{aligned} \Rightarrow & = \hat{S} \left\{ (n_{23})_s S(23)_s + 2 (n_{23})'_1 S(23)_1 + \frac{2}{3} (n_{23})'^2_2 S(23)_2 \right\}^{\frac{1}{2}} \\ & = \langle npp | DDD \rangle \end{aligned}$$

2. Hindsight: we know that the tensor form DDD is S-type. Therefore its projection into and sector must also be S. We could have known that at the start. Thus I needed only compute the S-row of the Clebsch equation.

But I didn't think of it so I did it out. First, I note that only the A and S survive cyclic; then I found that A=0. External factor is (1/6) as shown at the start and then x3 because cyclic just copies original twice.

$$\begin{aligned} \langle npp | DDD \rangle &= \frac{1}{2} S(23)_s (n_{23})_s \\ &+ S(23)_1 (n_{23})'_1 \\ &+ \frac{1}{3} S(23)_2 (n_{23})'^2_2 \end{aligned}$$

$$\text{where } S(23) \equiv -(1 \cdot 1) (n_{12} \times n_{13})$$

Repeat:

$$\langle npp | DDD \rangle = \frac{1}{2} S(23)_s (n_{23})_s + \left[S(23)_1 (n_{23})'_1 + \text{partners} \right]$$

This then is the precise answer to our original question: it tells you into exactly which of those 22 12 npp terms DDD overlaps.

3. Lets push this some more: (npp/DDD) just tells you ~~xxxxxx~~ the npp terms in DDD. If DDD appears with coefficient $A(123)$, then just multiply that in:

$$A(123) DDD \stackrel{npp}{=} \left\{ \frac{1}{2} A(123) S(23)_s \right\} (n23)_s + \left\{ A(123) S(23)_1 \right\} (n23)_1' + \left\{ A(123) S(23)_1 \right\} (\widetilde{n23})_1'$$

$$\begin{aligned} \text{Now: } [A(123) S(23)_1] &= X_1^2 [A(123) S(23)_1] \\ &= \left(\frac{\sqrt{3}}{6} \right) [-(13) + (23) + (125) - (152)] [A(123) S(23)_1] \\ &= \left(\frac{\sqrt{3}}{6} \right) = A(123) \left(\frac{\sqrt{3}}{6} \right) [+(15) - (25) + (115) - (152)] S(23)_1 \\ &\quad - X_2^1 \\ &= - A(123) S(23)_1' \end{aligned}$$

$$\Rightarrow A(123) \cdot DDD \stackrel{npp}{=} \left\{ \frac{1}{2} A(123) S(23)_s \right\} (n23)_s + \left\{ A(123) S(23)_1 \right\} (n23)_1' - \left\{ A(123) S(23)_1 \right\} (\widetilde{n23})_1'$$

$$\Rightarrow "A" = \frac{1}{2} A S(23)_s$$

$$"F_2" = A S(23)_1$$

$$\Rightarrow A + 2F_2 = \frac{1}{2} [S(23)_s + 4 S(23)_1] A(123)$$

$$\Rightarrow \text{embed in } [n23 + 1n3] (A + 2F_2) + \text{cyclic.}$$

So yes, this could be embedded in the $2 \times n23$ system. But that seems to just cloud the ~~xxxxxx~~ issue. The result on the past page in terms of irreducibles is clearer. You can replace one of the three tensor forms appearing by the DDD object.

A new Fraction Analysis of the BBB term.

1. I think writing in cyclic notation is useful here. There are 8 terms in BBB, all of the ppp class. You write them out and you find:

$$BBB = xyz(231-312) + [yz^2 \cdot 212 + c] + [-zy^2 \cdot 211 + c]$$

The first term is at once irreducible. The other terms require minimal work. Here is the first of the second two terms:

$$3[yz^2 \cdot 212]_2 = \text{only interest: (only 2 survive cyc; can be no 3 in BBB)}$$

Let

$$yz^2 = (yz^2)_s + (yz^2)_a + (yz^2)_1 + (yz^2)_2$$

$$6 \cdot 212 = (212)_s + (212)_a + (212)_1 + (212)_2$$

Then:

$$= \frac{1}{2} [(yz^2)_s + (yz^2)_a + (yz^2)_1 + (yz^2)_2] \left[\overset{(211)_s}{(212)_s} + \overset{-(211)_A}{(212)_a} + \overset{(211)_1}{(212)_1} + \overset{-(211)_2}{(212)_2} \right]$$

$$= \frac{1}{2} [(yz^2)_s + (yz^2)_a + (yz^2)_1 + (yz^2)_2] [(211)_s - (211)_A + (211)_1 - (211)_2]$$

$$= \frac{1}{2} \left\{ -(yz^2)_s (211)_A + (yz^2)_a (211)_s + \frac{1}{2} ((yz^2)_1 (211)_2 - p) + \frac{1}{2} ((yz^2)_2 (211)_1 - p) \right\}$$

$$\#^1 = \frac{1}{2} (yz^2)_a (211)_s + \frac{1}{4} [(yz^2)_2 (211)_1 - p]$$

$$- \frac{1}{2} (yz^2)_s (211)_A - \frac{1}{4} [(yz^2)_1 (211)_2 - p]$$

Second term has a) overall sign, b) $(yz^2) \rightarrow (zy^2)$ c) extra sign

$$\#^2 = -\frac{1}{2} (zy^2)_a (211)_s - \frac{1}{4} [(zy^2)_2 (211)_1 - p]$$

$$- \frac{1}{2} (zy^2)_s (211)_A - \frac{1}{4} [(zy^2)_1 (211)_2 - p]$$

Add them:

$$\begin{aligned} & \frac{1}{2} (211)_S \left[\overset{A(12)}{(y\bar{z}^2 - zy^2)_A} \right] + \frac{1}{2} \left[\overset{S(12)}{(zy^2 + y\bar{z}^2)_S} \right] (211)_A \\ & + \frac{1}{4} (211)_1 \left[(y\bar{z}^2 - zy^2)_2 \right] - \rho \\ & - \frac{1}{4} (211)_2 \left[(y\bar{z}^2 + zy^2)_1 \right] - \rho. \end{aligned}$$

2. So here, without having done too much work, we have a complete analysis into irreducibles of the BBB term: (irreducibles appearing in our list!)

B

$BBB = (231)_A \left\{ \frac{1}{3} x y z \right\}$	1. ✓
$+ (211)_S \left\{ \frac{1}{2} (y\bar{z}^2 - zy^2)_A \right\}$	2. ✓
$+ (211)_1 \left\{ \frac{1}{4} (y\bar{z}^2 - zy^2)_2 \right\} - \rho$	3.
$+ (211)_A \left\{ -\frac{1}{2} (zy^2 + y\bar{z}^2)_S \right\}$	4.
$+ (211)_2 \left\{ -\frac{1}{4} (y\bar{z}^2 + zy^2)_1 \right\} - \rho.$	5.

3. Conclusion: the BBB term overlaps into 5 of the 10 tensors on my list. (A pair now counts as one tensor because in the vertex that is true.) Last time thru I replaced term 1. with BBB, but I think I should have used this opportunity to get rid of a pair!

A new fraction analysis of the FB ward null term.

1. Just do this in the same way as you did the BBB.

$$F_{12}^{12} B_3^3 S(12) + c.$$

$$F_{12}^{12} B_3^3 = \frac{[(1,2) \delta^{12} - 2^1 1^2]}{x} [z 2^3 - x 3^3] = x z (\delta^{12} 2^3) - x^2 (\delta^{12} 3^3) - z (212) + x (213)$$

I think we are forced to do a complete analysis of this FB object. Well, $S(12)$ has an S and a 1 piece. We will Clebsch this with FB and look only at the a -line. Here $S(12)$ plays the role of g . There is a g_s and a g_1 . This means we will need $(FB)_a$ and $(FB)_2$ only. Our answer will then be:

$$3 \hat{a} \left\{ (FB)_a S(12)_s + \frac{1}{2} [(FB)_2 S(12)_1 - p] \right\}$$

2. So, let's take each of the four terms in FB and get their a -line and 2 -line.
Start with the third term since looks easy:

$$\begin{aligned} \#3 \quad -z(212) &= -[(z)_s + (z)_1 + (z)_2] [(212)_s + (212)_a + (212)_1 + (212)_2] \\ &= -\hat{a} \left\{ (z)_s (212)_2 + (z)_2 (212)_s + (z)_1 (212)_a + \frac{1}{2} [(z)_1 (212)_2 + p] + \frac{1}{2} [(z)_2 (212)_1 + p] \right\} \\ &\quad -\hat{a} \left\{ (z)_s (212)_a + \frac{1}{2} [(z)_1 (212)_2 - p] + \frac{1}{2} [(z)_2 (212)_1 - p] \right\} \end{aligned}$$

$$\begin{aligned} \#4 \quad &= +\hat{a} \left\{ (x)_s (213)_2 + (x)_2 (213)_s + (x)_1 (213)_a + \frac{1}{2} [(x)_1 (213)_2 + p] + \frac{1}{2} [(x)_2 (213)_1 + p] \right\} \\ &\quad +\hat{a} \left\{ (x)_s (213)_a + \frac{1}{2} [(x)_1 (213)_2 - p] + \frac{1}{2} [(x)_2 (213)_1 - p] \right\} \end{aligned}$$

too messy!

Go back and Re-analyze the non-n sector stuff with the idea of trying always to knock out pairs with the ward null terms.

1. Motivation: I know the answer there so I can watch what effect this has.

2. Here is our starting list:

$$(211)_S + [(211)_1^1, (211)_2^1]$$

$$(211)_A + [(211)_2^2, (211)_1^2]$$

$$(231)_S$$

$$(231)_A$$

$$(\delta^{12}3)_S + [(\delta^{12}3)_1^1, (\delta^{12}3)_2^1]$$

$$(\delta^{12}1)_A + [(\delta^{12}1)_2^2, (\delta^{12}1)_1^2]$$

$${}^{3p}L_1 + [{}^{3p}Q_1, \tilde{Q}_1]$$

$${}^{3p}G_1 + [\tilde{Q}_2, {}^{3p}Q_2]$$

$${}^{3p}L_2$$

$${}^{3p}G_2$$

$${}^{\delta p}L_1 + [{}^{\delta p}Q_1, \tilde{Q}_1]$$

$${}^{\delta p}G_1 + [\tilde{Q}_2, {}^{\delta p}Q_2]$$

There are four pair-tensors in here, and there will be four Ward null degrees. Maybe I can knock out all the pairs! Lets try it.

3. Now lets look at the BBB analysis one more time.

Now merge these new terms into the starting list.

1. Here is the starting list:

$${}^{3p}L_1 \cdot A(123)$$

$${}^{3p}G_1 \cdot S(123)$$

$${}^{3p}Q_1 \cdot F_2(123) - p.$$

BBB

$${}^{3p}Q_2 \cdot F_1(123) - p$$

BBB

$${}^{3p}L_2 \cdot A(123)$$

$${}^{3p}G_2 \cdot S(123)$$

$${}^{sp}L \cdot A(123)$$

$${}^{sp}G \cdot S(123)$$

$${}^{sp}Q_1 \cdot F_2(123) - p$$

$${}^{sp}Q_2 \cdot F_1(123) - p.$$

$S_3 \quad S_1 \quad S_2$

Restatement of the FB and F(1-2) analyses in terms of $S(23)_3$, $S(23)_1$ and $S(23)_2$.

1. Use the table just made to get these things:

FB $\cdot S(12) + \text{cyclic}$

$$\begin{aligned}
 &= \sum_{\text{sp}} L \cdot \left(-\frac{1}{12}\right) \left[S_3 \{xz+yx+zy+2x^2+2y^2+2z^2\} \right. \\
 &\quad \left. + S_1 \{-2xy+yx+zy-4x^2+2y^2+2z^2\} + S_2 \{yx-zy+2y^2-2z^2\} \right] \\
 &+ \sum_{\text{sp}} G \cdot \left(-\frac{1}{6}\right) \left[S_3 \{xz+yx+zy\} + S_1 \{-2xz+yx+zy\} + S_2 \{yx-zy\} \right] \\
 &+ \sum_{\text{sp}} Q_1 \cdot \left(-\frac{1}{24}\right) \left[S_3 \{2xy-yx-zy+4x^2-2y^2-2z^2\} \right. \\
 &\quad \left. + S_1 \{-4xz-yx-zy-8x^2-2y^2-2z^2\} + S_2 \{-yx+zy+2y^2-2z^2\} \right] - p \\
 &+ \sum_{\text{sp}} Q_2 \cdot \left(\frac{-1}{4\sqrt{3}}\right) \left[S_3 \{yx-zy\} + S_1 \{yx-zy\} + S_2 \{yx+zy\} \right] - p \\
 &+ \sum_{\text{sp}} L_1 \cdot \left(-\frac{1}{3}\right) \left[S_3 \{3(x+y+z)\} + S_1 \{3(z-y)\} + S_2 \{2x-y-z\} \right] \\
 &+ \sum_{\text{sp}} G_1 \cdot \left(\frac{1}{6}\right) \left[S_3 \{z+x+y\} + S_1 \{2z+x+y\} + S_2 \{x-y\} \right] \\
 &+ \sum_{\text{sp}} Q_1 \cdot \left(-\frac{1}{6}\right) \left[\quad \quad \quad \right] - p \\
 &+ \sum_{\text{sp}} Q_2 \cdot \left(\frac{1}{4\sqrt{3}}\right) \left[\quad \quad \quad \right] - p
 \end{aligned}$$

I have made some errors of putting 2's in the wrong way. But pause to figure out: what are you going to do with this thing????

2. Actually, there are two separate Ward null terms you can write like this:

$$FB \cdot S(12) + c_y = FB[S_3 - 2S_1] + c_y$$

$$= (FB + c_y) S_3 \quad \#1$$

$$+ \{S_1 [-2(FB)_2] - \text{partners}\} \quad \#2$$

Ie, you can consider these two terms completely separately; each is a totally anti symmetric ward null tensor. Each term "feeds" into all 8 terms on the previous page. A similar statement can be made about the $FB(1-2)$ term, except that one feeds only into four terms. Lets consider these together, schematically;

$$FB[S_3] \rightarrow \delta_p L + \delta_p G + \cancel{\delta_p Q_1} + \delta_p Q_2 + {}^3p L + {}^3p G + {}^3p Q_1 + {}^3p Q_2$$

$$FB[S_1] \rightarrow \delta_p L + \delta_p G + \delta_p Q_1 + \delta_p Q_2 + {}^3p L + {}^3p G + \cancel{{}^3p Q_1} + \cancel{{}^3p Q_2}$$

$$F(12)[S_3] \rightarrow \delta_p G + \cancel{\delta_p Q_2} + {}^3p G$$

$$F(1-2)[S_1] \rightarrow \delta_p G + \delta_p Q_2 + \cancel{{}^3p Q_2}$$

So this is our "feed list" to these terms.

3. Comment: Notice that $F(1-2)S_3$ does not feed into ${}^3p Q_2$. Why not? Because the S_3 object does not appear there. In general, if there are multiplying functions you seem to get all components in a given term; but if there are no multiplying functions you have to be careful because you might only get S_3 and not the $S_1 S_2$ grouping. Or vice versa. Thus see lists of feed above.

4. Well, I would like to replace the four irreducibles slashed in pencil with the four F-related terms. To really check the possiblitiy I have to face up to the "feedback problem" which is this: each time you have to project into the new list! It may, in other words, not be possible to slimate things you thought you could. This is all discussed clearly in the "spinlover theorem" section.

But suppose we just assume you can do it; probablities are with you.

Then in effect, you are using the F replacements to kill off both those -- all four mixed terms. Now use the other ward null, the BBB, to kill off some other term of your choice. Of course it has to be something that at least appears in BBB; lets take the G_1 . Then we are down to this:

$$\begin{array}{l}
 \text{null} \left\{ \begin{array}{l} \text{BBB} \\ \text{FB} \end{array} \right. \\
 + \\
 \text{non-null} \left\{ \begin{array}{l} F_{12}^{12} (1-2)^3 S(12) + \text{cyc.} \\ + L_1 \cdot A(123) \\ + L_2 \cdot A(123) \\ + L \cdot A(123) \\ + G \cdot S(123) \end{array} \right. \begin{array}{l} (2) \\ (1) \\ (1) \\ (1) \\ (1) \end{array}
 \end{array}$$

Time to end: I thought I was going to come up with a "better" list for the non-n sector. My plan was to always eliminate the mixed pair terms and work with irreducibles.

Now after 8 hours work that just seems not to pay off. I already have the best non-n list possible, I think.

the n's

(all classes considered, momentum conservation ignored)

Make Up A Complete Set of tensor forms for the n-terms.

1. I guess I will try and duplicate my previous method: ie, ignore momentum conservation and make all forms, then impose it and see what is independent.

2. $g^{ij}n^k$ class. This seems pretty simple. These objects have the same symmetry as the objects $d^{ij}n^k$ so I can read off the $(d^{12}n^3)$ table to get:

raw forms: $\delta^{12}n^3, \delta^{23}n^1, \delta^{31}n^2$

def sym: $\frac{1}{2}(\delta^{12}n^3)_s = [\delta^{12}n^3 + \text{cyclic}]$

$$\left\{ \begin{aligned} \frac{1}{2}(\delta^{12}n^3)_1 &= [2\delta^{12}n^3 - \delta^{23}n^1 - \delta^{31}n^2] \\ \frac{1}{2}(\delta^{12}n^3)_2 &= -\sqrt{3}[\delta^{23}n^1 - \delta^{31}n^2] \end{aligned} \right.$$

3. $N^i n^j n^k$ triple-n class. Also simple.

$$\frac{1}{6}(nnn)_s = nnn$$

/one raw, one def sym

4. The double-n class. The raw-terms are like nnp_i and have the same symmetry as the deltas. You can think of $n^i n^j$ as d^{ij} symmetrized. So here we are:

raw forms: $nn1, nn2, nn3; n1n, n2n, n3n; 1nn, 2nn, 3nn$.

def sym: $(nn1)_A = nn(1-2) + (2-3)n1 + n(3-1)n$

$$(nn1)_s = nn(1+2) + (2+3)n1 + n(3+1)n$$

order
wrong
but ok

$$\left\{ \begin{aligned} (nn1)_1 &= 2nn(1+2) - (2+3)n1 - n(3+1)n \\ (nn1)_2 &= -[(2+3)n1 - n(3+1)n]\sqrt{3} \end{aligned} \right.$$

$$\left\{ \begin{aligned} (nn1)_1^2 &= [(2-3)n1 - n(3+1)n]\sqrt{3} \\ (nn1)_2^2 &= 2nn(1-2) - (2-3)n1 - n(3-1)n \end{aligned} \right.$$

$$\frac{1}{2}(nn3)_s = nn3 + n2n + 1nn$$

$$\left\{ \begin{aligned} \frac{1}{2}(nn3)_1 &= [2nn3 - 1nn - n2n] \\ \frac{1}{2}(nn3)_2 &= -[1nn - n2n]\sqrt{3} \end{aligned} \right.$$

5. By the way, I am starting to write pairs in a standard notation like so:

$$\begin{cases} F_1^1 = [2A - B - C] \\ F_2^1 = -[B - C] \text{ SQRT}(3) \end{cases}$$

$$\begin{cases} F_1^2 = [B - C] \text{ SQRT}(3) \\ F_2^2 = [2A - B - C] \end{cases}$$

6. Now on to the npp terms. This category seems to be essentially new and requires some work. The raw terms are easy to list off, there are 27 of them:

$n_{11}, n_{12}, n_{13}, n_{21}, n_{22}, n_{23}, n_{31}, n_{32}, n_{33}$ and 9 with n in the middle
and 9 with n on the right.

Here are the effects of permutation operators on these things:

	<u>e</u>	<u>(12)</u>	<u>(13)</u>	<u>(23)</u>	<u>(123)</u>	<u>(132)</u>	
n_{11}	n_{11}	$2n_2$	$3n_3$	n_{11}	$2n_2$	$3n_3$	3
n_{22}	n_{22}	$1n_1$	$2n_3$	n_{33}	$3n_3$	$1n_1$	6
n_{23}	n_{23}	$1n_3$	$2n_2$	n_{23}	$1n_3$	$2n_2$	3
n_{12}	n_{12}	$2n_1$	$23n$	n_{31}	$3n_2$	$31n$	6
n_{21}	n_{21}	$1n_2$	$32n$	n_{13}	$2n_3$	$13n$	6
n_{32}	n_{32}	$3n_1$	$21n$	n_{32}	$3n_1$	$21n$	3

Each of our 27 raw forms appears at least once in this table, so
we know we have them all. We can now form the 27 tensor forms of
definite symmetry. No big problem, just do it.

27

Here follows a list of the 27 npp forms of definite symmetry:

$$\frac{1}{2}(n_{11})_S = n_{11} + 2n_2 + 3n_3 \quad \checkmark \quad = n_{11} + 2n_2 + 3n_3$$

$$\frac{1}{\sqrt{3}}(n_{11})_1^2 = -[2 \cdot 3n_3 - n_{11} - 2n_2] = n_{11} + 2n_2 - 2 \cdot 3n_3$$

$$\frac{1}{3}(n_{11})_2^2 = [n_{11} - 2n_2] = n_{11} - 2n_2$$

$$\frac{1}{2}(n_{23})_S = n_{23} + 1n_3 + 12n \quad \checkmark \quad = n_{23} + 1n_3 + 12n$$

$$\frac{1}{\sqrt{3}}(n_{23})_1^2 = -[2 \cdot 12n - n_{23} - 1n_3] = n_{23} + 1n_3 - 2 \cdot 12n$$

$$\frac{1}{3}(n_{23})_2^2 = [n_{23} - 1n_3] = n_{23} - 1n_3 \quad / \text{like } abc, abc \uparrow$$

$$\frac{1}{2}(n_{32})_S = n_{32} + 3n_1 + 21n \quad \checkmark \quad = n_{32} + 3n_1 + 21n$$

$$\frac{1}{\sqrt{3}}(n_{32})_1^2 = -[2 \cdot 21n - n_{32} - 3n_1] = n_{32} + 3n_1 - 2 \cdot 21n$$

$$\frac{1}{3}(n_{32})_2^2 = [n_{32} - 3n_1] = n_{32} - 3n_1$$

$$(n_{22})_S = n_{22} + 1n_1 + 22n + n_{33} + 3n_3 + 11n \quad \checkmark$$

$$(n_{22})_A = n_{22} - 1n_1 - 22n - n_{33} + 3n_3 + 11n$$

$$(n_{22})_1' = 2 \cdot n_{22} + 2 \cdot 1n_1 - 22n - n_{33} - 3n_3 - 11n$$

$$\frac{1}{\sqrt{3}}(n_{22})_2' = 0 \quad 0 \quad -22n + n_{33} - 3n_3 + 11n$$

$$\frac{1}{\sqrt{3}}(n_{22})_1^2 = 0 \quad 0 \quad -22n + n_{33} + 3n_3 - 11n$$

$$(n_{22})_2^2 = 2 \cdot n_{22} - 2 \cdot 1n_1 + 22n + n_{33} - 3n_3 - 11n$$

$$(n_{12})_S = n_{12} + 2n_1 + 23n + n_{31} + 3n_2 + 31n \quad \checkmark$$

$$(n_{12})_A = n_{12} - 2n_1 - 23n - n_{31} + 3n_2 + 31n$$

$$(n_{12})_1' = 2 \cdot n_{12} + 2 \cdot 2n_1 - 23n - n_{31} - 3n_2 - 31n$$

$$\frac{1}{\sqrt{3}}(n_{12})_2' = 0 \quad 0 \quad -23n + n_{31} - 3n_2 + 31n$$

$$\frac{1}{\sqrt{3}}(n_{12})_1'' = 0 \quad 0 \quad -23n + n_{31} + 3n_2 - 31n$$

$$(n_{12})_2'' = 2 \cdot n_{12} - 2 \cdot 2n_1 + 23n + n_{31} - 3n_2 - 31n$$

$$(n_{21})_S = n_{21} + 1n_2 + 32n + n_{13} + 2n_3 + 13n \quad \checkmark$$

$$(n_{21})_A = n_{21} - 1n_2 - 32n - n_{13} + 2n_3 + 13n$$

$$(n_{21})_1' = 2 \cdot n_{21} + 2 \cdot 1n_2 - 32n - n_{13} - 2n_3 - 13n$$

$$\frac{1}{\sqrt{3}}(n_{21})_2' = 0 \quad 0 \quad -32n + n_{13} - 2n_3 + 13n$$

$$\frac{1}{\sqrt{3}}(n_{21})_1'' = 0 \quad 0 \quad -32n + n_{13} + 2n_3 - 13n$$

$$(n_{21})_2'' = 2 \cdot n_{21} - 2 \cdot 1n_2 + 32n + n_{13} - 2n_3 - 13n$$

add:

$$(n_{11})_1'' = \sqrt{3}(n_{11})_1'$$

$$(n_{23})_1'' = \sqrt{3}(n_{23})_1'$$

$$(n_{32})_1'' = \sqrt{3}(n_{32})_1'$$

Application of Momentum Conservation to the npp class.

1. First, how many independent tensor forms of definite symmetry do we expect?

$n_{11} \ n_{12} \ n_{21} \ n_{22} / \ l_{n1} \ l_{n2} \ 2n_1 \ 2n_2 / \ 11n \ 12n \ 21n \ 22n$

Since there are 12 independent raw forms, we expect 12 independent tensors of definite symmetry. Thus each tensor is now a 12-component vector! (Get ready REDUCE!)

2. So I guess the first thing to do is classify each raw form into a 12-vector?

	n_{11}	n_{12}	n_{21}	n_{22}	l_{n1}	l_{n2}	$2n_1$	$2n_2$	$11n$	$12n$	$21n$	$22n$	
n_{11}	+												
n_{12}		+											
n_{13}	-	-											
n_{21}			+										
n_{22}				+									
n_{23}			-	-									
n_{31}	-		-										
n_{32}		-		-									
n_{33}	+	+	+	+									
l_{n1}					+								
l_{n2}						+							
l_{n3}					-	-							
$2n_1$							+						
$2n_2$								+					
$2n_3$							-	-					
$3n_1$					-		-	-					
$3n_2$						-		-					
$3n_3$					+	+	+	+					
$11n$									+			$11n$	
$12n$										+		$12n$	
$13n$									-	-		$13n$	
$21n$											+	$21n$	
$22n$												+	$22n$
$23n$											-	-	$23n$
$31n$									-		-		$31n$
$32n$										-		-	$32n$
$33n$									+	+	+	+	$33n$

-2-

add

$$(122)_S$$

$$= (1 \ 1 \ 1 \ 2 \ 2 \ 1 \ 1 \ 1 \ 1 \ 0 \ 0 \ 1)$$

$$(n22)_A$$

$$= (-1 -1 -1 \textcircled{0} \textcircled{0} 1 1 1 1 1 0 0 \textcircled{4})$$

$$(\pi_{22})_1$$

$$\rightarrow = \begin{pmatrix} -1 & -1 & -1 & 1 & 1 & -1 & -1 & -1 \\ 1 & 0 & 0 & -1 \end{pmatrix}$$

$$\frac{1}{\sqrt{3}}(n_{22})_2$$

$$\downarrow = \begin{pmatrix} 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 0 & 0 & 0 \end{pmatrix} +$$

$$\frac{1}{\sqrt{3}}(xyz)^2$$

$$= \begin{pmatrix} 1 & 1 & 1 & 1 & i & i & 1 & 1 & -1 & 0 & 0 \\ - & & & & & & & & & & \end{pmatrix}$$

$(n|z)_2^2$

2

-2

-4-

1

2

3

4

5

6

$\downarrow (-1 \ 2 \ -1 \ 0 \ 0 \ 1 \ -2 \ 1 \ 1 \ 0 \ 0 \ -1) \checkmark$

$(n|z)_s$

+

+

1. ✓

2. ✓

3. ✓

4. ✓

5. ✓

6. ✓

$\downarrow (-1 \ -1 \ 1 \ 0 \ 0 \ 1 \ -1 \ -1 \ -1 \ -2 \ 0 \ -1) \checkmark$

$(n|z)_A$

+

-

+

+

1

2

3

4

5

6

$\downarrow (1 \ 1 \ 1 \ 0 \ 0 \ -1 \ -1 \ -1 \ -1 \ 0 \ 0 \ 1) \checkmark$

$\frac{1}{\sqrt{3}}(n|z)_1^2$

-

+

+

3

4

5

6

$\downarrow (-1 \ -1 \ 0 \ 0 \ 0 \ 0 \ -1 \ -1 \ 1 \ 2 \ 0 \ 1) \checkmark$

$\frac{1}{\sqrt{3}}(n|z)_2^1$

-

+

+

3

4

5

6

$\downarrow (-1 \ -1 \ 0 \ 0 \ 0 \ 0 \ 1 \ 1 \ -1 \ 0 \ 0 \ 1) \checkmark$

$(n_{21})_1$

2

2

1

2

3

4

5

6

+

+

+

+

+

+

+

+

$$\rightarrow (1 \ 1 \ 2 \ 0 \ 0 \ 2 \ 1 \ 1 \ 1 \ 2 \ 0 \ 1)$$

✓

✓

$(n_{21})_2$

2

-2

1

2

3

4

5

6

-

-

+

+

+

+

$$\rightarrow (-1 \ -1 \ 2 \ 0 \ 0 \ -2 \ 1 \ 1 \ 1 \ 0 \ 0 \ -1)$$

✓

✓

$\frac{1}{2}(n_{23})_5$

-

-

1

2

3

-

-

+

$$\rightarrow (0 \ 0 \ -1 \ -1 \ -1 \ -1 \ 0 \ 0 \ 0 \ 1 \ 0 \ 0)$$

✓

$\frac{1}{\sqrt{3}}(n_{23})_1$

-

-

1

2

3

-

-

-2

$$\rightarrow (0 \ 0 \ -1 \ -1 \ -1 \ -1 \ 0 \ 0 \ 0 \ -2 \ 0 \ 0)$$

✓

✓

$\frac{1}{3}(n_{23})_2$

-

-

1

2

3

-

-

+

+

$$\rightarrow (0 \ 0 \ -1 \ -1 \ 1 \ 1 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0)$$

✓

✓

$$\frac{1}{2}(n_{11})_5 +$$

$$\downarrow (1 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 1 \ 1 \ 1 \ 1 \ 1)$$

1
2
3

$$\frac{1}{\sqrt{3}}(n_{11})_1^2 +$$

$$\downarrow (1 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 1 \ -2 \ -2 \ -2 \ -2)$$

1
2
3

$$\frac{1}{3}(n_{11})_2^2 +$$

$$\downarrow (1 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ -1 \ 0 \ 0 \ 0 \ 0)$$

1
2
3

$$\frac{1}{2}(n_{32})_5 -$$

$$\downarrow (0 \ -1 \ 0 \ -1 \ -1 \ 0 \ -1 \ 0 \ 0 \ 0 \ 1 \ 0)$$

1
2
3

$$\frac{1}{\sqrt{3}}(n_{32})_1^2 -$$

$$\downarrow (0 \ -1 \ 0 \ -1 \ -1 \ 0 \ -1 \ 0 \ 0 \ 0 \ -2 \ 0)$$

1
2
3

$$\frac{1}{3}(n_{32})_2^2 -$$

$$\downarrow (0 \ -1 \ 0 \ -1 \ 1 \ 0 \ 1 \ 0 \ 0 \ 0 \ 0 \ 0)$$

1
2
3

$(n_{22})_S$ ✓	1	1	1	2	2	1	1	1	0	0	1	L_1
$(n_{12})_S$ ✓	-1	1	-1	0	0	-1	1	-1	-1	0	-2	L_2
$(n_{21})_S$ ✓	-1	-1	1	0	0	1	-1	-1	-1	-2	0	L_3
$(n_{23})_S/2$ ✓	0	0	-1	-1	-1	-1	0	0	0	1	0	$(-L_1 - L_3)/2$
$(n_{32})_S/2$ ✓	0	-1	0	-1	-1	0	-1	0	0	0	1	$(-L_1 - L_2)/2$
$(n_{11})_S/2$ ✓	1	0	0	0	0	0	0	1	1	1	1	$(-L_2 - L_3)/2$
$(n_{22})_A$ ✓	-1	-1	-1	0	0	1	1	1	1	0	0	$+\bar{A}_1$
$(n_{12})_A$ ✓	1	1	1	0	0	-1	-1	-1	-1	0	0	$\equiv -\bar{A}_1$
$(n_{21})_A$ ✓	1	1	1	0	0	-1	-1	-1	-1	0	0	$\equiv -\bar{A}_1$
} only one A-type.												
$(n_{22})'_1$ ✓	-1	-1	-1	1	1	-1	-1	-1	-1	0	0	Q_1
$(n_{22})'^2_1/\sqrt{3}$ ✓	1	1	1	1	1	1	1	1	-1	0	0	$Q_1 + Q_3 - (Q_4/\sqrt{3})$
$(n_{12})'_1$ ✓	1	2	1	0	0	1	2	1	1	0	2	Q_2
$(n_{12})'^2_1/\sqrt{3}$ ✓	-1	0	-1	0	0	-1	0	-1	1	0	2	$Q_2 - Q_3 + (Q_4/\sqrt{3})$
$(n_{21})'_1$ ✓	1	1	2	0	0	2	1	1	1	2	0	Q_3
$(n_{21})'^2_1/\sqrt{3}$ ✓	-1	-1	0	0	0	0	-1	-1	1	2	0	$Q_4/\sqrt{3}$
$(n_{23})'^2_1/\sqrt{3}$ ✓	0	0	-1	-1	-1	-1	0	0	0	-2	0	$-Q_1 - Q_3$
$(n_{11})'^2_1/\sqrt{3}$ ✓	1	0	0	0	0	0	0	1	-2	-2	-2	$-Q_2 + \frac{1}{2}Q_3 - \frac{3}{2}(Q_4/\sqrt{3})$
$(n_{32})'^2_1/\sqrt{3}$ ✓	0	-1	0	-1	-1	0	-1	0	0	0	-2	$-Q_1 - Q_2$
✓ ✓ ✓ ✓ ✓ ✓ ✓ ✓												
$(n_{22})'^2_2/\sqrt{3}$ ✓	1	1	1	1	-1	-1	-1	-1	1	0	0	
$(n_{22})'^2_2$ ✓	1	1	1	3	-3	-1	-1	-1	-1	0	0	
$(n_{12})'^2_2/\sqrt{3}$ ✓	-1	0	-1	0	0	1	0	1	-1	0	0	
$(n_{12})'^2_2$ ✓	-1	2	-1	0	0	1	-2	1	1	0	0	
$(n_{21})'^2_2/\sqrt{3}$ ✓	-1	-1	0	0	0	0	1	1	-1	0	0	
$(n_{21})'^2_2$ ✓	-1	-1	2	0	0	-2	1	1	1	0	0	
$(n_{23})'^2_2/3$ ✓	0	0	-1	-1	1	1	0	0	0	0	0	$\tilde{O}_4$
$(n_{11})'^2_2/3$ ✓	1	0	0	0	0	0	0	-1	0	0	0	$\tilde{O}_3/3$
$(n_{32})'^2_2/3$ ✓	0	-1	0	-1	1	0	1	0	0	0	0	$\tilde{O}_2/3$
												$\tilde{O}_1/3$

New chart:

$$(n_{22})_S = L_1$$

$$(n_{12})_S = L_2$$

$$(n_{21})_S = L_3$$

$$(n_{23})_S = -L_1 - L_3$$

$$(n_{32})_S = -L_1 - L_2$$

$$(n_{11})_S = -L_2 - L_3$$

$$(n_{22})_A = -A_1$$

$$(n_{12})_A = +A_1$$

$$(n_{21})_A = +A_1$$

$$(n_{22})'_1 = Q_1$$

$$(n_{22})'^2_1 = \sqrt{3}Q_1 + \sqrt{3}Q_3 - Q_4$$

$$(n_{12})'_1 = Q_2$$

$$(n_{12})'^2_1 = \sqrt{3}Q_2 - \sqrt{3}Q_3 + Q_4$$

$$(n_{21})'_1 = Q_3$$

$$(n_{21})'^2_1 = Q_4$$

$$(n_{23})'^2_1 = -\sqrt{3}Q_1 - \sqrt{3}Q_3$$

$$(n_{11})'^2_1 = -\sqrt{3}Q_1 - (\sqrt{3}/2)Q_3 - (3/2)Q_4$$

$$(n_{32})'^2_1 = -\sqrt{3}Q_1 - \sqrt{3}Q_2$$

$$/(n_{22})'_2 = \tilde{Q}_1, \text{ etc.}$$

add: $(n_{11})'_1 = -Q_1 - \frac{1}{2}Q_3 - \frac{\sqrt{3}}{2}Q_4$

$$(n_{23})'_1 = -Q_1 - Q_3$$

$$(n_{32})'_1 = -Q_1 - Q_2$$

Apply Momentum Conservation to the Other Three Classes of n-Terms.

1. First; the triple-n class. Momentum conservation has no effect here since all nnn . The term is in a class by itself.

2. Now the delta-n class. Again no p's appear so stays "as is".

3. The nnp class. Make a vector table and do it:

	$nn1$	$nn2$	nnn	$n3n$	$1nn$	$2nn$
$(nn1)_S$	1	1	0	-1	-1	0
$(nn1)_A$	1	-1	-2	-1	-1	2
$(nn1)_1$	2	2	0	1	1	0
$\frac{1}{\sqrt{3}}(nn1)_2$	0	0	0	-1	1	0
$(nn1)_2^2$	2	-2	+2	+2	-1	-2
$\frac{1}{\sqrt{3}}(nn1)_1^2$	0	0	+2	+1	1	2
$\frac{1}{2}(nn3)_S$	-1	-1	0	1	1	0
$\frac{1}{2}(nn3)_1$	-2	-2	0	-1	-1	0
$\frac{1}{\sqrt{3}}\frac{1}{2}(nn3)_2$	0	0	0	+1	-1	0

So:

$(nn1)_S$	1	1	0	-1	-1	0	S_1
$(nn3)_S/2$	-1	-1	0	1	1	0	$-S_1$
$(nn1)_A$	1	-1	-2	-1	-1	2	A_1
$(nn1)_1$	2	2	0	1	1	0	O_1
$(nn1)_2^2/\sqrt{3}$	0	0	2	1	1	2	$O_2/\sqrt{3}$
$(nn3)_1/2$	-2	-2	0	-1	-1	0	$-O_1$
$(nn1)_2^2/\sqrt{3}$	0	0	0	-1	1	0	$\tilde{O}_1/\sqrt{3}$
$(nn1)_2^2$	2	-2	2	1	-1	-2	$\tilde{O}_2$
$(nn3)_2/(2\sqrt{3})$	0	0	0	1	-1	0	$-\tilde{O}_1/\sqrt{3}$

4. So, we now know exactly what independent tensors we are dealing with.
 Lets try to make a complet list using some kind of reasonable notation :

nnn class :

$$(nnn)_s$$

δn class :

$$(\delta n)_s \leftarrow \begin{cases} (\delta n)_1^1 \\ (\delta n)_2^2 \end{cases}$$

nnp class :

$$(nni)_s \leftarrow \begin{cases} (nni)_1^1 \\ (nni)_2^2 \end{cases} \quad \begin{cases} (nni)_1^2 \\ (nni)_2^2 \end{cases}$$

$$(nni)_A \leftarrow$$

npp class

$$\begin{array}{lll} (nii)_s & \begin{cases} (n32)_1^2 \\ (n32)_2^2 \end{cases} & \begin{cases} (n23)_1^2 \\ (n23)_2^2 \end{cases} \\ (n32)_s & & \\ (n23)_s & & \\ (n22)_A & \begin{cases} (nii)_1^2 \\ (nii)_2^2 \end{cases} & \begin{cases} (n2i)_1^2 \\ (n2i)_2^2 \end{cases} \end{array}$$

22 independent
n-terms

14 nnn-n

All in all, among the n-terms we are able to construct:

- 6 Symmetric tensor forms
- 2 Antisymmetric tensor forms
- 7 Mixed Pairs.

better npp choices

$$\begin{array}{lll} (n22)_s \leftarrow & \begin{cases} (n22)_1^1 \\ (n22)_2^2 \end{cases} & \begin{cases} (n12)_1^1 \\ (n12)_2^2 \end{cases} \\ (n12)_s \leftarrow & & \\ (n21)_s \leftarrow & & \\ (n21)_A \leftarrow & \begin{cases} (n21)_1^1 \\ (n21)_2^2 \end{cases} & \begin{cases} (n21)_1^2 \\ (n21)_2^2 \end{cases} \end{array}$$

indep. ok!

arrows show "absorption"

5. This concept of "absorption" allows you to "absorb" all the mixed symmetry terms into the S and A terms in a certain way. The key to this is shown on page 7 of the S_3 notes:

$$T_1^1 f_2 - \text{partners} = (\text{first two terms in T added}) f_2 + \text{cyclic}$$

$$T_2^2 f_1 - \text{partners} = (\text{first term} - \text{second term T}) f_1 + \text{cyclic}$$

By generalizing f_2 to be an arbitrary $A(12)$ type function, you include T_1

By generalizing f_1 to be an arbitrary $S(12)$ type function, you include T_2 .

6. So allowing for all the absorption, here is a shorter list of general n-terms vertex:

$$nnn \times A_1(123)$$

$$[\delta^{12} n^3 \times A_2(12) + \text{cyclic}]$$

$$[nn3 \times A_3(12) + \text{cyclic}]$$

$$S [nn(1-2) \times S_1(12) + \text{cyclic}]$$

$$(n22 + 1n1) A_4(12) + \text{cyclic}$$

$$(n12 + 2n1) A_5(12) + \text{cyclic}$$

$$(n21 + 1n2) A_6(12) + \text{cyclic}$$

$$S (n21 - 1n2) S_2(12) + \text{cyclic}$$

you can see the 22 terms

This is a form for the general n-terms vertex. Probably we will modify it to make Ward mulls fit nicely, but we have a reasonable result after all this work: there are 8 functions in there. Compare to 6 functions for the non-n-terms vertex.

Construction of Ward Null terms involving n.

1. We know all the ward nulls that have no n's. The idea here is to make up some objects like the F_{12}^{12} thing and vectors like the B thing and piece them together.

$$G_{12}^{12} \equiv (1 \cdot n) \delta^{12} - n^1 1^2 \quad 1^1 G_{12}^{12} = G_{12}^{12} n^2 = 0$$

This thing is only perp to one momenta, but it eats up two indices. Thus, even with a B_3 type thing you cannot make it so each term of the cyclic is perp to 3^3 . So lets not use this.

2. We can make some vectors howere:

$$D_1^i = (n \cdot 1) 1^i - (1 \cdot 1) n^i \quad D_1^i 1^i = 0$$

$$D_1^1 = (n \cdot 1) 1^1 - (1 \cdot 1) n^1 \quad D_1^1 1^1 = 0$$

$$D_2^2 = (n \cdot 2) 2^2 - (2 \cdot 2) n^2 \quad D_2^2 2^2 = 0$$

$$D_3^3 = (n \cdot 3) 3^3 - (3 \cdot 3) n^3 \quad D_3^3 3^3 = 0$$

This suggests

$$D_1^1 D_2^2 D_3^3 \times A(123) = \text{ward null.}$$

$$F_{12}^{12} D_3^3 \times A(12) + \text{cyclic} = \text{another one.}$$

Then there:

$$D_1^1 D_2^2 B_3^3 \times S(12) + \text{cyclic}$$

$$D_1^1 B_2^2 B_3^3 \times A(23) + \text{cyclic}$$

← agrees with
no old list.

So here are four new Ward Transverse terms, which means we now have a total of 6 of them. Hopefully this is all there are, we shall eventually find out.

3. Now the problem is one of recasting the general form of the vertex into a form that somehow includes these guys; series of spillover manoeuvres.

Insertion of Ward Transverse terms into our standard form.

1. Since we are going to have lots of spillover, we should know exactly what all our terms are at any given time, both n and non- n terms. Here is a starting list:

$$N F_{12}^{12} B_3^3 S(12) + c. \quad T_1$$

$$F_{12}^{12} (1-2)^3 S(12) + c. \quad T_2$$

$$123 \cdot A(123) \quad T_3$$

$$N \cdot B_1 B_2 B_3 \cdot S(123) \quad T_4$$

$$\delta^{12} 3^3 A(12) + c. \quad T_5$$

$$\delta^{12} (1-2)^3 S(12) + c. \quad T_6$$

$$[nnn \cdot A(123) \quad T_7] \rightarrow DDD \cdot A(123)$$

$$[\delta^{12} n^3 \cdot A(12) + c. \quad T_8] \rightarrow FDA(12) + c.$$

$$nn3 \cdot A(12) + c. \quad T_9$$

$$[nn(1-2) \cdot S(12) + c. \quad T_{10}] \rightarrow DDB \cdot S(12) + c.$$

$$(n_{22} + 1n_1) A(12) + c. \quad T_{11}$$

$$(n_{12} + 2n_1) A(12) + c. \quad T_{12}$$

$$(n_{21} + 1n_2) A(12) + c. \quad T_{13}$$

$$[(n_{21} - 1n_2) S(12) + c. \quad T_{14}]$$

$\delta^{12} DDB \cdot A(123) + c.$
not quite here, see App A

Note: this is equivalent to $(n_{22} - 1n_1) S(12)$
or $(n_{12} - 2n_1) S(12)$

+ serves as "partner" to T_{11}, T_{12}, T_{13} .

A rather extensive list. I have not bothered to give functions names since we are just doing overlap.

2. Lets first consider the DDD term since this is the only term which contains nnn . (the only Ward null that I know of). Expand to find that:

$$D_1 D_2^2 D_3^3 = (n \cdot 1)(n \cdot 2)(n \cdot 3) 123 \rightarrow T_3$$

$$+ (n \cdot 3)(1 \cdot 1)(2 \cdot 2) n n 3 + c. \rightarrow T_9$$

$$+ (n \cdot 2)(n \cdot 3)(1 \cdot 1) n 2(1+2) + c. \rightarrow ???$$

$$- (1 \cdot 1)(2 \cdot 2)(3 \cdot 3) n n n \rightarrow T_7$$

It is at once clear that this term overlaps into T_3, T_9 and T_7 ; it is not quite clear what to do with the ??? term.

Perhaps I should not worry. Obviously this is a single- n term, ie, an npp ter, and since we claim to have a complete basis for such terms, it must somehow be absorbed into T_{11} through T_{14} . Exactly how it is absorbed is not important (although it would be esthetically nice to check to make sure it is absorbable). I will save this for an appendix maybe and get on with it.

So we simply replace the nnn term by this DDD term. The overlaps into terms 3,9,7,11,12,13,14 simply redefine those functions (hence a good reason not to give them names yet!). Indicate this on the page:

3. Lets redo the above absorption more schematically. We had:

$$DDD = nnn + nnp + npp + ppp$$

Since our basis is complete, we know we can absorb each of these terms somewhere. We see clearly that term T_7 is in there as the nnn, so we can replace that term.

4. Now lets try the same schematic idea with:

$$DDB = nnp + npp + ppp$$

Again we know that each term can be absorbed somewhere. We have not changed our original 'nnp absorption capability' or that for npp or ppp. So far we have only fiddled with the nnn. Now, since no further nnp terms will arise from later Ward null terms, we would like to put this DDB into an nnp. But now we need more detail because we must know exactly which nnp terms are spilled into. We only need one of them, then we can replace that one.

5. Here is the explicit nnp piece of the DDB term:

$$\begin{aligned}
 & (1.1)(2.2)(3.2) \quad \textcircled{nn1} \rightarrow \frac{1}{2} [nn(1+2) + nn(1-2)] \\
 & - (1.1)(2.2)(3.1) \quad \textcircled{nn2} \rightarrow \frac{1}{2} [nn(1+2) - nn(1-2)] \\
 & = nn3 \left\{ -\frac{1}{2}(1.1)(2.2)(3.2) + \frac{1}{2}(1.1)(2.2)(3.1) \right\} \\
 & \quad + nn(1-2) \left\{ \frac{1}{2}(1.1)(2.2)(3.2) + \frac{1}{2}(1.1)(2.2)(3.1) \right\}
 \end{aligned}$$

Clearly this DDB term's nnp piece feeds into both T_9 and T_{10} so we could replace either of those by the DDB. There would be spillover in the npp and ppp classes which is just fine. I will choose to kill T_{10} , but may later decide on T_9 to kill.

6. Now here are how the remaining two terms look:

$$FD = d^{12}n^3 + d^{12}3^3 + 213 + 21n$$

$$DDB = npp + ppp$$

I can substitute FD for either $d^{12}n^3 = T_8$ or $d^{12}3^3 = T_5$. I have no idea which way is more efficient, so I will kill the T_8 on grounds that it leaves a term I have already considered once in non-n Ward analysis. Mark it in red.

7. That leaves us with only one more ward null to incorporate. Lets do an npp analysis of this term and then try to stuff this in the slot of one of our npp terms.

$$\begin{aligned}
 & - (1.1)(2.2)(3.2) \quad \textcircled{n11} \rightarrow \frac{1}{6}(n11)_S + \frac{1}{3\sqrt{3}}(n11)_1 + \frac{1}{6}(n11)_2 \\
 & - (1.1)(1.2)(3.1) \quad \textcircled{n22} \rightarrow \frac{1}{2}(n22+1n1) + \frac{1}{2}(n22-1n1) = \frac{1}{6}(n22)_S + \frac{1}{6}(n22)_A + \frac{1}{6}(n22)_1 + \frac{1}{6}(n22)_2 \\
 & + (1.1)(2.2)(3.1) \quad \textcircled{n12} \rightarrow \frac{1}{2}(n12+2n1) + \frac{1}{2}(n12-2n1) = \frac{1}{6}(n12)_S + \frac{1}{6}(n12)_A + \frac{1}{6}(n12)_1 + \frac{1}{6}(n12)_2 \\
 & + (1.1)(1.2)(3.2) \quad \textcircled{n21} \rightarrow \frac{1}{2}(n21+1n2) + \frac{1}{2}(n21-1n2) = \text{mix } T_{13} + T_{14}
 \end{aligned}$$

This is a difficult problem to see exactly which npp terms this analyses into. I think I will just guess that it goes into all of them so you can replace any one you want. This is very dangerous to assume and later I will have to come back and make sure. This will remain as the one loop hole in my argument.

Since DDB is (12) sym, I will replace one of the 3 sym term.

8. So, assuming that:

- the starting list on page 1 is a complete set for everything
- my DBB choice was OK

The general form of the vertex can be written like this:

1.	$F_{12}^{12} B_3 S(12) + c.$	3	$22 + 14 = 36$
2.	$B_1 B_2 B_3 S(123)$	1	
3.	$D_1 D_2 D_3 A(123)$	1	$36 - 14 = 22$ ✓
4.	$F_{12}^{12} D_3 A(12) + c.$	3	
5.	$DDB \cdot S(12) + c.$	3	
6.	$DBBA(23) + c.$	3	
		<u>19</u>	

non-null:

$$F_{12}^{12} (1-2)^3 S_2(12) + c.$$

$$(123) A(123)$$

$$\delta^{12} 3^3 A_1(12) + c.$$

$$\delta^{12} (1-2)^3 S_3(12) + c.$$

original non-n form

$$n^1 n^2 3^3 A_4(12) + c.$$

$$(n_1 2 + 2 n_1) A_5(12) + c.$$

$$(n_2 1 + 1 n_2) A_6(12) + c.$$

$$(n_2 2 + 1 n_1) A_7(12) + c.$$

new n-terms.

9. I count 22 functions of definite symmetry in this list. ($7 \times 3 + 1 = 22$). If there are to be no more ward null terms, there had better be 22 independent equations in the Ward identity; does not sound likely, that will be the next question to pursue.

22 still stands!

Appendix A: (not used, but same anyway)

1. Our goal here is to analyze the term $(n_{22}-n_{11})S(12) + \text{cyclic}$ into pieces which appear in our list of terms. My first thought on this is to analyze this thing by symmetry. Rather than make a long matrix inversion problem, you can easily analyse the tensor as follows:

$(n_{22}-n_{11})$ is like a $n A(12)$ object, thus it has only a and 2 pieces. So apply the X_a and X_{22} projectors to see what these pieces are. Look up in table 1es to see that $(n_{22})_a = -(n_{11})_a$ and similarly for the 22 projection. Thus, you got it.

Next remember that only the s and a pieces of something survive cyclicization. So you only need compute these pieces. Here is what I find:

$$(n_{22}-n_{11}) S(12) + c$$

$$= \frac{1}{2} \left\{ \underbrace{(n_{22})_A S(12)_s} + \frac{1}{2} \left(\underbrace{(n_{22})_2^2 S(12)_1} - \underbrace{\text{partners}} \right) \right\}$$

$$- (n_{21})_A$$

$$\begin{matrix} \underbrace{(n_{22})_2^2} & \underbrace{S(12)_1} \\ \downarrow & \downarrow \\ -\frac{2}{\sqrt{3}} S(23)_2 \end{matrix}$$

$$-(n_{23})_1^2 - (n_{21})_1^2 \leftarrow -O_3 - O_4 \leftarrow (n_{22})_1^2$$

So here is what seems to happen:

$$(n_{22}-n_{11}) S(12) + c$$

$$= (n_{21}-n_{12}) [-S(12)] + c$$

$$+ \frac{1}{2} \left((n_{23})_1^2 S(12)_1 - \underbrace{\text{partners}} \right)$$

"piece"

You can say that the $(n_{22}-n_{11})$ term "fits" into the $(n_{21}-n_{12})$ term, except you have this extra piece left over which must be stoked somewhere. Now consider:

$$(n_{23})_1^2 = O_3 \quad ; \quad (n_{22})_1' + (n_{21})_1' = -O_3/\sqrt{3} = -\frac{1}{\sqrt{3}} (n_{23})_1^2$$

$$\Rightarrow \text{piece} = -\frac{\sqrt{3}}{2} \left((n_{22})_1' S(12)_1 - p \right) - \frac{\sqrt{3}}{2} \left((n_{21})_1' S(12)_1 - p \right)$$

$$\Rightarrow \text{piece} = -\frac{\sqrt{3}}{2} \left\{ (2 \cdot n_{22} + 2 \cdot l_{n1}) \widetilde{S(12)}_1 \right\} + \text{cyc.}$$

$$- \frac{\sqrt{3}}{2} \left\{ (2 \cdot n_{21} + 2 \cdot l_{n2}) \widetilde{S(12)}_1 \right\} + \text{cyc.}$$

$$= (n_{22} + l_{n1}) \left[-\sqrt{3} \widetilde{S(12)}_1 \right] + c. = (n_{22} + l_{n1}) \left[2 S(23)_2 \right] + c.$$

$$+ (n_{21} + l_{n2}) \left[-\sqrt{3} \widetilde{S(12)}_1 \right] + c. \quad + "$$

$$= (n_{22} + l_{n1}) \left[S(23) - S(13) \right] + c.$$

$$+ (n_{21} + l_{n2}) \left[S(23) - S(13) \right] + c.$$

So here is our final theorem showing the analysis of this term:

$$(n_{22} - l_{n1}) S(12) + \text{cyc.}$$

$$= (n_{21} - l_{n2}) \left\{ -S(12) \right\} + \text{cyc.}$$

$$+ (n_{22} + l_{n1}) \left\{ S(23) - S(13) \right\} + \text{cyc.}$$

$$+ (n_{21} + l_{n2}) \left\{ S(23) - S(13) \right\} + \text{cyc.}$$

T_{14}

T_{11}

T_{13}

Status:

1. I have a tentative list of 8 terms (with 22 functions) for a general form of the vertex. However, I know that the Ward identity can give at most 20 equations. This seems to imply that there are at least 2 degrees of float. This in turn implies that there are at least 2 degrees worth of Ward nulls that I don't know about!! This certainly would be more interesting than having it come out like it did in the non-n' sector: all the ward nulls were obvious.

2. Counting.

non-n irreducible components 14

$$g_{uv}^p = 6$$

$$ppp = 8$$

n-term irreducible components 22

$$nnn = 1$$

$$g_{uv}^n = 3$$

$$nnp = 6$$

$$npp = 12$$

total irreducible comp's 36

Known Ward null degrees: - 14

: remaining components: 22

Ward identity: 20 equations max

There is a ward null hidden somewhere!!!!!!!!!!!!

but I counted wrong, so not here.

My starting list

OLD

$$F_{12}^{12} (1-2)^3 S_2(12) + F_{23}^{23} (2-3)^1 S_2(23) + F_{31}^{31} (3-1)^2 S_2(31)$$

$$(123) A(123)$$

$$\delta^{12} 3^3 A_1(12) + \delta^{23} 1^1 A_1(23) + \delta^{31} 2^2 A_1(31)$$

$$\delta^{12} (1-2)^3 S_3(12) + \delta^{23} (2-3)^1 S_3(23) + \delta^{31} (3-1)^2 S_3(31)$$

NEW

$$(nn3) A_4(12) + (1nn) A_4(23) + (n2n) A_4(31) \quad \checkmark$$

$$(n12 + 2n1) A_5(12) + (3n2 + 23n) A_5(23) + (31n + n31) A_5(31) \quad \checkmark$$

$$(n21 + 1n2) A_6(12) + (2n3 + 32n) A_6(23) + (13n + n13) A_6(31) \quad \checkmark$$

$$(n22 + 1n1) A_7(12) + (3n3 + 22n) A_7(23) + (11n + n33) A_7(31) \quad \checkmark$$

(see - 4 pages)

Dot 3^3 onto n-terms in the list (the last four terms).

1. I have already dotted and fractionated all terms in the non-n sector, so let's do the dotting for the n-sector.

2. First term:

$$3^3 [n^1 n^2 3^3 A_4(12) + 1^1 n^2 n^3 A_4(23) + n^1 2^2 n^3 A_4(31)]$$

$$= (3 \cdot 3) A_4(12) \cancel{n^1 n^2} + (3 \cdot n) A_4(23) \cancel{1^1 n^2} + (3 \cdot n) A_4(31) \cancel{n^1 2^2}$$

$$= (nn) \{ (3 \cdot 3) A_4(12) \}$$

$$+ (n2 + 1n) \left\{ \frac{(3 \cdot n) [A_4(31) + A_4(23)]}{2} \right\}$$

$$+ (n2 - 1n) \left\{ \frac{(3 \cdot n) [A_4(31) - A_4(23)]}{2} \right\}$$

3. The SEcond term (of the n-terms):

$$3^3 [(n^1 1^2 2^3 + 2^1 n^2 1^3) A_5(12) + (3^1 n^2 2^3 + 2^1 3^2 n^3) A_5(23)]$$

$$+ (3^1 1^2 n^3 + n^1 3^2 1^3) A_5(31)]$$

$$= (3 \cdot 2) A_5(12) \cancel{n^1 1^2} + (3 \cdot 1) A_5(12) \cancel{2^1 n^2}$$

$$+ (3 \cdot 2) A_5(23) \cancel{(3^1 n^2)} - [1^1 n^2 + 2^1 n^2]$$

$$+ (3 \cdot n) A_5(23) \cancel{(2^1 3^2)} - [2^1 1^2 + 2^1 2^2]$$

$$+ (3 \cdot n) A_5(31) \cancel{(3^1 1^2)} - [1^1 1^2 + 2^1 1^2]$$

$$+ (3 \cdot 1) A_5(31) \cancel{(n^1 3^2)} - [n^1 1^2 + n^1 2^2]$$

$$= n^1 1^2 [(3 \cdot 2) A_5(12) - (3 \cdot 1) A_5(31)] + n^1 2^2 [- (3 \cdot 1) A_5(31)]$$

$$+ 1^1 1^2 [-$$

next age.

$$11:) -(3 \cdot n) A_5(31) \checkmark$$

$$12:) \phi \checkmark$$

$$21:) -(3 \cdot n) A_5(23) \checkmark - (3 \cdot n) A_5(31) \checkmark$$

$$22:) -(3 \cdot n) A_5(23) \checkmark$$

$$3:) \phi \checkmark$$

= second term.

$$n1:) (3 \cdot 2) A_5(12) \checkmark - (3 \cdot 1) A_5(31) \checkmark$$

$$2n:) (3 \cdot 1) A_5(12) \checkmark - (3 \cdot 2) A_5(23) \checkmark$$

$$1n:) \quad \quad \quad - (3 \cdot 2) A_5(23) \checkmark$$

$$n2:) -(3 \cdot 1) A_5(31) \checkmark$$

$$nn:) \phi \checkmark$$

(6)

I do not like the way this term has scattered itself into 7 of the 10 possible equations. That means the next two terms are going to be equally messy.

4. This is going to be a nightmare! There will be too much cross-coupling between the 10 equation systems. I wonder if I should not have tried to kill these npp terms instead of the ones I did kill.....

5. So I just spent all day today trying to find a better way to do this, something without all that cross coupling: could not do it! This is the only route I have left, even though it is messy. Must therefore pursue it, or quit.

(continuation of dotting process on the four n-terms)

6. Term number three:

$$3^3 [(n^1 2^2 3^1 + 1^1 n^2 2^1) A_6(12) + (2^1 n^2 3^1 + 3^1 2^2 n^1) A_6(23) + (1^1 3^2 n^1 + n^1 1^2 3^1) A_6(31)] \checkmark$$

$$= (3 \cdot 1) A_6(12) \cancel{n^1 2^2} + (3 \cdot 2) A_6(12) \cancel{1^1 n^2} + (3 \cdot 3) A_6(23) \cancel{2^1 n^2} + (3 \cdot n) A_6(23) \cancel{3^1 2^2} - [1^1 2^2 + 2^1 2^2] + (3 \cdot n) A_6(31) \cancel{1^1 3^2} - [1^1 1^2 + 1^1 2^2] + (3 \cdot 3) A_6(31) \cancel{n^1 1^2}$$

③

11) $-(3 \cdot n) A_6(31)$ ✓

ni) $(3 \cdot 3) A_6(31)$ ✓

12) $-(3 \cdot n) [A_6(31) + A_6(23)]$ ✓

2n) $(3 \cdot 3) A_6(23)$ ✓

21) ϕ ✓

1n) $(3 \cdot 2) A_6(12)$ ✓

22) $-(3 \cdot n) A_6(23)$ ✓

n2) $(3 \cdot 1) A_6(12)$ ✓

3) ϕ ✓

nn) ϕ ✓

②

7. Term number four:

$$3^3 [(n^1 2^2 3^1 + 1^1 n^2 2^1) A_7(12) + (3^1 n^2 2^1 + 2^1 3^2 n^1) A_7(23) + (1^1 2^3 n^1 + n^1 1^2 3^1) A_7(31)] \checkmark$$

$$= (3 \cdot 2) A_7(12) \cancel{n^1 2^2} + (3 \cdot 1) A_7(12) \cancel{1^1 n^2} + (3 \cdot 3) A_7(23) \cancel{3^1 n^2} - [1^1 n^2 + 2^1 n^2] + (3 \cdot n) A_7(23) \cancel{2^1 3^2} + (3 \cdot n) A_7(31) \cancel{1^1 2^3} - [n^1 1^2 + n^1 2^2] + (3 \cdot 3) A_7(31) \cancel{n^1 3^2}$$

11) $(3 \cdot n) A_7(31) \checkmark$

11) $-(3 \cdot 3) A_7(31) \checkmark$

12) $\emptyset$

21) $-(3 \cdot 3) A_7(23) \checkmark$

21) $\emptyset$

12) $(3 \cdot 1) A_7(12) - (3 \cdot 3) A_7(23) \checkmark$

22) $(3 \cdot n) A_7(23) \checkmark$

22) $(3 \cdot 2) A_7(12) - (3 \cdot 3) A_7(31) \checkmark$

23) $\emptyset$

23) $\emptyset$

⑧

Redo the first term ✓

11) $\emptyset$

11) $\emptyset$

12) $\emptyset$

21) $\emptyset$

21) $\emptyset$

12) $(3 \cdot n) A_4(23) \checkmark$

22) $\emptyset$

22) $(3 \cdot n) A_4(31) \checkmark$

23) $\emptyset$

23) $(3 \cdot 3) A_4(12) \checkmark$

Composite of all terms:

VERIFY

$$11) + (3 \cdot n) [A_5(31) + A_6(31) - A_7(31)] - A_1(23) + 2S_3(31) - S_3(23) \quad \checkmark \checkmark$$

$$12) - (3 \cdot n) [A_6(31) + A_6(23)] - (y+z) A_1(123) - A_1(23) - A_1(31) + S_3(31) - S_3(23) \quad \checkmark \checkmark$$

$$21) - (3 \cdot n) [A_5(31) + A_5(23)] + (y-z) S_2(12) + 2S_3(31) - 2S_3(23) \quad \checkmark \checkmark$$

$$22) - (3 \cdot n) [A_5(23) + A_6(23) - A_7(23)] - A_1(31) + S_3(31) - 2S_3(23) \quad \checkmark \checkmark$$

$$8) \phi + x(z-y) S_2(12) - (y+z) A_1(12) + (z-y) S_3(12) \quad \checkmark \checkmark$$

$$m1) (3 \cdot 2) A_5(12) - (3 \cdot 1) A_5(31) + (3 \cdot 3) [A_6(31) - A_7(31)] \quad \checkmark$$

$$2n) (3 \cdot 1) A_5(12) - (3 \cdot 2) A_5(23) + (3 \cdot 3) [A_6(23) - A_7(23)] \quad \checkmark$$

$$1n) (3 \cdot n) A_4(23) - (3 \cdot 2) [A_5(23) - A_6(12)] + (3 \cdot 1) A_7(12) - (3 \cdot 3) A_7(23) \quad \checkmark$$

$$n2) (3 \cdot n) A_4(31) - (3 \cdot 1) [A_5(31) - A_6(12)] + (3 \cdot 2) A_7(12) - (3 \cdot 3) A_7(31) \quad \checkmark$$

$$nn) (3 \cdot 3) A_4(12) \quad \checkmark$$

This is the complete LHS of the Ward Identity, prior to pair symmetrizing which will be done soon. Terms in red are the old ones arising from the non-n terms. Terms in black are all from n-tensors.

Construction of the RHS Ward.

1. We are about to construct all the Ward equations; it would be nice to know the RHS now rather than later:

② Here are the new pieces arising from c :

$$11) -c(1) + b(1) \quad \checkmark \checkmark$$

$$n1) c(1) (1 \cdot 1) / (1 \cdot n) \quad \checkmark$$

$$12) \emptyset \quad \checkmark$$

$$2n) -c(2) (2 \cdot 2) / (2 \cdot n) \quad \checkmark$$

$$21) \emptyset \quad \checkmark$$

$$1n) c(1) (1 \cdot 1) / (1 \cdot n) \quad \checkmark$$

$$22) +c(2) -b(2) \quad \checkmark \checkmark$$

$$nn) -c(2) (2 \cdot 2) / (2 \cdot n) \quad \checkmark$$

$$\delta) \emptyset (2 \cdot 2) b(2) - (1 \cdot 1) b(1) \quad \checkmark$$

$$nn) -c(1) \frac{(1 \cdot 1)^2}{(1 \cdot n)^2} + c(2) \frac{(2 \cdot 2)^2}{(2 \cdot n)^2} \quad \checkmark$$

③ Symmetrize these:

$$[11+22] \quad [c(2) - c(1)] \quad [n1+2n] \quad c(1) \left(\frac{1 \cdot 1}{1 \cdot n} \right) - c(2) \left(\frac{2 \cdot 2}{2 \cdot n} \right)$$

$$[11-22] \quad -[c(2) + c(1)] \quad [n1-2n] \quad c(1) \left(\frac{1 \cdot 1}{1 \cdot n} \right) + c(2) \left(\frac{2 \cdot 2}{2 \cdot n} \right)$$

$$[12] \quad \emptyset \quad [n2+1n] \quad c(1) \left(\frac{1 \cdot 1}{1 \cdot n} \right) - c(2) \left(\frac{2 \cdot 2}{2 \cdot n} \right)$$

$$[21] \quad \emptyset \quad [n2-1n] \quad -c(1) \left(\frac{1 \cdot 1}{1 \cdot n} \right) - c(2) \left(\frac{2 \cdot 2}{2 \cdot n} \right)$$

$$[\delta] \quad \emptyset \quad [nn] \quad -c(1) \left(\frac{1 \cdot 1}{1 \cdot n} \right)^2 + c(2) \left(\frac{2 \cdot 2}{2 \cdot n} \right)^2$$

Now we have to break each of these RHS terms into its symmetry pieces.

	$V \overset{A}{\bigcirc}$	$S \overset{S_2}{}$	$T \overset{S_3}{}$	$R \overset{A_1}{}$	$J \overset{A_4}{\bigcirc}$	$K \overset{A_5}{}$	$L \overset{A_6}{}$	$M \overset{A_7}{}$	$\bigcirc$
11)			$2\bar{c} - c^+$	$-c^+$		$-n_3 \bar{c}$	$-n_3 \bar{c}$	$n_3 \bar{c}$	$b(1) - c(1)$
12)	$-(y+z)$		$\bar{c} - c^+$	$-c^+ - \bar{c}$			$-n_3 \bar{c}$ $-n_3 c^+$		ϕ
21)		$(y-z)$	$2\bar{c} - 2c^+$			$-n_3 \bar{c}$ $-n_3 c^+$			ϕ
22)			$\bar{c} - 2c^+$	$-c^-$		$-n_3 c^+$	$-n_3 c^+$	$n_3 c^+$	$c(2) - b(2)$
3)		$x(z-y)$	$(z-y)$	$-(y+z)$					$(2.2)b(2) - (1.1)b(1)$
n1)						y $-z \bar{c}$	$-(z+y) \bar{c}$	$(z+y) \bar{c}$	$\left(\frac{1.1}{1.n}\right) c(1)$
2n)		This chart checked against old one.				z $-y c^+$	$-(z+y) c^+$	$(z+y) c^+$	$-\left(\frac{2.2}{2.n}\right) c(2)$
1n)					$n_3 c^+$	$-y c^+$	y	z $+(z+y) c^+$	$\left(\frac{1.1}{1.n}\right) c(1)$
n2)					$n_3 \bar{c}$	$-z \bar{c}$	z	y $+(z+y) \bar{c}$	$-\left(\frac{2.2}{2.n}\right) c(2)$
nm)					$-(y+z)$				$c(2) \left(\frac{2.2}{n.2}\right)^2 - c(1) \left(\frac{1.1}{1.n}\right)^2$

	O	R	K	L	M	- RHS -	
(11)'	$(y+z)(1-2\bar{c}^+)$ ✓	$(y+z)$ ✓	$(y+z)\eta_2$ ✓	$(y+z)\eta_2$ ✓	$-(y+z)\eta_2$ ✓	$(3.3) [b(3) - c(3)]$ ✓	
(22)'	$(y+z)(2\bar{c}-1)$ ✓	$(y+z)$ ✓	$(y+z)\eta_1$ ✓	$(y+z)\eta_1$ ✓	$-(y+z)\eta_1$ ✓	$-(3.3) [b(3) - c(3)]$ ✓	
$\sqrt{8}$ '	$2x(\bar{c}-c^+)$ $+ (z-y)$ ✓	$-(y+z)$ ✓	$-x\eta_3(\bar{c}^+)$ ✓			$(2.2) b(2) - (1.1) b(1)$ ✓	
(n1)			$y-z\bar{c}$ ✓	$-(z+y)\bar{c}$ ✓	$(z+y)\bar{c}$ ✓	$c(1) \left(\frac{1.1}{1.n}\right)$ ✓	
(2n)			$z-y\bar{c}^+$ ✓	$-(z+y)\bar{c}^+$ ✓	$(z+y)\bar{c}^+$ ✓	$-c(2) \left(\frac{2.2}{2.n}\right)$ ✓	
(1n)''			$-\frac{z}{\eta_1}(y+z)$ ✓	$\frac{z}{\eta_1}(y+z)\bar{c}$ ✓	$\frac{x}{\eta_1}(y+z)\bar{c}$ $+\frac{(y+z)(x+z)}{\eta_1}$ ✓	$c(2) \left(\frac{2.2}{\eta.2}\right) \left\{ \left(\frac{2.2}{\eta.2}\right) - \left(\frac{3.3}{\eta.1}\right) \right\} - c(1) \left(\frac{1.1}{\eta.1}\right)^2$ ✓	
(n2)''			$-\frac{y}{\eta_2}(y+z)$ ✓	$\frac{y}{\eta_2}(y+z)\bar{c}^+$ ✓	$\frac{x}{\eta_2}(y+z)\bar{c}^+$ $+\frac{(y+z)(y+x)}{\eta_2}$ ✓	$c(2) \left(\frac{2.2}{\eta.2}\right)^2 - c(1) \left(\frac{1.1}{\eta.1}\right) \left\{ \left(\frac{1.1}{1.n}\right) - \left(\frac{3.3}{2.n}\right) \right\}$ ✓	

Processing of the Full Chart.

✓ 1) Regard the (12) equation as telling you what V is.

✓ 2) Replace (d) by: $(d)' = (d) + x(21)$

✓ 3) Regard (21) as telling you what S is.

✓ 4) Replace (11) by: $(11)' = -(y+z) C^- (11)$. Compute area 4.

✓ 5) Replace (22) by: $(22)' = -(y+z) C^+ (22)$. Compute area 5.

✓ 6) Regard (nn) as telling you what J is.

✓ 7) Replace (1n) by: $(1n)' = [(y+z)/n_1] C^+ (1n)$ Compute in area 7.

✓ 8) Replace (n2) by: $(n2)' = [(y+z)/n_2] C^- (n2)$ Compute in area 8.

✓ 9) Replace (1n)' by: $(1n)'' = (1n)' + (nn)$

✓ 10) Replace (n2)' by: $(n2)'' = (n2)' + (nn)$

***** make new chart ***** OK. [TRKLM]

✓ 11) Replace (11)' by $(11)'' = \left[\frac{(11)'}{(y+z)} + \frac{(d)'}{(y+z)} \right] (y+z)$

✓ 12) Replace (22)' by $(22)'' = \frac{(22)'}{(y+z)} \times \frac{(d)'}{(y+z)}$

✓ 13) Regard (d)' as telling you what R is.

✓ 14) Replace (1n)'' by $(1n)''' = (1n)'' \times [n_1/(y+z)]$

✓ 15) Replace (n2)'' by $(n2)''' = (n2)'' \times [n_2/(y+z)]$

***** make new chart *****

[TKLM, no overlap yet]

Execution of Procedures

(1) OK

(2)

$$(d') = \begin{array}{ccccccc} V & S & T & R & J & K & L & M \\ \phi, \phi, 2x(\bar{C}-\bar{C}^+) & -(y+z), \phi, -x\eta_3(\bar{C}+\bar{C}^+) & \phi, \phi, (2.2)b(2)-(1.1)b(1) \\ & + (2-y) & & & & & & \end{array}$$

(3) OK

$$(4) \bar{C}(11) = \phi, \phi, 2\bar{C}^+-1, -1, 0, -\underline{\eta_2}, -\underline{\eta_2}, \underline{\eta_2}, \parallel b(3)-c(3).$$

$$\bar{C}(2\bar{C}-C^+) = 2\bar{C}^+-1$$

$$\bar{C}[-\eta_3\bar{C}] = -\eta_2$$

$$\bar{C}[b(1)-c(1)] = b(3)-c(3)$$

Then $\otimes (y+z)$

$$(3.3)[b(3)-c(3)]$$

$$\Rightarrow (11)' = \begin{array}{ccccccc} V & S & T & R & J & K & L & M \\ \phi, \phi, (y+z)(1-2\bar{C}^+) & (y+z), \phi, (y+z)\eta_2, (y+z)\eta_2, -(y+z)\eta_2 \\ & & & & & & & \parallel \end{array}$$

$$(5) \bar{C}^+(22) = \phi, \phi, (1-2\bar{C}), -1, 0, -\eta_1, -\eta_1, \eta_1, \parallel c(3)-b(3)$$

$$C^+(\bar{C}-2C^+) = 1-2\bar{C}$$

$$C^+(-\eta_3C^+) = -\eta_1$$

Then $\otimes [-(y+z)]$

$$(3.3)[c(3)-b(3)]$$

$$\Rightarrow (22)' = \begin{array}{ccccccc} V & S & T & R & J & K & L & M \\ \phi, \phi, (y+z)(2\bar{C}-1) & (y+z), 0, (y+z)\eta_1, (y+z)\eta_1, -(y+z)\eta_1 \\ & & & & & & & \parallel \end{array}$$

(6) OK

$$\textcircled{7} \quad C^+(1n) = \phi, \phi, \phi, \phi, \underline{n_1}, \underline{-z}, \underline{z\bar{C}}, \underline{x\bar{C} + (x+z)} \parallel \left(\frac{2.2}{2.n} \right) c(2) \checkmark$$

$$C^+ n_3 C^+ = n_1 \checkmark$$

$$-C^+ y C^+ = -z \checkmark$$

$$C^+ y = (C^+ y C^+ \bar{C} = z\bar{C} \checkmark$$

$$C^+ z = (C^+ z C^+) \bar{C} = x\bar{C} \checkmark$$

$$C^+(z+y)C^+ = (x+z) \checkmark$$

$$\Rightarrow (1n)' = \phi, \phi, \phi, \phi, \underline{(y+z)}, \underline{\frac{-(y+z)z}{n_1}}, \underline{\frac{(y+z)z\bar{C}}{n_1}}, \underline{\frac{(y+z)x\bar{C} + (y+z)(x+z)}{n_1}} \parallel \left(\frac{-3.3}{n.1} \right) \left(\frac{2.2}{2.n} \right) c(2) \checkmark$$

$$\textcircled{9} \quad (1n)'' = \phi, \phi, \phi, \phi; \phi, \underline{\frac{-(y+z)z}{n_1}}, \underline{\frac{(y+z)z\bar{C}}{n_1}}, \underline{\frac{(y+z)x\bar{C} + (y+z)(x+z)}{n_1}} \checkmark$$

$$-\left(\frac{3.3}{n.1} \right) \left(\frac{2.2}{2.n} \right) c(2) + c(2) \left(\frac{2.2}{n.2} \right)^2 - c(1) \left(\frac{1.1}{1.n} \right)^2 \checkmark$$

$$c(2) \left(\frac{2.2}{n.2} \right) \left[\left(\frac{2.2}{n.2} \right) - \left(\frac{3.3}{n.1} \right) \right] - c(1) \left(\frac{1.1}{1.n} \right)^2 \checkmark$$

$$\textcircled{8} \quad \bar{C}(n2) = \phi\phi\phi\phi; \underline{n_2}, \underline{-y}, \underline{y\bar{C}^+}, \underline{x\bar{C}^+ + (y+x)} \parallel - \left(\frac{1.1}{1.n} \right) c(1) \checkmark$$

$$\bar{C}z = \bar{C}z\bar{C}C^+ = yC^+$$

$$\bar{C}y = xC^+$$

$$C(z+y)\bar{C} = (y+x)$$

then:

$$(n_2)' = \phi \phi \phi \phi; \underbrace{(y+z)}_{n_2}, \underbrace{-y(y+z)}_{n_2} \underbrace{(y+z)c^+}_{n_2}, \underbrace{(y+z)x c^+ + (y+z)(y+x)}_{n_2} \parallel$$

$$\left[\frac{-(3.3)}{(n.2)} \right] \left[\frac{-(1.1)}{(1.n)} \right] c(1) = \frac{(1.1)(3.3)}{(1.n)(2.n)} c(1) \checkmark$$

$$(10) \quad (n_2)'' = \phi, \phi, \phi, \phi; \underbrace{\phi}_{n_2}, \underbrace{-y(y+z)}_{n_2}, \underbrace{y(y+z)c^+}_{n_2}, \underbrace{(y+z)x c^+ + (y+z)(y+x)}_{n_2} \parallel$$

$$\frac{c(1)(1.1)}{(1.n)} \left[\frac{(3.3)}{(2.n)} - \frac{(1.1)}{(1.n)} \right] + c(2) \frac{(2.2)^2}{(2.2)} \checkmark$$

$$(11) \quad (H)''$$

$$(y+z) \sim 2(y+z)$$

$$\frac{(11)'}{y+z} = (1-2c^+), 1, n_2, n_2, -n_2 \parallel [c(3) - b(3)]$$

$$\frac{(d)'}{y+z} = \frac{2x(\bar{c}-c^+)(z-y)}{y+z}, -1, \frac{-x n_3(\bar{c}+c^+)}{y+z}, \phi, \phi, \parallel$$

$$\frac{(2.2)b(2) - (1.1)b(1)}{-(3.3)}$$

$$\Rightarrow (11)'' = \underbrace{(1-2c^+)}_{+ 2x(\bar{c}-c^+)(y+z)}, \phi, n_2, \underbrace{-x n_3(\bar{c}+c^+)}_{+ (z-y)/(y+z)}, n_2, -n_2 \parallel \frac{(1.1)b(1) - (3.3)b(2)}{+ c(3) - b(3)}$$

$$X(y+z) \cdot (11)'' = \underbrace{(1-2c^+)(y+z)}_{+ 2x(\bar{c}-c^+)(y+z)}, \phi, \underbrace{-x n_3(\bar{c}+c^+)}_{+ (z-y)/(y+z)}, n_2(y+z), -n_2(y+z) \parallel \frac{(2.2)b(2) - (1.1)b(1) + (3.3)[b(3) - c(3)]}{2z}$$

(12)

$$\frac{(22)'}{y+z} = (2\bar{c}-1), 1, n_1, n_1, -n_1 \parallel b(3) - c(3)$$

$$\frac{- (11)'}{y+z} = (2\bar{c}-1), -1, -n_2, -n_2, n_2 \parallel b(3) - c(3)$$

$$\Rightarrow (22)'' = 2(\bar{c}+c-1), \phi, n_1-n_2, n_1-n_2, n_2-n_1 \parallel 2b(3) - 2c(3).$$

Repeat:

$$(11)'' = \{ -2\bar{c}(x+y+z) + 2xc + 2z \}, etc. = \{ -2\bar{c}(x+y+z) \}$$

$$(14) \text{ RHS} = \left(-\frac{n \cdot 1}{3 \cdot 3} \right) \times \text{old RHS} =$$

$$= -c(2) \left(\frac{2 \cdot 2}{n \cdot 2} \right)^2 \left(\frac{n \cdot 1}{3 \cdot 3} \right) + c(1) \left(\frac{1 \cdot 1}{n \cdot 1} \right)^2 \left(\frac{n \cdot 1}{3 \cdot 3} \right) + c(2) \left(\frac{2 \cdot 2}{n \cdot 2} \right)$$

$$(16) \text{ RHS} = \left(-\frac{n \cdot 2}{3 \cdot 3} \right) \times \text{old RHS} = -c(2) \left(\frac{2 \cdot 2}{n \cdot 2} \right)^2 \left(\frac{n \cdot 2}{3 \cdot 3} \right)$$

$$+ c(1) \left(\frac{1 \cdot 1}{n \cdot 1} \right)^2 \left(\frac{n \cdot 2}{3 \cdot 3} \right) - c(1) \left(\frac{1 \cdot 1}{n \cdot 1} \right)$$

Second Processing of the Chart.

1) Apply $\frac{1}{2}(C^+ + C^-)$ to (22)" and replace that row with the result:

	T	K	L	M	RHS.
a	1	$\frac{1}{4}(\bar{C} + \bar{C})(\eta_1 - \eta_2)$	same	- same	$\frac{1}{2}(\bar{C} + \bar{C})[b(3) - c(3)]$

Call this row "a". Now multiply this by C^+ and call that row b.

Multiply a by C^- and call that row c:

b	C^+	$\frac{1}{4}(\bar{C} + 1)(\eta_1 - \eta_2)$	S	-S	$\frac{1}{2}(\bar{C} + 1)[b(3) - c(3)]$
c	C^-	$\frac{1}{4}(1 + \bar{C})(\eta_1 - \eta_2)$	$\tilde{S}$	$-\tilde{S}$	$\frac{1}{2}(1 + \bar{C})[b(3) - c(3)]$

Now compute this combination:

$$\{-2z \text{ a} + 2(x+y+z) \text{ b} - 2x \text{ c}\} =$$

-2z	$-\frac{1}{2}z(\bar{C} + \bar{C})(\eta_1 - \eta_2)$	$\rightarrow "$	$\rightarrow -"$	-z $-z(\bar{C} + \bar{C})[b(3) - c(3)]$
$+2(x+y+z)\bar{C}^+$	$+\frac{1}{2}(x+y+z)(\bar{C} + 1)(\eta_1 - \eta_2) \rightarrow "$	$\rightarrow -"$	$\rightarrow -"$	$+ (x+y+z)(\bar{C} + 1)[b(3) - c(3)]$
$-2x C^-$	$-\frac{1}{2}x(1 + \bar{C})(\eta_1 - \eta_2) \rightarrow "$	$\rightarrow -"$	$\rightarrow -"$	$-x(1 + \bar{C})[b(3) - c(3)]$

Now the idea is to add this mess to Row 1 of the chart, thus cancelling the T coefficient. We then regard "a" above as telling us what T is. Our problem is now to compute up and simplify the new first row, then make a new table See (X)

overlay #1

① $[(z)(e) - (x+y+z) c^+ + x c^-]$

$a = z$

$b = -(x+y+z)$

$d = x$

$\det = z^3 - (x+y+z)^3 + x^3 + 3xz(x+y+z)$

$= \cancel{z^3} - [\cancel{x^3} + \cancel{y^3} + \cancel{z^3} + \cancel{3x^2y} + \cancel{3y^2x} + \cancel{3z^2x} + \cancel{3xz^2} + \cancel{3y^2x} + \cancel{3z^2y}] + \cancel{x^3} + \cancel{3xz^2} + \cancel{3xyx} + \cancel{3xz^2}$

$= -y^3 - 3x^2y - 3y^2x - 3y^2x - 3z^2y + 3xyz$

$= y(-y^2 - 3x^2 - 3yz - 3yx - 3z^2 + 3xz)$

$\alpha = \frac{\text{NUM}}{\text{DEN}} \quad a^2 - bd = z^2 + (x+y+z)x = z^2 + x^2 + xy + xz$

So $\alpha =$ messy ratio, no cancel!

② Now look at $[-1 + c^+ + c^-] \Rightarrow$

$a = -1$

$b = 1$

$d = 1$

$\det = -1 + 1 + 1 + 3 \cdot 1 \cdot 1 = 4.$

$\alpha = [1-1]/4 = 0$

$\beta = [1+1]/4 = \frac{1}{2} \Rightarrow [c^+ + c^- - 1]^{-1} = \frac{1}{2}(c^+ + c^-)$

$\gamma = [1+1]/4 = \frac{1}{2}$

$\Rightarrow [c^+ + c^- - 1] \frac{1}{2}(c^+ + c^-) = 1$

So $\boxed{\frac{1}{4}(c^+ + c^-) \cdot [2c^+ + 2c^- - 2] = 1.}$

⊗ Let us compute that messy second entry of (d).

$$\frac{1}{2} \left\{ \begin{array}{l} (-z-x) C^+(n_1-n_2) \cdot C^+ C^- \\ + (-z+x+y+z) C^-(n_1-n_2) \cdot C^- C^+ \\ + (x+y+z-x)(n_1-n_2) \end{array} \right\} = \frac{1}{2} \left\{ \begin{array}{l} (-z-x)(n_2-n_3) C^- \\ + (x+y)(n_3-n_1) C^+ \\ + (y+z)(n_1-n_2) \end{array} \right\}$$

$$= \frac{1}{2} \left\{ (z+x)(n_3-n_2) C^- + (x+y)(n_3-n_2) C^+ + (y+z)(n_1-n_2) \right\}$$

Add this first to K-entry of row 1 which is:

$$\frac{1}{2} \left\{ 2(y+z)n_2 - 2xn_3 C^- - 2xn_3 C^+ \right\}$$

Sum is:

$$\frac{1}{2} \left\{ C^- [zn_3 - zn_2 + xn_3 - xn_2 - 2xn_3] + C^+ [xn_3 - xn_2 + yn_3 - yn_2 - 2xn_3] + 1 [yn_1 - yn_2 + zn_1 - zn_2 + 2yn_2 + 2zn_2] \right\}$$

$$= \frac{1}{2} \left\{ C^- [zn_3 - zn_2 - xn_3 - xn_2] + C^+ [-xn_3 - xn_2 + yn_3 - yn_2] + [yn_1 + yn_2 + zn_1 + zn_2] \right\}$$

$$= \frac{1}{2} \left\{ C^- [n_3(z-x) - n_2(z+x)] + C^+ [n_3(y-x) - n_2(y+x)] + [n_1(y+z) + n_2(y+z)] \right\}$$

So:

K-entry of new Row-1 is:

$$\frac{1}{2} \left\{ [n_3(z-x) - n_2(z+x)] C^- + [n_3(y-x) - n_2(y+x)] C^+ + (n_1+n_2)(y+z) \right\}$$

L-entry of new Row-1 is:

$$\frac{1}{2} \left\{ (z+x)(n_3-n_2) C^- + (x+y)(n_3-n_2) C^+ + (y+z)(n_1+n_2) \right\}$$

$$(M \text{ entry}) = -(L \text{ entry})$$

RHS of (d) is:

$$\left[(-z-x)C^+ + (-\cancel{z}+x+y+\cancel{z})\tilde{C} + (x+y+\cancel{z}-x) \right] [b(3) - c(3)]$$

$$= -(z+x) [b(1) - c(1)] + (x+y) [b(2) - c(2)] + (y+z) [b(3) - c(3)].$$

Now add this to RHS of row 1:

$$\rightarrow = (1.1) [\cancel{b(1)} - \cancel{c(1)}] - (2.2) [\cancel{b(2)} - \cancel{c(2)}] - (3.3) [\cancel{b(3)} - \cancel{c(3)}]$$

$$\text{Row 1} = -(\cancel{1.1}) \cancel{b(1)} + (\cancel{2.2}) \cancel{b(2)} + (\cancel{3.3}) [\cancel{b(3)} - \cancel{c(3)}]$$

$$\boxed{\text{new Row 1 RHS} = -(1.1)c(1) + (2.2)c(2).}$$

Third Chart Processing

1. With the overlay I am now down to three variables. The duplication of information in the L and M columns suggests to me that I really should convert to L+M and L-M as variables since this will ~~output~~ three big fat zeros in the chart for no work!!! I also find that K-L is a nice replacement for K because it simplifies that messy K entry in the first row, and doesn't complicate the other rows much.
2. Notice by the way that b, the Ward function, no longer appears in our dynamic equations. Of course it gets fed in by the kinematic sleeper equations that we have removed.
3. Here is how you convert columns to sum and difference. :

$$\begin{array}{c|c} L & M \\ \hline a & b \end{array} \quad \rightarrow \quad \begin{array}{c|c} L-M & L+M \\ \hline \frac{1}{2}(a-b) & \frac{1}{2}(a+b) \end{array}$$

(2 terms) (2 terms)

and same:

$$\begin{array}{c|c} K & L \\ \hline a & b \end{array} \quad \rightarrow \quad \begin{array}{c|c} K-L & \\ \hline \frac{1}{2}(a-b) & \end{array} \quad \underline{\underline{NO!}}$$

NO, not correct. You cannot replace K by K-L in this way because $aK+bL \neq \frac{1}{2}(a-b)(K-L)$. Just leave the K column the way it was.

4. Check results visually, red checks. Notice that, for the first three entries, in the L-M column $\frac{1}{2}$ the difference just duplicates L. And in the L+M column you get 0 for these entries. The last two entries do out by hand. (attached) See on back. This is the second overlay.

5. Third overlay: add row three to row four; add row 2 to row 5. Reason: makes the RHS's look nice, doesn't do too much harm. May undo this later!????

$$\frac{1}{2} [z\bar{c} - x\bar{c} - (x+z)]$$

$$= \frac{(z-x)}{2} \bar{c} - \frac{1}{2}(x+z)$$

$$\frac{1}{2} [z\bar{c} + x\bar{c} + (x+z)]$$

$$= \frac{1}{2}(z+x)(\bar{c}+1)$$

$$\frac{1}{2} [y c^+ - x c^+ - (y+x)]$$

$$= \frac{1}{2}(y-x) c^+ - \frac{1}{2}(y+x)$$

$$\frac{1}{2} [y c^+ + x c^+ + (y+x)]$$

$$= \frac{1}{2}(y+x)(c^++1)$$

Fourth Process on Chart

① First, consider

$$[1 + \bar{c}] \Rightarrow \begin{matrix} a = 1 \\ b = 0 \\ d = 1 \end{matrix} \Rightarrow \det = 1 + 1 + 0 - 0 = 2 \Rightarrow \begin{matrix} \alpha = 1/2 \\ \beta = 1/2 \\ \delta = -1/2 \end{matrix}$$

$$\Rightarrow [\frac{1}{2} + \frac{1}{2}c^+ - \frac{1}{2}c^-][1 + \bar{c}] = 1$$

Next:

$$\begin{matrix} \uparrow \\ (c^- c^+) \\ \downarrow \end{matrix} \Rightarrow [\frac{1}{2}c^- + \frac{1}{2} - \frac{1}{2}c^+][c^+ + 1] = 1$$

Therefore:

$\{1 + c^+ - c^-\}$	$\frac{1}{2}(1 + \bar{c}) = 1$
$\{1 - c^+ + c^-\}$	$\frac{1}{2}(1 + c^+) = 1$

② Now, divide Row-4 by $(x+z) = -(1 \cdot 1)$
divide Row-5 by $(x+y) = -(2 \cdot 2)$

K	L-M	L+M	RHS
$\frac{-y c^+}{(x+z)} A_1$	$\frac{1}{2} \left(\frac{z-x}{x+z} \right) c^-$ $= -\frac{1}{2}$ $-\left(\frac{z+y}{x+z} \right) c^+ A_2$	$\frac{1}{2}(1 + \bar{c})$	$-\frac{1}{(3 \cdot 3)} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \left[\text{same} \right] : A_3$
$\frac{-z c^-}{(x+y)} B_1$	$\frac{1}{2} \left(\frac{y-x}{x+y} \right) c^+$ $= -\frac{1}{2}$ $-\left(\frac{z+y}{x+y} \right) \bar{c} B_2$	$\frac{1}{2}(1 + c^+)$	$-\frac{1}{(3 \cdot 3)} \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix} \left[\text{same} \right] : B_3$

③ Now apply $[1 + c^+ - \bar{c}]$ to top row $\uparrow$ to get +1 in L+M column.

apply $[-1 + c^+ - \bar{c}]$ to bottom row. to get -1 in L+M column.

Compute new rows. Need more symbols. We have:

$[1+c^+-\bar{c}]A_1$	$[1+c^+-\bar{c}]A_2$	$+1$	$[1+c^+-\bar{c}]A_3$
$[-1+c^+-\bar{c}]B_1$	$[-1+c^+-\bar{c}]B_2$	-1	$[-1+c^+-\bar{c}]B_3$

Now, let top row tell you $(L+M)$. Then add rows and apply $\frac{1}{2}(1+\bar{c})$ on the left. You have:

$$\frac{1}{2}(1+\bar{c}) \cdot [1+c^+-\bar{c}] = 1$$

$$\begin{aligned} \frac{1}{2}(1+\bar{c}) \cdot [-1+c^+-\bar{c}] &= \frac{1}{2}(1+\bar{c})[(1+c^+-\bar{c})-2] \\ &= 1 - (1+\bar{c}) = -\bar{c}. \end{aligned}$$

Thus our new entry is:

$A_1 - \bar{c}B_1$	$A_2 - \bar{c}B_2$	$\emptyset$	$A_3 - \bar{c}B_3$
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Comment: a new trick here: once you have a zero where you want it, you can apply any operator you want, so you might as well apply an operator which makes the other entries simple! Now compute these entries:

$$\bar{c}(B_1) = \bar{c} \cdot \left[\left(\frac{-2}{x+y} \right) \bar{c} \right] = \left(\frac{-y}{2+x} \right) \Rightarrow A_1 - \bar{c}B_1 = \left(\frac{-y}{2+x} \right) \bar{c}^+ - \left(\frac{-y}{2+x} \right)$$

$$\Rightarrow A_1 - \bar{c}B_1 = \left(\frac{y}{2+x} \right) (1 - \bar{c}^+)$$

$$-\bar{c}(B_2) = -\bar{c} \left\{ \frac{1}{2} \left(\frac{y-x}{y+x} \right) \bar{c} \bar{c} - \frac{1}{2} - \left(\frac{2+y}{x+y} \right) \bar{c} \right\}$$

$$= -\frac{1}{2} \left(\frac{x-2}{x+2} \right) \bar{c} + \frac{1}{2} \bar{c} + \left(\frac{y+x}{2+x} \right)$$

$$A_2 = -\frac{1}{2} \left(\frac{x-2}{x+2} \right) \bar{c} - \frac{1}{2} - \left(\frac{2+y}{x+2} \right) \bar{c}^+$$

$$\Rightarrow A_2 - \bar{c}B_2 = -\left(\frac{x-2}{x+2} \right) \bar{c} + \frac{1}{2} \bar{c} - \left(\frac{2+y}{x+2} \right) \bar{c}^+ + \left(\frac{y+x}{2+x} - \frac{1}{2} \right)$$

$$-C B_3 = +C \left\{ \frac{1}{(3 \cdot 3)} \left(\frac{n \cdot 2}{2 \cdot 2} \right) [\{1\} - \{2\}] \right\} \checkmark$$

$$= \frac{1}{(2 \cdot 2)} \left(\frac{n \cdot 1}{(1 \cdot 1)} \right) [\{3\} - \{1\}] \checkmark$$

$$A_3 = - \frac{1}{(3 \cdot 3)} \left(\frac{n \cdot 1}{(1 \cdot 1)} \right) [\{1\} - \{2\}] \checkmark$$

$$\text{So } A_3 - C B_3 = \left(\frac{n \cdot 1}{1 \cdot 1} \right) \{1\} \left[\frac{1}{(3 \cdot 3)} - \frac{1}{(2 \cdot 2)} \right] \checkmark + \frac{1}{(3 \cdot 3)} \left(\frac{n \cdot 1}{1 \cdot 1} \right) \{2\} \checkmark$$

$$+ \frac{1}{(2 \cdot 2)} \left(\frac{n \cdot 1}{1 \cdot 1} \right) \{3\} \checkmark$$

$$= \left(\frac{n \cdot 1}{1 \cdot 1} \right) \{ \}$$

here.

$$= \frac{(n \cdot 1)}{(1 \cdot 1)(2 \cdot 2)(3 \cdot 3)} \left\{ -[(2 \cdot 2) + (3 \cdot 3)] \{1\} + (2 \cdot 2) \{2\} + (3 \cdot 3) \{3\} \right\}$$

$$\text{Let } \{1\} = \textcircled{1} - \textcircled{2}$$

$$= \frac{(n \cdot 1)}{(1 \cdot 1)(2 \cdot 2)(3 \cdot 3)} \left\{ -[(2 \cdot 2) + (3 \cdot 3)] [\textcircled{1} - \textcircled{2}] + (2 \cdot 2) [\textcircled{2} - \textcircled{3}] + (3 \cdot 3) [\textcircled{3} - \textcircled{1}] \right\}$$

$$= \frac{(n \cdot 1)}{(1 \cdot 1)(2 \cdot 2)(3 \cdot 3)} \left\{ -2[\textcircled{1} - \textcircled{2}] - 3[\textcircled{2} - \textcircled{3}] + 2[\textcircled{2} - \textcircled{3}] + 3[\textcircled{3} - \textcircled{1}] \right\}$$

$$\frac{y}{(z+x)} (1 - C^+)$$

$$- \left(\frac{z+y}{x+z} \right) C^+$$

$$\left[\frac{1}{2} - \left(\frac{x-z}{x+z} \right) C^+ \right]$$

$$\left[\frac{y+x}{z+x} = \frac{1}{2} \right]$$

• ϕ

$$\frac{(n \cdot 1)}{(1 \cdot 1)(2 \cdot 2)(3 \cdot 3)} \left\{ -[(2 \cdot 2) + (3 \cdot 3)] \{1\} \right.$$

$$\left. + (2 \cdot 2) \{2\} + (3 \cdot 3) \{3\} \right\}$$

replacement for last two rows.

$$(z+x) = -(1 \cdot 1)$$

Might as well clear out this $(z+x)$ junk, so $\times [-(1.1)]$ both.

$y - y c^+$	$-(z+y) c^+$ $[\frac{1}{2}(x+z) - (x-z)] c^-$ $+ [y+x - \frac{1}{2}(z+x)]$	ϕ	$\frac{(n.1)}{(2.2)(3.3)} \{ [(2.2) + (3.3)] \{1\}$ $- (2.2) \{2\} - (3.3) \{3\}$
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Repeat:

K	L-M	RHS
$y - y c^+$	$-(z+y) c^+$ $+ \frac{1}{2}[3z-x] c^-$ $+ \frac{1}{2}[2y+x-z]$	$\frac{(n.1)}{(2.2)(3.3)} \{ [(2.2) + (3.3)] c(1) \left(\frac{1.1}{1.n}\right)^2$ $- (2.2) c(2) \left(\frac{2.2}{2.n}\right)^2 - (3.3) c(3) \left(\frac{3.3}{3.n}\right)^2 \}$

→ overlay 4

	K	L-M	L+M	RHS
(R1)	$\begin{aligned} & \frac{1}{2} [\eta_3(z-x) - \eta_2(z+x)] \bar{C} \\ & \frac{1}{2} [\eta_3(y-x) - \eta_2(y+x)] \bar{C}^+ \\ & + (\eta_1 + \eta_2)(y+z) \end{aligned}$	$\begin{aligned} & \frac{1}{2} (z+x)(\eta_3 - \eta_2) \bar{C} \\ & + \frac{1}{2} (x+y)(\eta_3 - \eta_2) \bar{C}^+ \\ & + (\eta_1 + \eta_2)(y+z) \end{aligned}$	.	$(2,2) c(2) - (1,1) c(1)$
(n1)	$y - z \bar{C}$	$-(z+y) \bar{C}$	.	$c(1) \left(\frac{1,1}{1,n} \right)$
(2n)	$z - y C^+$	$-(z+y) C^+$	.	$-c(2) \left(\frac{2,2}{2,n} \right)$
(4)	$y - y C^+$	$\begin{aligned} & -(z+y) C^+ \\ & + \frac{1}{2} [3z-x] \bar{C} \\ & + \frac{1}{2} [2y+x-z] \end{aligned}$	.	$\frac{(n,1)}{(2,2)(3,3)} \left\{ [(2,2) + (3,3)] c(1) \left(\frac{1,1}{1,n} \right)^2 - (2,2) c(2) \left(\frac{2,2}{2,n} \right)^2 \right. \\ \left. - (3,3) c(3) \left(\frac{3,3}{3,n} \right)^2 \right\}$

	K	L-M	L+M	RHS
(R1)	$\frac{1}{2}[\eta_3(z-x) - \eta_2(z+x)] \bar{C}$ $\frac{1}{2}[\eta_3(y-x) - \eta_2(y+x)] \bar{C}^{\dagger}$ $+ (\eta_1 + \eta_2)(y+z)$	$\frac{1}{2}(z+x)(\eta_3 - \eta_2) \bar{C}$ $+ \frac{1}{2}(x+y)(\eta_3 - \eta_2) \bar{C}^{\dagger}$ $+ (\eta_1 + \eta_2)(y+z)$	.	$(2,2)c(2) - (1,1)c(1)$

(1n) ^{III}	-z	$\frac{1}{2}(z-x)\bar{C}$ $-\frac{1}{2}(z+x)$	$\frac{1}{2}(x+z)(1+\bar{C})$	$-c(2)\left(\frac{2,2}{n,2}\right)^2\left(\frac{n,1}{3,3}\right) + c(1)\left(\frac{1,1}{n,1}\right)^2\left(\frac{n,1}{3,3}\right) + c(2)\left(\frac{2,2}{n,2}\right)$
(n2) ^{III}	-y	$\frac{1}{2}(y-x)\bar{C}^{\dagger}$ $-\frac{1}{2}(y+x)$	$\frac{1}{2}(x+y)(1+\bar{C}^{\dagger})$	$-c(2)\left(\frac{2,2}{n,2}\right)^2\left(\frac{n,2}{3,3}\right) + c(1)\left(\frac{1,1}{n,1}\right)^2\left(\frac{n,2}{3,3}\right) - c(1)\left(\frac{1,1}{n,1}\right)$

	K	L-M	L+M	RHS
(R1)	$\begin{aligned} & \frac{1}{2} [\eta_3(z-x) - \eta_2(z+x)] \bar{C} \\ & \frac{1}{2} [\eta_3(y-x) - \eta_2(y+x)] C^+ \\ & + (\eta_1 + \eta_2)(y+z) \end{aligned}$	$\begin{aligned} & \frac{1}{2} (z+x)(\eta_3 - \eta_2) \bar{C} \\ & + \frac{1}{2} (x+y)(\eta_3 - \eta_2) C^+ \\ & + (\eta_1 + \eta_2)(y+z) \end{aligned}$	.	$(2,2)c(2) - (1,1)c(1)$

(1n) ^{III}	-z	z C ⁻	$\begin{aligned} & x C^- \\ & + (x+z) \end{aligned}$	$-c(2) \left(\frac{2 \cdot 2}{n \cdot 2} \right)^2 \left(\frac{n \cdot 1}{3 \cdot 3} \right) + c(1) \left(\frac{1 \cdot 1}{n \cdot 1} \right)^2 \left(\frac{n \cdot 1}{3 \cdot 3} \right) + c(2) \left(\frac{2 \cdot 2}{n \cdot 2} \right)$
(n2) ^{III}	-y	y C ⁺	$\begin{aligned} & x C^+ \\ & + (y+x) \end{aligned}$	$-c(2) \left(\frac{2 \cdot 2}{n \cdot 2} \right)^2 \left(\frac{n \cdot 2}{3 \cdot 3} \right) + c(1) \left(\frac{1 \cdot 1}{1 \cdot n} \right)^2 \left(\frac{n \cdot 2}{3 \cdot 3} \right) - c(1) \left(\frac{1 \cdot 1}{n \cdot 1} \right)$

Sidetrack: what can you learn from the (n1) and (2n) equations/?

1. These equations were never modified from the very beginning, they were just carried along with each manipulation of the other rows. When you go to the L+M and L-M variables, you could tell right from the start that you were going to get these simple equations.:

K	L-M	RHS.
$\frac{y-z}{(3.3)} \bar{C}$ ✓	$\bar{C}$ ✓	$\frac{c(1)(1.1)}{(3.3)(1.n)} \checkmark$
$\frac{z-y}{(3.3)} C^+$ ✓	C^+ ✓	$-\frac{c(2)(2.2)}{(3.3)(2.n)} \checkmark$

So take C^+ on the top and $\bar{C}$ on the bottom and see what you get:

$$\left\{ \begin{aligned} C^+ \left[\frac{y}{(3.3)} - \frac{z}{(3.3)} \bar{C} \right] &= \underbrace{C^+ \left[\frac{y}{(3.3)} \right]}_{\left(\frac{y}{(1.1)} \right) \bar{C}} \bar{C} - \underbrace{C^+ \left(\frac{z}{(3.3)} \right)}_{\left(\frac{x}{(1.1)} \right) C^+} C^+ \\ &= \left(\frac{y}{(1.1)} \right) \bar{C} - \left(\frac{x}{(1.1)} \right) C^+ \quad \checkmark \\ C^+ \left[\frac{c(1)(1.1)}{(3.3)(1.n)} \right] &= \frac{c(2)(2.2)}{(1.1)(2.n)} \quad \checkmark \end{aligned} \right.$$

$$\left\{ \begin{aligned} C^- \left[\frac{z}{(3.3)} - \frac{y}{(3.3)} C^+ \right] &= \underbrace{C^- \left[\frac{z}{(3.3)} \right]}_{\left(\frac{y}{(2.2)} \right) C^+} \bar{C} C^+ - \underbrace{C^- \left(\frac{y}{(3.3)} \right)}_{\left(\frac{x}{(2.2)} \right) \bar{C}} C C \\ &= \left(\frac{y}{(2.2)} \right) C^+ - \left(\frac{x}{(2.2)} \right) \bar{C} \quad \checkmark \\ -\bar{C} \left[\frac{c(2)(2.2)}{(3.3)(2.n)} \right] &= -\frac{c(1)(1.1)}{(2.2)(1.n)} \quad \checkmark \end{aligned} \right. \quad \text{THUS:}$$

$\frac{1}{(1.1)}(z \bar{C} - x C^+)$ ✓	1	$\frac{c(2)(2.2)}{(1.1)(2.n)}$ ✓	B_1 ✓
$\frac{1}{(2.2)}[y C^+ + x \bar{C}]$ ✓	-1	$+\frac{c(1)(1.1)}{(2.2)(1.n)}$ ✓	B_2 ✓
$(A_1 + A_2)(1.1)(2.2)$ ✓	0 ✓	$(B_1 + B_2)(1.1)(2.2)$ ✓	

K		L-M	
$(2.2) [z\bar{c} - x c^+] \checkmark$ $+ (1.1) [x\bar{c} - y c^+] \checkmark$	$\Downarrow$ $\checkmark$	ϕ $\checkmark$	$\frac{c(2)(2.2)^2}{(2 \cdot n)} + \frac{c(1)(1.1)^2}{(1 \cdot n)}$

$$\Rightarrow \left\{ [(2.2)(3.1) + (1.1)(1.2)] \bar{c} - [(2.2)(1.2) + (1.1)(2.3)] c^+ \right\} K$$

$$= \frac{c(2)(2.2)^2}{(2 \cdot n)} + \frac{c(1)(1.1)^2}{(1 \cdot n)}$$

This is an equation K must satisfy. Probably that thing is a projection, because if it isn't then I know K right away! Analyze:

$$a = 0 \checkmark$$

$$b = -[(2.2)x + (1.1)y] \checkmark = +[(y+x)x + (x+z)y] \checkmark = x^2 + 2yx + yz \checkmark$$

$$d = +[(2.2)z + (1.1)x] \checkmark = -[(y+x)z + (x+z)x] \checkmark = -(x^2 + 2zx + yz) \checkmark$$

$$\Rightarrow \det = b^3 + d^3 \checkmark = (x^2 + 2yx + yz)^3 - (x^2 + 2zx + yz)^3 \checkmark$$

$$= [(x^2)^3 + (2yx)^3 + (yz)^3 + 3 \text{ stop.}]$$

$$= [(x^2 + yz) + 2yx]^3 - [(x^2 + yz) + 2zx]^3$$

But $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$, so

$$= \cancel{(x^2 + yz)^3} + 3(x^2 + yz)^2(2yx) + 3(x^2 + yz)(2yx)^2 + (2yx)^3 - \cancel{(x^2 + yz)^3} - 3(x^2 + yz)^2(2zx) - 3(x^2 + yz)(2zx)^2 - (2zx)^3$$

$$= 3(x^2 + yz)^2 2x(y-z) + 12(x^2 + yz)x^2(y^2 - z^2) + 8x^3(y^3 - z^3)$$

$$= 6(x^2 + yz) \underbrace{[(x^2 + yz)x(y-z) + 2x^2(y^2 - z^2)]}_{(x+y)(x+z)} + 8x^3(y^3 - z^3)$$

$$(x+y)$$

$$= 6x(y-z)(x^2 + yz) [x^2 + yz + 2x(y+z)] + 8x^3(y^3 - z^3)$$

$$* x^2 + yz + 2xy + 2xz$$

$$\begin{aligned}
 & 2x(y-z) \left\{ 3(x^2+yz)(x^2+yz+2xy+2xz) + 4x^2(y^2+yz+z^2) \right\} \\
 &= 2x(y-z) \left\{ 3x^4 + 3x^2yz + 6x^3y + 6x^3z + 3y^2x^2 + 3y^2z^2 + 6xy^2z + 6xz^2y + 4x^2y^2 + 4x^2yz + 4x^2z^2 \right\} \\
 &= (2x)(y-z) \left\{ x^4 + 2x^2(y+z) + x^2(2yz+4yz+4y^2+4z^2) + x(2y^2z+2z^2y) + y^2z^2 \right\} \\
 &= (2x)(y-z) \left\{ x^4 + x^2 2(y+z) + x^2(6yz+4y^2+4z^2) + 2yzx(y+z) + y^2z^2 \right\}
 \end{aligned}$$

This det seems not to vanish, I must have goofed. If it is true that it does not vanish, then, I have the solution for K! And Ward null implies $K=0$.

This is extremely important in the overall scheme. Check it again.

(see next page).

$\alpha \neq 0$

Backup Calc on det:

$$\det = (x^2 + 2yx + yz)^3 - (x^2 + 2zx + yz)^3 \quad A = x^2 + 2yx + yz$$

$$= A^3 - B^3 = (A-B)(A^2 + AB + B^2) \quad B = x^2 + 2zx + yz$$

$$= 2x(y-z) \left\{ \begin{array}{l} x^4 + 4x^3y + y^2z^2 + 4x^3y + x^2yz + 4y^2xz \\ + \text{stop} \end{array} \right.$$

$$= (2x)(y-z) [A^2 + AB + B^2] \quad \begin{array}{l} A = C + 2xy \\ B = C + 2xz \end{array} \quad C = x^2 + yz$$

$$= (2x)(y-z) [(C+2xy)^2 + (C+2xy)(C+2xz) + (C+2xz)^2]$$

$$= [C^2 + 4x^2y^2 + 4xyC + C^2 + 2x(y+z)C + 4x^2yz + C + 4xzC + 4x^2z^2]$$

$$= 3C^2 + C[4xy + 4xy + 4xz + 4xz] + [4x^2y^2 + 4x^2yz + 4x^2z^2]$$

$$= 3C^2 + 4xC(y+y+z+z) + 4x^2[y^2 + yz + z^2]$$

$$= 3C^2 + 8x(y+z)C + 4x^2(y^2 + yz + z^2)$$

$$= 3(x^2 + yz)^2 + 8x(y+z)(x^2 + yz) + 4x^2(y^2 + yz + z^2)$$

$$= 3x^4 + 6x^2yz + 3y^2z^2 + 8x^3(y+z) + 8xyz(y+z) + 4x^2(y^2 + yz + z^2)$$

$$\text{Poly} = 3x^4 + 8x^3(y+z) + 2x^2(2y^2 + 2yz + 2z^2 + 3yz)$$

$$+ 8x(yz(y+z)) + 3y^2z^2$$

$\Rightarrow$

$$\det = 2x(y-z) \cdot \text{Poly} \neq 0!$$

Now:

$$a=0$$

$$b=A$$

$$d=-B$$

$$\alpha = AB/(A^3 - B^3)$$

$$\beta = B^2/(A^3 - B^3)$$

$$\delta = A^2/(A^3 - B^3)$$

$$\det = A^3 - B^3$$

$$\Rightarrow \left\{ \begin{array}{l} \alpha \\ \beta \\ \delta \end{array} \right\}^{-1} = \frac{1}{(A^3 - B^3)} [AB + B^2\alpha + A^2\delta]$$

Therefore:

$$K = \frac{1}{(A^3 - B^3)} \{ AB + B^2 C^+ + A^2 C^- \} \cdot \left[\frac{c(2)(2 \cdot 2)^2}{(2 \cdot n)} + \frac{c(1)(1 \cdot 1)^2}{(1 \cdot n)} \right]$$

where $A = (x^2 + yz + 2xy)$

$B = (x^2 + yz + 2xz)$

$$\frac{1}{A^3 - B^3} \left\{ \frac{c(2)(2 \cdot 2)^2}{(2 \cdot n)} [AB + B^2] + \frac{c(1)(1 \cdot 1)^2}{(1 \cdot n)} [AB + A^2] + \frac{c(3)(3 \cdot 3)^2}{(3 \cdot n)} [B^2 + A^2] \right\}$$

$$\Rightarrow K = \frac{A(A+B)}{A^3 - B^3} \frac{c(1)(1 \cdot 1)^2}{(1 \cdot n)} + \frac{B(A+B)}{A^3 - B^3} \frac{c(2)(2 \cdot 2)^2}{(2 \cdot n)} + \frac{B^2 + A^2}{A^3 - B^3} \frac{c(3)(3 \cdot 3)^2}{(3 \cdot n)}$$

Conclusion: In the general inhomogeneous problem where you are trying to find a vertex, the function $K(12) = A_5(12)$ must be what is given above! If a solution exists, then this must be K . If other equations violate this, then there is no solution starting from my starting point.

But this is truly a real mess. I guess I have a poor choice of basis, but OK.

For the WARD NULL problem, however, I may conclude that $K=0!!!!$

The general Ward-Null Problem: , Reverse Trace starting with $K=0$.

1. In section called "Sidetrack", I found that, by studying only equations (n1) and (2n), that $c=0$ implies $K=0$. (Ward null of course). Lets take it backwards from these and see what gives.

2. Obviously from the same equations you may conclude that $(L-M)=0$, actually see the final overlay for quick view of why this is true. In fact, any of those equations on the last overlay tell you this, but especially the (n1) and (2n) equations (either one) because they contain invertible operators! Or look on first page of sidetrack. I have solved exactly for $(L-M)R$ there, and if $c=0$ and $K=0$, then $(L-M)=0$, no doubt about it!

3. Now undo the fourth overlay. Since the operator in the $L+M$ column is invertable in the last two rows (I actually solved this in fourth process section_), you may conclude that $L+M = 0$ also.

So far:***** $K=L=M=0$ for the Ward Null

4. Now look at top of "second process". Since $K,L,M=0$ and $c=0$ and $b=0$, conclude that $T=0$ also!

***** $K,L,M,T = 0$

5. Now just look at the TRKIM chart. Any of the first three rows tells you that $R=0$.

***** $K,L,M,T,R = 0$

6. Finally go back to the original chart. Second row tells you $V=0$. Third row tells you that $S=0$.

***** $K,L,M,T,R,V,S = 0$ and $J=0$ from the start. , last row.

Conclusion: If the RHS of the Ward is null, (ie, if $b=0$ and $c=0$), then the only solution of the Ward given my starting point is: everything zero. Ie, there are no hidden solutions.

n - oldway

Restate all Results

① (3.3) [A(23) + A(31)]

$$= \hat{2} \{ -4d_s A_2 - 4d_1 A_a - 2d_1 A_2 + \frac{2}{3} d_2 A_1 \} \\ + \hat{a} \{ -4d_s A_a - 2d_1 A_2 - \frac{2}{3} d_2 A_1 \}$$

② (3.3) [A(23) - A(31)]

$$= \hat{s} \{ -2d_1 A_1 + 2d_2 A_2 \} \\ + \hat{1} \{ -4d_s A_1 - 2d_1 A_1 - 2d_2 A_2 \}$$

③ (3.3) A(12) = - (y+z) A(12)

$$= \hat{2} \{ 4d_s A_2 - 2d_1 A_a + 2d_1 A_2 - \frac{2}{3} d_2 A_1 \} \\ + \hat{a} \{ -2d_s A_a + 2d_1 A_2 + \frac{2}{3} d_2 A_1 \}$$

④ (y-z) A(12)

$$= \hat{s} \{ -2d_2 A_2 + 2d_1 A_1 \} \\ + \hat{1} \{ 2d_2 A_a - 2d_2 A_2 - 2d_1 A_1 \}$$

(over)

data: $(y-z) = 2d_2$

$(y+z) = 2d_s + 2d_1$

$$\text{Let } \left[\begin{array}{l} (1 \cdot n)_1 = \frac{f_1}{f_2} = -\frac{1}{2}(n \cdot 3) \\ (1 \cdot n)_2 = \frac{f_1}{f_2} = \frac{1}{2}[(n \cdot 1) - (n \cdot 2)] \end{array} \right] \quad (n \cdot 3) = -2f_1$$

$$\textcircled{5} \quad \underline{(3 \cdot n) [A(23) + A(31)]}$$

$$= \hat{2} \left\{ -4f_1 A_0 - 2f_1 A_2 + \frac{2}{3} f_2 A_1 \right\} \\ + \hat{a} \left\{ -2f_1 A_2 - \frac{2}{3} f_2 A_1 \right\}$$

$$\textcircled{6} \quad \underline{(3 \cdot n) [A(23) - A(31)]}$$

$$= \hat{2} \left\{ -2f_1 A_1 + 2f_2 A_2 \right\} \\ + \hat{1} \left\{ -2f_1 A_1 - 2f_2 A_2 \right\}$$

$$\textcircled{7} \quad \underline{[y A(23) + z A(31)]}$$

$$= \hat{2} [2d_3 A_2 + 2d_1 A_0] \\ + \hat{a} [2d_3 A_0 + 2d_1 A_2 + 2d_2 A_1]$$

$$\textcircled{8} \quad \underline{[y A(23) - z A(31)]}$$

$$= \hat{1} [2d_3 A_1 + 2d_2 A_0 + 2d_1 A_1 + 2d_2 A_2]$$

Write down the LH Sides of each of the 20 Ward equations.

1. Later I can put in the RHS's, or if looking for ward null's can just set

RHS = 0 for all terms.

Translate Code :

$$S_2 = S$$

$$S_3 = T$$

$$A = V$$

$$A_1 = R$$

$$A_4 = J$$

$$A_5 = K$$

$$A_6 = L$$

$$A_7 = M$$

[11+22]^a ①

$$-2R_a$$

ϕ

$$2f_1 K_2 + \frac{2}{3} f_2 K_1$$

$$2f_1 L_2 + \frac{2}{3} f_2 L_1$$

$$-2f_1 M_2 - \frac{2}{3} f_2 M_1$$

ϕ

[11+22]² ②

$$-2R_2$$

$$-6T_2$$

$$4f_1 K_a + 2f_1 K_2 - \frac{2}{3} f_2 K_1$$

$$4f_1 L_a + 2f_1 L_2 - \frac{2}{3} f_2 L_1$$

$$-4f_1 M_a - 2f_1 M_2 + \frac{2}{3} f_2 M_1$$

ϕ

[11-22]¹ ③

$$-2R_1$$

$$+2T_1$$

$$-2f_1 K_1 - 2f_2 K_2$$

$$-2f_1 L_1 - 2f_2 L_2$$

$$+2f_1 M_1 + 2f_2 M_2$$

ϕ

[11-22]^s ④

ϕ

$$+2T_s$$

$$-2f_1 K_1 + 2f_2 K_2$$

$$-2f_1 L_1 + 2f_2 L_2$$

$$+2f_1 M_1 - 2f_2 M_2$$

ϕ

$$\frac{1}{a} [12] \textcircled{4}$$

$$-2d_s V$$

$$-2R_a$$

ϕ

$$2f_1 L_2 + \frac{2}{3} f_2 L_1$$

ϕ

$$\frac{1}{2} [12]^2$$

$$-2d_1 V$$

$$-2R_2$$

$$-2T_2$$

$$4f_1 L_a + 2f_1 L_2 - \frac{2}{3} f_2 L_1$$

ϕ

$$\frac{1}{a} [21]$$

$$-2d_2 S_1 + 2d_1 S_2$$

ϕ

$$+2f_1 K_2 + \frac{2}{3} f_2 K_1$$

ϕ

$$\frac{1}{2} [21]$$

$$-2d_2 S_1 - 2d_1 S_2 + 2d_2 S_3$$

$$-4T_2$$

$$4f_1 K_a + 2f_1 K_2 - \frac{2}{3} f_2 K_1$$

ϕ

$$\frac{1}{a} [3]$$

$$-x(z-y) S_1$$

$$+(xy+xz-2zy) S_2$$

$$-2d_s R_a + 2d_1 R_2 + \frac{2}{3} d_2 R_1$$

$$-2d_1 T_2 + 2d_2 T_1$$

ϕ

$$\frac{1}{2} [3]^2$$

$$-x(z-y) S_1 + x(z-y) S_3$$

$$-(xy+xz-2zy) S_2$$

$$-2d_1 R_a - \frac{2}{3} d_2 R_1 + 2d_1 R_2 + 4d_s R_2$$

$$+2d_2 T_1 + 2d_1 T_2 - 2d_2 T_3$$

ϕ

$$\underline{[m_1 + 2n]^a}$$

$$2d_s K_a - 2d_1 K_2 - \frac{2}{3} d_2 K_1$$

$$- 2d_s K_a - 2d_1 K_2 - 2d_2 K_1$$

$$- 4d_s L_a - 2d_1 L_2 - \frac{2}{3} d_2 L_1$$

$$+ 4d_s M_a + 2d_1 M_2 + \frac{2}{3} d_2 M_1$$

$\emptyset$

$$\underline{[n_1 + 2n]^2}$$

$$- 4d_s K_2 + 2d_1 K_a - 2d_1 K_2 + \frac{2}{3} d_2 K_1$$

$$- 2d_s K_2 - 2d_1 K_a$$

$$- 4d_s L_2 - 4d_1 L_a - 2d_1 L_2 + \frac{2}{3} d_2 L_1$$

$$+ 4d_s M_2 + 4d_1 M_a + 2d_1 M_2 - \frac{2}{3} d_2 M_1$$

$\emptyset$

$$\underline{[m_1 - 2n]^1}$$

$$2d_2 K_a - 2d_2 K_2 - 2d_1 K_1$$

$$+ 2d_s K_1 + 2d_2 K_a + 2d_1 K_1 + 2d_2 K_2$$

$$4d_s L_1 + 2d_1 L_1 + 2d_2 L_2$$

$$- 4d_s M_1 - 2d_1 M_1 - 2d_2 M_2$$

$\emptyset$

$$\underline{[n_1 - 2n]^s}$$

$$- 2d_2 K_2 + 2d_1 K_1$$

$\emptyset$

$$2d_1 L_1 - 2d_2 L_2$$

$$- 2d_1 M_1 + 2d_2 M_2$$

$\emptyset$

$$\underline{[nn]^a}$$

$$- 2d_s J_a + 2d_1 J_2 + \frac{2}{3} d_2 J_1$$

$\emptyset$

$$\underline{[nn]^2}$$

$$4d_s J_2 - 2d_1 J_a + 2d_1 J_2 - \frac{2}{3} d_2 J_1$$

$\emptyset$

$$\underline{[n_2 + 1n]^a}$$

$$-2f_1 J_2 - \frac{2}{3} f_2 J_1$$

$$-2d_s K_a - 2d_1 K_2 - 2d_2 K_1$$

$$2d_s L_a - 2d_1 L_2 - \frac{2}{3} d_2 L_1$$

$$2d_s M_a - 2d_1 M_2 - \frac{2}{3} d_2 M_1$$

$$+4d_s M_a + 2d_1 M_2 + \frac{2}{3} d_2 M_1$$

ϕ

$$\underline{[n_2 - 1n]^1}$$

$$2f_1 J_1 + 2f_2 J_2$$

$$2d_s K_1 + 2d_2 K_a + 2d_1 K_1 + 2d_2 K_2$$

$$-2d_2 L_a + 2d_2 L_2 + 2d_1 L_1$$

$$2d_2 M_a - 2d_2 M_2 - 2d_1 M_1$$

$$-4d_s M_1 - 2d_1 M_1 - 2d_2 M_2$$

ϕ

$$\underline{[n_2 + 1n]^2}$$

$$-4f_1 J_a - 2f_1 J_2 + \frac{2}{3} f_2 J_1$$

$$-2d_s K_2 - 2d_1 K_a$$

$$-4d_s L_2 + 2d_1 L_a - 2d_1 L_2 + \frac{2}{3} d_2 L_1$$

$$-4d_s M_2 + 2d_1 M_a - 2d_1 M_2 + \frac{2}{3} d_2 M_1$$

$$+4d_s M_2 + 4d_1 M_a + 2d_1 M_2 - \frac{2}{3} d_2 M_1$$

ϕ

$$\underline{[n_2 + 1n]^s}$$

$$2f_1 J_1 - 2f_2 J_2$$

ϕ

$$2d_2 L_2 - 2d_1 L_1$$

$$-2d_2 M_2 + 2d_1 M_1$$

$$-2d_1 M_1 + 2d_2 M_2$$

ϕ

Prelim Conclusions

$$[11+22]^a \Rightarrow 2R_a = 2f_1(K_2+L_2-M_2) + \frac{2}{3}f_2(K_1+L_1-M_1)$$

$$[11+22]^c \Rightarrow 2R_2+6T_2 = 2f_1(K_2+L_2-M_2) - \frac{2}{3}f_2(K_1+L_1-M_1) \\ + 4f_1(K_a+L_a-M_a)$$

$$[11-22]^t \Rightarrow 2R_1-2T_1 = -2f_1(K_1+L_1-M_1) - 2f_2(K_2+L_2-M_2)$$

$$[11-22]^s \Rightarrow +2T_s = 2f_1(K_1+L_1-M_1) - 2f_2(K_2+L_2-M_2)$$

We can combine these to get:

$$2R_1-2T_1+2T_s = -4f_2(K_2+L_2-M_2)$$

$$2R_1-2T_2-2T_s = -4f_1(K_1+L_1-M_1)$$

$$[12]^a \Rightarrow 2d_s V + 2R_a = 2f_1 L_2 + \frac{2}{3}f_2 L_1$$

$$[12]^c \Rightarrow 2d_1 V + 2R_2 + 2T_2 = 4f_1 L_a + 2f_1 L_2 - \frac{2}{3}f_2 L_1$$

$$[20]^t \Rightarrow (2d_2 S_1 - 2d_1 S_2) = 2f_1 K_2 + \frac{2}{3}f_2 K_1$$

etc etc, so what!

$$[n1+2n]^a \Rightarrow \frac{8}{3}K_1 = 4d_s(M_a-L_a) + 2d_1(M_2-L_2) + \frac{2}{3}d_2(M_1-L_1)$$

$$[n1+2n]^2 \Rightarrow K_2(4d_s+2d_1+2d_s) - \frac{2}{3}d_2 K_1$$

$$= (2d_1+4d_s)(M_2-L_2) + 4d_1(M_a-L_a) - \frac{2}{3}d_2(M_1-L_1)$$

$$[n1-2n]^t \Rightarrow -4d_2 K_a - 2d_s K_1$$

$$= -(2d_1+4d_s)(M_1-L_1) - 2d_2(M_2-L_2)$$

$$[n1-2n]^s \Rightarrow 2d_2 K_2 - 2d_1 K_1$$

$$= -2d_1(M_1-L_1) + 2d_2(M_2-L_2)$$

	S_1	S_2	S_3	R_1	R_2	R_3	T_1	T_2	T_3	K_1	K_2	K_3	L_1	L_2	L_3	M_1
y^a	0	0	0	0	0	-2	0	0	0	0	0	0	$\frac{2}{3}f_2$	$2f_1$	0	0
y^c	0	0	0	0	-2	0	0	-2	0	0	0	0	$-\frac{2}{3}f_2$	$2f_1$	$4f_1$	0
$[n]^a$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$m]^c$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$m2-[n]^s$	0	0	0	0	0	0	0	0	0	0	0	0	$-2d_1$	$2d_2$	0	0
$m2-[n]^1$	0	0	0	0	0	0	0	0	0	$2d_3+2d_1$	$2d_2$	$2d_2$	$2d_1$	$2d_2$	$-2d_2$	$-4d_1-4d_3$
$m2+[n]^a$	0	0	0	0	0	0	0	0	0	$-2d_2$	$-2d_1$	$-2d_3$	$-\frac{2}{3}d_1$	$-2d_1$	$2d_3$	0
$m2+[n]^c$	0	0	0	0	0	0	0	0	0	0	$-2d_3$	$-2d_1$	$\frac{2}{3}d_1$	$-4d_2-2d_1$	$2d_1$	0
$[n1+2n]^a$	0	0	0	0	0	0	0	0	0	$-\frac{8}{3}d_2$	0	0	$-\frac{2}{3}d_2$	$-2d_1$	$-4d_3$	$\frac{2}{3}d_2$
$[n1+2n]^c$	0	0	0	0	0	0	0	0	0	$\frac{2}{3}d_2$	$-2d_1-6d_3$	0	$\frac{2}{3}d_2$	$-2d_1-4d_3$	$-4d_1$	$-\frac{2}{3}d_2$

¹¹ [$\pi_1 - 2\pi$] ¹	0	0	0	0	0	0	0	0	0	2d ₅	0	4d ₂	2d ₁ +4d ₃	2d ₂	0
¹² [$\pi_1 - 2\pi$] ⁵	0	0	0	0	0	0	0	0	0	2d ₁	-2d ₂	0	2d ₁	-2d ₂	0
¹³ [$11+22$] ^a	0	0	0	0	0	-2	0	0	0	$\frac{2}{3}f_2$	2f ₁	0	$\frac{2}{3}f_2$	2f ₁	0
¹⁴ [$11+22$] ²	0	0	0	0	-2	0	0	-6	0	$-\frac{2}{3}f_2$	2f ₁	4f ₁	$-\frac{2}{3}f_2$	2f ₁	4f ₁
¹⁵ [$11-22$] ¹	0	0	0	-2	0	0	2	0	0	-2f ₁	-2f ₂	0	-2f ₁	-2f ₂	0
¹⁶ [$11-22$] ⁵	0	0	0	0	0	0	0	0	2	-2f ₁	2f ₂	0	-2f ₁	2f ₂	0
¹⁷ [21] ²	-2d ₂	-2d ₁	2d ₂	0	0	0	0	-4	0	$-\frac{2}{3}f_2$	2f ₁	4f ₁	0	0	0
¹⁸ [21] ^a	-2d ₂	2d ₁	0	0	0	0	0	0	0	$\frac{2}{3}f_2$	2f ₁	0	0	0	0
¹⁹ δ^a	-x(z-y)	(xy+xz-2zy)	0	$\frac{3}{2}d_2$	2d ₁	-2d ₅	2d ₂	-2d ₁	0	0	0	0	0	0	0
²⁰ δ^2	-x(z-y)	-(xy+xz-2zy)	x(z-y)	$-\frac{3}{2}d_2$	2d ₁ +4d ₅	-2d ₁	2d ₂	2d ₁	-2d ₂	0	0	0	0	0	0
	S ₁	S ₂	S ₅	R ₁	R ₂	R _a	T ₁	T ₂	T ₅	K ₁	K ₂	K _a	L ₁	L ₂	L _a

$-2d_1 - 4d_2$	$-2d_2$	0	0	0	0
$2d_2$	0	0	0	0	0
$-2f_1$	0	0	0	0	0
$-2f_1$	$-4f_1$	0	0	0	0
$2f_2$	0	0	0	0	0
$-2f_2$	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
M_2	M_2	J_1	J_2	J_2	V

M_z	M_a	J_1	J_2	J_a	V
0	0	0	0	0	$-2d_5$
0	0	0	0	0	$-2d_1$
0	0	$\frac{2}{3}d_2$	$+2d_1$	$-2d_5$	0
0	0	$-\frac{2}{3}d_2$	$4d_5+2d_1$	$-2d_1$	0
0	0	$2f_1$	$-2f_2$	0	0
$-4d_2$	$2d_2$	$2f_1$	$2f_2$	0	0
0	$6d_5$	0	0	0	0
0	$6d_1$	0	0	0	0
$2d_1$	$4d_5$	0	0	0	0
$2d_1+4d_5$	$4d_1$	0	0	0	0

Status Report.

1. I have now before me explicitly a system of 20 homogeneous equations in 22 variables. Question: given the power of Reduce and so on, is there not some automatic way to analyze this thing to find the hidden null term?

2. Suppose you had 2 equations in 4 variables. What could you do?

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

I am very unfamiliar with non-square systems of equations. Clearly this equation above has a trivial solution, but I think it may have two? nontrivial solutions. Matrix could be rank 1 or rank 2. But my notes show nothing on this.

How would you solve this? Either these two equations are "coupled" or they are uncoupled. I.e., if x, y appear only in the first, and x and t appear only in the second, then you have something like this:

$$\begin{pmatrix} a & b & 0 & 0 \\ 0 & 0 & c & d \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow$$

$$\begin{aligned} ax + by &= 0 \Rightarrow y = -\frac{a}{b}x \\ cx + dt &= 0 \Rightarrow t = -\frac{c}{d}x \end{aligned}$$

So if they decouple pairwise, you get only a trivial solution? NO. You get two solutions:

$$\begin{pmatrix} x \\ -ax/b \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ z \\ -cz/d \end{pmatrix}$$

$$\begin{pmatrix} b \\ -a \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ d \\ -c \end{pmatrix}$$

Since things are homogeneous, any multiple of a solution is a solution, there is no scale set. So you could take these solutions to be: Even if $a=0$, say, you still have these two solutions.

3. Now suppose the equations are coupled in at least one variable. Then you can use one equation to solve for that variable and substitute that solution into the second equation to get 1 equation in 3 variables. :

$$ax + by + cz + dt = 0, \quad a \neq 0 \Rightarrow x = -\frac{b}{a}y - \frac{c}{a}z - \frac{d}{a}t$$

$$a'x + b'y + c'z + d't = 0 \Rightarrow a'[-\frac{b}{a}y - \frac{c}{a}z - \frac{d}{a}t] + b'y + c'z + d't = 0$$

$$(b' - \frac{a'b}{a})y + (c' - \frac{a'c}{a})z + (d' - \frac{a'd}{a})t = 0$$

$$\Rightarrow t = t(y, z)$$

$$\begin{pmatrix} x(y, z, t) \\ y \\ z \\ t(y, z) \end{pmatrix}$$

You can set y and z independently, so again there are two independent solutions.

Remember that a solution means an un-normalized quad (x, y, z, t) .

4. Is there some standard way to "reduce" a system like this?

$$\begin{bmatrix} x & x & x & x & x \\ x & x & x & x & \otimes \\ x & x & x & x & \otimes \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Since homo, linear combination of LHS's is a valid new equations. Thus, you can take multiple of one row and add it to another, just as in determinant evaluation equivlanceing. (But dont combine columns!)

So lets assume that x_5 appears in at least one of the above equations. (If not then we fill it and start over with lower system). We can combine rows to get matrix into new form: (zero in row).

Now x_5 occurs only in the first equation, so we can set that equation aside and later use it to compute x_5 from other stuff. In other words we make this reduction:

$$\begin{bmatrix} a & b & c & d & e \\ f & g & h & i & \emptyset \\ j & k & l & m & \emptyset \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} \rightarrow \begin{bmatrix} a & b & c & d \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -ex_5 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} f & g & h & i \\ j & k & l & m \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Now we repeat. Assume $m=0$ (a lin comb at start to make it so).

$$\begin{bmatrix} f & g & h & i \\ j & k & l & \emptyset \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \rightarrow$$

$$\begin{bmatrix} f & g & h \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -ix_4 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} j & k & l \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} j & k \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -lx_3 \\ 0 \end{bmatrix}$$

At this point we see that x_1, x_2 may be arbitrarily set, you then compute x_3 , then x_4 , then x_5 and you get your general solution with its two degrees of freedom.

5. Generalize:

Consider this undetermined system:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} & a_{17} & a_{18} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} & a_{27} & \emptyset \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} & \emptyset & \emptyset \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & \emptyset & \emptyset & \emptyset \\ a_{51} & a_{52} & a_{53} & a_{54} & \emptyset & \emptyset & \emptyset & \emptyset \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4 + a_{15}x_5 + a_{16}x_6 + a_{17}x_7 &= -a_{18}x_8 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + a_{24}x_4 + a_{25}x_5 + a_{26}x_6 &= -a_{27}x_7 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + a_{34}x_4 + a_{35}x_5 &= -a_{36}x_6 \\ a_{41}x_1 + a_{42}x_2 + a_{43}x_3 + a_{44}x_4 &= -a_{45}x_5 \\ a_{51}x_1 + a_{52}x_2 + a_{53}x_3 &= -a_{54}x_4 \end{aligned}$$

So you first get the matrix into "semitriangular form", then you set x_1, x_2, x_3 any way you like and compute x_4 through x_8 . There are three solutions.

No sweat, very clear how you do it!

6. By the way, if you can get the matrix to semitriangular form, then you can see that the equations must all be independent.

Conversely, if they are not all independent, then you could have zeroed out an entire row at the beginning and you would then have a different and lower rank problem.

You could check all subdeterminants for independence: ie, if all dubbed vanish, then you do have linear dependence.